MATH 239 Assignment 3

- This assignment is due on Friday, October 5th, 2012, at 10 am in the drop boxes in St. Jerome's (section 1) or outside MC 4067 (the other two sections).
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.

1. (a) Let S be a set of configurations, and w be a weight function on S. By definition, the number of elements in S with weight exactly n is just $[x^n]\Phi_S(x)$. Prove that the number of elements in S with weight *at most* n is $[x^n]\frac{\Phi_S(x)}{1-x}$.

   (b) Determine the number of k-tuples $(a_1, ..., a_k)$ of positive integers that satisfy the inequality
$$a_1 + ... + a_k \leq n.$$

Solution:

(a) Let $\Phi_S(x) = \sum_{i\geq 0} a_i x^i$. Then

$$\begin{aligned}
[x^n]\frac{\Phi_S(x)}{1-x} &= [x^n]\left(\sum_{i\geq 0} a_i x^i\right)\left(\sum_{j\geq 0} x^j\right) \\
&= [x^n]\sum_{i\geq 0}\sum_{j\geq 0} a_i x^{i+j} \\
&= \sum_{i=0}^{n} a_i.
\end{aligned}$$

This is exactly the number of elements of S with weight n or less, as required.

(b) Let S be the set of all k-tuples $(a_1, ..., a_k)$ of positive integers. Let $w(a_1, ..., a_k) = a_1 + ... + a_k$ be the standard weight function for considering compositions. Then $\Phi_S(x) = \frac{x^k}{(1-x)^k}$ and $[x^n]\Phi_S(x)$ is the number elements of weight exactly n. By part a), the number of elements of S with weight at most n is

$$\begin{aligned}
[x^n]\frac{\Phi_S(x)}{1-x} &= [x^n]\left(\frac{x^k}{(1-x)^k}\right)\left(\frac{1}{1-x}\right) \\
&= [x^{n-k}](1-x)^{-(k+1)} \\
&= \binom{(n-k)+(k+1)-1}{(k+1)-1} \\
&= \binom{n}{k} \quad .
\end{aligned}$$

Since the number of elements $(a_1, ..., a_k)$ of S with weight at most n are precisely the solutions to $a_1 + ... + a_k \leq n$, we conclude that there are $\binom{n}{k}$ of them.

2. (a) Let A_k be the set of all compositions of n with k parts, and let B_{k-1} be the set of binary strings of length $n-1$ which have exactly $k-1$ elements equal to 0. Describe and justify a bijection between A_k and B_{k-1}.

   (b) Using part a), give an alternate proof that there are 2^{n-1} total compositions of n.

Solution:

(a) Let $a = (a_1, ..., a_k) \in A_k$ be a composition of n with k parts. Define a map $f(a) = b$, where b is a binary string defined by

$$b = 1^{a_1-1}01^{a_2-1}0\cdots 1^{a_{k-1}-1}01^{a_k-1}.$$

Since $a_1 + ... + a_k = n$, b is seen to have length $n-1$. As well, b clearly has $k-1$ elements equal to 0, and therefore $b \in B_{k-1}$. Now let $b' \in B_{k-1}$. We can write $b' = 1^{c_1}01^{c_2}0\cdots 1^{c_{k-1}}01^{c_k}$ where each c_i is a non-negative integer, by decomposing b' about its $k-1$ elements that are equal to 0. Now define $g(b') = a'$, where

$$a' = (c_1 + 1, c_2 + 1, ..., c_k + 1).$$

Since b' had length $n-1$, the weight of a' is n, and so $a' \in A_k$. Clearly the functions f and g are inverses of each other, so they are bijections, as required.

(b) Let A be the set of all compositions of n, and let B be the set of all binary strings of length $n-1$. Clearly $|B| = 2^{n-1}$. Now,

$$\begin{aligned}|A| &= \left|\bigcup_{k\geq 1} A_k\right| = \sum_{k\geq 1} |A_k| \\ &= \sum_{k\geq 1} |B_{k-1}| = \left|\bigcup_{k\geq 1} B_{k-1}\right| \\ &= |B| = 2^{n-1}\end{aligned}$$

where we have used the fact that the A_k for $k \geq 1$ partition A, and the B_k for $k \geq 0$ partition B. Thus there are 2^{n-1} total compositions of n, as required.

