LIHORNE.COM

# MATH 239

## Introduction to Combinatorics

Dr. Kevin Purbhoo • Fall 2013 • University of Waterloo

Last Revision: November 29, 2013

## Table of Contents

**Abstract**

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

# 1 Combinatorial Analysis

## 1.1 Binomial Coefficients

**Example 1.1.** Consider the subsets of a three element set $\{1, 2, 3\}$. We have,

$$\underbrace{\emptyset}_{0 \text{ elements}}, \underbrace{\{1\}, \{2\}, \{3\}}_{1 \text{ element}}, \underbrace{\{1,2\}, \{1,3\}, \{2,3\}}_{2 \text{ elements}}, \underbrace{\{1,2,3\}}_{3 \text{ elements}}$$

**Definition 1.1.** $\binom{n}{k}$ is the number of $k$-element subsebts of $\{1, 2, 3, \cdots, n\}$ read "$n$ choose $k$", for example $\binom{3}{2} = 3$.

**Example 1.2.** Consider binary strings of length 3.

$$\underbrace{000}_{0 \times 1s}, \underbrace{001, 010, 100}_{1 \times 1}, \underbrace{011, 101, 110}_{2 \times 1's}, \underbrace{111}_{3 \times 1s}$$

This is essentially the same as our last example.

**Definition 1.2.** $\binom{n}{k}$ is the number of binary strings of length $n$ with exactly $k$ digits of "1".

**Definition 1.3.**

$$\binom{n}{k} := \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 1} = \frac{n!}{(n-k)!k!}$$

This can be thought of as the numerator being the number of ways to list $k$ different elements of a set of $n$ elements and the denominator being the number of ways these lists can be ordered.

**Theorem 1.1.**

$$\binom{n}{k} = \binom{n}{n-k}$$

*Algebraic Proof.*

$$LHS = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$$RHS = \binom{n}{n-k} = \frac{n!}{(n-(n-k))!(n-k)!} = \frac{n!}{k!(n-k)!}$$

□

**Definition 1.4.** Let $S$ be a set. Then $|S_k|$ is defined to be the **cardinality** of $S$. That is, it is the number of elements in $S$.

*Combinatorial Proof.* Let $S_k$ be the set of all $k$-element subsets of $\{1, 2, \cdots, n\}$. Then, according to Definition 1,

$$LHS = \binom{n}{k} = |S_k|$$

$$RHS = \binom{n}{n-k} = |S_{n-k}|$$

To do this, we need to find a bijection between $S_k$ and $S_{n-k}$. We need to define the complement.

**Definition 1.5.** If $A \subseteq \{1, \cdots, n\}$, let $A^{\complement} = \{i \in \{1, \cdots, n\} | i \notin A\}$

Note firstly that if $A \in S_k$ then $A^{\complement} \in S_{n-k}$ and the map $S_k \to S_{n-k}$ is a bijection.
This shows that $|S_k| = |S_{n-k}|$ and therefore

$$\binom{n}{k} = \binom{n}{n-k}$$

□

In this example I can describe a $k$-element subset of $\{1, \cdots, n\}$ in two ways. First, what elements are in it, and what elements are not in it.

**Definition 1.6.** If $A_1, \cdots, A_k$ are sets, we say that they are **pairwise disjoint** if $A_i \cap A_j = \emptyset$ for $i \neq j$. If this is the case,

$$\left| \bigcup_{i=1}^{k} A_i \right| = \sum_{i=1}^{k} |A_i|$$

**Example 1.3.** Prove that

$$\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$$

*Proof.* Let $\mathscr{S}$ = set of $n$-element subsets of $\{1, 2, \cdots, n+k\}$. Then,

$$\mathscr{S} = \binom{n+k}{n}$$

Let $\mathscr{S}_i$ be the set of $n$-element subsets of $\{1, \cdots, n+k\}$ whose largest element is $n+i, i = 0, 1, 2, \cdots, k$. Note that $\mathscr{S}_0, \cdots, \mathscr{S}_k$ are pairwise disjoint. Then,

$$|\mathscr{S}| = \sum_{i=0}^{k} |S_i|$$

Now, if $A \in \mathscr{S}$ then $n+i \in A$ and $A \setminus \{n+i\}$ is an $(n-1)$-element subset of $\{1, \cdots, n+i-1\}$. Then,

$$|\mathscr{S}_i| = \binom{n+i-1}{n-1}$$

□

**Definition 1.7** (cartesian product)**.** Let $A$ and $B$ be sets, then the **Cartesian Product** of $A$ and $B$ is

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

and then

$$|A \times B| = |A| \cdot |B|$$

**Example 1.4.** Prove that

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}$$

*Proof.* Let $M = \{1, \cdots, m\}$ and $N = \{m+1, \cdots, m+n\}$, then $M \cup N = \{1, \cdots, m+n\}$. If $X$ is a set, let $S_k(X)$ denote the set of $k$-element subsets of $X$.

$$\begin{aligned} LHS &= \binom{m+n}{k} \\ &= |S_k(M \cup N)| \end{aligned}$$

We break it down according to how many elements are from $M$. So,

$$RHS = \left| \bigcup_{i=0}^{k} S_i(M) \times S_{k-i}(N) \right|$$

□

Proof left as an exercise to the reader.

Consider $f(y_1, y_2, y_3) = (1+y_1)(1+y_2)(1+y_3)$, multiplying together we get

$$\begin{aligned} (1+y_1)(1+y_2)(1+y_3) &= (1 + y_2 + y_1 + y_1 y_2)(1+y_3) \\ &= 1 + y_2 + y_2 + y_3 + y_1 y_2 + y_1 y_3 + y_2 y_3 + y_1 y_2 y_3 \end{aligned}$$

Note that the subsets of $\{1, 2, 3\}$ are in a way built into this expansion. Now we consider $y_1 = y_2 = y_3 = x$, then

$$f(x, x, x) = (1+x)^3 = \underbrace{1} + \underbrace{3}x + \underbrace{3}x^2 + \underbrace{1}x^3$$

Hence we can restate $(1+x)^3$ as

$$(1+x)^3 = \binom{3}{0}x^0 + \binom{3}{1}x^2 + \binom{3}{2}x^2 + \binom{3}{3}x^3$$

Generalizing this reasoning lets us define something new.

**Definition 1.8** (binomial theorem)**.** The **Binomial Theorem** states that for $n \in \mathbb{N}$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

**Example 1.5.** Prove that

$$\binom{n}{k} = \binom{n}{n-k}$$

*Proof.* First consider the fact that $(1+x) = x\left(1+\frac{1}{x}\right)$, then

$$\begin{aligned}
(1+x)^n &= \sum_{j=0}^{n} \binom{n}{j} x^j \\
\implies \left(x\left(1+\frac{1}{x}\right)\right)^n &= x^n\left(1+\frac{1}{x}\right)^n \\
&= x^n\left[\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{x}\right)^k\right] \\
&= \sum_{k=0}^{n} x^n \binom{n}{k}\left(\frac{1}{x}\right)^k \\
&= \sum_{k=0}^{n}\binom{n}{k} x^{n-k}
\end{aligned}$$

Now, we substitute $j = n-k$, thus $k = n-j$. So,

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k} = \sum_{j=0}^{n}\binom{n}{n-j}x^j$$

Therefore

$$(1+x)^n = \sum_{j=0}^{n}\binom{n}{j}x^j = \sum_{j=0}^{n}\binom{n}{n-j}x^j$$

□

**Example 1.6.** Prove that

$$\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$$

The hint is that $(1+x)^{m+n} = (1+x)^m(1+x)^n$. So,

$$LHS = (1+x)^{m+n} = \sum_{k=0}^{m+n}\binom{m+n}{k}x^k$$

$$RHS = (1+x)^m(1+x)^n = \left[\sum_{i=0}^{m}\binom{m}{i}x^i\right]\left[\sum_{j=0}^{n}\binom{n}{j}x^j\right]$$

Expanding the right hand side we get,

$$\left[\sum_{i=0}^{m}\binom{m}{i}x^i\right]\left[\sum_{j=0}^{n}\binom{n}{j}x^j\right] = \sum_{j=0}^{n}\sum_{i=0}^{m}\binom{m}{i}\binom{n}{j}x^{i+j}$$

Now we substitute $k = i+j$ to eliminate $j$, hence $j = k-i$,

$$\sum_{j=0}^{n}\sum_{i=0}^{\min(k,m)}\binom{m}{i}x^i\binom{n}{j}x^j = \sum_{k=0}^{m+n}\sum_{j=0}^{k}\binom{m}{i}x^i\binom{n}{k-i}x^k$$

Note that

$$(1+x)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k = \sum_{k=0}^{m+n} \sum_{i=0}^{\min(k,m)} \binom{m}{i}\binom{n}{k-i} x^k$$

We conclude that

$$\binom{m+n}{k} = \sum_{i=0}^{\min(k,m)} \binom{m}{i}\binom{n}{k-i}$$

So finally note that

$$\sum_{i=0}^{\min(k,m)} \binom{m}{i}\binom{n}{k-i} = \sum_{i=0} k \binom{m}{i}\binom{n}{k-i}$$

Because additional $i$-values on RHS are $i > m$ and $\binom{m}{i} = 0$ for these. Note since $\binom{n}{k} = 0$, for $k > n$ we sometimes write the binomial theorem as

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

There is a third $\sum$ notation trick, that is turning a sum into a sum of sums or split of terms. For example

$$\sum_{k=0}^{n} a_k = a_0 + \sum_{k=1}^{n} a_k = a_0 + a_1 + \sum_{k=2}^{n} a_k = \cdots$$

## 1.2 Generating Functions

The idea of a generating function is that we can take enumeration word problems that are either hard or easy, and translate them into the language of generating functions and using the tools of generating functions to turn a hard problem to an easy problem at which point they are again translatable to an easy word problem.

The idea is to phrase all problems in the same way.

How many $x$'s are there with property $y$?

**Definition 1.9** (weight function)**.** Let $S$ be a set of objects. A **weight function** on $S$ is a function $w : S \to \mathbb{N} = \{0, 1, 2, 3, \ldots\}$ which assigns to each $\sigma \in S$ a non-negative integer $w(\sigma)$ called the **weight** of $\sigma$.

**Example 1.7.** Take $S =$ binary strings of length 4. Also, the weight function $w(\sigma) =$ number of 1's in $\sigma$. For example, $w(0110) = 2$.

In this setup, the general counting problem is: how many elements of $S$ are there of weight $n$?

**Definition 1.10** (generating function)**.** Let $w$ be a weight function on a set $S$. The **generating function** for $S$ with respect to $w$ is

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$$

**Example 1.8.** Consider the same set $S$ of binary strings of length 4, where $w(\sigma)$ is the number of 1's in $\sigma$. Find the generating function $\Phi_S(x)$.

To solve this problem, we create a table:

| $\sigma$ | $w(\sigma)$ | $x^{w(\sigma)}$ | $\sigma$ | $w(\sigma)$ | $x^{w(\sigma)}$ |
|---|---|---|---|---|---|
| 0000 | 0 | 1 | 1000 | 1 | x |
| 0001 | 1 | x | 1001 | 2 | $x^2$ |
| 0010 | 1 | x | 1010 | 2 | $x^2$ |
| 0100 | 1 | x | 1100 | 2 | $x^2$ |
| 0011 | 2 | $x^2$ | 1011 | 3 | $x^3$ |
| 0101 | 2 | $x^2$ | 1101 | 3 | $x^3$ |
| 0110 | 2 | $x^2$ | 1110 | 3 | $x^3$ |
| 0111 | 3 | $x^3$ | 1111 | 4 | $x^4$ |

So,

$$\Phi_S(x) = 1 + x + x + x^2 + \cdots = 1 + 4x + 6x^2 + 4x^3 + x^4$$

We can make the following observations from this example:

(i) The coefficient of $x^i$ is the number of elements of weight (see weight function) $i$ in $S$

(ii) $\Phi_S(x) = (1+x)^4$

(iii) $\Phi_S(1) = 16 = |S|$

(iv) $\frac{d\Phi_S(x)}{dx} = 4 + 12x + 12x^2 + 4x^3$ and $\frac{d\Phi_S(1)}{dx} = 32 = \sum_{\sigma\in S} w(\sigma)$

We can turn this into a general answer.

**Theorem 1.2.** Let $S$ be a set of objects with a weight function $w$. Let

$$\Phi_S(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then $a_n$ is the number of elements of $S$ having weight $n$.

*Proof.*

$$\begin{aligned}
\Phi_S(x) &= \sum_{\sigma\in S} x^{w(\sigma)} \\
&= \sum_{n=0}^{\infty} \sum_{\substack{\sigma\in S \\ w(\sigma)=n}} x^{w(\sigma)} \\
&= \sum_{n=0}^{\infty} \sum_{\substack{\sigma\in S \\ w(\sigma)=n}} x^n \\
&= \sum_{n=0}^{\infty} \left( \sum_{\substack{\sigma\in S \\ w(\sigma)=n}} 1 \right) x^n
\end{aligned}$$

Since $\Phi_S(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left( \sum_{\substack{\sigma\in S \\ w(\sigma)=n}} 1 \right) x^n$

Therefore, $a_n = \sum_{\substack{\sigma\in S \\ w(\sigma)=n}} 1$ which is the number of elements of $S$ having weight $n$. $\square$

**Notation 1.1.** We call $a_n$ the coefficient of $x^n$ in $\Phi_S(x)$ and write this as

$$[x^n]\Phi_S(x)$$

That is, the answer to our general question is always $[x^n]\Phi_S(x)$.

**Theorem 1.3.** Let $S$ be a finite set with weight function $w$. Let $\Phi_S(x)$ be the generating function. Then,

(i) $\Phi_S(1) = |S|$

(ii) $\Phi'_S(1) = \sum_{\sigma \in S} w(\sigma) =$ "sum of the weights of elements in $S$"

(iii) $\dfrac{\Phi'_S(1)}{\Phi_S(1)} = \dfrac{\sum_{\sigma \in S} w(\sigma)}{|S|} =$ "average of weights"

See the course notes for the proof.

**Example 1.9.** Let $S$ be the set of all binary strings (infinite set). For $\sigma \in S$, define weight function $w(\sigma) =$ "length of $\sigma$"

Let $S = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$ with $\epsilon$ denoting the string of length 0. In this context, our question is how many $\underbrace{\text{binary strings}}_{\text{elements of } S}$ are there of weight $n$. The answer from intuition is $2^n$. So we conclude that

$$[x^n]\Phi_S(x) = 2^n$$

and

$$\Phi_S(x) = \sum_{n=0}^{\infty} (2x)^n$$

where these statements are equivalent.

**Note.** In general,

$$[x^n]\Phi_S(x) = a_n \iff \Phi_S(x) = \sum_{n=0}^{\infty} a_n x^n$$

Also, we notice that our sum is a geometric series, so we can expand

$$\Phi_S(x) = \sum_{n=0}^{\infty} (2x)^n = 1 + (2x) + (2x)^2 + \cdots = \frac{1}{1-2x}$$

We can question this result; does it make sense?

A1. **Yes**, if $-\frac{1}{2} < x < \frac{1}{2}$.

A2. **Yes**, in the context of formal power series.

## 1.3 Formal Power Series

The idea is to think about power series in terms of the operations you are **allowed** to perform on them. We want to get rid of the concept of radius of convergence.

**Definition 1.11** (formal power series)**.** A formal power series is a power series such as

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

but $A(x)$ should not be thought of as a function. *We disallow the ability to plug in numbers* (some exceptions to be discussed). The following are the allowed operations:

- Coefficient Extraction: $[x^n]A(x) = a_n$.
- Arithematic Operations: Consider $A(x) = \sum_{n=0}^{\infty}, B(x) = \sum_{n=0}^{\infty}$, then

  i. $A(x) + B(x) = \displaystyle\sum_{n=0}^{\infty}(a_n + b_n)x^n$

  ii. $A(x) - B(x) = \displaystyle\sum_{n=0}^{\infty}(a_n - b_n)x^n$

  iii. $cA(x) = \displaystyle\sum_{n=0}^{\infty}(ca_n)x^n$

  iv. $A(x)B(x) = \left(\displaystyle\sum_{j=0}^{\infty} a_i x^j\right)\left(\displaystyle\sum_{k=0}^{\infty} b_k x^k\right) = \displaystyle\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} b_k a_i x^{j+k}$, then substitute $n = j + k$ to get

  $$A(x)B(x) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_j b_{n-j}\right)x^n$$

  Another way to think about this is

  | | $a_0+$ | $a_1x+$ | $a_2x^2+$ | $a_3x^3+\cdots$ |
  |---|---|---|---|---|
  | $b_0$ | $a_0b_0$ | $a_1b_0x$ | $a2b_0x^2$ | $a_3b_0x^3$ |
  | $+b_1x$ | $a_0b_1x$ | $a_1b_1x^2$ | $a2b_1x^3$ | |
  | $+b_2x^2$ | $a_0b_2x^2$ | $a_1b_2x^3$ | | |
  | $+b_3x^3$ | $a_0b_3x^3$ | | | |
  | $+\vdots$ | | | | |

- Division / Inverses : If $A(x)B(x) = 1 = 1 + x + 0x^2 + 0x^3 + \cdots$ then we say $B(x)$ is the inverse of $A(x)$ and

  $$B(x) = \frac{1}{A(x)} \quad \text{or} \quad B(x) = A(x)^{-1}$$

  This obeys the usual properties of division or inverses. Note that not every formal power series has an inverse.

**Theorem 1.4.** Let $A(x)$ be a formal power series

$$A(x) = \sum_{a=0}^{\infty} a_n x^n$$

then $A(x)$ has an inverse if and only if $a_0 \neq 0$.

Proof in the course notes.

**Example 1.10.** For some problem we can take the formal power series.

$$A(x) = x = 0 + 1x + 0x^2 + 0x^3 + \cdots$$

We know that $\frac{1}{A(x)} = \frac{1}{x}$ which is not a formal power series, so there is no inverse. Consider again $B(x)A(x) = 1$, then since $B(x)A(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + \cdots \implies a_0b_0 = 1$. However, $a_0 = 0$ here, so there is no inverse.

- Composition / Substitution $A(B(x))$ doesn't always make sense, so it is allowed in two cases.
  i. $b_0 = 0$
  ii. $A(x)$ is a polynomial

To summarize what we know of Formal Power Series; we are allowed to use operators of addition, multiplication, substraction, and scalar multiplciation. As well, we can extract coefficients, deal with inverses (sometimes), and substitutions (sometimes). We disallow plugging in numbers, limits in the traditional sense, and infinite sums of numbers. Radius of convergence is also not allowed.

