MATH 239 - Assignment 4

DUE: 10am Friday Feb. 8th in the drop boxes opposite the Math Tutorial Centre MC 4067.

Exercise 1 (20pts)**.**

The C&O Zoo has a fire sale and the following animals are purchasable at the following unit prices:

- lion (7 gold coins),
- kangaroo (5 gold coins),
- moose (4 gold coins),
- rabbit (1 gold coin).

Customers are not allowed to buy more than four animals of the same type, with the exception of rabbits (customers can buy as many rabbits as they wish).

(a) Find a formal power series $\Phi(x)$ such that $[x^n]\Phi(x)$ is the number of ways you can spend exactly n gold coins buying animals from the zoo.

(b) Since lions find rabbits very tasty the zoo has a new policy: no customer is allowed to purchase both rabbits and lions. Under this new policy find a formal power series $\Phi'(x)$ such that $[x^n]\Phi'(x)$ is the number of ways you can spend exactly n gold coins buying animals from the zoo.

Solution:

Give 10pts for (a), 10pts for (b). For (a): setting up sets $U_1, \ldots, U_4$ 2pts, choosing weights $w_1, \ldots, w_4$ 2pts, writing generating function for each set 4pts (if no simplification 2/4), invoking the product lemma 2pts. For (b): correct partition 4pts, writing generating function for each set 3pts, invoking sum lemma 3pts.

(a) Represent each purchase of animals as a 4-tuple (a_1, a_2, a_3, a_4) where a_1 is the number of lions purchased, a_2 is the number of kangaroos purchased, a_3 is the number of moose purchased, and a_4 is the number of rabbits purchased. Let $\mathcal{S} = \{(a_1, a_2, a_3, a_4) : 0 \leq a_i \text{ integer, for } i = 1, 2, 3, 4 \text{ and } a_i \leq 4 \text{ for } i = 1, 2, 3\}$. Define the weight $w[(a_1, a_2, a_3, a_4)]$ as the cost of the animals namely, $7a_1 + 5a_2 + 4a_3 + a_4$. Then $\Phi(x)$ is the power series for set S and weights w. Define $U_1 = U_2 = U_3 = U_4 = \{0, 1, 2, 3\}$ and $U_4 = \{0, 1, 2, 3, \ldots\}$ with weights, $w_1(a) = 7a$, $w_2(a) = 5a$, $w_3(a) = 4a$, $w_4(a) = a$. For $i = 1, 2, 3, 4$ denote the generating function for U_i and w_i by $\Phi_i(x)$, then

$$\Phi_1(x) = 1 + x^7 + x^{14} + x^{21} + x^{28} = 1 + (x^7) + (x^7)^2 + (x^7)^3 + (x^7)^4 = \frac{1 - (x^7)^5}{1 - x^7} = \frac{1 - x^{35}}{1 - x^7},$$

$$\Phi_2(x) = 1 + x^5 + x^{10} + x^{15} + x^{20} = \frac{1 - x^{25}}{1 - x^5},$$

$$\Phi_3(x) = 1 + x^4 + x^8 + x^{12} + x^{16} = \frac{1 - x^{20}}{1 - x^4},$$

$$\Phi_4(x) = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1 - x}.$$

Since $S = U_1 \times U_2 \times U_3 \times U_4$ and $w[(a_1, a_2, a_3, a_4)] = w(a_1) + w(a_2) + w(a_3) + w(a_4)$, we can apply the product lemma, thus,

$$\Phi(x) = \Phi_1(x)\Phi_2(x)\Phi_3(x)\Phi_4(x) = \frac{(1 - x^{35})(1 - x^{25})(1 - x^{20})}{(1 - x^7)(1 - x^5)(1 - x^4)(1 - x)}.$$

(b) Denote by A the set of tuples $(a_1, a_2, a_3, a_4) \in \mathcal{S}$ with $a_1 = 0$ (no lion purchased) and $a_4 \geq 1$ (at least one rabbit purchased). Denote by B the set of tuples $(a_1, a_2, a_3, a_4) \in \mathcal{S}$ with $a_1 \geq 1$ (at least one lion purchased) and $a_4 = 0$ (no rabbit purchased). Denote by C the set of tuples $(a_1, a_2, a_3, a_4) \in \mathcal{S}$ with $a_1 = a_4 = 0$ (no lion or rabbit purchased). Then $\Phi(x)$ is the power series for set $A \cup B \cup C$ and weights w. Proceeding similarly as in (a) (replacing $\Phi_1(x)$ by 1 and $\Phi_4(x)$ by $\frac{x}{1-x}$), we obtain the generating function for A and w,

$$\Phi_A(x) = \frac{(1-x^{25})(1-x^{20})x}{(1-x^5)(1-x^4)(1-x)}.$$

Proceeding similarly as in (a) (replacing $\Phi_4(x)$ by 1 and $\Phi_1(x)$ by $x^7\frac{1-x^{28}}{1-x^7}$) we obtain the generating function for B and w,

$$\Phi_B(x) = \frac{x^7(1-x^{28})(1-x^{25})(1-x^{20})}{(1-x^7)(1-x^5)(1-x^4)}.$$

Similarly the generating function for C and w is,

$$\Phi_C(x) = \frac{(1-x^{25})(1-x^{20})}{(1-x^5)(1-x^4)}.$$

Since A, B, C are disjoint, by the sum lemma we have that,

$$\Phi'(x) = \Phi_A(x) + \Phi_B(x) + \Phi_C(x) = \frac{(1-x^{25})(1-x^{20})}{(1-x^5)(1-x^4)}\left[\frac{x}{(1-x)} + \frac{x^7(1-x^{28})}{(1-x^7)} + 1\right].$$

Note, as an alternate proof you can find the generating function $\Phi''(x)$ for the number of ways you can spend exactly n gold coins buying animals from the zoo where at least one lion and at least one rabbit is purchased. Then $\Phi'(x) = \Phi(x) - \Phi''(x)$ using the sum lemma.

