UNIVERSITY OF WATERLOO
FINAL EXAMINATION
SPRING TERM 2009

Last Name: ____________________

First Name: ____________________

Signature: ____________________

Id.#: ____________________

Indicate your instructor:

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David Jao (Section 2)	
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Course Number	MATH 239
Course Title	Introduction to Combinatorics
Date of Exam	August 6, 2009
Time Period	9:00–11:30 am
Number of Exam Pages (including this cover sheet)	14
Exam Type	Closed Book
Additional Materials Allowed	NONE
Additional Instructions	Write your answers in the space provided. If the space is insufficient, use the back of the page and indicate clearly where your solution continues. Show all your work.

Problem	Value	Mark Awarded	Problem	Value	Mark Awarded
1	6		6	11	
2	5		7	12	
3	15		8	6	
4	6		9	7	
5	12		Total	80	

1. **[6 marks]** Determine the following coefficients as summations, where m and n are nonnegative integers.

(a)

$$[x^n]\frac{(1+2x)(1+x^2)^m}{(1-x)^3}$$

(b)

$$[x^n]\frac{1}{1-x-x^m+x^{m+1}}$$

2. **[5 marks]** Alice and Bob have stolen a total of \$10000 from the bank, in the form of fifty \$20 bills, forty \$50 bills, and seventy \$100 bills. How many ways are there to split the loot evenly between them? (For the purposes of this problem, any two bills of different denominations are considered different, and any two bills of the same denomination are considered indistinguishable.) Express your answer as a coefficient of a generating function.

3. (a) **[2 marks]** Describe in words the set of binary strings S that is represented by the following decomposition:

$$S = \{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^* \setminus \{11\}\{11\}^*\{0\}\{00\}^*)^*\{1\}^*.$$

(b) **[1 mark]** Explain why elements of S are uniquely created in the decomposition in (a).

(c) **[4 marks]** Determine the generating function with respect to length for S.

(d) [**4 marks**] Let S' be the set of all binary strings in which no block of 0's is followed by a block of 1's of greater length. For example 1100100011100100 is in S' but 000111011 is not. Prove that the generating function for S' with respect to length is

$$\frac{1+x}{1-x-2x^2+x^3}.$$

(e) [**4 marks**] Find a recurrence relation with initial conditions that uniquely specifies the sequence of coefficients of the generating function in (d).

4. **[6 marks]** Let G be a connected graph. Suppose that the number of vertices in G of degree one is n_1, the number of vertices of degree three is n_3, and that there are no vertices of any other degree in G. Prove that G is a tree if and only if the total number of vertices of G is $p = 2n_3 + 2$.

5. (a) [**4 marks**] Find a breadth-first search tree in the graph G shown, rooted at the vertex 1. When considering the vertices adjacent to the vertex being examined, take them in increasing order of their labels.

