Instructions

1. **EXPLANATIONS ARE ALWAYS REQUIRED.**
   SHOW ALL YOUR WORK.
   STATE ANY THEOREMS YOU ARE USING.
2. No aids of any kind are permitted.
3. Please write your solutions in the space provided. If you need more space, please use the back of a page. Clearly indicate where your solution continues.

Question	Mark	Question	Mark
1 8 points		5 8 points	
2 6 points		6 8 points	
3 4 points		7 10 points	
4 9 points		8 7 points	
		Total 60 points	

Question 1. [8 = 2+2+2+2 points] Consider the following formal power series:

$$\Phi(x) = \frac{1}{1-x} \qquad \Psi(x) = \frac{x}{(1-x)(1+2x)}$$
$$\Lambda(x) = 1 + 10x + 100x^2 + \cdots \qquad \Upsilon(x) = x + 2x^2 + 3x^3 + \cdots .$$

For each of the following, does the expression represent a well-defined formal power series? Be sure to justify your answers. Answers given without justification will receive a mark of zero.

(a) $\Phi(x) + \Psi(x)$

Solution: Yes. The sum of any two formal power series is a formal power series.

(b) $A(x)$ satisfying $\Lambda(x)A(x) = \Upsilon(x)$

Solution: Yes. The constant term of $\Lambda(x)$ is nonzero, so $(\Lambda(x))^{-1}$ exists and $A(x) = (\Lambda(x))^{-1}\Upsilon(x)$ is just the product of two formal power series.

(c) $\Upsilon(x)^{-1}$

Solution: No. The constant term of $\Upsilon(x)$ is zero, so it is not invertible.

(d) $\Phi(\Psi(x))$

Solution: Yes. The constant term of $\Psi(x)$ is zero, so it can be substituted into another formal power series.

Question 2. [6 points] Let n be a positive integer. Give a combinatorial proof that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

Solution: Let $A = \{1, ..., n\}$ and $B = \{n+1, ..., 2n\}$. Let $\mathcal{P}$ be the set of all unordered pairs of elements from $A \cup B$. Clearly $|\mathcal{P}| = \binom{2n}{2}$.

Suppose $\{x, y\}$ is an element of $\mathcal{P}$. There are three possibilities: either both x and y are in A, both are in B, or one is in A and the other in B. There are $\binom{n}{2}$ pairs $\{x, y\}$ such that both x and y are in A. Likewise there are $\binom{n}{2}$ pairs $\{x, y\}$ such that both are in B. Finally there are n^2 pairs with one of x, y in A and the other in B. Hence,

$$\binom{2n}{2} = |\mathcal{P}| = \binom{n}{2} + \binom{n}{2} + n^2 = 2\binom{n}{2} + n^2,$$

as required.

Question 3. [**4 points**] Let A and B be sets, and let w be a weight function defined on $A \cup B$. Suppose $A \cap B \neq \emptyset$. Give an expression for the generating series $\Phi_{A\cup B}(x)$ with respect to w, in terms of generating series for smaller sets. Prove your expression is correct.

Solution: By definition we have

$$\Phi_{A\cup B}(x) = \sum_{\sigma\in A\cup B} x^{w(\sigma)}.$$

The set of those $\sigma \in A \cup B$ that are in both A and B is $A \cap B$. Therefore

$$\sum_{\sigma\in A\cup B} x^{w(\sigma)} = \sum_{\sigma\in A} x^{w(\sigma)} + \sum_{\sigma\in B} x^{w(\sigma)} - \sum_{\sigma\in A\cap B} x^{w(\sigma)}.$$

Therefore by definition we find

$$\Phi_{A\cup B}(x) = \Phi_A(x) + \Phi_B(x) - \Phi_{A\cap B}(x).$$

Question 4. [**9 = 2+3+4 points**] Let k be a fixed positive integer. Let S denote the set of compositions into k parts, where each part is even and at least 6.

(a) [**2 points**] Express S as a Cartesian product of suitable sets.

Solution: Let $A = \{6, 8, 10, ...\}$ be the set of all even integers which are at least 6. Then A is the Cartesian product of k copies of A, which we can write as

$$S = A^k.$$

(b) [**3 points**] Find the generating function for S, with respect to the usual weight function (the weight of a composition is the sum of its parts). Be sure to state all theorems you use. Express your solution as a rational function.

