$(a+b)$

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Q. Find remainder when $1^5+2^5+\ldots+100^5 \div 4$

- Either $a=2n$ or $a=2n+1$

i) $a^5=(2n)^5$

$=32n^5 \equiv 0 \pmod 4$

ii) $a^5=(2n+1)^5$

Consider

$a^2=(2n+1)^2$

$=4n^2+4n+1$

$a^2 \equiv 1 \pmod 4$

$a^5 = a \pmod 4$

$1^5+2^5+\ldots+100^5 \equiv 1+0+3+0+5+\ldots+99+0 \pmod 4$

$\equiv \frac{50}{2}(1+99) \pmod 4$

$= 2500 \pmod 4$

$\div 0 \pmod 4$

Ans: 0

Linear Diophantine Equation

$\gcd(a,b)=d$

$x = x_0 + \frac{b}{d}t \qquad y = y_0 - \frac{a}{d}t$

Q→ $18x+5y=48$

$\gcd(18,5)$

$\Rightarrow 18 = 5\times3+3$

$5 = 3\times1+2$

$3 = 2\times1+1$

$1 = 1\times1+0$

$1 = 3-2(1)$

$1 = 3-1(5-3(1))$

$1 = 3-5+1(3) \;=\; 1 = 2(3)-5$

$1 = 2(18-3(5))-5$

$1 = 2(18)-7(5)$

~~$1 = (18) \times 2 - 7(5)$~~

$1 = 2(18) - 7(5)$

$\times 48 \quad 48 = 96(18) - 336(5)$

$x_0 = 96 \quad ; \quad y_0 = -336$

$x = 96 + \frac{5}{1} \times t \qquad y = -336 - \frac{18}{1} \times t$

Linear Congruences

Consider $ax \equiv b \pmod{n}$ having a solution iff $d \mid b$ where $d = \gcd(a,n)$ then solution is

$$x = x_0 + \frac{n}{d} t$$

solve till $(d-1)$.

Q. Solve $18x \equiv 30 \pmod{42}$

$\gcd(18, 42)$

$42 = 18 \times 2 + 6$

$18 = 6 \times 3 + 0$

$d = 6.$

6 divides 30 $\therefore$ solution exist

find 1^{st} solution by giving values $1, 2, 3 \ldots$

$x = 4$ satisfies $\therefore x_0 = 4$

$x = 4 + \frac{42}{6} t$

put values $0, 1, 2, 3, 4, 5$ for t.

Q $9x \equiv 21 \pmod{33}$

gcd$(9, 33) = 3$

3 divides 21 ∴ solutions exists.

for $x_0 = 6$

$x = 6 + \frac{33}{3}t$ $\quad x = 6 + 11t$

$\underline{\underline{x_1 = 6}} \quad \underline{\underline{x_2 = 28}} \quad x_3 = 39.$

Chinese Remainder Theorem

Let $n_1, n_2, n_3 \dots n_r$ be relatively prime non 0 integers

Consider linear congruences $x \equiv a_1 \pmod{n_1}$

$x \equiv a_2 \pmod{n_2}$

$\vdots$

$x \equiv a_r \pmod{n_r}$

Let $n = n_1 \times n_2 \times n_3 \times \dots \times n_r$

$N_k = \frac{n}{n_k}$

then gcd$(n_k, N_k) = 1$

∴ linear congruence $N_k x \equiv 1 \pmod{n_k}$ has a solution x_k

Then simultaneous sol of linear congruence is given by

$$\bar{x} = a_1 N_1 x_1 + a_2 N_2 x_2 + \dots + a_r N_r x_r \pmod{n}$$

Q. Solve $x \equiv 2 \bmod 3$ $\quad n_1 = 3 \quad n_2 = 5 \quad n_3 = 7.$

$x \equiv 3 \bmod 5$ $\quad a_1 = 2 \quad a_2 = 3 \quad a_3 = 2.$

$x \equiv 2 \bmod 7$

$n = n_1 \times n_2 \times n_3$

$n = 3 \times 5 \times 7 \quad n = 105$

$N_1 = \dfrac{n}{n_1} = \dfrac{3\times5\times7}{3} = 5\times7 = 35$

$N_2 = \dfrac{n}{n_2} = 3\times7 = 21 \qquad N_3 = \dfrac{n}{n_3} = 3\times5 = 15$

Consider $N_1 x \equiv 1 (\bmod n_1)$

$35x \equiv 1 \bmod 3 \quad \therefore x_1 = 2.$

$N_2 x \equiv 1 (\bmod n_2) \qquad N_3 x \equiv 1 (\bmod n_3)$

$21 x \equiv 1 (\bmod 5) \quad x_2 = 1 \qquad 15x \equiv 1 (\bmod 7) \quad x_3 = 1$

$\bar{x} = a_1 N_1 x_1 + a_2 N_2 x_2 + a_3 N_3 x_3 \quad (\bmod n)$

$= 2\times35\times2 + 3\times21\times1 + 2\times15\times1 \; (\bmod 105)$ 210.

