MODULE - 3

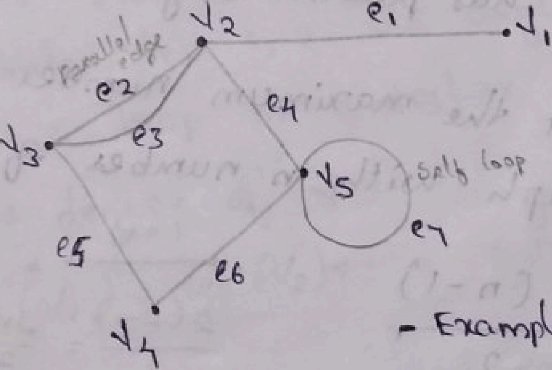
18/9/2023 Monday

* Graph Theory

Let V be a finite non empty set and let $E \subset V \times V$ the pair (V, E) is then called a directed graph or digraph on V, where V is the set of vertices or nodes and E is it's edges or arches we write $G = (V, E)$ to denote such a graph.

~~Line~~

When there is no concern about the direction of any edge, G is called an undirected graph.



//For self loop the edges is considered as 2.

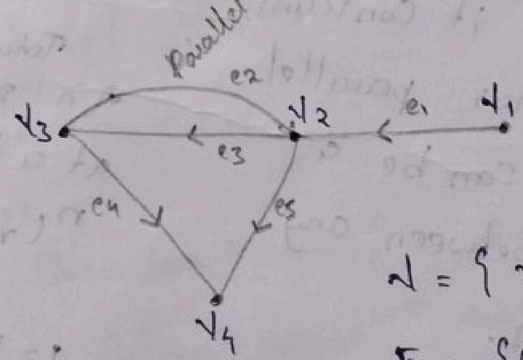
v_1 can be connected in both direction ie; edges of self loop is always 2.

- Example of undirected graph -

$G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

Subset of $(V \cdot V)$



- Example of directed graph -

$V = \{v_1, v_2, v_3, v_4\}$

$E = \{(v_1, v_2), (v_2, v_3), (v_3, v_2), (v_3, v_4), (v_2, v_4)\}$

For any edge (a,b) in a graph G, we say that the edge is incident with the vertices a and b; a is said to be adjacent to b. The edge (a,a) is called a self loop. If a vertex 'a' has no incident edges then it is called an isolated vertex.

If more than one edge associated with a given pair of vertices, the edges are referred to as parallel edges.

A graph has neither self loops nor parallel edges is called a simple graph.

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A graph that has parallel edges is called a multi-graph.

Q. Prove that the maximum number of edges in a simple graph with n number of vertices is

$$^nC_2 = \frac{n(n-1)}{2}$$

$\frac{2(2-1)}{2} = \frac{2\times1}{2} = 1$

$^nC_r =$

ans). Let 'G' be a simple graph with n vertices.

Since 'G' is simple it contains neither self loops nor parallel edges, hence there can be a maximum 1 edge between any two distinct vertices.

Hence the no. of edges with n vertices $= \frac{n(n-1)}{2}$

when we select 'r' objects from 'n' objects, each selection is called a combination.

Total no. of selections of 'n' objects taken r at a time is denoted by

$$^nC_r = \frac{n(n-1)\dots(n-(r-1))}{r!}$$

$1 = 1!$
$0! = 1$
$2! = 1\times2$
$3! = 1\times2\times3$
$n! = 1\times2\times\dots\times n$

The number of edges incident on a vertex with self loops counted twice, is called degree of the vertex.

Q. Prove that the sum of degrees of all vertices of a graph is twice the number of edges. [Handshake theorem]

ans) Let 'G' be a graph with 'n' vertices $v_1, v_2 \dots v_n$ and m edges

we have to compute:

$$deg(v_1) + deg(v_2) + \dots deg(v_n).$$

Each edge 'e' is either a loop or incident with two distinct vertices.

If 'e' is incident with two distinct vertices 'u' and 'v', then 'e' contributes 1, 2 the degree of each vertex.

If 'e' is a loop on a vertex 'v', then e contributes two(2) to the degree of 'v'.

$$\therefore deg(v_1) + deg(v_2) + \dots + deg(v_n) = 2m$$

$^5C_2 \Rightarrow \frac{5\times4}{1\times2} = 10$

$\Rightarrow {}^7C_3 = \frac{7\times6\times5}{1\times2\times3} = 35$

(loop gives 2 degree)

Q. Prove that the no. of vertices of odd degree is always even.

ans) Let $v_1, v_2 \dots v_k$ be ~~any~~ k vertices with even degree and $u_1, u_2 \dots u_p$ are the vertices with odd degree.

Then we know that:

$$\underbrace{deg(v_1) + deg(v_2) \dots + deg(v_k)} + \underbrace{deg(u_1) + deg(u_2) \dots + deg(u_p)} = 2m$$

where, m is the number of edges.

Sum of even numbers is always even.

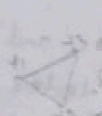
Therefore $\deg(u_1) + \deg(u_2) \cdots + \deg(u_p) = 2m$ - even number

$\therefore$ This is possible only when the number of terms in the left side is even.

$\therefore p$ must be even

Q. How many edges are there in a graph with 10 vertices each of degree is 6.

ans) I know that sum of the degrees of the vertices equal to twice the numbers of edges.



$10 \times 6 = 2m$

$m = 60/2 = \underline{30}$ edges.

Q. A graph has 21 edges, three vertices of degree 4 and other vertices of degree 3. Find the number of vertices in G.

ans) Let n be the number of vertices.

We know that the sum of the degrees of the vertices equal to twice the number of edges.

$\therefore 3 \times 4 + (n-3)3 = 2 \times 21 \quad (2m)$

$12 + 3n - 9 = 42$

$3n = 49 + 9 - 12$

$3n - 9 = 30 \quad , 3n = 30 + 9 = 39$

$n = \frac{39}{3} = \underline{\underline{13}}$.

Q. A graph has 8 edges find the number of vertices if the degree of each vertex is two.

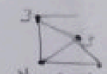
ans) Let vertex be n

then $2 \times n = 2 \times 8$

$2n = 16$

$n = 16/2 = \underline{8}$

Q. Does there exist a simple graph with degree sequence 2, 2, 4, 6.



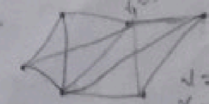
ans) Here there are 4 vertices

Since 'G' is simple, it contains neither loop nor parallel edges. Hence there can be a maximum 1 edge between any two distinct vertices. Hence the maximum number of degree of a vertex can be 3.

So the graph with degree sequence 2, 2, 4, 6 not possible.

Q. Does there exist a simple graph with degree sequence, 3, 3, 4, 5, 6 6.

ans) We know that the number of vertices of odd degree is always even, Here only three vertices have odd degree. So a graph with degree sequence, 3, 3, 4, 5, 6 6 is not possible.



Q. Does there exist a simple graph with 7 vertices having degree sequence 1,2,3,4,5,6,6

ans)

Here there are two vertices with degree 6. This is possible only when, these two vertices are adjacent to other 5 vertices, Hence the graph cannot contain a vertex of degree 1.

★ Incidence Matrix

Let 'G' be a graph with n vertices, $v_1, v_2, \ldots v_n$ and m edges $e_1, e_2 \cdots e_m$.

Define an n by m $(n \times m)$ matrix.

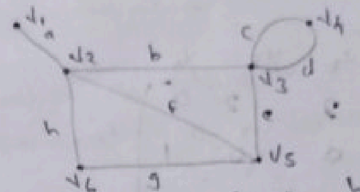
$A = [a_{ij}]$

whose 'n' rows corresponds to 'n' vertices, and the 'm' column corresponds to 'm' edges, as follows.

~~a[ij]~~

$$a_{ij} = \begin{cases} 1 & \text{if } j^{th} \text{ edge } e_j \text{ incident on the } i^{th} \text{ vertex } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Q. Write the incidence matrix of the following graph.

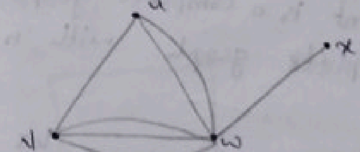


$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

★ Adjacency Matrix.

Let 'G' be a graph with 'n' vertices is adjacency matrix 'A' is the $n \times n$ matrix whose $(i,j)^{th}$ entry is the number of edges joining vertex 'i' to vertex 'j'.

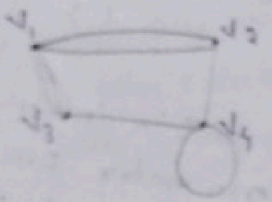
Q. Find the adjacency matrix of the following graph:



$$\begin{array}{c} \\ u \\ v \\ w \\ x \end{array} \begin{array}{c} \begin{array}{cccc} u & v & w & x \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Q. Draw a graph a represented by the adjacency matrix.

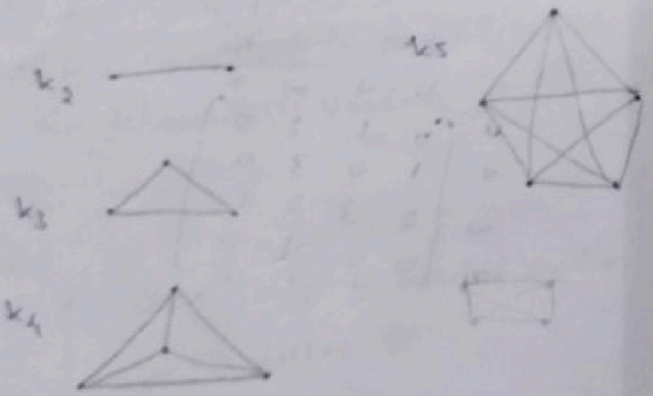
$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 2 & 2 & 0 \\ v_2 & 2 & 0 & 0 & 1 \\ v_3 & 2 & 0 & 0 & 1 \\ v_4 & 0 & 1 & 1 & 1 \end{array}$$



(degrees of all vertices are same)

* Complete Graph (each vertex is connected to all other vertex)

A simple graph in which each pair of distinct vertices are adjacent is a complete graph. we denote a complete graph with 'n' vertices by K_n



- A complete graph is always a regular graph).

Q. Prove that the no. of edges in a complete graph K_n is $\frac{n(n-1)}{2}$. ~~the sum~~

The number of edges in a complete graph is the number combination of n vertices taken 2 at a time

ie; $nC_2 = \frac{n(n-1)}{2}$ ∴

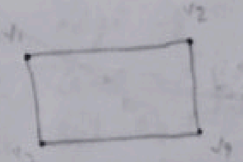
* Regular graph.

A graph in which each vertex has the same degree is a regular graph.

If each vertex has degree 'r', the graph is regular of degree 'r' or 'r' regular.

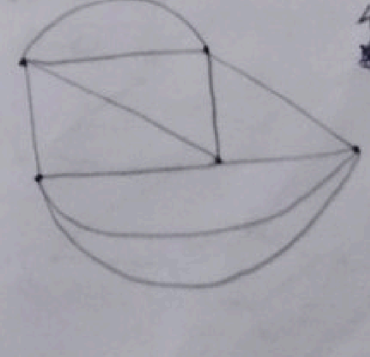
Eg: K_n is a regular graph with degree $n-1$

Q. Give an example of a regular graph which is not complete.



2-regular. Not all vertex are connected to each other

(v_1 - v_3 no connection)



4 regular. not (complete graph).

Q. Is any 4 - regular graph with 15 edges.

ans) Suppose there are 'n' vertices we know that the sum of the degrees of the vertices equal to twice the number of edges.

$\therefore 4 \times n = 2 \times 15$

$n = 15/2 = 7.5$

Such a graph does not exist.

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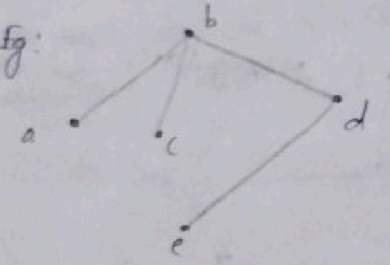
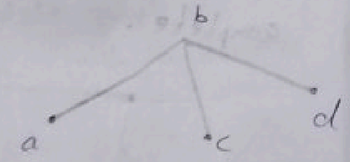
★ Sub graph

A graph G' is said to be a sub graph of a graph G if all the vertices and all the edges of G' are in G, and each edge of G' has the same end vertices in G' as in G.

Eg:



- G -



(Sub graph)

- G' -

(Subgraph by removing vertex)

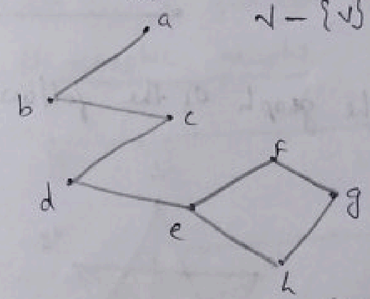
⇒ Let v be a vertex in a graph $G = (V, E)$. The sub graph of G, denoted by $G - v$ has the vertex set;

$V_1 = V - \{v\}$

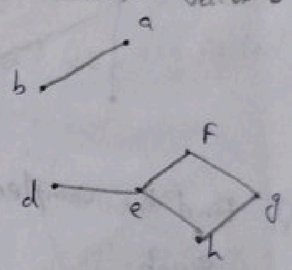
and the edge set;

$E_1 \subset E$, where E_1 contains all the edges in E except for those that are incident with vertex v.

G　　$G - v$, $V - \{v\}$



$G - c$ (Subtracting vertex 'c')

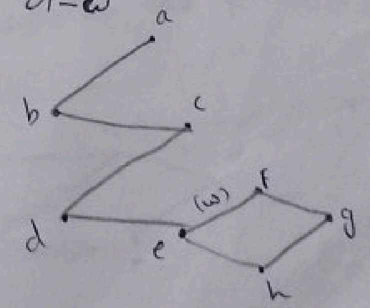


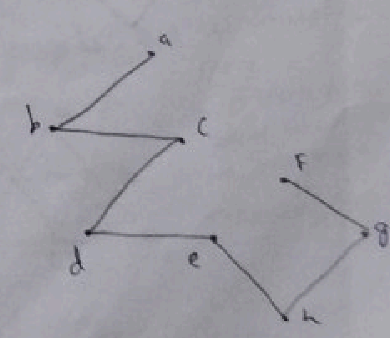
- Subgraph -

(Subgraph by removing edges)

⇒ If 'e' is an edge of a graph $G = (V, E)$, we obtain a subgraph $G - e = (V_1, E_1)$ of G. where the set of edges $E_1 = E - \{e\}$ and the vertex set is unchanged.

Eg: $G - \omega$



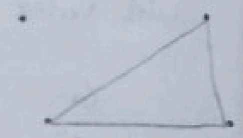


* <u>Complement of a graph</u>

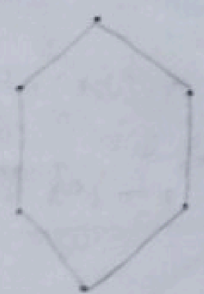
Let G be a loop free graph on 'n' vertices. The complement of G, denoted by $\overline{G}$, is the sub graph of K_n consisting of the n vertices in G and all edges that are not in G.

G

$\overline{G}$



Q. Find the complement of the graph of the following graph.



(To form a complete graph)

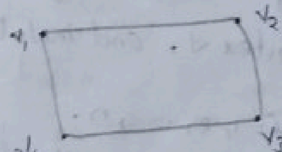
* <u>Isomorphism between graphs</u>

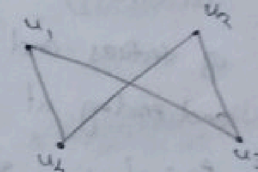
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, a function from $f: V_1 \rightarrow V_2$ is called a graph isomorphism and if; (a) f is one-to-one and onto

(b) for all $a, b \in V_1$,
edge $\{a, b\} \in E_1$ if and only if
$\{f(a), f(b)\} \in E_2$. (2nd graph)

when such a function exists G_1 and G_2 are called <u>isomorphic graphs</u>.

Eg:



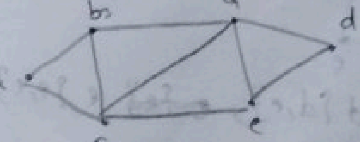


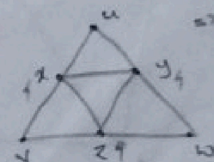
$- G_1 -$

$f(v_1) = u_1$, $f(v_2) = u_3$,
$f(v_4) = u_4$, $f(v_3) = u_2$

Q. Check whether the following graphs are isomorphic.

G_1





G_2

CHECK
=> No. of vertices & edges are same in both graph
=> check degree

In the first Graph there exist two vertices 'c' and 'f' with degree 4.

But in the second graph there are three vertices with degree 4.

Hence the graphs are not isomorphic.

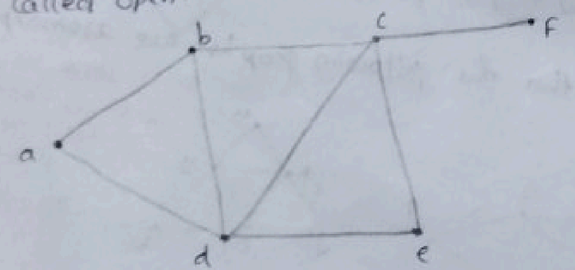
26/9/2023 Tuesday

★ walk, trail, circuit, path.

Let u, v be two vertices in an undirected graph $G = (V, E)$. And u-v walk in G is a finite alternating sequence, $u = v_0, e_1, v_1, e_2 \cdots, e_n, v_n = v$ of vertices and edges from G, starting at the vertex u and ending at the vertex v, and involving n edges

$e_i = \{v_{i-1}, v_i\}$, $i = 1, 2, \ldots, n$.

The length of this walk is 'n', the number of edges in the walk. Any u-v walk where u = v is called as closed walk. Otherwise the walk is called open.



1. a, {a,b}, b {b,d}, d {d,e}, ~~e {e}~~ e {e,c}, c {c,d}

d {d,b}, b

$a \to b \to d \to e \to c \to d \to b$

a-b walk of length 6 in which the edge {b d} - {d b} repeats.

2) $F \to c \to e \to d \to c \to b$

F-b walk of length 5, in which the vertex c repeats.

3) $a \to d \to e \to c \to b$ (trail)

a-b walk of length 4, in which no edge or no vertex repeats.

Consider any u-v walk in an undirected graph $G = (V, E)$;

(a) If no edge in u-v walk is repeated, then the walk is called an u-v trail.

A closed u-v trail is called a Circuit.

(b) If no vertex of the u-v walks occur more than once, then the walk is called an u-v path.

When u = v, the term cycle is used to describe such a closed path.

walk → trail (no repeat edge) → circuit (closed trail)

→ path → cycle

*** Result**

i) Let $G = (V, E)$ be an undirected graph, with a, b belongs to V $(a, b \in V)$, $a \neq b$.
If there is a trail (edges not repeated) from a to b, then there is a path from a to b.

\- component.

*** Connected graph** (A connected graph is a complete graph) $\triangle\ \triangledown$ (not connected)

Let $G = (V, E)$ be an undirected graph. We call G connected if there is a path between any two distinct vertices of G. Otherwise G is disconnected. A disconnected graph consists of two or more connected sub graphs. Each of these connected subgraphs is called a component. For any graph G, the number of components of G is denoted by $k(G)$.

*** Result**

ii) If G is a graph in which the degree of each vertex is atleast two then G contains a cycle (closed path).

*** Euler Graph**

Let $G = (V, E)$, be an undirected graph or multi-graph with no isolated vertices. The G is said to have an **Euler Circuit** (no edges are repeated, vertex can be repeated), if there is a circuit in G that traverses every edge of the graph exactly once.
If there is an open trail from a to b in G and this trail traverses each edge in G exactly once, the trail is called an **Euler trail**.

$\Rightarrow$ implies
$\Leftarrow$ implied by
$\Leftrightarrow$ if and only if (iff)

*** Result**

imp Ques.

iii) Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. The G has an Euler Circuit if ~~any~~ and only if G is connected and every vertex in G has even degree.

$\Rightarrow$ **Proof**

3/10/2023 Tuesday

Let G be a graph with an euler circuit. Then for all a, b $\in V$ (belongs to) there is a trail from a to b ~~wit~~ which is the path of the euler circuit that starts at a and terminate at b.
Hence there exist a path from a to b. Thus connecting any two vertices in G; there exist a path in G, and hence G is connected.
Let 's' be the starting vertex of the euler circuit. For any other vertex 'v' of G, each time the circuit comes to the, it then departs from the vertex. Thus the circuit has traversed either two edges that are incident ~~in~~ with 'v' or a loop at 'v'.

In either case a count of two is contributed to degree of v. Since v is the not the starting point and each edge incident to v is traversed only once, a count of two is obtained each time the circuit passes through v, So the degree of v is even.

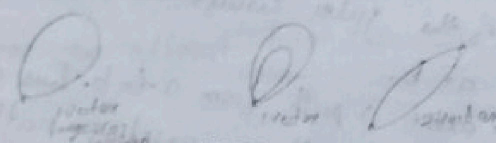
As for the starting vertex S, the first edge of the circuit must be distinct from the last edge, ~~but~~ and because of any other visit to S result in a count of two, for degree of S, we have degree of S is even.

Conversely, Let G be a connected graph with every vertex of even degree.

n = 1 (edge)

n = 2 (edge)

If the no. of vertex in G is one or two, the graph contains an euler circuit.