Recall that

$$\sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}.$$

Interpeted in formal power series we see that $1 - 2x$ is the formal power series inverse of $\sum_{n=0}^{\infty} 2^n x^n$ which means that

$$(1-2x)\left(\sum_{n=0}^{\infty} 2^n x^n\right) = 1 = 1 + 0x + 0x^2 + 0x^3 + \cdots$$

Check

| | $1+$ | $2x+$ | $4x^2+$ | $8x^3+\cdots$ |
|---|---|---|---|---|
| $1$ | $1$ | $2x$ | $4x^2$ | $8x^3$ |
| $-2x$ | $-2x$ | $-4x^2$ | $-8x^3$ | $-16x^4$ |
| $+0x^2$ | | | | |
| $+0^3$ | | | | |
| $+\vdots$ | | | | |

When substituion is allowed, consider

$$A(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

then

$$\begin{aligned} A(1+x) &= 1 + (1+x) + (1+x)^2 + (1+x)^3 + \cdots \\ &= 1 \\ &\quad + 1 + x \\ &\quad + 1 + 2x + x^2 \\ &\quad + 1 + 3x + 3x^2 + x^3 \end{aligned}$$

The problem is that we have an infinite sum of ones, which isn't something we can understand. This is bad and is an example of why infinite sums are disallowed.
Another way,

$$\begin{aligned} A(x+x^2) = 1 & \\ & + x + x^2 \\ & \quad + x^2 + 2x^3 + x^4 \\ & \quad\quad + x^3 + 3x^4 + 3x^5 + x^6 \end{aligned}$$

Collecting like terms does not involve infinite sums.

**Theorem 1.5** (negative binomial theorem)**.** For $n \in N$,

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

**Note.** Consider that $\frac{1}{(1-x)^n}$ is the inverse of $(1-x)^n$ and

$$(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$$

Therefore it is clear that

$$\left(\sum_{k=0}^{n} \binom{n}{k} (-1)^n x^n\right) \left(\sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k\right) = 1$$

Also note that the binomial expansion for $(1-x)^n$ can be a series up to infinity since $\binom{n}{k} = 0$ for $k > n$.

If we use the interpretation for $n \in \mathbb{N}$,

$$\binom{n}{k} = \frac{n(n-1)(n-1)\cdots(n-k+1)}{k!}$$

then

$$\binom{n+k-1}{k} = \binom{-n}{k}(-1)^k$$

## 1.4 Generating Function Tools

**Lemma 1.1** (sum lemma)**.** Let $S$ be a set with a weight function. If $S = A \cup B$, whose $A, B$ disjoint, then

$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$$

*Proof.*

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{\sigma \in A} x^{w(\sigma)} + \sum_{\sigma \in B} x^{w(\sigma)} = \Phi_A(x) + \Phi_B(x)$$

□

**Note.** i. If $S$ is a finite set, then

$$\Phi_S(1) = \Phi_A(1) + \Phi_B(1) \implies |S| = |A| + |B|$$

ii. If $A$ and $B$ are not dsjoint, then

$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x) - \Phi_{A\cap B}(x)$$

iii. If $A_1, \ldots, A_n$ are pairwise disjoint and $S = \bigcup_{i=0}^{n} A_i$, then

$$\Phi_S(x) = \sum_{i=0}^{n} \Phi_{A_i}(x)$$

"This next example is the heart and soul of this course." - Kevin Purhboo

**Example 1.11.** Suppose you have 5 loonies and 4 toonies. How many ways are there to make \$n?
Let $S$ be the set of all possbible combinations of up to 5 loonies and 4 toonies. Let the weight function be its dollar value. Compute $\Phi_S(x)$.

| | 0 loonies | 1 loonies | 2 loonies | 3 loonies | 4 loonies | 5 loonies |
|---|---|---|---|---|---|---|
| 0 toonies | $x^0$ | $x^1$ | $x^2$ | $x^3$ | $x^4$ | $x^5$ |
| 1 toonies | $x^2$ | $x^3$ | $x^4$ | $x^5$ | $x^6$ | $x^7$ |
| 2 toonies | $x^4$ | $x^5$ | $x^6$ | $x^7$ | $x^8$ | $x^9$ |
| 3 toonies | $x^6$ | $x^7$ | $x^8$ | $x^9$ | $x^{10}$ | $x^{11}$ |
| 4 toonies | $x^8$ | $x^9$ | $x^{10}$ | $x^{11}$ | $x^{12}$ | $x^{13}$ |

Then

$$\Phi_S(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x^2 + x^4 + x^6 + x^8) = 1 + x + x^2 + 2x^2 + 2x^3 + \cdots + 3x^9 + \cdots + x^{13}$$

So the number of ways that we can make \$$n$ is

$$[x^n]\Phi_S(x)$$

For example, the number of ways to make \$9 is $[x^9]\Phi_S(x) = 3$.

**Remark 1.1.** Note that for these types of problems there are two keys features:

- An element of $S$ is represented as a pair (e.g., loonies and toonies, that is, $S = L \times T$). A set is represented as a cartesian product.
- The weight of a combination is the sum of the individual weights from the two items in the pair (e.g., loonies and toonies). That is, $\sigma = (l, t) \in S \implies w(\sigma) = w(l) + w(t)$.

**Lemma 1.2** (product lemma)**.** Let $A$ be a set with weight function $w_A$.
Let $B$ be a set with weight function $w_B$.
Let $S = A \times B$.
Define a weight function on $S$: For $\sigma = (a, b) \in S$, $w(\sigma) = w_A(\sigma) + w_b(\sigma)$. Then,

$$\Phi_S(x) = \Phi_A(x)\Phi_B(x)$$

When using this, there are two hypotheses that need to be checked ($S = A \times B$ and $w(\sigma) = w_A(\sigma) + w_b(\sigma)$).

*Proof.* By definition,

$$\begin{aligned}
\Phi_S(x) &= \sum_{\sigma\in S} x^{w(\sigma)} \\
&= \sum_{(a,b)\in A\times B} x^{w(a,b)} \\
&= \sum_{(a,b)\in A\times B} x^{w_A(a)+w_B(b)} \\
&= \sum_{a\in A}\sum_{b\in B} x^{w_A(a)+w_B(b)} \\
&= \sum_{a\in A}\sum_{b\in B} x^{w_A(a)} x^{w_B(b)} \\
&= \left(\sum_{a\in A} x^{w_A(a)}\right)\left(\sum_{b\in B} x^{w_B(b)}\right) \\
&= \Phi_A(x)\Phi_B(x)
\end{aligned}$$

□

**Example 1.12.** You have 5 loonies and 4 toonies. Alive has 1 loonie and 2 five dollar bills. Bob has 2 loonies, 1 toonie, and 1 five dollar bill. How many ways are there for you (the reader) and your friends Alice and Bob to make \$20? Express your answer as a coefficient of a formal power series.
Let $S$ be the set of all combinations I can make. Now let $L$ be the set of all combinations that you (the reader) can make. Similarly, let $A$ and $B$ be the set of all combinations Alive and Bob can make, respectively.
Then, $S = L \times A \times B$. This is because to specify a combination the three of you can make, we need to specify three things (my contribution, Alice's contribution, Bob's contribution).

In each case, define the weight to be the dollar value. Since the weight of the total contribution is the sum of the weights of the individual contributions, the product lemma can be used.

$$\Phi_S(x) = \Phi_L(x)\Phi_A(x)\Phi_B(x)$$

From the previous example we can see that $\Phi_L(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x^2 + x^4 + x^6 + x^8)$, and now

$$\Phi_A(x) = (1 + x)(1 + x^5 + x^{10})$$

$$\Phi_B(x) = (1 + x + x^2)(1 + x^2)(1 + x^5)$$

We are looking for

$$[x^{20}]\Phi_S(x) = [x^{20}](1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4+x^6+x^8)(1+x)(1+x^5+x^{10})(1+x+x^2)(1+x^2)(1+x^5)$$

**Example 1.13.** Let $S$ the set of binary strings of length $n$. We can define the weight function to be the number of 1's in $\sigma \in S$. Compute the generating function $\Phi_S(x)$ in two ways.

(1) Using first principles / definition

(2) Using product lemma

For (1), the number of strings in $S$ of weight $k$ in $\sigma$ is $\binom{m}{k}$. So,

$$\Phi_S(x) = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

For (2); We can also think of a binary string as an $n$-tuple. A binary string of length $m$ is an $m$-tuple of elements from $T = \{0,1\}$. Thus,

$$S = \underbrace{T \times T \times \cdots \times T}_{m}$$

Define the weight function on $T$ as

$$w_T(0) = 0, w_T(1) = 1$$

If $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in T \times T \times \cdots \times T$, then $w(\sigma) =$ number of 1's in $\sigma = w_T(\sigma_1) + w_T(\sigma) + \cdots + w_T(\sigma_m)$ We can therefore use the product lemma,

$$\Phi_S(x) = \underbrace{\Phi_T(x)\Phi_T(x)\cdots\Phi_T(x)}_{m} = (\Phi_T(x))^m = (1+x)^m$$

Combining (1) and (2) we obtain,

$$(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

This is a proof of the binomial theorem.

**Note.** We have used a generalization. If $A_1, \ldots, A_m$ are sets with weight functions, $w_{A_1}, w_{A_2}, \ldots, w_{A_m}$ and $S = A_1 \times A_2 \times \cdots \times A_m$ with weight function satisfying for $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$, $w(\sigma) = w_{A_1}(a_1) + \cdots + w_{A_m}(a_m)$ then $\Phi_S(x) = \Phi_{A_1}(x) \cdots \Phi_{A_m}(x)$.

### 1.5 How to Solve Combinatorics Problems with Generating Functions in 10 Easy Steps

The following ten steps are given by Kevin Purbhoo's years of experience solving combinatorics problems with generating functions.

1. Identify paramters and any constants that you might want to treat as parameters (e.g., in a binary strings problem, $n$ being the length of a binary string and $k$ being the number of ones).

2. Create a set of objects $S$ by taking out one of the parameters.

3. Give a mathematical description of $S$, in terms of unions and cartesian products, using simpler sets $A_1, A_2, \ldots$.

4. Reintroduce the missing parameter as the weight function on $S$.

5. Define weight functions on our simpler sets $A_1, A_2, \ldots$.

6. Check that our weight functions behave correctly for the product lemma.

7. Compute the generating functions $\Phi_{A_1}(x), \Phi_{A_2}(x), \ldots$ (usually done by first principles).

8. Use the sum lemma and product lemma to get a formula for $\Phi_S(x)$.

9. Simplify (often geometric series formula comes in)

10. Answer is $[x^n]\Phi_S(x)$. Compute this.

Computation is usally done with one of

- Brute force (binomial theorem, sigma notation tricks)
- Partial Fractions
- Find a recurrence
- S.E.P. approach (somebody else's problem)

# 2 Compositions and Strings

## 2.1 Compositions of an Integer

**Definition 2.1** (composition)**.** Let $n, k \in \mathbb{N}$, a **composition** of $n$ with $k$ parts is a $k$-tuple $(c_1, c_2, \ldots, c_k)$ whose each $c_i$ is a positive integer greater than or equal to 1 and $c_1 + c_2 + \cdots + c_k = n$. The empty composition is a composition with 0 parts.

**Definition 2.2** (parts)**.** The numbers $c_i$ in a composition $(c_1, \ldots, c_k)$ are called the **parts**.

**Example 2.1.** List the compositions of 5 with 3 parts: $(1,2,2),(2,1,2),(2,2,1),(1,1,3),(1,3,1),(3,1,1)$.

**Question.** How many composition of $n$ with $k$ parts are there?
**Solution.** We apply the ten steps listed earlier.
Our parameters are $n =$"size of composition" and $k =$"number of parts". Let $S =$"the set of compositions with $k$ parts". That is,

$$S = \underbrace{\mathbb{N}_{\geq 1} \times \mathbb{N}_{\geq 1} \times \cdots \times \mathbb{N}_{\geq 1}}_{k} = (\mathbb{N}_{\geq 1})^k$$

where $\mathbb{N}_{\geq 1} = \{1, 2, 3, 4, \ldots\}$. Define the weight function $w : S \longrightarrow \mathbb{N} = w(c_1, c_2, \ldots, c_k) = c_1 + c_2 + \cdots + c_k$.
Define a weight function $\alpha$ on $\mathbb{N}_{\geq 1}$:

$$\alpha : \mathbb{N}_{\geq 1} \longrightarrow \mathbb{N} \qquad \alpha(i) = i$$

Then the conditions of the product lemma are met, because

$$w(c_1, \ldots, c_k) = c_1 + c_2 + \cdots + c_k \text{ and } \alpha(c_1) + \cdots + \alpha(c_k) = c_1 + c_2 + \cdots + c_k$$

Then,

$$\begin{aligned} \Phi_{\mathbb{N}_{\geq 1}}(x) &= x^1 + x^2 + x^3 + x^4 + \cdots \\ &= \sum_{i=1}^{\infty} x^i \\ &= \frac{x}{1-x} \end{aligned}$$

By the product lemma,

$$\begin{aligned} \Phi_S(x) &= \underbrace{\Phi_{\mathbb{N}_{\geq 1}} \Phi_{\mathbb{N}_{\geq 1}} \cdots \Phi_{\mathbb{N}_{\geq 1}}}_{k} \\ &= \left(\frac{x}{1-x}\right)^k \end{aligned}$$

Our answer is

$$\begin{aligned}
[x^n]\left(\frac{x}{1-x}\right)^k &= [x^n]x^k(1-x)^{-k} \\
&= [x^n]x^k\sum_{j=0}^{\infty}\binom{k+j-1}{j}x^j \qquad \text{(negative binomial theorem)} \\
&= [x^n]\sum_{j=0}^{\infty}\binom{k+j-1}{j}x^{j+k} \\
&= [x^n]\sum_{m=k}^{\infty}\binom{m-1}{m-k}x^m \\
&= \begin{cases} \binom{n-1}{n-k} & \text{if } n \geq k \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

The alternate approach is

$$[x^n]x^k(1-x)^{-k} = [x^{n-k}](1-x)^{-k} = \begin{cases} \binom{k+(n-k)-1}{n-k} & \text{if } n \geq k \\ 0 & \text{otherwise} \end{cases}$$

There are $\binom{n-1}{k-1}$ compositions of $n$ with $k$ parts ($k \geq 1$). If $k = 0$ there is a composition with no parts ( ), this is a composition of 0.

**Note.** This question is equivalent to asking how many solutions are there to the equation $c_1 + c_2 + \cdots + c_n = n$ where $c_1, c_2, \ldots, c_k$ are positive integers.

**Note.** Example 1.6.4 in the course notes is very similar except that $c_i = 0$ permissed with compositions $c_i \geq 1$.

**Example 2.2.** Let $n, k$ be positive integers. Find the number of solutions to $x_1 + x_2 + \cdots + x_k = n$ where $x_i \geq i$ is a positive integer. This is essentially a composition problem asking the number of compositions $(x_1, \ldots, x_n)$ with $k$ parts such that the $i$-th part is greater than or equal to $i$.

Let $S =$ "set of all composotions with $k$ parts such that the $i$-th part is greater than or equal to $i$.", then

$$S = \mathbb{N}_{\geq 1} \times \mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 3} \times \cdots \times \mathbb{N}_{\geq k}$$

where $\mathbb{N}_{\geq i} = \{i, i+1, i+2, \ldots,\}$. Define the weight of a $k$-tuple as just the sum of the numbers. The required answer is

$$[x^n]\Phi_S(x)$$

now using the product lemma,

$$\begin{aligned}
\Phi_S(x) &= \Phi_{\mathbb{N}_{\geq 1}}(x)\Phi_{\mathbb{N}_{\geq 2}}(x)\cdots\Phi_{\mathbb{N}_{\geq k}}(x) \\
&= \left(\frac{x}{1-x}\right)\left(\frac{x^2}{1-x}\right)\cdots\left(\frac{x^k}{1-x}\right) \\
&= \frac{x^{1+2+\cdots+k}}{(1-x)^k} \\
&= \frac{x^{\frac{k(k+1)}{2}}}{(1-x)^k}
\end{aligned}$$

Now we want

$$\begin{aligned}
[x^n]\Phi_S(x) &= [x^n]\frac{x^{\frac{k(k+1)}{2}}}{(1-x)^k} \\
&= [x^{n-\frac{k(k+1)}{2}}]\frac{1}{(1-x)^k} \\
&= [x^{n-\frac{k(k+1)}{2}}](1-x)^{-k} \\
&= \binom{n-\frac{k(k+1)}{2}+k-1}{n-\frac{k(k+1)}{2}} \\
&= \binom{n-\frac{k(k+1)}{2}+k-1}{k-1}
\end{aligned}$$

**Example 2.3.** Find the number of solutions to the equation

$$x_1 + x_2 + \cdots + x_k + 2x_{k+1} + 2x_{k+2} + \cdots + 2x_{2k} = n$$

where $x_1, \ldots, x_k$ are odd positive integers and $x_{k+1}, \ldots, x_{2k}$ are even positive integers. That is, the composition $(c_1, \ldots, c_k, c_{k+1}, \ldots, c_{2k})$ of $n$ where $c_1, \ldots, c_k$ odd, $c_{k+1}, \ldots, c_{2k}$ even.

