MATH 239 - Assignment 7

DUE: 10am Friday March. 15th in the drop boxes opposite the Math Tutorial Centre MC 4067.

Exercise 1 (20pts).
Let $G = (V, E)$ be a graph and suppose that every vertex has degree at least $k \geq 2$.

(a) Show that G has a path with at least k edges.
(b) Show that G has a cycle with at least k edges.

Exercise 2 (20pts).
A walk is *closed* if the first vertex and the last vertex of the walk are the same.

(a) Show that in a bipartite graph, every closed walk has an even number of edges.
(Note, edges are counted as many time as they appear in the walk.)
(b) Show that every closed walk that does not repeat an edge contains a cycle.

Exercise 3 (20pts).
Prove that the following statements are equivalent for a graph $G = (V, E)$,

(i) G is connected and G has exactly one cycle,
(ii) G is connected and $|E| = |V|$,
(iii) G has exactly one cycle and $|E| = |V|$.

Exercise 4 (20pts).
Let $G = (V, E)$ be a graph with distinct vertices s and t. We say that a set of st-paths are *internally disjoint* if no two of these paths share a common vertex aside from s and t. A set of vertices X is a *vertex st-cut* if $X \subseteq V \setminus \{s, t\}$ and the graph obtained from G by removing all vertices in X has no path from s to t. Show that statement (i) implies statement (ii).

(i) There exists k internally disjoint paths from s to t.
(ii) Every vertex st-cut contains at least k vertices.

Note, these statements are in fact equivalent but you are not asked to prove this.

Exercise 5 (20pts).
Let $G = (V, E)$ be a graph that is k-regular. Denote by $\delta(S)$ the set of edges with exactly one endpoint in S and by $\gamma(S)$ the set of edges with two endpoints in S.

(a) Show that for every $S \subseteq V$ we have

$$\sum_{v \in S} deg(v) = |\delta(S)| + 2|\gamma(S)|.$$

(b) Using (a) show that if a connected graph is k-regular where k is even then G has no bridge.

Exercise 6 (20pts).

(a) Show that if a tree has a vertex of degree r then it has at least r vertices of degree 1.
(b) Show that if a tree with at least two vertices has k vertices of degree r then it has at least $k(r-2)+2$ vertices of degree 1.