MATH 239 Spring 2012: Assignment 3

Due: 9:29 AM, Friday, May 25 2012 in the dropboxes outside MC 4066

Note: In questions asking for the number of certain objects with certain properties, you may represent your answers as coefficients of rational expressions. As always, simplify these expressions as much as possible. When creating a generating series, always write down the set you are enumerating.

Last Name:

First Name:

I.D. Number:

Section:

Mark (For the marker only): /50

Acknowledgments:

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1. {8 marks} At an intergalactic yard sale, there are three distinct planets costing 5, 7 and 9 gold coins respectively, one comet costing 12 gold coins, 120 identical stars selling for 3 gold coins each, and an unlimited supply of star bits selling for 2 gold coins each. For a positive integer n, how many ways can one spend n gold coins in this sale?

2. {8 marks} Let k be a fixed integer. How many compositions of n with k parts are there where each part is congruent to 1 modulo 5? Determine an explicit formula.

3. {8 marks} Let m be a fixed integer. How many compositions of n are there where the largest part is at most m?

4. {Extra credit: 5 marks} How many compositions of n into three parts are there where $n = a_1 + a_2 + a_3$ and $a_1 < a_2 < a_3$?

5. {8 marks} For any $n \in \mathbb{N}_0$, let E_n be the set of all compositions of n with even number of parts, and let O_n be the set of all compositions of n with odd number of parts. Prove that for $n \geq 2$, $|E_n| = |O_n|$.

6. {8 marks} Let $\{a_n\}$ be the sequence where

$$\sum_{n \geq 0} a_n x^n = \frac{1 + 2x^3}{1 - 2x + x^3 - 3x^4}.$$

Determine a recurrence relation that $\{a_n\}$ satisfies, with sufficient initial conditions to uniquely specify $\{a_n\}$.

7. {10 marks}

(a) Let a_n denote the number of compositions of n. In class, we found out that for $n \geq 1$, $a_n = 2^{n-1}$. This tells us that for $n \geq 2$, a_n satisfies the recurrence $a_n = 2a_{n-1}$. Give a combinatorial interpretation of this recurrence.

(b) Let b_n denote the total number of parts of all possible composition of n. For example, compositions of 3 are $(3), (1,2), (2,1), (1,1,1)$, so $b_3 = 1 + 2 + 2 + 3 = 8$. Determine a recurrence relation that $\{b_n\}$ satisfies, with sufficient initial conditions to uniquely specify $\{b_n\}$. Use your recurrence to generate b_5.