Fuzzy Sets and Fuzzy Logic

Theory and Applications

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Part one: theory

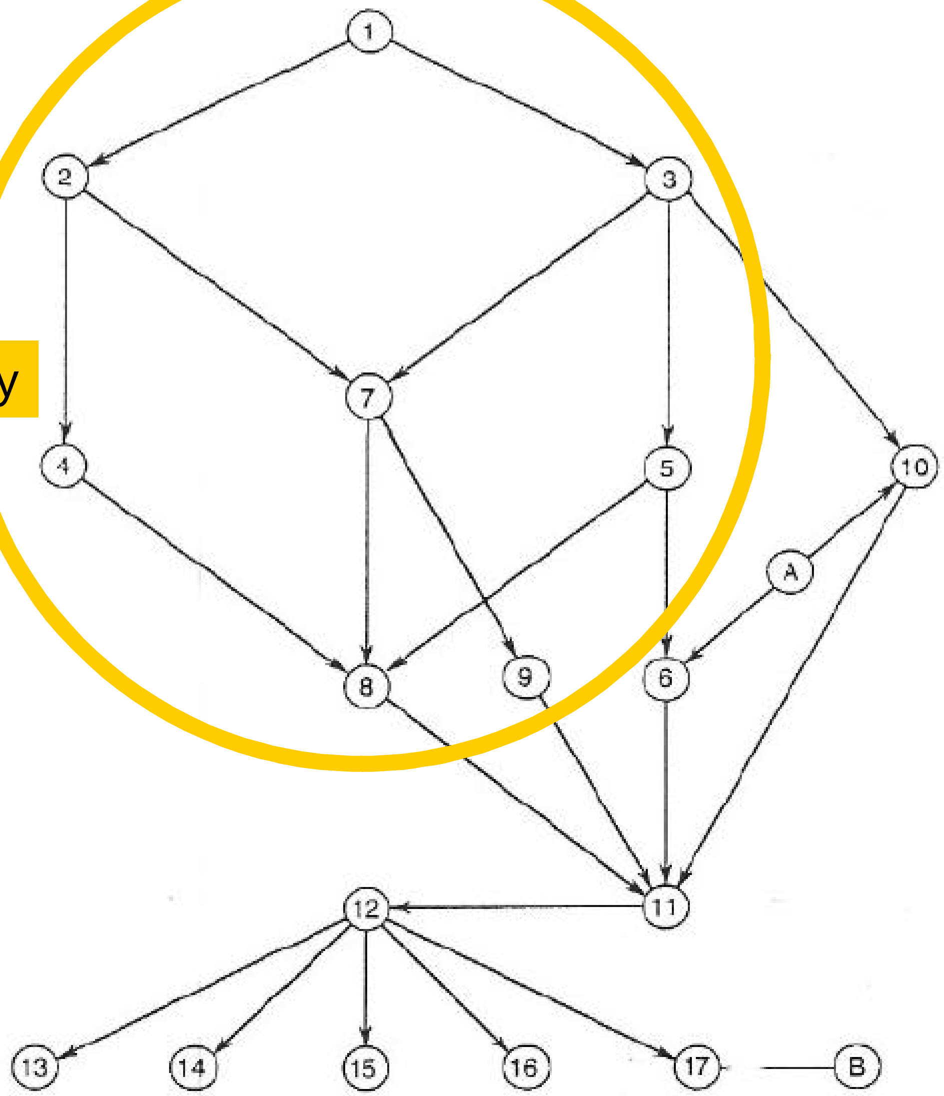

Figure P.1 Prerequisite dependencies among chapters of this book.

Part one: Theory

Chapter 1
From Crisp Sets to Fuzzy Sets

1.1 Introduction

- Uncertainty
  - Probability theory is capable of representing only one of several distinct types of uncertainty.
  - When A is a fuzzy set and x is a relevant object, the proposition "x is a member of A" is not necessarily either true or false. It may be true only to some degree, the degree to which x is actually a member of A.
  - For example: the weather today
    - Sunny: If we define any cloud cover of 25% or less is sunny.
    - This means that a cloud cover of 26% is not sunny?
    - "Vagueness" should be introduced.

1.1 Introduction

- The crisp set v.s. the fuzzy set
  - The crisp set is defined in such a way as to dichotomize the individuals in some given universe of discourse into two groups: members and nonmembers.
    - However, many classification concepts do not exhibit this characteristic.
    - For example, the set of tall people, expensive cars, or sunny days.

  - A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set.
    - For example: a fuzzy set representing our concept of sunny might assign a degree of membership of 1 to a cloud cover of 0%, 0.8 to a cloud cover of 20%, 0.4 to a cloud cover of 30%, and 0 to a cloud cover of 75%.

1.2 Crisp sets: an overview

- The theory of crisp set
  - The following general symbols are employed throughout the text:

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (the set of all integers),

$\mathbb{N} = \{1, 2, 3, \ldots\}$ (the set of all positive integers or natural numbers),

$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ (the set of all nonnegative integers),

$\mathbb{N}_n = \{1, 2, \ldots, n\}$,

$\mathbb{N}_{0,n} = \{0, 1, \ldots, n\}$,

$\mathbb{R}$: the set of all real numbers,

$\mathbb{R}^+$: the set of all nonnegative real numbers,

$[a, b], (a, b], [a, b), (a, b)$: closed, left-open, right-open, open interval of real numbers between a and b, respectively,

$(x_1, x_2, \ldots, x_n)$: ordered n-tuple of elements $x_1, x_2, \ldots, x_n$.

1.2 Crisp sets: an overview

- Three basic methods to define sets:
  - The list method: a set is defined by naming all its members.

    $$A = \{a_1, a_2, \ldots, a_n\}$$

  - The rule method: a set is defined by a property satisfied by its members.

    $$A = \{x \mid P(x)\}$$

    where '|' denotes the phrase "such that"

    $P(x)$: a proposition of the form "x has the property P "

  - A set is defined by a characteristic function.

    $$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

    the characteristic function $\chi_A : X \to \{0,1\}$

1.2 Crisp sets: an overview

- A family of sets: a set whose elements are sets
  - It can be defined in the form:

    $$\{A_i \mid i \in I\}$$

    where i and I are called the set index and the index set, respectively.
  - The family of sets is also called an indexed set.
  - For example: $\mathcal{A}$

    $$\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$$
- A is a subset of B: $A \subseteq B$
- A, B are equal sets: $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$
- A and B are not equal: $A \neq B$
- A is proper subset of B: $A \subset B \Leftrightarrow A \subseteq B$ and $A \neq B$
- A is included in B: $A \subseteq B$

1.2 Crisp sets: an overview

- The power set of A ($P(A)$): the family of all subsets of a given set A.
  - The second order power set of A: $P^2(A) = P(P(A))$
  - The higher order power set of A: $P^3(A), P^4(A), \ldots$
- The cardinality of A ($|A|$): the number of members of a finite set A.
  - For example: $|P(A)| = 2^{|A|}$, $|P^2(A)| = 2^{2^{|A|}}$
- $B - A$: the relative complement of a set A with respect to set B

  $$B - A = \{x \mid x \in B, x \notin A\}$$

  - If the set B is the universal set, then $B - A = \overline{A}$.
  - $\overline{\overline{A}} = A$
  - $\overline{\phi} = X$
  - $\overline{X} = \phi$

1.2 Crisp sets: an overview

- The union of sets A and B:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

- The generalized union operation: for a family of sets,

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$

- The intersection of sets A and B:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

- The generalized intersection operation: for a family of sets,

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

1.2 Crisp sets: an overview

TABLE 1.1 FUNDAMENTAL PROPERTIES OF CRISP SET OPERATIONS

Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotence	$A \cup A = A$
	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$
	$A \cap (A \cup B) = A$
Absorption by X and $\varnothing$	$A \cup X = X$
	$A \cap \varnothing = \varnothing$
Identity	$A \cup \varnothing = A$
	$A \cap X = A$
Law of contradiction	$A \cap \overline{A} = \varnothing$
Law of excluded middle	$A \cup \overline{A} = X$
De Morgan's laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
	$\overline{A \cup B} = \overline{A} \cap \overline{B}$

1.2 Crisp sets: an overview

- The partial ordering of a power set:
  - Elements of the power set $\mathcal{P}(A)$ of a universal set can be ordered by the set inclusion $\subseteq$.

    $$A \subseteq B \text{ iff } A \cup B = B \text{ (or } A \cap B = A) \text{ for any } A, B \in \mathcal{P}(X)$$

- Disjoint: any two sets that have no common members

    $$A \cap B = \phi$$

1.2 Crisp sets: an overview

- A partition on A ($\pi(A)$):
  - A family of pairwise disjoint nonempty subsets of a set A is call a partition on A if the union of these subsets yields the original set A.

    $\pi(A) = \{A_i | i \in I, A_i \subseteq A\}$,

    where $A_i \neq \varnothing$, is a partition on A iff $A_i \cap A_j = \varnothing$

    for each pair $i, j \in I, i \neq j$, and $\bigcup_{i \in I} A_i = A$.

  - Members of a partition $\pi(A)$ are usually referred to as blocks of the partition.
  - Each member of A belongs to one and only one block of $\pi(A)$.

1.2 Crisp sets: an overview

Given two partitions $\pi_1(A)$ and $\pi_2(A)$, we say that $\pi_1(A)$ is a *refinement* of $\pi_2(A)$ iff each block of $\pi_1(A)$ is included in some block of $\pi_2(A)$. The refinement relation on the set of all partitions of A, $\Pi(A)$, which is denoted by $\leq$ (i.e., $\pi_1(A) \leq \pi_2(A)$ in our case), is a partial ordering. The pair $\langle \Pi(A), \leq \rangle$ is a lattice, referred to as the *partition lattice* of A.

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a family of sets such that

$$A_i \subseteq A_{i+1} \text{ for all } i = 1, 2, \ldots, n-1.$$

Then, $\mathcal{A}$ is called a *nested family*, and the sets A_1 and A_n are called the *innermost set* and the *outermost set*, respectively. This definition can easily be extended to infinite families.

1.2 Crisp sets: an overview

The *Cartesian product* of two sets—say, A and B (in this order)—is the set of all ordered pairs such that the first element in each pair is a member of A, and the second element is a member of B. Formally,

$$\boxed{A \times B = \{\langle a, b\rangle \mid a \in A, b \in B\},}$$

where $A \times B$ denotes the Cartesian product. Clearly, if $A \neq B$ and A, B are nonempty, then $A \times B \neq B \times A$.

The Cartesian product of a family $\{A_1, A_2, \ldots, A_n\}$ of sets is the set of all n-tuples $\langle a_1, a_2, \ldots, a_n\rangle$ such that $a_i \in A_i (i = 1, 2, \ldots, n)$. It is written as either $A_1 \times A_2 \times \ldots \times A_n$ or $\underset{1\leq i\leq n}{\times} A_i$. Thus,

$$\boxed{\underset{1\leq i\leq n}{\times} A_i = \{\langle a_1, a_2, \ldots, a_n\rangle \mid a_i \in A_i \text{ for every } i = 1, 2, \ldots, n\}.}$$

The Cartesian products $A \times A$, $A \times A \times A, \ldots$ are denoted by $A^2, A^3, \ldots,$ respectively.

1.2 Crisp sets: an overview

- A set whose members can be labeled by the positive integers is called a countable set.
- If such labeling is not possible, the set is called uncountable.
- For example, $\{ a \mid a$ is a real number, $0 < a < 1\}$ is uncountable.
- Every uncountable set is infinite.
- Countable sets are classified into finite and countable infinite.

