MATH 239 Tutorial 4

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PROBLEMS

1. For each of the two following sets S, give an unambiguous decomposition for S and find its generating series with respect to length.

   (a) S is the set of binary strings where all blocks of 0s must have even length and all blocks of 1 must have odd length.

   (b) S is the set of binary strings where every 0 has an even number of 1s after it.

2. Let P be the set of palindromic binary strings.

   (a) Give a recursive decomposition for P.

   (b) Find the generating series of P with respect to length.

   (c) Determine how many binary strings of length n are palindromes.

3. For each of the following sets A, prove either that A^* is unambiguous or that A^* is ambiguous.

   (a) $A = \{\epsilon, 00, 01, 10, 11\}$

   (b) $A = \{11, 10, 01110\}$

   (c) $A = \{0, 10, 11, 01\}$

SOLUTIONS

1. (a) Starting with the block decomposition

$$\{0\}^* \left(\{1\}\{1\}^*\{0\}\{0\}^*\right)^* \{1\}^*$$

we restrict it so that every block of 0s has even length and every block of 1s has odd length:

$$\{00\}^* \left(\{1\}\{11\}^*\{00\}\{00\}^*\right)^* \left(\epsilon \cup \{1\}\{11\}^*\right)$$

Since it's a restriction of the block decomposition, which is known to be unambiguous, it is also unambiguous. We can therefore use the Product Lemma, *-Lemma, and Sum Lemma to find the generating series:

$$\begin{aligned}
\Phi_{\{00\}^*}(x) &= \frac{1}{1-x^2} \\
\Phi_{\{1\}\{11\}^*\{00\}\{00\}^*}(x) &= x\left(\frac{1}{1-x^2}\right)x^2\left(\frac{1}{1-x^2}\right) = x\left(\frac{x}{1-x^2}\right)^2 \\
\Phi_{\epsilon\cup\{1\}\{11\}^*}(x) &= x^0 + x\left(\frac{1}{1-x^2}\right) = 1 + \frac{x}{1-x^2} = \frac{1+x-x^2}{1-x^2}
\end{aligned}$$

Combining these three parts and applying the *-Lemma, we get:

$$\begin{aligned}
\Phi_S(x) &= \left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x(\frac{x}{1-x^2})^2}\right)\left(\frac{1+x-x^2}{1-x^2}\right) \\
&= \frac{1+x-x^2}{(1-x^2)^2-x^3} \\
\Phi_S(x) &= \frac{1+x-x^2}{1-2x^2-x^3+x^4}
\end{aligned}$$

(b) We'll show two different ways to obtain an umabiguous decomposition for the set of strings where every 0 has an even number of 1s after it.

First, we could start with the zero decomposition

$$\{1\}^*\left(\{0\}\{1\}^*\right)^*$$

and restrict it so that every block of 1s after a zero has even length:

$$\{1\}^*\left(\{0\}\{11\}^*\right)^*$$

Because it's a restriction of the unambiguous zero-decomposition, it is also unambiguous. We can therefore use the Product Lemma and *-Lemma to find the generating series:

$$\begin{aligned}
\Phi_{\{1\}^*}(x) &= \frac{1}{1-x} \\
\Phi_{\{0\}\{11\}^*}(x) &= x\left(\frac{1}{1-x^2}\right) = \frac{x}{1-x^2}
\end{aligned}$$

Combining these two parts and applying the *-Lemma, we get:

$$\begin{aligned}
\Phi_S(x) &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-\frac{x}{1-x^2}}\right) \\
&= \frac{1}{1-x-\frac{x(1-x)}{(1-x)(1+x)}} \\
&= \frac{1}{1-x-\frac{x}{1+x}} \\
\Phi_S(x) &= \frac{1+x}{1-x-x^2}
\end{aligned}$$

For the second method of obtaining an unambiguous decomposition, we think of every block of 1s occurring after a block of 0s as having even, positive length. If there is an even number of 1s after every 0, then there is an even number of 1s after every block of 0s. Starting with the block decomposition

$$\{1\}^* \left(\{0\}\{0\}^*\{1\}\{1\}^*\right)^* \{0\}^*$$

we restrict it so that every block of 1s after a block of zeros has even length:

$$\{1\}^* \left(\{0\}\{0\}^*\{11\}\{11\}^*\right)^* \{0\}^*$$

Since it's a restriction of the unambiguous block decomposition, it is also unambiguous and we can therefore apply the Product Lemma and *-Lemma to get the generating series:

$$\begin{aligned}
\Phi_{\{1\}^*}(x) &= \frac{1}{1-x} \\
\Phi_{\{0\}\{0\}^*\{11\}\{11\}^*}(x) &= x\left(\frac{1}{1-x}\right)x^2\left(\frac{1}{1-x^2}\right) = \frac{x^3}{(1-x)(1-x^2)} \\
\Phi_{\{0\}^*}(x) &= \frac{1}{1-x}
\end{aligned}$$