Let $S = \underbrace{N_{\in O} \times \cdots \times N_{\in O}}_{k} \times \underbrace{N_{\in E} \times \cdots \times N_{\in E}}_{k}$ where $N_{\in O}$ are the odd positive integers, and $N_{\in E}$ are the even positive integers not including 0. By the same reasoning from the previous example,

$$\begin{aligned}
\Phi_S(x) &= \left(\Phi_{N_{\in O}}(x)\right)^k \left(\Phi_{N_{\in E}}(x)\right)^k \\
&= \left(\frac{x}{1-x^2}\right)^k \left(\frac{x^2}{1-x^2}\right)^k \\
&= \frac{x^{3k}}{(1-x^2)^{2k}}
\end{aligned}$$

Therefore the number of solutions is

$$\begin{aligned}
[x^n]\frac{x^{3k}}{(1-x^2)^{2k}} &= [x^{n-3k}]\frac{1}{(1-x^2)^{2k}} \\
&= [x^{n-3k}](1-x^2)^{-2k} \\
&= [x^{n-3k}]\sum_{m\geq 0}\binom{m+2k-1}{m}(x^2)^m \\
&= [x^{n-3k}]\sum_{m\geq 0}\binom{m+2k-1}{m}x^{2m} \\
&= \binom{m+2k-1}{m} \quad \text{when } 2m = n-3k\text{, 0 otherwise} \\
&= \begin{cases} \binom{\frac{n-3k}{2}+2k-1}{\frac{n-3k}{2}} & n-3k \text{ is even and } n-3k \geq 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

**Example 2.4.** Find the number of compositions of $n$ (with any number of parts).

A composition of $n$ could have any number of parts. Let $S$ be the set of all compositions. So,

$$S = (\mathbb{N}_{\geq 1})^0 \cup (\mathbb{N}_{\geq 1})^1 \cup (\mathbb{N}_{\geq 1})^2 \cup \cdots$$

here $(\mathbb{N}_{\geq 1})^k$ is the set of compositions with $k$ parts. And, $(\mathbb{N}_{\geq 1})^k = \underbrace{(\mathbb{N}_{\geq 1}) \times \cdots (\mathbb{N}_{\geq 1})}_{k}$. So,

$$S = \bigcup_{k=0}^{\infty} (\mathbb{N}_{\geq 1})^k$$

by the sum lemma,

$$\Phi_S(x) = \sum_{k=0}^{\infty} \Phi_{\mathbb{N}_{\geq 1}}(x)$$

and by the product lemma,

$$\begin{aligned}\Phi_{(\mathbb{N}_{\geq 1})^k}(x) &= (\Phi_{\mathbb{N} \geq 1}(x))^k \\ &= \left(\frac{x}{1-x}\right)^k\end{aligned}$$

thus,

$$\begin{aligned}\Phi_S(x) &= \sum_{k=0}^{\infty} \left(\frac{x}{1-x}\right)^k \\ &= 1 + \left(\frac{x}{1-x}\right) + \left(\frac{x}{1-x}\right)^2 + \cdots \\ &= \frac{1}{1 - \frac{x}{1-x}} \\ &= \frac{1-x}{1-2x}\end{aligned}$$

so our answer is

$$\begin{aligned}[x^n]\left(\frac{1-x}{1-2x}\right) &= [x^n](1-x)(1-2x)^{-1} \\ &= [x^n](1-2x)^{-1} - [x^n]x(1-2x)^{-1} \\ &= [x^n](1-2x)^{-1} - [x^{n-1}](1-2x)^{-1} \\ &= \begin{cases} 2^n - 2^{n-1} & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}\end{aligned}$$

**Example 2.5.** Determine the number of compositions of $n$, with an odd number of parts where each part is congruent to $1 \mod 3$.

Let $S$ be the set of all compositions with an odd number of parts, each congruent to 1 mod 3. Then,

$$S = A \cup A^3 \cup A^5 \cup \cdots$$

where $A$ is the set of positive integers congruent to 1 mod 3 ($\{1, 4, 7, 10, \ldots\}$). The weight function on $S$ is $w(c_1, \ldots, c_k) = c_1 + \cdots + c_k$. We define the weight function on $A$ to be $\alpha(c) = c$ for $c \in A$. Since $w(c_1, \ldots, c_k) = \alpha(c_1) + \cdots + \alpha(c_k)$ the product lemma applies. By the sum lemma,

$$\begin{aligned}\Phi_S(x) &= \Phi_A(x) + \Phi_{A^3}(x) + \Phi_{A^5}(x) + \cdots \\ &= \sum_{k=0}^{\infty} \Phi_{A^{2k+1}}(x)\end{aligned}$$

By the product lemma,

$$\Phi_{A^{2k+1}}(x) = (\Phi_A(x))^{2k+1} = (x + x^4 + x^7 + x^{10} + \cdots)^{2k+1} = \left(\frac{x}{1-x^3}\right)^{2k+1}$$

Therefore,

$$\begin{aligned}\Phi_S(x) &= \sum_{k=0}^{\infty} \left(\frac{x}{1-x^3}\right)^{2k+1} \\ &= \left(\frac{x}{1-x^3}\right) + \left(\frac{x}{1-x^3}\right)^3 + \left(\frac{x}{1-x^3}\right)^5 + \cdots \\ &= \frac{\left(\frac{x}{1-x^3}\right)}{1 - \left(\frac{x}{1-x^3}\right)^2} \cdot \frac{(1-x^3)^2}{(1-x^3)^2} \\ &= \frac{x - x^4}{1 - x^2 - 2x^3 + x^6}\end{aligned}$$

Thus our answer is

$$[x^n]\left(\frac{x - x^4}{1 - x^2 - 2x^3 + x^6}\right)$$

figuring out the numerical value is somebody else's problem. (unless problem states otherwise)

## 2.2 Binary Strings

**Example 2.6.** How many $\{0, 1\}$-strings (binary strings) of length $n$ are there in which every 0 is followed by at least two 1s, for example "1110111011011111".

First we identify the theory, and then return to the example.

**Definition 2.3** (concatenation)**.** Let $a$ and $b$ be binary strings, then we write $ab$ as the concatenation of $a$ and $b$. For example, if $a = 101$ and $b = 00110$, then $ab = 10100110$.

**Definition 2.4** (string product)**.** Let $A, B$ be sets of binary strings. Then, $AB = \{ab | a \in A, b \in B\}$.

Note the similarity of the string product to the cartesian product. Are they the same concept? Take the example of $A = \{1, 01\}$ and $B = \{1, 10\}$, then $AB = \{11, 110, 011, 0110\}$, whereas $A \times B = \{(1, 1), (1, 10), (01, 1), (01, 10)\}$.

Now let's look at $BA = \{11, 101, 1001\}$. In this case there is a difference between it and the cross product $B \times A = \{(1,1),(1,01),(10,1),(10,01)\}$. So we conclude that the two product types are "sometimes" the same.

**Definition 2.5** (unambiguous)**.** We say that $AB$ is **unambiguous** if for every $s \in AB$, there is a unique $a \in A$ and $b \in B$ such that $s = ab$.

In the example above, $AB$ is unambiguous and $BA$ is ambiguous. We can think of this as though unambiguous meand that $AB$ and $A \times B$ are essentially the same concept.

We use the same terminology for unions of sets and strings. That is, $A \cup B$ is unambiguous if $A \cap B = \emptyset$. We also use this for complicated expressions build out of these. For example, $AB \cup BA$ is unambiguous if all operations built out of these are unambiguous. That is,

- $AB$ is unambiguous
- $BA$ is unambiguous
- $AB \cap BA = \emptyset$

**Definition 2.6** (weight function on strings)**.** The weight function on a string $s$ is the length of $s$.

**Theorem 2.1** (sum lemma on strings)**.** If $A$ and $B$ are sets of strings, and $A \cup B$ is unambiguous, then

$$\Phi_{A \cup B}(x) = \Phi_A(x) + \Phi_B(x)$$

**Theorem 2.2** (product lemma on strings)**.** If $A$ and $B$ are sets of strings, and $AB$ is unambiguous, then

$$\Phi_{AB}(x) = \Phi_A(x)\Phi_B(x)$$

*Proof.* (this proof is sketchy)

$$\Phi_{AB}(x) = \Phi_{A \times B}(x)$$

then by ordinary product lemma it is true. □

**Definition 2.7** (empty string)**.** The empty string $\epsilon$ has length 0 and has the property that

$$\varepsilon a = a = a\varepsilon$$

**Definition 2.8** (star)**.**

$$A^* = \{\varepsilon\} \cup A \cup AA \cup AAA \cup \cdots$$

that is, $A^*$ is obtained by containing 0 or more strings from $A$. Note that $A^*$ is built out of concatenation and unions. $A^*$ is unambiguous is sets $\{\varepsilon\}, A, AA, AAA, \ldots$ are pairwise disjoint and $\underbrace{AA \cdots A}_{k}$ is unambiguous for all $k$.

A simple way to look back at what we've learned so far is that concatenation is to the product lemma on strings, unions is to the sum lemma on strings, and $*$ is to the finite string lemma, all only if unambiguous.

$$A* = \bigcup_{k \geq 0} A^k$$

where $A^0 = \{\varepsilon\}$, $A^1 = A$, $A^2 = AA$, etcetera. unambiguous informally means that we can't get the same string in two different ways.

**Example 2.7.** The set $\{0\}^*$ being the set of all binary strings with only 0s ($\{\varepsilon, 0, 00, 000, \ldots\}$) is unambiguous. Additionally, $\{0,1\}^*$ being the set of all binary strings is unambiguous. Because $\{0,1\}^* = \{\varepsilon\} \cup \{0,1\} \cup \{0,1\}^2 \cup \cdots$ is disjoint because $\{0,1\}^k$ has only length $k$ strings. $\{0,1\}^k$ is unambiguous because if $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ then $\sigma_1, \ldots, \sigma_k$ must be the digits of $\sigma$.

$\{0,1\}\{0,1\}^*$ is the set of all strings of length $\geq 1$ is unambiguous.

$\{\varepsilon, 0, 1\}^*$ being all binary strings, including the empty string, is ambiguous because for $0 \in \{\varepsilon, 0, 1\}^*$, $0\varepsilon = \varepsilon 0 = 0$. Additionally, $(\{0\}^*)^*$ is all strings with only 0s is ambiguous for the same reason.

**Note.** If $\varepsilon \in A$, then $A^*$ is **ambiguous** because $\varepsilon = (\varepsilon) = (\varepsilon)(\varepsilon)$. The moral is to never $*$ something if the empty string is there.

Consider $\{100, 101, 010\}$, the set of binary strings of length a multiple of 3. This is unambiguous. Consider $\{10, 101, 010\}$, then this is ambiguous because $(10)(10)(10) = (101)(010)$.

**Note.** If $A$ is a set of strings all of length $k$ (where $k \geq 1$) then $A^*$ is unambiguous.

Consider $\{1000, 101, 010\}^*$; unambiguous.

**Theorem 2.3** (finite string lemma)**.** If $A$ is a set of binary strings and $A^*$ is unambiguous, then

$$\Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)}$$

*Proof.*

$$\begin{aligned}
A^* &= \{\varepsilon\} \cup A \cup AA \cup A^3 \cup \cdots \\
\Phi_{A^*}(x) &= \Phi_{\{\varepsilon\}}(x) + \Phi_A(x) + \Phi_{A^2}(x) + \cdots && \text{(sum lemma on strings)} \\
&= \sum_{k\geq 0} \Phi_{A^k}(x) \\
&= \sum_{k\geq 0} (\Phi_A(x))^k && \text{(product lemma on strings)} \\
&= \frac{1}{1 - \Phi_A(x)}
\end{aligned}$$

note that in the world of formal power series this is only allowed if $[x^0]B(x) = 0$. $\square$

**Remark 2.1.** For the geometric series formula,

$$\sum_{n\geq 0} x^n = \frac{1}{1-x}$$

we substitute $B(x)$ into this.

$$\sum_{n\geq 0} B(x)^n = \frac{1}{1 - B(x)}$$

In this situation, $[x^0]\Phi_A(x) = 0$ so this substitution is indeed allowed because $A^*$ is unambiguous which means that $\epsilon \notin A$, so there are no strings of length 0.

Recall the problem from Example 2.6, "How many $\{0,1\}$-strings (binary strings) of length $n$ are there in which every 0 is followed by at least two 1s, for example 1110111011011111."

1. Describe the set of strings $S$ in which each 0 is followed by at least two 1s.
   Think about a string $\sigma \in S$. We can write it as $\sigma = a0b_1 0b_2 0 \cdots 0b_k$ where $a_1, b_1, \ldots, b_k$ are strings of 1s where $k \geq 0$ and $b_i$ has at least two 1s, for $i = 1, \ldots, k$, that is $a \in \{1\}^*$ and $b \in \{11\}\{1\}^*$. Then

   $$S = \{1\}^* \left(\{0\}\{11\}\{1\}^*\right)^*$$

   and now

   $$\begin{aligned}\Phi_S(x) &= \Phi_{\{1\}^*}(x)\Phi_{(\{0\}\{11\}\{1\}^*)^*}(x)\\ &= \left(\frac{1}{1-\Phi_{\{1\}}(x)}\right)\left(\frac{1}{1-\Phi_{\{0\}\{11\}\{1\}^*}(x)}\right)\\ &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-\Phi_{\{0\}}(x)\Phi_{\{11\}}(x)\Phi_{\{1\}^*}(x)}\right)\\ &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^1x^2\left(\frac{1}{1-x}\right)}\right)\\ &= \frac{1}{1-x-x^3}\end{aligned}$$

**Remark 2.2.** For $S = \{(a,b,c), a \leq b, a \leq c\}$ we have

$$\sum_{(a,b,c)\in S} x^{a+b+c} = \sum_{a=1}^{\infty}\sum_{b=a}^{\infty}\sum_{c=a}^{\infty} x^a x^b x^c = \sum_{a=1}^{\infty} x^a \left(\sum_{b=a}^{\infty} x^b\right)\left(\sum_{c=a}^{\infty} x^c\right) = \sum_{a=1}^{\infty} x^a \left(\frac{x^a}{1-x}\right)\left(\frac{x^a}{1-x}\right)$$

which then simplifies to

$$\sum_{a=1}^{\infty} x^a \left(\frac{x^a}{1-x}\right)\left(\frac{x^a}{1-x}\right) = \sum_{a=1}^{\infty} \frac{x^{3a}}{(1-x)^2} = \frac{\frac{x^3}{(1-x)^2}}{1-x^3}$$

## 2.3 Decomposition of $\{0,1\}-$Strings

**Definition 2.9** (block)**.** Given a string $\sigma \in \{0,1\}^*$, a **block** of $\sigma$ is a **maximal** non-empty substring consisting of only 0s or 1s. "Maximal" means that it can't be extended.

For example, 00011111010001 has 6 blocks, $000, 11111, 0, 1, 000, 1$. Note that a block must haver at least one digit.

Now the set of blocks of 0s is $\{0\}\{0\}^*$ and similarly the set of blocks of 1s is $\{1\}\{1\}*$. Additionally, all $\{0,1\}$-strings are of the form $\{0\}^* \left(\{1\}\{1\}^*\{0\}\{0\}^*\right)^* \{1\}^*$, this is called the **block decomposition**. These are unambiguous.

**Principle.** If we replace any part of this by a subset (given by an unambiguous expession), this will still be unambiguous. In other words, we can **specialize.** There are some other decompositions:

- **1-decomposition** $\{0\}^*(\{1\}\{0\}^*)^* = (\{0\}^*\{1\})^*\{0\}^*$
- **0-decomposition** $\{1\}^*(\{0\}\{1\}^*)^* = (\{1\}^*\{0\})^*\{1\}^*$

Recall the example asking for an expression that describes all binary strings where every 0 is followed by at least two 1s. We found $\{1\}^*(\{0\}\{11\}\{1\}^*)^*$ to be this expresion. This is a specialization of the the 0-decomposition replaced by $\{1\}^*$ by subset $\{11\}\{1\}^*$. Note that $\{0,011\}^*$ is the same expression but it is less obviously unambiguous and so would require a more clever argument.

**Example 2.8.** Determine the number of binary strings of length $n$ with the property that every even block of 0s is followed by an odd block of ones. Express your answer as the coefficient of a rational function. An "even block" means a block of length 2,4,6,8,... and an "odd block" means a block of length 1,3,5,7,....

We'll start by using the block decomposition that looks like $\{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$ since it has 0s followed by 1s similar to the problem statement. Now to specialize for our problem note there are two cases. Either

(a) even block of 0s, followed by an odd block of 1s i.e., $\{00\}\{00\}^*\{1\}\{11\}^*$

(b) odd block of 0s, followed by any block of 1s i.e., $\{0\}\{00\}^*\{1\}\{1\}^*$

Let $S$ be the set of all strings in which every even block of 0s is followed by an odd block of 1s.

$$S = \{1\}^*(\{00\}\{00\}^*\{1\}\{11\}^* \cup \{0\}\{00\}^*\{1\}\{1\})^*(\{\varepsilon\} \cup \{0\}\{00\}^*)$$

So,

$$\begin{aligned}
\Phi_S(x) &= \Phi_{\{1\}^*}(x)\Phi_{(\{00\}\{00\}^*\{1\}\{11\}^*\cup\{0\}\{00\}^*\{1\}\{1\})^*}(x)\Phi_{(\{\varepsilon\}\cup\{0\}\{00\}^*)}(x)\\
&= \frac{1}{1-x}\left(\frac{1}{1-\left(\Phi_{\{00\}\{00\}^*\{1\}\{11\}^*}(x)+\Phi_{\{0\}\{00\}^*\{1\}\{1\}^*}(x)\right)}\right)\left(1+\frac{x}{1-x^2}\right)\\
&= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-\left(\frac{x^2}{1-x^2}\frac{x}{1-x^2}\frac{x}{1-x^2}\frac{x}{1-x}\right)}\right)\left(1+\frac{x}{1-x^2}\right)
\end{aligned}$$

and our answer is $[x^n]$ of whatever that thing simplified is.

**Example 2.9.** A quick example of recursion (see section 2.8 of course notes). $S = \{0,1\}^*$ is the set of all binary strings. We could also write this as $S = \{\varepsilon\} \cup \{0,1\}S$. So,

$$\Phi_S(x) = \Phi_{\{\varepsilon\}}(x) + \Phi_{\{0,1\}}(x)\Phi_S(x) = 1 + 2x\Phi_s(x) \implies (1-2x)\Phi_S(x) = 1 \implies \Phi_S(x) = \frac{1}{1-2x}$$

which is the same answer we get using $\Phi_S(x) = \frac{1}{1-\Phi_{\{0,1\}}(x)}$

An expression for a set of strings made from finite sets, union, concatenation, and star (*) is called a **regular expression**.

# 3 Recurrences

[Chapter 3]
Let $c_n$ be the number of compositions of $n$ with 3 parts where the first part is even.

(a) Find a formula for $c_n$ without $\Sigma$-notation.

(b) Find a recurrence (linear homogenous recurrence) for the numbers $c_n$.

