DUE: 10am THURSDAY Mar 28 in the drop boxes opposite the Math Tutorial Centre MC 4067.

1. For each of the graphs shown, determine whether it is planar. If the graph is planar, exhibit a planar embedding. If the graph is not planar, exhibit a subdivision of K_5 or $K_{3,3}$.

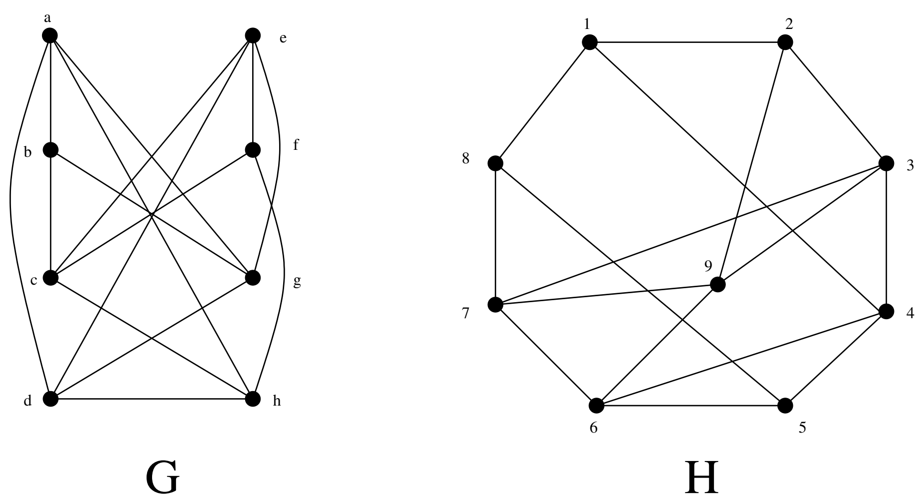

Figure 1:

SOLUTION. The graph G is planar, as shown by the planar drawing in Figure 2. The graph H is not planar, as is shown by the subdivision of $K_{3,3}$ with branch vertices $\{7, 5, 2\}$ in one vertex class and $\{8, 6, 3\}$ in the other. The subdivision shown also contains the vertices $4, 1$ and 9.

2. For each of the graphs shown, determine whether it is planar. If the graph is planar, exhibit a planar embedding. If the graph is not planar, exhibit a subdivision of K_5 or $K_{3,3}$.

SOLUTION. The graph G is planar, as shown by the planar drawing in Figure 4. The graph H is not planar, as is shown by the subdivision of $K_{3,3}$ with branch vertices $\{a, i, d\}$ in one vertex class and $\{b, h, f\}$ in the other. The subdivision shown also contains the vertices g, c, e, j and k.

3. Let G be a bipartite graph with at least 9 vertices. Prove that the complement $\bar{G}$ of G is not planar.

SOLUTION. Since G is bipartite there is a partition of $V(G)$ into sets A and B such that no edge of G joins two vertices in A or two vertices in B. Since G has at least 9 vertices we may assume without loss of generality that $|A| \geq 5$. Therefore in $\bar{G}$, all edges joining two vertices of A are present. In particular this means that $\bar{G}$ contains a subgraph isomorphic to K_5. Therefore by (the easy direction of) Kuratowski's Theorem $\bar{G}$ is not planar.