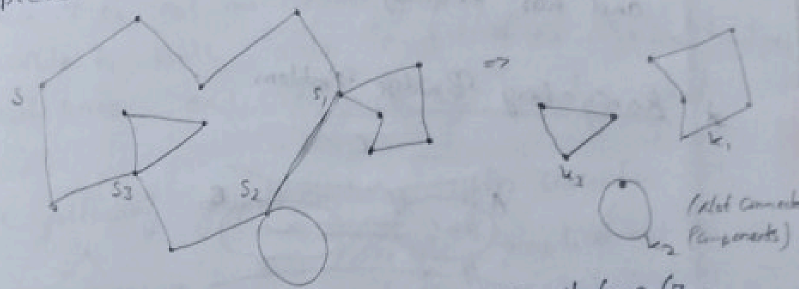


We proceed now by induction of the no. of edges of G.
Let 'n' be a positive integer.
Assume that any connected graph with 'k' edges, $1 \le k < n$, in which every vertex ~~has~~ with even degree has an euler circuit.

~~Let S be the~~

If G has 'n' edges, select a vertex 'S' in G as a starting point to build an euler circuit. The graph G is connected and each vertex has even degree, there exist a circuit 'C' containing 'S'. If 'C' contains every edge of G, the proof is complete.

by removing edges & isolated vertices



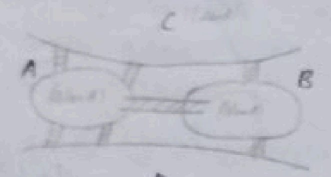
If not, remove the edges of the circuit from G, making sure to remove any vertex that would become isolated.
The remaining subgraph 'K' has all vertices of even degree, it but it may not be connected.
By the induction hypothesis, each component of 'K' is connected and will have euler circuit.
Each of these euler circuit has a vertex that is on 'C'. Consequently, starting at 'S' we travel on 'C' until we arrive at a vertex 'S_1' that is on the euler circuit of the component 'K_1' of 'K', continue on 'C' until we reach a ~~vertices~~ vertex to that is on the euler circuit of the component 'K_2' of 'K'.

Since the graph is finite, as we continue this process, we construct an euler circuit for G.

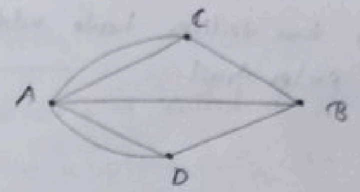
*** Result**

If G is an undirected graph or multi graph with no isolated vertex, then we can construct an euler trail on G if and only if G is connected and has exactly two vertices of odd degree.

*** Konigsberg Bridge Problem**



Two islands, C and D, formed by Pregel river in konigsberg where connected to each other and to the banks A and B with 7 bridges as shown in the figure, the problem was to start at any four land areas of the city A, B, C, D, walk over each of the 7 bridges exactly once, and return to the starting point.

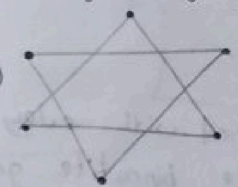


Konigsberg Problem is equivalent to asking whether the graph in the above figure has an euler circuit, we find that not all these vertices are of even degree.

Hence it is not an euler graph, thus it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

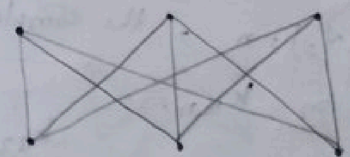
*** Q. Is the following graph contains euler circuit.**

a)



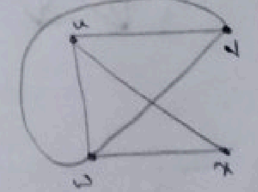
ans) Since the graph is not connected it does not contain ~~eat~~ euler circuit.

b)



ans) All vertices are of odd degree, hence this graph does not contain an euler circuit.

Q. Prove that the following graph, contain no euler circuit, but an euler trail.



ans) Here $deg(u) = 3$, $deg(v) = 3$
$deg(w) = 4$, $deg(x) = 2$

Since sum of the vertices have odd degree, there is no euler circuit.

Since exactly two vertices have odd degree the graph has an euler trail.

★ Bipartite Graph

A graph $G = (V, E)$ is called bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$, and every edge of G is of the form $\{a, x\}$ where a belongs to V_1 and x belongs to V_2.

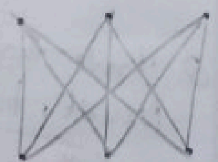
$a \in V_1$
$x \in V_2$.

Eg:

If each vertex in V_1 is joined with every vertex in V_2 we have a complete bipartite graph, if $|V_1| = m$ and $|V_2| = n$, the complete bipartite graph is denoted by $K_{m,n}$.

Eg: K_{32}

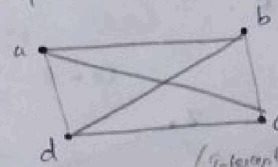
K_{33}

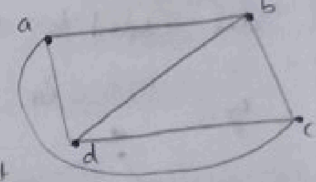


★ Planar Graphs

A graph G is called a planar graph if G can be drawn in a plane with it's edges intersecting only at vertices of G, such a drawing is called an embedding of G in the plane.

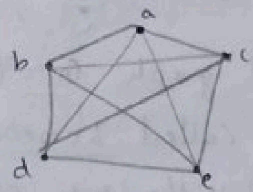
Eg. K_4 is planar

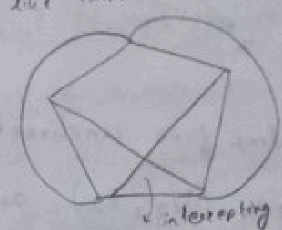




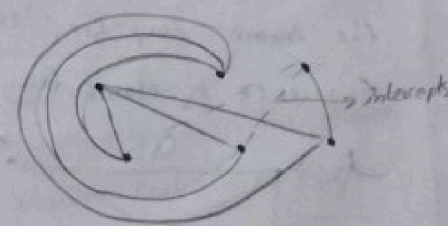
(Intercepting but we can draw edge like this)

$\Rightarrow K_5$ is not planar





$\Rightarrow K_{33}$



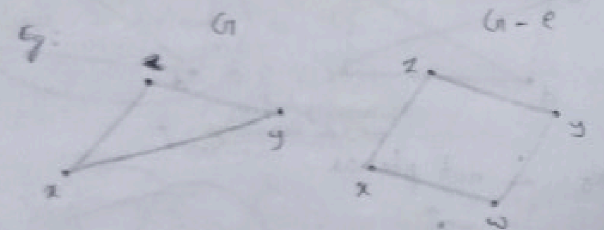
So K_{33} is non planar.

9/10/2023
Monday

* Elementary Sub division of a graph.

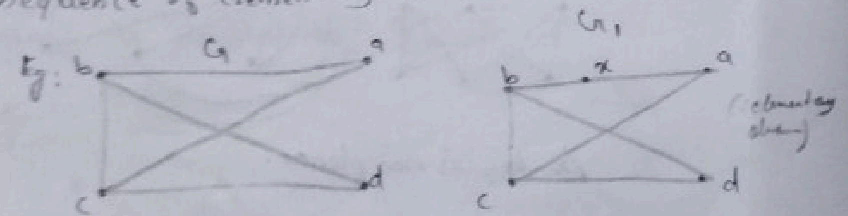
Let $G = (V, E)$ be a loop free undirected graph, where $E \neq \phi$ and elementary subdivision of G results when an edge $e = \{x, y\}$ is removed from G and then the edges $\{x, w\}$ and $\{w, y\}$ are added to $G - e$.

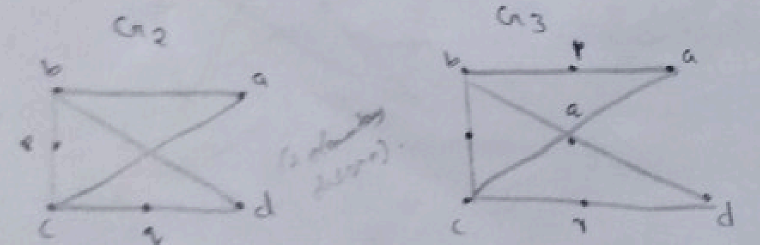
Eg:



The loop free undirected graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic if they are isomorphic or if they are obtained from the same loop free undirected graph G, by a sequence of elementary division.

Eg:





Here G_1, G_2, G_3 are homeomorphic, as they are derived (obtained from G), by a sequence of elementary sub division.

* Kuratowski's Theorem

5 marks Ques

A graph is non planar if and only if it contains a subgraph that is homeomorphic to either K_5 or K_{33}

* Region

The planar representation of a graph divides the plane of paper into several regions called windows or phases.

* Euler's theorem

Let $G = (V, E)$ be a connected planar graph with cardinality of V_2 (edges of vertices & regions)

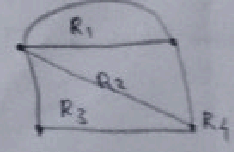
$|V| = v$ and $|E| = e$

Let R be the number of regions in the plane determined by the planar embedding of G.

Then,

$$\boxed{v - e + r = 2}$$

Eg:



$v = 4$
$e = 6$
$r = 4$

$\Rightarrow 4 - 6 + 4 = 2$

★ **Result**

Let $G=(V,E)$ be a loop free connected planar graph with cardinality of $V = v$

ie; $|V| = v$ and $|E| = e > 2$

and r regions.

Then;

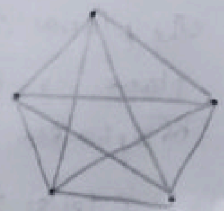
i) $3r \le 2e$

ii) $e \le 3v - 6$

Q. Prove that k_5 is non-planar.

(Contradiction)

k_5



$e = 10,\ v = 5$

In k_5 we have

we know that for a planar graph $\underline{e \le 3v - 6}$

here $3v - 6 = 3 \times 5 - 6 = 9$

$e = 10$, $= 10 \nleq 9$

ie; $e \nleq 3v - 6$

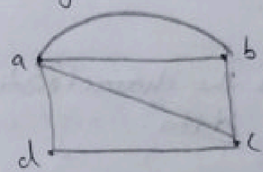
Therefore k_5 is not planar.

★ Hamilton Circuit and Hamilton Cycle

A hamilton's circuit or cycle in a connected graph is defined as a closed walk that traverses every vertex of G exactly once, except the starting vertex at which the walk also terminates.

A hamiltonian graph is a graph which contains, a hamiltonian cycle.

Eg:



$a \to b \to c \to d \to a$

write

Q. Give an example of a connected graph that has.

i) neither an euler circuit nor a hamilton cycle.

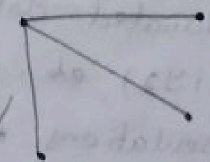
ii) An euler circuit but no hamilton cycle.

iii) A hamilton cycle ~~by~~ but no euler circuit.

iv) Both hamilton cycle and euler circuit. □ △

ans) i)

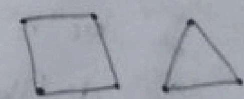
Connected but no circuit/cycle



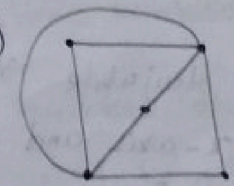
iii)

odd degree No euler

iv)



ii)



[even degree]

euler circuit.

4/10/2023 Saturday

★ Module - 5

=> Bivariated data.

$X \rightarrow x_1, x_2 \dots x_n$

$Y \rightarrow y_1, y_2 \dots y_n$.

★ Correlation and regulation

• Bivariated data.

Data based on the characteristics of two variables are called bivariated data.

• Correlation

Correlation refers to a process for establishing the relationships between variables.

• Scattered diagram

It is a simplest way of the diagramatic representation of bivariated data.

imp 6 mark ques

Let $(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$ be 'n' pair of observations of a bivariated distribution x, y.

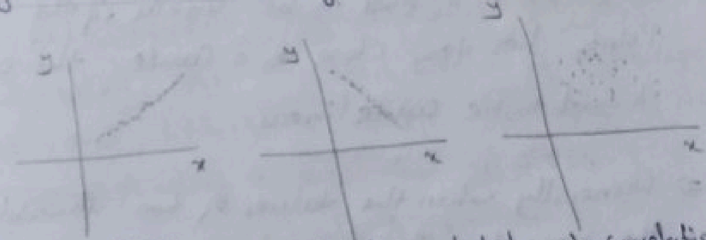
If the values of the variable X and Y are marked along x-axis and y-axis respectively and if we plot the points $x_1, y_1, x_2 y_2 \dots x_n y_n$ in the xy plane, the diagram of ~~so~~ points so obtained is known as Scattered diagram

If these points show some trends either upward or downward these two variables are ~~said to be~~ correlated. If the ~~dotted~~ plotted points do not show any trend, these two variables are not correlated.

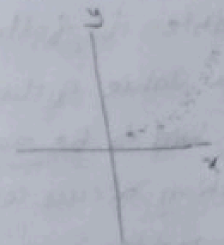
This method is not suitable if the number of observations are large.

imp 6 mark ques

★ Interpretation of nature and degree of relation using scattered diagram.



Positively correlated   negatively correlated   No correlation.



Curve linear Correlation

i) In this scattered diagram if the plotted points are distributed from lower left corner to upper right corner the correlation is said to be positive.

ii) If the plotted points are distributed from upper left corner to lower right corner the correlation is said to be negative.

iii) If the points are scattered over the plane and do not show any specific pattern, then there is no correlation between the variables.

* Linear and Curve linear Correlation

If the plotted points lies around a line the correlation is said to be linear, if the plotted points lies very close to a curve the correlation is said to be curve linear.

=> Generally when the values of two variables move in the same direction, so that an increase or decrease in the value of one variable is followed by an increase or decrease in the value of the other variable, the correlation is said to be positive.

increase decrease in the same direction

Similarly negative correlation occurs when the values of two variables move is opposite direction, so that an increase or decrease in the value of one variable is followed by decrease or increase in the value of the other variable.

* Curve fitting => finding the relation between two related variables (related - have to find the relation)

Let (x_1, y_1) (x_2, y_2) ... (x_n, y_n) be n pair of observations of two related variables X and Y. The general problem of finding a relation of the form $y = F(x)$ which fits (suitable) best to the given data is called curve fitting.

Imp

* Principle of least squares.

Let (x_1, y_1), (x_2, y_2) ... (x_n, y_n) be n pair of observations of two related variables x and y, and let $y = F(x)$ be the best fitted curve. y_i is called the observed value of y corresponding to x_i and $F(x_i)$ is called the expected value of y_i corresponding to x_i then $y_i - F(x_i)$ is called the error or residual for y_i.

The principle of least squares states that for a best fit curve the sum of the squares of the residuals is a minimum.

ie; $\boxed{E = \sum_{i=1}^{n} \left[y_i - F(x_i)\right]^2}$ is a minimum

* Filling of straight line

Straight line: $\boxed{y = ax+b}$

Assume that the observations $(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$ are linearly correlated.

Let $y = ax+b$ be the best fitted straight line for the given data.

By the principle of least square, we get

$$\boxed{\begin{aligned} \Sigma y &= a\Sigma x + nb \longrightarrow (1) \\ \Sigma xy &= a\Sigma x^2 + b\Sigma x \longrightarrow (2) \end{aligned}}$$

The above 2 equations are called normal equations for fitting of the straight line.

Solving normal equations we get the best values of a and b.

Q. Using the principle of least squares fit a straight line to the following data.

x: 1 5 10 15 20

y: 8 12 16 20 26

ans) Let $y = ax+b$ be the best fitted straight line. The normal equations are

$$\Sigma y = a\Sigma x + nb \longrightarrow (1)$$

$$\Sigma xy = a\Sigma x^2 + b\Sigma x \longrightarrow (2)$$

x	y	xy	x^2
1	8	8	1
5	12	60	25
10	16	160	100
15	20	300	225
20	26	520	400
Σx 51	Σy 82	Σxy = 1048	Σx^2 = 751

$n = 5$ (no. of elements)

sum

$ax + by = c$

$51a + 5b = 82 \longrightarrow 1$

$751a + 51b = 1048 \longrightarrow 2$

$a = \frac{5240 - 4182}{2601 - 3755} = \frac{1058}{1154} = 0.9$

$\boxed{\begin{aligned} a &= 0.9 \\ b &= 7.04 \end{aligned}}$

Use calculator to solve.

Q. Fit a straight line of the form $y = \cancel{ax+b}$ $a+bx$ for the following data.

x: 1 3 5 7 8 10

y: 8 12 15 17 18 20

The normal equations are

$$\Sigma y = na + b\Sigma x \longrightarrow (1)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \longrightarrow (2)$$

$b = 51 \times 0.9 + 5b = 82$
$45.9 + 5b = 82$
$5b = 82 - 45.9$
$5b = 36.1$
$b = 36.1/5 = 7.22$

x	y	xy	x^2
1	8	8	1
3	12	36	9
5	15	75	25
7	17	119	49
8	18	144	64
10	20	200	100
34	90	582	248

$n = 6$ observation.

$6a + 34b = 90 \longrightarrow (1)$

$34a + 248b = 532 \longrightarrow (2)$

a = 7.62

b = 1.30

* Fitting of parabola

Let $y = ax^2 + bx + c$ be the best fitted parabola for the given data.

Using principle of least squares we get the normal equations as:

n → observations

$$\boxed{\begin{aligned} \sum y &= a\sum x^2 + b\sum x + nc \longrightarrow (1) \\ \sum xy &= a\sum x^3 + b\sum x^2 + c\sum x \longrightarrow (2) \\ \sum x^2 y &= a\sum x^4 + b\sum x^3 + c\sum x^2 \longrightarrow (3) \end{aligned}}$$

2. Fit a second degree parabola of the form $y = a + bx + cx^2$ to the following data.

x : 1 2 3 4 5

y : 5 12 26 60 97.

ans) The fitted parabola is $y = a + bx + cx^2$

The normal equations are:

$$\sum x = na + b\sum x + c\sum x^2 \longrightarrow (1)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \longrightarrow (2)$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \longrightarrow (3)$$

x	y	x^2	x^3	x^4	xy	x^2y
1	5	1	1	1	5	5
2	12	4	8	16	24	48
3	26	9	27	81	78	234
4	60	16	64	256	240	960
5	97	25	125	625	485	2425.
15	200	55	225	979	832	3672.

$5a + 15b + 55c = 200 \longrightarrow (1)$.

$15a + 55b + 225c = 832 \longrightarrow (2)$.

$55a + 225b + 979c = 3672 \longrightarrow (3)$.

a = 10.4

b = -11.08

c = 5.7

Q. Using principle of least squares fit a parabola of the form $y = ax^2 + bx + c$ for the following data.

x : 1 3 4 7 9

y : 2 7 8 11 9

ans) the fitted parabola is $y = ax^2 + bx + c$.

The normal equations are;

$$\Sigma y = a\Sigma x^2 + b\Sigma x + nc \longrightarrow ①$$

$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x \longrightarrow ②$$

$$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2 \longrightarrow ③$$

	x	y	x^2	x^3	x^4	xy	x^2y
	1	2	1	1	1	2	2
	3	7	9	27	81	21	63
	4	8	16	64	256	32	128
	7	11	49	343	2401	77	539
	9	9	81	729	6561	81	729
Σ	24	37	156	1164	9300	213	1461

$n = 5$

$$156a + 24b + 5c = 24 \longrightarrow ①$$

$$1164a + 156b + 24c = 213 \longrightarrow ②$$

$$9300a + 1164b + 156c = 1461 \longrightarrow ③$$

-1.36

16.27

-30.7

15/10/2023 Sunday

Normal equations for fitting of the parabola of the form $y = a + bx^2$ are:

$$\Sigma y = na + b\Sigma x^2 \longrightarrow ①$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^4 \longrightarrow ②$$

Q. Fit a parabola of the form $y = a + bx^2$ to the following data.

x : 1 2 4 5 8

y : 2 6 8 10 9

The normal equations for fitting of the parabola $y = a + bx^2$ are:

$$\Sigma y = na + b\Sigma x^2 \longrightarrow ①$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^4 \longrightarrow ②$$

x	y	x^2	x^4	x^2y
1	2	1	1	2
2	6	4	16	24
4	8	16	256	128
5	10	25	625	250
8	9	64	4096	576
20	35	110	4994	980

$5a + 110b = 35 \longrightarrow (1)$.

$110a + 4994b = 980 \longrightarrow (2)$.

$b = 0.08$

* $x_1, x_2, \ldots x_n$ of a variable

Mean $= \bar{x} = \left[\frac{x_1+x_2+\ldots+x_n}{n}\right] = \frac{\sum x_i}{n}$ (average)

Variance $= Var(x) = (\sigma_x^2) = \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\ldots+(x_n-\bar{x})^2}{n}$

$= \frac{\sum (x_i-\bar{x})^2}{n}$

Standard deviation (S.D) $= \sigma_x = \sqrt{Var(x)}$

* <u>Covariance for 2 variables x & y.</u>

The covariance is defined as

$$\frac{(x_1-\bar{x})(y_1-\bar{y})+(x_2-\bar{x})(y_2-\bar{y})+\ldots(x_n-\bar{x})(y_n-\bar{y})}{n}$$

$$\boxed{Cov(x,y) = \sum \frac{(x_i-\bar{x})(y_i-\bar{y})}{n}}$$

* For a data $x_1, x_2, \ldots x_n$ of a variable x.

we have mean $=$

$\bar{x} = \frac{[x_1+x_2\ldots+x_n]}{n}$

$\boxed{= \frac{\sum x_i}{n}}$

Variance $= Var(x)$

$= \sigma_x^2 = \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\ldots+(x_n-\bar{x})^2}{n}$

$\boxed{= \frac{\sum(x_i-\bar{x})^2}{n}}$

Standard deviation

$\boxed{= \sigma_x = \sqrt{Var(x)}}$

* <u>Correlation</u> (Linear Correlation Regression - Karl Pearson's coefficient of correlation)

define

The intensity or degree of linear relationship between two variables can be measured using a formula called <u>Correlation coefficient</u> developed by Karl pearson.