Solution: By the product lemma and sum of geometric series,

$$\begin{aligned}\Phi_S(x) &= (\Phi_A(x))^k = (x^6 + x^8 + x^{10} + ...)^k \\ &= \left(\frac{x^6}{1-x^2}\right)^k.\end{aligned}$$

(c) [**4 points**] Let n be a positive integer. Find the number of compositions of n into k parts, where each part is even and at least 6. Express your solution as a closed-form expression in terms of n and k.

Solution: The number we want is the n^{th} coefficient of $\Phi_S(x)$. By the binomial theorem,

$$\begin{aligned}[x^n]\Phi_S(x) &= [x^n]x^{6k}(1-x^2)^{-k} = [x^{n-6k}](1-x^2)^{-k} \\ &= [x^{n-6k}]\sum_{i\geq 0}\binom{k+i-1}{k-1}x^{2i} \\ &= \begin{cases} 0 & \text{if } n \text{ is odd or } n < 6, \\ \binom{\frac{n}{2}-2k-1}{k-1} & \text{if } n \text{ is even and } n \geq 6. \end{cases}\end{aligned}$$

Question 5. [**8 points**] Let S be the set of all binary strings which start with a 1 and do not contain the substring 00111. Show that the generating series of S is equal to

$$\frac{x}{1-2x+x^5},$$

where the weight of a string is its length.

Solution: Consider the blocks of an element of S. If a block of 0's has size two or more, it cannot have a block of 1's of size more than two following it. Restricting the block decomposition (with blocks of 1's after blocks of 0's) to avoid these substrings and force the string to start with 1, we have

$$S = \{1\}\{1\}^*(\{0\}\{1\}\{1\}^* \cup \{00\}\{0\}^*\{1,11\})^*\{0\}^*.$$

This decomposition is unambiguous since it is just a restriction of the block decomposition. Now, using the sum, product, and $*$ lemmas, the generating series is

$$\begin{aligned}\Phi_S(x) &= \left(\frac{x}{1-x}\right)\left(\frac{1}{1-\left(\frac{x^2}{1-x}+\frac{x^3+x^4}{1-x}\right)}\right)\left(\frac{1}{1-x}\right)\\ &= \left(\frac{x}{1-x}\right)\left(\frac{1}{1-x-x^2-x^3-x^4}\right)\\ &= \frac{x}{1-2x+x^5},\end{aligned}$$

as required.

Solution: Alternatively, using a recursive decomposition, let M be the set of strings that start with a 1 and contain 00111 exactly once, as a suffix. Then we claim that

$$\begin{aligned}\{1\} \cup S\{0,1\} &= S \cup M\\ S\{00111\} &= M.\end{aligned}$$

The former is proved in the notes in a way that does not depend on the forbidden string; the only change is that since the string must start with 1, we have $\{1\}$ in place of $\{\epsilon\}$. The latter is proved using only the fact that a string ending in 00111 cannot be identical to a string that contains 00111 in an overlapping location, by considering possible overlaps. These expressions are unambiguous, so by the sum and product lemmas, we have

$$\begin{aligned}x + 2x\Phi_S(x) &= \Phi_S(x) + \Phi_M(x)\\ x^5\Phi_S(x) &= \Phi_M(x).\end{aligned}$$

Substituting the second expression into the first gives

$$x + 2x\Phi_S(x) = (1+x^5)\Phi_S(x)$$

and solving for $\Phi_S(x)$ gives

$$\Phi_S(x) = \frac{x}{1-2x+x^5}$$

as claimed.

Question 6. [**8 points**] Suppose the sequence $\{a_n\}_{n\geq 0}$ satisfies the homogeneous recurrence relation

$$a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0 \quad (n \geq 3),$$

along with $a_0 = 3, a_1 = 10, a_2 = 52$. Solve this recurrence to find an explicit formula for a_n in terms of n, for all $n \geq 0$.

Solution: The characteristic polynomial for this recurrence is

$$x^3 - 6x^2 + 12x - 8 = (x-2)^3.$$

The characteristic polynomial has a single root, 2, of multiplicity 3. Therefore by a theorem we know that

$$a_n = (A + Bn + Cn^2)2^n,$$

where A, B, C are some real numbers. The initial conditions each give an equation:

$$\begin{aligned} 3 &= a_0 = A \\ 10 &= a_1 = 2A + 2B + 2C \\ 52 &= a_2 = 4A + 8B + 16C. \end{aligned}$$

Solving this system gives $A = 3, B = -1, C = 3$. Thus,

$$a_n = (3 - n + 3n^2)2^n \quad (n \geq 0).$$

Question 7. [**10 = 4+3+3 points**] Let n be a positive integer. Let T_n denote the set of all *ternary* strings of length n. Here a ternary string of length n is a string $\sigma = a_1 a_2 \cdots a_n$ where each $a_i \in \{0, 1, 2\}$. We define a graph H_n as follows. The vertex set of H_n is T_n. Two vertices σ and σ' are joined by an edge of H_n if and only if σ and σ' differ in exactly one position. (For example 012 is adjacent to 022 in H_3 but not to 121.)