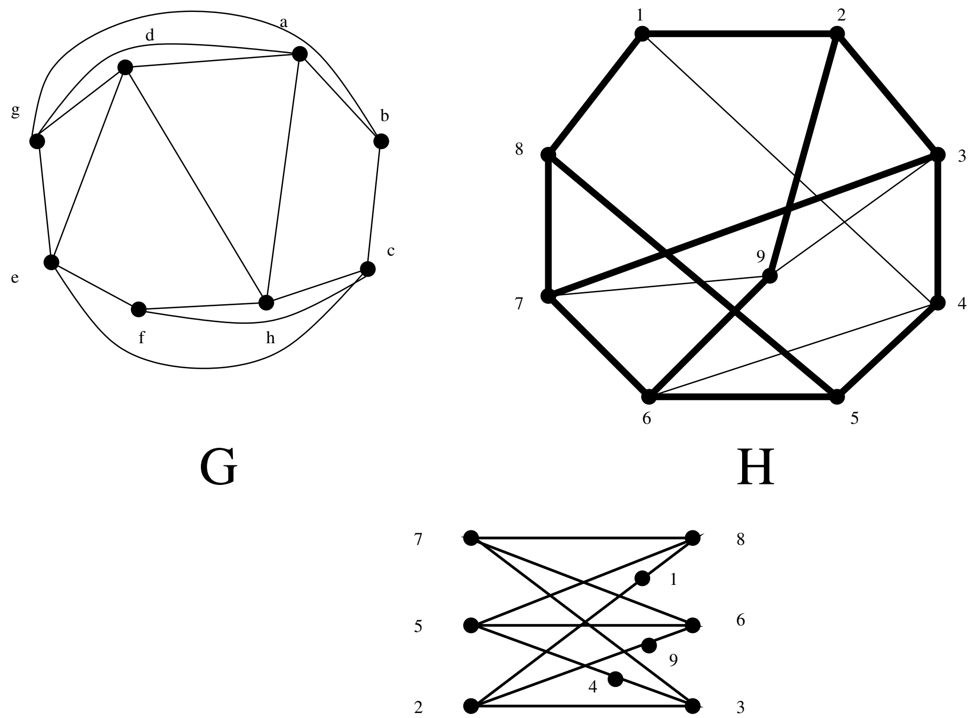

Figure 2:

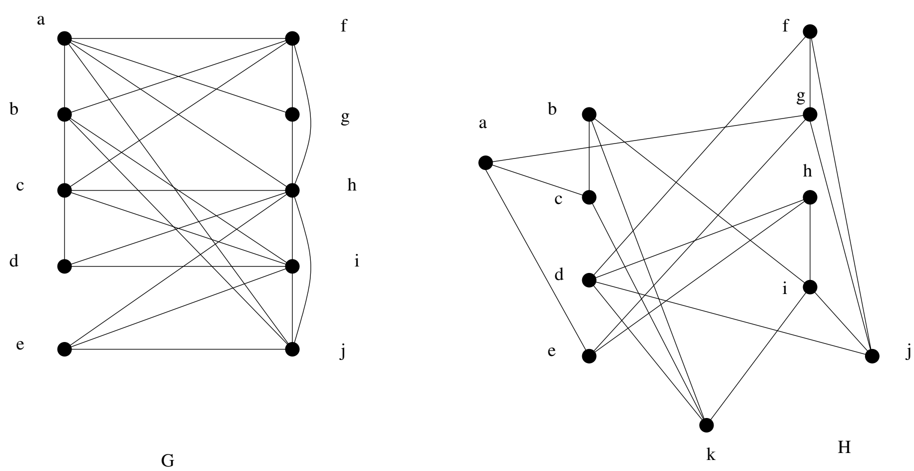

Figure 3:

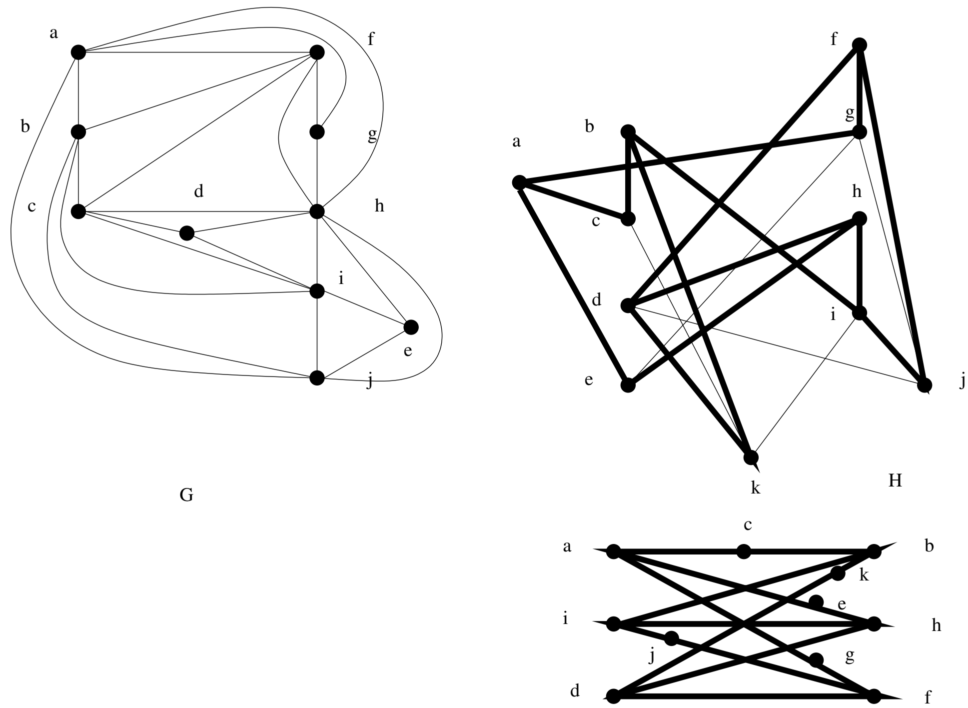

Figure 4:

4. Let G be a planar graph that does not contain any cycles of length 3.

   (a) Prove that G contains a vertex of degree at most 3.