Let $S$ be the set of all compositions with three parts where the first is even. Then,

$$S = \mathbb{N}_{\in E} \times \mathbb{N}_{\geq 1} \times \mathbb{N}_{\geq 1}$$

where $\mathbb{N}_{\in E}$ is the set of even numbers. We use the usual weight function $w(a, b, c) = a + b + c$ and now we can use the product lemma to find the generating function,

$$\Phi_S(x) = \Phi_{\mathbb{N}_{\in E}}(x)\Phi_{\geq 1}(x)\Phi_{\geq 1}(x) = \left(\frac{x^2}{1-x^2}\right)\left(\frac{x}{1-x}\right)^2 = \frac{x^4}{(1+x)(1-x)^3} = \frac{x^4}{1-2x+2x^3-x^4}$$

(a) Use partial fractions. We divide $x^4$ into $1 - 2x + 2x^3 - x^4$ using long division to get

$$\begin{aligned}\Phi_S(x) &= -1 + \frac{2x^3-2x+1}{(1+x)(1-x)^3} \\ &= -1 + \frac{A}{1+x} + \frac{B+Cx+Dx^2}{(1-x)^3}\end{aligned}$$

Then,

$$\frac{2x^3-2x+1}{(1+x)(1-x)^3} = \frac{A}{1+x} + \frac{B+Cx+Dx^2}{(1-x)^3}$$

Clear denominators,

$$2x^3 - 2x + 1 = (1-x)^3 A + (1+x)(B + Cx + Dx^2)$$

Expand and solve for $A$, $B$, $C$, $D$,

$$A = \frac{1}{8}, B = \frac{7}{8}, C = \frac{-5}{2}, D = \frac{17}{8}$$

Then,

$$\Phi_S(x) = -1 + \frac{\frac{1}{8}}{1+x} + \frac{\frac{7}{8} - \frac{5}{2}x + \frac{17}{8}x^2}{(1-x)^3}$$

now using the binomial theorem we can extract the coefficients.

$$\begin{aligned}\Phi_S(x) &= -1 + \frac{1}{8}\sum_{n\geq 0}(-1)^n x^n + \left(\frac{7}{8} - \frac{5}{2}x + \frac{17}{8}x^2\right)\sum_{n\geq 0}\binom{n+2}{2}x^n \\ &= -1 + \frac{1}{8}\sum_{n\geq 0}(-1)^n x^n + \frac{7}{8}\sum_{n\geq 0}\binom{n+2}{2}x^n - \frac{5}{2}\sum_{n\geq 0}\binom{n+2}{2}x^{n+1} + \frac{17}{8}\sum_{n\geq 0}\binom{n+2}{2}x^{n+2} \\ &= -1 + \frac{1}{8}\sum_{n\geq 0}(-1)^n x^n + \frac{7}{8}\sum_{n\geq 0}\binom{n+2}{2}x^n - \frac{5}{2}\sum_{n\geq 1}\binom{n+1}{2}x^n + \frac{17}{8}\sum_{n\geq 2}\binom{n}{2}x^n\end{aligned}$$

Therefore,

$$c_n = [x^n]\Phi_S(x) = \begin{cases} \frac{1}{8}(-1)^n + \frac{7}{8}\binom{n+2}{2} - \frac{5}{2}\binom{n+1}{2} + \frac{17}{8}\binom{n}{2} & \text{if } n \geq 2 \\ 0 & \text{if } n = 0, 1\end{cases}$$

(b)

$$\Phi_S(x) = \sum_{n\geq 0} c_n x^n$$

then

$$\sum c_n x^n = \frac{x^4}{1-2x+2x^3-x^4}$$
$$(1-2x+2x^3-x^4)\left(\sum c_n x^n\right) = x^4$$

Multiplying the left hand side out using a table, we are able to see that there is a set of equations that can be used to find any $c_n$. They are

$$\begin{aligned} c_5 - 2c_4 + 2c_2 - c_1 &= 0 \\ c_6 - 2c_5 + 2c_3 - c_2 &= 0 \\ &\vdots \\ c_n - 2c_{n-1} - 2c_{n-3} - c_{n-4} &= 0 \end{aligned}$$

for $n \geq 5$.

- Let $a_n = [x^n]\frac{x^2+1}{2x^2+3x+4}$, find a linear recurrence for $c_n$ and initial conditions.

$$\begin{aligned} \sum_{n=0}^{\infty} c_n x^n &= \frac{x^2+1}{2x^2+3x+4} \\ (2x^2+3x+4)\left(\sum_{n=0}^{\infty} c_n x^n\right) &= x^2+1 \\ \sum_{n=0}^{\infty} 2c_n x^{n+2} + \sum_{n=0}^{\infty} 3c_n x^{n+1} + \sum_{n=0}^{\infty} 4c_n x^n &= x^2+1 \\ \sum_{k=2}^{\infty} 2c_{k-2} x^k + \sum_{k=1}^{\infty} 3c_{k-1} x^k + \sum_{k=0}^{\infty} 4c_k x^k &= x^2+1 \\ \left(\sum_{k=2}^{\infty} 2c_{k-2} x^k\right) + 3c_0 x^1 + \left(\sum_{k=2}^{\infty} 3c_{k-1} x^k\right) + 4c_0 x^0 + 4c_1 x^1 + \left(\sum_{k=2}^{\infty} 4c_k x^k\right) &= x^2+1 \\ 3c_0 x^1 + 4c_0 x^0 + 4c_1 x^1 + \left(\sum_{k=2}^{\infty} 2c_{k-2} x^k + 3c_{k-1} x^k + 4c_k x^k\right) &= x^2+1 \end{aligned}$$

Comparing coefficients of $x^k$ on both sides:

- $k=0 \implies 4c_0 = 1 \implies c_0 = \frac{1}{4}$.
- $k=1 \implies 4c_1 + 4c_0 = 0 \implies c_1 = \frac{-3}{16}$.
- $k=2 \implies 2c_0 + 3c_1 + 4c_2 = 1 \implies c_2 = \frac{17}{64}$
- $k \geq 2 \implies 2c_{k-2} + 3c_{k-1} + 4c_k = 0 \implies c_k = -\frac{1}{2}c_{k-2} - \frac{3}{4}c_{k-1}$

Notice that these steps can be performed in the other order. For example, suppose we're given a reccurrence with initial conditions, we could work backwards to find $c_n$ as a coefficient of some rational function. By pattern recognition, consider that the coefficients in the recurrence relation describe the denominator in the original equation, and the coefficients of the inital conditions relate bck to the coefficients of the numerator.

**Example 3.1.** Let $c_0 = 1$, $c_1 = 0$, $c_2 = 2$ and $c_n = 7c_{n-1} - 16c_{n-2} + 12c_{n-3}$ for $n \geq 3$. Solve for $c_n$.

We start by letting $c(x) = \sum_{n\geq 0} c_n x^n$, then we write

$$-12c_{n-3} + 16c_{n-2} - 7c_{n-1} + c_n = 0$$

then we want denominator $-12x^3 + 16x^2 - 7x + 1$.

Now consider

$$\begin{aligned}(12x^3 + 16x^2 - 7x + 1)\sum_{n\geq 0} c_n x^n = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots \\ -7c_0x - 7c_1x^2 - 7c_2x^3 - 7c_4x^4 - 7c_5x^5 + \cdots \\ 16c_0x^2 + 16c_1x^3 + 16c_2x^4 + 16c_3x^5 + \cdots \\ -12c_0x^3 - 12c_1x^4 - 12c_2x^5 + \cdots\end{aligned}$$

which simplifies to

$$(12x^3 + 16x^2 - 7x + 1)\sum_{n\geq 0} c_n x^n = c_0 + (c_1 - 7c_0) + (c_2 - 7c_1 + 16c_0)x^2 + 0x^3 + 0x^4 + 0x^5 + \cdots = 1 - 7x + 18x^2$$

then

$$\sum_{n\geq 0} c_n x^n = \frac{1 - 7x + 18x^2}{1 - 7x + 16x^2 - 12x^3}$$

Now we use partial fractions.

$$\frac{1 - 7x + 18x^2}{1 - 7x + 16x^2 - 12x^3} = \frac{1 - 7x + 18x^2}{(1-2x)^2(1-3x)}$$

(use high school factoring methods for this step), so

$$\frac{1 - 7x + 18x^2}{(1-2x)^2(1-3x)} = \frac{Ax + B}{(1-2x)^2} + \frac{C}{1-3x} \implies 1 - 7x + 18x^2 = (1-3x)(Ax + b) + (1-2x)^2 C$$

Then $A = 2$, $B = -5$, $C = 6$. Then,

$$\begin{aligned}
\sum c_n x^n &= \frac{2x-5}{(1-2x)^2} + \frac{6}{1-3x} \\
&= (2x-5)\sum_{k\geq 0}\binom{k+1}{1}(2x)^k + 6\sum_{k\geq 0}(3x)^k \\
&= (2x-5)\sum_{k\geq 0}(k+1)2^k x^k + 6\sum_{k\geq 0}3^k x^k \\
&= \sum_{k\geq 0} -5(k+1)2^k x^k + \sum_{k\geq 0} 2(k+1)2^k x^{k+1} + \sum_{k\geq 0} 6\cdot 3^k x^k \\
&= \sum_{k\geq 0} -5(k+1)2^k x^k + \sum_{k\geq 1} 2k2^{k-1} x^k + \sum_{k\geq 0} 6\cdot 3^k x^k
\end{aligned}$$

Finally,

$$\begin{aligned}
c_n &= [x^n]\sum_{k\geq 0} -5(k+1)2^k x^k + \sum_{k\geq 1} 2k2^{k-1} x^k + \sum_{k\geq 0} 6\cdot 3^k x^k \\
&= \begin{cases} -5(n+1)2^n + n2^n + 6\cdot 3^n & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases} \\
&= 6\cdot 3^n - (4n+5)\cdot 2^n
\end{aligned}$$

Note the form of the answer:

$$c_n = \alpha\cdot 3^n + (\beta n + \gamma)\cdot 2^n$$

3 and 2 came from factoring $Q(x)$ and the linear coefficient came from the double root. The easier approach knowing in advance that $c_n = \alpha\cdot 3^n + (\beta n + \gamma)\cdot 2^n$ would have been to plug in $n = 0$, $n = 1$, and $n = 2$ to get three equations

$$\begin{aligned}
1 &= c_0 = \alpha + \gamma \\
0 &= c_1 = \beta + 2\beta + 2\gamma \\
2 &= c_2 9\alpha + 8\beta + 4\gamma
\end{aligned}$$

Solve the system of equations and find $\alpha = 6$, $\beta = -4$, and $\gamma = -5$.

## 3.1 Homogeneous Linear Recurrence

A sequence $\{a_n\}_{n\geq 0}$ is defined by a **homogeneous linear recurrence** if for $n \geq k$, $a_n + q_1 a_{n-1} + \cdots + q_k a_{n-k} = 0$ and **initial conditions** $a_0, a_1, \ldots, a_{k-1}$ are given.

**Definition 3.1** (characteristic polynomial)**.** The **characteristic polynomial** of this recurrence $C(y) = y^k + q_1 y^{k-1} + \cdots + q_{k-1}y + q_k$.

**Theorem 3.1.** Let $\beta_1, \ldots, \beta_j$ be the distinct roots of $C(y)$, where $\beta_i$ has multiplicity $m_i$, $C(y) = (y-\beta_1)^{m_1}\cdots(y-\beta_j)^{m_j}$. Then the solution to the recurrence is

$$a_n = P_1(n)\beta_1^n + \cdots + P_j(n)\beta_j^n$$

where, for each $i$, $P_i(n)$ is a polynomial in $n$ of degree less than or equal to $m_i$ whose coefficients are determined by the values $a_0, a_1, \ldots, a_{k-1}$.

**Example 3.2.** $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$ for all $n \geq 3$. Also, $a_0 = 4$, $a_1 = 9$, and $a_2 = 17$. Then, by theorem 3.1

$$a_n - 4a_{n-1} + 5a_{n-2} - 2a_{n-3} = 0$$

and the characteristic polynomial is

$$C(y) = y^3 - 4y^2 + 5y - 2$$

observe that $y = 1$ is a root. Then, by long division,

$$\frac{y^3 - 4y^2 + 5y - 2}{y - 1} = y^2 - 3y + 1 \implies C(y) = (y-1)(y^2 - 3y + 2) = (y-1)(y-1)(y-2) = (y-1)^2(y-2)$$

Then the solution to the linear recurrence is

$$a_n = (A + Bn)(1)^n + C(2^n)$$

where $A, B, C$ satisfiy that

$$4 = a_0 = (A + B(0)) + C = A + C$$
$$9 = a_1 = (A + B(1) + C \cdot 2 = A + B + 2C$$
$$17 = a_2 = (A + 2b) + C(4) = A + 2B + 4C$$

Solving this linear system gets $A = 1$, $B = 2$, $C = 3$.

So $a_n = 1 + 2n + 3(2^n)$ for all $n \geq 0$.

## 3.2 Nonhomogeneous Recurrences

A sequence $\{b_n\}_{n \geq 0}$ can also be defined by a **nonhomogeneous recurrence**:

$$b_n + q_1 b_{n-1} + \cdots + q_k b_{n-k} = f(n), n \geq k \qquad (*)$$

with given initial conditions $b_0, \ldots, b_{k-1}$, where $f(n)$ is a function of $n$. Suppose we can find a sequence $a_n$ that satisfies $(*)$ for all $n \geq k$ (ignore initial conditions). Let $c_n$ be the solution to the homogenous linear recurrence $c_n + q_1 c_{n-1} + \cdots + q_k c_{n-k} = 0$, then $b_n = a_n + c_n$ which satisfies $(*)$. Then we can choose the coefficients in $c_n$ so that $a_0 + c_0 = b_0, \ldots, a_{k-1} + c_{k-1} = b_{k-1}$. Then, $b_n = a_n + c_n$ is the unique solution to the recurrence.

**Example 3.3.**

$$(*) \qquad b_n - 4b_{n-1} + 5b_{n-2} - 2b_{n-3} = 24(-1)^n$$

for $n \geq 3$, where $b_0 = -1$, $b_1 = -3$, $b_2 = 2$. We try $a_n = A(-1)^n$ where $A$ is a constant.

Then,

$$\begin{aligned} a_n - 4a_{n-1} + 5a_{n-2} - 2a_{n-3} &= A(-1)^n - 4A(-1)^{n-1} + 5A(-1)^{n-2} - 2A(-1)^{n-3} \\ &= A(-1)^{n-3}((-1)^3 - 4(-1)^2 + 5(-1) - 2) \\ &= A(-1)^{n-3}(-12) \\ &= 12A(-1)^{n-2} \end{aligned}$$

So we can choose $A = 2$ to get

$$a_n - 4a_{n-1} + 5a_{n-2} - 2a_{n-3} + = 24(-1)^n$$

Set $a_n = 2(-1)^n$. To solve the homogeneous recurrence we find the characteristic polynomial $C(y) = y^3 - 4y^2 + 5y - 2$. We already found the roots of this: 1 with multiplicity 2 and 2 with multiplicity 1. The solution is

$$b_n = 2(-1)^n + (A + Bn)(1)^n + C(2^n)$$

We want to find $A, B, C$ satisfying

$$-1 = b_0 = 2(-1)^0 + (A + B(0)) + C = 2 + A + C$$

$$-3 = b_1 = 2(-1) + (A + B) + 2C = -2 + A + B + 2C$$
$$2 = b_2 = 2 + (A + 2B) + 4C = 2 + A + 2B + 4C$$

Solving this linear system returns $A = -2$, $A = 3$ and $C = -1$.

The solution is $b_n = 2(-1)^n + (-2 + 3n) - 2^n$ for all $n \geq 0$. What if guessing $a_n = A(-1)^n$ didn't work? Then try $a_n = (An - B)(-1)^n$ and in fact you may as well take $B = 0$ in this attempt. Then try $a_n = Cn^2(-1)^n$, etc.

**Example 3.4.** $b_n - 3b_{n-1} + 4b_{n-3} = 3(2^n)$ for all $n \geq 3$. Try $a_n = A(2^n)$. So,

$$\begin{aligned} a_n - 3a_{n-1} + 4a_{n-3} &= A(2^n) - 3A(2^{n-1}) + 4A(2^{n-3}) \\ &= A(2^{n-3})(8 - 3(4) + 4) \\ &= 0 \end{aligned}$$

We cannot choose $A$ to make this $3(2^n)$. Next try $a_n = (Bn + A)(2^n)$, then

$$\begin{aligned} a_n - 3a_{n-1} + 4a_{n-3} &= (Bn + A)(2^n) - 3(B(n-1) + A)(2^{n-1}) + 4(B(n-3) + A)(2^{n-3}) \\ &= B_n(2^n) - 3B(n-1)(2^{n-1}) + 4B(n-3)(2^{n-3}) + \underbrace{A(2^n) - 3A(2^{n-1}) + 4A(2^{n-3})}_{0} \\ &\vdots \\ &= 0 \end{aligned}$$

so we may as well forget $A$. Next try $a_n = Cn^2(2^n)$, this one does work and gives $3C(2^n)$ so we choose $C = 1$.

**Kevin Purbhoo's Favourite Problem**
This problem is called the Crazy Dice Problem. Suppose you have an ordinary pair of 6-sided dice, and they're fair. We can draw a probability table,

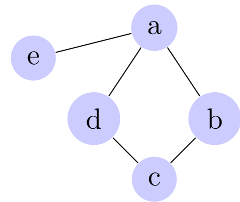
| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(n)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

Is it possible to change the numbers on the dice (still positive integers) so that you get the exact same probabilities as an ordinary pair of dice? This problem is left as an exercise to the reader.

# 4 Graph Theory

**Definition 4.1** (graph)**.** A **graph** $G$ is a finite set $V(G)$ of elements called **vertices**, together with a set $E(G)$ of unordered pairs of distinct vertices.

**Example 4.1.** Here's a graph, $V(G) = \{a, b, c, d, e\}$ and $E(G) = \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{a, e\}\}$.



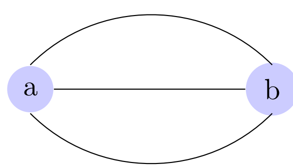

**Definition 4.2** (planar graph)**.** A planar graph is a graph that can be drawn with no edges crossing.

**Definition 4.3** (terminology)**.** If $e = \{u, v\} \in E(G)$, we say

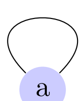
- $u$ and $v$ are **adjacent**
- $e$ is **incident** with $u$
- $e$ is **incident** with $v$
- $e$ **joins** $u$ and $v$
- $v$ is a **neighbour** of $u$

Some notes about the definition of a graph.

- $V(G)$ is finite, we have no infinite graphs
- $E(G)$ is a set, we have no notion of multiple edges, e.g., we'll never have this



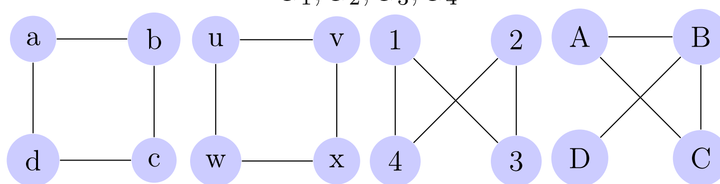

- Edges and unordered pairs of verticies - edges don't have a direction (we don't draw arrows or anything on graphs)
- Edges join **distinct** vertices
- $E(G)$ is a set, we have no notion of multiple edges, e.g., we'll never have this




People do study alternate / more general notions of graphs where these things are allowed, but not in this course.

**Example 4.2.** Consider,

$G_1, G_2, G_3, G_4$




Note that $G_1, G_2$ and $G_3$ are essentially the same, but not equal, and $G_4$ is fundamentally different.

