DUE: NOON Friday 18 November 2011 in the drop boxes opposite the Math Tutorial Centre MC 4067 or next to the St. Jerome's library for the St. Jerome's section.

1. Let T be a tree having at least two vertices of degree 2, at least three vertices of degree 3, exactly two vertices of degree 5, and at least three vertices of degree 6. Show that T has at least 33 vertices and find an example of such a tree with precisely 33 vertices.

   SOLUTION. Let n_i be the number of vertices of T that have degree i. We have seen that $n_1 = 2 + \sum_{i\geq 3}(i-2)n_i$ and that $|V(T)| = \sum_{i\geq 1} n_i$. We are told that $n_2 \geq 2$, $n_3 \geq 3$, $n_5 = 2$ and $n_6 \geq 3$. The first equation implies

$$n_1 = 2 + \sum_{i\geq 3}(i-2)n_i \geq 2 + (3-2)3 + (5-2)2 + (6-2)3 = 23\,.$$

   Now the second equation implies

$$|V(T)| = \sum_{i\geq 1} n_i \geq 23 + 2 + 3 + 2 + 3 = 33\,.$$

   Variation: Substitute the first equation into the second to get $|V(T)| = 2 + \sum_{i\geq 2}(i-1)n_i$. In this case, we get $|V(T)| \geq 2 + 2 + 2(3) + 4(2) + 5(3) = 33$.

   There are many different examples. One is illustrated in the figure.

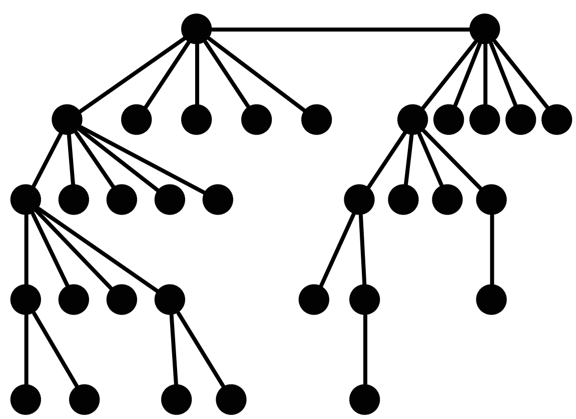

Figure 1: A tree with specified degrees and number of vertices.

2. Let $p \geq 2$ be an integer. A sequence $d_1, d_2, \ldots, d_p$ of non-negative integers is *graphic* if there is a graph with p vertices whose degrees are $d_1, d_2, \ldots, d_p$.

   Show that if $d_1, d_2, \ldots, d_p$ is graphic and n is any integer with $0 \leq n \leq p$, then $n, d_1+1, d_2+1, \cdots, d_n+1, d_{n+1}, d_{n+2}, \cdots, d_p$ is also graphic.

   SOLUTION. Because $d_1, d_2, \ldots, d_k$ is graphic, there is a graph G having p vertices $v_1, v_2, \ldots, v_p$ so that v_i has degree d_i. We create a new graph G' by adding a new vertex w to G and join w by edges to each of $v_1, v_2, \ldots, v_n$. Because $n \leq p$, this is possible.

We note that w has degree n, for $i = 1, 2, \ldots, n$, v_i has degree $d_i + 1$, and, for $i = n+1, n+2, \ldots, p$, v_i has degree d_i. Thus, G' has degree sequence $n, d_1 + 1, d_2 + 1, \ldots, d_n + 1, d_{n+1}, d_{n+2}, \ldots, d_p$, showing this sequence is graphic.

3. (a) Suppose G is a bipartite graph with p vertices. Show that G has at most $\frac{p^2}{4}$ edges.

SOLUTION. Let (X, Y) be a bipartition of G. Since every edge has one end in X and one end in Y, $|E(G)| \leq |X|\,|Y|$. Moreover, $|X| + |Y| = p$. This now reduces to a problem in calculus: maximize xy under the condition that $x + y = p$.

Solving for y, we get $y = p - x$, so $xy = x(p - x) = px - x^2$. The derivative is $p - 2x$, so the only critical point is $x = p/2$. Since the second derivative is $-2 < 0$, this is a local maximum. In fact it is a global maximum, since the function increases from $x = -\infty$ to $x = p/2$ and then decreases as x goes to $+\infty$. It follows that the maximum is $(p/2)(p - p/2) = p^2/4$, as required.

(b) Suppose G is a bipartite graph with p vertices. Show that if G has at least $\frac{(p-1)^2+1}{4}$ edges, then G is connected.

SOLUTION. Suppose by way of contradiction that G is not connected. Let H be one component of G (say with fewest vertices) and let K be the union of the remaining components of G. We suppose H has h vertices and k has k vertices, so that $h + k = p$.

From (a), H has at most $h^2/4$ edges, while k has at most $k^2/4$ edges. Since every edge of G is either in H or in K, but not both, $|E(G)| = |E(H)| + |E(K)| \leq (h^2/4) + (k^2/4)$. The choice of H implies $k \geq h$ and, since $h + k = p$, we have $p - 1 \geq k \geq p/2$.

The number of edges in G is at most $(h^2 + k^2)/4$ and, using $h = p - k$, this is $(2k^2 - 2kp + p^2)/4$. Again, we use calculus to find the maximum. The derivative is $k - p/2$, so the only critical point is $p/2$. The second derivative is 1, so $k = p/2$ is a minimum. The maximum must occur at the other end point, namely $k = p - 1$.

At this maximum, G has at most $(1^2 + (p-1)^2)/4$ edges. Since $(2k^2 - 2kp + p^2)/4$ is a strictly increasing function of k, for $p \geq 2$, G has fewer than $(1^2 + (p-1)^2)/4$ edges, the desired contradiction. If $p = 1$, then G in fact has only $|E(K)|$ edges (as $h = 1$ implies H has no edges). Since K has at most $(p-1)^2/4$ edges, we again have the desired contradiction.

4. (a) In the graph illustrated below, find a breadth-first search tree beginning at the vertex 1. As new vertices come in to the tree, bring them in least label first. (To indicate the progress of the algorithm, make a table with the vertices at each level on separate lines, and indicate the predecessor of each vertex.)

SOLUTION. NOTE: because of the requirement to use least label first, this is the only correct solution.

Predecessor in parentheses.

1(*)
5(1) 7(1) 8(1) 10(1)
4(5) 11(5) 3(7) 2(8) 6(10)
9(4) 12(6)

The tree consists of the thick edges in the figure; the arrows point to the predecessor.

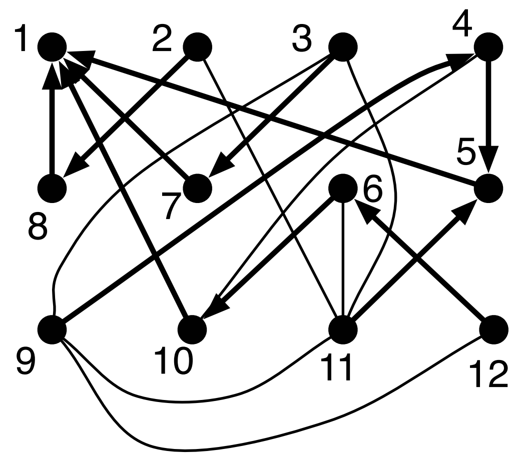

Figure 2: The breadth-first search tree

(b) Determine whether the graph is bipartite.

SOLUTION. Not bipartite, as 11 and 2 are at the same level and adjacent.