   SOLUTION. We may assume G is connected. Otherwise, we may add edges joining the components of G until the resulting graph is connected, while keeping the drawing planar and not introducing any 3-cycles. If we can find a vertex of degree at most 3 in this graph then it has degree at most 3 in G as well.

   First note that if G is a tree then we know either that $p = |V(G)| = 1$ (in which case G has a vertex of degree 0) or G has a leaf (a vertex of degree 1). Therefore we may assume that G contains a cycle. Then by Lemma 7.5.1, in any planar drawing of G, every face contains a cycle and hence has degree at least 3. But since G doesn't contain any 3-cycles, every face has degree at least 4.

   Let $\tilde{G}$ be a planar drawing of G. Let $q = |E(G)|$. Then we know $2q = \sum_{f \in F(\tilde{G})} deg(f) \geq 4s$, where $F(\tilde{G})$ denotes the set of faces of $\tilde{G}$ and $s = |F(\tilde{G})|$. By Euler's Formula $s = 2 + q - p$, so we find

   $$2q \geq 4(2 - q + p),$$

   which implies that $q \leq 2p - 4$. But if every vertex of G has degree at least 4 then $2q = \sum_{v \in V(G)} deg(v) \geq 4p$, which implies $q \geq 2p$ which is a contradiction. Therefore G must have a vertex of degree at most 3.

   (b) Prove that G is 4-colourable. (Do not use the Four-Colour Theorem.)

SOLUTION. We prove that G is 4-colourable by induction on p.

If $p \leq 4$ then G is 4-colourable because we can give every vertex a different colour.

IH: assume $p \geq 5$ and that every planar graph with $p-1$ vertices that does not contain any cycles of length 3 is 4-colourable.

Consider G with p vertices. By (a) we know G has a vertex x of degree at most 3. By erasing x from a planar drawing of G, we obtain a planar drawing of $G-x$, which shows that $G-x$ is planar. Moreover $G-x$ does not contain any 3-cycles. Therefore by IH $G-x$ is 4-colourable. Let f be a 4-colouring of $G-x$. Then we can extend f to a 4-colouring of G by setting $f(x) \in \{1,2,3,4\}$ to be a colour that is none of $\{f(y) : xy \in E(G)\}$. This is possible since x has a most 3 neighbours. This completes the induction step, and so we conclude that G is 4-colourable.

5. Let G' be a subdivision of a bipartite graph G. Prove that G' is 3-colourable.

SOLUTION.

We call the vertices of G' that are also vertices of G the *branch* vertices of G'. Vertices of G' that are not vertices of G are called *path* vertices. Since G is bipartite, the branch vertices of G' can be partitioned into two classes A and B such that there is no edge of G' joining two vertices of A or two vertices of B.

We give a 3-colouring of G' as follows. Give colour 1 to all vertices in A and colour 2 to all vertices in B. Now for each edge ab of G, colour the path vertices of G' on the path P_{ab} joining a to b in G' as follows: if $P_{ab} = av_1v_2 \dots v_rb$ has odd length $r+1$ then colour v_i 1 if i is even and 2 if i is odd. If the length $r+1$ of P_{ab} is even, give colour 3 to v_1, then colour 1 to v_i for all odd $i \geq 3$ and colour 2 to v_i for all even i. Then this gives a colouring of G' with 3 colours as required. (Note in particular that if $r=0$ then P_{ab} is just the edge ab, and a is coloured 1 and b is coloured 2.)

Alternate solution:

We prove the claim by induction on the number n of path vertices in G'.

If $n=0$ then $G'=G$ is bipartite, hence 2-colourable, hence 3-colourable by definition.

IH: Assume that $n \geq 1$ and every subdivision of G with fewer than n path vertices is 3-colourable.

Consider G with n path vertices. Let x be a path vertex of G'. Then by definition of subdivision x has degree 2 in G'. Let y be a neighbour of x. Then the graph G'/xy obtained by contracting the edge xy is a subdivision of G with $n-1$ vertices, so by IH there exists a 3-colouring f of G'/xy. Let z denote the vertex that is the image of xy in G'/xy. Define a colouring f' of G' as follows:

if $w \notin \{x,y\}$ let $f'(w) = f(w)$,
let $f'(y) = f(z)$,
let $f(x) \in \{1,2,3\}$ be a colour different from $f(y)$ and from $f(v)$, where v is the other neighbour of x in G'. This is possible since $|\{y,z\}| = 2$ and $|\{1,2,3\}| = 3$.

Then f' is a 3-colouring of G' by construction and the fact that f is a 3-colouring of G'/xy. Therefore G' is 3-colourable, completing the induction step.