[The <u>Karl pearson's</u> correlation coefficient between two variables x and y is defined as]

$$\underset{(gamma)}{r}(x,y) = r_{xy} = \frac{Cov(x,y)}{\sigma_x \sigma_y}$$

$$= \frac{\frac{1}{n}\sum(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\frac{1}{n}\sum(x_i-\bar{x})^2}\sqrt{\frac{1}{n}\sum(y_i-\bar{y})^2}}$$

Imp. ques.

$$r = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum(x_i-\bar{x})^2}\sqrt{\sum(y_i-\bar{y})^2}}$$

$$= \frac{\sum xy - \frac{1}{n}(\sum x)(\sum y)}{\sqrt{\sum x^2 - \frac{1}{n}(\sum x)^2}\sqrt{\sum y^2 - \frac{1}{n}(\sum y)^2}}$$

* Properties of correlation coefficient

1) The correlation coefficient lies between -1 and 1

ie; $-1 \leq r \leq 1$

r-game

=> For any n pair of real numbers

$(a_1, b_1)\ (a_2, b_2) : \dots (a_n, b_n)$

we have $\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$

above inequality is called as Schwatz inequality.

By Schwatz inequality we have

$(\sqrt{a})^2 = a$

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

Let $a_i = x_i - \bar{x}$, $b_i = y_i - \bar{y}$

Then $\left(\sum (x_i - \bar{x})(y_i - \bar{y})\right)^2 \leq \left(\sum (x_i - \bar{x})^2\right)\left(\sum (y_i - \bar{y})^2\right)$

$$= \left(\frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2}\sqrt{\sum (y_i - \bar{y})^2}}\right)^2 \leq 1$$

$r^2 \leq 1$ -1, 0, 1 square ≤ 1 $(\sqrt{a})^2 = a$

ie; $-1 \leq r \leq 1$

2) If $r = 1$, the correlation is positive and perfect.

3) If $r = -1$, the correlation is negative and perfect.

4) If $r = 0$, ~~the co~~ there is no correlation between the variable.

5) Correlation coefficient is independent of change of origin and change of scale.

ie; if $u_i = \frac{x_i - a}{h}$ and $v_i = \frac{y_i - b}{k}$

where a, b, h and k are suitable choice of numbers, (any number which can smaller the value.)

then $r(x,y) = r(u,v)$.

Q. Compute the correlation coefficient for the following data.

x :	65	66	67	67	68	69	70	72
y :	67	68	65	68	72	72	69	71

The correlation coefficient is given by.

$$r(x,y) = \frac{\sum xy - \frac{1}{n}(\sum x)(\sum y)}{\sqrt{\sum x^2 - \frac{1}{n}(\sum x)^2}\sqrt{\sum y^2 - \frac{1}{n}(\sum y)^2}}$$

x	y	x^2	y^2	xy
65	67	4225	4489	4355
66	69	4356	4624	4489
67	65	4489	4225	4355
67	~~67~~	4489	4356	4556
68	68	4624	5194	4896
69	72	4761	5184	4968
70	72	4900	4761	4930
72	69			
	71	5184	5041	5112
544	552	37028		37560

$$r = \frac{37560 - \frac{1}{8}(544)(552)}{\sqrt{37028 - \frac{1}{8}(544)^2}\sqrt{\ldots - \frac{1}{8}(552)^2}}$$

Q. The following table gives the verbal reasoning test score, x, and English test score y of a random sample of 8 children, who took both tests.

Child	A	B	C	D	E	F	G	H
x	112	113	110	113	112	114	109	113
y	69	65	75	70	70	75	68	76

Calculate the Correlation Coefficient ~~of the~~ between scores of verbal reasoning & English score. Comment briefly on the result obtained.

ans) $$r(x,y) = \frac{\sum xy - \frac{1}{n}(\sum x)(\sum y)}{\sqrt{\sum x^2 - \frac{1}{n}(\sum x)^2}\sqrt{\sum y^2 - \frac{1}{n}(\sum y)^2}}$$

Convenient number (u, v)

Subtract 100.

since not a multiple of any number $\frac{x-100}{1}$

x	y	$u = \frac{x-100}{1}$	$v = \frac{y-65}{1}$	u^2	v^2	uv
112	69	12	4	144	16	48
113	65	13	0	169	0	0
110	75	10	10	100	100	100
113	70	13	5	169	25	65
112	70	12	5	144	25	60
114	75	14	10	196	100	140
109	68	9	3	81	9	27
113	76	13	11	169	121	143
×	×	96	48	1172	396	583

not needed

$(n=8)$

$$r = \frac{\sum uv - \frac{1}{n}(\sum u)(\sum v)}{\sqrt{\sum u^2 - \frac{1}{n}(\sum u)^2}\sqrt{\sum v^2 - \frac{1}{n}(\sum v)^2}}$$

$r = \dfrac{573 - \frac{1}{8}(96)(49)}{\sqrt{1172 - \frac{1}{8}(96)^2}\sqrt{396 - \frac{1}{8}(49)^2}}$

$\boxed{r(u,v) = r(x,y)}$ Property.

28/10/2025 Saturday

* Rank Correlation

Instead of taking the values of the variables, if the ranks of the observations are considered, the correlation coefficient so obtained is called the rank correlation coefficient.

* Spearman's rank correlation coefficient

Let (x_i, y_i) be the ranks of the ith individual in two characteristics A and B respectively. Assume that no two observations are bracketed equal in either classification, each of the variables X and Y takes the values $1, 2, \ldots, n$. Then the Spearman's correlation coefficient is given by:

$$\boxed{\rho(x,y) = 1 - \frac{6\sum d^2}{n(n^2-1)}}$$

where, $d_i = x_i - y_i$

2. Calculate the rank correlation coefficient for the following data.

x: 68 66 75 50 65 80 75 40 55 64

y: 62 58 68 45 81 60 65 48 50 70

(rank not repeating)

x	y	r_x	r_y	$d = r_x - r_y$	d^2
68	62	4	5	-1	1
66	58	5	7	-2	4
75	68	3	3	0	0
50	45	9	10	-1	1
65	81	6	1	5	25
80	60	1	6	-5	25
76	65	2	4	-2	4
40	48	10	9	1	1
55	50	8	8	0	0
64	70	7	2	5	25

n=10 observation

$\sum d^2 = 86$

$\dfrac{1-6\times 86}{10(10^2-1)} = \dfrac{1-6\times 86}{990} = \dfrac{1-86}{165} = \underline{\underline{0.47}}$

$\leftarrow \rho(x,y) = 1 - \dfrac{6\times 86}{10(10^2-1)} = \dfrac{1-516}{10(100-1)} = \dfrac{1-516}{10\times 99} = \underline{\underline{0.47}}$

=> When two values has same rank average rank calculated

65 65 70 65

$= \dfrac{5+6+7}{3} = 6$ no. of repetition

Rank correlation when there are tied rank (the rank repeats).

In this case the formula is modified as:

$$\rho(x,y) = 1 - \frac{6\left[\sum d^2 + \frac{m_1(m_1^2-1)}{12} + \frac{m_2(m_2^2-1)}{12} \dots + \frac{m_k(m_k^2-1)}{12}\right]}{n(n^2-1)}$$

where $m_1, m_2 \dots m_k$ are the no. of repetion of the rank.

2. Calculate the rank correlation coefficient for the following data.

x: 65 63 67 64 68 62 70 66 68 67

y: 68 66 68 65 69 66 68 65 71 67

x	y	r_x	r_y	$d = r_x - r_y$	d^2
65	68	7	4	3	9
63	66	9	7.5	2.5	6.25
67	68	4.5	4	0.5	0.25
64	65	8	9.5	-1.5	2.25
68	69	2.5	2	.5	0.25
62	66	10	7.5	2.5	6.25
70	68	1	4	-3	9
66	65	6	9.5	-3.5	12.25
68	71	2.5	1	1.5	2.25
67	67	4.5	6	-1.5	2.25

$\sum d^2 = 50$ (46)

$m_1 = 2\ (x = 68)$ $\quad \frac{3+4+5}{3} = 4$ $\quad m_4 = 2(y = 65)$

$m_2 = 2\ (x = 67)$ $\quad m_3 = 3\ (y = 68)$ $\quad m_5 = 2(y = 66)$.

$$\frac{m_1(m_1^2-1)}{12} = \frac{2(2^2-1)}{12} = \frac{2\times3}{12} = \frac{1}{2}$$

$$\frac{m_3(m_3^2-1)}{12} = \frac{3(3^2-1)}{12} = \frac{3\times8}{12} = 2$$

$$\rho(x,y) = 1 - \frac{6\left[\sum d^2 + \frac{m_1(m_1^2-1)}{12} + \frac{m_2(m_2^2-1)}{12} \dots + \frac{m_k(m_k^2-1)}{12}\right]}{n(n^2-1)}$$

$$= 1 - \frac{6\left[46 + 0.5 + 0.5 + 2 + 0.5 + 0.5\right]}{10\cdot(10^2-1)} = \frac{1-6\times50}{990} = 0.696$$

$$= 1 - \frac{6\times50}{10\times99} = \frac{1-300}{990} = 0.696$$

0.714

* **Regression** (predicting value of one variable) [one already known]

If two variables x, and y are correlated the process of predicting the value of one variable when the other variable is known is called regression.

$x \longrightarrow y$ (predict)
(given)

$x \longleftarrow y$
(predict) (given)

$y = ax + b$
$x = cy + d$ } Minimizing

Two regression lines, suppose a two variables x and y are linearly related (linear correlation). The line which is used predict, the value of y when x is given is called the regression line of y on x and is given by

$$\boxed{y - \bar{y} = b_{yx}(x - \bar{x})}$$

where b_{yx} is called the regression coefficient of y on x computed by the formula:

$$b_{yx} = \frac{Cov(x,y)}{\sigma_x^2}$$

$$= \frac{\sum xy - \frac{1}{n}(\sum x)(\sum y)}{\sum x^2 - \frac{1}{n}(\sum x)^2}$$

Similarly, the line which is used to predict the value of x, when y is given is called the regression line of x on y, which is given by:

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

where $b_{xy} = \frac{Cov(x,y)}{\sigma_y^2}$

$$\boxed{= \frac{\sum xy - \frac{1}{n}(\sum x)(\sum y)}{\sum y^2 - \frac{1}{n}(\sum y)^2}}$$

b_{xy} is called the regression coefficient of x on y

*Note 1: We have $b_{yx} = \frac{Cov(X,y)}{\sigma_x^2}$ and $b_{xy} = \frac{Cov(X,Y)}{\sigma_y^2}$

hence b_{yx} and b_{xy} have the same sign.

*Note 2: $b_{yx} \times b_{xy} = \frac{(Cov(x,y))^2}{\sigma_x^2 \sigma_y^2} = \left(\frac{Cov(x,y)}{\sigma_x \sigma_y}\right)^2 = r^2$

$$= r = \pm\sqrt{b_{yx} + b_{xy}}$$

*Note 3: The two regression lines always passes through the point $(\bar{x}, \bar{y})$.

*Note 4: We have $b_{yx} = \frac{Cov(x,Y)}{\sigma_x^2}$

$$= \frac{\sigma_y}{\sigma_x} \frac{Cov(x,Y)}{\sigma_x \cdot \sigma_y} = r$$

$$= \frac{\sigma_y}{\sigma_x} r$$

$$\Rightarrow b_{yx} = \frac{\sigma_y}{\sigma_x} r$$

Similarly $b_{xy} = \frac{\sigma_x}{\sigma_y} r$

31/10/2023
Tuesday

Q. The grades of a students on a mid term report, (X). and the final examination, (Y). are as follows:

x:	77	50	71	72	81	94	96	89	99	67
y:	82	66	78	64	67	85	99	75	99	65

Estimate the final examination grade of a student who received a grade of 85 on the midterm report.

ans) To predict the value of Y, when $x = 85$ we have to construct, the regression line of Y on X

The regression line of y on x is given by:

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

where b_{yx} is given by $= \dfrac{\sum xy - \frac{1}{n}(\sum x)(\sum y)}{\sum x^2 - \frac{1}{n}(\sum x)^2}$

x	y	xy	x^2
77	82	6314	5929
50	66	3300	2500
71	78	5538	5041
72	64	4608	5184
81	67	5427	6561
94	85	7990	8836
96	99	9504	9216
99	99	9801	9801
67	75	5025	4489
$\sum$ 707	715	57,507	57557

$$\bar{x} = \frac{\sum x}{n} = \frac{707}{9} = 78.55$$

$$\bar{y} = \frac{\sum y}{n} = \frac{715}{9} = 79.44$$

$$b_{yx} = \frac{57507 - \frac{1}{9}(707)(715)}{57557 - \frac{1}{9}(707)^2} = 0.6638$$

$\therefore$ Regression line of y on x is

$$y - 79.4 = 0.6638\,(x - 78.5)$$

when $x = 85$

$$y - 79.4 = 0.6638\,(85 - 78.5)$$

$$y = 79.4 + 0.6638 \times 6.5$$

$$= \underline{83.7147}$$

Q. The following data where available $\bar{x} = 970$, $\bar{y} = 18$, and correlation coefficient $r = 0.92$

$\sigma_x = 38$, $\sigma_y = 2$

Find the value of x when $y = 20$.

To predict the value of x, we have to construct the regression line of x on y, which is given by:

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$b_{yx} = \frac{\mathrm{cov}(x,y)}{\sigma_x^2}$$

$$b_{xy} = \frac{\sigma_x r}{\sigma_y} = \cancel{\frac{970}{18} \times 0.92}$$

$$= \frac{38}{2} \times 0.92 = \underline{17.48}$$

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 970 = 17.48\,(y - 18)$$

when $y = 20$

$$x - 970 = 17.48\,(20 - 18)$$

$$x = \underline{1004.96}$$

Q. In a partially destroyed laboratory record of an analysis of correlation data. The following results only are legible.

$Var(x) = 9$, regression equations $8x - 10y + 66 = 0$,

$40x - 18y = 214$.

regression line
x on y: $x - \bar{x} = b_{yx}(x - \bar{x})$
y on x: $x - \bar{x} = b_{xy}(y - \bar{y})$

what are:

1) The mean value of X and Y

ii) Correlation coefficient between X and Y.

iii) standard deviation of ~~X~~ Y.

ans).

i) I know that the two regression lines always passes through the point $(\bar{x}, \bar{y})$

$\therefore 8\bar{x} - 10\bar{y}$ that must be equal to $-66 \rightarrow (1)$.

$40\bar{x} - 18\bar{y} = 214 \rightarrow (2)$.

$\bar{x} = 13$

$\bar{y} = 17$

ii) y on x $\quad y - \bar{y} = b_{yx}(x - \bar{x})$

x on y $\quad x - \bar{x} = b_{xy}(y - \bar{y})$.

Assume that the regression line of x on y be:

$8x - 10y + 66 = 0$.

$8x = 10y - 66$

$x = \frac{10}{8}y - \frac{66}{8}$

Then $b_{xy} = \frac{10}{8}$

No,

$40x - 18y = 214$

$-18y = -40x + 214$

$y = \frac{40x}{18} - \frac{214}{18}$

Then $b_{yx} = \frac{40}{8}$

$\therefore r = \pm\sqrt{b_{yx} + b_{xy}}$

$= \pm\sqrt{\frac{40}{8} \times \frac{10}{8}}$ positively correlated $= +5/3 = +1.66$

We know that $-1 \leq r \leq 1$ (correlation coefficient), hence our assumption is wrong.

$\therefore$ The regression line of y on x is:

$8x - 10y + 66 = 0$.

$10y = 8x + 66$

$y = \frac{8}{10}x + \frac{66}{10}$

$b_{yx} = \frac{8}{10}$

$40x - 18y = 214$

$40x = 18y + 214$

$x = \frac{18}{40}y + \frac{214}{40}$ denominator larger

$b_{xy} = \frac{18}{40}$

$\therefore r = \pm\sqrt{\frac{8}{10} \times \frac{18}{40}} = \pm 0.6$

iii) $b_x = \frac{\sigma_y}{\sigma_x} r$ $\quad \sigma_x^2 = Var(x)$

$= \frac{8}{10} = \frac{\sigma_y}{\sqrt{9}}(0.6)$ $\quad \sqrt{9} = 3$

$\sigma_y = \frac{3 \times 8}{10 \times 0.6} = \underline{4}$

Q. If the two regression lines are $3x + 2y = 26$ and $6x + y = 31$, find the correlation coefficient.

$2y = -3x$ $\qquad 6x = -y$

$y = \frac{-3}{2}$ $\qquad x = \frac{-1}{6}y$

Let $3x + 2y = 26$ be the regression line of y on x

$2y = -3x + 26$

$y = \frac{-3x}{2} + \frac{26}{2}$

$\underline{b_{yx} = \frac{-3}{2}}$

$6x + y = 31$

$6x = -y + 31$

$x = \frac{-1}{6y} + \frac{31}{y}$

$b_{xy} = \underline{\frac{-1}{6}}$

$r = \pm\sqrt{b_{yx} + b_{xy}}$

$= \pm\sqrt{\frac{-3}{2} \times \frac{-1}{6}} = \underline{-0.5}$

Since $-1 \leq r \leq 1$ our assumption is right

$\underline{r = -0.5}$

Q. ~~Prove~~ Prove ~~to~~ the equations $y = 2x + 3$ and $x + 3y = 4$ represents the regression line, of a set of pairs of random data, why?

ans). The given two lines are;

$y = 2x + 3$

$x + 3y = 4$

~~$3x = -3y$~~ $\qquad b_{yx} = 2$

~~$yx = 2$~~ $\qquad b_{xy} = -3.$ [Not same sign]

We know that, the two regression coefficient have the same sign, hence the given lines cannot represent the regression lines.

* Module - I.

Set relations and Functions

* Set Theory

A set is a well defined collection of objects. These objects are called elements and are said to be members of the set.

For a set we write $x \in A$ if x is an element of A; $y \notin A$ indicates that y is not a member of A.

Eg: Let $A = \{1, 3, 6, 8\}$

Here $1 \in A$ but $2 \notin A$

* Representation of Sets

There are two methods of representing a set:

i) Roaster or tabular form

ii) set-builder form.

⇒ In roaster form, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces $\{\}$.

Eg: The set of all even positive integers less than 7 is described in roaster form as $\{2, 4, 6\}$

Eg: The set of all vowels in the English alphabet is described in roaster form as $\{a, e, i, o, u\}$

* In roaster form, the order in which the elements are listed is immaterial and an element is not generally repeated.

⇒ In set-builder form, all the elements of a set posses a single common property which is not possessed by any element outside the set.

Eg: The set of all even positive integers less than 7 is described in set-builder form as;

$A = \{x : x \text{ is a positive even number and } 1 < x < 7\}$

Eg: The set of all vowels in the English alphabet is described in set builder form as;

$B = \{y : y \text{ is a vowel in the English alphabet}\}$

* A set which does not contain any element is called the empty set or null set or the void set.

The empty set is denoted by the symbol ϕ or $\{\}$

* A set which is empty or consists of a definite number of elements is called finite otherwise, the set is called infinite. For any finite set A, $|A|$ or $n(A)$ denotes the number of element in A and is referred to as the cardinality or size, of A.

* **Universe or Universal set**

Usually, in a particular context, we have to deal with the elements of a basic set which is relevant to that particular context. This basic set is called the Universe. The universal set is usually denoted by U

* **Subset**

If A and B are sets from a universe U, we say that A is a subset of B and write $A \subseteq B$, or $B \supseteq A$, if every element of A is an element of B. If, in addition, B contains an element that is not in A, then A is called a proper subset of B, and is denoted by $A \subset B$ or $B \supset A$. If A is not a subset of B, we denote this by $A \nsubseteq B$.

Eg: $A = \{1,2,3\}$, $B = \{1,2,3,4\}$

$A \subseteq B$ & $B \supseteq A$

$A \subset B$ & $B \supset A$

$$A \subseteq B \text{ if } x \in A \Rightarrow x \in B$$

$$A \subset B \Rightarrow A \subseteq B$$

* Every set A is a subset of itself. Also, the empty set ϕ is a subset of every set.
* For a given universe U, the sets A and B are said to be equal and we write $A = B$, when $A \subseteq B$ and $B \subseteq A$.

$A = \{4,5,6\}$ and $B = \{6,5,4\}$, $A = B$

$A \subseteq B$ & $B \subseteq A$

* **Power set**

If A is a set from universe U, the power set of A, denoted by $\wp(A)$, is the collection of all subsets of A.

Eg: Let $A = \{1,2,3\}$, Then

$$\wp(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

$\boxed{2^n}$.

* **Set Operations**

For $A, B \subseteq U$, we define the following.

i) $A \cup B$ (the union of A and B) $= \{x \mid x \in A \text{ or } x \in B\}$

ii) $A \cap B$ (the intersection of A and B) $= \{x \mid x \in A \text{ and } x \in B\}$

Eg: Let $A = \{1,2,5,6\}$ and $B = \{2,3,4,5,9\}$, Find $A \cup B$ and $A \cap B$.

$$A \cup B = \{1,2,3,4,5,6,9\}$$

$$A \cap B = \{2,5\}$$

* For a set $A \subseteq U$, the complement of A, denoted by $\overline{A}$ or A^c or A', is given by

$$U - A = \{x \mid x \in U \text{ and } x \notin A\}$$

Eg: Let $U = \{1,2,3,4,5,6,7\}$ and $A = \{1,2,5,6\}$. Find $\overline{A}$

$$\overline{A} = \{3,4,7\}$$

* For $A, B \subseteq C$, the relative complement of B in A or simply difference between A and B is denoted by $A \sim B$ or $A - B$, given by $\{x \mid x \in A \text{ and } x \notin B\}$

Eg: Let $A = \{1,2,3,4,5\}$ and $B = \{3,4,5,6,7\}$ Find $A - B$ & $B - A$

$$A - B = \{1,2\}, \quad B - A = \{6,7\}.$$

* For $A, B \subseteq C$, the symmetric difference of A and B, denoted by $A \Delta B$, is given by.

$$\boxed{A \Delta B = (A \cup B) - (A \cap B) = (A - B) \cup (B - A)}$$

Eg: Let $A = \{1,2,3\}$ and $B = \{2,3,4\}$, Find $A \Delta B$.

$$A - B = \{1\}, \quad B - A = \{4\}, \quad A \Delta B = (A - B) \cup (B - A) = \{1,4\}$$

* **Demorgen's Laws.**

For any sets A and B,

i) $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and

ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

* **Distributive laws.**

For any sets A, B and C.

i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Eg: Verify Demorgen's law for the following sets.

