MATH 239 Assignment 10

DUE: 10am Friday April 5 in the drop boxes opposite the Math Tutorial Centre MC 4067.

1. Prove the converse to Lemma 8.1.1 by following these steps:

   (a) Let G be a graph and let M be a matching in G. Let M^* be a maximum matching in G. Prove that every component of the graph H with vertex set $V(G)$ and edge set $M \cup M^*$ is a path or an even cycle.

   (b) Prove that if M is not a maximum matching in G then some path component of H has more edges of M^* than edges of M.

   (c) Prove that if M is not a maximum matching of G then there exists an M-augmenting path in G.

SOLUTION.

1. Since M and M^* are both matchings, each vertex v of G is incident to at most one edge of M and at most one edge of M^*. Therefore every vertex v has degree at most 2 in H. Therefore each component of H is a path or a cycle. To see that each cycle component C has even length, note that since every vertex of C has degree 2 in H, every edge is incident to an edge of M and an edge of M^*. So the edges of C must alternate between an edge of M and an edge of M^*, since they are both matchings. Thus C has even length.

2. Since M is not a maximum matching we know $|M| < |M^*|$. Every cycle component of H has the same number of M-edges as M^*-edges by (b). Thus since some component of H must have more M^*-edges than M-edges, there must be a path component with this property.

3. Let P be a path component of H with more M^*-edges than M-edges. Then since P cannot be a single edge in $M \cap M^*$ (which would have the same number of M-edges as M^*-edges), if P has length greater than 1 then the edges of P must alternate between an edge of M^* and an edge of M. Thus the only possibilities are

   - P has length one and is an edge of M^* but not an edge of M,
   - P has odd length at least 3, starts and ends with an M^*-edge, and has one more M^*-edge than M-edge.

   Note that since P is a component of H, the endpoints of P are M-exposed. But then P is an M-augmenting path (in which each edge not in M is in M^*).

2. Recall that a vertex colouring of a graph is an assignment of colours to the vertices such that adjacent vertices are assigned distinct colours. A face colouring of a graph is an assignment of colours to the faces of a planar embedding such that any two faces that share an edge in the boundary are assigned distinct colours. A planar graph is *triangulated* if there exists a planar embedding where every face boundary is a cycle of length three. A graph is *cubic* if every vertex has degree three.

   Prove that the following statements are equivalent,

(i) every planar graph has a 4-colouring of the vertices,

(ii) every triangulated planar graph has a 4-colouring of the vertices,

(iii) every planar embedding of a cubic graph has a 4-colouring of the faces.

SOLUTION. (i) implies (ii) as every triangulated planar graph is a planar graph. Let us show the following claim:

Claim: If G is a planar graph then there exists a triangulated planar H obtained from G by adding edges.

Proof. Consider G with a planar embedding. If the boundary walk of any face F_1 has length more than three, add an edge between two vertices of the boundary walk, to divide the face into F into a faces F_1, F_2 of length greater or equal to three. Repeat the process until the graph is triangulated.

Suppose (ii) holds. Let G be a planar graph with a planar embedding. Let H be a triangulated planar graph obtained from G by adding edges. Since (ii) holds, there exists a 4-colouring of the vertices of H. But the same colouring of the vertices is a valid colouring of G (as adjacent vertices in G are adjacent in H).

Suppose that (ii) holds. Let G be a cubic graph with a planar embedding. Let G^* be the planar dual of G. Since every vertex of G has degree 3 every boundary walk of G^* is a cycle of length 3, i.e. G^* is triangulated. It follows from (ii) that there exists a 4-colouring of the vertices of $G*$. Thus there exists a 4-colouring of the faces of G (each vertex F of G is assigned the colour of face F of G^*). Suppose that (iii) holds. Let G be a triangulated planar graph. Let G^* be the planar dual of G. Since every face of G is a cycle of length 3, G^* is cubic. It follows from (iii) that there exists a 4-colouring of the faces of G^*. Thus there exists a 4-colouring of the vertices of G.

3. A planar graph is maximal if it is planar but adding any edge to it makes it non-planar. Show that a planar graph is maximal if and only if has exactly $3p-6$ edges where p is the number of vertices.

SOLUTION. We proved that if G is a planar graph then $q \leq 3p-6$ where p denotes the number of vertices and q denotes the number of edges. It particular, if $q = 3p-6$ and G is planar, then it is maximal. Conversely, suppose G is a maximal planar graph. Then G is triangulated (see the claim in the previous exercise). Then $deg(F) = 3$ for every face of G, and

$$2q = \sum_{F \text{ face of } G} deg(F) = 3f = 3(2-p+q),$$

where f denotes the number of faces; and the last equation follows from Euler's formula (which states that $p - q + f = 2$). Thus $2q = 6 - 3p + 3q$ or equivalently, $q = 3p-6$ as required.

4. This problem involves finding specific matchings and coverings.

   (a) Show that the n-cube O_n has a perfect matching by exhibiting one. Please show that the set of edges that you find is a matching.

(b) Find a cover C of O_n with $|C| = 2^{n-1}$.

SOLUTION.

(a) There are many ways to do this problem, but here is one. Recall that the vertices of O_n correspond to binary strings of length n while edges correspond to pairs of binary strings that differ in exactly one place. We let M be the set of edges of the form uv where

$$u = 0a_2a_3 \dots a_n, \quad v = 1a_2a_3 \dots a_n$$

for $a_i \in \{0, 1\}$. That is, we consider the edges where the binary strings differ in the first bit. To show that M is a perfect matching, we must show that each vertex v belongs in exactly one edge in M. Let

$$w = b_1b_2 \dots b_n.$$

Then w belongs to the edge uv for

$$u = 0b_2b_3 \dots b_n, \quad v = 1b_2b_3 \dots b_n$$

(b) Again, there are many ways to do this problem. This way comes from the bipartite nature of O_n. Let C be the cover consisting of all vertices $v = a_1a_2 \dots a_n$ with $a_1 + a_2 + \cdots + a_n$ equal to an even number. This sum is called the binary digit sum. Because one end-point of an edge in O_n has an even binary digit sum, C is a cover.

5. Let M be a matching in a graph G. The end-point set $P(M) \subseteq V(G)$ associated to M is the set of all end-points of edges of M. A matching is said to be maximal if it is not properly contained in a larger matching. Show that a matching is maximal if and only if $P(M)$ is a cover. Note: A maximum matching is maximal, but not the converse.

SOLUTION. Let M be a matching of G. An edge $e = uv$ can be added to a matching M to obtain a larger matching if and only if neither u nor v is incident to an edge in M. Therefore, if $e = uv$ cannot be added to M, then either u or v is incident to an edge in M.

Note that if M is maximal, then one cannot add an edge to M and have it remain a matching. So M is maximal if and only if no edge can be added to M. This occurs if and only for every edge e not in M, one end-point is incident to an edge in M. This, in turn, occurs if and only if $P(M)$ is a cover of G.