Exercise 2 (10 pts)**.**
Your friend Tony Montana, proposes the following game: flip a coin n times, if tail appears 5 times in a row you loose, otherwise you win. Define a power series $\Phi(x)$, such that $[x^n]\Phi(x)$ is the number of winning sequences of n coin flips.

Solution:

Representation as string 3pts, express string 4pts, generating function 3pts.

We can encode a sequence of n coin flips as a binary string of length n where 0's represent tail, and 1's represent head. Let $\mathcal{S}$ be the set of all binary strings that have at most 4 consecutive 0's. Then the number of winning sequences of n coin flips is equal to the number of strings in $\mathcal{S}$ of length n. Hence, it suffices to find the generating function for $\mathcal{S}$. Let us first find an unambiguous representation for $\mathcal{S}$,

$$\mathcal{S} = A^*B \quad \text{where} \quad A = \{1, 01, 001, 0001, 00001\} \quad \text{and} \quad B = \{e, 0, 00, 000, 0000\}.$$

Thus

$$\Phi_A(x) = x + x^2 + x^3 + x^4 + x^5 = x\frac{1-x^5}{1-x} \quad \text{and} \quad \Phi_B(x) = 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}.$$

It follows that,

$$\Phi_{A^*}(x) = \frac{1}{1-\Phi_A(x)} = \frac{1}{1-\frac{x-x^6}{1-x}} = \frac{1-x}{1-2x+x^6}.$$

Finally, by the product lemma we obtain,

$$\Phi(x) = \Phi_{A^*}(x)\Phi_B(x) = \frac{1-x^5}{1-2x+x^6}.$$

Exercise 3 (20pts)**.**
For each of the following sets of binary strings, determine an unambiguous expression which generates every string in that set.

(a) The set of binary strings where the length of each block is divisible by 3.
(b) The set of binary strings which do not contain 01111 as a substring.
(c) The set of binary strings where each block of 1's must be followed by a block of 0's of length at least 3.
(d) The set of binary strings which do contain 1111000 as a substring.

Solution:

Give 5pts for each part. No justification is required.

(a) $\{000, 111\}^*$. **(b)** $\{1\}^*(\{0\}\{e, 1, 11, 111\})^*$. **(c)** $\{0\}^*(\{1\}\{1\}^*\{000\}\{0\}^*)^*$. (Note that there is no $\{1\}^*$ at the end, since it is either an empty string, or a block of 1s which must have another block of 0's following it.) **(d)** $\{0,1\}^* \setminus \{0\}^*(\{1, 11, 111\}\{0\}\{0\}^*)^*\{1111\}\{1\}^*\{0, 00\})^*\{1\}^*$. (The idea is that we start with strings that do not contain 1111000, which means that any block of 1's with length 3 or less can be followed by any number of 0's. However, any block of 1's with length 4 or more can be followed by at most 2 0's.)

Exercise 4 (10pts)**.**

(a) Let $s_1, \ldots, s_k$ be distinct binary strings all of the same length $m \geq 1$. Prove that $\{s_1, \ldots, s_k\}^*$ is an unambiguous expression.
(b) Prove that for any choice of positive integers m and n where $m \neq n$, there exist binary strings s and t of lengths m and n respectively where $\{s, t\}^*$ is an ambiguous expression.

Solution:

Give 5pts for each parts.

(a) We prove this using strong induction on the length n of a string $s \in \mathcal{S}$. When $n = 0$, $s = e$, and this can only be generated once from $\{s_1, \ldots, s_k\}$. Suppose s has length $n > 0$. Since $s_1, \ldots, s_k$ are distinct and each string has same length (say m), the first m bits of s is uniquely generated, say it is s_i for some i. Then we can decompose s as $s = s_i t$ where $t \in \{s_1, \ldots, s_k\}^*$. Since t has shorter length than s, by induction, there is only one way to generate t from $\mathcal{S}$. Therefore, $\mathcal{S}$ can be generated only once from $\mathcal{S}$, and s is unambiguous.
(b) Let s be the string of m 1's, and let t be the string of n 1's. Then $s^n = t^m$ which is the string of mn 1's, and both $s^n, t^m \in \{s, t\}^*$. Hence this is ambiguous.

Exercise 5 (10pts)**.** A binary string is a palindrome if it reads the same forwards and backwards. Examples of palindromes include 01100110, 11011, 1. Let $\mathcal{P}$ be the set of all binary strings that are palindromes. Determine a recursive definition of $\mathcal{P}$, and use it to find the generating series for $\mathcal{P}$.

Solution:

Writing expression 6pts, obtaining generating function 4pts.

For a palindrome with length at least 2, the first and last bits must be the same, either they are both 0 or both 1, and the remaining string is another palindrome. Palindromes of length 1 or 0 are $\{e, 1, 0\}$. This gives us the recursion

$$\mathcal{P} = \{0\}P\{0\} \cup \{1\}\mathcal{P}\{1\} \cup \{e, 1, 0\}.$$

This gives us the equation

$$\phi(x) = x^2\Phi(x) + x^2\Phi(x) + 1 + 2x.$$

Solving for $\Phi(x)$ and we get,

$$\Phi(x) = \frac{1 + 2x}{1 - 2x^2}.$$