MATH 239 Tutorial 9 Problems

1. Let G be a weighted connected graph, and let T be a minimum spanning tree of G. Prove that if C is a cycle in G, then some edge in C of maximum weight is not in T.

   Solution. By way of contradiction, assume that every maximum weight edge in C is also in T. Let e be one of those edges of maximum weight. Since e is in T, it is a bridge of T (as every edge of a tree is a bridge), so $T - e$ has components C_1 and C_2. Since e is part of cycle C in G, it is not a bridge in G, so there must be some other edge of C, call it e', that goes from the vertices of C_1 to C_2. Since T is acyclic, $e' \notin T$.

   But recall our assumption that every maximum weight edge in C is also in T. This means that e' is not of maximum weight, i.e. $w(e') < w(e)$. Now consider the graph $T' = T + e' - e$. Adding e' to T creates exactly one unique cycle, and deleting e removes that cycle, so the graph is still acyclic. Since e was in that cycle, it was not a bridge of $T + e'$, so $T + e' - e$ is still connected and spanning. Thus we have created a new spanning tree T', and since $w(e') < w(e)$, T' has a lower total weight than T. But this is a contradiction, since T is a Minimum Spanning Tree.

2. Suppose we want to find a spanning tree whose largest weight is as small as possible. Prove that Kruskal's algorithm produces such a tree.

   Solution. We know that Kruskal's algorithm returns a Minimum Spanning Tree T. We will show that, of all spanning trees, it also minimizes the value $\max_{e \in T} w(e)$, i.e. the heaviest edge is as small as possible.

   By way of contradiction, assume that T does not minimize this value. Then there must be some other spanning tree T^* that does minimize the value, i.e.

   $$\max_{e' \in T^*} w(e') < \max_{e \in T} w(e)$$

   Consider the heaviest edge e in T. Since T is a tree, e is a bridge, so $T - e$ is two connected components C_1 and C_2. Since e is the heaviest edge of T, and all edges of T^* are lighter than the heaviest edge of T, $e \notin T^*$. Thus, there must be some other edge $e' \in T^*$ that connects the vertices of C_1 to C_2, since T^* is a spanning tree. Additionally,

   $$w(e') \leq \max_{e' \in T^*} w(e') < \max_{e \in T} w(e) = w(e)$$

   But now consider $T' = T + e' - e$. As in Question 1, we see that T' is a spanning tree with lower total weight than T. But this is a contradiction, since T was returned by Kruskal's Algorithm, and is thus a Minimum Spanning Tree.

3. Prove the more general version of Euler's formula: If G is a graph with a planar embedding with n vertices, m edges, s faces and c components, then

   $$n - m + s = 1 + c.$$

Solution. We first recall Euler's Identity for connected graphs with a planar embedding:

$$V - E + F = 2$$

We will now proceed to prove the general version by Induction on c, the number of components.

Base Case: c = 1. Then by Euler's Identity for connected graphs, we know that $V - E + F = 2 = 1 + c$, so it is true for our base case.

Induction Hypothesis: Assume the statement is true when $c = k$, where k is some positive integer.

Induction Step: Prove the statement is true when $c = k + 1$. Consider a graph G with components $C_1, C_2, ..., C_k, C_{k+1}$ and a planar embedding. We will split G into two graphs: G_1 will consist of the components $C_1, C_2, ..., C_k$ while G_2 will consist only of component C_{k+1}.

Since G has a planar embedding, clearly G_1 and G_2 must have planar embeddings. We also note that the total number of vertices n is equal to the number of vertices in G_1 plus the number of vertices in G_2, i.e.

$$n = |V(G_1)| + |V(G_2)|$$

Similarly, we can make the same statement about the edges.

$$m = |E(G_1)| + |E(G_2)|$$

However, we cannot make the same statement about the faces. We notice that in our planar embedding of G_1, we have an outer face, and similarly G_2 has an outer face. However, these two faces are the same in G, so we are counting it twice (once in G_1 and once in G_2). Thus,

$$s = F_1 + F_2 - 1$$

where F_1 and F_2 are the number of faces in G_1 and G_2 respectively.