1.2 Crisp sets: an overview

- $\mathbb{R}^n$: the n-dimensional Euclidean vector space for some $n \in \mathbb{N}$

A set A in $\mathbb{R}^n$ is called $\boxed{convex}$ iff, for every pair of points

$$\mathbf{r} = \langle r_i \mid i \in \mathbb{N}_n \rangle \text{ and } \mathbf{s} = \langle s_i \mid i \in \mathbb{N}_n \rangle$$

in A and every real number $\lambda \in [0, 1]$, the point

$$\mathbf{t} = \langle \lambda r_i + (1 - \lambda) s_i \mid i \in \mathbb{N}_n \rangle$$

is also in A.

- A set A in $\mathbb{R}^n$ is convex iff, for every pair of points r and s in A, all points located on the straight-line segment connecting r and s are also in A.
- For example, A=[0,2]U[3,5] is not convex.
  - Let r = 1, s = 4, and λ=0.4; then $\lambda r + (1-\lambda)s = 2.8$ and $2.8 \notin A$.

1.2 Crisp sets: an overview

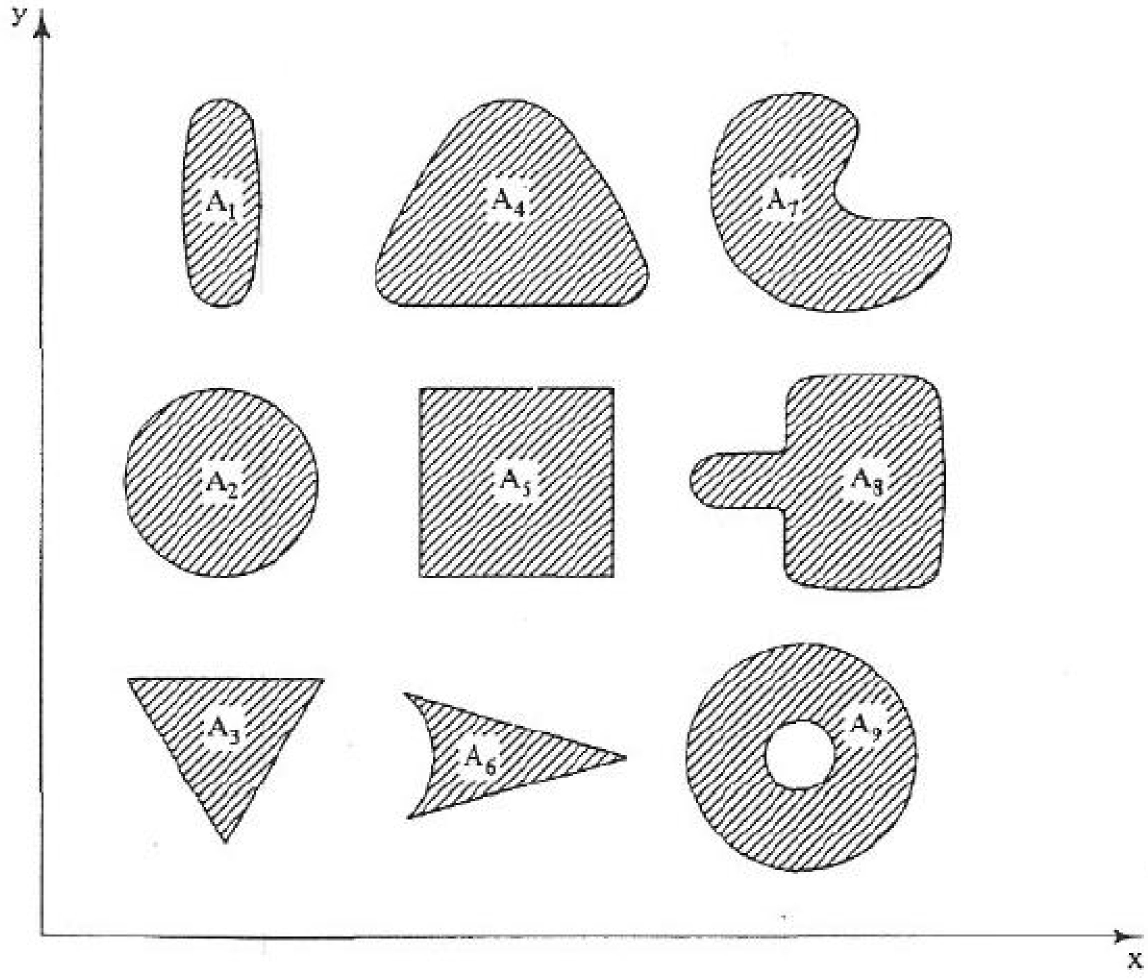

Figure 1.1 Example of sets in $\mathbb{R}^2$ that are convex (A_1–A_5) or nonconvex (A_6–A_9).

1.2 Crisp sets: an overview

- Let R denote a set of real number.
  - If there is a real number r such that $x \le r$ for every $x \in R$, then r is called an upper bound of R, and R is bounded above by r.
  - If there is a real number s such that $x \ge s$ for every $x \in R$, then s is called an lower bound of R, and R is bounded below by s.

- For any set of real numbers R that is bounded above, a real number r is called the supremum of R (write $r = \sup R$) iff:

  (a) r is an upper bound of R;

  (b) no number less than r is an upper bound of R.

- For any set of real numbers R that is bounded below, a real number s is called the infimum of R (write $s = \inf R$) iff:

  (a) s is an lower bound of R;

  (b) no number greater than s is an lower bound of R.

1.3 Fuzzy sets: basic types

- A membership function:
  - A characteristic function: the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of their elements in the set.
  - Larger values denote higher degrees of set membership.
- A set defined by membership functions is a fuzzy set.
- The most commonly used range of values of membership functions is the unit interval [0,1].
- We think the universal set X is always a crisp set.
- Notation:
  - The membership function of a fuzzy set A is denoted by μ_A:

    $$\mu_A : X \rightarrow [0,1]$$

  - In the other one, the function is denoted by A and has the same form

    $$A : X \rightarrow [0,1]$$

  - In this text, we use the second notation.

1.3 Fuzzy sets: basic types

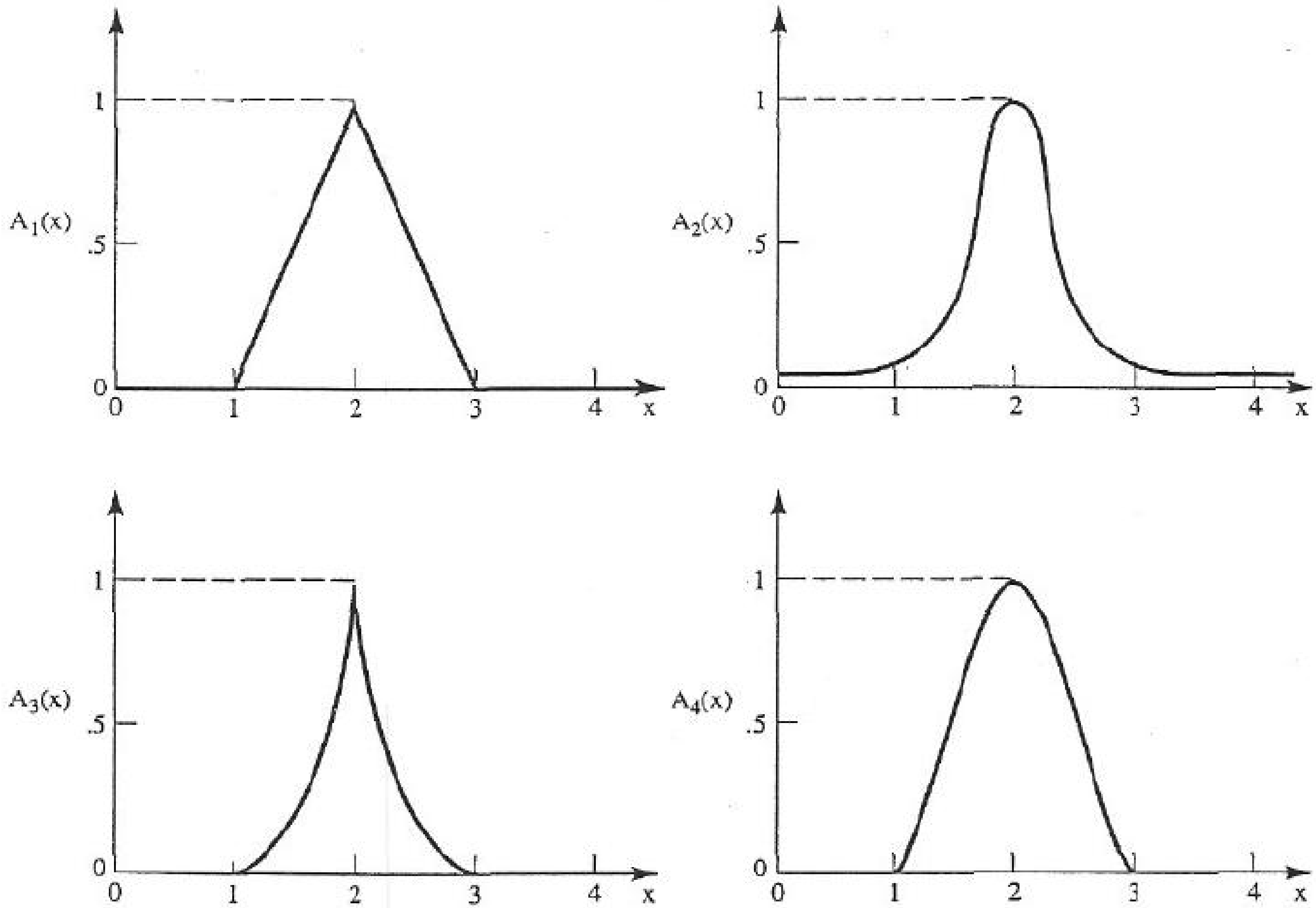

Figure 1.2 Examples of membership functions that may be used in different contexts for characterizing fuzzy sets of real numbers close to 2.

1.3 Fuzzy sets: basic types

- The four fuzzy sets are similar in the sense that the following properties are possessed by each $A_i (i \in \mathbb{N}_4)$:

(i) $A_i(2) = 1$ and $A_i(x) < 1$ for all $x \neq 2$;
(ii) A_i is symmetric with respect to $x = 2$, that is $A_i(2 + x) = A_i(2 - x)$ for all $x \in \mathbb{R}$;
(iii) $A_i(x)$ decreases monotonically from 1 to 0 with the increasing difference $|2 - x|$.

- Each function in Fig. 1.2 is a member of a parameterized family of functions.

$$A_1(x) = \begin{cases} p_1(x - r) + 1 & \text{when } x \in [r - 1/p_1, r] \\ p_1(r - x) + 1 & \text{when } x \in [r, r + 1/p_1] \\ 0 & \text{otherwise} \end{cases}$$

$$A_2(x) = \frac{1}{1 + p_2(x - r)^2}$$

$$A_3(x) = e^{-|p_3(x-r)|}$$

$$A_4(x) = \begin{cases} (1 + \cos(p_4 \pi (x - r)))/2 & \text{when } x \in [r - 1/p_4, r + 1/p_4] \\ 0 & \text{otherwise} \end{cases}$$

Can you find the values of parameters p_1, p_2, p_3, and p_4?

