MATH 239 Assignment 10

DUE: 10am Friday April 5 in the drop boxes opposite the Math Tutorial Centre MC 4067.

1. Prove the converse to Lemma 8.1.1 by following these steps:

   (a) Let G be a graph and let M be a matching in G. Let M^* be a maximum matching in G. Prove that every component of the graph H with vertex set $V(G)$ and edge set $M \cup M^*$ is a path or an even cycle.

   (b) Prove that if M is not a maximum matching in G then some path component of H has more edges of M^* than edges of M.

   (c) Prove that if M is not a maximum matching of G then there exists an M-augmenting path in G.

2. Recall that a vertex colouring of a graph is an assignment of colours to the vertices such that adjacent vertices are assigned distinct colours. A face colouring of a graph is an assignment of colours to the faces of a planar embedding such that any two faces that share an edge in the boundary are assigned distinct colours. A planar graph is *triangulated* if there exists a planar embedding where every face boundary is a cycle of length three. A graph is *cubic* if every vertex has degree three.

   Prove that the following statements are equivalent,

   (i) every planar graph has a 4-colouring of the vertices,

   (ii) every triangulated planar graph has a 4-colouring of the vertices,

   (iii) every planar embedding of a cubic graph has a 4-colouring of the faces.

3. A planar graph is maximal if it is planar but adding any edge to it makes it non-planar. Show that a planar graph is maximal if and only if has exactly $3p - 6$ edges where p is the number of vertices.

4. This problem involves finding specific matchings and coverings.

   (a) Show that the n-cube O_n has a perfect matching by exhibiting one. Please show that the set of edges that you find is a matching.

   (b) Find a cover C of O_n with $|C| = 2^{n-1}$.

5. Let M be a matching in a graph G. The end-point set $P(M) \subseteq V(G)$ associated to M is the set of all end-points of edges of M. A matching is said to be maximal if it is not properly contained in a larger matching. Show that a matching is maximal if and only if $P(M)$ is a cover. Note: A maximum matching is maximal, but not the converse.