$\bar{x} = 233 (\bmod 105)$

$\Rightarrow \bar{x} = 23 (\bmod 105)$

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Q. $x \equiv 1 \bmod 3$ $\qquad a_1 = 1 \quad a_2 = 2 \quad a_3 = 3$

$x \equiv 2 \bmod 5$

$x \equiv 3 \bmod 7$

$n = 105$

$N_1 = 75 \qquad N_2 = 21 \qquad N_3 = 15$

$N_1 x \equiv 1 \bmod n_1$

$35x \equiv 1 \pmod 3 \qquad \underline{\underline{x_1 = 2.}}$

$N_2 x \equiv 1 \bmod n_2 \qquad\qquad N_3 x \equiv 1 \bmod n_3$

$21x \equiv 1 \bmod 5 \quad \underline{\underline{x_2 = 1.}} \qquad 15x \equiv 1 \bmod 7 \quad \underline{\underline{x_3 = 1}}$

$\bar{x} = a_1 N_1 x_1 + a_2 N_2 x_2 + a_3 N_3 x_3 \pmod n$

$1 \times 35 \times 2 + 2 \times 21 \times 1 + 3 \times 15 \times 1 \bmod 105$

$\bar{x} = 157 \bmod 105$

$\bar{x} = 52 \bmod 105$

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First order linear recurrence relation

- A first order linear recurr relation is of the form $a_{n+1} = d a_n + c$ where d & c are constants.

Q Find a recurr relation based on

i) 2, 6, 18 . . . .

$a_0 = 2$

$a_1 = 3a_0$

$a_2 = 3a_1$

$$\left[\begin{array}{l} a_{n+1} = 3a_n \quad ; n = 0, 1, 2 \\ a_0 = 2 \end{array}\right]$$

ii) 2, 10, 50 . . .

$$\left[\begin{array}{l} a_{n+1} = 5a_n \; ; n = 0, 1, 2 \\ a_0 = 2 \end{array}\right]$$

iii) $7, \frac{14}{5}, \frac{28}{25} \ldots$

$\Rightarrow a_{n+1} = \frac{2}{5} a_n \qquad a_0 = 7.$

→ Solution of recurr. relation

- Consider $a_{n+1} = d a_n$

$a_1 = d a_0 \qquad a_2 = d a_1 = d^2 a_0 \qquad a_3 \Rightarrow d(a_2) = d^3 a_0.$

$\therefore \underline{a_n = d^n a_0}$

Q. Solve $a_n = 7a_{n-1}$ $n \geq 1$, $a_2 = 98.$

$d = 7.$

$a_2 = d^2 a_0$

$98 = 7^2 a_0 \qquad a_0 = \frac{98}{49}.$

$a_n = 7^n \times \frac{98}{49} \quad = \quad a_n = 2 \times 7^n$

Q. Solve $3a_{n+1} - 4a_n = 0 \quad n \geq 0 \; ; \quad a_1 = 5$

$= \cancel{3}a_{n+1} = \frac{4a_n}{3}$

$d = \frac{4}{3} \qquad a_1 = d a_0 \qquad a_0 = \frac{15}{4}$

$a_n = \left(\frac{4}{3}\right)^n \times \frac{15}{4}.$

Q. $2a_n - 3a_{n-1} = 0 \quad n \geq 1 \quad a_4 = 81$

$a_n = \frac{3a_{n-1}}{2}$

$a_4 = d^4 a_0$

$a_0 = \frac{a_4}{d^4} = \frac{81 \times 16}{81}$.