1.3 Fuzzy sets: basic types

- An example:
  - Define the seven levels of education:

0 – no education
1 – elementary school
2 – high school
3 – two-year college degree
4 – bachelor's degree
5 – master's degree
6 – doctoral degree

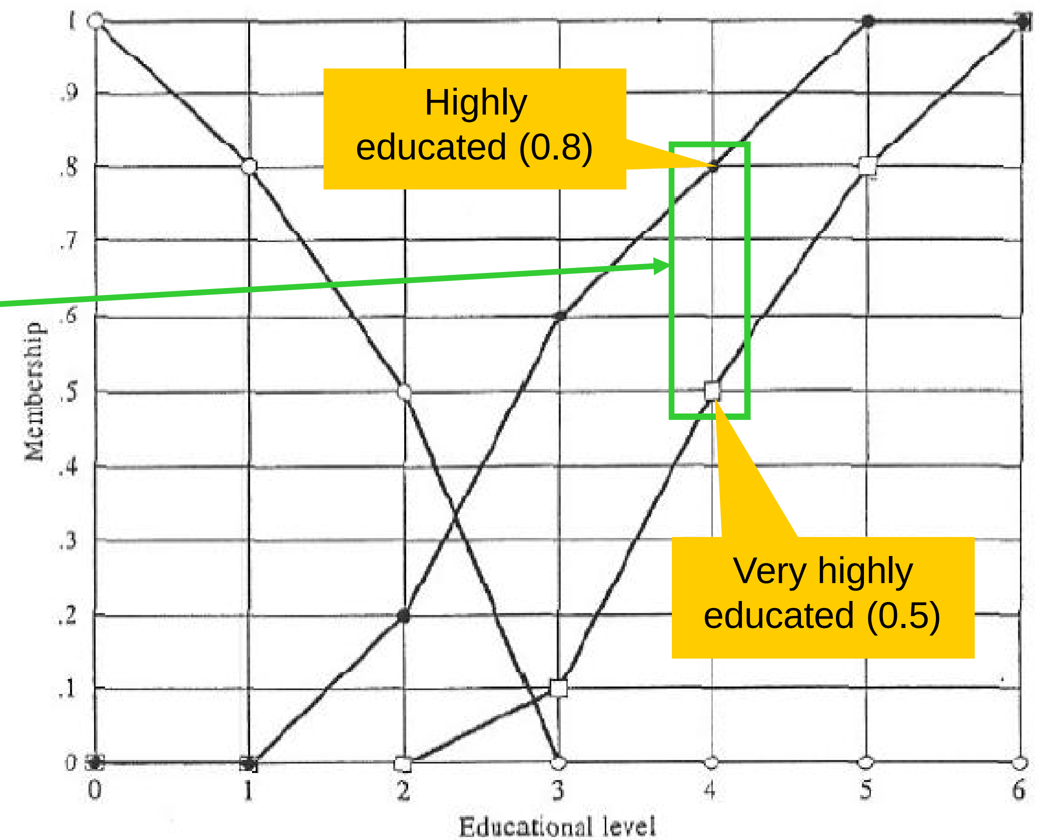

Figure 1.3 Examples of fuzzy sets expressing the concepts of people that are little educated (○), highly educated (●), and very highly educated (□).

1.3 Fuzzy sets: basic types

- Several fuzzy sets representing linguistic concepts such as low, medium, high, and so on are often employed to define states of a variable. Such a variable is usually called a fuzzy variable.
- For example:

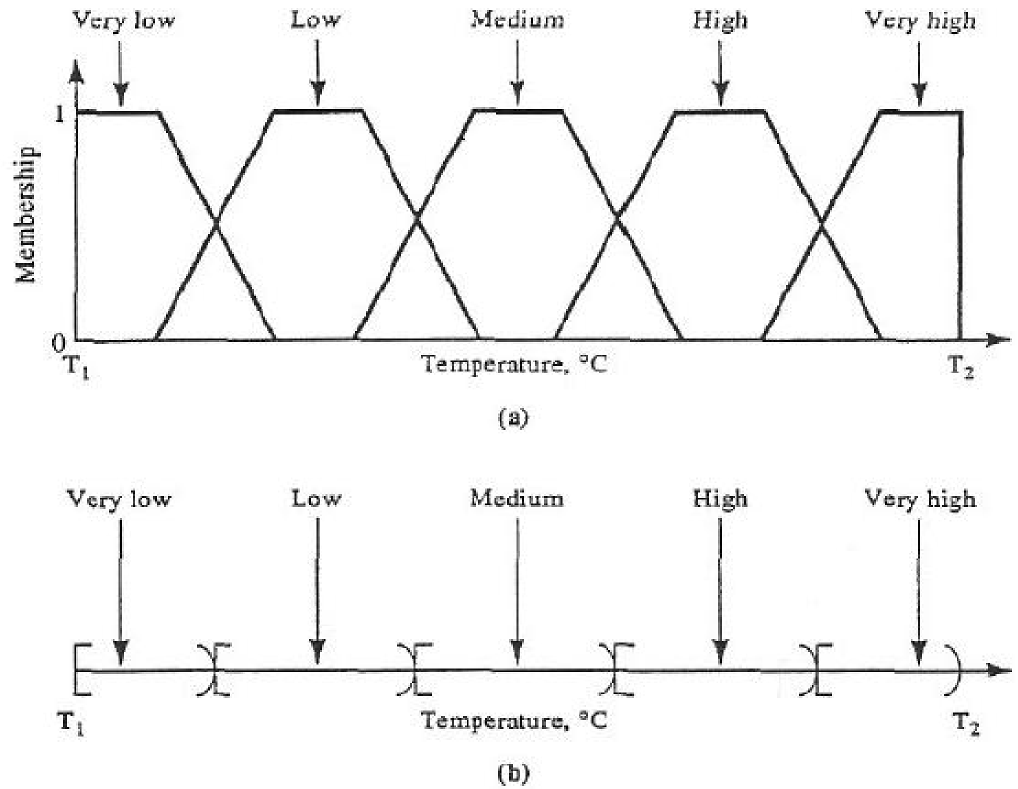

Figure 1.4 Temperature in the range $[T_1, T_2]$ conceived as: (a) a fuzzy variable; (b) a traditional (crisp) variable.

Fuzzy sets: basic types

- Certain properties are given to fuzzy sets, like ”increasing”, ”decreasing” or ”convex”, according to properties of a membership function that characterizes the fuzzy set.
- Increasing fuzzy sets are used for expressing such linguistic concepts as ”big”, ”hot”, ”old” etc., un universes of length, temperature, age etc. Similarly examples of decreasing concepts in these universes would be ”small”,”cold”, ”young”.

Some essential types of fuzzy sets

- Next we introduce some commonly used fuzzy set:
- Γ-shaped fuzzy set: A function with one variable and two parameters Γ:X→[0,1] is defined by

$$\Gamma(x;\alpha,\beta)=\begin{cases}0 \quad if \quad x<\alpha \\ \dfrac{x-\alpha}{\beta-\alpha} \quad if \quad \alpha\le x\le\beta \\ 1 \quad if \quad x>\beta\end{cases}$$

Some essential types of fuzzy sets

- S-shaped fuzzy set is defined by

$$S(x;\alpha,\beta,\gamma)=\begin{cases}0 \quad if \quad x<\alpha \\ 2\left(\dfrac{x-\alpha}{\gamma-\alpha}\right)^2 \quad if \quad \alpha\le x\le\beta \\ 1-2\left(\dfrac{x-\gamma}{\gamma-\alpha}\right)^2 if \quad \beta\le x\le\gamma \\ 1 \quad if \quad x>\gamma\end{cases}$$

- where $\beta=\dfrac{\alpha+\gamma}{2}$

Some essential types of fuzzy sets

- L-shaped fuzzy set is decreasing piecewise continuous function L: :X→[0,1] is defined by

$$L(x;\alpha,\beta)=\begin{cases}1 \quad if \quad x<\alpha \\ \dfrac{\beta-x}{\beta-\alpha} \quad if \quad \alpha\le x\le\beta \\ 0 \quad if \quad x>\beta\end{cases}$$

Some essential types of fuzzy sets

Λ-shape fuzzy set is defined as:

$$\Lambda(x;\alpha,\beta,\gamma)=\begin{cases}0 \quad if \quad x<\alpha \\ \dfrac{x-\alpha}{\beta-\alpha} \quad if \quad \alpha\le x\le\beta \\ \dfrac{\gamma-x}{\gamma-\beta} \quad if \quad \beta<x\le\gamma \\ 0 \quad if \quad x>\gamma\end{cases}$$

Some essential types of fuzzy sets

- Bell-shaped fuzzy set:

$$\pi(x;\beta,\gamma)=\begin{cases} S(x;\gamma-\beta,\gamma-\beta/2,\gamma) & if \quad x\le\gamma \\ 1-S(x;\gamma,\gamma+\beta/2,\gamma+\beta) & if \quad x>\gamma \end{cases}$$

- Trapezoidal fuzzy set:

$$\Pi(x;\alpha,\beta,\gamma,\delta)=\begin{cases} 0 & if \quad x<\alpha \\ \dfrac{x-\alpha}{\beta-\alpha} & if \quad \alpha\le x\le\beta \\ 1 & if \quad \beta<x\le\gamma \\ \dfrac{\delta-x}{\delta-\gamma} & if \quad \gamma<x\le\delta \\ 0 & if \quad x>\delta \end{cases}$$

1.3 Fuzzy sets: basic types

- Now, we introduced only one type of fuzzy set. Given a relevant universal set X, any arbitrary fuzzy set of this type is defined by a function of the form

  $$A : X \rightarrow [0,1]$$

  This kind of fuzzy sets is called ordinary fuzzy sets.
- Interval-valued fuzzy sets:
  - The membership functions of ordinary fuzzy sets are often overly precise.
  - We may be able to identify appropriate membership functions only approximately.
  - Interval-valued fuzzy sets: a fuzzy set whose membership functions does not assign to each element of the universal set one real number, but a closed interval of real numbers between the identified lower and upper bounds.

    $$A : X \rightarrow \varepsilon([0,1]), \quad \mathcal{E}([0, 1]) \subset \mathcal{P}([0, 1]).$$