3. Show that the generating series for the set of all compositions which have an even number of parts, and each part congruent to 1 modulo 5, is equal to

$$\frac{1 - 2x^5 + x^{10}}{1 - x^2 - 2x^5 + x^{10}}.$$

Solution: Let $P = \{1, 6, 11, ...\}$ be the set of all positive integers congruent to 1 modulo 5. For each $i = 1, 2, ...$, let S_{2i} be the set of all $(2i)$-tuples $(a_1, ..., a_{2i})$ such that each a_j is in P. Finally let $S = \cup_{i\geq 0} S_{2i}$. Then

$$S = \bigcup_{i\geq 0} P^{2i}.$$

Now we compute the generating series:

$$\begin{aligned}
\Phi_S(x) &= \sum_{i\geq 0}(\Phi_P(x))^{2i} \\
&= \sum_{i\geq 0}(x^1 + x^6 + ...)^{2i} \\
&= \sum_{i\geq 0}\left(\frac{x}{1-x^5}\right)^{2i} \\
&= \frac{1}{1-\left(\frac{x}{1-x^5}\right)^2} \\
&= \frac{(1-x^5)^2}{(1-x^5)^2 - x^2}.
\end{aligned}$$

The result now follows by expanding the terms above.

4. Determine the exact number of compositions $(a_1, ..., a_k)$ of n which have k parts, and satisfy $a_i \geq i$ for $i = 1, ..., k$.

   (Give a closed-form expression which does not involve any summation.)

   Solution: Let S be the set of all k-tuples $(a_1, ..., a_k)$ that satisfy $a_i \geq i$ for $i = 1, ..., k$. Then

$$S = N_{\geq 1} \times N_{\geq 2} \times \cdots \times N_{\geq k}.$$

   Therefore, with the usual weight function for compositions we have

$$\begin{aligned}
\Phi_S(x) &= \Phi_{N_{\geq 1}}(x)\cdots\Phi_{N_{\geq k}}(x) \\
&= \left(\frac{x}{1-x}\right)\cdots\left(\frac{x^k}{1-x}\right) \\
&= x^{1+...+k}\frac{x^k}{(1-x)^k} \\
&= x^{\frac{1}{2}(k+1)(k)}(1-x)^{-k}.
\end{aligned}$$

   Note that $1 + ... + k = \frac{1}{2}(k+1)(k)$ is simply an arithmetic series. Lastly the number we are looking for is

$$\begin{aligned}
[x^n]\Phi_S(x) &= [x^n]x^{\frac{1}{2}(k+1)(k)}(1-x)^{-k} \\
&= [x^{n-\frac{1}{2}(k+1)(k)}](1-x)^{-k} \\
&= \binom{n-\frac{1}{2}(k+1)(k)+k-1}{k-1}.
\end{aligned}$$

5. Let $A = \{101, 00, 1011\}$ and let $B = \{101, 01, 0111\}$. Find the sets AB and BA and determine their generating series. Are the generating series for AB and BA the same? If not, try to explain why they wouldn't be.

Solution: We explicitly construct every element of AB and BA.

$$AB = \{101101, 10101, 1010111, 00101, 0001, 000111, 1011101, 10110111\}$$
$$BA = \{101101, 10100, 1011011, 01101, 0100, 011011, 0111101, 011100, 01111011\}.$$

And now by the definition of generating series with standard weights for binary strings,

$$\Phi_{AB}(x) = x^4 + 2x^5 + 2x^6 + 2x^7 + x^8,$$
$$\Phi_{BA}(x) = x^4 + 2x^5 + 3x^6 + 2x^7 + x^8.$$

We easily see that they are not the same. From examining the generating series, it appears that AB is "missing" a string of length 6. This happened because the string 101101 is actually constructed in AB in two different ways; once as (101)(101) and once as (1011)(01). This shows that AB is an ambiguous expression. On the other hand, BA is an unambiguous expression.