MATH 239 - Tutorial 4

Feb. 6, 2013

1. Show that $\{01, 011, 101\}^*$ is an ambiguous expression.

   Solution. $01101 = 011/01 = 01/101$

2. For each of the following sets of binary strings, determine an unambiguous expression which generates every string in that set.

   (a) Strings where each block of 1s is followed by a block of exactly two 0s.
   (b) Strings without three consecutive 0s.
   (c) Strings where each block of 0s has even length.
   (d) Strings that contain the substring 01.
   (e) Strings that do not contain the substring 010.

   Solution. Note: expressions in (a), (b), (c) are unambiguous, since (by course note pg 36) decomposing a string after each occurrence of a given digit, or after each block of a given digit, is unambiguous.

   (a) $\{0\}^*(\{1\}\{1\}^*\{00\})^*$ (Decompose after each block of 0s.)
   (b) $\{1\}^*(\{0, 00\}\{1\}\{1\}^*)^*\{\varepsilon, 0, 00\}$ (Decompose after each block of 1s) or $\{\varepsilon, 0, 00\}\{1, 10, 100\}^*$ (Decompose after each occurrence of 1.)
   (c) $\{00\}^*(\{1\}\{1\}^*\{00\}\{00\}^*)^*\{1\}^*$ (Decompose after each block of 0s)
   (d) $\{0, 1\}^* \setminus \{1\}^*\{0\}^*$ (All strings except for those that do not contain 01. Both parts of the expression are obviously unambiguous, and the second part is a subset of the first part.)
   (e) $\{1\}^*(\{0\}\{0\}^*\{11\}\{1\}^*)^*(\{0\}^* \cup \{0\}^*\{01\})$ (The first part A of the expression follows the decomposition after each block of 1s. The second part B is unambiguous too, as it is a union of strings ending in 0 and strings ending in 1. Note that A contains only (1) strings ending in at least two 1s, or (2) only consisting of 1s, so no non-empty suffix of a string in A is a prefix of a string in B, as (1) no strings in B contain two consecutive 1s, and (2) all non-empty strings in B start with 0. Suffix/prefix here means substring at the end/beginning of a string.) Note that the strings in this set are the strings that do not contain a block of a single 1, except for possibly at the beginning or end, which is intuitively what the expression describes.

3. Let $S = \{0\}^*(\{1\}\{11\}^*\{00\}\{00\}^* \cup \{11\}\{11\}^*\{0\}\{00\}^*)^*$.

   (a) Describe in words the strings that belong to S.
   (b) Find the generating series for S.

   Solution

   (a) Inside the brackets - the first set contains strings composed of an odd number of 1's followed by an even number of 0's. The second set contains strings with an even number (≥ 2) of 1's followed by an odd number of 0's. So the union of these sets is the set of binary strings beginning with 1 in which each block of 1's is followed by a block of 0's with a different parity. Since S may begin either with a block of 0's of any length or without a block of 0's, S is the set of binary strings in which every block of 1's is followed by a block of 0's of different parity.

(b) Let the weight of a string be equal to its length. Denote $A = \{1\}\{11\}^*\{00\}\{00\}$, $B = \{11\}\{11\}^*\{0\}\{00\}^*$. Then

$$\begin{aligned}
\Phi_{\{0\}}(x) &= \Phi_{\{1\}}(x) = x \\
\Phi_{\{11\}}(x) &= x^2 \\
\Phi_{\{00\}}(x) &= x^2 \\
\Phi_A(x) &= \Phi_{\{1\}}(x) \cdot \Phi_{\{11\}^*}(x) \cdot \sum_{k\geq 1} \left[\Phi_{\{00\}}(x)\right]^k \text{ (Sum Rule and Product Rule, part (a))} \\
&= x(1-\Phi_{\{11\}}(x))^{-1} \sum_{k\geq 1} x^{2k} \text{ (Product Rule part (b))} \\
&= x(\frac{1}{1-x^2})(\frac{x^2}{1-x^2}) \\
&= \frac{x^3}{(1-x^2)^2}, \\
\Phi_B(x) &= \frac{x^3}{(1-x^2)^2}, \text{ (by symmetry; same algebra as previous)} \\
\Phi_{A\cup B} &= \Phi_A(x) + \Phi_B(x) \text{(Sum rule, since } A \cap B = \emptyset), \\
&= \frac{2x^3}{(1-x^2)^2} \\
\Phi_S(x) &= \Phi_{\{0\}^*}(x) \cdot \Phi_{(A\cup B)^*}(x) \text{(Product Rule)} \\
&= (1-x)^{-1}(1-\frac{2x^3}{(1-x^2)^2})^{-1} \\
&= \frac{1}{1-x}\left(\frac{(1-x^2)^2-2x^3}{(1-x^2)^2}\right)^{-1} \\
&= \frac{(1-x^2)^2}{(1-x)((1-x^2)^2-2x^3)}.
\end{aligned}$$

This gives the recurrence relation.