Power set

1.3 Fuzzy sets: basic types

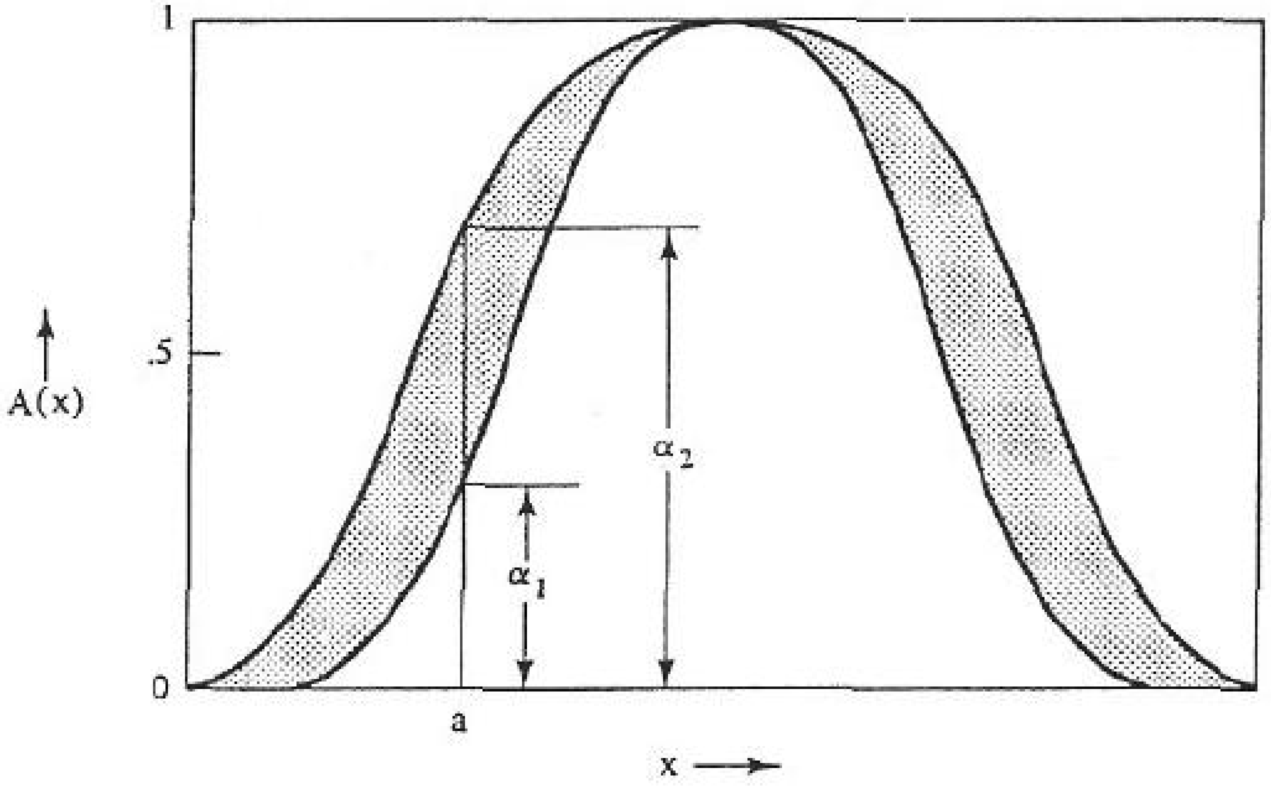

Figure 1.5 An example of an interval-valued fuzzy set $\boxed{(A(a) = [\alpha_1, \alpha_2])}$.

1.3 Fuzzy sets: basic types

- Fuzzy sets of type 2:

  $$A : X \to \mathcal{F}([0,1]),$$

  - $\mathcal{F}([0,1])$: the set of all ordinary fuzzy sets that can be defined with the universal set [0,1].
  - $\mathcal{F}([0,1])$ is also called a fuzzy power set of [0,1].

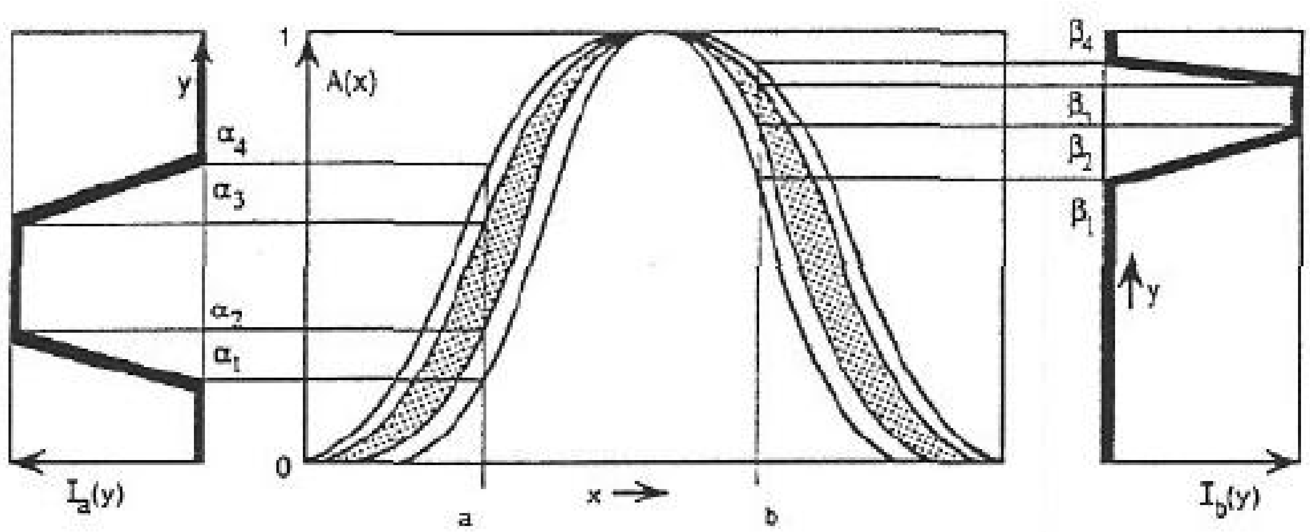

Figure 1.6 Illustration of the concept of a fuzzy set of type 2.

1.3 Fuzzy sets: basic types

- Discussions:
  - The primary disadvantage of interval-value fuzzy sets, compared with ordinary fuzzy sets, is computationally more demanding.
  - The computational demands for dealing with fuzzy sets of type 2 are even greater then those for dealing with interval-valued fuzzy sets.
  - This is the primary reason why the fuzzy sets of type 2 have almost never been utilized in any applications.

1.3 Fuzzy sets: basic types

- L-fuzzy set:

$$A : X \to L$$

  - The membership grades is represented by symbols of an arbitrary set L that is at least partially ordered.
  - L-fuzzy sets are very general. They capture all the other types introduced thus far as special cases.

1.3 Fuzzy sets: basic types

- Level 2 fuzzy sets:

$$A : \mathcal{F}(X) \rightarrow [0, 1],$$

  - $\mathcal{F}(X)$: the fuzzy power set of X.
  - Level 2 fuzzy sets allow us to deal with situations in which elements of the universal set cannot be specified precisely, but only approximately.
  - For example:
    - Assuming that the proposition "x is close to r" is represented by an ordinary fuzzy set B, the membership grade of a value of x that is known to be close to r in the level 2 fuzzy sets A is given by $A(B)$.

1.3 Fuzzy sets: basic types

- Fuzzy sets of type 2 and level 2:

$$A : \mathfrak{F}(X) \to \mathfrak{F}([0, 1]).$$

  - $\mathfrak{F}(X)$: the fuzzy power set of X.
- Other combinations are also possible.

1.3 Fuzzy sets: basic types

- Discussions:
  - These generalized types of fuzzy sets have not as yet played a significant role in applications of fuzzy set theory.
  - Two reasons to introduce the generalized fuzzy sets in this section:
    - The reader can understand that fuzzy set theory does not stand or fall with ordinary fuzzy sets.
    - The practical significance of some of the generalized types will increase.

1.4 Fuzzy sets: basic concepts

- Consider three fuzzy sets that represent the concepts of a young, middle-aged, and old person. The membership functions are defined on the interval [0,80] as follows:

$$A_1(x) = \begin{cases} 1 & \text{when } x \leq 20 \\ (35 - x)/15 & \text{when } 20 < x < 35 \\ 0 & \text{when } x \geq 35 \end{cases}$$

$$A_2(x) = \begin{cases} 0 & \text{when either } x \leq 20 \text{ or } \geq 60 \\ (x - 20)/15 & \text{when } 20 < x < 35 \\ (60 - x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \leq x \leq 45 \end{cases}$$

$$A_3(x) = \begin{cases} 0 & \text{when } x \leq 45 \\ (x - 45)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } x \geq 60 \end{cases}$$

1.4 Fuzzy sets: basic concepts

$$A_2(x) = \begin{cases} 0 & \text{when either } x \le 20 \text{ or } \ge 60 \\ (x-20)/15 & \text{when } 20 < x < 35 \\ (60-x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \le x \le 45 \end{cases}$$

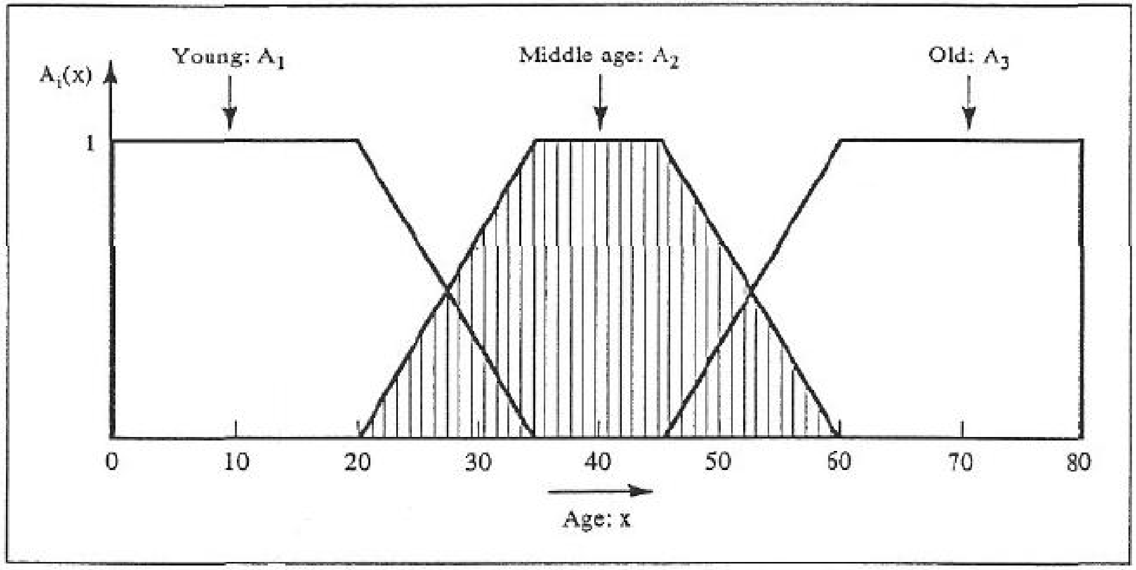

Figure 1.7 Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation D_2 of A_2 is defined numerically in Table 1.2.

TABLE 1.2 DISCRETE APPROXIMATION OF MEMBERSHIP FUNCTION A_2 (FIG. 1.7) BY FUNCTION D_2 OF THE FORM:
$D_2 : \{0, 2, 4, \ldots, 80\} \rightarrow [0, 1]$

x	$D_2(x)$
$x \notin \{22, 24, \ldots, 58\}$	0.00
$x \in \{22, 58\}$	0.13
$x \in \{24, 56\}$	0.27
$x \in \{26, 54\}$	0.40
$x \in \{28, 52\}$	0.53
$x \in \{30, 50\}$	0.67
$x \in \{32, 48\}$	0.80
$x \in \{34, 46\}$	0.93
$x \in \{36, 38, \ldots, 44\}$	1.00

1.4 Fuzzy sets: basic concepts

- α-cut and strong α-cut
  - Given a fuzzy set A defined on X and any number $\alpha \in [0,1]$, the α-cut and strong α-cut are the crisp sets:

$$^{\alpha}A = \{x | A(x) \geq \alpha\}$$
$$^{\alpha+}A = \{x | A(x) > \alpha\}.$$

  - The α-cut of a fuzzy set A is the crisp set that contains all the elements of the universal set X whose membership grades in A are greater than or equal to the specified value of α.
  - The strong α-cut of a fuzzy set A is the crisp set that contains all the elements of the universal set X whose membership grades in A are only greater than the specified value of α.