$d = \frac{3}{2}.$

$a_0 = 16.$

$a_n = d^n a_0$

$= \left(\frac{3}{2}\right)^n \times 16$

Q: Given $a_{n+1} - da_n = 0 \quad a_3 = \frac{153}{49} \quad a_5 = \frac{1377}{2401}$ find $\to d$

$a_{n+1} = da_n \qquad a_n = d^n a_0$

$a_3 = d^3 a_0 \qquad \frac{153}{49} = d^3 a_0$ — (1)

$a_5 = d^5 a_0 \qquad \frac{1377}{2401} = d^3 a_0.$ — (2)

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Dividing 2/1

$d^2 = \frac{1377}{2401} \times \frac{49}{153}$

$d^2 = \frac{3}{7}$

Q. A bank pays 6% annual interest on saving coumpounding monthly. If you deposit 1000 on first day of may How much will you get after on year.

Ans. Let P_0 be amt deposited.

monthly interest $= \frac{6}{12} = 0.5\% = 0.005$

$P_1 = P_0 + 0.005 P_0$

$= P_0 (1 + 0.005)$

~~[illegible]~~

$P_n = (1.005)^n P_0$

$P_{12} = (1.005)^{12} 1000.$

$P_{12} =$

$z = x + iy \quad r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1} y/x$

$z = r\cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

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Second order linear homogenous recurrence relation with constant coefficient

A linear recurrence relation of order K with const coefficient is of the form

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots C_K a_{n-K} = f(n)$$

order 1: $C_0 a_n + C_1 a_{n-1} = f(n)$

order 2: $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n)$

~~…~~ For recurrence relation to be homogenous $f(n) = 0$

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0 \quad -①$$

Suppose $a_n = K r^n$ is a solution.

$$C_0 K r^n + C_1 K r^{n-1} + C_0 K r^{n-2} = 0$$

$\boxed{\div K r^{n-2}} \Rightarrow \boxed{C_0 r^2 + C_1 r + C_2 = 0} \Rightarrow$ characteristic eqn

$\Rightarrow$ i) if r is real & distinct

$$r = r_1, r_2$$

$$a_n = K(r_1^n), K(r_2^n)$$

General solution: $a_n = \boxed{A_1(r_1^n) + A_2(r_2^n)}$

$\Rightarrow$ ii) Distinct complex roots

$a_n = A(r_1)^n + B(r_2)^n \Rightarrow r_1, r_2$ are complex then change complex

$\Rightarrow$ iii) Real & equal roots

$a_n = A(r)^n + B\,n(r)^n$

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Q. Solve $a_n + a_{n-1} - 6a_{n-2} = 0$; where $n >= 2$

$a_0 = -1 \quad a_1 = 8$

Characteristic eqn $\Rightarrow r^2 + r - 6 = 0$

$r^2 + 3r - 2r - 6 = 0$

$(r+3)(r-2) = 0$

$r = -3, 2$ [real & distinct]

a_n General sol $= A_1 r_1^n + A_2 r_2^n$

$a_n = A_1(-3)^n + A_2(2)^n.$

$a_0 = -1$	$a_1 = 8.$
$-1 = A_1(-3)^0 + A_2(2)^0$	$8 = A_1(-3)^1 + A_2(2)^1$
$A_1 + A_2 = -1$	$-3A_1 + 2A_2 = 8$

$$\begin{array}{r} 3A_1 + 3A_2 = -3 \\ -3A_1 + 2A_2 = 8 \\ \hline 5A_2 = 5 \end{array} \qquad A_2 = 1 \qquad A_1 = -2$$

$\underline{a_n = -2(-3)^n + 1(2)^n}$

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a. Solve

$a_n = 5a_{n-1} + 6a_{n-2}$ $n \geq 2$ $a_0 = 1$ $a_1 = 3.$

$a_n - 5a_{n-1} - 6a_{n-2} = 0.$

Charecteristic eqn = $r^2 - 5r - 6 = 0.$

$r^2 + r - 6r - 6 = 0$

$r(r+1) - 6(r+1) = 0$

$(r+1)(r-6) = 0$

$r = -1, 6$

$a_n = A_1(-1)^n + A_2(6)^n.$ $a_0 = 1$ $a_1 = 3$

$1 = A_1 + A_2$ $3 = A_1(-1) + A_2(6)$

$3 = -A_1 + 6A_2$

$$\begin{array}{r} A_1 + A_2 = 1 \\ -A_1 + 6A_2 = 3 \\ \hline 7A_2 = 4. \end{array}$$

$A_2 = 4/7.$

$A_1 + 4/7 = 1$

$A_1 = \frac{7-4}{7} = 3/7.$

$a_n = \frac{3}{7}(-1)^n + \frac{4}{7}(6)^n$

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Q. Solve $a_n = 2(a_{n-1} - a_{n-2})$ $n \geq 2$ $a_0 = 1$, $a_1 = 2$

$a_n - 2a_{n-1} + 2a_{n-2} = 0$

C.E: $r^2 - 2r + 2 = 0$.

