MATH 239 Tutorial 4 Problems

1. Prove that $\{01, 011, 101\}^*$ is an ambiguous expression.

2. Prove that $\{101, 110\}^*$ is an unambiguous expression.

3. Determine an unambiguous decomposition for each of the following sets of strings. Then determine the generating series of each set with respect to the length of the string.

   (a) The set of binary strings that begin and end with the same bit.

   (b) The set of binary strings where the length of each block of 0's is not divisible by 3.

   (c) The set of binary strings where every block of 1's of even length cannot be followed by at least 5 0's.

   (d) The set of binary strings that begins with a 1, and other than the last 2 bits, the i-th bit is different from the $(i+2)$-th bit.

4. Let k be a fixed positive integer. Let S be the set of binary strings with no k consecutive 1's, and let b_n be the number of strings in S of length n. Prove that for $n \geq k$,

$$b_n = \sum_{i=1}^{k} b_{n-i}.$$

   Give a combinatorial proof of this recurrence.

Additional exercises

1. Determine the generating series for the set of binary strings where every block of 0's cannot be followed by a block of 1's of equal or greater length.

2. Prove that $\{00, 101, 11\}^*$ is an unambiguous expression.

3. Question 4 above suggests that if S is the set of all binary strings with no 2 consecutive 1's, then the number of strings of length n satisfies the Fibonacci recurrence $b_n = b_{n-1} + b_{n-2}$ with initial conditions $b_0 = 1, b_1 = 2, b_2 = 3$. From class, the number a_n of compositions of n where each part is odd also satisfies the same recurrence, with different initial conditions $a_0 = a_1 = a_2 = 1$. By comparing the two sequences, we can then conclude that for $n \geq 0$, the number of binary strings in S with length n is equal to the number of compositions of $n+2$ where each part is odd. Find a bijection between these two sets of objects.