1.4 Fuzzy sets: basic concepts

- For example:

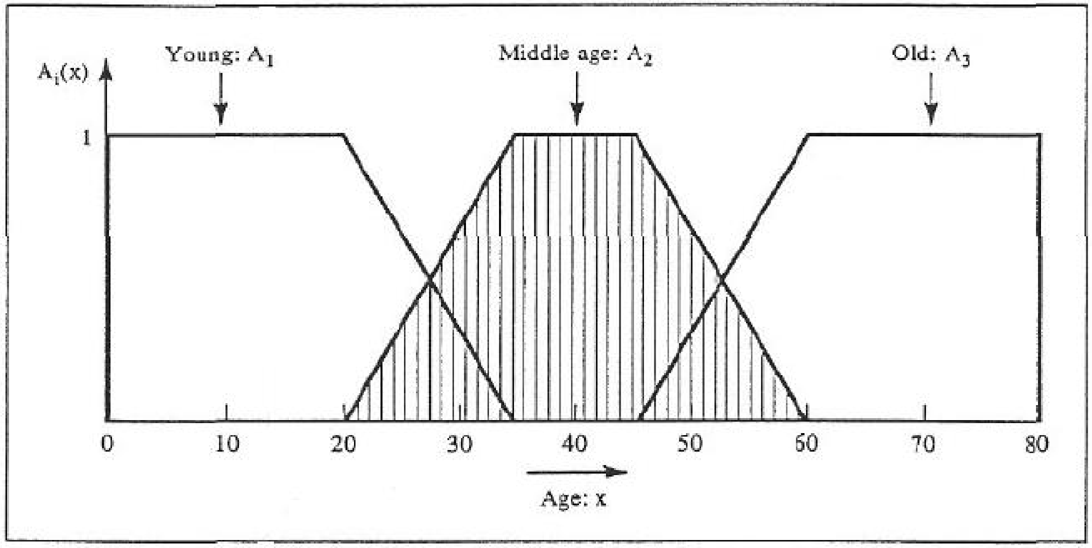

Figure 1.7 Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation D_2 of A_2 is defined numerically in Table 1.2.

$$^0A_1 = {}^0A_2 = {}^0A_3 = [0, 80] = X;$$

$$^{\alpha}A_1 = [0, 35 - 15\alpha], {}^{\alpha}A_2 = [15\alpha + 20, 60 - 15\alpha], {}^{\alpha}A_3 = [15\alpha + 45, 80] \text{ for all } \alpha \in (0, 1];$$

$$^{\alpha+}A_1 = (0, 35 - 15\alpha), {}^{\alpha+}A_2 = (15\alpha + 20, 60 - 15\alpha), {}^{\alpha+}A_3 = (15\alpha + 45, 80) \text{ for all } \alpha \in [0, 1);$$

$$^{1+}A_1 = {}^{1+}A_2 = {}^{1+}A_3 = \varnothing.$$

1.4 Fuzzy sets: basic concepts

- A level set of A:
  - The set of all levels $\alpha \in [0,1]$ that represent distinct α-cuts of a given fuzzy set A.

$$\Lambda(A) = \{\alpha \mid A(x) = \alpha \text{ for some } x \in X\},$$

  - For example:

$$\Lambda(A_1) = \Lambda(A_2) = \Lambda(A_3) = [0, 1], \text{ and}$$
$$\Lambda(D_2) = \{0, 0.13, 0.27, 0.4, 0.53, 0.67, 0.8, 0.93, 1\}.$$

$$\boxed{A_2(x) = \begin{cases} 0 & \text{when either } x \le 20 \text{ or } \ge 60 \\ (x-20)/15 & \text{when } 20 < x < 35 \\ (60-x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \le x \le 45 \end{cases}}$$

TABLE 1.2 DISCRETE APPROXIMATION OF MEMBERSHIP FUNCTION A_2 (FIG. 1.7) BY FUNCTION D_2 OF THE FORM: $D_2 : \{0, 2, 4, \ldots, 80\} \rightarrow [0, 1]$

x	$D_2(x)$
$x \notin \{22, 24, \ldots, 58\}$	0.00
$x \in \{22, 58\}$	0.13
$x \in \{24, 56\}$	0.27
$x \in \{26, 54\}$	0.40
$x \in \{28, 52\}$	0.53
$x \in \{30, 50\}$	0.67
$x \in \{32, 48\}$	0.80
$x \in \{34, 46\}$	0.93
$x \in \{36, 38, \ldots, 44\}$	1.00

1.4 Fuzzy sets: basic concepts

- The properties of α-cut and strong α-cut
  - For any fuzzy set A and pair $\alpha_1, \alpha_2 \in [0,1]$ of distinct values such that $\alpha_1 < \alpha_2$, we have

$$^{\alpha_1}A \supseteq {}^{\alpha_2}A \quad \text{and} \quad {}^{\alpha_1+}A \supseteq {}^{\alpha_2+}A$$

$$^{\alpha_1}A \cap {}^{\alpha_2}A = {}^{\alpha_2}A, \quad {}^{\alpha_1}A \cup {}^{\alpha_2}A = {}^{\alpha_1}A$$

$$^{\alpha_1+}A \cap {}^{\alpha_2+}A = {}^{\alpha_2+}A, \quad {}^{\alpha_1+}A \cup {}^{\alpha_2+}A = {}^{\alpha_1+}A$$

  - All α-cuts and all strong α-cuts of any fuzzy set form two distinct families of nested crisp sets.

1.4 Fuzzy sets: basic concepts

- For example: consider the discrete approximation D_2 of fuzzy set A_2

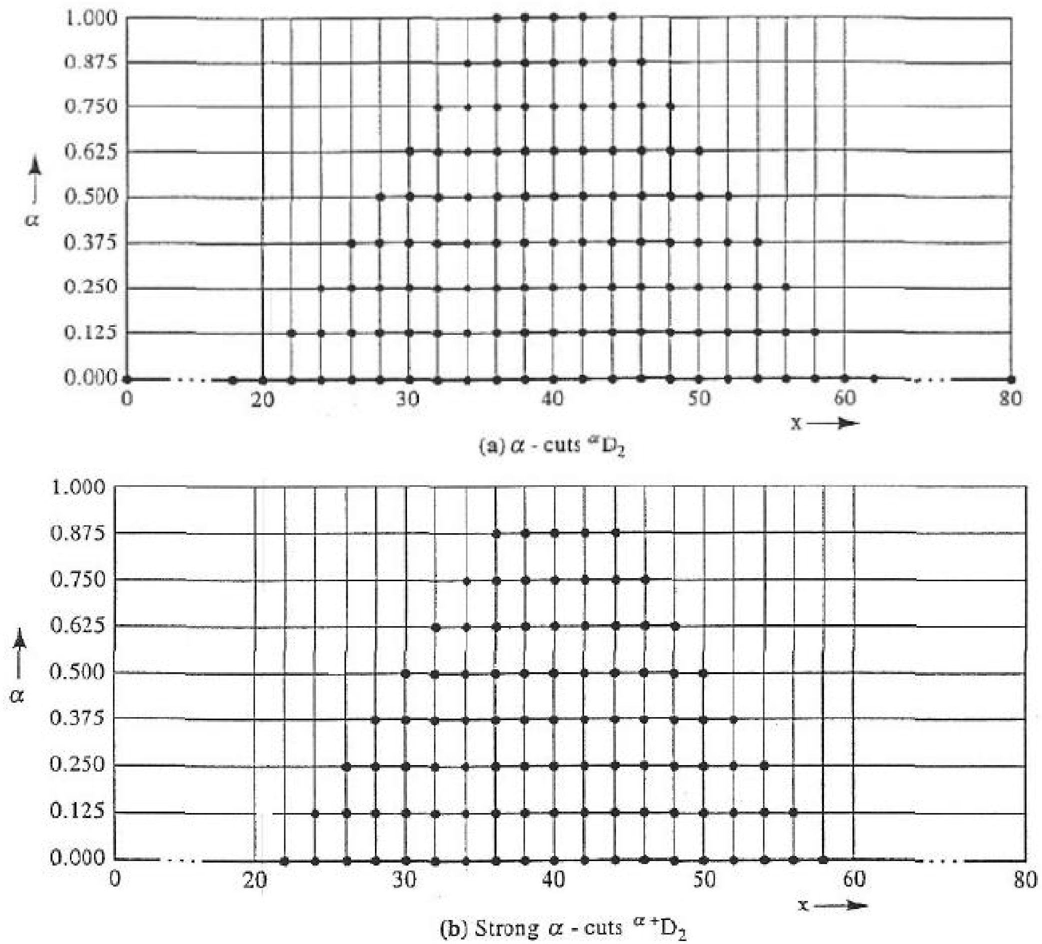

TABLE 1.2 DISCRETE APPROXIMATION OF MEMBERSHIP FUNCTION A_2 (FIG. 1.7) BY FUNCTION D_2 OF THE FORM: $D_2 : \{0, 2, 4, \ldots, 80\} \to [0, 1]$

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$x \in \{32, 48\}$	0.80
$x \in \{34, 46\}$	0.93
$x \in \{36, 38, \ldots, 44\}$	1.00

1.4 Fuzzy sets: basic concepts

- The support of a fuzzy set A:
  - The support of a fuzzy set A within a universal set X is the crisp set that contains all the elements of X that have nonzero membership grades in A.
  - The support of A is exactly the same as the strong α-cut of A for $\alpha = 0$.
  - $S(A)$ or supp(A) = ${}^{0+}A$.
  - Definition: The support of a fuzzy set A in X is a set

$$\text{supp}(a) = \{x \in X \mid A(x) > 0\}$$

- The core of A:
  - The 1-cut of A (${}^{1}A$) is often called the core of A.
  - Definition: The core of a fuzzy set A is a crisp set

$$core(A) = \{x \in X \mid A(x) = 1\}$$

1.4 Fuzzy sets: basic concepts

- The height of a fuzzy set A:
  - The height of a fuzzy set A is the largest membership grade obtained by any element in that set.

$$h(A) = \sup_{x \in X} A(x)$$

  - A fuzzy set A is called normal when $h(A) = 1$.
  - It is called subnormal when $h(A) < 1$.
  - The height of A may also be viewed as the supremum of α for which ${}^{\alpha}A \neq \phi$.