$r = \dfrac{2 \pm \sqrt{4-8}}{2}$

$= 1 \pm i$

$a_n = A(1+i)^n + B(1-i)^n$

$1 \pm i$ $r = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1} 1$

$1+i = \sqrt{2}\cos \pi/4 + \sqrt{2}\sin \pi/4$

$1-i$

$\downarrow$

4th quadrant

$\theta = \tan^{-1}(-1) = -\tan^{-1}(1)$

$= -\pi/4$

$\therefore 1-i = \sqrt{2}\cos -\pi/4 + i\sqrt{2}\sin -\frac{\pi}{4}$

$a_n = A\left(\sqrt{2}\cos \pi/4 + i\sqrt{2}\sin \pi/4\right)^n +$

$B\left(\sqrt{2}\cos -\pi/4 + i\sqrt{2}\sin -\pi/4\right)^n$

$\cos -\theta = \cos\theta$

$\sin -\theta = -\sin\theta$

$= A\left(\sqrt{2}\cos \pi/4 + i\sqrt{2}\sin \pi/4\right)^n + B\left(\sqrt{2}\cos \pi/4 - i\sqrt{2}\sin \pi/4\right)^n$

$= A(\sqrt{2})^n\left[\cos n\frac{\pi}{4} + i\sin n\frac{\pi}{4}\right] + B(\sqrt{2})^n\left[\cos n\frac{\pi}{4} - i\sin n\frac{\pi}{4}\right]$

$= \left[A(\sqrt{2})^n + B(\sqrt{2})^n\right]\cos n\frac{\pi}{4} + \left[A(\sqrt{2})^n - B(\sqrt{2})^n\right] i\sin n\frac{\pi}{4}.$

$a_n = \underbrace{(\sqrt{2})^n K_1}_{A+B} \cos n\frac{\pi}{4} + \underbrace{(\sqrt{2})^n K_2}_{A-B \text{ with complex}} \sin n\frac{\pi}{4}.$

$a_0 = 1$

$1 = (\sqrt{2})^0 K_1 \cos 0 \times \frac{\pi}{4} + (\sqrt{2})^0 K_2 \sin 0 \times \frac{\pi}{4}$

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$1 = K_1 \cos 0 + K_2 \sin 0$

$\boxed{1 = K_1}$

$a_1 = 2$

$2 = \sqrt{2} K_1 \cos \frac{\pi}{4} + \sqrt{2} K_2 \sin \frac{\pi}{4}$

$2 = K_1 \sqrt{2} \times \frac{1}{\sqrt{2}} + \sqrt{2} K_2 \times \frac{1}{\sqrt{2}}.$

$2 = K_1 + K_2$

$2 = 1 + K_2$

$K_2 = 1.$

$\therefore a_n = (\sqrt{2})^n \left(\cos \frac{n\pi}{4} + \sin \frac{n\pi}{4}\right)$

iii) Q. Solve $a_n - 4a_{n-1} + 4a_{n-2} = 0$ $n \geq 2$ $a_0 = 1$ $a_1 = 3$

C.E: $r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2, 2.$

$a_n = A(2)^n + Bn(2)^n$

$a_0 = 1$

$1 = A + 0$

$\boxed{A = 1}$

$a_1 = 3$

$3 = A \times 2 + 2B$

$3 = 2 + 2B$ $\boxed{B = 1/2}$

$a_n = 2^n + \frac{1}{2} n 2^n$

Q Solve $a_n - 6a_{n-1} + 9a_{n-2} = 0$ $\quad n \geq 2 \quad a_0 = 5 \quad a_1 = 12$

C.E: $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3, 3.$

$a_n = A(3)^n + Bn(3)^n.$

$a_0 = 5$

$\boxed{5 = A}$

$a_1 = 12$

$12 = 5 \times 3 + 3B$

$12 = 15 + 3B$

$\boxed{B = -1}$

$a_n = 5(3)^n - n(3)^n$

Non homogenous recurrence relation

Method of undetermined coefficients

$a_n + C_1 a_{n-1} = f(n)$ $\qquad a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n)$

Q. Solve $a_n - 3a_{n-1} = 5(7)^n$ $\quad n \geq 2, \quad a_0 = 2$

⇒ Consider homogenous.