$A = \{1,2,3\}$, $B = \{2,3,4\}$ and $U = \{1,2,3,4,5,6\}$

$\overline{A} = \{4,5,6\}$ $\overline{B} = \{1,5,6\}$

$\overline{A \cup B} = \{5,6\}$

$\overline{A} \cap \overline{B} = \{5,6\}$

$\overline{A \cap B} = \{1,4,5,6\}$

$\overline{A} \cup \overline{B} = \{1,4,5,6\}$

6/11/2023
Monday

Q. For any 3 sets A, B, C, Prove that :

i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

[Distributive property].

$A = B$
$A \subseteq B$
$B \subseteq A$

ans) Let $x \in A \cap (B \cup C)$

i)
$x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and $x \in B \cup C$

$\Leftrightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Leftrightarrow x \in A$ and $x \in B$ or $x \in A$ and $x \in C$

$\Leftrightarrow x \in A \cap B$ or $x \in A \cap C$

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ — (1)

ii) Let $x \in A \cup (B \cap C)$

$x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ or $x \in (B \cap C)$

$\Rightarrow x \in A$ or $x \in B$ and $x \in C$

$\Leftrightarrow x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

$\Leftrightarrow x \in A \cup B$ and $x \in A \cup C$

$\Leftrightarrow x \in (A \cup B) \cap x \in (A \cup C)$ (Finding both part) if and only if

$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Q. State and prove Demorgen's law.

ans): For any two sets A and B.

i) $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$

ii) $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

i) Let $x \in \overline{A \cup B}$

$\Leftrightarrow x \notin A \cup B$ (not a member)

$\Leftrightarrow x \notin A$ and $x \notin B$ (if x is not in A & B then it is in $\bar{A}$, $\bar{B}$)

$\Leftrightarrow x \in \bar{A}$ and $x \in \bar{B}$

$\Leftrightarrow x \in \bar{A} \cap \bar{B}$

$\therefore \overline{A \cup B} = \bar{A} \cap \bar{B}$

ii) Let $x \in \overline{A \cap B}$

$\Rightarrow x \notin A \cap B$

$\Rightarrow x \in A$ and $x \notin B$ or
$x \notin A$ and $x \in B$ or
$x \notin A$ and $x \notin B$.
} take any 1 choice.

$x \notin A$. and $x \notin B$.

$\Rightarrow x \notin A$ or $x \notin B$, $x \in \bar{A}$ or $x \in \bar{B}$

$x \in \bar{A} \cup \bar{B}$

$\therefore \overline{(A \cap B)} = \bar{A} \cup \bar{B}$

Case 1: $x \in A$ and $x \notin B$
or

Case 2: $x \notin A$ and $x \in B$

Case 3 $\Rightarrow$ $x \notin A$ and $x \notin B$

$\Leftrightarrow x \notin \bar{A}$ and $x \in \bar{B}$ or
or $x \in \bar{A}$ and $x \notin \bar{B}$ or
$x \in \bar{A}$ and $x \in \bar{B}$

$\Rightarrow x \in \bar{A} \cup \bar{B}$ → case 1. (if $x \in \bar{B}$) then must be present in all cases

or

$x \in \bar{A} \cup \bar{B}$

or

$x \in \bar{A} \cup \bar{B}$

$\Rightarrow x \in \bar{A} \cup \bar{B}$

$\therefore \overline{A \cap B} = \bar{A} \cup \bar{B}$

Q. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{1, 2, 6\}$, $B = \{2, 4, 6, 7\}$

verify Demorgen's laws.

Ans): Demorgan's laws are:

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

$A \cup B = \{1, 2, 4, 6, 7\}$

$\overline{A \cup B} = \{3, 5, 8\}$

$\bar{A} = \{3, 4, 5, 7, 8\}$

$\bar{B} = \{1, 3, 5, 8\}$

$\bar{A} \cap \bar{B} = \{3, 5, 8\}$

$\therefore \overline{A \cup B} = \bar{A} \cap \bar{B}$

$A \cap B = \{2, 6\}$

$\overline{A \cap B} = \{1, 3, 4, 5, 7, 8\}$

$\bar{A} \cup \bar{B} = \{1, 3, 4, 5, 7, 8\}$

$\therefore \overline{A \cap B} = \bar{A} \cup \bar{B}$

Q. Let $A = \{1, 2, 3, 5\}$, $B = \{2, 4, 5, 7\}$ and $C = \{1, 5, 7, 8\}$

verify that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$B \cap C = \{5, 7\}$

$A \cup B = \{1, 2, 3, 4, 5, 7\}$

$A \cup (B \cap C) = \{1, 2, 3, \cancel{4}, 5, 7\}$

$A \cup C = \{1, 2, 3, 5, 7, 8\}$

~~A∩B~~ $(A \cup B) \cap (A \cup C) = \{1, 2, 3, 5, 7\}$

* Crossproduct of two sets

Let A and B be two nonempty sets. The the crossproduct of A and B is denoted by $A \times B$ and is defined as

~~A×B =~~

$A \times B = \{(a, b) \mid a \in A ; b \in B\}$ such that

Eg: Let $A = \{1, 2, 3\}$ and $B = \{7\}$

$\Rightarrow$ Find $A \times B$, $A \times A$ ~~and~~, $B \times A$ and $B \times B$.

$A \times B = \{(1, 7), (2, 7) (3, 7)\}$

$A \times A = \{(1,1) (1,2) (1,3) (2,2) (2,1) (2,3) (3,1) (3,2) (3,3)\}$

$B \times A = \{(7,1),(7,2),(7,3)\}$

$B \times B = \{(7,7)\}$

*Note

$A \times B \neq B \times A$

Q. For any three sets A, B and C, i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

ans): i) Let $x \in A \times (B \cup C)$.

$\Rightarrow x = (a,b)$

where $a \in A$ and $b \in (B \cup C)$

$\Rightarrow a \in A$ and ($b \in B$ or $b \in C$)

$\Rightarrow a \in A$ and $b \in B$ or $a \in A$ and $b \in C$

$\Rightarrow (a,b) \in A \times B$ or $(a,b) \in A \times C$.

$\Rightarrow x \in A \times B$ or $x \in A \times C$.

$\Rightarrow x \in (A \times B) \cup (A \times C)$

ii) Let $x \in A \times (B \cap C)$

$\Rightarrow x = (a,b)$ where $a \in A$ and $b \in B \cap C$

$\Leftrightarrow a \in A$ and $b \in B$ and $b \in C$

$\Leftrightarrow (a,b) \in A \times B$ and $(a,b) \in A \times C$.

$\Leftrightarrow (a,b) \in (A \times B) \cap (A \times C)$

$\Leftrightarrow x \in (A \times B) \cap (A \times C)$

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Relations

Let A and B be two non-empty sets, a relation R from A to B is a subset of $A \times B$.

If $(a,b) \in R$ we say that a related to b by the relation R and is denoted by aR_b.

Eg: Let $A = \{1,2,3,4\}$ and $B = \{4,6\}$

Let R be the relation from A to B defined as follows

$(a,b) \in R$ if and only if a divides b (a/b)

list all elements in R

$R = \{(1,4),(1,6),(2,4),(2,6),(3,6),(4,4)\}$

A relation R from ~~A to~~ on a set A is the subset of $A \times A$

Eg: Let $A = \{1,2,3\}$

$R = \{(a,b) \mid a+b=4, ab \in A\}$

set of all ordered pairs (a,b) such that

List the relation R.

$R = \{(1,3),(2,2),(3,1)\}$

subsets

$A \times A = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ $2^3 = 8$

* Properties of relations

i) A relation R on set A is said to be reflexive if $(a,a) \in R$ for all $a \in A$.

Q. Let $A = \{0,1,2,3\}$ and $R_1 = \{(0,0),(1,1),(2,2),(3,3),(1,2)\}$

Eg: $R_2 = \{(1,1),(2,2),(3,3),(1,3)\}$

The relation R_1 is reflexive but R_2 is not reflexive

$(0,0)$ and $(2,2) \notin R_2$

(does not belong to)

ii) A relation R on a set A is called a Symmetric relation $(b,a) \in R$ when $(a,b) \in R$

Eg: Let $A = \{-1, -2, 3\}$

$R_1 = \{(-1,3)(-2,3), (3,-1), (3,-2)(3,3)\}$

$R_2 = \{(-1,-1)(-2,3), (3,-2), (-1,3)\}$

R_1 is symmetric but R_2 is not symmetric

$\rightarrow (-1,3) \in R_2$ but $(3,-1) \notin R_2$

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A relation R on a set A is antisymmetric if for all a, b belongs to A, if $(a,b) \in R$ and $a \neq b$, then (b,a) not an element of R. $(b,a) \notin R$

Eg: Consider the relation divides on the set of all non zero Z^* of all non zero integers. $\{-2, -1, 0, 1, 2 \ldots\}$

Here 2 divides 4 implies $(2,4) \in R$

• N - natural numbers $\{0,1,2 \cdots \infty\}$

$2/4 \Rightarrow (2,4) \in R$

But $4/2 \Rightarrow (4,2) \notin R$ (4 doesn't divides 2)

$\therefore$ Hence R is not symmetric

Q - rational

R - real number

both rational & irrational

$2/_{-2} \Rightarrow (2,-2) \in R$, $-2/2 \Rightarrow (-2,2) \in R$

$\therefore$ Hence R is not antisymmetric

iii) Transitive relation

A relation on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ for all $a, b, c \in A$.

Eg: Let $A = \{(1,2), (2,2), (2,1), (3,3), (2,3), (1,3)\}$

$(1,2) \in R$ $(2,1) \in R$ but $(1,1) \notin R$

$(a,b) \in R$ $(b,a) \in R$ then $(a,a) \in R$

$\therefore A$ is not transitive.

★ Equivalence relation

imp definition

A relation on a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

Eg: $A = \{1\}$

$R = \{(1,1)\}$

Eg: Let $A = \{1, 2, 3\}$

$R_1 = \{(1,1)(2,2)(3,3)(1,3)(1,2)\}$

$R_2 = \{(1,1)(2,2)(1,3)(3,1)\}$

$R_3 = \{(1,1), (2,2), (3,3)(1,2)(2,1)(2,3)(3,2)\}$

Which of the following relations are equivalence relation.

ans) • R_1 is reflexive $(1,1)(2,2)(3,3)$, $(1,3) \in R_1$ but $(3,1) \notin R_1$ hence not symmetric.

$\therefore R_1$ is not an equivalence relation.

• R_2 is not reflexive $(3,3) \notin R_2$,

Hence R_2 is not an equivalence relation.

• R_3 is reflexive and symmetric.

$(1,2)(2,3) \in R$ but $(1,3) \notin R$.

hence it is not a transitive relation, hence not an equivalence relation.

exam

Q. Let R be a relation on a set of integers defined by $R = \{(x,y) \mid x-y$ is divisible by $6\}$, check whether R an equivalence relation.

ans) for any integer x

$x - x = 0$ is divisible by 6.

hence $(y,x) \in R$

hence R is reflexive.

Let $(x,y) \in R$

$\Rightarrow x-y$ is divisible by 6

$\Rightarrow -(y-x)$ is divisible by 6

$\Rightarrow y-x$ is divisible by 6

$\Rightarrow (y,x) \in R$.

Suppose $(x,y) \in R$ and $(y,z) \in R$

$\Rightarrow x-y$ divisible by 6.

$\Rightarrow y-z$ divisible by 6.

$\Rightarrow (x-y)+(y-z)$ is divisible by 6.

$x-z$ is divisible by 6

$\Rightarrow \underline{(x,z) \in R}$

$\therefore$ Hence R is transitive.

★ <u>Closure Operations on Relations</u>

Given a relation R on a set A the reflexive closure of R can be formed by adding to R, all pairs of form (a,a) with $a \in A$, but not already in R.

Let $A = \{1,2,3\}$ and $R = \{(1,1)(2,2)(1,3)(3,2)\}$

Q. find the reflexive closure of R.

Ans) <u>Reflexive closure</u> of $R = \{(1,1)(2,2)(1,3)(3,2)\boxed{(3,3)}\}$

- <u>Symmetric closure</u>

The symmetric closure of a relation R on a set A, can be constructed by adding all ordered pairs of the form (b,a), where (a,b) is in the relation R, that are not already present in R.

Eg: Let $A = \{1,3,5,7\}$

$R = \{(1,1),(3,5),(1,7),(5,7)\}$ $(a,b) \to (b,a) \to$ add if not present

Symmetric closure of $R = \{(1,1)(3,5)(5,3),(1,7)(7,1),(5,7)(7,5)\}$

- <u>Transitive closure</u>

The transitive closure of a relation R on a set A can be produced by adding all pairs of the form (a,c) where $(a,b)(b,c)$ are already in the relation.

Constructing a transitive closure of a relation is more complicated than constructing either reflexive, or symmetric closure, so need algorithm for constructing transitive closure.

★ <u>Matrix representation of a relation.</u>

A relation between two finite sets can be represented using a 0,1 matrix. Suppose R is a relation from $A = \{a_1, a_2 \dots a_m\}$ to $B = \{b_1, b_2 \dots b_n\}$. Then the relation R can be represented by a matrix.

$M_R = [m_{ij}]$ where $m_{ij} = 1$ if $(a_i, b_j) \in R$

$m_{ij} = 0$ if $(a_i, b_j) \notin R$

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Eg: Let $A = \{1,2,3\}$, $B = \{4,5\}$ and $R = \{(2,4)(3,4)(3,5)\}$

Write the matrix associated with the matrix R.

$$M_R = \begin{array}{c|cc} & 4 & 5 \\ \hline 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{array}$$

Q. Let $A = \{1,2,3\}$ and $R = \{(1,2),(1,3),(2,3)\}$, find the matrix associated with the relation R.

$$M_R = \begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Q. Let $A = \{a,b,c\}$ and $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, then write the relation R.

$R = \{(a,a),(a,b),(b,c)\}$

* **Warshall's Algorithm**

To find the transitive closure warshall's algorithm is used. warshall's algorithm is based on the construction of a sequence of 0,1 matrices. These matrices are $W_0, W_1, \ldots, W_n$, where $W_0 = M_R$ is the 0/1 matrix of the relation.

⇒ Step 1: Transfer all 1's in W_{k-1} to W_k

Step 2: List the locations $P_1, P_2, \ldots$, in the column k of W_{k-1}, where the entry is 1, and locations $q_1, q_2, \ldots$ in row k of W_{k-1} where the entry is 1.

Step 3: put 1 in all positions (P_i, q_j) of W_k if they are not already there.

Eg: Let $A = \{1,2,3,4\}$ and $R = \{(1,2)(2,3)(3,4)(2,1)\}$ using warshall's algorithm find the transitive closure of R.

i) $W_0 = M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1$

ii) set $k=1$

1 is in the position 2 of Column C_1 and position 2 of R_1

∴ Insert 1 in the position (2,2)

$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2$

⇒ set $k=2$

1 is in the positions 1 and 2 of C_2, and positions 1, 2 and 3 of R_2.

Then insert 1 in the positions $\{(1,1)(1,2)(1,3)(2,1)(2,2)(2,3)\}$

$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3$

⇒ set $k=3$

1 and 2 of C_3 and 4th of R_3. $\{1,2\}\{4\} = (1,4)(2,4)$

(1,4) & (2,4)

$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4$

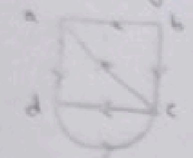
⇒ set $k=4$

1, 2, 3 of C_4 and no position in R_4

∴ Hence $W_4 = W_3$

Transitive closure of $R = \{(1,1)(1,2)(1,3)(1,4)(2,1)(2,2)(2,3)(2,4)(3,4)\}$

Q. Find the transitive closure of the relation which is expressed in the following diagram.



i) $W_0 = M_R = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 0 & 1 & 0 \end{array}$

(d,c)
(c,d) but no (d,d)
not transitive

ii) Set $k=1$

$2,3 \to C_1$, $4 \to R_1$

(2,4) (3,4) insert 1

$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

⇒ Set $k=2$

$C_2 \to$ No 1's.

W_2 is W_1 itself.

$W_2 = W_1$

⇒ Set $k=3$.

$2,4 \to C_3$ and $1,4 \to R_3$.

(2,1) (2,4) (4,1) (4,4) insert 1 in that positions.

$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

⇒ Set $k=4$

$1,2,3,4 \to C_4$

$1,3,4 \to R_4$

(1,1) (1,3) (1,4) (2,1) (2,3) (2,4) (3,1) (3,3) (3,4) (4,1) (4,3) (4,4)

insert 1 in these positions.

$W_4 \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

Hence the transitive closure $R = \{(1,1)\ (1,3)\ (1,4)\ (2,1)\ (2,3)\ (2,4)\ (3,1)\ (3,3)\ (3,4)\ (4,1)\ (4,3)\ (4,4)\}$

★ Partial Ordering

(defn) A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric and transitive.

A set S together with a partial ordering R is called a partially ordered set or Poset

Eg: The inclusion relation $\subseteq$ (subset) is a partial ordering on the powerset of a set S.

• Powerset - set of all subset of

ans) Powerset of S = P(S) = All subsets of S

Eg: $S = \{1,2,3\}$
$P(S) = \{\{1,2,3\}\ \{1,2\}\ \{1,3\}\ \{2,3\}\ \{1\}\ \{2\}\ \{3\}, \phi\}$

I know that any set is subset of itself.

So for any $A \in P(S)$, A is always a subset of A ($A \subseteq A$).

Hence '$\subseteq$' (inclusive relation) is reflexive

Suppose $A \subseteq B$ and $B \subseteq A$, then $A = B$

∴ '$\subseteq$' is antisymmetric.

Suppose $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ [A always subset of C]

Hence inclusion '$\subseteq$' is a partial ordering on the power set of S.

★ n-ary relation

The relationship among elements from more than two sets are called n-ary relations.

★ Functions

Let A and B be non-empty sets, a function F from A to B is an assignment of exactly one element of B to each element of A.

We write $F(a) = b$ if b is the unique element of B assigned by the function F, to the element a of A.

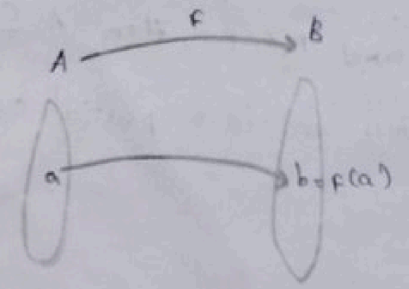
If F is a function from $A \xrightarrow{(to)} B$, we write this by

$F : A \longrightarrow B$

Functions are sometimes also called mappings or transformations.

Here A is called the domain of F and B is called the Codomain of F.

If $F(a) = b$, we say that b is the image of a, & a is called the pre image of b under the mapping F.



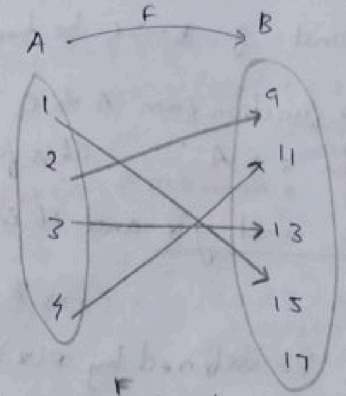
★ Types of functions

A function F is said to be one to one or an injection, if and only if $a \neq b$ implies $F(a) \neq F(b)$.

A function F from A to B, $F : A \to B$ is called onto or a surjection if and only if for every element $b \in B$ there exist an element $a \in A$ such that $F(a) = b$.

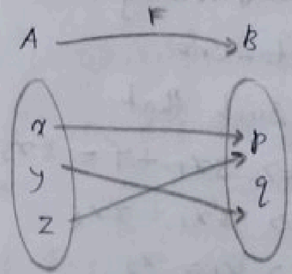
A function which is one-one and onto is called a bijective function.

Eg.



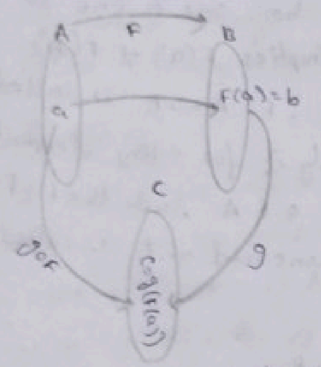
F is one-one, not onto
(17 has no pre image)

Eg:



F is Onto but not one-one.
(x, z has same preimage)

+ Composition of ~~matrix~~ Functions/mapping.



Let $F: A \to B$ and $g: B \to C$ be two functions.

Then we can define a function from A to C in such a way that $g(F(a)) = c$ for $a \in A$. This function is called the composition of F and g, and it is denoted by gof :

a element of A.