1.4 Fuzzy sets: basic concepts

- The convexity:
  - α-cuts of a convex fuzzy set should be convex for all $\alpha \in (0,1]$.
  - For example:
    - Fig. 1.9 illustrates a subnormal convex fuzzy set.
    - Fig. 1.10 illustrates a normal fuzzy set that is not convex.
    - Fig. 1.11 illustrates a normal fuzzy set defined on $\mathbb{R}^2$ by all its α-cuts for $\alpha > 0.$

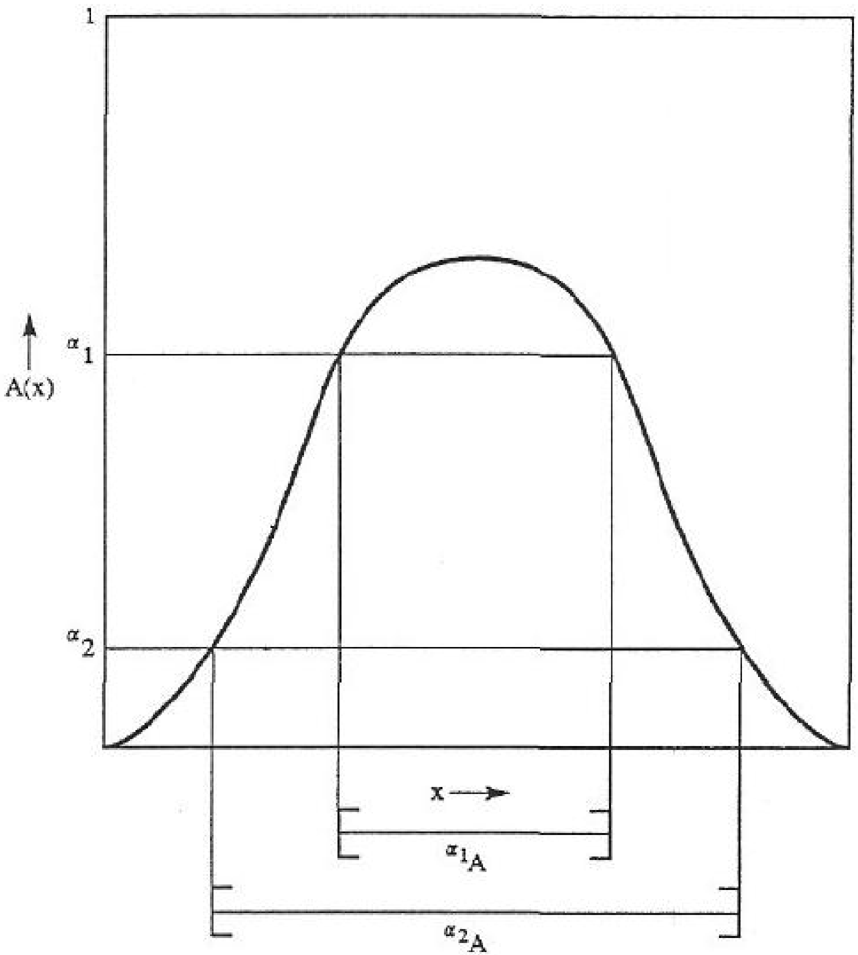


Figure 1.9 Subnormal fuzzy set that is convex.

1.4 Fuzzy sets: basic concepts

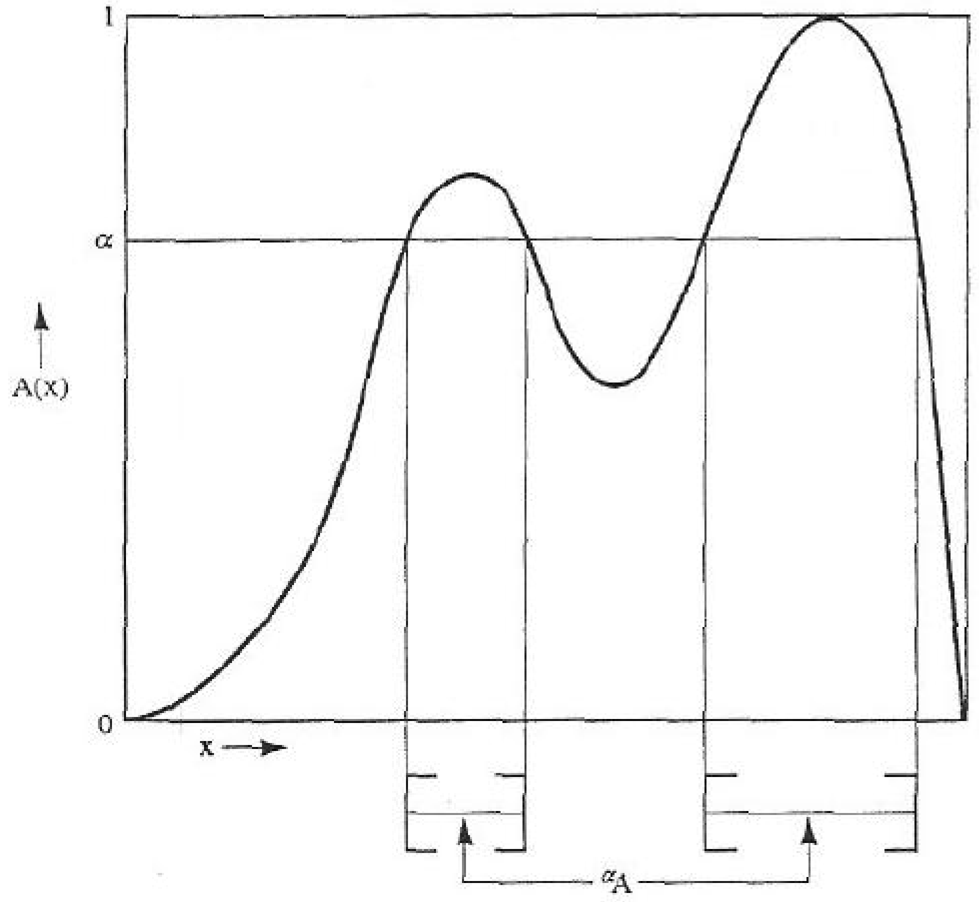


Fig. 1.10 Normal fuzzy set that is not convex.

1.4 Fuzzy sets: basic concepts

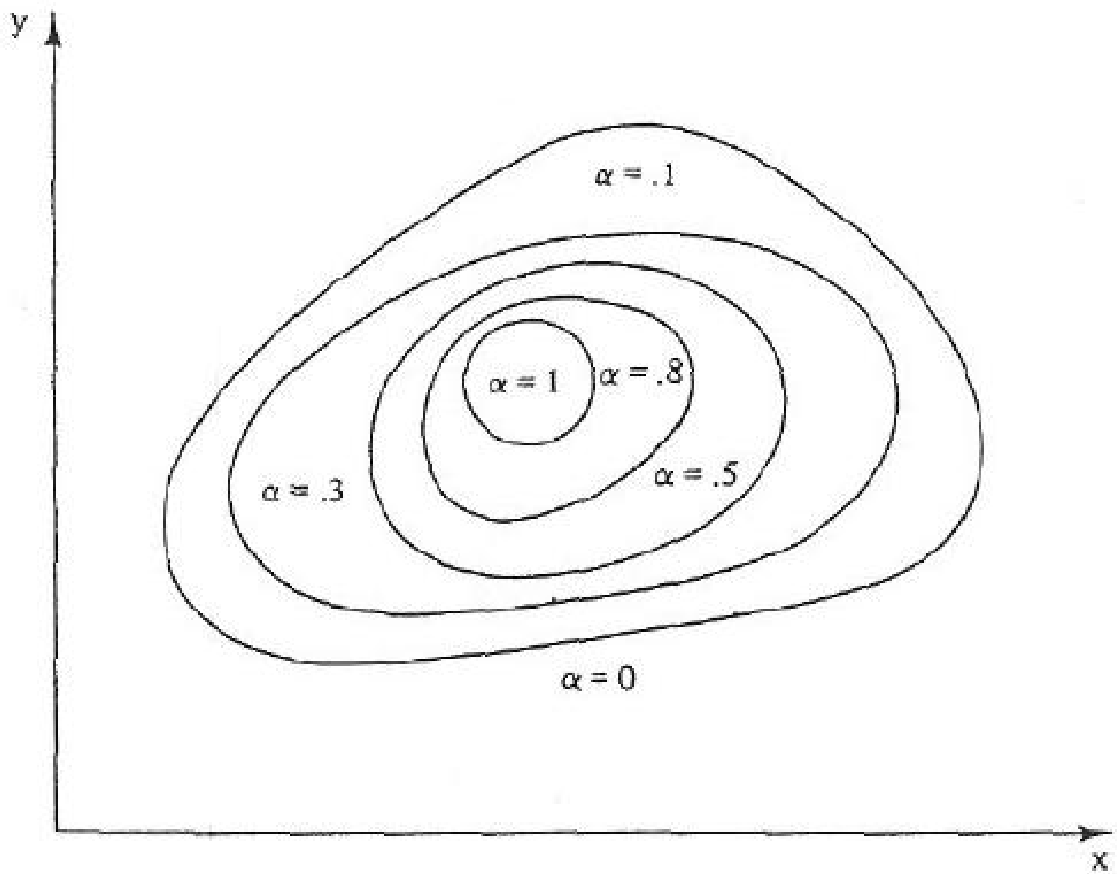


Figure 1.11 Normal and convex fuzzy set A defined by its α-cuts ${}^{.1}A$, ${}^{.3}A$, ${}^{.5}A$, ${}^{.8}A$, ${}^{1}A$.

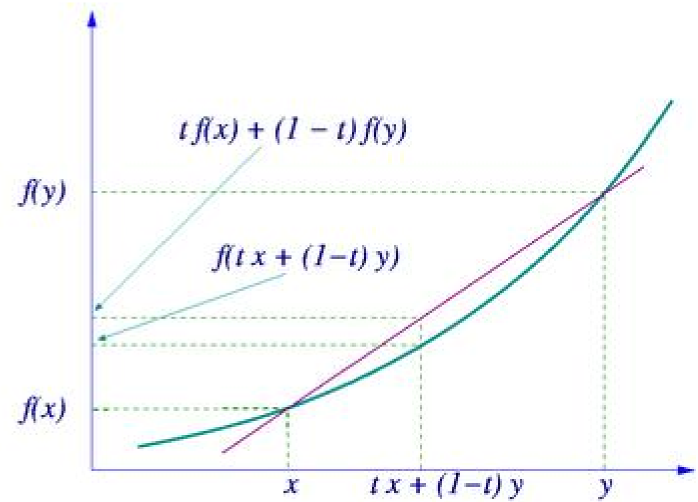
1.4 Fuzzy sets: basic concepts

- Discussions:
  - The definition of convexity for fuzzy sets does not mean that the membership function of a convex fuzzy set is a convex function.
  - In fact, membership functions of convex fuzzy sets are concave functions, not convex ones.

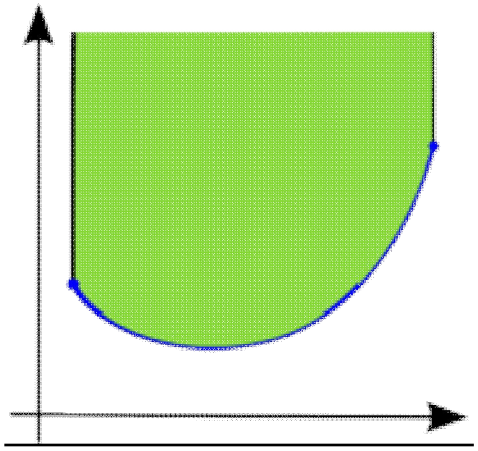
Convex function

- In mathematics, a real-valued function *f* defined on an interval is called **convex**, if for any two points *x* and *y* in its domain *C* and any *t* in [0,1], we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$



- In other words, a function is convex if and only if its epigraph (the set of points lying on or above the graph) is a convex set.



http://en.wikipedia.org/wiki/Convex_function

Concave function

- In mathematics, a **concave function** is the negative of a convex function.
- Formally, a real-valued function *f* defined on an interval is called **concave**, if for any two points *x* and *y* in its domain *C* and any *t* in [0,1], we have

$$f(tx + (1-t)y) \geq t f(x) + (1-t) f(y).$$

http://en.wikipedia.org/wiki/Concave_function

1.4 Fuzzy sets: basic concepts

Theorem 1.1. A fuzzy set A on $\mathbb{R}$ is convex iff

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min[A(x_1), A(x_2)] \tag{1.13}$$

for all $x_1, x_2 \in \mathbb{R}$ and all $\lambda \in [0, 1]$, where min denotes the minimum operator.