$a_n - 3a_{n-1} = 0$

$r - 3 = 0 \qquad r = 3$

$a_n^{(h)} = A(3)^n$

$a_n - 3a_{n-1} = 5(7)^n$ — ①

while assuming if it is similar to homogenous. eg if $a_n^{(h)} = A(7)^n$ the assume $Bn(7)^n$

Assume $a_n^{(p)} = B(7)^n$ be a solution of ①

$\therefore B(7)^n - 3B(7)^{n-1} = 5(7)^n$

$\div 7^{n-1} \quad 7B - 3B = 5 \times 7$

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$4B = 35$

$B = \frac{35}{4}$

$\therefore \cancel{\ } \ a_n^{(p)} = \frac{35}{4}(7)^n$

$\therefore a_n = a_n^{(h)} + a_n^{(p)}$

$= A(3)^n + \frac{35}{4}(7)^n$

$a_0 = 2.$

$2 = A + \frac{35}{4}$

$A = \frac{8 - 35}{4}$ $\qquad$ $A = \frac{27}{4}$

$\therefore a_n = \frac{27}{4}(3)^n + \frac{35}{4}(7)^n$

Q. $a_n - 3a_{n-1} = 5(3)^n$

$a_n^{(h)} = A(3)^n$

$a_n - 3a_{n-1} = 5(3)^n$

Assume $a_n^{p} = Bn(3)^n$

$Bn(3)^n - 3B(n-1)(3)^{n-1} = 5(3)^n$

$\div 3^{n-1} \Rightarrow Bn3 - 3Bn + 3B = 5 \times 3$

$\Rightarrow Bn = 15/3$

$a_n^{p} = \frac{15}{3} n 3^n$

$a_n = A3^n + \frac{15}{3} n 3^n.$

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f_n	$a_n^{(P)}$
k^n	Bk^n
	Bnk^n. if same as $a_n^{(h)}$
n^2	Bn^2+Cn+D
	Bn^3+Cn^2+Dn if same as $a_n^{(h)}$
n	$Bn+C$
	Bn^2+C if same as $a_n^{(h)}$

Q. Solve $a_{n+2} - 4a_{n+1} + 3a_n = -200$ $n \geq 0$ $a_0 = 3000$

$a_1 = 3300$

CF. $r^2 - 4r + 3 = 0$

$r^2 - r - 3r + 3 = 0$

$r(r-1) - 3(r-1) = 0$

$(r-1)(r-3) = 0$

$r = 1, 3$

$a_n^{(h)} = A + B(3)^n$

Assume $a_n^{(p)} = C$ [redacted] since we have a const in $a_n^{(h)}$ $\boxed{a_n^{p} = Cn.}$

$a_{n+2} - 4a_{n+1} + 3a_n = -200$

$\cancel{a_n}\ C(n+2) - 4C(n+1) + 3Cn = -200$

$Cn + 2C - 4Cn - 4C + 3Cn = -200.$

$2C - 4C = -200.$

$2C = 200$

$C = 100n$

$a_n^{(p)} = 100n.$

$a_n = a_n^{(h)} + a_n^{(p)} = A + B(3)^n + 100n$

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$a_0 = 3000$ $a_1 = 3300$

$3000 = A + B$ $3300 = A + 3B + 100$

$3200 = A + 3B$

$B = 100 \quad A = 2900.$

$a_n = 2900 + 100(3)^n + 100n.$

Q. Solve $a_{n+1} - 2a_n - 1 = 0 \quad a_0 = 0 \quad a_1 = 1$

$a_{n+1} - 2a_n = 1$

$C.E = \quad r - 2 = 0$

$r = 2$

$a_n^{(h)} = A(2)^n$

Assume $\Rightarrow a_n^{(p)} = B$

$a_{n+1} - 2a_n - 1 = 0$

$B - 2B - 1 = 0$

$B = -1$

$a_n^{(p)} = -1$

$a_n = a_n^h + a_n^p$

$= A(2)^n - 1$

$a_1 = 1$

$1 = 2A - 1$

$A = 1$

$a_n = (2)^n - 1$