$\Rightarrow \mathbb{R}$ - Real numbers

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Eg: Let $F: \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = 3x+7$

Prove that F is one-to-one.

ans) Let $x_1, x_2 \in \mathbb{R}$ such that

$F(x_1) = F(x_2) \Rightarrow 3x_1 + \cancel{7} = 3x_2 + \cancel{7}$

$\Rightarrow \cancel{3}x_1 = \cancel{3}x_2$

$\Rightarrow \underline{x_1 = x_2}$

$\therefore$ which means F is one-to-one.

Q. Find the domain and range of the functions $F(x) = 3x^2+5$

ans) clearly the domain of F is $\mathbb{R}$ (Real numbers).

for any real number x, x^2 is always greater than or equal to 0.

$x \in \mathbb{R} \Rightarrow x^2 \geq 0$

$\Rightarrow 3x^2 \geq 0$

$\Rightarrow 3x^2 + 5 \geq 0 + 5$

$\Rightarrow F(x) \geq 5$

Range $= [5, \infty]$ (take values greater than 5)

Q. Let $F, g: R \to R$ defined by $F(x) = 2x^2+3$ and $g(x) = 5x+7$.

Find i) Fog (composite g) ii) goF (composite F) iii) FoF iv) gog

i) $(Fog)(x) \Rightarrow F(g(x))$ $x = 5x+7$

$\Rightarrow F(5x+7) \Rightarrow 2(5x+7)^2 + 3$

$F(x) \Rightarrow 2(25x^2 + 70x + 49) + 3$

$\Rightarrow 50x^2 + 140x + 98 + 3$

$= \underline{\underline{50x^2 + 140x + 101}}$

ii) $(goF)(x) = g(F(x))$

$= g(2x^2+3) \Rightarrow 5(2x^2+3)+7$

$\Rightarrow \underline{\underline{10x^2 + 22}}$

iii) $FoF(x) = F(F(x))$

$= F(2x^2+3) = 2(2x^2+3)^2 + 3$

$\Rightarrow 2[4x^4 + 12x^2 + 9] + 3$

$\Rightarrow 8x^4 + 24x^2 + 18 + 3$

$= \underline{\underline{8x^4 + 24x^2 + 21}}$

$2x^2 \times 3 \times 2 = 2x^2 \times 6 = 12x^2$

iv) $(gog)(x) = g(5x+7)$

$= 5(5x+7)+7$

$\Rightarrow 25x + 35 + 7 = \underline{\underline{25x + 42}}$

Q. If $A = \{1,2,3,4,5\}$ $B = \{1,2,3,8,9\}$ and the function $F: A \to B$ and $g: A \to A$ defined by $F = \{(1,8)(3,9)(4,3),(2,1)(5,2)\}$ and $g = \{(1,2)(3,1)(2,2)(4,3)(5,2)\}$

Find Fog and gog.

ans) : Fog is from $A \to B$

So $(Fog)(1) = F(g(1)) = F(2) = 1$

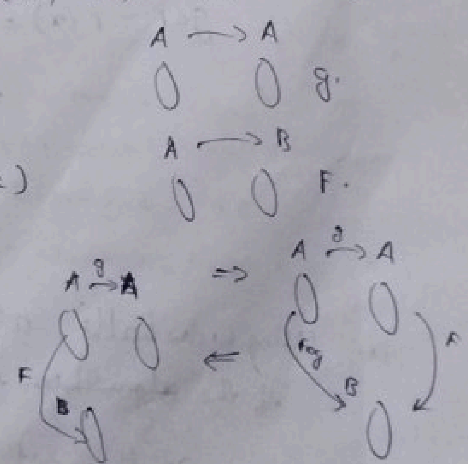
$(Fog)(2) = F(g(2))$

$= F(2)$ (1,2) image

$= 1$

$(Fog)(3) = 1$ (3,1)

$= F(1) = \underline{\underline{8}}$ (1,8)



$fog(4) = f(g(4))$
$= f(3)$
$= \underline{9}$

$fog(5) = f(g(5))$
$= f(2)$
$= \underline{1}$

Hence our $fog = \{(1,1)\,(2,1)\,(3,9)\,(4,9)\,(5,1)\}$

$(gog)(1) = g(g(1))$
$= g(2) = 2$

$(gog)(2) = g(g(2)) = 2$

$(gog)(3) = g(g(3)) = 1$
$g(1) = \underline{\underline{2}}$

$(gog)(4) = g(g(4)) = g(3)$
$= \underline{\underline{1}}$

$gog(5) = g(g(5)) = g(2) = 2$

$\Rightarrow gog = \{(1,2),(2,2)\,(3,2)\,(4,1),(5,2)\}$

Q. Let $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x-5$ and $g(x) = 3x^2+7$.
find fog and gof.

ans): fog and gof $: \mathbb{R} \to \mathbb{R}$ $\quad x = 3x^2+7$

$fog = g(x) = 2(3x^2+7)-5$
$= 6x^2+14-5$
$= \underline{\underline{6x^2+9}}$

$x = 2x-5$

$gof = f(x) = 3(2x-5)^2+7$
$= 3(4x-25)+7$
$= \underline{12x^2\,75+7}$

Q. using warshall's algorithm find the transitive closure of the ~~algorithm~~ ratio $\{(1,3)\,(3,2)\,(2,4)\,(3,1)\,(4,1)\}$ on the set $\{1,2,3,4\}$

ans):

$$W_0 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

set $k=1$ $\quad C_1 \Rightarrow 3,4$, $R_1 \Rightarrow 3$.

$W_1 = (3,3)\,(4,3)$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Set $k=2$.
$C_2 \Rightarrow 3$, $R_2 \Rightarrow 4$
$(3,4)$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

set $k=3$
$C_3 \Rightarrow 1,3,4$, $R_3 \Rightarrow 1,2,3,4$
$(1,1)\,(1,2)\,(1,3)\,(1,4)\,(3,1)\,(3,2)\,(3,3)\,(3,4)$
$(4,1)\,(4,2)\,(4,3)\,(4,4)$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

set $k=4$
$W_4 \Rightarrow$
$C_4 \Rightarrow (1,2,3,4)$ $\quad R_4 \Rightarrow (1,2,3,4)$
$(2,1)\,(2,2)\,(2,3)\,(2,4)$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence the transitive closure is $R = \{(1,1)\,(1,2)\,(1,3)\,(1,4)\,(2,1)\,(2,2)\,(2,3)\,(2,4)\,(3,1)\,(3,2)\,(3,3)\,(3,4)\,(4,1)\,(4,2)\,(4,3)\,(4,4)\}$

Q. what kind of Conic Section is given by the quadratic equation
$4x_1^2 + 6x_1x_2 - 4x_2^2 = 10$

Let $q = 4x_1^2 + 6x_1x_2 - 4x_2^2$

The associated matrix is $\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$

The characteristic equation is given by $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 3 \\ 3 & -4-\lambda \end{vmatrix} = 0$$

$\lambda^2 - (4+-4) + |A| = 0$
$\lambda^2 - 0 + (-16-9) = 0$

$\lambda = \underline{\underline{\pm 5}}$

$\therefore$ The principal axes form $= 5y_1^2 - 5y_2^2$

$\therefore 4x_1^2 + 6x_1x_2 - 4x_2^2 = 10 \Longrightarrow$

$$5y_1^2 - 5y_2^2 = 10$$

$x^2+y^2=a^2 \rightarrow$ circle

$y^2=4ax$

$x^2=4ay$

$$\frac{y_1^2}{2} - \frac{y_2^2}{2} = 1 \Rightarrow \text{hyperbola.}$$

$\frac{x^2}{a^2}+\frac{y^2}{b^2} \longrightarrow$ ellipse

$\frac{x^2}{a^2}-\frac{y^2}{b} \rightarrow$ hyperbola

15/5/2023 Monday

Module-4

Linear System of Equations

Consider the system of equations $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

with n unknowns $x_1, x_2, \dots, x_n$ in m equations, the above system of equations can be written in the matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\Longrightarrow AX = B$

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Here A is called the coefficient matrix of the system.

Our aim is to find the number of independent equations this number is ~~named~~ termed as the rank of the system of equations.

The matrix:

$$(AB) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$ is augmented matrix.

The following operations in a matrix is called elementary transformations.

i) Interchange of two rows (equations)

ii) Multiplication of a row with a non zero constant.

iii) Add a constant multiple of a row to another row

These operations does not affect the solution of the system of equations.

The number of linearly independent rows in a matrix is ~~named~~ termed as the rank of the matrix.

* Solution of System of equations

- Gauss elimination method.

Q. Using Gauss elimination method, solve the following system of equations.

$$x + 2y + 3z = 1$$
$$2x + 3y + 2z = 2$$
$$3x + 3y + 4z = 1$$

$$[AB] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 2 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix}$$

make the first equation coefficient as 1 (here already 1)
$\frac{x+2y+3z=1}{1}$ take the first equation else divide to make 1 eg: $\frac{2x}{2}=x$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & 0 \\ 0 & -3 & -5 & -2 \end{bmatrix}$$

• $R_2 \to R_2 - 2R_1$: $2-2$, $3-4$, $2-6$, $2-2$

• $R_3 \to R_3 - 3R_1$: $3-3$, $3-6$, $4-9$, $1-3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & -3 & -5 & -2 \end{bmatrix}$$

$R_2 \leftarrow (-1)R_2$

now the other 2 rows (R_2 & R_3) make coefficient y as 1

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 7 & -2 \end{bmatrix}$$

• $R_3 \to R_3 + 3R_2$: $0+0$, $-3+(-3)$, $-5+12$, $-2+0$

$x + 2y + 3z = 1$

$y + 4z = 0$

$7z = -2$

$z = \frac{-2}{7}$

$y + 4\left(\frac{-2}{7}\right) = 0$

$y - \frac{8}{7} = 0$, $y = \frac{8}{7}$

$\frac{7-10}{7} = \frac{-3}{7}$

$x + 2\left(\frac{8}{7}\right) + 3\left(\frac{-2}{7}\right) = 1$

$x = 1 - \frac{16}{7} + \frac{6}{7}$

$= \cancel{\frac{7-16+6}{7}}$

$= \frac{-3}{7}$

$\frac{7-32}{7} = \frac{-29}{7}$

$x + 2\left(\frac{8}{7}\right) + 3\left(\frac{-2}{7}\right) = 1$

$= x + \frac{16}{7} - \frac{6}{7} = 1$, $x + \frac{10}{7} = 1$, $x = 1 - \frac{10}{7} =$

Q. Solve the system of equation using gauss elimination method.

$x + y + z = 6$

$x + 2y - 3z = -4$

$-x - 4y + 9z = 18$, (already 1, make it 1 otherwise)

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & -3 & -4 \\ -1 & -4 & 9 & 18 \end{bmatrix}$$

remove x

• $R_2 \to R_2 - R_1$: $1-1$, $2-1$, $-3-1$, $-4-6$

• $R_3 \to R_3 + R_1$: $-1-1$, $-4-1$, $9-1$, $18-6$

$$(AB) = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & -3 & 10 & 24 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -2 & -6 \end{bmatrix}$$

already 1

remove y

$R_3 \to R_3 + 3R_2$: $0+0$, $-3+3$, $10+-12$, $24+-30$

$x + y + z = 6$

$y - 4z = -10$

$-2z = -6$

$z = -6/-2 = 3$

$y - 4 \times 3 = -10$

$y = -10 + 12 = 2$

$x + 2 + 3 = 6$

$x = 6 - 5$, $\underline{\underline{x = 1}}$

14/11/2023 Tuesday

Q. Using gauss elimination method find the solution of the system of equations

$-2x + 4y - 6z = 40$

$8y + 6z = -6$

$x + y - z = 9$

$-x + 5y - 7z = 49$

$$[AB] = \begin{bmatrix} -2 & 4 & -6 & 40 \\ 0 & 8 & 6 & -6 \\ 1 & 1 & -1 & 9 \\ -1 & 5 & -7 & 49 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ -2 & 4 & -6 & 40 \\ -1 & 5 & -7 & 49 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

(interchanges)

To make coefficient of 1

$$\sim \begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 6 & -8 & 58 \\ 0 & 6 & -8 & 58 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 1 & 3/4 & -3/4 \\ 0 & 6 & -8 & -58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to 1/8\, R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 1 & 3/4 & -3/4 \\ 0 & 0 & \frac{-25}{2} & \frac{125}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 6R_2$$

$6 - 6 \times 1 = 0$

$-8 - 6 \times 3/4 = -25/2$

$58 - 9/2 = \frac{125}{2}$

$x + y - z = 9$

$y + \frac{3}{4} z = -3/4$

$\frac{-25z}{2} = \frac{125}{2}$

$z = -5$

$y + \frac{3}{4}(-5) = -3/4$

$y = -3/4 + \frac{15}{4} = \frac{12}{4}$

$= 3$

$x + 3 - (-5) = 9$

$x = 1$ $(9 - 8)$

$\Rightarrow$ No. of non zero rows are called rank of the matrix

Q. Using gauss elimination method find the solution of the system of equations
$2x + 3y = 9$
$x - 2y = -13$
$x + y = 7$

$$(AB) = \begin{bmatrix} 1 & -2 & -13 \\ 1 & 1 & 7 \\ 2 & 3 & 9 \end{bmatrix}$$

eliminate x

$$\sim \begin{bmatrix} 1 & -2 & -13 \\ 0 & 3 & 20 \\ 0 & 7 & 35 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -13 \\ 0 & 1 & 5 \\ 0 & 3 & 20 \end{bmatrix} \quad \begin{matrix} R_2 \leftrightarrow R_3 \text{ (interchanges)} \\ R_2 \to 1/7\, R_2 \end{matrix}$$

equations can be interchanged

eliminate y

$$\sim \begin{bmatrix} 1 & -2 & -13 \\ 0 & 1 & 5 \\ 0 & 0 & 5 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

rank = 2 but rank of augmented matrix $[AB]$ is 3

$x - 2y = -13$

$y = 5$

$0 = 5$, which is wrong.

rank of coefficient $\neq$ rank of $[AB]$

This system has no solution. [No solution]

Q. Using gauss elimination method solve the following system of equations
$2x + z = 3$
$x - y - z = 1$
$3x - y = 4$.

ans)

$$[AB] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 0 & 1 & 3 \\ 3 & -1 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & 3 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

duplication should be avoided

$$\sim \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x - y - z = 1$

$2y + 3z = 1$

Take $\boxed{z = t}$

$2y + 3t = 1$

$2y = \cancel{3t}\ 1 - 3t$

$y = \dfrac{1-3t}{2}$

$x = 1 + y + z$

$= 1 + \dfrac{1-3t}{2} + t$

$= \dfrac{2 + 1 - 3t + 2t}{2}$ $-3t + 2t = -t$

$= \dfrac{3 - t}{2}$ $\dfrac{3-t}{2}$

rank of coefficient = rank of $[AB]$

$R = 2$

* Echelon matrix.

A matrix 'A' is an Echelon matrix if the number of zeros proceeding the first non-zero element of a row increases row by row until only zero rows remain.

Eg: $A = \begin{bmatrix} 2 & -1 & 5 & 6 & 7 \\ 0 & 5 & 0 & 8 & -1 \\ 0 & 0 & 0 & 7 & 11 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ number of zero increases. "Echelon matrix"

* Result.

The non zero rows of an Echelon matrix are linearly independent.

Rank of a matrix.

Rank of a matrix 'A' is the number of non zero rows in the Echelon form of the matrix, which is equivalent to the given matrix A.

Q. Find the rank of the matrix

3 mark Ques

ans)

$$\begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -1 & 2 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 4 & 1 & -8 & -1 \\ 0 & 8 & 2 & -15 & -2 \end{bmatrix} \begin{array}{l} \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - 9R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & 1/4 & -7/4 & -1/4 \\ 0 & 4 & 1 & -8 & -1 \\ 0 & 8 & 2 & -15 & -2 \end{bmatrix} R_2 \to 1/4 R_2$$

remove 8.

$$\begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & 1/4 & -7/4 & -1/4 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{array}{l} \\ \\ R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 - 8R_2 \end{array}$$

duplicate

R_3: $0-0$, $4-4$, $1-1$, $-8 - -7$, $-1 - -1$

R_4: $0-0$, $8-8$, $2-2$, $-15 - -14$, $-2 - -2$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & 1/4 & -7/4 & -1/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to -1 \times R_3$$

$-1 \times -1 = 1$

Rank = 3 (No. of nonzero rows).

* Result

Consider the linear system of equation $AX = B$, consisting of m equations in n unknowns.

i) If rank $[AB]$ = rank $A = n$, the system is consistent and has unique solutions. (n → no. of unknowns)

ii) If rank $[AB]$ = rank $A = r < n$, then the system is consistent and has infinite number of solutions. (r = rank)

In this case we can choose $\underline{n-r}$ variables arbitarly

iii) If rank $[AB] \neq$ rank A, the system is inconsistent (No solution).

Q. Prove that the system of equations
$$x + 2y + z = 3$$
$$2x + 3y + 2z = 5$$
$$3x - 5y + 5z = 2$$
$$3x + 9y - z = 4$$
is consistent and hence solve it.

ans)

$$[AB] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{bmatrix} R_2 \to (-1)R_2 \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -8 \end{bmatrix} \begin{array}{l} R_3 \to R_3 + 11R_2 \\ R_4 \to R_4 - 3R_2 \end{array}$$

multiple

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to \tfrac{1}{2} R_3$$

A AB

rank$[AB] = 3$

rank$[A] = 3 = n$ (x, y, z) 3 variable)

The system has unique solution

$$x + 2y + z = 3$$
$$y = 1$$
$$z = 2$$
$$x + 2\times 1 + 2 = 3 \qquad x + 2 + 2 = 3$$
$$\qquad\qquad x + 4 = 3$$
$$x = \underline{\underline{-1}} \qquad x = 3 - 4$$
$$\qquad = \underline{\underline{-1}}$$

Q. Prove that the system of equations
$$-x + 2y + 3z = -2$$
$$2x - 5y + z = 2$$
$$3x - 8y + 5z = 2$$
$$5x - 12y - z = 6$$
consistent and hence solve it.

ans)
$$[AB] = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix} R_1 \to R_1(-1) \sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

make zero.

3 rows are identical

$$\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 5R_1 \end{array} \sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & -7 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_2 \to \\ R_2(-1) \end{array}$$

A AB

Rank $[AB] = 2$

Rank $A = 2$

$\therefore$ System is Consistent

(rank)

rank$[AB]$ = rank$[A] = 2 < 3 = n$

no. of variable (x, y, z)

$\Rightarrow n - r = 3 - 2 = 1$ variable arbitary.

$$x - 2y - 3z = 2$$
$$y - 7z = 2$$

Take $z = t$.

$$y - 7t = 2 \qquad , \qquad x = 2 + 2y + 3z$$
$$\underline{y = 2 + 7t} \qquad = 2 + 2(2 + 7t) + 3t$$
$$(2 - 7t) \qquad = 2 + 4 + 14t + 3t$$
$$\qquad x = \underline{\underline{6 + 17t}}$$

Q. Find the value of λ and μ for which the following system of equations
$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$
has i) No solution
ii) A unique solution
iii) A one parameter family of solutions.

ans. $[AB] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$ $R_1 \to \frac{1}{2}R_1$

$\sim \begin{bmatrix} 1 & 3/2 & 5/2 & 9/2 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix} \sim \begin{bmatrix} 1 & 3/2 & 5/2 & 9/2 \\ 0 & \frac{-15}{2} & \frac{-39}{2} & \frac{-47}{2} \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix}$ $R_2 \to R_2 - 7R_1$, $R_3 \to R_3 - 2R_1$

$3 - 21/2 = -15/2$
$-2 - 35/2 = -39/2$
$9 - 63/2 = -55/2$

i) No solution $\Rightarrow$ rank $[AB] \neq$ rank A.

$\Rightarrow$ rank $A = 2$ and rank $[AB] = 3$

$\Rightarrow \lambda - 5 = 0$ and $\mu - 9 \neq 0$.

$\Rightarrow \lambda = 5$ and $\mu \neq 9$

ii) For unique solution $\Rightarrow$ rank $[AB]$ = rank $A = n = 3$.

$\Rightarrow \lambda - 5 \neq 0$ and μ can have any value.

iii) for one parameter family of solutions

~~rank [A]~~ rank A = rank $[AB] = 2$.

$\lambda - 5 = 0$ and $\mu - 9 = 0$.

$\lambda = 5$ and $\mu = 9$.

H-W

Q. Solve the system of equations $4y + 3z = 8$
$2x - z = 2$
$3x + 2y = 5$.

ans): $[AB] = \begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{bmatrix}$ $R_1 \leftrightarrow R_2$ (interchange)

$\sim \begin{bmatrix} 1 & 0 & -1/2 & 1 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{bmatrix}$ $R_1 \to \frac{1}{2}R_1$

$\sim \begin{bmatrix} 1 & 0 & -1/2 & 1 \\ 0 & 4 & 3 & 8 \\ 3 & -2 & 0 & 5 \end{bmatrix}$ $R_2 \to \frac{1}{4}R_2$ $\sim \begin{bmatrix} 1 & 0 & -1/2 & 1 \\ 0 & 1 & 3/4 & 2 \\ 3 & -2 & 0 & 5 \end{bmatrix}$ $R_3 \to R_3 - 3R_1$

$3 - 3\times1 = 0$
$-2 - 3\times0 = -2$
$0 - 3\times -1/2 = 3/2$
$5 - 3\times1 = 2$

$\begin{bmatrix} 1 & 0 & -1/2 & 1 \\ 0 & 1 & 3/4 & 2 \\ 0 & -2 & 3/2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 & 1 \\ 0 & 1 & 3/4 & 2 \\ 0 & 0 & 3 & 6 \end{bmatrix}$ $R_3 \to R_3 + 2R_2$

• $0 + 2\times0 = 0$
• $-2 + 2\times1 = 0$
• $3/2 + 2\times3/4 = \frac{3}{2} + \frac{6}{4} = \frac{12+12}{8} = \frac{24}{8} = 3$
• $2 + 2\times2 = 2+4 = 6$

• $3z = 6$
$z = 6/3 = 2$

• $y + \frac{3}{4}z = 2$
$y + \frac{3}{4} \times 2 = 2$
$y + \frac{6}{4} = 2$, $y = \frac{2}{1} - \frac{6}{4} = \frac{8-6}{4} = \frac{2}{4} = \frac{1}{2}$

• $x + -\frac{1}{2}z = 1$
$x + \frac{-1}{2} \times 2 = 1$
$x + \frac{-2}{2} = 1$
$x - 1 = 1$
$x = 1 + 1 = 2$

[Continuation on ASE backside]

[Continuation $\to$ ASE Notebook].

