**DUE: NOON Friday 2 December 2011** in the drop boxes opposite the Math Tutorial Centre MC 4067 or next to the St. Jerome's library for the St. Jerome's section.

1. For each of the graphs shown, determine whether it is planar. If the graph is planar, exhibit a planar embedding. If the graph is not planar, exhibit a subdivision of $K_5$ or $K_{3,3}$.

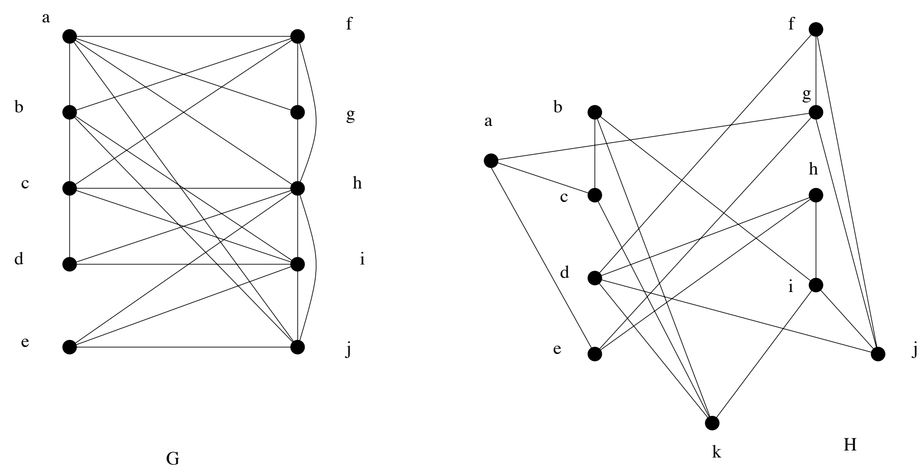


Figure 1: Are they planar?

**SOLUTION.** The graph $G$ is planar, as shown by the planar drawing in Figure 2. The graph $H$ is not planar, as is shown by the subdivision of $K_{3,3}$ with branch vertices $\{a, i, d\}$ in one vertex class and $\{b, h, f\}$ in the other. The subdivision shown also contains the vertices $g, c, e, j$ and $k$.

2. (a) A graph $G$ is said to be *d-degenerate* if there is an ordering $v_1, v_2, \ldots, v_p$ of its vertex set such that each $v_i$ has at most $d$ neighbours in $\{v_{i+1}, \ldots, v_p\}$. Prove that every $d$-degenerate graph is $(d+1)$-colourable.

**SOLUTION.** We prove the statement by induction on $p = |V(G)|$.

If $G$ has at most $d+1$ vertices then we can colour every vertex a different colour to get a $(d+1)$-colouring of $G$.

Assume $G$ has at least $d+2$ vertices and the statement is true for all $d$-degenerate graphs with fewer vertices than $G$.

Let $G$ be given. Let $v_1, v_2, \ldots, v_p$ be an ordering of its vertex set such that each $v_i$ has at most $d$ neighbours in $\{v_{i+1}, \ldots, v_p\}$. Then $v_2, \ldots, v_p$ is an ordering which shows that $G - v_1$ is also $d$-degenerate. By the induction hypothesis we know $G - v_1$ has a $(d+1)$-colouring $f$. Since the degree of $v_1$ in $G$ is at most $d$, there is a colour $c$ in $\{1, 2, \ldots, d+1\}$ that is not used by $f$ on the neighbourhood of $v_1$. Therefore $f$ can be extended to a $(d+1)$-colouring of $G$ by giving $v_1$ colour $c$. Thus $G$ is $(d+1)$-colourable.

