Chinese Remainder theorem

Let $n_1, n_2 \ldots n_r$ relatively prime non-zero integers. Consider the linear congruence

$$x \equiv a_1 \pmod{n_1}$$
$$x \equiv a_2 \pmod{n_2}$$
$$\vdots$$
$$x \equiv a_r \pmod{n_r}$$

$$\begin{cases} n = n_1, n_2, \ldots n_r \\ N_k = \dfrac{n}{n_k}. \end{cases}$$

then $\gcd(n_k, N_k) = 1$

$\therefore$ linear congruence $N_k x \equiv 1 \pmod{n_k}$ has a solution and be x_k.

Simultaneous solution of the linear congruences is denoted by $\bar{x} = a_1 N_1 x_1 + a_2 N_2 x_2 + \cdots + a_r N_r x_r$.

Q. $x \equiv 2 \pmod 3$

$x \equiv 3 \pmod 5$

$x \equiv 2 \pmod 7$

$a_1 = 2$; $a_2 = 3$; $a_3 = 2$

$n_1 = 3$; $n_2 = 5$; $n_3 = 7$;

$n = n_1 \times n_2 \times n_3 = 3 \cdot 5 \cdot 7$

$N_1 = \frac{n}{n_1} = 35$

$N_2 = \frac{n}{n_2} = 21$

$N_3 = \frac{n}{n_3} = 15$

Consider the equation

$N_1 x \equiv 1 \pmod{n_1}$

$35x \equiv 1 \pmod 3$

$\underline{\underline{x_1 = 2}}$

$N_2 x \equiv 1 \pmod{n_2}$

$21x \equiv 1 \pmod 5$

$x_2 = 1$ //

$N_3 x \equiv 1 \pmod{n_3}$

$15x \equiv 1 \pmod 7$ $\quad \underline{\underline{x_3 = 1}}$

$\bar{x} = (2 \times 35 \times 2) + (1 \times 21 \times 5) + (1 \times 15 \times 7)$

$\underline{\underline{\bar{x} = 233 \pmod{105}}}$

Q Solve $x \equiv 1 \pmod 3$ $x \equiv 2 \pmod 5$ $x \equiv 3 \pmod 7$

$N = 3 \times 5 \times 7 = 105$

$N_1 = 35$ $N_2 = 21$ $N_3 = 15$

Consider $N_1 x \equiv 1 \pmod{n_1}$

$35x \equiv 1 \pmod 3$

$\underline{\underline{x_1 = 2}}$

$\cancel{35 \equiv 2 \pmod 5}$ $\quad 21 \equiv 2 \pmod 5$

$\underline{\underline{\cancel{x_2 = 5}}}$ $\quad \underline{\underline{x_2 = 1}}$

$\overline{35}$.

$15 \equiv 3 \pmod 7$

$x_3 = 1$ //

$$\bar{x} = a_1 n_1 x_1 + a_2 n_2 x_2 + a_3 n_3 x_3$$
$$= 157 \pmod{105}$$

First Order linear Recurrence Relation

A first order linear recurrence relation is of the form

$$a_{n+1} = d a_n + e$$

Q Find a recurrence relation corresponding to following sequence.

$2, 6, 18, \ldots$

$a_0 = 2 \quad a_1 = 3a_0 \quad a_2 = 3a_1$

$\ldots$

$a_{n+1} = 3a_n \;;\; n = 0, 1, 2 \ldots$

$a_0 = 2$

Q $2, 10, 50, \ldots$

$a_{n+1} = 5a_n \;;\; n = 0, 1, \ldots$

$a_0 = 2$

Q $7, 14/5, 28/25, \ldots$

$a_0 = 7$

$a_{n+1} = 2/5 \times a_n$

Solution of recurrence Relation

Consider the recurrence relation,

$a_{n+1} = d a_n, \; a_0$

$a_1 = d a_0$

$a_2 = d a_1 = d^2 a_0$

$a_3 = d a_2 = d^3 a_0$

$$\boxed{a_n = d^4 a_0}$$

Q $a_n = 7a_{n-1} \;;\; n \geq 1 \;:\; a_2 = 98 \quad d = 7$

$a_2 = 7^2 \cdot a_0 \qquad 98 = 49 \cdot a_0$

$a_0 = 2$ //

$a_n = 7^n \times 2$

Q Solve $3a_{n+1} - 4a_n = 0$; $n \geq 0$; $a_1 = 5$

$3a_{n+1} = 4a_n$

$a_{n+1} = \frac{4}{3} a_n \qquad d = 4/3$

Sol^n is $a_n = d^n a_0$

$a_1 = d^1 a_0$

$5 = \frac{4}{3} \cdot a_0$

$a_0 = \frac{15}{4}$

$\therefore \ a_n = \cancel{\frac{15}{4}} (4/3)^n \cdot \frac{15}{4}$

$2a_n - 3a_{n-1} = 0$; $n \geq 1 \quad a_4 = 81$

$2a_n = 3a_{n-1}$

$a_n = \frac{3}{2} a_{n-1}$

Sol^n is

$a_4 = d^4 a_0$

$81 = \left(\frac{3}{2}\right)^4 a_0$

$81 = \cancel{\frac{81}{16}} \cancel{a_0} \ \frac{16}{81} a_0$

$a_0 = 16$

$a_n = (3/2)^n \cdot 16$

Q Given $a_{n+1} - d a_n = 0 \quad a_3 = \frac{153}{49} \quad a_5 = \frac{1377}{2401}$

Find d.