Since G_1 and G_2 are planar embeddings, and have either k or 1 component, we can use our induction hypothesis.

$$|V(G_1)| - |E(G_1)| + F_1 = 1 + k$$

and

$$|V(G_2)| - |E(G_2)| + F_2 = 1 + 1$$

Adding these two equations together, we get

$$|V(G_1)| + |V(G_2)| - |E(G_1)| - |E(G_2)| + F_1 + F_2 = 3 + k$$

Which simplifies to

$$V - E + F + 1 = 3 + k$$

$$V - E + F = 1 + (1 + k)$$

As required.

4. Let G be a 3-regular connected planar graph which has an embedding where each face has degree either 4 or 6, and no two faces of degree 4 are adjacent. Determine an example of such a graph with the fewest number of vertices.

   Solution. We could start drawing graphs and see if we get the desired properties, but first it might be better to use some identities based on what we already know. The graph only has faces of degree 4 or 6; we will let the number of degree 4 faces be S, and the number of degree 6 faces be H.

   Since the graph is planar, we can use Euler's Identity.

$$V - E + (S + H) = 2$$

   Since the graph is 3-regular, we can use Handshaking.

$$2E = \sum_{v \in V} \deg(v) = 3V$$

   Since the only faces are of degree 4 or 6, we can use Faceshaking.

$$2E = \sum_{f \in F} \deg(f) = 4S + 6H$$

   Additionally, we have the fact that no degree 4 face is adjacent to another degree 4 face. Each degree 4 face has exactly 4 vertices on its perimeter, and since the graph is 3-regular and no two degree 4 faces are adjacent, a vertex can be in at most one degree 4 face. Thus,

$$4S \leq V$$

   Using these four identities, we can determine that $S = 6$, and thus $V \geq 24$. Since the question is looking for the smallest graph possible, we will try to draw it with 24 vertices.

5. In class, we have proved that any planar graph with n vertices have at most $3n - 6$ edges, i.e. $m \leq 3n - 6$. What can you say about those graphs with exactly $3n - 6$ edges? Draw one where $n = 8$.

   Solution. First, we will recall how we determined this identity. Since the graph is planar, we can use Euler's Identity

$$n - m + F = 2$$

   Second, we know that each face contains a cycle, and any cycle must have length at least 3. Thus, each face of a planar graph must have degree at least 3. By Faceshaking,

$$2m = \sum_{f \in F} \deg(f) \geq 3F$$

   Multiplying the first equation by 3 and then substituting in the second equation, we get

$$3n - 3m + 2m \geq 6$$

And thus $m \leq 3n - 6$. We now wish to understand what G looks like if this identity is satisfied by equality, i.e. $m = 3n - 6$. If this identity held by equality, then each of the preceding equations must have held by equality as well. In particular,

$$2m = \sum_{f \in F} \deg(f) = 3F$$

Thus, each face of our graph G must have degree exactly 3, so every face is a triangle.

Additional exercises

1. Let T be a tree with n vertices, and let x be a vertex in T. For any vertex v in T, define $d(v)$ to be the length of the unique v, x-path in T. Prove that

$$\sum_{v \in V(G)} d(v) \leq \binom{n}{2}.$$

   When does eqauality hold?

   Solution. Induction on n. Equality holds only when T is a path of length $n - 1$.

2. Suppose we want to find a minimum spanning path in a weighted complete graph. We use a greedy algorithm as follows: Start with an edge of minimum weight. At each stage, we consider the edges incident with the two ends of our current path, pick one that has the smallest weight and extend our path by one edge. We repeat until we get a spanning path. Does this algorithm work? Either prove that this algorithm works, or provide an example of a weighted K_n for each n for which this does not work.

   Solution. This algorithm does not work.