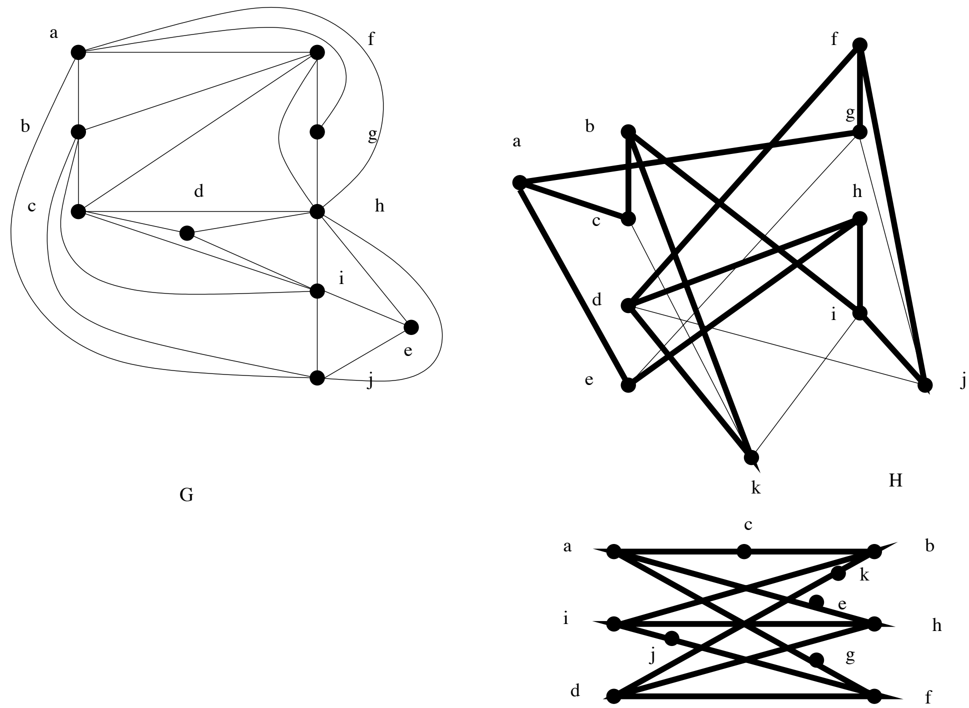


Figure 2: Yes and No.

(b) Let $G$ be a planar graph of girth at least 4. Prove that $G$ is 3-degenerate. (Hint: look at Theorem 6.4.6.)

**SOLUTION.** We prove the statement by induction on $p = |V(G)|$.

If $G$ has at most 4 vertices then every vertex has degree at most 3, and so $G$ is 3-degenerate.

Assume $G$ has at least 5 vertices and the statement is true for all planar graphs of girth at least 4 with fewer vertices than $G$.

Let $G$ be given. We know that any planar graph with girth at least 4 has at most $2p-4$ edges, where $p = |V(G)|$. If every vertex of $G$ had degree at least 4 then $q = \frac{1}{2}\sum_{v \in V(G)} deg(v) \geq 2p$ which is a contradiction. Thus $G$ has a vertex $v_1$ of degree at most 3. Then $G - v_1$ is a planar graph with $p-1$ vertices and girth at least 4, which by the induction hypothesis is 3-degenerate. So there is an ordering $v_2, \ldots, v_p$ of the vertices of $G - v_1$ such that each $v_i$ has at most 3 neighbours in $\{v_{i+1}, \ldots, v_p\}$. Thus since $deg(v_1) \leq 3$ we get that $v_1, \ldots, v_p$ is an ordering that shows $G$ is 3-degenerate.

(c) Conclude (without using the 4-colour theorem) that every planar graph $G$ of girth at least 4 is 4-colourable.

**SOLUTION.** Since $G$ is 3-degenerate, by (a) it is 4-colourable.

3. Let $G$ be a connected planar graph embedded on the sphere, in which every face is bounded by a cycle. The *truncation* $T(G)$ of $G$ is obtained by replacing each vertex

$v$ by a face bounded by a cycle of length $deg(v)$. Intuitively, we can think of forming $T(G)$ by "slicing off" each "corner" of $G$. Shown is the graph $K_4$ (the tetrahedron) and the truncation $T(K_4)$.

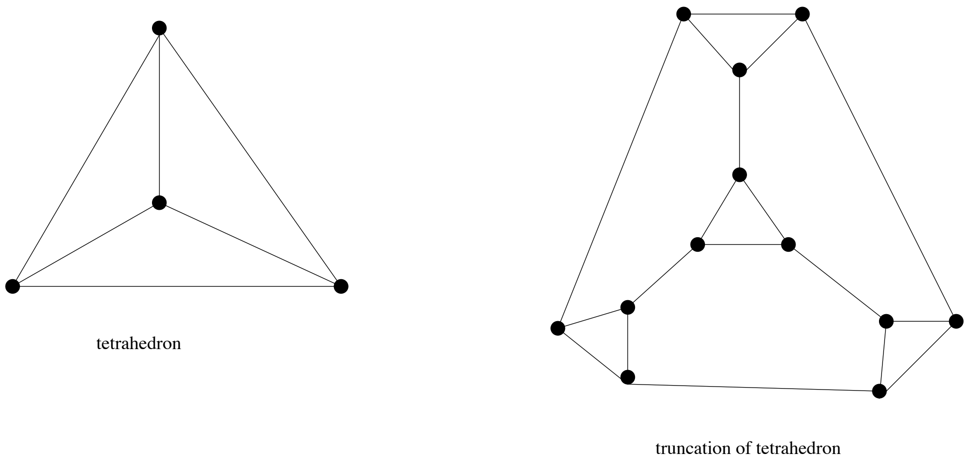


Figure 3: Tetrahedron and truncation

(a) Draw the planar dual of the graph $T(K_4)$.

**SOLUTION.** Shown in Figure 4.

(b) Let $G$ be a connected planar graph with $p$ vertices, embedded on the sphere, in which every vertex has degree $d$ and every face is bounded by a cycle. Find the number of vertices, edges and faces of $T(G)$ in terms of $d$ and $p$.

**SOLUTION.** Let $q = |E(G)|$ and $s = |F(G)|$, then we know $q = \frac{1}{2}\sum_{v \in V(G)} deg(v) = \frac{pd}{2}$. By Euler's Formula we know $s = 2 - p + q = 2 - p + \frac{pd}{2}$.

For each vertex of $G$ there are $d$ vertices of $T(G)$. So the number of vertices of $T(G)$ is $dp$.