Sol^n is $a_n = d^n a_0$

$a_3 = d^3 a_0$

$a_5 = d^5 a_0$

$d^2 = \frac{1377}{2401} \times \frac{49}{153}$

$d = 3/7$

Q A bank pays 6% annual ~~percente~~ interest compounding montly if you deposit 1000 dollars on first day of may. How much will you get after an year?

Let P_0 be the amount deposited.

Montly Interest is given as $6/12 = .5\%$

$= .5/100 = .005$

Montly rate $= .005$

$P_1 = P_0 + .005 P_0 = (1.005) P_0$

$P_2 = P_1 + .005 P_1 = (1.005) P_1$

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Recurrence relation is

$P_n = d^n P_0$

$P_n = (1.005) P_0$

$P_{12} = (1.0005)^{12} \times 1000$

$= 1061.677$

Second order Linear Homogeneous recurrence relation with constant coefficient

A linear recurrence relation of order k with constant coefficients is of the form $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = f(n)$

Eg:

order 1 : $C_0 a_n + C_1 a_{n-1} = f(n)$

order 2 : $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n)$

if $f(n) = 0$

$C_0 a_{n-1} + C_1 a_{n-1} + C_2 a_{n-2} = 0$

which is homogeneous second order linear equation

Suppose $a_n = kr^n$ is a soln —— (1)

$C_0 kr^n + C_1 kr^{n-1} + C_2 kr^{n-2} = 0$ ——

$\div kr^{n-2}$ $\boxed{C_0 r^2 + C_1 r + C_2 = 0}$

characteristics equation of (1)

(i) Real distinct

$r = r_1, r_2$ be real & distinct

Then sol^n are $a_n = kr_1^n$; $a_n = kr^n$

General sol^n is $a_n = Ar_1^n + Br_2^n$

Q Solve $a_n + a_{n-1} - 6a_{n-2} = 0$ $n \geq 2$; $a_0 = -1, a_1 = 8$

Characteristics equation is $r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

$r = -3, 2$

General sol^n is $a_n = A(-3)^n + B(2)^n$

$a_0 = -1 = A + B \longrightarrow (1)$

$a_1 = 8 = 3A + 2B \longrightarrow (2)$

$3A + 3B = -3$

$-3A + 2B = 8$

$5B = 5$

$B = 1$

$A + 1 = -1$

$A = -1 - 1$

$A = -1 - 1$

$= -2$

Q $a_n = 5a_{n-1} + 6a_{n-2}$ $n \geq 2$; $a_0 = 1$; $a_1 = 3$

Characteristics equation is

$a_n - 5a_{n-1} - 6a_{n-2} = 0$

$r^2 - 5r - 6 = 0$

$\cancel{r^2 - 2r - 3r - 6}$

$\cancel{r(r-2) - 3(r-2)}$

$\cancel{r = 3, 2}$

$r^2 - 6 + 1 - 6$

$r(r-6) + 1(r-6)$

$r = 6, -1$

General sol^n is $a_n = A(6)^n + B(-1)^n$

$a_0 = 1 = A + B \longrightarrow (1)$

$a_1 = 3 = 6A - B \longrightarrow (2)$

$A + B = 1$

$6A - B = 3$

$A = 4/7$

$B = 3/7$

$6A + 6B = 6$

$6A - B = 3$

$7B = 9$

$B = 9$

Case (ii) Distinct complex roots

Solve $a_n = 2(a_{n-1} - a_{n-2})$; $n \geq 2$, $a_0 = 1$, $a_1 = 2$

$$a_n - 2a_n - 1 + 2a_{n-2} = 0$$

Characteristic eqn is $r^2 - 2r + 2 = 0$

$$r = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \mp 2i}{2} = 1 \pm i$$

$$a_n = A(1+i)^n + B(1-i)^n$$

$$a_n = A\left[\sqrt{2}\cos\frac{\pi}{4} + i\sqrt{2}\sin\frac{\pi}{4}\right]^n +$$

$$B\left[\sqrt{2}\cos(\pi/4) - \sqrt{2}\sin(\pi/4)\right]^n$$

$$= A(\sqrt{2})^n\left[\cos n\pi/4 + i\sin n\pi/4\right] + B(\sqrt{2})^n$$

$$\left[\cos n\pi/4 - \sin n\pi/4\right]$$

$3 = x + iy$

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1} y/x$

$3 = r\cos\theta + ir\sin\theta$

$1 + i = \sqrt{2}\cos \pi/4 + i\sqrt{2}\sin\frac{\pi}{4}$

$1 - i = \sqrt{2}\cos -(\pi/4) + i\sqrt{2}\sin -\frac{\pi}{4}$

$(\cos\theta + i\sin\theta)^n =$

$\cos n\theta + i\sin n\theta$

$$= \left[-1(\sqrt{2})^n + B(\sqrt{2})^n\right]\cos n\pi/4 + A(\sqrt{2})^n -$$

$$B(\sqrt{2})^n]\, i\sin\frac{n\pi}{4}$$

$$= (\sqrt{2})^n k_1 \cos\frac{n\pi}{4} + (\sqrt{2})^n k_2 \sin\frac{n\pi}{4}$$

$$= (\sqrt{2})^n\left[k_1\cos\frac{n\pi}{4} + k_2\sin\frac{n\pi}{4}\right]$$

$a_0 = 1 \quad a_1 = 2$

$a_0 = 1 = k_1\cos 0 + k_2\sin 0 = 1 = k_1$

$a_1 = 2 = \sqrt{2}\left[k_1\cos\pi/4 + k_2\sin\pi/4\right]$

$2 = \sqrt{2}\left[k_1 \cdot 1/\sqrt{2} + k_2 \cdot 1/\sqrt{2}\right]$

$2 = k_1 + k_2 \implies k_2 = 1$