**Definition 4.4** (isomorphic)**.** Two graphs $G_1$ and $G_2$ are **isomorphic** if there exists a bijection $f : V(G_1) \longrightarrow V(G_2)$ that preserved adjacencies, that is

$$\{u, v\} \in E(G_1) \iff \{f(u), f(v)\} \in E(G_2)$$

THe bijection $f$ is called an **isomorphism**.

**Example 4.3.** From the above example, $f : V(G_1) \longrightarrow V(G_3)$ where $f(a) = 1,\ f(b) = 3,\ f(c) = 2,\ f(d) = 4$ is an isomorphism, it is isomorphic. Can you find a different isomorphism?

If $G_1$ and $G_2$ are isomorphic, they have all the same **features**. (anything you can define that doesn't involve specific vertex names)

**Definition 4.5** (degree)**.** If $G$ is a graph $u \in V(G)$, then the set of all neighbours of $u$ is denoted $N(u)$. The **degree** of $u$ is

$$\deg(u) = |N(u)|$$

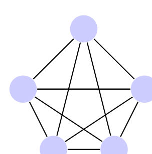
**Definition 4.6** (degree sequence)**.** The degree sequence of $G$ is the list of the degrees of the verticies of $G$ in decreasing order.

**Example 4.4.** The degree sequence of $G_1 : 2, 2, 2, 2$, the degree sequence of $G_4 : 3, 2, 2, 1$. This shows that $G_1$ and $G_4$ are **not** isomorphic.

To prove that two graphs are isomorphic, first state the isomorphism, then to find it line up the features. To prove that two graphs are not isomorphic, find some features that distinguish them. Another possibility is to use proof by contradiction; try to line up features and find none of the options work.

## 4.1 Special Families of Graphs

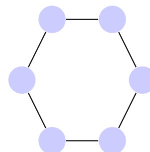
**Definition 4.7** (complete graph)**.** A complete graph $K_p$ for $p \in \mathbb{N}$ is a graph with $p$ vertices, and every pair of verticies is an adjacency. For example,



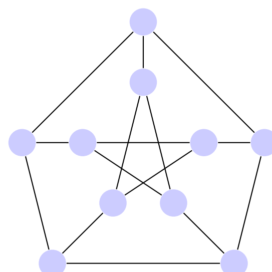
so $|V(K_p)| = p$ and $|E(K_p)| = \binom{p}{2} = \frac{p(p-1)}{2}$.

There's also a graph at the other extreme with $p$ vertices and no edges.

**Definition 4.8** ($k$-regular)**.** A $k$-regular graph is a graph where every vertex has degree $k$. For example, $K_p$ is a $(p-1)$-regular graph. A 2-regular graph example,

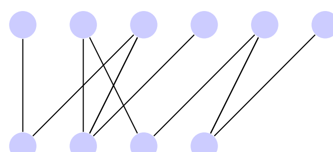


Also, the **Petersen** graph (3 regular)



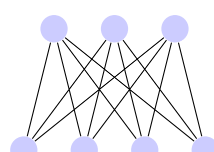
This can be drawn in another way, check the course notes.

**Definition 4.9** (bipartite)**.** A bipartite graph is a graph where the vertices can be partitioned into two sets $A$ and $B$ where each edge is incident to one vertex in $A$ and one vertex in $B$. That is, each edge is of the form $\{a, b\}$ for $a \in A, b \in B$. For example,



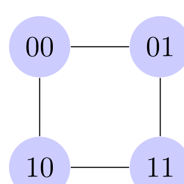
where the top row is $A$ and bottom row $B$. The pair $(A, B)$ is called a **bipartition**.

**Definition 4.10** (complete bipartite)**.** A complete bipartite graph $K_{m,n}$ has a vertex set partitioned into $(A, B)$ where $|A| = m$ and $|B| = n$ and every vertex in $A$ is adjacent to every vertex in $B$. For example, $K_{3,4}$

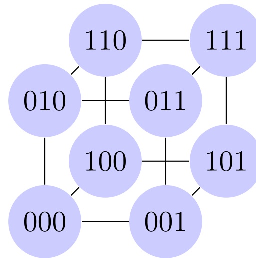


where $|V(K_{m,n})| = m + n$ and $|E(K_{m,n})| = mn$

**Definition 4.11** ($n$-cube)**.** A graph where vertices are $\{0, 1\}$ strings of length $n$. Two strings are adjacent if they differ in exactly one position.




$O_2$




$O_3$

$O_n$ is $n$-regular (each vertex has $n$ positions that could be changed to get an incident edge).

**Theorem 4.1.** $O_n$ is also bipartite.

*Proof.* Let $A$ = strings with an even number of 1s, and $B$ the same with odd number of 1s. THen every edge connects two strings, one of which has $k$ 1s andthe other has $k+1$ 1s, so one is in $A$ and the other is in $B$. □

**Theorem 4.2.** If a graph $G$ has $q$ edges, then

$$\sum_{v \in V(G)} \deg(v) = 2q$$

*Proof.* Each edge is incident with verticies. So we sum the degrees of the verticies, we count each edge twice. □

**Corollary 4.1.** Every graph has an even number of vertices of odd degree.

How many edges, verticies does $O_n$ have?

$$|V(O_n)| = 2^n$$

$$|E(O_n)| = q$$

We apply the theorem,

$$\sum_{v \in V(G)} \deg(v) = 2q$$

Since $\deg(v) = n$ for all $v \in V(O_n)$, $2^n \cdot n = 2q$ therefore $q = n2^{n-1}$.

**Example 4.5.** The Petersen graph is a 3-regular graph with 10 verticiesl. Can you find a 3-regular graph with 11 verticies?

The answer is that you can't becuase if $G$ were such a graph it would mean that

$$\sum_{v \in V(G)} \deg(v) = 3 \cdot 11 = 33 \neq 2q$$

## 4.2 Paths and Walks

**Definition 4.12** (walk)**.** Let $x$ and $y$ be vertices in a graph $G$. A **walk** from $x$ to $y$ is an alternating sequence of vertices and edges

$$v_0 e_1 v_1 e_2 v_2 e_3 v_3 \cdots v_{n-1} e_n v_n$$

where

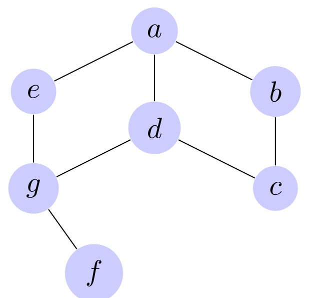
- $v_0, \ldots, v_n \in V(G)$ and
- $e_1, \ldots, e_n \in E(G)$, and

- $e_i = \{v_{i-1}, v_i\}$ joins $v_{i-1}$ and $v_i$
- $x = v_0, y = v_n$

Sometimes this becomes cumbersome and a walk is described by listing vertices:

$$v_0 v_1 v_2 \cdots v_n$$

**Definition 4.13** (length)**.** In the above definition, $n$ is called the **length** of the walk. (It is the number of edges in the sequence). For example,




where $abcbcdgfgdae$ is a walk from $a$ to $e$ of length 11.

**Definition 4.14** (path)**.** A **path** in $G$ from $x$ to $y$ is a walk in which no vertices are repeated.

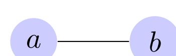
**Theorem 4.3.** If there is a walk from $x$ to $y$ in $G$ then there is a path from $x$ to $y$ in $G$.

*Proof.* Let $v_0 v_1 v_2 \cdots v_n$ be a walk from $x$ to $y$. Perform the following algorithm on this walk:
If this is a path, STOP.
Otherwise, there must be a vertex repeated, say $v_i = v_j$ for $i \neq j$. This means $v_0 v_1 \cdots v_i v_{j+1} v_{j+2} \cdots v_n$ is a walk from $x$ to $y$.
Repeat, with this walk.
Since the walk gets shorter each time we run through the loop, the algorithm must stop. But, it stops when we have a path from $x$ to $y$. Therefore, there is a path from $x$ to $y$. □

Here is a question, what happens in this algorithm if the walk is something like this:

$$v_0 v_1 v_2$$

where $v_0 = a$, $v_1 = b$, $v_2 = a$ and we have a graph



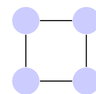

**Corollary 4.2.** If there is a path from $x$ to $y$, and a path from $y$ to $z$, there is a path from $x$ to $z$.

*Proof.* Let $u_0 u_1 \cdots u_m$ be a path from $x$ to $y$ and let $v_0 v_1 \cdots v_n$ be a path from $y$ to $z$. Then $u_0 u_1 \cdots u_m v_1 v_2 \cdots v_n$ is a walk from $x$ to $z$. Therefore since there is a walk from $x$ to $z$ there's a path from $x$ to $z$. □

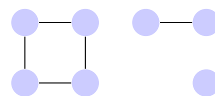
**Theorem 4.4.** The relation on $V(G)$ given by $x \sim y$ if there is a walk (or a path) from $x$ to $y$ iis an equivalence relation.

**Note.** Think of an equivalence relation as putting the elements of a set into groups. (equivalence classes)

**Definition 4.15** (connected)**.** We day that a graph is **connected** if this equivalence relation has one equivalence class. That is, for any two vertices $x, y \in V(G)$ there is a walk/path from $x$ to $y$.

**Example 4.6.** 

This is an example of a connected graph.

**Example 4.7.** 

This graph is not connected.

**Theorem 4.5.** Suppose there is a vertex $v \in V(G)$ such that for every vertex $u \in V(G)$ there is a path from $u$ to $v$ in $G$. Then $G$ is connected. Kevin Purbhoo calls this the "Hub Model".

*Proof.* Let $x$ and $y$ be any two vertices of $V(G)$, sicne there is a walk from $x$ to $y$, and a walk from $v$ to $y$, there is a walk from $x$ to $y$. This proves that $G$ is connected. ☐

**Note.** If there is a walk from $x$ to $y$, then there is a walk from $y$ to $x$.

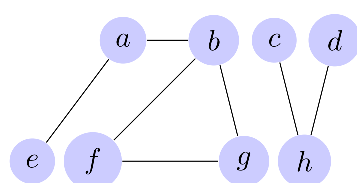
**Example 4.8.** Prove that the $n$-cube is connected.

*Proof.* Let $v = \underbrace{000\ldots0}_{n} \in V(O_n)$. Let $x$ be any vertex of $O_n$. Let $i_1, i_2, \ldots, i_k$ be the positions of the 1s in $x$. For $j = 0, \ldots, k$, let $v_j$ be the $\{0,1\}$-string that has 1s in positions $i_1, i_2, \ldots, i_j$ and 0s elsewhere. Then $v_0 = 00\ldots0 = v$ and $v_k = x$. And so $v_0 v_1 v_2 \cdots v_k$ is a path from $v$ to $x$. By theorem 4.5, this proves that the $n$-cube is connected. ☐

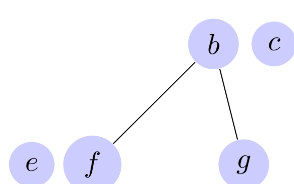
### 4.3 Subgraphs

**Definition 4.16** (subgraph)**.** Let $G$ be a graph. A **subgraph** of $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

**Example 4.9.** A graph.



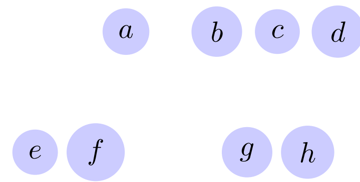

A subgraph example could be A graph.




Note that every edge in $E(H)$ must have both ends in $V(H)$.

**Definition 4.17** (spanning)**.** A subgraph $H$ of $G$ is **spanning** if $V(H) = V(G)$

For example,

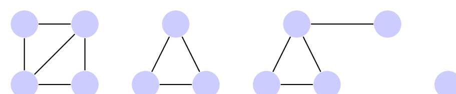



is a spanning subgraph of the graph above.

Also, not relevant to this course, but a subgraph where $E(H)$ uses all edges that make sense is called **induced.**

**Definition 4.18** (component)**.** A **component** of a graph $G$ is a subgraph $H$ such that $H$ is connected. Any subgraph of $G$ that properly contains $H$ is not connected.

**Example 4.10.** A graph with four components,



**Fact.** A graph is connected if it has exactly one component. This follows from definitions.

**Definition 4.19** (cut)**.** Let $G$ be a graph and let $X \subseteq V(G)$. The **cut** on $X$ is the set of all edges $e \in E(G)$ that have exactly one vertex in $X$.

**Note.** Drawing these graphs is a little tedious for me, I'll add them later, check the course notes for now.

**Theorem 4.6.** Let $G$ be a graph. If there is a proper, non-empty subset $X \subset V(G)$, such that the cut on $X$ is empty, then $G$ is not connected.

That is, how to prove a graph is not connected.

- Find a proper, non-empty subset $X \subset V(G)$
- Check that for every edge $e \in E(G)$, either $e$ joins two vertices in $X$, or $e$ joins two vertices in $V(G) \setminus X$.
- How to get $X$? Take $X$ to be all vertices in one component. If $X$ has more than one component this works, because $X$ is proper ($X \neq V(G)$) and $X = \emptyset$.

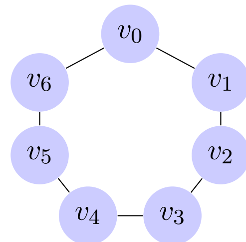
*Proof.* (of Theorem 4.6) Let $x \in X$, $y \in V(G) \setminus X$. We show that if the cut on $X$ is empty, there is no path from $x$ to $y$. Suppose to the contrary that we had such a path $v_0 v_1 v_2 \cdots v_n$, where $v_0 = x$, $v_n = y$.
Let $i$ be the the largest index such that $v_i \in X$. Note that $i < n$ because $v_n \notin X$ so $v_{i+1} \in V(G) \setminus X$, and $\{v_i, v_{i+1}\} \in E(G)$. So, $\{v_i, v_{i+1}\}$ belongs to the cut on $X$. This contradicts our assumption that cut on $X$ is empty. □

The converse is also true: If $G$ is not connected, then we can find a proper non-empty subset $X \subset V(G)$, such that the cut on $X$ is empty.

**Idea:** Take $X = V(H)$, where $H$ is a component, argue that this works.

## 4.4 Cycles and Bridges

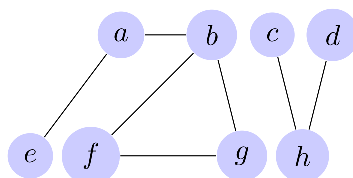
**Definition 4.20** (cycle)**.** A **cycle** is a graph $C$ with $n$ vertices $V(C) = \{v_0, v_1, \dots, v_{n-1}\}$ and $m$ edges $E(C) = \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-2}, v - n - 1\}, \{v_{n-1}, v_0\}\}$. For example, a cycle with 7 vertices and edges - also called a 7-cycle.




Note that a cycle can't have length 1 or 2.

- Length 1 : requires edge $\{v_0, v_0\}$ (not allowed)
- Length 2 : requires 2 edges $\{\{v_0, v_1\}, \{v_1, v_0\}\}$ (just one edge)

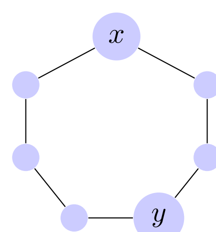
**Definition 4.21** (subcycle)**.** If $G$ is a graph, a cycle in $G$ is a subgraph of $G$ that is a cycle. For example,




There is a subcycle between vertices $b, f$, and $g$.

An easier way to specify a cycle is to write down a walk around the cycle. For example, $v_0 v_1 v_2 \cdots v_{n-1} v_0$. This is a **closed walk**, meaning that it starts and ends at the same vertex.

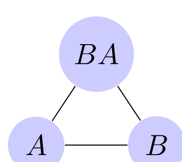
Cycles are "more than connected". Consider



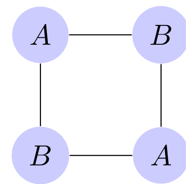

There are almost completely disjoint paths from $x$ to $y$.

### Cycles and Bipartite Graphs

If $G$ is a bipartite graph, then any subgraph of $G$ is a bipartite graph. Every cycle in $G$ is a bipartite graph. When is a cycle bipartite? Consider this graph,



This is not bipartite (like we have shown by labelling $BA$ on the top node). Additionally, a graph with 5 vertices is not bipartite. Consider,




This graph is bipartite, so we conclude that for a cycle to be bipartite, it must have an even number of vertices.

**Theorem 4.7.** Even cycles are bipartite. Odd cycles are not bipartite.

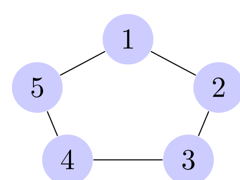
*Proof.* Let $C$ be an $n$-cycle and let $v_0v_1 \cdots v_{n-1}v_0$ be a walk around $C_1$. If $n$ is even, let $A = \{v_0, v_2, \ldots, v_{n-2}\}$ and $B = \{v_1, v_3, \ldots, v_{n-1}\}$ then $(A, B)$ is a bipartition. If $n$ is odd, we can try to cosntruct a bipartition. Without loss of generality, let $v_0 \in A$. Then $v_1 \in B$, $v_2 \in A, v_3 \in B$, and in general we can easily show that $v_1 \in A$ if $i$ is even and $v_i \in B$ if $i$ is odd. But since $n$ is odd, $v_0, v_{n-1} \in A$ and $\{v_0, v_{n-1}\} \in E(C)$. So this is not a bipartition. Therefore, there is no bipartition. $\square$

**Corollary 4.3.** If $G$ has an odd cycle, then $G$ is not bipartite.

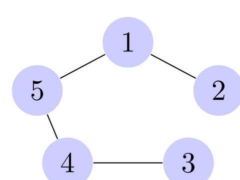
*Proof.* We just discuss how we can't have a non-bipartite subgraph of a bipartite graph. If we had an odd cycle in a bipartite graph, we'd have a massive contradiction. $\square$

**Definition 4.22** (edge deletion)**.** Let $G$ be a graph and $e \in E(G)$. Then $G - e$ is the subgraph $G$ with $V(G-e) = V(G)$ and $E(G-e) = E(G) \setminus \{e\}$.

**Example 4.11.** Consider $G$,

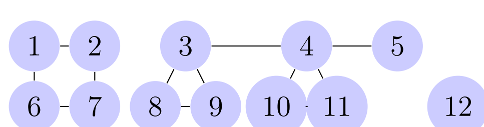



where $e = \{2, 3\}$. Then $G - e$ looks like

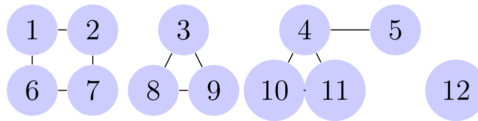



**Definition 4.23** (bridge)**.** Let $e \in E(G)$. We say that $e$ is a **bridge** if $G - e$ has more components than $G$.