From the definition we see that $T(G)$ has two types of faces, Type 1 which correspond to vertices of $G$ and Type 2 which correspond to faces of $G$. The number of Type 1 faces is $p$ and the number of Type 2 faces is $s$. Thus the number of faces of $T(G)$ is $p + (2 - p + \frac{pd}{2}) = \frac{pd}{2} + 2$.

Using Euler's Formula we get that the number of edges is $dp + (\frac{pd}{2} + 2) - 2 = \frac{3pd}{2}$.

(c) What are the face degrees of the truncation of the icosahedron? (See p 167 for a picture of the icosahedron, Figure (d). Unfortunately this is not a correct drawing - the top vertex should be adjacent to the one directly below it, and the rightmost vertex should be adjacent to the vertex north-west of it!) How many faces are there of each degree?

**SOLUTION.** The Type 1 faces have degree 5 and the Type 2 faces have degree 2(3)=6. Since the icosahedron has 12 vertices (all of degree 5) and 20 faces (all of degree 3), the truncation has 12 faces of degree 5 and 20 faces of degree 6.

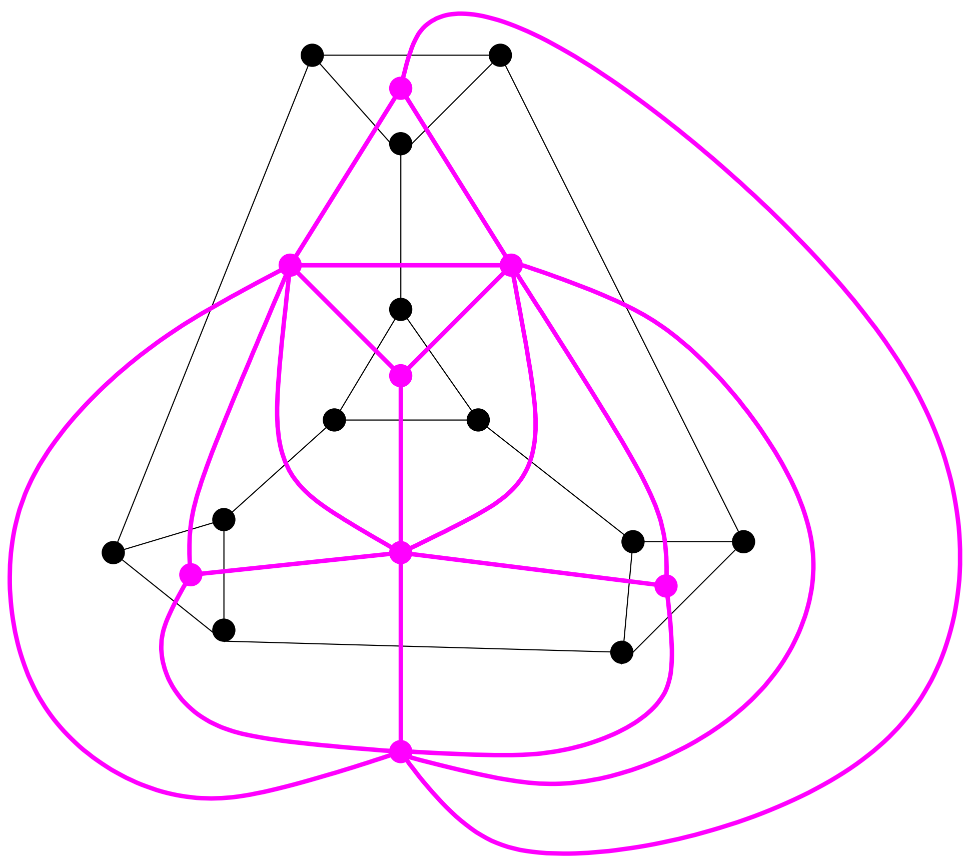

Figure 4: The dual is in pink (gray in BW printer).

(d) The truncation of the icosahedron is more commonly known as what everyday object?

**SOLUTION.** The standard soccer ball is the truncation of the icosahedron.

4. Let $G$ be a graph with $p$ vertices, with minimum degree $d$. Suppose $d \leq \frac{p}{2}$. Prove that $G$ has a matching of size at least $d$.

**SOLUTION.** Let $M$ be a matching of maximum size in $G$, and suppose on the contrary that $|M| \leq d-1$. Then the set $V(M)$ of vertices saturated by $M$ has size at most $2d-2 \leq p-2$. Let $u$ and $v$ be vertices that are not saturated by $M$. Then by maximality of $M$ we know that all neighbours of $u$ are in $V(M)$, and similarly all neighbours of $v$ are in $V(M)$. Since $|M| < d$ and there are at least $2d$ edges with one end in $\{u, v\}$ and the other end in $V(M)$, by the pigeonhole principle there must be some edge $xy$ of $M$ such that there are 3 edges between $\{x, y\}$ and $\{u, v\}$. We may assume without loss of generality that $xu$, $xv$ and $yv$ are all edges of $G$. But then $M \setminus \{xy\} \cup \{xu, yv\}$ is a bigger matching in $G$, contradicting the maximality of $M$. Thus we must have $|M| \geq d$.