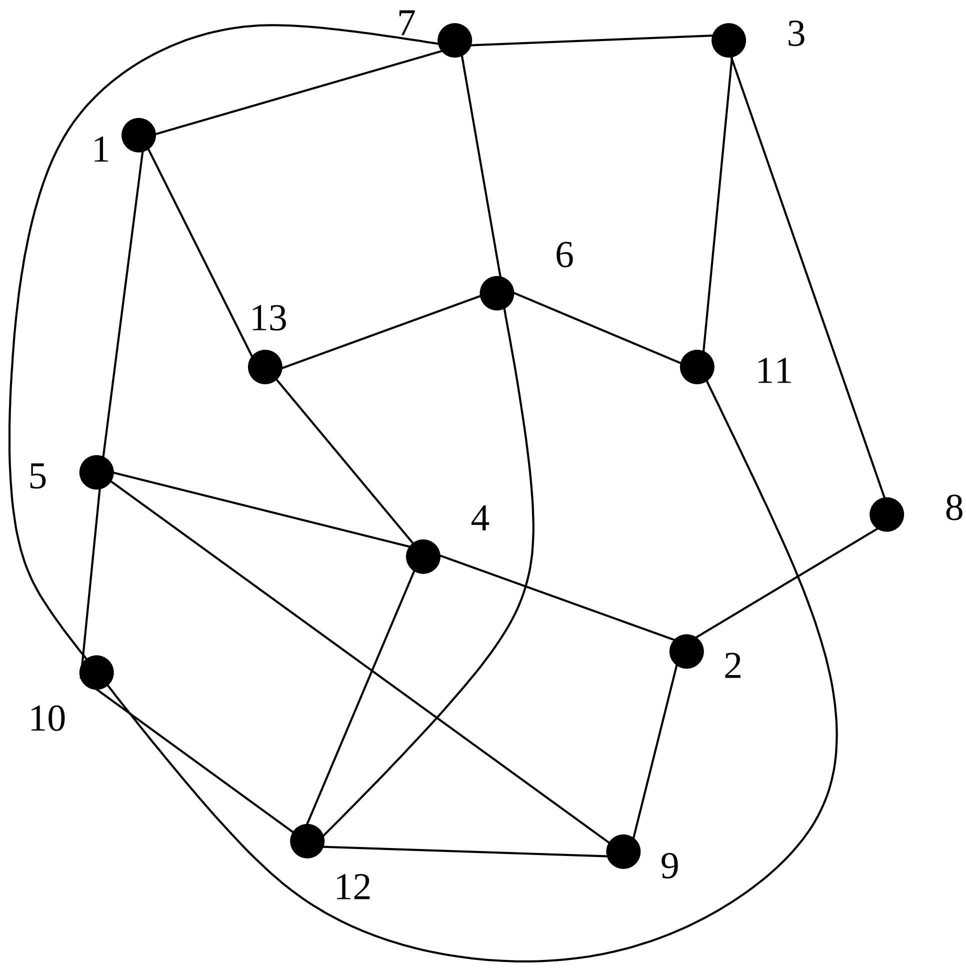


Figure 1: The graph G

(b) **[2 marks]** Determine the set of vertices whose distance to vertex 1 is exactly two.

(c) **[2 marks]** Determine whether or not G is bipartite, and prove your assertion.

(d) **[4 marks]** Let H be a connected graph. Suppose H has a breadth first search tree T where the number of edges of H that join vertices at the same level in T is exactly one. Prove that H is 3-colourable.

6. The matching M in the graph G shown is indicated by the bold edges.

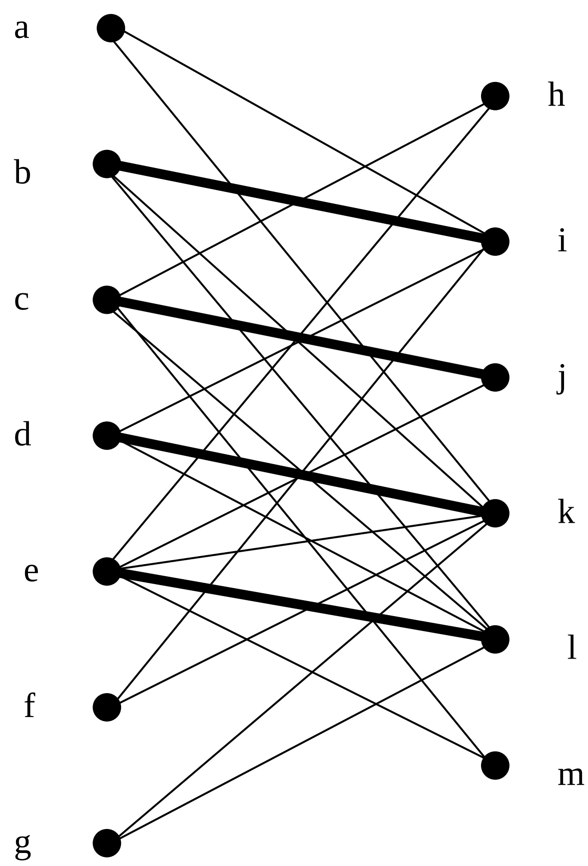


Figure 2: The graph G with matching M

(a) **[2 marks]** Find an M-augmenting path P in G starting at the vertex a.

(b) **[2 marks]** Let M^* be the matching of size $|M| + 1$ obtained by switching on the M-augmenting path P (so $M^* = M \Delta E(P) = M \setminus (E(P) \cap M) \cup (E(P) \setminus M)$). List the edges of the matching M^*.

(c) **[4 marks]** Let X_0 be the M^*-unsaturated vertices in A. Let X be the set of vertices in A that are reachable from a vertex in X_0 by an M^*-alternating path, and let Y be the set of vertices in B that are reachable from a vertex in X_0 by an M^*-alternating path. Find X_0, X and Y. (Another copy of G is shown here to assist you.)