Proof: (i) Assume that A is convex and let $\alpha = A(x_1) \leq A(x_2)$. Then, $x_1, x_2 \in {}^{\alpha}A$ and, moreover, $\lambda x_1 + (1-\lambda)x_2 \in {}^{\alpha}A$ for any $\lambda \in [0, 1]$ by the convexity of A. Consequently,

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha = A(x_1) = \min[A(x_1), A(x_2)].$$

(ii) Assume that A satisfies (1.13). We need to prove that for any $\alpha \in (0, 1]$, ${}^{\alpha}A$ is convex. Now for any $x_1, x_2 \in {}^{\alpha}A$ (i.e., $A(x_1) \geq \alpha$, $A(x_2) \geq \alpha$), and for any $\lambda \in [0, 1]$, by (1.13)

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min[A(x_1), A(x_2)] \geq \min(\alpha, \alpha) = \alpha;$$

i.e., $\lambda x_1 + (1-\lambda)x_2 \in {}^{\alpha}A$. Therefore, ${}^{\alpha}A$ is convex for any $\alpha \in (0, 1]$. Hence, A is convex. ■

1.4 Fuzzy sets: basic concepts

- Cutworthy property:
  - Any property generalized from classical set theory into the domain of fuzzy set theory that is preserved in all α-cuts for $\alpha \in (0,1]$.
  - Convexity of fuzzy sets is an example of a cutworthy property.

- Strong cutworthy property
  - Any property generalized from classical set theory into the domain of fuzzy set theory that is preserved in all strong α-cuts for $\alpha \in [0,1]$.

1.4 Fuzzy sets: basic concepts

- The standard complement of fuzzy set A with respect to the universal set X is defined for all $x \in X$ by the equation

  $$\overline{A}(x) = 1 - A(x)$$

  - Elements of X for which $\overline{A}(x) = A(x)$ are called equilibrium points of A.
  - For example, the equilibrium points of A_2 in Fig. 1.7 are 27.5 and 52.5.

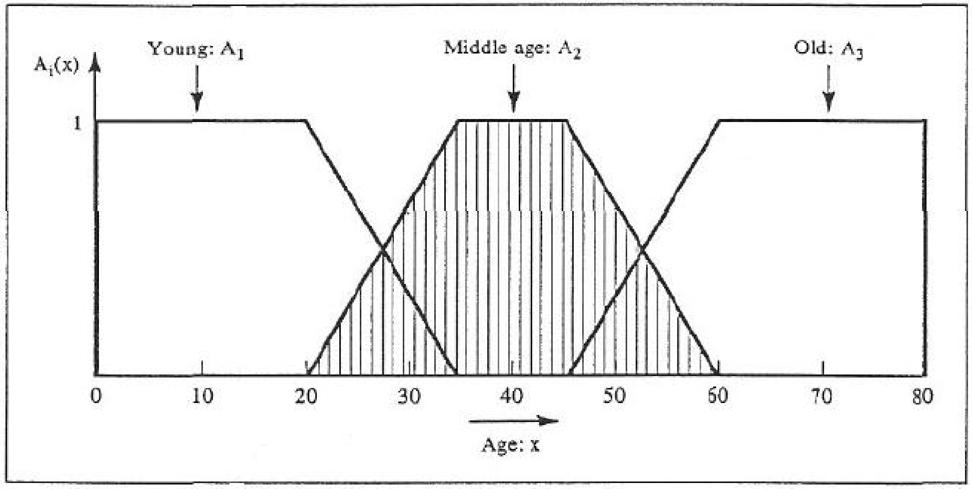
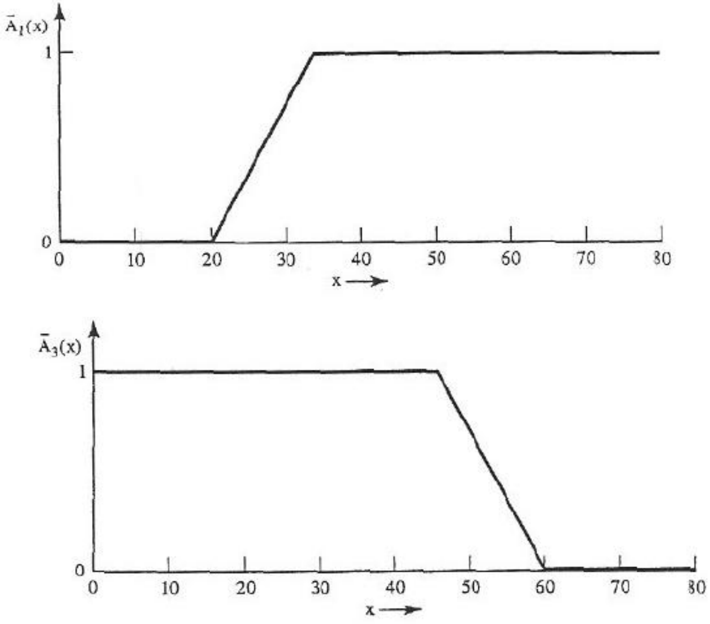


Figure 1.7 Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation D_2 of A_2 is defined numerically in Table 1.2.

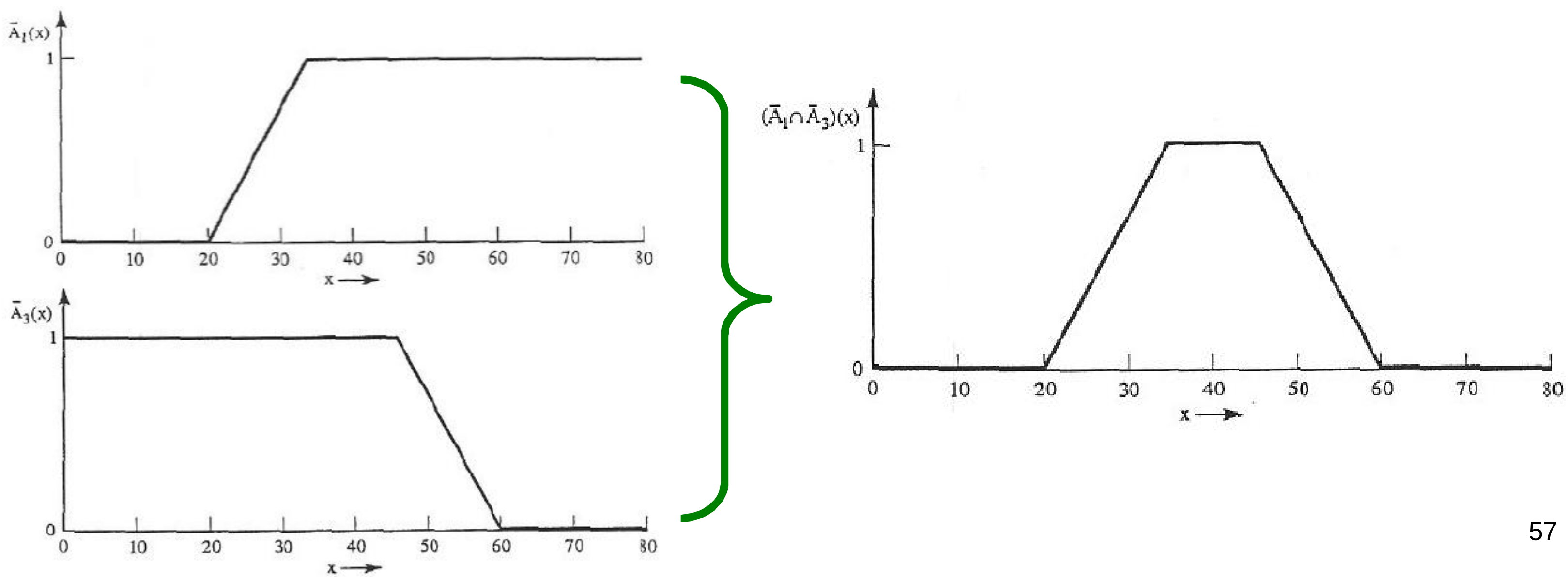


1.4 Fuzzy sets: basic concepts

- Given two fuzzy sets, A and B, their standard intersection and union are defined for all $x \in X$ by the equations

$$(A \cap B)(x) = \min[A(x), B(x)],$$
$$(A \cup B)(x) = \max[A(x), B(x)],$$

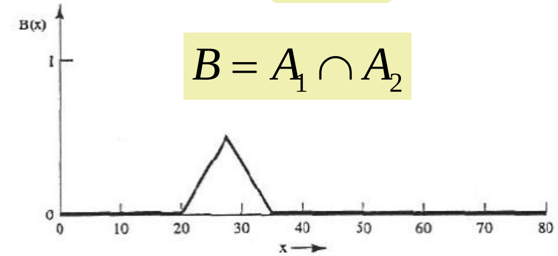
where min and max denote the minimum operator and the maximum operator, respectively.



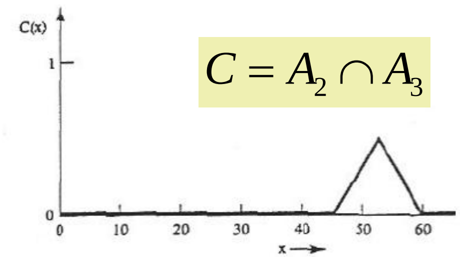
1.4 Fuzzy sets: basic concepts

- Another example:
  - A_1, A_2, A_3 are normal.
  - B and C are subnormal.
  - B and C are convex.
  - $B \cup C$ and $\overline{B \cup C}$ are not convex.

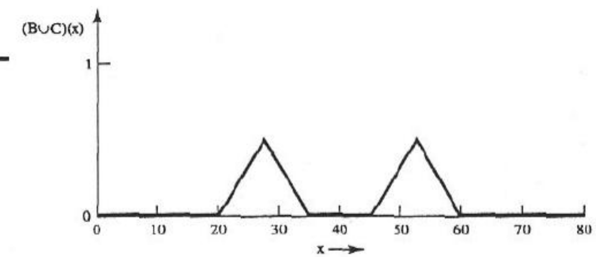
$B = A_1 \cap A_2$

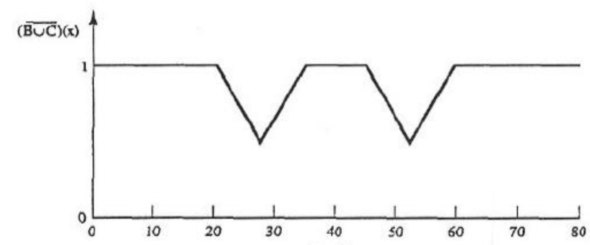


$C = A_2 \cap A_3$



Normality and convexity may be lost when we operate on fuzzy sets by the standard operations of intersection and complement.





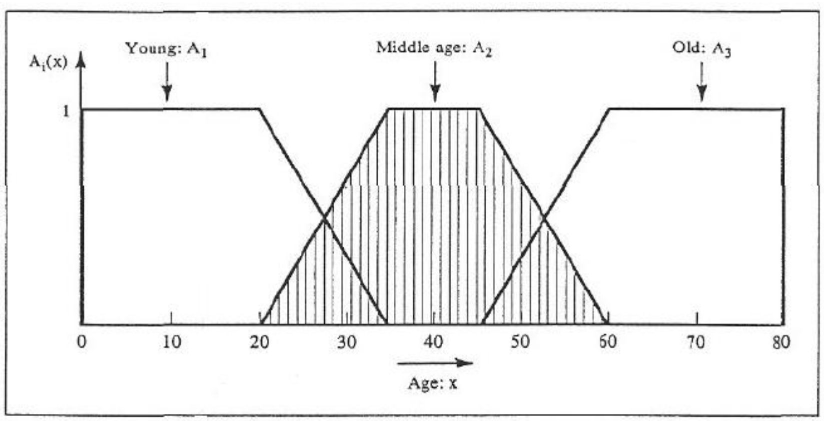


Figure 1.7 Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation D_2 of A_2 is defined numerically in Table 1.2.