* Homogeneous System of Equations — Mathematics

17/11/2023 Friday

→ Continuation

Consider the system of equations $AX = B$; if $B = 0$ the system is called homogeneous system.

For a homogeneous system of equations $AX = 0$, the system is always Consistent $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is always a solution. This solution is called trivial solution. A solution other than the trivial solution is called non-trivial solution.

For non-trivial solution, rank A must be less than the number of unknowns. (x, y, z)

Here rank [AB] always equal to rank A.

Q. Do the equation
$$x - 3y - 8z = 0$$
$$3x + y = 0$$
$$2x + 5y + 6z = 0.$$
have a non trivial solution. why?

A solution other than $x = 0, y = 0, z = 0$.

$$A = \begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 11 & 22 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$5 - 2 \times -3$
$= 5 - -6 = \underline{11}$

$6 - 2 \times -8$
$= 6 - -16 = \underline{22}$

$$\sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 10 & 24 \end{bmatrix} \begin{matrix} R_3 \to 1/11\, R_3 \\ R_2 \leftrightarrow R_3 \end{matrix}$$

third equation interchanged.

$$\sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} R_3 \to R_3 - 10R_2$$

x, y, z unknowns

Rank of $A = 3 =$ No. of unknowns

$\therefore$ the system has only trivial solution.

Q. Show that the system of equations
$$x + 2y - z = 0$$
$$3x + y - z = 0$$
$$2x - y = 0,$$
have non-trivial solution and find them.

ans) Coefficient matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & -5 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

identical

Rank of $A = 2$, which is less than no. of unknowns

$\therefore$ the system has non trivial solution.

n - no. of unknowns
r - Rank.

$$x + 2y - z = 0$$
$$-5y + 2z = 0$$

Choose $n - r = 3 - 2 = 1$ variable as arbitrary.

take $z = t$

$-5y = -2z$

$-5y = -2t$

$y = \frac{2}{5}t$.

$x + 2y - z = 0$

$\cancel{x + 2y \frac{2}{5} t}$

$x = -2\left(\frac{2}{5}t\right) + t$

$x = \frac{-4}{5}t + t$

★ Eigen Values and Eigen Vectors.

Let A be a square matrix, a non-zero vector X such that:

$AX = \lambda X$, where λ is a scalar, is called an Eigen vector corresponding to the Eigen value λ.

increase/decrease

eg: $AI = A$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Any matrix multiplied with I gives that matrix

$AX = \lambda X \Rightarrow AX - \lambda X = 0$

$\Rightarrow (A - \lambda I) X = 0$

For $X \neq 0$, (for non-trivial) we must have rank of $A - \lambda I$ must be less than number of unknowns, this happens only when $|A - \lambda I| = 0$, this equation is called characteristic equation of A. Solving the characteristic equation we get the Eigen values. Then solving the equation $(A - \lambda I) X = 0$, we get: the eigen vector X.

If Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}$$

$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

then, $|A - \lambda I| = 0 \Rightarrow$

$\lambda^3 -$ (sum of the diagonal elements) λ^2

$+ \left\{ \begin{vmatrix} e & f \\ h & i \end{vmatrix} + \begin{vmatrix} a & c \\ g & i \end{vmatrix} + \begin{vmatrix} a & b \\ d & e \end{vmatrix} \right\} \lambda - |A| = 0$

20/11/2023
Monday

Q Find the eigen values and eigen ~~matrix~~ vectors of the matrix

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

the characteristic equation is $|A - \lambda I| = 0$

$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$

$\lambda^3 - (-2+1+0)\lambda^2 + \left\{ \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} \right\} \lambda$

$- |A| = 0$

$\lambda^3 + \lambda^2 + \{ (0-12) + (0-3) + (-2-4) \} \lambda - |A| = 0$

$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = (-2-\lambda) \begin{vmatrix} 1-\lambda & -6 \\ -2 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & -6 \\ -1 & -\lambda \end{vmatrix} + (-3) \begin{vmatrix} 2 & 1-\lambda \\ -1 & -2 \end{vmatrix} = 0$

$= -2-\lambda \left[(1-\lambda)(-\lambda) - -2\times-6 \right] - 2(-2\lambda - 6) - 3\left(-4 - (-1)(1-\lambda)\right) = 0$

$= -2-\lambda \left[(1-\lambda)(-\lambda) + 12 \right] - 2[-2\lambda - 6] - 3[-4 - (-1)(1-\lambda)]$

$= \quad \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\lambda = -3, -3, 5$

The eigen values are $-3, 5, -3$.

The eigen vector corresponding to,

$\lambda = 5$ is obtained by solving the equation,

$(A - \lambda I)X = 0 \qquad (A - 5I)X = 0$

$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & 0-5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$R_3 \to R_3 + -1$
$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & 5 \\ 1 & -2 & -3 \\ -7 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 7R_1 \end{array}$$

reduced

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -4 & -8 \\ 0 & 16 & 32 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x + 2y + 5z = 0$$
$$y + 2z = 0$$

$x = -2(-2t) + 5t$
$= x + -4t + 5t$
$x - 4t + 5t$
$x = -t$
$x + 1 = x = -1$

$r = 2,\ n = 3$

choose $n - r = 3 - 2 = 1$ variable arbitrary.

Take $z = t$,

$y = -2t$

$x = -2(-2t) - 5t$
$= -t$.

$$X = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix}$$

$$t = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \text{ multiplied by } -1 \qquad X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

when $\lambda = -3$ $\qquad (A - \lambda I)X = 0$, $\lambda = -3$

$$\begin{bmatrix} -2--3 & 2 & -3 \\ 2 & 1--3 & -6 \\ -1 & -2 & 0--3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

reduced

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad \begin{array}{l} R_2 \to \text{multiple (2) of } R_1 \\ R_3 \to -1 \times \text{ of } R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$x + 2y - 3z = 0$$

$r = 1,\ n = 3$

$n - r = 3 - 1 = 2$ variable are arbitrary

Take $y = s$ and $z = t$ (arbitrary values (any value))

$x = -2s + 3t$

$$X = \begin{bmatrix} -2s+3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 3t \\ 0 \\ t \end{bmatrix}$$

$$= s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Other eigen vectors are, $X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $x_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Q. Find all eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

(-) (on diagonal elements only)

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (2+1+4)\lambda^2 + \left\{ \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \right\}\lambda - |A| = 0$$

$$\lambda^3 - 7\lambda^2 + \{(4+2) + (8-0) + 2\}\lambda - 2(4+2) = 0$$

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0.$$

$$\lambda = 2, 3, 2$$

when $\underline{\lambda = 3}$, the eigen vectors is given by

$$(A - 3I)X = 0$$

$$\begin{bmatrix} 2-3 & 1 & 0 \\ 0 & 1-3 & -1 \\ 0 & 2 & 4-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$x - y = 0$

$2y + z = 0$

rank = 2

$n = 3$ (no. of unknowns)

Choose $n - r = 3 - 2 = 1$ variable arbitrary

take $y = t$

$z = -2t$, $x = t$.

$x - t = 0$

$x = t$

$2t + z = 0$

$z = -2t$

$$X_1 = \begin{pmatrix} t \\ t \\ -2t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$\underline{\lambda = 2}$, the eigen vector is obtained by;

$$(A - 2I)X = 0$$

$R_3 \to R_3 + 2R_2$ identical

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad R_2 \to R_2 - R_1$$

$y = 0$

$z = 0$

$x = t$

Hence our $X = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

2/11/2023 Tuesday

Q. Find the Spectrum of the matrix $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$ also find the Corresponding Eigen vectors.

The Spectrum of a matrix is the collection of all eigen values of the matrix.

characteristics equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 10\lambda + 24$$

$$= \lambda^2 - (8+2)\lambda + |A| = 0$$

$$\lambda^2 - 10\lambda + (16 - -8) = 0$$

$$\lambda^2 - 10\lambda + 24 = 0$$

$4+6=10$, $4\times6=24$

$$(\lambda-4)(\lambda-6) = 0$$

$$\lambda = 4, 6$$

Spectrum of $A = \{4, 6\}$

Eigen vectors corresponding to $\lambda = 4$ is given by

$$(A - 4I)X = 0$$

$$\begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$x - y = 0$

$\therefore r = 1$ (rank)

$n = 2$ (no. of unknowns)

$R_2 \to 2$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

choose $n - r = 2 - 1 = 1$ variable is free

Let $y = t$

$x = t$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 6$ given by

$$(A - 6I)X = 0$$

$$\begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$x - 2y = 0$

$r = 1$, $n = 2$ (no. of unknowns)

rank

$n - r = 1$ variable is free

take $y = t$

$x = 2t$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

★ Diagonalization

Let A be a 3×3 matrix. Let ~~$X_1 =$~~.

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \quad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}.$$

be the three independent eigen vectors corresponding to the eigen values, λ_1, λ_2 and λ_3 of A.

Define $P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

Then

$$\boxed{P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}$$

The matrix P is called model matrix

Q. Diagonalize the matrix $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ i) Find eigen values & vectors first

characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (1+-2)\lambda + |A| = 0$$

$$\lambda^2 + 1\lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0$$

$$\lambda = -3, 2.$$

$3 \times -2 = -6$ constant

$3 + -2 = \underline{1}$ middle term

$3, -2$

$-3+2 = -1$

$-3 \times 2 = -6$

when $\underline{\lambda = -3}$

$$(A - -3I)X = 0$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0, \quad \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$3, -2$

$2x + y = 0$

$r = 1, \; n = 2$

$n - r = 1$ free variable

Put $x = t$

$y = -2t$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -2t \end{pmatrix} = t\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \underline{X_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

when $\underline{\lambda = 2}$

$$(A - 2I)X = 0$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0, \quad \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$-x + 2y = 0$

$r = 1, \; n = 2$

$n - r = 2 - 1 = 1$ free variable

Take $y = t$

$x = 2t$

$$X = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \underline{X_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}.$$

Define

$$P = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

Then $P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

★ **Quadratic form**

An expression of the form $q(x) = ax^2 + by^2 + cxy$ is called a quadratic expression in two variables x and y.

An expression of the form $q(x) = ax^2 + by^2 + cz^2 + dxy + exz + fyz$ is called a quadratic expression in three variables x, y, z.

The above quadratic forms can be expressed in matrix form as

$$q = ax^2 + by^2 + cxy$$

$$= [x \;\; y] \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \underline{\underline{X^T C X}}$$

Where C is a symmetric matrix.

$$q = ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

$$= [x \;\; y \;\; z] \begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ → for symmetric matrix (note: eigen values / vectors for n unknowns)

Here the matrix C is the matrix associated with the quadratic form.

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of ~~the ma~~ C then using a suitable transformation:

$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$, the quadratic expression q can be written in the form:

$$\boxed{q(Y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2}$$

This form is called principle axes form or Canonical form of the quadratic expression.

3 mark Qn

Q. Write the principle axes form of the quadratic equation

$$q(x) = 3x^2 + 5y^2 + 3z^2 - 2xy + 2xz - 2yz$$

ans) The matrix associated with the quadratic form is

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 3_{x^2} & -1 & 1 \\ -1 & 5_{y^2} & -1 \\ 1 & -1 & 3_{z^2} \end{bmatrix} \end{array}$$

xz: $2 \div 2 = 1$ (half)

$xy = 2$ (half) $= -1$

The characteristic equation is $|C-\lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - \underbrace{(3+5+3)}_{\text{sum of diagonal elements}}\lambda^2 + \left\{ \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix} \right\}\lambda - |C| = 0.$$

$$\lambda^3 - 11\lambda^2 + \{(15-1) + (9-1) + (15-1)\}\lambda - 36 = 0$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = \underline{6, 3, 2}$$

$\therefore$ Principal axes form $= 2y_1^2 + 3y_2^2 + 6y_3^2$

Q. Find the principal axes form $q = x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$.

associated matrix
$$= \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}$$

The characteristic equation is

$$|C - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (1+2+1)\lambda^2 + \{1+1+1\}\lambda - \underbrace{[1(1) + 1(-1)]}_{\text{determinant}} = 0$$

$$\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$$

$$\lambda(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda(\lambda-3)(\lambda-1) = 0$$

$$\lambda = 0, 1, 3$$

Principal axes form $= 0y_1^2 + y_2^2 + 3y_3^2$

$= \underline{y_2^2 + 3y_3^2}$.

★ MODULE - 2.

★ Division Algorithm

Given integers a and b with $b > 0$ there exist unique integers q and r satisfying $a = q(b) + r$, where $0 \le r < b$, here the integers q and r called quotient and remainder respectively.

★ **Result**

If a and b are integers with b not equal to 0 $b \neq 0$ there exist unique integers q and r, such that $\boxed{a = q(b) + r}$; $0 \le r < |b|$

Q. Find q and r.

i) $a = 10$, $b = 3$.

$$10 = \underset{q}{\underline{3}}(3) + \underset{r}{\underline{1}}$$

$$\begin{array}{r} 3 \\ 3 \overline{)10} \\ 9 \\ \hline 1 \end{array}$$

ii) $a = 17$, $b = 5$

$$17 = \underline{3}(5) + \underline{2}$$

iii) $a = 10$, $b = -3$

$$10 = -3(-3) + 1$$

$q = -3$, $r = 1$

$r < |b|$

$r < |-3| \Rightarrow \underline{1 < 3} \checkmark$

iv) $a = 17$, $b = -5$

$$17 = -3(-5) + 2$$

v) $a = -10$, $b = 3$

$$-10 = -4(3) + 2$$

vi) $a = -17$, $b = -5$

$$-17 = 4(-5) + 3$$

$4 \times -5 = -20 + 3 = \underline{-17}$

* Note

Any integer 'a' can be written in the form $a = 2q$ or $a = 2q+1$ $(b=2)$ $b<2$ $\Rightarrow 0,1$

(Put $b=2$ in division algorithm).

Similarly any integer 'a' can be written in the form

$a = 3q$ $(b=3)$ $b<3$ $0,1,2$
$a = 3q+1$
$a = 3q+2$

Q. Show that the square of any odd integer can be represented of the form $8k+1$.

Any integer 'a' can be written as;

$a = 4q,\ 4q+1,\ 4q+2,\ 4q+3$

Since a is odd;

$\Rightarrow a = 4q+1$ or $4q+3$

Case i) $a = 4q+1$

$a^2 = (4q+1)^2$

$= 4q^2 + 8q + 1$

$= 16q^2 + 8q + 1$

$= 8(2q^2+q)+1$

ie; $8k+1$

Case ii)

$a = 4q+3$

$a^2 = (4q+3)^2$

$= (4q)^2 + 24q + 3^2$

$= 16q^2 + 24q + 9$

$= 16q^2 + 24q + (8+1)$

$\cancel{8(2q^2+24q)+1}$

$\Rightarrow 8(2q^2+3q+1)$

$\Rightarrow 8(2q^2+3q)+1$

$\Rightarrow 8k+1$ ie;

Q. Prove that the expression $\frac{a(a^2+2)}{3}$ is an integer $b<3$ $0,1,2$

$\forall a$ (for all) ≥ 1 ie; $\forall a \geq 1$

any integer a can be written as;

any integer $3q+2$ $(b=3) \Rightarrow$ $b<3$ $\Rightarrow 0,1,2$

$a = 3q,\ 3q+1,\ 3q+2$

i) $a = 3q$

$\frac{a(a^2+2)}{3} = \frac{\cancel{3}q\,(9q^2+2)}{\cancel{3}} = \frac{\cancel{3}q\,(9q^2+2)}{\cancel{3}}$

$= q(9q^2+2)$, an integer.

ii) $a = 3q+1$

$\frac{a(a^2+2)}{3} = 3q+1\,\frac{((3q+1)^2+2)}{3}$

$= \frac{3q+1\,(9q+2)+2}{3}$

$= \frac{(3q+1)(9q^2+6q+3)}{3} = \frac{\cancel{3}(3q+1)(3q^2+2q+1)}{\cancel{3}}$

$= (3q+1)(3q^2+2q+1)$.

iii) $a = 3q+2$

$$a = \frac{a(a^2+2)}{3}$$

$$= \frac{3q+2((3q^2+2)^2+2)}{3}$$

$$= \frac{3q+2(9q^2+12q+4)+2}{3}$$

$$= \frac{3q+2(9q^2+12q+6)}{3}$$

$$\frac{(\cancel{3}q+2)\times\cancel{3}(3q^2+4q+2)}{\cancel{3}} , \underline{3q+2(3q^2+4q+2)}$$

H.W. Q. The cube of any integer is of the form $9k$, $9k+1$ or $9k+8$.

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

Q. Prove that no integer in the sequence, 11, 111, 1111, 11111 . . . . is a perfect square.

ans) any integer 'a' is of the form
$a = 2q$
$a = 2q+1$

i) $a^2 = 4q^2 = \underline{4k}$ $\Rightarrow (2q)^2 = 2^2q^2 = \underline{4q^2}$

ii) $a^2 = (2q+1)^2 = 4q^2+4q+1$
$= 4(q^2+q)+1$
$= \underline{4k+1}$.

$(a+b)^2 = a^2+2ab+b^2$

now
1111 . . . 11111
= 1111 . . . 1108 + 3
$= \underline{4k+3}$

★ GCD (Greatest Common Divisor)

$\Rightarrow$ 8, 12
8 -> 1, 2, 4, 8
12 -> 1, 2, 3, 4, 6, 12
=> common divisors (1, 2, 4)
greatest => gcd (8, 12) = 4

Let 'a' and 'b' be given integers with atleast one of them not equal to 0. The GCD and 'a' and 'b' is the positive integer c satisfying the following.

i) a, b
c/a, c/b

ii) if d/a and d/b
then $d \le c$, then c is the GCD.

Q. gcd of (12, -30)
12 - 1, 2, 3, 4, 6, 12
-30 - 1, 2, 3, 5, 6, 10, 15, 30
gcd (12, -30) = 6

Q. gcd (12, -30) = 6

Theorem given integers 'a' and 'b' not both of which are zeros, there exist integers x and y such that

$\boxed{\gcd(a,b) = ax + by}$

Eg: $\gcd(12,-30) = 6$

$6 = (-2)(12) + (-1)(-30)$

H.W Q. The cube of any integer is of the form $9k$, $9k+1$, $9k+8$.

$\begin{cases} 3q \\ 3q+1 \\ 3q+2 \end{cases}$ cube

$\boxed{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$

$a = 3q$

$(3q)^3 = 9q \Rightarrow \underline{9k}$

$a = 3q+1$

$(3q+1)^3 = (3q)^3 + 3(3q)^2 \times 1 + 3(3q) \cdot 1^3 + 1^3$

$= 27q^3 + 3(9q^2) \times 1 + 9q \times 1 + 1$

$\Rightarrow 27q^3 + 27q^2 + 9q + 1$

$\Rightarrow 9(3q^3 + 3q^2 + 1q) + 1$

$= \underline{\underline{9k+1}}$

$\frac{27 \times 2}{54}$

$a = 3q+2$

$(3q+2)^3 = (3q)^3 + 3(3q)^2 \times 2 + 3(3q) \cdot 2^2 + 2^3$

$= 27q^3 + 3(9q^2) \times 2 + 9q \times 4 + 8$

$= 27q^3 + 54q^2 + 36q + 8$

$= 9(3q^3 + 6q^2 + 4q) + 8$

$= \underline{\underline{9k+8}}$

5/9/2023 Monday

Q. Show that for any integer a, $\gcd(2a+1, 9a+4) = 1$

$\Rightarrow \gcd(a,b) = d$

$d = ax + by$

$\gcd(2a+1, 9a+4) = 1$

Let

$1 = k_1(2a+1) + k_2(9a+4)$

$1 = (2k_1 + 9k_2)a + (k_1 + 4k_2)$

Equation of 'a' and constant terms.