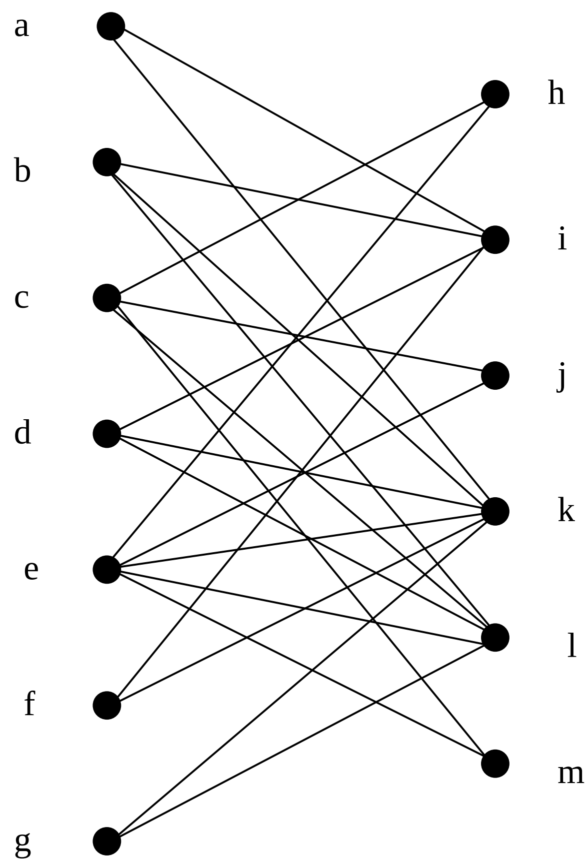


Figure 3: The graph G

(d) **[3 marks]** Prove that M^* is a maximum matching in G by finding a cover C with $|C| = |M^*|$. List the elements of C.

7. (a) **[2 marks]** State Euler's Formula for connected planar graphs.

(b) **[4 marks]** Let G be a connected 5-regular embedded planar graph, in which every face has the same degree. Find the number of faces of G.

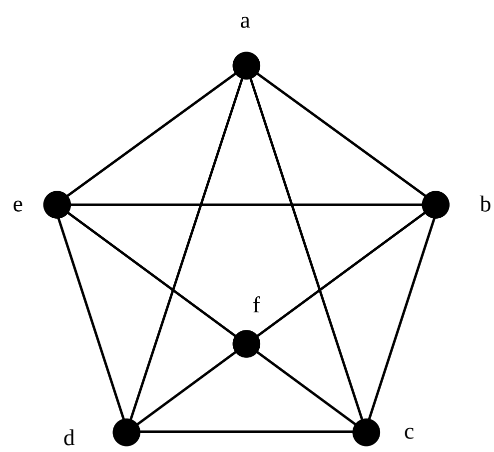
(c) **[6 marks]** For each of the following descriptions of graphs, draw an example of such a graph and briefly explain why it satisfies the description.

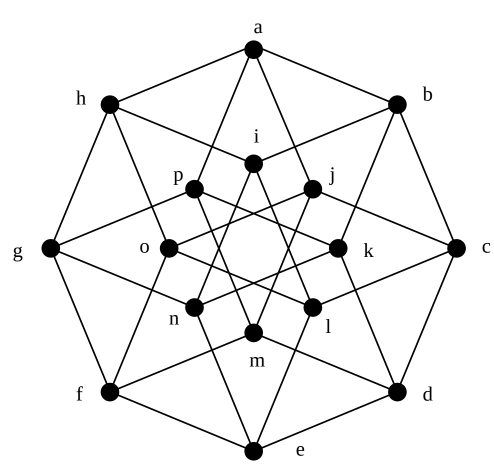
i. A 3-regular planar graph that is not 3-colourable.

ii. A 3-regular planar graph that is 3-colourable but not 2-colourable.

iii. A 3-regular planar graph that is 2-colourable.

8. **[6 marks]** Determine if the following graphs are planar or not planar. In each case, exhibit either a planar embedding of the graph, or a subgraph that is a subdivision of K_5 or $K_{3,3}$.





9. Let G be a graph, and suppose P is a path in G of greatest possible length.

   (a) **[2 marks]** Prove that the endvertices x and y of P satisfy $N(x) \subseteq V(P)$ and $N(y) \subseteq V(P)$.

   (b) **[5 marks]** Prove that if every vertex of G has degree at least k then P has length at least k.