Figure 1.13 Illustration of standard operation on fuzzy sets $B = A_1 \cap A_2$ and $C = A_2 \cap A_3$ (A_1, A_2, A_3 are given in Fig. 1.7).

1.4 Fuzzy sets: basic concepts

- Discussions:
  - Normality and convexity may be lost when we operate on fuzzy sets by the standard operations of intersection and complement.
  - The fuzzy intersection and fuzzy union will satisfies all the properties of the Boolean lattice listed in Table 1.1 except the law of contradiction and the law of excluded middle.

TABLE 1.1 FUNDAMENTAL PROPERTIES OF CRISP SET OPERATIONS

Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotence	$A \cup A = A$
	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$
	$A \cap (A \cup B) = A$
Absorption by X and $\varnothing$	$A \cup X = X$
	$A \cap \varnothing = \varnothing$
Identity	$A \cup \varnothing = A$
	$A \cap X = A$
Law of contradiction	$A \cap \overline{A} = \varnothing$
Law of excluded middle	$A \cup \overline{A} = X$
De Morgan's laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
	$\overline{A \cup B} = \overline{A} \cap \overline{B}$

1.4 Fuzzy sets: basic concepts

- The law of contradiction

$$A \cap \overline{A} = \phi$$

- To verify that the law of contradiction is violated for fuzzy sets, we need only to show that

$$\min[A(x), 1 - A(x)] = 0$$

is violated for at least one $x \in X$.
  - This is easy since the equation is obviously violated for any value $A(x) \in (0,1)$, and is satisfied only for $A(x) \in \{0,1\}$.

1.4 Fuzzy sets: basic concepts

- To verify the law of absorption,

  $A \cup (A \cap B) = A$

  - This requires showing that $\max[A(x), \min[A(x), B(x)]] = A(x)$ is satisfied for all $x \in X$.
  - Consider two cases:

    (1) $A(x) \le B(x)$

    $\Rightarrow \max[A(x), \min[A(x), B(x)]] = \max[A(x), A(x)] = A(x)$

    (2) $A(x) > B(x)$

    $\Rightarrow \max[A(x), \min[A(x), B(x)]] = \max[A(x), B(x)] = A(x)$

$\Rightarrow \max[A(x), \min[A(x), B(x)]] = A(x)$

1.4 Fuzzy sets: basic concepts

- Given two fuzzy set $A, B \in \mathcal{F}(X)$,

  we say that A is a subset of B and write $A \subseteq B$ iff

  $$A(x) \le B(x)$$

  for all $x \in X$.

  - $A \subseteq B$ iff $A \cap B = A$ and $A \cup B = B$ for any $A, B \in \mathcal{F}(X)$.

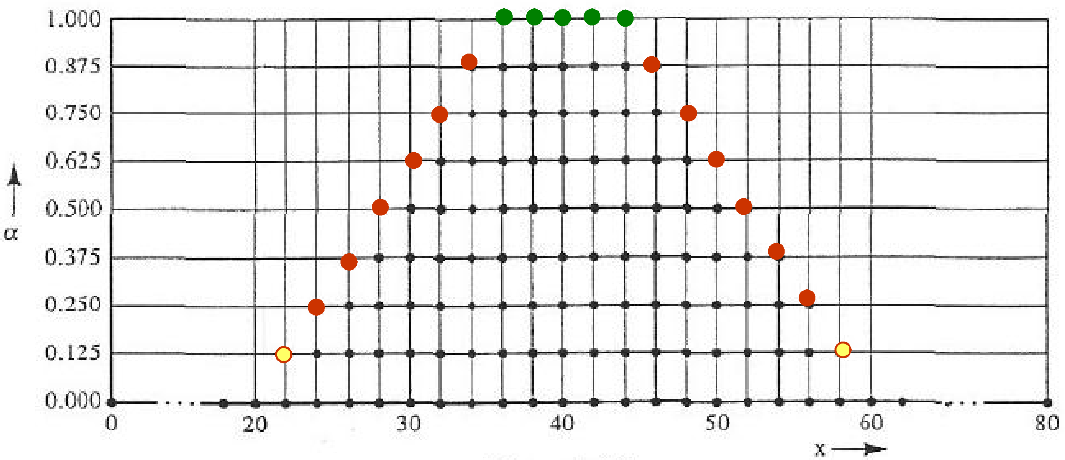
1.4 Fuzzy sets: basic concepts

For any fuzzy set A defined on a finite universal set X, we define its *scalar cardinality*, $|A|$, by the formula

$$|A| = \sum_{x \in X} A(x). \tag{1.18}$$

For example, the scalar cardinality of the fuzzy set D_2 defined in Table 1.2 is

$$|D_2| = 2(.13 + .27 + .4 + .53 + .67 + .8 + .93) + 5 = 12.46.$$



(a) α - cuts ${}^{\alpha}D_2$

TABLE 1.2 DISCRETE APPROXIMATION OF MEMBERSHIP FUNCTION A_2 (FIG. 1.7) BY FUNCTION D_2 OF THE FORM: $D_2 : \{0, 2, 4, \ldots, 80\} \rightarrow [0, 1]$

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$x \in \{32, 48\}$	0.80
$x \in \{34, 46\}$	0.93
$x \in \{36, 38, \ldots, 44\}$	1.00

1.4 Fuzzy sets: basic concepts

For any pair of fuzzy subsets defined on a finite universal set X, the *degree of subsethood*, $S(A, B)$, of A in B is defined by the formula

$$S(A, B) = \frac{1}{|A|}\left(|A| - \sum_{x \in X} \max[0, A(x) - B(x)]\right). \tag{1.19}$$

The Σ term in this formula describes the sum of the degrees to which the subset inequality $A(x) \leq B(x)$ is violated, the difference describes the lack of these violations, and the cardinality $|A|$ in the denominator is a normalizing factor to obtain the range

$$0 \leq S(A, B) \leq 1. \tag{1.20}$$

It is easy to convert (1.19) to the more convenient formula

$$\boxed{S(A, B) = \frac{|A \cap B|}{|A|},} \tag{1.21}$$

where $\cap$ denotes the standard fuzzy intersection.

1.4 Fuzzy sets: basic concepts

- Given a fuzzy set A defined on a finite universal set X let $x_1, x_2, \ldots, x_n$ denote elements of the support ${}^{0+}A$ of A and let a_i denote the grade of membership of x_i in A for all $i \in \mathbb{N}_n$.

$$A = a_1/x_1 + a_2/x_2 + \ldots + a_n/x_n$$

  - If the universal set is finite or countable:

$$A = \sum_{i=1}^{n} a_i/x_i \text{ or } A = \sum_{i=1}^{\infty} a_i/x_i .$$

  - If X is an interval of real numbers:

$$A = \int_X A(x)/x .$$

the integral sign indicates that all the pairs of x and $A(x)$ in the interval X collectively form A.

1.4 Fuzzy sets: basic concepts

It is interesting and conceptually useful to interpret ordinary fuzzy subsets of a finite universal set X with n elements as points in the n-dimensional unit cube $[0,1]^n$. That is, the entire cube represents the fuzzy power set $\mathcal{F}(X)$, and its vertices represent the crisp power set $\mathcal{P}(X)$.

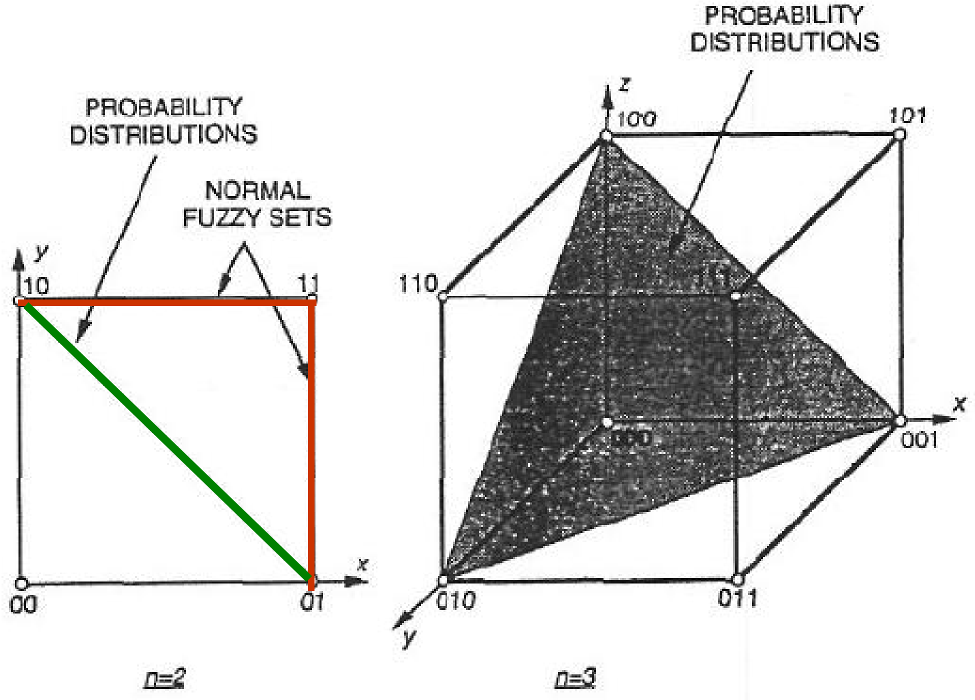

Figure 1.14 Examples illustrating the geometrical interpretation of fuzzy sets.

1.4 Fuzzy sets: basic concepts

This interpretation suggests that a suitable distance be defined between fuzzy sets. Using, for example, the concept of the Hamming distance, we have

$$d(A, B) = \sum_{x \in X} |A(x) - B(x)|. \tag{1.22}$$

The cardinality $|A|$ of a fuzzy set A, given by (1.18), can be then viewed as the distance $d(A, \varnothing)$ of A from the empty set. Observe that probability distributions are represented by sets whose cardinality is 1.

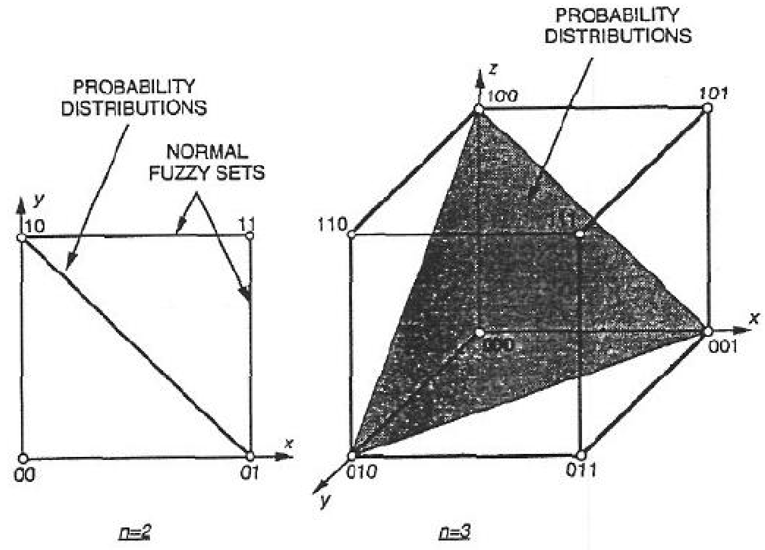

Figure 1.14 Examples illustrating the geometrical interpretation of fuzzy sets.