$2k_1 + 9k_2 = 0$ (1)

$k_1 + 4k_2 = 1$ (2)

(2)×2 $\Rightarrow 2k_1 + 8k_2 = 2$ (3)

(1)−(3)

$2k_1 + 9k_2 = 0$

$2k_1 + 8k_2 = 2$

$0 \quad \underline{\underline{k_2 = -2}}$, $k_1 = 9$

$2k_1 + 8 \times -2 = 2$
$2k_1 + -16 = 2$
$2k_1 - 16 = 2$
$2k_1 = 16+2$
$k_1 = 18/2 = 9$

Q. So Let $\gcd(5a+2, 7a+3) = 1$

Let $1 = k_1(5a+2) + k_2(7a+3)$

$1 = (5k_1 + 7k_2)a + (2k_1 + 3k_2)$ *

$\frac{15 \times 5}{75}$

$5k_1 + 7k_2 = 0 \rightarrow$ (1)

$2k_1 + 3k_2 = 1 \rightarrow$ (2)

(1)×2 $\quad 10k_1 + 14k_2 = 0$

(2)×5 $\quad 10k_1 + 15k_2 = 5$

$0 + -k_2 = -5$

$\underline{\underline{k_2 = 5}}$

$10k_1 + 15k_2 = 5$
$10k_1 + 15 \times 5 = 5$
$10k_1 + 75 = 5$
$10k_1 = 75 + 5$
$= -70$

$\underline{\underline{k_1 = -7}}$

$10k_1 = -70$
$k_1 = -70/10$
$= \underline{-7}$

★ Euclidean algorithm (used to find gcd of two numbers)

Q. gcd (12, 28) = 4

$a = 28,\ b = 12$ $\quad a = bq + r$, $0 \le r < b$

$28 = 2(12) + 4 \rightarrow$ gcd

$12 = 3(4) + 0$

Q. $a = 12378,\ b = 3054$ (Find the gcd). $\quad b\overline{)a}^{q}$ ... r

$12378 = 4(3054) + 162$

$3054 = 18(162) + 138$

$162 = 1(138) + 24$

$138 = 5(24) + 18$

$24 = 1(18) + 6 \Rightarrow$ gcd (6)

$18 = 3(6) + 0$

Q. Find the gcd of (143, 227).

$227 = 1(143) + 84$

$143 = 1(84) + 59$

$84 = 1(59) + 25$

$59 = 2(25) + 9$

$25 = 2(9) + 7$

$9 = 1(7) + 2$

$\cancel{7} = 3(2) + 1 \rightarrow$ gcd(1)

$\cancel{2} = \cancel{1 \times (2) + 0}$ $\cancel{1(2)}$

$2 = 2(1) = 0$

⇒ Euclidean Algorithm (explanation).

Let a, b be two integers with $a > b$.

By division algorithm, there exist q_1 and r_1

$a = q_1(b) + r_1$; $0 \le r_1 < b$.

If $r_1 = 0$, then gcd $(a, b) = b$

Eg: 12, 24

$24 = 2(12) + 0 \rightarrow$ gcd(12)

If $r_1 \ne 0$ again apply division algorithm

$b = q_2(r_1) + r_2$; $0 \le r_2 < r_1$

$r_1 = q_3(r_2) + r_3$

⋮

$r_{n-2} = q_n(r_{n-1}) + r_n$

$r_{n-1} = q_{n+1}(a_n) + 0$

Then gcd $(a, b) = r_n$

Q. gcd (143, 227) = 1

gcd $(a, b) = d$

$d = ax + by$

imp. ⇒ Express gcd (143, 227) as a linear combination of the numbers.

Division algorithm $a = bq + r$ $[0 \le r < b]$

⇒ $1 = 143k_1 + 227k_2$

$1 = 7 - 3(2)$ $\quad \rightarrow 2 = 9 = 1(7) + 2$, $9 - 1(7)$

$= 7 - 3[9 - 1(7)]$

$= 7 - 3(9) + 3(7)$

$= -3(9) + 4(7)$

$= -3(9) + 4[25 - 2(9)]$

$= -3(9) + 4(25) - 8(9)$

$= -11(9) + 4(25)$

$= -11[59 - 2(25)] + 4(25)$

$= -11(59) + 22(25) + 4(25)$

$= -11(59) + 26(25)$

$= -11(59) + 26[84 - 1(59)]$

$= -11(59) + 26(84) - 26(59)$

$= -37(59) + 26(84)$

$= -37[143 - 1(84)] + 26(84)$

$= -37(143) + 37(84) + 26(84)$

$= -37(143) + 63(84)$

$= -37(143) + 63[227 - 1(143)]$

$= -37(143) + 63(227) - 63(143)$

$= -100(143) + 63(227)$

$227 = 1(143) + 84$

$143 = 1(84) + 59$

$84 = 1(59) + 25$

$59 = 2(25) + 9$

$25 = 2(9) + 7$

$9 = 1(7) + 2$

$7 = 3(2) + 1$

$2 = 2(1) + 0$

227×26

1362

454

5702

Q. Find gcd(306 and 657) and express it as a linear combination of the numbers.

$\boxed{a = qb + r}$

$\Rightarrow 657 = 2(306) + 45$

$306 = 6(45) + 36$

$45 = 1(36) + 9$

$36 = 4(9) + 0 \Rightarrow gcd(306, 657)$ is $\underline{9}$

$\Rightarrow 1 = 306 k_1 + 657 k_2$

$1 = 45 - 1(36)$

$= 45 - 1[306 - 6(45)]$

$= 45 - 306 + 6(45)$

$= 7(45) - 306$

$= 7[657 - 2(306)] - 306$

$= 7(657) - 14(306) - 306$

$= 7(657) - 15(306)$

Q. 153, 24

$153 = 6(24) + 9$

$24 = 2(9) + 6$

$9 = 1(6) + 3$

$6 = 2(3) + 0$

3 GCD

$= 9 - 1(6)$

$= 9 - 1[24 - 2(9)]$

$= 9 - 1(24) + 2(9)$

$= 3(9) - 1(24)$

$= 3[153 - 6(24)] - 1(24)$

$= 3(153) - 18(24) - 1(24)$

$= 3(153) - 19(24)$

9/10/2023
Monday

* Congruences

$a \equiv (b \bmod n)$
$8 \equiv 1 \pmod 7$
$8-1 \bmod 7$
$7 \bmod 7$
$8 \equiv -1 \pmod 9$
$8-(-1) \bmod 9$
$9 \bmod 9$

A number is congruent;

$a \equiv b \pmod n$ $\begin{cases} 7 \equiv 4 \pmod 3 \\ 7 \equiv -2 \pmod 3 \end{cases}$

Let n should be any positive integer and a, b be any integer.

$a \equiv b \pmod n$ if $a-b$ is divisible by n or $a-b$ is the multiple of n.

Eg: $8 \equiv 15 \pmod 7$

$8 \equiv -1 \pmod 9$

$8 \equiv 8 \pmod 9$.

* Properties

1) $a \equiv a \pmod n$

2) if $a \equiv b \pmod n$, then $b \equiv a \pmod n$ — Symmetric

$a-b = kn$ multiple of n $\qquad b-a = -(a-b)$

$= -kn$

3) If $a \equiv b \pmod n$, $b \equiv c \pmod n$ — transitive

then $a \equiv c \pmod n$.

$a-b = k_1 n$
$b-c = k_2 n$
$a-c = (a-b)+(b-c)$
$= k_1 n + k_2 n$
$= (k_1+k_2)n$

$a-b = k_1 n$
$b-c = k_2 n$
$a-c = (a-b)+(b-c)$
$= k_1 n + k_2 n = \underline{(k_1+k_2)n}$ multiple of n.

4) If $a \equiv b \pmod n$, then

$a+c \equiv b+c \pmod n$

$ac \equiv bc \pmod n$

given $a-b = kn$

Consider $(a+c)-(b+c) = a-b = kn$ $\Big\}$ proof

Consider $ac - bc = c(a-b)$
$= \underline{c \cdot kn}$

5) If $a \equiv b \pmod n$,

$c \equiv d \pmod n$ then, $a+c \equiv b+d \pmod n$

$ac \equiv bd \pmod n$.

$a-b = k_1 n$
$c-d = \underline{k_2 n}$ (proof)

$(a+c)-(b+d)$ proof
$= (a-b)+(c-d)$
$= k_1 n + k_2 n$
$= \underline{(k_1+k_2)n}$

6) If $a \equiv b \pmod{}$
then $a^k \equiv b^k \pmod n$

Q. Show that 41 divides ~~$2^{20}+1$~~ $2^{20}-1$.

ans) we have to prove $2^{20}-1 \equiv 0 \pmod{41}$

$2^{20} \equiv 1 \pmod{41}$

$2^5 = 32$ a number closer to 41

$2^5 = 32 \equiv -9 \pmod{41}$

$2^5 \equiv -9 \pmod{41}$

$(2^5)^4 \equiv (-9)^4 \pmod{41}$

$2^{20} \equiv 81 \times 81 \pmod{41}$ ⟶ ①.

$81 \equiv -1 \pmod{41}$ $\Rightarrow 82$ is a multiple of 41

$81^2 \equiv 1 \pmod{41} \longrightarrow ①$.

Q Find the remainder when 16^{53} is divided by 7

ans) $16 \equiv 2 \pmod 7$ $16-2=14$ (multiple of 7)

$16^{53} \equiv 2^{53} \pmod 7 \longrightarrow ①$.

$3\overline{)53}$ quotient 17, $\frac{3}{23}$, $\frac{21}{②}$

$2^3 \equiv 1 \pmod 7$

$(2^3)^{17} \equiv 1^{17} \pmod 7$

$2^{51} \equiv 1 \pmod 7$

$2^{51} \times 2^2 \equiv 2^2 \pmod 7$

$2^{53} \equiv 4 \pmod 7 \rightarrow ②$.

Q. Find the remainder when 41^{65} is divided by 7.

$41 \equiv 6 \pmod 7$

$41^{65} \equiv -1^{65} \pmod 7$ $41 \equiv -1 \pmod 7$

$41 - -1$

$7\overline{)41}$ quotient 5, $\frac{35}{6}$

$(41)^{65} \equiv 6^{65} \pmod 7$ $41 \equiv -1 \pmod 7$

$6^{65} \pmod 7$ $\therefore 41^{65} \equiv -1 \pmod 7$

$\Rightarrow 6^2 \equiv 1 \pmod 7$ Remainder $= \underline{\underline{6}}$

$(6^2)^{32} \equiv 1^{32} \pmod 7$

$6^{64} \equiv 1^{32} \pmod 7$

$6^{64} \cdot 6 \equiv 1 \cdot 6 \pmod 7$

$6^{65} \equiv 6 \pmod 7$

Q. Find the remainder when 2^{50} is divided by 7.

Find the remainder.

$3\overline{)50}$ quotient 16, $\frac{3}{20}$, $\frac{18}{2}$

ans) $2^3 \equiv 1 \pmod 7$

$(2^3)^{16} \equiv 1^{16} \pmod 7$

$16 \times 3 = 48$

$2^{48} \times 2^2 \equiv 2^2 \pmod 7$ $2^{48} \equiv 1 \pmod 7$

$= 2^{50} \equiv 4 \pmod 7$ $2^{50} \equiv 4 \pmod 7$

Q. Find the remainder when $1! + 2! + \dots + 99! + 100!$ is divided by 12, find remainder.

$1! \equiv 1$ $4! \equiv 0 \pmod{12} \equiv$

$2! \equiv 2$ $1 + 2 + 6 + \dots 0 + 0 + 0 \pmod{12}$

$3! \equiv 6$

$4! \equiv 0$ $\equiv 9 \pmod{12}$

$\vdots$

0

Q. $1! + 2! + \cdot + 100!$ is divided by 15.

$1! \equiv 1 \pmod{15}$ $= 1 + 2 + 6 + \cancel{24}\, 9 \pmod{15}$

$2! \equiv 2 \pmod{15}$ $= 18 \pmod{15}$

$3! \equiv 6 \pmod{15}$ $\equiv 3 \pmod{15}$

$4! = 24$ ~~(mod 15~~

$\equiv 9 \pmod{15}$ $\Rightarrow$ remainder is $\underline{\underline{3}}$

$5! \equiv 0 \pmod{15}$

$120 \equiv 0 \pmod{15}$

multiple of 15.

$\vdots$

Q. Find the remainder when $1^5 + 2^5 + 3^5 \dots + 100^5$ is divided by 4.

i) $a = 2n$ (even)
ii) $a = 2n+1$ (odd)

i) $a = 2n$

$a^5 = (2n)^5$

$= 32n^5$ (32 multiple of 4)

$\equiv 0 \pmod 4$ (for even).

iii) $a = 2n+1$

$a^2 = (2n+1)^2$ $[(a+b)^2]$

$= (2n)^2 + 2 \times 2n \times 1 + 1^2$ multiple of 4

$= \underline{4n^2} + \underline{4n} + 1$

$\equiv \underline{1} \pmod 4$

$a^5 = a^2 \times a^2 \times a \equiv \cancel{1 \times 1}$

$a^5 = 2+2+1$ $1 \times 1 \times a \pmod 4$

$1^5 + 2^5 + \dots + 99^5 + 100^5$

$\equiv 1 + 0 + 3 + 0 + \dots + 99 + 0 \pmod 4$

$\equiv \frac{50}{2}(1 + 99) \pmod 4$

$= 2500 \pmod 4$

$= 0 \pmod 4$

28/10/2025 Saturday

★ **Linear Diophantine Equations** (integer coefficient)

Often we are interested in integral solutions of equations with integral coefficients. Such equations are called diophantine equations.

Eg: $\underline{2x} + \underline{3y} = 4$ Variables x and y have integer coefficient

$y = \frac{1}{3}(4 - 2x)$

$x = 1$; $y = \frac{2}{3}$
$x = 2$, $y = 0$

by giving different values of x, we get different corresponding values for y. [integer solutions]

The simplest class of diophantine equations is the class of linear diophantine equations (LDEs). A linear diophantine equation in two variables x and y is diophantine equation of the form $ax + by = c$.

degree of x & y is 1
$ax^1 + by^1 = c$
no square.

While solving a LDE's

- Does every LDE have a solution?
  No eg: $2x + 4y = 5$
- If not, under what condition does an LDE have a solution
- If an LDE is solvable, what is the maximum number of solutions it can have?

★ **Theorem**: The LDE $ax + by = c$ is solvable if and only if $d \mid c$ where $d = (a,b)$. If x_0, y_0
→ d divides c   gcd
is a particular solution of the LDE, then all it's solutions are given by:

$x = x_0 + \left(\frac{b}{d}\right)t$ and $y = y_0 - \left(\frac{a}{d}\right)t$

where t is an arbitary integer.

Eg: $2x + 4y = 5$

The gcd of (2 and 4) is 2

2 is not a multiple of 5 (So it can't be solved).

Q. Check whether the given Diophantine equation can be solved or not.

1) $6x + 51y = 22$.

Here $\gcd(6, 51) = 3$

Since $3 \nmid 22$

3 doesn't divides 22

$\therefore 6x + 51y = 22$ has no integer solution.

2) $33x + 14y = 115$.

Here $\gcd(33, 14) = 1$

Since $1 \mid 115$,

This equation has solution.

Q. Determine all solutions of the positive integer of the given diophantine equations.

1. $18x + 5y = 48$

To find $\gcd(18, 5)$:

Using euclidean algorithm.

$18 = 3(5) + 3$

$5 = 1(3) + 2$

$3 = 1(2) + 1$

$2 = 2(1) + 0$

$\gcd(18, 5) = 1$

1 divides 48

$\therefore$ this diophantine eqn has solution.

To express $\gcd(18, 5)$ as a linear combination of 18 and 5

$1 = 3 - 1(2)$

$= 3 - 1[5 - 1(3)]$

$= 2(3) - 1(5)$

$= 2[18 - 3(5)] - 1(5)$

$= 2(18) - 7(5)$

$\therefore$ $18(2) + 5(-7) = 1$ → multiply with 48 as right hand side of $\underset{a}{18}x + \underset{b}{5}y = 48$.

$(\times 48)$

$18(96) + 5(-336) = 48$

one solution is $x_0 = 96$, $y_0 = -336$

$\therefore$ All solutions are $x = 96 + \frac{5}{1}t$

$y = -336 - \frac{18}{1}t$

$$\boxed{x = x_0 + \frac{b}{d}t \quad y = y_0 - \frac{a}{d}t}$$

$a = 18$, $b = 5$, $d = 1$ (gcd)

For positive solutions;

$96 + 5t > 0$ and $-336 - 18t > 0$

ie; $5t > -96$ and $18t > -336$

ie; $t > \frac{-96}{5} = -19\frac{1}{5}$ and $t < \frac{-336}{18} = -18\frac{12}{18}$

Combining these two inequalities we get,

$$-19\tfrac{1}{5} < t < -19\tfrac{12}{18}$$

$\therefore t = -19$

-19 $-19\tfrac{1}{5}$ $-19\tfrac{12}{18}$

Correspondingly. Solution is

$x = 96 + 5(-19) = 1$ $x = 96 + 5t$

$y = -336 - 18(-19) = 6$ $y = -336 - 18t$.

ie; $x = 1, y = 6$

2. Solve $172x + 20y = 1000$

ans): Apply euclidean algorithm to find gcd of (172, 20)

i) $172 = 8(20) + 12$

$20 = 1(12) + 8$

$12 = 1(8) + 4$

$8 = 2(4) + 0$

$\therefore \gcd(172, 20) = 4$

ii) Now express the gcd (172, 20) as a linear combination of 172 and 20.

$4 = 12 - 8$

$= 12 - [20 - 12]$

$= 2(12) - 20$

$= 2[172 - 8(20)] - 20$

$= (2)(172) + (-17)(20)$

ie; $172(2) + 20(-17) = 4$ make this 4 as 1000

x by 250, (x250) // $172x + 20y = 1000$ right hand side

$\Rightarrow 172(500) + 20(-4250) = 1000$

$\therefore x_0 = 500,\ y_0 = -4250$ is a solution of

$172x + 20y = 1000$

$\therefore$ All solutions are given by $x = x_0 + \frac{b}{d}t$

$y = y_0 + \frac{a}{d}t$

ie; $x = 500 + \frac{20}{4}t$ $\quad y = -4250 - \frac{172}{4}t$

$= 500 + 5t$ $\quad = -4250 - 43t$.

For positive solutions ~~are~~ we have to choose t such that $500 + 5t > 0$ and $-4250 - 43t > 0$

ie; $t > -100$ and $t < \frac{-4250}{43} = -98\frac{36}{43}$

Since t must be an integer, $t = -99$

$\therefore$ linear diophantine equation has a unique notation.

$x = 500 + 5(-99) = 5$

$y = -4250 - 43(-99) = 7$.

Q. A farmer purchased 100 head of livestock for a total cost of Rs. 4000/- . Prices were as follows: Calves @ Rs. 120, Lamb @ Rs. 50, Piglets @ Rs. 25. If the farmer obtained at least one animal of each type, how many of each did he buy..?

ans): Let the no. of calves be 'c', number of lambs be 'l' and number of piglets be 'p'.

$$120c + 50l + \cancel{20} 25p = 4000 \rightarrow ①$$

$$c + l + p = 100 \Rightarrow p = 100 - c - l$$

Subsititute in eqn - ①.

$$120c + 50l + 25(100 - c - l) = 4000$$

$$\underline{95}c + 25l = 1500.$$

$(120c - 25c)$ $(50l - 25l)$ $(4000 - 2500)$

=> Find gcd (95, 25)

$95 = 3(25) + 20$

$25 = 1(20) + 5$

$20 = 4(5) = 0$

gcd (95, 25) = 5.

=> Express in linear combination.

$5 = 25 - 1(20)$

$= 25 - 1[95 - 3(25)]$

$= (-1)95 + 4(25)$

=> $95(-1) + 25(4) = 5$ $(\times 300)$

=> $95(\overset{c}{-300}) + 25(\overset{l}{1200}) = 1500$

$c_0 = -300$; $l_0 = 1200$ (one of the soln)

$c = c_0 + \frac{b}{d}t$; $l = l_0 + \frac{a}{d}t$ $\quad d = 5$, $a = 95$, $b = 25$

$c = -300 + 5t$ $\quad l = 1200 - \frac{95}{5}t$

$l = 1200 - 19t$

for positive soln:

$-300 + 5t > 0$; $1200 - 19t > 0$

$t > 60$; $t < \frac{1200}{17} = 63.16$.

$t = 61$ or 62

(to get +ve soln of the given diophantine eqn).

60 — 63.16

when $t = 61$, $c = 5$; $l = 41$, $p = 54$ $\quad p = 100 - c - l$

when $t = 62$, $c = 10$; $l = 22$; $p = 68$

30/10/2023
Monday

Linear diophantine Equation

$\gcd(a,b) = d$

$$\boxed{\begin{aligned} x &= x_0 + \frac{b}{d}t \\ y &= y_0 - \frac{a}{d}t \end{aligned}}$$

x_0, y_0 are one solution of x and y.

Q. $18x + 5y = 48$

$\gcd(18, 5)$

$18 = 3(5) + 3$

$5 = 1(3) + 2$

$3 = 1(2) + 1 \Rightarrow \underline{\underline{GCD}}$

$2 = 2(1) + 0$

express gcd in linear combination

$1 = 3 - 1(2)$

$= 3 - 1[5 - 1(3)]$

$= 2(3) - 1(5)$

$= 2[18 - 3(5)] - 1(5)$

$= 2(18) - 6(5) - 1(5)$

$\underline{1 = 2(18) - 7(5)}$

multiply the eqn by $\times(48)$

$\Rightarrow 48 = 96(18) - 336(5)$

$x_0 = 96,\ y_0 = -336$ (take it as one solution)

All other solutions are given by:

$\overset{a}{18}x + \overset{b}{5}y + \overset{c}{48}$

$d = 1$ (gcd).

$x = 96 + 5t \qquad x = x_0 + \frac{b}{d}t$

$\underline{y = -336 - 18t \qquad y = y_0 - \frac{a}{d}t}$

★ **Linear Congruences**

Consider the linear congruence:

$ax \equiv b \pmod{n}$ — used solve linear congruence

$ax - b = ny$

$ax - ny = b$

$$\boxed{\begin{aligned} x &= x_0 + \frac{n}{d}t \\ d &= \gcd(a, n). \end{aligned}}$$

$x \equiv 2 \pmod{7}$

$x = 9$

$9 - 2$ multiple of 7

$\Rightarrow$ It has a solution if and only if ~~b~~ $\underline{\underline{d/b}}$ where $d = \gcd(a, n)$ then the solution is;

$x = x_0 + \frac{n}{d}t \quad ; d = \gcd(a, n)$

Q. Solve $18x \equiv 30 \pmod{42}$.

$ax \equiv b \pmod{n}$

$x_0 = 4$

gcd of $(18, 42) \Rightarrow 6$; $d = 6$

$18 \times 4 \equiv 36$

$36 - 30 = 6$ which is a multiple of 42.