4. Let k be a fixed positive integer. Let S be the set of binary strings with no k consecutive 1's, and let b_n be the number of strings in S of length n. Prove that for $n \geq k$,

$$b_n = \sum_{i=1}^{k} b_{n-i}.$$

Solution. Let $M = \bigcup_{i=0}^{k-1} \{1\}^i$. Then the decomposition is

$$M(\{0\}M)^*.$$

Generating series is

$$(1+x+\cdots+x^{k-1})\frac{1}{1-x(1+x+\cdots+x^{k-1})} = \frac{1+x+\cdots+x^{k-1}}{1-x-x^2-\cdots-x^k}.$$

Denote $\Phi_M(x)$ by $\sum_{i\geq 0} b_i x^i$. Then

$$\begin{aligned}
&(1-x-\cdots-x^k)\sum_{i\geq 0} b_i x^i = 1+x+\cdots+x^{k-1}, \text{ so}\\
&[x^n](1-x-\cdots-x^k)\sum_{i\geq 0} b_i x^i = [x^n](1+x+\cdots+x^{k-1})\\
&[x^n](1-x-\cdots-x^k)\sum_{i\geq 0} b_i x^i = 0, \text{ since } n\geq k\\
&[x^n](\sum_{i\geq 0} b_i x^i - x\sum_{i\geq 0} b_i x^i - \cdots - x^k\sum_{i\geq 0} b_i x^i) = 0\\
&[x^n](\sum_{i\geq 0} b_i x^i - \sum_{i\geq 1} b_{i-1} x^i - \cdots - \sum_{i\geq k} b_{i-k} x^i) = 0 \text{ (reindex)}
\end{aligned}$$

Since $n \geq k$,

$$[x^n]\sum_{i\geq j} b_{i-j} x^i = b_{i-j} \tag{1}$$

for all $j = 1, 2, .., k$. So

$$[x^n](\sum_{i\geq 0} b_i x^i - \sum_{i\geq 1} b_{i-1} x^i - \cdots - \sum_{i\geq k} b_{i-k} x^i) = b_n - b_{n-1} - ... - b_{n-k} = 0$$

.

Rearranging gives

$$\begin{aligned}
b_n &= b_{n-1} - b_{n-1} - \cdots - b_{n-k}\\
&= \sum_{i=1}^{n} b_{n-i},
\end{aligned}$$

as required.

5. Define $a_k = 1\underbrace{00...0}_{k}$ to be the string starting at 1 and followed by k 0's (e.g. $a_0 = 1, a_1 = 10, a_2 = 100$).

   Denote $S = \{a_0, a_i, ..., a_n\}$. Show that the set $\{a_0, a_1, a_2, ..., a_n\}^*$ is an unambiguous expression for any non-negative integer n.

   Solution. Let $x \in \{a_0, a_1, a_2, ..., a_n\}^*$. Lets prove by induction on the length of x, that x has a unique decomposition $x = c_1c_2...c_s$, where $c_i \in \{a_0, a_1, a_2, ..., a_n\}$ for each $i = 1, 2, ..., s$. If $x = \epsilon$, then x has a unique decomposition (the empty one). Suppose that $x = c_1'c_2'...c_r'$, where $c_j' \in \{a_0, a_1, a_2, ..., a_n\}$ for all $j = 1, ..., n$. Thus $c_1c_2...c_s = c_1'c_2'...c_r'$, hence $c_1 = c_1'$ (because the first block of 0's should be of the same length in LHS than in RHS). Therefore $y := c_2...c_s = c_2'...c_r'$, and by induction, the decomposition of y is unique, then $r = s$ and $c_2 = c_2'$, ..., $c_r = c_r'$. Therefore the decomposition is unique for x, because $c_1 = c_1', c_2 = c_2', ..., c_r = c_s'$.

   Another Solution. Denote $S = \{a_0, a_1, a_2, ..., a_n\}$. Let $x \in S^* = \{a_0, a_1, a_2, ..., a_n\}^*$. We will use induction to prove that the decomposition of x into elements of S is unambiguous. (i.e. we'll prove that x can only be generated one way by concatenating elements of S.)

   <u>Base case of inductive proof:</u> If $x = \epsilon$, then x has a unique decomposition (the empty decomposition).

   <u>Inductive Hypothesis:</u> Suppose $x \neq \epsilon$, and that the decomposition of s is unique for all strings $s \in S$ that are shorter than x.

   <u>Inductive Step:</u> Suppose the first block of 0's in x has length i. Then the first element of S in the decomposition of x must be a_i. (No other element of S contains a sequence of exactly i zeros. Each

sequence of consecutive 0's in x must be contained in a single element of S in any decomposition of x.) So we can write $x = a_i y$, where $y \in S$. Then y has a unique decomposition into elements of S, by the inductive hypothesis. Because a_i and y are both uniquely determined, the decomposition of x is also unique. Thus the result holds.