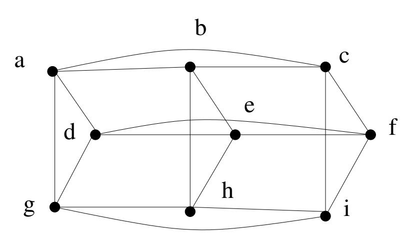
(a) [**4 points**] Find the number of vertices and the number of edges of H_n.

Solution: H_n has $|T_n| = 3^n$ vertices, clearly. To determine the number of edges, we first show that H_n is a regular graph. Indeed, if $\sigma = a_1 a_2 \cdots a_n$ is a vertex, then σ is adjacent to exactly $2n$ other vertices, since there are n possible positions a_i for another vertex to differ with, and 2 ways in which it can differ. Therefore H_n is $(2n)$-regular.

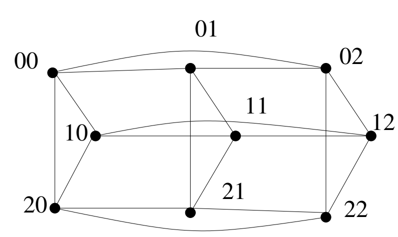
Now by the degree-sum formula,

$$|E(H_n)| = \frac{1}{2} \sum_{v \in V(H_n)} \deg(v) = \frac{1}{2} 2n3^n = n3^n.$$

(b) [**3 points**] Prove that H_2 is isomorphic to the graph shown below.



Solution: There are many possible isomorphisms that work. One of them is described by the following drawing:



(c) [**3 points**] Prove that H_n is connected for every n.

Solution: We will show that for every vertex $\sigma = a_1 a_2 \cdots a_n$ of H_n, there is a path joining σ to the vertex $v = 0 \cdots 0$ with all n positions equal to 0. By a theorem from class this shows that H_n is connected. Consider the sequence of vertices $\sigma_0, \ldots, \sigma_n$ in H_n where

$$\sigma_i = 0 \cdots 0 a_{i+1} \cdots a_n.$$

Then $\sigma_0 = \sigma$ and $\sigma_n = v$, and for each $i \leq n-1$, either $\sigma_i = \sigma_{i+1}$ or σ_i and σ_{i+1} differ in eactly one position and are therefore joined by an edge. Then removing repeated vertices from this sequence (which can only occur if they are consecutive) gives a path from σ to v in H_n, as required.

Question 8. [**7 = 2+3+2 points**] Let G be a graph, and let P be a path of maximum length in G. Let x be the first vertex of P.

(a) [**2 points**] Prove that all neighbours of x are vertices of P.

Solution: Let P be $x_0, \cdots, x_n$, where $x_0 = x$. If x is adjacent to some vertex y such that $y \neq x_i$ for any i, then $y, x_0, \cdots, x_n$ is a path of length one larger than the length of P. Since P is a path of maximum length, we conclude that if x is adjacent to y, we must have $y = x_i$ for some i. That is, all of x's neighbours are in P.

(b) [**3 points**] Prove that if $k \geq 2$ and all vertices in a graph G have degree at least k, then G contains a cycle of length at least $k+1$.

Solution: Let P equal to $x_0, \cdots, x_n$ be a path of maximum length. Since x_0 has at least k neighbours, and each of those neighbours must be in P (by part a), x_0 must have a neighbour x_i with $i \geq k$. Then $x_0, x_1, \cdots, x_i, x_0$ is a cycle of length $i+1 \geq k+1$, as required.

(c) [**2 points**] For each $k \geq 2$, give an example of a graph in which all vertices have degree at least k and all cycles have length at most $k+1$.

Solution: The complete graph K_{k+1} has all vertices having degree at least k (it's k-regular, after all), and moreover no cycle can have length exceeding $k+1$ since there are only $k+1$ vertices in K_{k+1}.