**Example 4.12.** Consider $G$ with $e = \{3, 4\}$.




Which has 3 components. Then $G - e$,




has 4-components.

**Lemma 4.1.** Let $G$ be a connected graph and let $e = \{x, y\}$ be an edge. If $e$ is a bridge, then $G - e$ has **exactly** 2 components and $x$ and $y$ are in different components.

*Proof.* Let $z \in V(G) = V(G - e)$. We will show that there is either a path from $x$ to $z$ in $G - e$ or a path from $y$ to $z$ in $G - e$. Since $G$ is connected, there is a path from $x$ to $z$ in $G$. If $e$ is not in this path, then this is a path in $G - e$ from $x$ to $z$. Otherwise, the path is of the form

$$x \cdots e v_k \cdots z$$

Since $e$ can't appear twice, $v_k \cdots z$ is a path in $G - e$ and $v_k \in \{x, y\} = e$.
So in either case, we have either a path from $x$ to $z$ or from $y$ to $z$ in $G - e$. This shows that every vertex of $G - e$ is either in the component of $x$ or the component of $y$. Therefore $G - e$ has at most 2 components.
Since $e$ is a bridge, $G - e$ has at least two components. The result follows. □

**Generalization.** Let $G$ be any graph. If $e = \{x, y\} \in E(G)$ is a bridge, then $G - e$ has exactly one more component than $G$, and $x$ and $y$ are in different components of $G - e$.

*Proof.* Component of $e$ splits in two other componenets unchanged in $G - e$. □

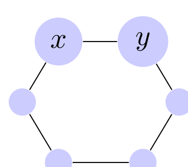
**Theorem 4.8.** Let $e \in E(G)$ be an edge of a graph $G$. Then $e$ is a bridge if and only if $e$ is not contained in any cycle.

*Proof.* Suppose to the contrary, that $e$ is contained in a cycle, say the cycle is given by the walk:

$$xeye_1v_1e_2v_2 \cdots x$$

Then $ye_1v_1e_2v_2 \cdots x$ is a path in $G - e$ from $y$ to $x$. So $y$ and $x$ are in the same component of $G - e$. By the lemma, $e$ cannot be a bridge.

In the other direction, suppose $e$ is not a bridge, then $x$ and $y$ are in the same component of $G - e$, therefore there is a path $ye_1v_1e_2v_2 \cdots x$ in $G - e$. Then, $xeye_1v_1e_2v_2 \cdots x$ is a walk around a cycle. Therefore $e$ is contained in a cycle. □




where the edge between $x$ and $y$ is $e$.

**Theorem 4.9.** Let $G$ be a graph. If there are two vertices $u$ and $v$ of $G$ such that there are two different paths from $u$ to $v$, then $G$ contains a cycle.

*Proof.* Let $P_1 = ux_1x_2x_3 \cdots x_{k-1}v$ and $P_2 = uy_1y_2y_3 \cdots y_{l-1}v$. Since a path from $u$ to $v$ is determined by the set of edges it uses, $P_1$ and $P_2$ can't use the same set of edges. These must be an edge $e$ that appears in one but not the other. Without loss of generality, say $P_1 = u \cdots x_i e x_{i+1} \cdots v$ and $e$ not in $P_2$. Then $x_i x_{i-1} \cdots u y_1 y_2 \cdots v y_{k-1} \cdots x_{i+1}$ is a walk from $x_i$ to $x_{i+1}$ that does not use $e$. Therefore $x_i$ and $x_{i+1}$ are in the same component of $G - e$. Therefore $e$ is not a bridge, $e$ is in a cycle, and so $G$ has a cycle. □

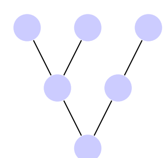
Essentially we have shown, to summarize,

$$e \text{ is a bridge} \iff e \text{ is not a cycle}$$

$$\text{cycle} \iff \text{two paths between } u \text{ and } v$$

## 4.5 Trees

**Definition 4.24** (tree)**.** A **tree** is a connected graph with no cycles. For example,



Some properties of trees,

1. There is a unique path between any two vertices.

   *Proof.* Let $T$ be a tree. Since $T$ is connected there is at least one path between any two vertices. If there were two, one would have a cycle. □

2. Every edge of $T$ is a bridge.

   *Proof.* If $T$ had an edge that was not a bridge, that edge would be in a cycle. □

3. If a tree has $p$ vertices, then it has $q = p - 1$ edges.

   *Proof.* By strong induction on $p$, the number of vertices. If $p = 1$ then any graph with one vertex has 0 edges, so the result is true. Fix $p > 1$, assume the result is true for all trees with fewer than $p$ vertices. Let $T$ be a tree with $p$ vertices, we want to prove that $T$ has $p - 1$ edges. Let $e \in E(T)$. Then $e$ is a bridge. So, $T - e$ has two components, call them $T_1$ and $T_2$. $T_1$ and $T_2$ are connected (because they're components) and have no cycles, because they are subgraphs of $T$. Therefore $T_1$ and $T_2$ are both trees. Let $p_1 = |V(T_1)|$ and $p_2 = |V(T_2)|$, then $p_1 \geq 1$ and $p_2 \geq 2$, because $T_1$ and $T_2$ are components. Since $p_1 + p_2 = p$, $p_1 < p$ and $p_2 < p$. Therefore by our inductive hypothesis, $T_1$ has $q_1 = p_1 - 1$ edges and $T_2$ has $q_2 = p_2 - 1$ edges. But $E(T) = E(T_1) \cup E(T_2) \cup \{e\}$. Therefore, $q = |E(T)| = q_1 + q_2 + 1 = (p_1 - 1) + (p_2 - 1) + 1$. □

   If you believe in the empty graph with 0 vertices, the empty graph is **not** connected.

**Theorem 4.10.** A tree with at least two vertices has at least two vertices of degree 1.

**Definition 4.25** (leaf)**.** A vertex of degree 1 in a tree is called a **leaf**.

**Note.** There may not be more than 2 leaves.

(only two) 

To help with our proof, first we consider the following:

Let $T$ be a tree. Let $n_i$ = number of vertices of degree $i$ in $T$ for $i \geq 0$. Assume $p = |V(T)| \geq 2$. So that $n_0 = 0$ (any vertex of degree 0 is in a component by itself). Also $n_i = 0$ for $i \geq p$ (next possible degree for a vertex is $p-1$).

Now we know that $|E(T)| = q = p - 1$, and

$$\sum_{v \in V(T)} \deg(v) = 2q$$

we can rewrite this as

$$1n_1 + 2n_2 + 3n_3 + \cdots + (p-1)n_{p-1} = 2(p-1) \tag{1}$$
$$n_1 + n_2 + n_3 + \cdots + n_{p-1} = p \tag{2}$$

then 2×(2) - (1) is,

$$n_1 + 0 - n_3 - 2n_4 - 3n_5 - \cdots - (p-3)n_{p-1} = 2$$
$$n_1 = (n_3 + 2n_4 + 3n_5 + \cdots) + 2 \tag{$\star$}$$

Now we start the proof since we have ($\star$)

*Proof.* From ($\star$) we see that

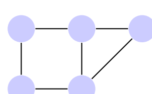
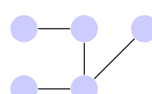
$$n_1 \geq 2$$

so $T$ has at least 2 leaves. □

**Note.** $n_2$ does not appear in ($\star$).

### 4.6 Spanning Trees

**Definition 4.26** (spanning tree)**.** Let $G$ be a graph. A **spanning tree** in $G$ is a spanning subgraph that is also a tree.

$G$:  $H$: 

$H$ is a spanning tree of $G$.

**Theorem 4.11.** A graph hs a spanning tree if and only if it is connected.

*Proof.* ($\Longrightarrow$) Let $G$ be a graph, and let $T$ be a spanning tree of $G$. To prove $G$ is connected, take $x, y \in V(G)$. Since $T$ is a tree there is a path from $x$ to $y$ in $T$. This is also a path in $G$. Done.

($\Longleftarrow$) Let $G$ be a connected graph. If $G$ has no cycles, then $G$ is a tree, and hence $G$ is a spanning tree of itself. If $G$ has a cycle let $e$ be an edge in a cycle. Consider $G - e$, since $e$ is in a cycle, $e$ is not a bridge. So, $G - e$ is connected and has fewer edges. Repeat until there are no cycles left. The result must be a connected spanning subgraph with no cycles, that is a spanning tree. □

**Corollary 4.4.** If $G$ is a connected graph with $p$ vertices and $q = p - 1$ edges, then $G$ is a tree.

*Proof.* Assume $G$ is connected. Then $G$ has a spanning tree, $T$. We know

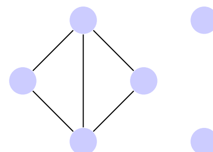
$$V(T) = V(G)$$

which means that

$$|E(T)| = |V(T)| - 1 = p - 1 = q$$

but $|E(G)| = q$ which implies that $E(T) = E(G)$. $\square$

**Note. WARNING!** $G$ must be connected. Consider,



which has 6 vertices and 5 edges, but is not a tree.

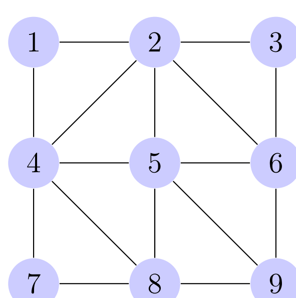
## 4.7 Breadth First Search Trees

**Input.** A graph $G$ with $p$ vertices and a vertex $r \in V(G)$.

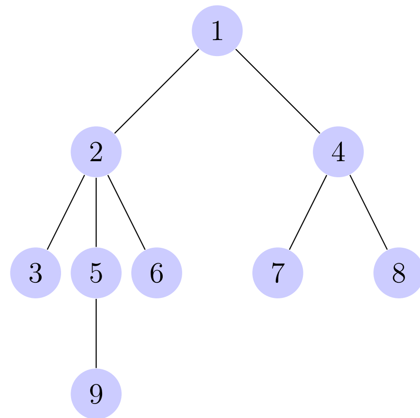
**Output.** A Breadth First Search Tree (BFST) **rooted** at $r$.

1. Begin $T$ with vertex $r$ ($r$ is called the **root** of $T$).
   - Define $\underbrace{pr(r)}_{\text{parent}} = \phi \longleftarrow$ null
   - Begin a queue of **unexhausted vertices** with $r$.
2. While the queue is non-empty do:
   - Let $x$ be the vertex at the head of the queue ($x$ is called the **active vertex**)
   - While there is an edge $e = \{x, y\}$ where $y \notin V(T)$.
     * Add $y$ and $e$ to $T$
     * define $pr(y) = x$
     * Add $y$ to the queue
   - Delete $x$ from the queue
3. Output $(T, pr)$

**Example 4.13.** Consider the graph,



Then the following Breadth First Search Tree is produced,




**Note.** There may be choices. When there are multiple edges to add to an active vertex, the algorithm does not say what order to add them in.

**Applications of Breadth First Search Trees**

**Theorem 4.12.** Let $G$ be a graph and let $T$ be a Breadth First Search Tree of $G$. That is, let $T$ be the output of this algorithm.

(i) $T$ is always a tree

(ii) $T$ is a spanning tree if and only if $G$ is connected.

*Proof.* (i) We will show that $T$ is connected and if $T$ has $k$ vertices, then $T$ has $k-1$ edges.

Note that for any vertex $v \in V(T)$, there is a path

$$v \to pr(v) \to pr(pr(v)) \to \cdots pr^l(v) = r$$

Therefore $T$ is connected. Also, $T$ begins with 1 vertex and 0 edges. In the algorithm, we always add 1 vertex and 1 edge together. Therefore, $|V(T)| = |E(T)| + 1$ at all points in the algorithm. This shows that $T$ is a tree.

(ii) ($\Longrightarrow$) If $T$ is a spanning tree then $G$ is connected.
($\Longleftarrow$) If $T$ is not a spanning tree, then $V(T)$ is a proper non-empty subset of $V(G)$. The algorithm terminates when the cut on $V(T)$ is empty. Therefore $G$ is not connected. □

So the Breadth First Search Algorithm gives us a way to test whether graphs are connected.
Another application is a way of finding the distance between 2 vertices. Let $T$ be a Breadth First Search Tree rooted at $r$. For any $v \in V(T)$ there is a path which goes from $v$ to its parent, to its parent's parent and so forth until we reach the root,

$$v \to pr(v) \to pr(pr(v)) \to \cdots pr^l(v) = r$$

**Definition 4.27** (level)**.** The length of this path is the **level**.

For example, on the top of this page the tree drawn has 1 in level, 2 in level 2, 5 in level 3, and 1 in level 4.

**Fact.** In the breadth first search tree algorithm, the vertices of $T$ are added in non-decreasing order of level.

**Theorem 4.13** (Fundamental propert of BFSTs)**.** Let $G$ be a connected graph, and let $T$ be a Breadth First Search Tree of $G$. For any edge $e = \{x, y\} \in E(G)$, the vertices $x$ and $y$ are at most one level apart.

*Proof.* Suppose without loss of generality that $i = level(x) \leq level(y)$. We show that $level(y) \leq i + 1$. When $x$ becomes the active vertex in the breadth first search tree algorithm, there are two cases.

(1) $y$ is already in the tree. Then $pr(y)$ must precede $x$ in the queue. (None of the vertices after $x$ have had their children added yet). Then $\text{level}(pr(y)) \leq \text{level}(x)$. Therefore $\text{level}(y) = \text{level}(pr(y)) + 1 \leq \text{level}(x) + 1 = i + 1$.

(2) $y$ is not already in the tree. Then $y$ gets added to the tree now, and $pr(y) = x$. Therefore $\text{level}(y) = \text{level}(x) + 1$.

□

**Definition 4.28** (distance)**.** The **distance** $d(u, v)$ between two vertices $u$ and $v$ in a graph $G$ is the length of a shortest path from $u$ to $v$.

If there is no path from $u$ to $v$, $d(u, v)$ is undefined (or $d(u, v) = \infty$).

**Theorem 4.14.** Let $G$ be a connected graph and let $u, v \in V(G)$. Let $T$ be a breadth first search tree rooted at $u$. Then, $d(u, v) = \text{level}(v)$.

*Proof.* We'll prove two things:

(1) $d(u, v) \geq \text{level}(v)$

(2) $d(u, v) \leq \text{level}(v)$

So,

(1) Consider a shortest path from $u$ to $v$:
$$u = x_0 x_1 x_2 \cdots x_k = v$$
We know that $\text{level}(v) = \text{level}(x_k) \leq \text{level}(x_{k-1}) + 1$ by Fundamental propert of BFSTs, we can repeat this so,
$$\begin{aligned} \text{level}(x_k) &\leq \text{level}(x_{k-1}) + 1 \\ &\leq \text{level}(x_{k-2}) + 2 \\ &\vdots \\ &\leq \text{level}(x_0) + k \\ &= k && (\text{since } u = x_0 \text{ is the root}) = d(u, v) \end{aligned}$$
Then we have shown that $d(u, v) \geq \text{level}(v)$.

(2) Note that $v, pr(v), pr^2(v), \ldots, pr^k(v) = u$ is a path from $u$ to $v$ whose distance is $\text{level}(v)$. Either this is a shortest path, or there's a shorter one. Therefore, $\text{level}(v) \geq d(u, v)$.

□

This proves that
$$v, pr(v), pr^2(v), \ldots, pr^k(v) = u$$
is a shortest path from $u$ to $v$ if $u$ is the root. There be other shortest paths.
**WARNING!** This only works if one of the two vertices involved is the root of $T$.
Another way that the algorithm helps, is finding bipartite graphs.

**Theorem 4.15.** Let $G$ be a connected graph. Then the following are equivalent.

i. $G$ is bipartite

ii. $G$ has no odd cycles

iii. Let $T$ be a breadth first search tree of $G$. There are no edges of $G$ joining two vertices of the same level.

*Proof.* We will prove that (1) $i. \implies ii.$ and (2) $ii. \implies iii.$ and (3) $iii. \implies i.$. We have already shown $i. \implies ii.$.

(2) In contrapositive form: If there is an edge, $\{x, y\} \in E(G)$ such that $\text{level}(x) = \text{level}(y)$, then $G$ has an odd cycle. Consider the subgraph formed by the edge $e = \{x, y\}$ and the path

$$x, pr(x), pr^2(x), \ldots, pr^t(x)$$

$$y, pr(y), \ldots, pr^t(y)$$

where $pr^t(x) = pr^t(y)$ is the first common ancestor of $x$ and $y$. This is a cycle of length $2t + 1$.

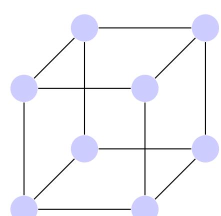
(3) Let $A = \{v \in V(G) \mid \text{level}(v) \text{ is odd}\}$ and $B = \{v \in V(G) \mid \text{level}(v) \text{ is even}\}$. Since there are no edges of $G$ joining two vertices of the same level, every pair of adjacent vertices is exactly one level apart. That means, one is in $A$ and one is $B$, therefore $(A, B)$ is a bipartition, therefore the graph $G$ is bipartite.

□

## 4.8 Planar Graphs

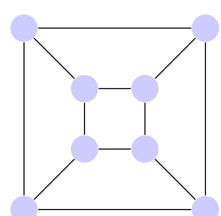
**Definition 4.29** (planar)**.** A graph is **planar** if it can be drawn in the plane with no edges crossing.

For example,



$O_3$

is planar, redrawn it looks like:



$O_3$

Some other planar graphs include $K_4$. $K_4$ is not planar (try it). $K_{3,3}$ is also not planar. So how do we prove this?

**Note.** A graph is planar if and only if all of its components are planar.

**Definition 4.30** (planar embedding)**.** A **planar embedding** is a specific drawing of a graph in the plane with no edges crossing.

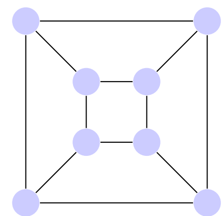
**Definition 4.31** (face)**.** A planar embedding divides the plane into regions called **faces**.

**Definition 4.32** (adjacent)**.** Two faces are **adjacent** if they are **incident** with a common edge.

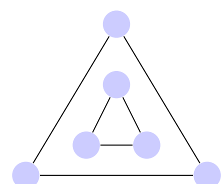
**Definition 4.33** (boundary)**.** The **boundary** of a face is the subgraph consisting of all vertices and edges incident with the face.

**Definition 4.34** (degree)**.** The **degree** of a face is the number of edges in the boundary with bridges counted twice.