Combining these three results and applying the *-Lemma, we get:

$$\begin{aligned}
\Phi_S(x) &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-\frac{x^3}{(1-x)(1-x^2)}}\right)\left(\frac{1}{1-x}\right) \\
&= \frac{1}{(1-x)^2 - \frac{x^3(1-x)^2}{(1-x)(1-x)(1+x)}} \\
&= \frac{1}{1-2x+x^2-\frac{x^3}{1+x}} \\
&= \frac{1+x}{(1-2x+x^2)(1+x)-x^3} \\
&= \frac{1+x}{1-2x+x^2+x-2x^2+x^3-x^3} \\
\Phi_S(x) &= \frac{1+x}{1-x-x^2}
\end{aligned}$$

2. (a) Palindromic strings are symmetric about the middle bit (if the string has odd length) or between the middle two bits (if the string has even length). A recursive decomposition for the set of palindromic strings P is:

$$P = \{\epsilon\} \cup \{0\} \cup \{1\} \cup \{0\}P\{0\} \cup \{1\}P\{1\}$$

(b) Before finding the generating series, we should check that the decomposition is unambiguous. Let σ be a palindromic string in P. We'll prove by induction on the length of σ that it can be generated in only one way from P.

If the length of σ is 0, then it must be the empty string, ϵ. If the length of σ is 1, then either it is 0 or it is 1. Each of these possibilities can happen in only one way.

Suppose any palindromic string of length strictly less than k for some $k \geq 2$ is created in only one way from this recursive decomposition of P. Let σ now be a binary string of length k. There are two possiblities: either the first bit is 0, or the first bit is 1. If the first bit is 0, then σ must be $0\sigma'0$ for some palindromic σ' of length $k-2 \geq 0$. By the induction hypothesis, σ' is unambiguous. The case where the first bit is 1 is similar.

Since this recursive decomposition is unambiguous, we can apply the Sum Lemma to find the generating series for P:

$$\begin{aligned}
\Phi_P(x) &= x^0 + x + x + x^2\Phi_P(x) + x^2\Phi_P(x) \\
\Phi_P(x) &= 1 + 2x + 2x^2\Phi_P(x) \\
\Phi_P(x) - 2x^2\Phi_P(x) &= 1 + 2x \\
\Phi_P(x) &= \frac{1+2x}{1-2x^2}
\end{aligned}$$

(c) To determine how many palindromic binary strings of length n there are, we find $[x^n]\Phi_P(x)$:

$$\begin{aligned}
[x^n]\Phi_P(x) &= [x^n]\frac{1+2x}{1-2x^2} \\
&= [x^n]\left(\frac{1}{1-2x^2} + 2x\frac{1}{1-2x^2}\right) \\
&= [x^n]\left(\Sigma_{j\geq 0}(2x^2)^j + 2x(\Sigma_{j\geq 0}(2x^2)^j)\right) \\
&= [x^n]\left(\Sigma_{j\geq 0}2^j x^{2j} + \Sigma_{j\geq 0}2^{j+1}x^{2j+1}\right) \\
[x^n]\Phi_P(x) &= \begin{cases} 2^{n/2} & \text{if } n \text{ is even} \\ 2^{(n+1)/2} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

3. (a) The set A contains the empty string ϵ so A^* can't be unambiguous because there are infinitely many ways to write it: $\epsilon, \epsilon\epsilon, \epsilon\epsilon\epsilon, \ldots$.

(b) Let σ be a string in A^*. We'll show that it is unambiguous by induction on the length of σ. If σ has length 0, then there is only one way to write it — as ϵ. Suppose that every string in A^* of length strictly less than k for some $k \geq 1$ is generated unambiguously. Consider any string σ of length k. Since $k \geq 1$, σ must start with an element of A, say $\sigma = a_1 \cdots a_m$ for some $m \geq 1$. If a_1's first bit is 0, then a_1 must be 01110. Otherwise, look at the second bit of a_1. If it is 0, then a_1 must be 10. If it is 1, then a_1 must be 11. In each of these three cases, we can write $\sigma = a_1\sigma'$ for some σ' in A^* with length strictly less than k. By the induction hypothesis, it can be expressed uniquely as an element of A^*. Therefore, by induction, A^* is unambiguous.

(c) The set A is not unambiguous because we can write 010 as (0)(10) or (01)(0).