Check whether $6/30$

$\therefore$ one has a solution

$x = 4 + \frac{42}{6}t$

here $t = 0, 1, 2, 3, \ldots 5$

$\boxed{t = 0, 1, 2 \ldots (d-1)}$

$\frac{\overset{3}{18} \times 4}{72}$ (42)

Q. $9x \equiv 21 \pmod{30}$.

$x_0 = 9 \times 3 = 27 - 21 = 6/30$

find $d = \gcd(9, 30) \Rightarrow 3$

Check $3/30$ (d/b)

$9 \times 9 = 81 - 21 = 60$

one has solution.

$x_0 = 9$

$x = 9 + \frac{30}{3} t; \quad t = 0, 1, 2 \dots$

$= \underline{9 + 10t}; \quad t = 0, 1, 2.$

★ Chinese Remainder Theorem

Let $n_1, n_2 \dots n_r$ be relatively prime non-zero integers, consider the linear congruences

$x \equiv a_1 \pmod{n_1}$

$x \equiv a_2 \pmod{n_2}$

$\vdots$

$x \equiv a_r \pmod{n_r}$

Let, $n = n_1, n_2 \dots \cancel{n_r}\, n_r$ and

$N_k = \frac{n}{n_k}$

then $\gcd(n_k, N_k) = 1$

Therefore the linear congruence

$n_k x \equiv 1 \pmod{n_k}$ has a solution and let it be x_k.

Then the simultaneous solution of the linear congruences is given by.

$\bar{x} = a_1 N_1 x_1 + a_2 N_2 x_2 + \dots + a_r N_r x_r \pmod{n}$

Q. $x \equiv 2 \pmod 3$

$x \equiv 3 \pmod 5$

$x \equiv 2 \pmod 7$.

$a_1 = 2 \quad n_1 = 3$

$a_2 = 3 \quad n_2 = 5$

$a_3 = 2 \quad n_3 = 7$

$n = n_1 \times n_2 \times n_3 = 3 \times 5 \times 7 = \underline{105}$

$N_1 = \frac{n}{n_1} = \frac{\cancel{3} \times 5 \times 7}{\cancel{7}} = 35$

$N_2 = \frac{n}{n_2} = 21 \quad \left(\frac{3 \times \cancel{5} \times 7}{\cancel{7}}\right)$

$N_3 = \frac{n}{n_3} = 15 \quad \left(\frac{3 \times 5 \times \cancel{7}}{\cancel{7}}\right)$

Consider $N_1 x \equiv 1 \pmod{n_1}$ $k = 1$

$= 35x \equiv 1 \pmod 3$

$35 \times 2 = 70 - 1 = 69$ divides 3. multiple

$\underline{x_1 = 2}$

$N_2 x \equiv 1 \pmod{n_2}$

$21x \equiv 1 \pmod 5$

$\underline{x_2 = 1}$

$N_3 x \equiv 1 \pmod{n_3}$

$15x \equiv 1 \pmod 7$

$\underline{x_3 = 1}$

$\bar{x} = (2 \times 35 \times 2) + (3 \times 21 \times 1) + (2 \times 15 \times 1) = 233$

$\bar{x} = 233 \pmod{105}$ mod n

$= 23 \pmod{105}$

$\bar{x} = 233$

$233 - 2 = 231$

$x \equiv 2 \pmod 3$

$x \equiv 3 \pmod 5$

$x \equiv 2 \pmod 7$

$(105 \times 2) = 210$

$233 - 210 = \underline{23}$.

6/11/2023 Monday

Q. Solve $x \equiv 1 \pmod 3$ (Could be mutually prime) (No common divisor)

$x \equiv 2 \pmod 5$

$x \equiv 3 \pmod 7$.

$N = 3 \times 5 \times 7 = 105$

$N_1 = 5 \times 7 = 35$

$N_2 = 3 \times 7 = 21$

$N_3 = 15 \ (3 \times 5)$

Consider $N_1 x \equiv 1 \pmod{n_1}$

$35x \equiv 1 \pmod 3$

$x_1 = 2$

$x_2 = 1 \qquad 21x \equiv 1 \pmod 5$

$x_3 = 1 \qquad 15x \equiv 1 \pmod 7$

$\bar{x} = a_1 N_1 x_1 + a_2 N_2 x_2 + a_3 N_3 x_3$.

$\bar{x} = (1 \times 35 \times 2) + (2 \times 21 \times 1) + (3 \times 15 \times 1)$

$\bar{x} = 157 \pmod{105}$

$\bar{x} \quad 52 \pmod{105}$

$157 - 105$

$21 \times 3 = 63$, $35 \times 2 = 70$, $21 \times 2 = 42$, $15 \times 3 = 45$

$70 + 42 + 45 = 157$

$157 - 105 = 52$

First order linear recurrence relation

A first order linear recurrence relation is of the form $a_{n+1} = d\,a_n + e$ (constant)

order = 1 (no raised)

$2, 6, 18$; a_0, a_1, a_2; $a_1 = 3a_0$; $a_2 = 3a_1$; $a_{n+1} = 3a_n$; $a_1 = 3a_0 = 6$; $a_2 = 3a_1 = 18$; $n = 0, 1, 2, \dots$; $a_0 = 2$

Q. Find a recurrence relation corresponding to following sequences.

i) $2, 6, 18, \dots$

$a_0 = 2$

$a_1 = 3a_0$

$a_2 = 3a_1$

$\vdots$

$a_{n+1} = 3a_n$; where $n = 0, 1, 2, \dots$

$a_0 = 2$

ii) $2, 10, 50, \dots$

$a_0 = 2$

$a_1 = 5a_0$

$a_2 = 5a_1$

$\vdots$

$a_{n+1} = 5a_n$

$n = 0, 1, 2, \dots$

$a_0 = 2$

iii) $7, \frac{14}{5}, \frac{28}{25}, \dots$

$a_0 = 7$

$a_1 = \frac{2}{5} a_0$

$a_2 = \frac{2}{5} a_1$

$\vdots$

$a_{n+1} = \frac{2}{5} a_n$

$n = 0, 1, 2, \dots$

$a_0 = 7$

$\Rightarrow a_{n+1} = d\,a_n$

$a_1 = d\,a_0$

$a_2 = d\,a_1 = d^2 a_0$

$a_3 = d(a_2) = d^3 a_0$

$a_n = d^n a_0$ (Solution for recurrence relation)

* Solution of recurrence relations

Consider the recurrence relation

$a_{n+1} = d a_n$

a_0

$a_1 = d a_0$

$a_2 = d a_1 = d^2 a_0$

$a_3 = d(a_2) = d^3 a_0$

$\boxed{a_n = d^n a_0}$

Q. Solve $a_n = 7a_{n-1}$; $n \geq 1$; $a_2 = 98$

$d = 7$

$a_2 = 7^2 a_0$

$98 = 49\, a_0$, $a_n = 7^n \times 2$

$a_0 = \frac{98}{49} = 2$; $a_0 = 2$

Q. Solve $3a_{n+1} - 4a_n = 0$, $n \geq 0$; $a_1 = 5$.

$\Rightarrow 3a_{n+1} = 4a_n$ or

$a_{n+1} = \frac{4}{3} a_n$

Now, we can see that $d = 4/3$

Solution is $a_n = d a_0$ (given $a_1 = 5$)

$a_1 = 5$

$\cancel{5 = d^5}$ $a_1 = d^1 a_0$ $a_n = \left(\frac{4}{3}\right)^n \times \frac{15}{4}$

$5 = \frac{4}{3} a_0$

$a_0 = \frac{15}{4}$

Q. $2a_n - 3a_{n-1} = 0$; $n \geq 1$, $a_4 = 81$.

$2a_n = 3a_{n-1}$

$a_n = \frac{3}{2} a_{n-1}$

$a_4 = d^4 a_0$

$81 = \left(\frac{3}{2}\right)^4 a_0$

$a_0 = \frac{81 \times 16}{9 \times 9} = 16$

Q. Given $a_{n+1} - d a_n = 0$

$a_3 = \frac{153}{49}$ Find d.

$a_5 = \frac{1377}{2401}$

Solution is $\boxed{a_n = d^n a_0}$

$a_3 = d^3 a_0$

$a_5 = d^5 a_0$

$= \frac{153}{49} = d^3 a_0 \longrightarrow (1)$

$\frac{1377}{2401} = d^5 a_0 \longrightarrow (2)$.

(eqn 2 × eqn 1)

$d^2 = \cancel{d^2 \times} \frac{1377}{2401} \times \frac{49}{153}$

$d = 3/7$.

Q. A bank pays 6% annual interest on savings compounding the interest monthly, if you deposit \$1000 dollars on first day of May, how much will you get after an year.

ans) Let P_0 be the amount deposited.

monthly interest is $= 6/12 = 1/2\%$ or $.5\%$

$\frac{.5}{100} = .005$ to convert percentage to number.

Monthly rate is $= .005$

$P_1 = P_0 + .005P_0 = (1.005)P_0$ $\quad P_0(1+.005)$

$P_2 = P_1 + 0.005P_1 = (1.005)P_1$

Recurrence relation is

$P_n = d^n P_0$ $\qquad P_0 = 1000$

$P_n = (1.005)^n P_0$

$P_{12} = d^{12} P_0$

$P_{12} = (1.005)^{12} P_0$

$= (1.005)^{12} \times 1000$

$= 1061.677$

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Second order linear homogeneous recurrence relations with constant coefficient:

A linear recurrence relation of order k with constant coefficients is of the form:

$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = F(n)$

⇒ For example

order 1: $C_0 a_n + C_1 a_{n-1} = F(n)$

order 2: $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = F(n)$.

⇒ If $F(n) = 0$, $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0$ which is, homogeneous second order linear recurrence relation.

Suppose $a_n = k r^n$ is a solution of ①

$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0 \longrightarrow ①$

Substituting k_r^n in equation 1.

$C_0 k_r^n + C_1 k_r^{n-1} + C_2 k r^{n-2} = 0$

$\boxed{C_0 r^2 + C_1 r + C_2 = 0}$ characteristic eqn of eqn ①.

Case i) Real distinct

Let $r = r_1, r_2$

then the solutions are $a_n = k r_1^n$, $a_n = k r_2^n$

then the general solution is; $\boxed{a_n = A r_1^n + B r_2^n}$

Q. Solve $a_n + a_{n-1} - 6a_{n-2} = 0$; $n \geq 2$; $a_0 = -1, a_1 = 8$

Characteristic equation is $r^2 + r - 6 = 0$ $(r^2 + r - 6 = 0)$

$(r+3)(r-2) = 0$ $\quad 3 \times -2 = -6$ sum $\quad 3 + -2 = 1$ multiple $= -6$

$r = -3, 2$

∴ general solution in $a_n = A(-3)^n + B(2)^n$ $\quad 1^0 = 1$

$a_0 = -1 = A(-3)^0 + B(2)^0 = A + B \rightarrow$ ④
$(n=0)$

$a_1 = 8 = 3A + 2B \longrightarrow$ ⑤

$A(-3)^1 + B(2)^1$

$= 3A + 2B$.

$3A + 3B = -3$
$-3A + 2B = 8$

$5B = 5, \quad B = 5/5 = 1$

$A = -1 - B, \quad -1 - 1 = \underline{-2}$

$a_n = -2(-3)^n + (2)^n$

Q. $a_n = 5a_{n-1} + 6a_{n-2} \quad n \geq 2; \quad a_0 = 1; \quad a_1 = 3.$

ans) characteristics equation

$a_n - 5a_{n-1} - 6a_{n-2} = r^2 - 5r - 6$

$\Rightarrow (r-6)(r+1)$

$r = 6, -1$

general solution is $a_n = A(6)^n + B(-1)^n$

$a_0 = A(6)^0 + B(-1)^0 \longrightarrow (1)$
$= \underline{A + B}$

$a_1 = 3 = A(6)^1 + B(-1)^1 \longrightarrow (2)$
$= 6A - B$

$A + B = 1$
$6A - B = 3$

$7A = 4$
$A = 4/7$

$6\,4/7 - B = 3$

$B = 6\,4/7 - 3 = \underline{3/7}$

$7A = 4$
$A = 4/7$
$A + B = 1$
$4/7 + B = 1$
$B = 1 - 4/7 = 3/7$

$6\frac{4}{7} - 3 = \frac{24}{7} - \frac{3}{1} = \frac{24-21}{7} = 3/7$

Case ii) Distinct Complex Roots. (complex number) (i imaginary)

Q. Solve $a_n = 2(a_{n-1} - a_{n-2}); \quad n \geq 2, \ a_0 = 1, a_1 = 2$

$a_n - 2a_{n-1} + 2a_{n-2} = 0$

characteristic equation is $r^2 - 2r + 2 = 0$

$r = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$
$= \underline{1 + i}$

Solutions are:

$a_n = A(1+i)^n + B(1-i)^n$

$1 + i = \sqrt{2}\cos\frac{\pi}{4} + i\sqrt{2}\sin\frac{\pi}{4}$

$a_n = A\left[\sqrt{2}\cos\frac{\pi}{4} + i\sqrt{2}\sin\frac{\pi}{4}\right]^n +$

$1 - i = \sqrt{2}\cos\left(\frac{-\pi}{4}\right) + i\sqrt{2}\sin\left(\frac{-\pi}{4}\right)$

$B\left[\sqrt{2}\cos(\pi/4) - \sqrt{2}\sin(\pi/4)\right]^n$

$\boxed{\cos\theta + i\sin\theta = \cos n\theta + i\sin\theta}$

$= A(\sqrt{2})^n\left(\cos n\,\pi/4 + i\sin n\,\pi/4\right) +$

$r = \sqrt{2}$
$\theta = \tan^{-1} 1 = \pi/4$

$B(\sqrt{2})^n\left[\cos n\pi/4 - i\sin n\pi/4\right]$

$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$= \left[A(\sqrt{2})^n + B(\sqrt{2})^n\right]\cos n\,\pi/4 + \left[A(\sqrt{2})^n - B(\sqrt{2})^n\right] i\sin n\pi/4$

$\theta = \tan^{-1}(-1) = -\tan^{-1} 1 = \frac{-\pi}{4}$

Constant k is used

$= (\sqrt{2})^n k_1 \cos n\,\pi/4 + (\sqrt{2})^n k_2 \sin n\,\pi/4$

$a_n = \underline{(\sqrt{2})^n\left[k_1 \cos n\pi/4 + k_2 \sin n\pi/4\right]}$

$a_0 = 1 = \left[k_1 \cos 0\pi/4 + k_2 \sin 0\,\pi/4\right]$

$\cos 0 = 0$
$\sin 0 = 0$

$\underline{1 = k_1}$

$a_1 = 2 = \sqrt{2}\left[k_1 \cos\pi/4 + k_2 \sin\pi/4\right]$
$= 2 = \sqrt{2}\left[k_1\, 1/\sqrt{2} + k_2\, 1/\sqrt{2}\right]$

$\cos\pi/4 = 1/\sqrt{2}$
$\sin\pi/4 = 1/\sqrt{2}$

$2 = k_1 + k_2$ $\quad$ $\underline{k_1 = 1}$

$k_2 = 2 - 1 = \underline{1}$ $\quad$ $\underline{k_2 = 1}$

$a_n = (\sqrt{2})^n \left(\cos n\pi/4 + \sin n\pi/4\right)$

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case iii) Real and equal roots.

Q. Solve $a_n - 4a_{n-1} + 4a_{n-2} = 0$; $n \geq 2$, $a_0 = 1$, $a_1 = 3$.

characteristic equation be $\quad r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$, $\underline{r = 2, 2}$.

∴ general solution is

$\underline{a_n = A(2)^n + Bn(2)^n}$.

$n=0$ $\quad a_0 = 1$; $1 = A + 0$ $\quad \underline{A = 1}$

$n=1$ $\quad a_1 = 3$; $3 = 2A + 2B$ $\quad a_n = 2^n + 1/2\, n 2^n$

$\underline{B = 1/2}$

Q. $a_n - 6a_{n-1} + 9a_{n-2} = 0$; $n \geq 2$, $a_0 = 5$, $a_1 = 12$.

characteristic equation be : $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$ $\quad$ $\underline{r = 3, 3}$.

∴ general solution is

$a_n = A3^n + Bn3^n$.

$n=0$ $\quad a_0 = 5$; $5 = A + 0$, $\underline{A = 5}$ $\quad$ $\underline{a_n = 5 3^n - n3^n}$

$n=1$ $\quad a_1 = 12$; $12 = 3A + 3B$

$12 - 15 = 3B$

$B = \underline{-1}$

$12 = 3 \times 5 + 3B$

$12 = 15 + 3B$

$3B = 12 - 15$

$-3 = 3B$

$B = -3/3 = \underline{-1}$

★ Non homogeneous Recurrence Relation

Eg: Solve $a_n - 3a_{n-1} = 5(7)^n$; $n \geq 2$, $a_0 = 2$.

(1) not zero (so non homogeneous).

Consider $a_n - 3a_{n-1} = \underline{0}$ (homogeneous) //first order so take r.

$r - 3 = 0$, $\underline{r = 3}$ → (2) $\quad$ i) $\quad$ (h) → homogeneous

Solution is $a_n^{(h)} = \underline{A(3)^n}$.

ii) $a_n - 3a_{n-1} = \underline{5(7)^n}$ [first order]

$a_n^{(p)}$ → particular solution

we assume $a_n^{(p)} = B(7)^n$ be a solution of (1).

constant

∴ $B7^n - 3\,B7^{n-1} = 5(7)^n$

$a_n \quad a_{n-1}$

(7^{n-1}) common. $\quad 7B - 3B = 5 \times 7$

$4B = 35$, $B = 35/4$

∴ $\underline{a_n = 35/4\, 7^n}$

∴ $a_n = a_n^{(h)} + a_n^{(p)}$

$= A3^n + \frac{35}{4} 7^n$

$a_0 = 2$; $2 = A + 35/4$

$A = 2 - \frac{35}{4} = \frac{-27}{4}$

Q. Solve $a_n - 3a_{n-1} = 5(3)^n$; $n \geq 2$, $a_0 = 2$ → (1) $\quad$ //Method of undetermined coefficient

Consider $a_n - 3a_{n-1} = 0$ → (2)

$r - 3 = 0$, $r = 3$

Solution is $a_n(h) = \underline{A(3)^n}$ $\quad B(3)^n$

$a_n - 3a_{n-1} = 5(3)^n$ $\quad$ As same multiply by (n)

Assume $a_n(p) = \underline{Bn(3)^n}$ be solution of (1).

$Bn3^n - 3B(n-1)3^{n-1} = 5 3^{n-1}$

$3\cancel{n}B - 3\cancel{n}B + 3B = 15$

$B = 15/3$

$a_n(p) = \frac{15}{3} n 3^n$

$a_n = a_n^{(h)} + a_n^{(p)}$

$= A3^n + 15/3\, n3^n$.

$a_0 = 2$ $\quad 2 = A + 15/3$

$\underline{A = -5/3}$.

F^n	$a_n^{(p)}$
k^n	Bk^n Bnk^n
n^2	$Bn^2 + Cn + D$
n	$Bn + C$ $Bn^2 + Cn$

Q. Solve $a_{n+2} - 4a_{n+1} + 3a_n = -200$ (2nd order)

$n \geq 0$, $a_0 = 3000$, $a_1 = 3300$

$a_{n+2} - 4a_{n+1} + 3a_n = 0$ (homogeneous)

$r^2 - 4r + 3 = 0$ $\times 3$, $+ -4$ $1 \times 3 = 3$, $1 + 3 = 4$

$(r-3)(r-1) = 0$

$r = 1, 3$.

$a_n^{(h)} = A(1)^n + B(3)^n \rightarrow ①$.

$a_{n+2} - 4a_{n+1} + 3a_n = -200$

constant

$A(1)^n = A$ Assume $a_n^{(p)} = Cn$

$C(n+2) \cancel{+4C} - 4C(n+1) + 3Cn = -200$

C constant so multiply by n. $a_n = C$ so $a_{n+2} = C(n+2)$ $-4C(n+1)$

$2C - 4C = -200$

$-2C = -200$, $C = \frac{-200}{-2} = 100$

$C = \underline{\underline{100}}$

$\underline{\underline{a_n^{(p)} = 100n}}$

$a_n = A + B3^n + 100n$. $(n=0)$

$a_0 = 3000$; $3000 = A + B$

$a_1 = 33000$; $33000 = A + 3B + 100$

$A + B = 3000$

$A + 3B = 3200$

$2B = 200$

$B = \underline{\underline{100}}$ $A = \underline{\underline{2900}}$

$A + 100 = 3000$

$A = 3000 - 100$

$= 2900$

$\underline{= 100 + 2900 3^n + 100n}$

Q. $a_{n+1} - 2a_n - 1 = 0$; $a_0 = 0$; $a_1 = 1$ (first order)

constant

$a_{n+1} - 2a_n = 1$

$r - 2 = 0$, $r = \underline{\underline{2}}$

Solution $\Rightarrow a_n^{(h)} = \underline{A(2)^n} \rightarrow ①$.

no constant in eqn ①.

$a_{n+1} - 2a_n = 1$

$\cancel{a(n)}$ $a_n^{(p)} = B$

$B - 2B = 1$

$-B = 1$

$B = \underline{\underline{-1}}$.

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n^{(p)} = \underline{\underline{-1}}$.

$a_n = A2^n - 1$

$a_1 = 1$; $1 = 2A - 1$, $A = \underline{\underline{1}}$

$a_n = \underline{\underline{2^n - 1}}$