**Example 4.14.** Some examples,

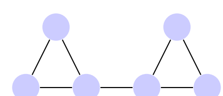


Each empty space within this figure is a face, and the empty space outside of it is also a face. So there are $p = 8$ vertices, $q = 12$ edges, $s = 6$ faces. Aditionally, for each face $f_i$ from $1 \leq i \leq 6$, $\deg(f_i) = 4$. For the graph



has 6 vertices, 6 edges, and 3 faces. The degree of the face in the inner triangle is 3, of the face between the boundaries of both triangles is 6, and the degree of the face of the outside is 3.

also,



then there are 6 vertices, 7 edges, and 3 faces. The degree of the face of the left triangle is 3, the degree of the face in the right triangle is 3, and the degree of the outside face is 8.

**Key Observation.** In a planar embedding,

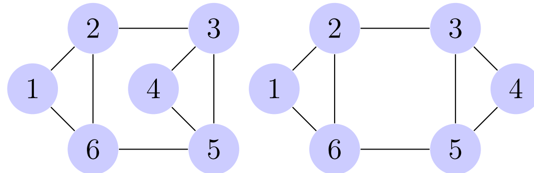
- a bridge is incident with just one face
- a non-bridge is incident with two different faces.

**Theorem 4.16.** For a planar embedding with $q$ edges and faces $f_1, f_2, \ldots, f_s$, then

$$\sum_{i=1}^{s} \deg(f_i) = 2q$$

*Proof.* Each edge that is not a bridge is incident with 2 faces, so gets counted twice on the LHS. Each bridge is incident with one face, but gets counted twice in the degree of that face. Therefore the LHS counts each edge twice. $\square$

A graph can have different planar embeddings. For example, both of these graphs are equal




Their face degree sequences are (respectively), $5, 5, 3, 3$ and $6, 4, 3, 3$. Not a feature of graph itself.

**Theorem 4.17** (Euler's Formula)**.** For a planar embedding with $p$ vertices, $q$ edges, $s$ faces and $c$ components, then

$$p - q + s = c + 1$$

**Corollary 4.5.** For a connected planar embedding with $p$ vertices, $q$ edges, and $s$ faces then

$$p - q + s = 2$$

*Proof.* By induction on $q$, the number of edges.

**Base Case.** Prove it for all planar graphs with $q = 0$ edges. If there are $p$ vertices, then $c = p$ and $s = 1$. Check $p - q + s = p - 1$ and $c + 1 = p + 1$. Thus, $p - q + s = c + 1$.

**Inductive Hypothesis.** Fix $p > 0$ and assume Euler's Formula holds for graphs with $q - 1$ edges.

**Inductive Step.** Let $P$ be a planar embedding with $p$ vertices, $q$ edges, $s$ faces, and $c$ components. Let $e$ be an edge of $P$ and consider $P - e$. $P - e$ has $p$ vertices, $q' = q - 1$ edges, $s'$ faces, and $c'$ components. By the inductive hypothesis, $p' - q' + s' = c' + 1$.

- **Case 1**. If $e$ is a bridge, then $c' = c + 1$ (Lemma 4.1). Also $e$ is incident with only one face (the face on one side of $e$ is the same face as the face on the other side). So when we delete $e$, we do not create any new faces. Therefore $s' = s$. This implies that $p - (q - 1) + s = (c + 1) + 1$, and therefore $p - 1 + s = c + 1$.

- **Case 2**. If $e$ is not a bridge, then $c' = c$ (by definition). Also $e$ is incident with 2 different faces, when we delete $e$, these become 1 face, and therefore $s' = s - 1$. Then $p - (q - 1) + (s - 1) = c + 1$ which implies $p - q + s = c + 1$.

□

The proof in the course notes is fairly similar, but the case base is a tree, with $p$ vertices, $p - 1$ edges, and 1 face. How do we know there's only 1 face though? This proof also skips case 1, because the graph is connected. To prove a tree has one face is to prove it using induction using the argument in case 1.

## 4.9 Platonic Solids

In Platonic Solids,

- Faces are regular polygons.
- There are the same number of faces meeting at every vertex. ($d$ faces at each vertex)
- Can be modeled by a planar graph.

Then let $d^*$ be the degree of every face, and then $d$ the degree of every vertex. If we want to study Platonic Solids, we should study connected planar graphs, where every face has degree $d^*$ and every vertex has degree $d$.

We have three equations: ($p$ vertices, $q$ edges, $s$ faces)

$$\sum_{v \in V(G)} \deg(v) = 2q \implies pd = 2q \tag{1}$$

$$\sum_{i=1}^{s} \deg(f_i) = 2q \implies sd^* = 2q \tag{2}$$

$$p + q - s = 2 \tag{3}$$

**Question.** What are the possible values of $d$ and $d^*$?
We solve the system of equations, first by eliminating $p$ and $s$.

$$p = \frac{2q}{d} \qquad\qquad s = \frac{2q}{d^*}$$

Then Euler's formula becomes,

$$\frac{2q}{d} - q + \frac{2q}{d^*} = 2$$

Move all the $q$'s to one side

$$\frac{2q}{d} + \frac{2q}{d^*} = \frac{2+q}{q}$$

since $\frac{2+q}{q} > 1$, we have

$$\frac{2}{d} + \frac{2}{d^*} > 1$$

Multiply by $dd^*$ to get

$$\begin{aligned} 2d^* + 2d &> dd^* \\ dd^* - 2d^* - 2d &< 0 \\ dd^* - 2d^* - 2d + 4 &< 4 \\ (d-2)(d^*-2) &< 4 \end{aligned}$$

The possibilities are $d = 2$, which is some $d^*$-cycle, or $d > 2$ (to get a 3-dimensional object), in which case $(d, d^*) \in \{(3,3), (4,3), (3,4), (5,3), (3,5)\}$. These pairs correspond to the 5 platonic solids.

- $(3,4)$ is the cube
- $(4,3)$ is the octahedron
- $(3,3)$ is the tetrahedron
- $(5,3)$ is the icosahedron
- $(3,5)$ is the dodecahedron

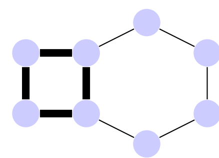
The next goal is to prove that $K_5$ is **not** planar.

- Assume it is

- Use these equations
- Get a contradiction

In order to do this we need a general property about faces of planar graphs.

**Definition 4.35** (girth)**.** If $G$ is a graph with a cycle, the **girth** of $G$ is the length of a shortest cycle. For example, the following graph has girth 4.



**Theorem 4.18.** $K_5$ is not planar.

*Proof.* Suppose it were planar. Then we would have a planar embedding with $p = 5$vertices, $q = 10$ edges, $s$ faces, and every face has degree greater than or equal to 3. Thus $\text{Girth}(K_5) = 3$. By Euler's Formula,

$$s = q - p + 2 = 7$$

Let $f_1, f_2, \ldots, f_7$ be the faces. Then

$$\sum_{i=1}^{s} \deg(f_i) = 20$$

but $\deg(f_i) \leq 3$ for $1 \leq i \leq 7$, so $\sum \deg(f_i) \geq 21$. This is a contradiction. □

**Theorem 4.19.** $K_{3,3}$ is not planar.

*Proof.* Suppose it were. Then we would have a planar embedding with $p = 6$, $q = 9$ and $s$ faces. By Euler's Formula, $s = q - p + 2 = 5$. Let $f_1, \ldots, f_5$ be the faces, then since $\text{Girth}(K_{3,3}) = 4$, $\deg(f_i) \geq 4$. However,

$$\sum \deg(f_i) = 2q = 18$$

$$\sum \deg(f_i) \geq 4s = 20$$

contradiction. □

**Exerciese:** Prove that the Petersen graph is not planar.

**Note.** You cannot use this method to prove that a graph is planar.

**Theorem 4.20.** Let $G$ be a graph with a cycle. Suppose $\text{Girth}(G) = k$. If a planar embedding of $G$ exists, then every face of that planar embedding has degree at least $k$.

In the next little bit we will

- Prove this theorem
- Streamline the method (new equation)
- What to do if this method doesn't work
- Applications to graph colouring

**Lemma 4.2.** Let $P$ be a planar embedding of a graph with a cycle. Then the boundary of every face of $P$ also has a cycle.

*Proof.* Let $f$ be a face of $P$. Let $H$ be the boundary of $f$. $H$ is a subgraph of $P$ embedded in the plane and $f$ is also a face of $H$. Since $P$ contains a cycle, there is an edge of $P$ that is not an edge. This edge is incident with 2 faces.

- Therefore $P$ has at least 2 faces.
- Therefore $f$ is not the whole plane.
- Therefore $H$ has another face.

Therefore there is an edge along which two faces are adjacent. This edge cannot be a bridge which implies that $H$ is a cycle. □

*Prooof of Theorem 4.20.* Let $P$ be a planar embedding of $G$ and let $f$ be a face. By the lemma, the boundary of $f$ has a cycle $C$. Then

$$k \leq length(C) \leq \text{the number of edges in the boundary of } f \leq \deg(f)$$

□

**Theorem 4.21.** Suppose $G$ is a planar embedding with $p$ vertices, $q$ edges, and suppose every face has degree at least $d^* \geq 3$. Then,

$$q = \frac{d^*(p-2)}{d^*-2}$$

*Proof.* Let $'f_1, f_2, \ldots, f_s'$ be the faces of $G$.

$$2q = \sum_{i=1}^{2} \deg(f_i) \geq d^* s \implies s \leq \frac{2q}{d^*}$$

By Euler's Formula,

$$p - q + s = c + 1 \qquad \text{where } c \text{ is the number of components}$$

Since $c \geq 1$, $p - q + s \geq 2$,

$$\begin{aligned} s &\geq 2 + q - p \\ \implies 2d^* + qd^* - pd^* &\leq 2q \\ \implies 2q - qd^* &\geq 2d^* - pd^* \\ \implies (2-d^*)q &\geq d^*(2-p) \\ \implies q &\leq \frac{d^*(2-p)}{2-d^*} \\ \implies q &\leq \frac{d^*(p-2)}{d^*-2} \end{aligned}$$

□

**Applications.** In any planar graph with $p \geq 3$ vertices and $q$ edges, we have

$$q \leq 3p - 6$$

This is the maximum possible number of edges in a planar graph.

*Proof.* If $G$ has a cycle, then by Lemma 4.2, every face $f_i$ of a planar embedding of $G$ has $\deg(f_i) \geq \text{Girth}(G) \geq 3$. Therefore by Theorem 4.21 with $d^* = 3$ implies

$$q \leq \frac{3(p-2)}{3-2} = 3p - 6$$

Otherwise, if $G$ does not have a cycle, then $G$ is a forest (every component is a tree), therefore

$$q \leq p - 1 \leq 3p - 6$$

$\square$

**Example 4.15.** Use this to show that $K_5$ is not planar.

For $K_5$, $p = 5$ and $q = 10$. Is $q \leq 3p - 6$? No. Therefore $K_5$ is not planar.

Note that this doesn't work for $K_{3,3}$. In this case we have $p = 6$ and $q = 9$, but $q \leq 3p - 6$ is a valid inequality so we have no conclusion.

**Application 2.** Suppose $G$ is a graph with a cycle, and $\text{Girth}(G) \geq k$. If $G$ is planar then

$$q \leq \frac{k(p-2)}{k-2}$$

*Proof.* We know that every face of a planar embedding of $G$ has degree greater than or equal to $k$. $\square$

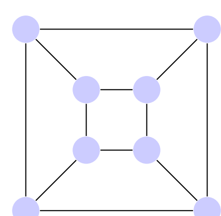
Main Point: The girth of $G$ is always an acceptable value for $d^*$.

**Example 4.16.** Prove $K_{3,3}$ is not planar ($p = 6, q = 9, k = \text{Girth}(K_{3,3}) = 4$). Is it true that

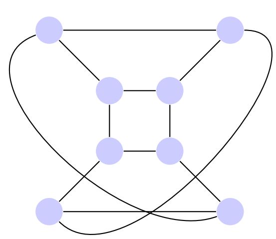
$$q \leq \frac{k(p-2)}{k-2}$$

No. $LHS = 9$, and $RHS = 8$. So, $K_{3,3}$ is not planar.

**Example 4.17.** Consider

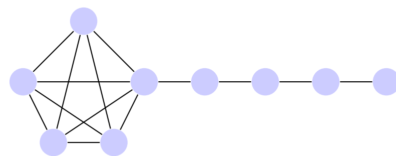


then redrawn as



This has the same $p = 8$, $q = 12$, and girth $k = 4$. However, it is not planar.

**Example 4.18.** Consider



with $p = 2$, $q = 17$ and $k = 3$. And

$$q \leq \frac{k(p-2)}{k-2}$$

has $LHS = 17$, $RHS = 30$. By adding stuff to a non-planar graph, the equation $q \leq \frac{k(p-2)}{k-2}$ might become satisfied.

**Definition 4.36** (edge subdivision)**.** An edge subdivision of a graph $G$ is obtained by applying the following operation, independently, to each edge of $G$: replace the edge by a path of length 1 or more; if the path has length $m > 1$, then there are $m - 1$ new vertices and $m - 1$ new edges created; if the path has length $m = 1$, then the edge is unchanged.

**Theorem 4.22** (Kuratowski's Theorem)**.** A graph is non-planar if and only if it has a subgraph that is either an edge-subdivision of $K_{3,3}$ or an edge-subdivision of $K_5$.

*Proof.* CO 342 □

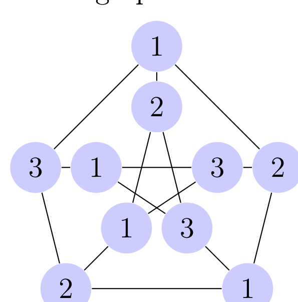
The best way to determine if a graph is planar:

- First, make a guess about whether or not it is planar
  - (a) If yes, rewdraw it without crossings. Check out this website: `planarity.net`
  - (b) If not, count the vertices $p$ and edges $q$, is $q > 3p - 6$?
    - i. If yes, conclude that the graph is not planar.
    - ii. If not, let $k$ be the girth of the graph; is $q > \frac{k(p-2)}{k-2}$?
      - · If yes, then conclude the graph is not planar.
      - · If not, find an edge-subdivision $K_5$ or $K_{3,3}$. Do this by identifying that it might be an edge-subdivision of $K_5$, since that one is more likely, and then note that there **must** be five distinct vertices that have degree 5.

## 4.10 Graph Colouring

**Definition 4.37** ($k$-colouring)**.** A **$k$-colouring** of a graph $G$ is an assignment of $k$ (or fewer) colours to the vertices of $G$, such that no pair of adjacent vertices has the same colour. If a $k$-colouring exists, we say the graph is $k$-colourable.

**Example 4.19.** Find a 3-colouring of the petersen graph:



This means that the Petersen graph is 3-colourable. Is it 4-colourable? Yes. Is it 2-colourable? No.

**Note.** 2-colourable is the same thing as bipartite. If a bipartition is bipartite with $A$ and $B$ then vertices in $A$ get colour 1 and in $B$ they get colour 2.

The 2-colourable is easy because there is an efficient algorithm. 3-colourable is a totally different story, it is hard, and there is no efficient algorithm.

**Theorem 4.23** (4 colour theorem)**.** Every planar graph is 4-colourable.

*Proof.* The proof is not human-readable. Check this out: `http://research.microsoft.com/en-us/um/people/gonthier/4colproof.pdf`. □

**Theorem 4.24** (6-colour theorem)**.** Every planar graph is 6-colourable.

**Lemma 4.3.** Every planar graph has a vertex $v$ with $\deg(v) \leq 5$.

*Proof.* Suppose to the contrary that $G$ is a planar graph with $p$ vertices and $q$ edges, and $\deg(v) \geq b$ for all $v \in V(G)$. This means that

$$2q = \sum_{v \in V(G)} \deg(v) \geq 6p$$

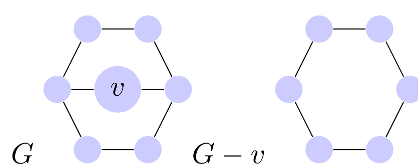
which implies $q \geq 3p$. But since $G$ is planar, $q \leq 3p - 6$. □

*Proof of 6-colour theorem.* By induction on the number of vertices $p$.

**Base Case.** A graph with 1 vertex is 6-colourable.

**Inductive Hypothesis.** Let $p \geq 2$ and assume that every planar graph with $p-1$ vertices is 6-colourable.

**Inductive Step.** Let $G$ be a planar graph with $p$ vertices. Choose a vertex $v \in V(G)$ such that $\deg(v) \leq 5$. (possible, by Lemma 4.3). Let $G - v$ be the subgraph of $G$ with vertex set $V(G) \setminus \{v\}$ and all edges of $G$ that are not incident with $v$. For example,

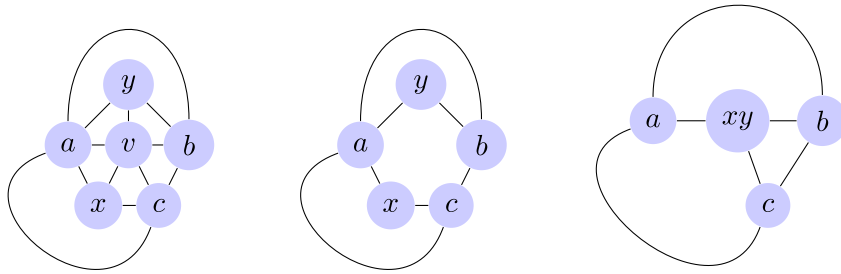



Since $G - v$ is a subgraph of a planar graph, $G - v$ is planar, and it has $p - 1$ vertices. By the inductive hypothesis $G - v$ has a 6-colouring. We can extend this to a 6-colouring of $G$, by assigning $v$ one the colours not used by its neighbours. ($v$ has at most 5 neighbours, so there's a colour left.). Therefore $G$ is 6-colourable as required. □

**Theorem 4.25** (5-colour theorem)**.** Every planar graph is 5-colourable.

*Proof.* Basically the same as the 6-colour theorem. But here's a case where that proof doesn't work. If $v$ has 5 neighbours, and all have different colours, we're stuck.

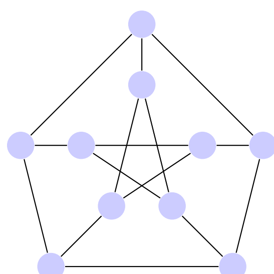
**Note.** The neighbours of $v$ can't all be adjacent to each other; there must be two neighbours of $v$, say $x$ and $y$ that are not adjacent in $G$. (why? if they were all adjcent, the nighbours of $v$ would form a $K_5$-subgraph, which can't appear in a planar graph).



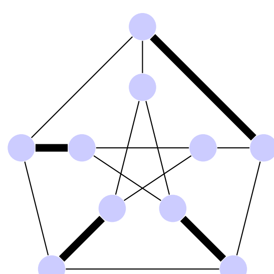

If $G$ is a planar graph with $p-2$ vertices, then the graph on the right is 5-colourable. The middle graph has a 5-colouring where $x$ and $y$ have the same colour. This avoids the problem which happened if all neighbours of $v$ have different colours. $\square$

## 4.11 Matchings

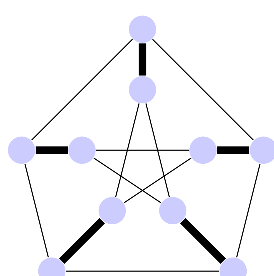
**Definition 4.38** (matching)**.** A **matching** $M$ in a graph $G$ is a subset of the edges such that no two edges in $M$ have a common vertex.



$\varphi$ (empty graph) is always a matching.



a matching of size 4.



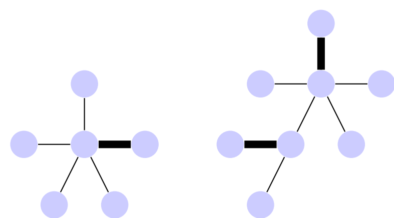
a perfect matching (size 5).

**Definition 4.39** (saturated)**.** A vertex of $G$ is said to be **saturated** by $M$ if it is incident with an edge in $M$. In a perfect matching, every vertex is saturated, which implies $|M| = \frac{p}{2}$ where $p = |V(G)|$.

**Definition 4.40** (maximum)**.** A matching $M$ is called a **maximum** matching if it has the largest possible number of edges among all matchings.

If a perfect matching exists, it is automatically a maximum matching. If not, a maximum matching has fewer than $\frac{p}{2}$ edges.

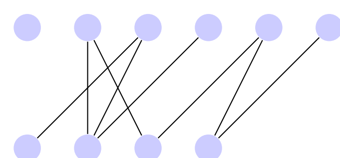
**Example 4.20.** Find a maximum matching:



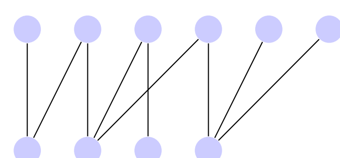
**Motivation.** The job assignment problem:

- you have some people
- you have some jobs

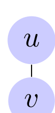
Fill as many jobs as possible. Consider a bipartition where the top vertices are people and the bottom vertices are jobs, and an edge connecting a vertex from either side means that the person is qualified for the job.



Also, in **scheduling**; such as



where

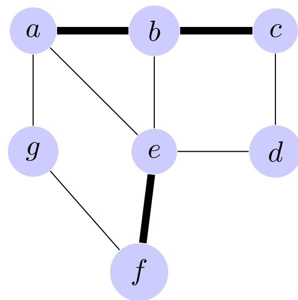



means person $u$ is available for timeslot $v$.

Note that a matching is similar to scheduling people and timeslots. These examples involve bipartite graphs. We'll study matchings in bipartite graphs as a special case where theory is particularly nice.

**Definition 4.41** (alternating path)**.** An **alternating path** relative to matching $M$ is a path whose edges alternate being in $M$ and not in $M$.

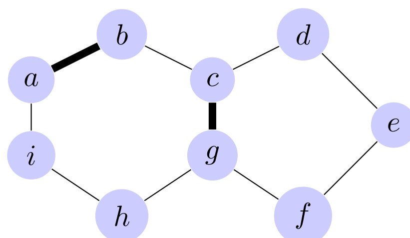
**Example 4.21.**




$gfebcd$ is an alternating path.

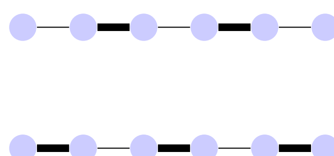
**Definition 4.42** (augmenting path)**.** An **augmenting path** is an alternating path of length greater than or equal to 1, where both ends are unsaturated by $M$.

**Example 4.22.**




$hgcd$ is an augmenting path. $fgcbai$ is an augmenting path.

Augmenting paths let me do this:



Given a matching $M$ and augmenting path $P = v_0 e_1 v_1 e_2 \cdots e_{2k+1} v_{2k+1}$, then

- $e_2, e_4, \ldots, e_{2k} \in M$.
- $e_1, e_3, e_5, \ldots, e_{2k+1} \notin M$.

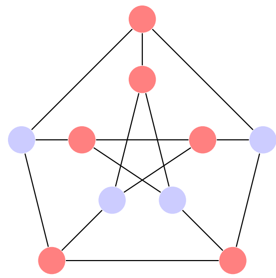
Let $M' = (M \setminus \{e_2, e_4, \ldots, e_{2k}\}) \cup \{e_1, e_3, \ldots, e_{2k+1}\}$. Then $M'$ has one more edge than $M$.

**Lemma 4.4.** Suppose $G$ is a graph, $M$ a matching in $G$. If there exists an augmenting path relative to $M$, then $M$ is **not** a maximum matching.

*Proof.* In this situation, $M'$ can be defined as outlined, and $|M'| > |M|$. $\square$

**Definition 4.43** (cover)**.** A **cover** of a graph $G$ is a subset of $C \subseteq V(G)$ with the property that every edge of $G$ is incident with at least one vertex in $C$.

**Example 4.23.**

 This is a minimum cover of the petersen graph.

**Find a minimum cover.**

**Theorem 4.26.** Suppose $G$ is a graph, $C$ is a cover of $G$, $M$ is a matching of $G$. If $|C| = |M|$, then $M$ is a maximum matching and $C$ is a minimum cover.

**Lemma 4.5.** In any graph $G$, if $M$ is a matching and $C$ is a cover then $|M| \leq |C|$.

*Proof.* Let $M = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}\}$, then $|M| = k$. Since $C$ is a cover, every edge in $M$ is incident to at least one vertex in $C$. Without loss of generality, assume $u_i \in C$. Since $M$ is a matching, $u_i \neq u_j$ for $i \neq j$. $C$ has at least $k$ elements, $|C| \geq k = |M|$. ∎

*Theorem 4.26.* Suppose $M'$ is a maximum matching and $C'$ is a minimum cover.

$$|M| \underbrace{\leq}_{M' \text{ is a max match}} |M'| \underbrace{\leq}_{\text{lemma}} |C'| \underbrace{\leq}_{C' \text{ is a min cover}} |C|$$

Therefore if $|M| = |C|$, all of them must be equal. In particular, $|M| = |M'|$. Therefore $M$ is also a maximum matching and $|C| = |C'|$. So, $C$ is also a minimum cover. ∎

**Theorem 4.27** (König's Theorem)**.** In a bipartite graph, a maximum matching and a minimum cover have the same size.

**Note.** This is not always true in non-bipartite graphs.

**Idea of Proof.**

- Start with a matching $M$
- Search for augmenting paths
- In the process of searching we'll also produce a cover $C$
- There are only two possibilities:
  - Either we find an augmenting path
  - or $|C| = |M|$

**Proof Preamble.** Let $(A, B)$ be a bipartition of $G$. Since augmenting paths have odd length, an augmenting path (if it exists) must hav one end in $A$ and the other end in $B$. To search for augmenting paths, start at an unsaturated vertex in $A$.
Let $X_0$ be the set of unsaturated vertices in $A$.
Let $X$ be the set of reachable vertices in $A$; note $X_0 \subseteq X$.
Let $Y$ be the set of reachable vertices in $B$.
where reachable means reachable by an alternating path starting at a vertex in $X_0$.

Let $Y_0$ be the set of unsaturated vertices in $Y$.

If a vertex $u$ is reachable, let $P(u)$ denote an alternating path to $u$ starting at some vertex in $X_0$.

**Note.** If $u \in Y_0$, then $u$ is reachable and unsaturated which implies $P(u)$ is an augmenting path. If $Y_0 \neq \emptyset$ then we have an augmenting path.

**Note.** If $u \in X$, then the last edge in $P(u)$ must be in $M$.

**Lemma 4.6.** If $u \in X$ and $e = \{u, v\} \in E(G)$ then $v \in Y$.

*Proof.* Let $u \in X$, let $P(u)$ be as defined.

- **Case 1.** Suppose $v \in P(u)$, then $P(u) = \underbrace{x \ldots v}_{\text{alternating path from } x \text{ to } v} \ldots u$ where $x \in X_0$. This implies $v$ is reachable, and hence $v \in Y$.
- **Case 2.** Suppose $v \notin P(u)$. I claim $\{u, v\} \notin M$. Why? Suppose $\{u, v\} \in M$. The last edge in $P(u)$ must be in $M$ since $M$ has at most 1 edge incident with $u$, this edge must be $\{u, v\}$, but if $\{u, v\}$ is in $P(u)$ then $v \in P(u)$. Contradiction. Therefore $P(u)v$ is an alternating path. Therefore $v \in Y$.

□

**Lemma 4.7.** If $v \in Y$ and $e = \{u, v\} \in M$, then $u \in X$.

*Proof.* Claim: $P(v)u$ is an alternating path from a vertex in $X_0$ to $u$.
Why is it alternting? Since $P(v)$ is an alternating path and $v \in Y$, the last edge of $P(v) \notin M$. Following this by an edge in $M$ gives an alternating path.
Why is it a path? The only reason this might not be a path is if $u$ is already in $P(v)$. But since the edge before $u$ in $P(v)$ must be in $M$ and since there is only one edge in $M$ incident with $v$, this edge must be $\{u, v\}$.

$$P(v) = \cdots vu \cdots u$$

$v$ appears twice in $P(v)$ which is a contradiction. □

**Lemma 4.8.** $C = Y \cup (A \setminus X)$ is a cover.

*Proof.* By Lemma 4.6, every edge in $G$ joins either:

- a vertex in $X$ and a vertex in $Y$
- a vertex in $A \setminus X$ and a vertex in $Y$
- a vertex in $A \setminus X$ and a vertex in $B \setminus Y$

In any casem the edge is incident with a vertex of $C$, so $C$ is a cover. □

**Lemma 4.9.** $|C| = |M| + |Y_0|$.

*Proof.* Every matching edge is incident with a vertex in $A \setminus X$ or a vertex in $Y \setminus Y_0$, but (by Lemma 4.7), not both. Every vertex in $A \setminus X$ or $Y \setminus Y_0$ is saturated, so it is incident with a unique edge in $M$. Thus we have a bijective correspondance between $M$ and $(A \setminus X) \cup (Y \setminus Y_0) = C \setminus Y_0$. Therefore

$$|M| = |C \setminus Y_0| = |C| = |Y_0|$$

□

*Proof of König's Theorem.* Suppose $M$ is a maximum matching, then there is no augmenting path, so $Y_0 = \emptyset$ (if we had a verex $v \in Y_0$) then $P(v)$ would be an augmenting path. Therefore $|C| = |M|$ by Lemma 4.9. $\square$

**Maximum Matching Algorithm.** (finds a maximum matching and minimum cover in a bipartite graph)

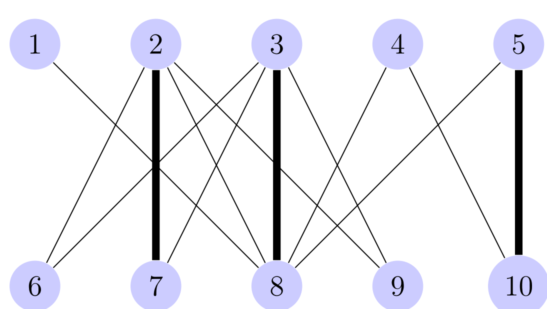
- Find a bipartition $(A, B)$
- Compute $X_0, X, Y, Y_0$
- If $Y_0 \neq \emptyset$ then we get an augmenting path, and repeat the algorithm with a bigger matching
- If $Y_0 = \emptyset$ then output: $M$ is a maximum matching and $C = Y \cup (A \setminus X)$ is a minimum cover.

**Details.** Use a variation of Breadth-First Search,

1. We begin constructing $F$ with vertices $V(F) = X_0 = \{x_1, \dots, x_k\}$ and no edges, and define $pr(x_i) = \emptyset$. Begin a queue with $x_1, \dots, x_k$.
2. While the queue is non-empty, let $U$ be the vertex at the head of the queue.
   - If $u \in A$ : while there is a non-matching edge $\{u, v\} \in E(G) \setminus M$ with $v \notin V(F)$ do:
     * Add vertex $v$ and edge $e$ to $F$.
     * Add $v$ to the queue.
     * Define $pr(v) = u$.
   - If $u \in B$ : if there is a vertex $v \in A$ such that $e = \{u, v\} \in M$ then
     * Add vertex $v$ and edge $e$ to $M$.
     * Add $v$ to the queue.
     * Define $pr(v) = u$.

Output: $X = V(F) \cap A$ and $Y = V(F) \cap B$ and $Y_0 =$ unsaturated vertices in $Y$.

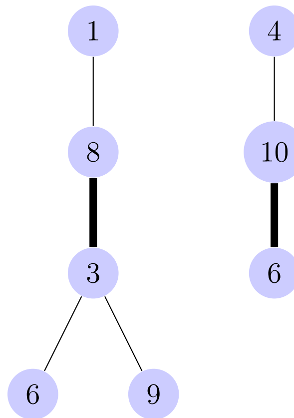
**Kevin Purhboo's Advice on Matchings**



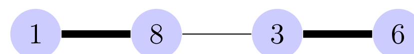

Find a maximum matching and minimum cover.

**Start with the matching shown**

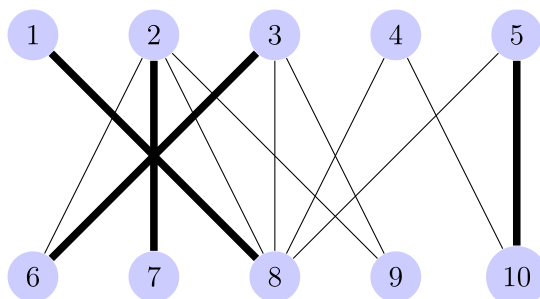
Follow the algorithm, starting by adding the unsaturated vertices in $A$ (the top row) to the queue. So we first add 1 and 4 to the queue. Then, follow the first thing in the queue (1) and add it's first (and only) adjacent vertex 8 to the queue, and also draw a line connecting it via a non-matching edge to 8. Then, go to the next item in the queue (4), and do the same thing, we already added 8 so now add 10. Follow this process (following the above algorithm) and we produce:



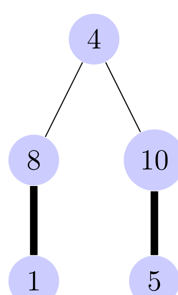

where $X_0 = \{1, 4\}$, $X = \{1, 3, 4, 5\}$, $Y = \{6, 8, 9, 10\}$ and $Y_0 = \{6, 9\}$. From the trees, I can see that 1836 is an augmenting path and we can use it to get a bigger matching:

change to



then the new matching is:




Now, we repeat the exact same thing with the new matching. The algorithm then produces:




where $X_0 = \{4\}$, $X = \{1, 4, 5\}$, $Y = \{8, 10\}$ and $Y_0 = \emptyset$. Since $Y_0$ is empty, **there are no augmenting paths**! We conclude that $M = \{\{1, 8\}, \{2, 7\}, \{3, 6\}, \{5, 10\}\}$ is a maximum matching, and $C = Y \cup (A \setminus X) = \{2, 3, 8, 10\}$.

### 4.12 Hall's Marriage Theorem

**Question.** Let $G$ be a bipartite graph with bipartition $(A, B)$; when does $G$ have a matching of size $|A|$? Equivalently, when does $G$ have a matching in which every vertex of $A$ is saturated?

- If $|B| < |A|$, not possible

- Every vertex in $A$ must have a neighbour
- What generalizes both of these statements?

If $D \subseteq A$, let $N(D) = \{v \in B \mid v \text{ is adjacent to some vertex in } D\}$.

**Example 4.24.** In the above matching algorithm example, $N(\{2,3\}) = \{6,7,8,9\}$. For a matching of size $|A|$ to exist, we must have $|N(D)| \geq |D|$ for every subset $D \subseteq A$.

**Theorem 4.28** (Hall's Marriage Theorem)**.** *$G$ has a matching of size $|A|$ if and only if we have $|N(D)| \geq |D|$ for every subset $D \subseteq A$.*

*Proof.* ($\Rightarrow$) If a matching $M$, saturating every vertex in $A$ exists, let $D = \{a_1, \dots, a_k\} \subseteq A$. Since $a_1, \dots, a_k$ are saturated, there exists edges $\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_k, b_k\}$ in $M$ where $b_1, \dots, b_k$ are distinct.
Note that $\{b_1, b_2, b_3, \dots, b_k\} \in N(D)$. Therefore $|D| = k$ and $|N(D)| \geq k$.

($\Leftarrow$) Suppose condition $|N(D)| \geq |D|$ holds for all $D \subseteq A$. Let $M$ be a maximum matching and let $C$ be a minimum cover. By König's Theorem there exists a cover $C$ of $G$ with $|C| \leq |A| - 1$. Suppose to the contrary that $|M| < |A|$. Let $D = A \setminus C$. Then, $N(D) \subseteq C \cap B$. So,

$$|N(D)| \leq |B \cap C| = |C| - |C \cap A| = |C| - (|A| - |D|) = |D| - (|A| - |C|) =\leq |D| - 1$$

So $|N(D)| \leq |D| - 1$; contradicting our assumption. Therefore the maximum size of a matching in $G$ is $|A|$. □

Note that if $G$ does **not** have a matching of size $|A|$ then a "bad" set (i.e., $D$ where $|N(D)| < |D|$) is given by $A \setminus C$, where $C$ is a minimum cover of $G$. In the bipartite matching algorithm, we know $C = Y \cup (A \setminus X)$ is a minimum cover. So $A \setminus C = X$.

**Corollary 4.6.** A bipartite graph $G$ with vertex classes $A$ and $B$ has a **perfect matching** (a matching that saturates every vertex of the graph) if and only if $|N(D)| \geq |D|$ for all $D \subseteq A$, and $|A| = |B|$.

**Corollary 4.7.** Let $G$ be a bipartite graph that is $k$-regular with $k \geq 1$. Then $G$ has a perfect matching.

*Proof.* To show $|A| = |B|$, note that $|E(G)| = k|A|$, but also $|E(G)| = k|B|$. So $k|A| = k|B| \implies |A| = |B|$. To verify Hall's Condition, let $D$ be an arbitrary subset of $A$. Let $E(D, N(D))$ denote the set of edges incident to $D$. Then $|E(D, N(D))| = k|D|$. Since each vertex in $N(D)$ has at most $k$ edges going to $D$. So, $|E(D, N(D))| \leq k|N(D)|$, then $k|N(D)| \geq k|D| \implies |N(D)| \geq |D|$. □