UNIVERSITY OF WATERLOO
FINAL EXAMINATION
FALL TERM 2010

Last Name: ______________________

First Name: ______________________

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Indicate your instructor:

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Course Number	MATH 239
Course Title	Introduction to Combinatorics
Date of Exam	December 17, 2010
Time Period	4:00pm – 6:30pm
Number of Exam Pages (including this cover sheet)	14
Exam Type	Closed Book
Additional Materials Allowed	NONE
Additional Instructions	Write your answers in the space provided. If the space is insufficient, use the back of the page and indicate clearly where your solution continues. Show all your work.

Problem	Value	Mark Awarded	Problem	Value	Mark Awarded
1	10		6	9	
2	12		7	11	
3	9		8	13	
4	10		9	8	
5	8		Total	90	

1. [10 marks]

   Find the generating function with respect to length for the set of nonempty binary strings in which every block of 0's has odd length and the first digit in the string is 1. Express your answer as a rational function.

2. [12 marks]

(a) [3 marks]
Let $\Phi(x) = \sum_{n\geq 0} a_n x^n$ be a formal power series. Set $b_n = \sum_{i=0}^{n} a_i$. Prove that

$$\sum_{n\geq 0} b_n x^n = \frac{\Phi(x)}{1-x}.$$

(b) [4 marks]
The Fibonacci numbers f_n are given by the recurrence

$$\begin{aligned} f_0 &= 0, \\ f_1 &= 1, \\ f_n &= f_{n-1} + f_{n-2}, \quad \text{for all } n \geq 2. \end{aligned}$$

Prove that

$$\sum_{n\geq 0} f_n x^n = \frac{x}{1-x-x^2}.$$

(c) [2 marks] Let $b_n = \sum_{i=0}^{n} f_i$. Compute

$$\sum_{n\geq 0} b_n x^n$$

as a rational function. You may use the result of parts (a) and (b), even if you did not solve parts (a) or (b).

(d) [3 marks] Let $c_n = f_{n+2} - 1$. Compute

$$\sum_{n\geq 0} c_n x^n$$

as a rational function, and show that $b_n = c_n$ for all n, where b_n is as in part (c).

3. [9 marks]

Let t be a positive integer. Prove that

$$\sum_{i=0}^{t} 2^{2i}\binom{2t}{2i} = 1 + \sum_{i=0}^{t-1} 2^{2i+1}\binom{2t}{2i+1}.$$

Hint: consider $(1+y)^n$ for suitable y and n.

4. [10 marks]

(a) Find a connected 4-regular graph with a bridge, or prove that no such graph exists.

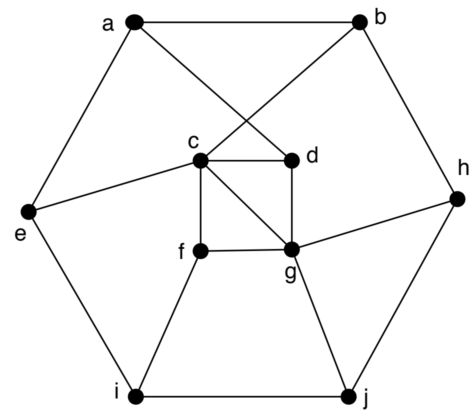
(b) Find a connected 3-regular graph with a bridge, or prove that no such graph exists.

5. [8 marks]

Suppose a connected graph G has a cycle C of length n. Prove that in any breadth-first search tree of G, any two vertices in C are at most $\lfloor n/2 \rfloor$ levels apart.

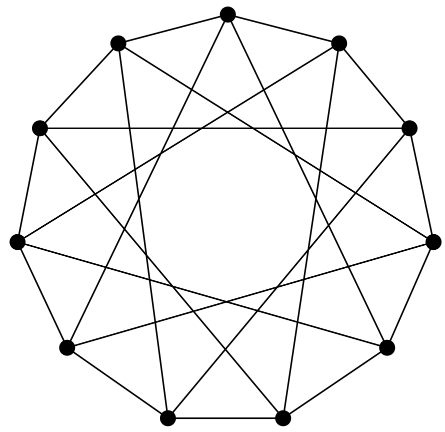
6. [9 marks]

(a) [4 marks]
Prove the following graph is nonplanar, or give a planar embedding for it:



(b) [5 marks]

The *Möbius ladder graph* shown below has 11 vertices, 22 edges, and girth 4. Prove that if any three edges are removed, the resulting graph is still nonplanar.



7. [11 marks]

Let G be a connected graph with p vertices, and let T_1 and T_2 be two spanning trees of G. Define the spanning subgraph $H = T_1 \cup T_2$ of G to be the spanning subgraph with $E(H) = E(T_1) \cup E(T_2)$.

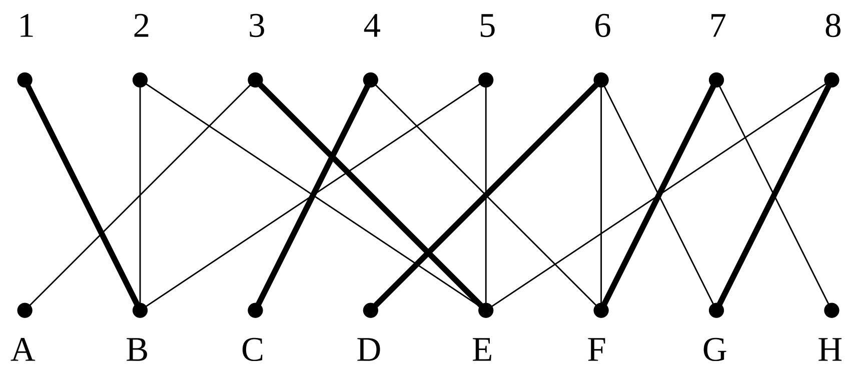
(a) [4 marks]
Prove that H has a vertex v with $\deg(v) \leq 3$.

(b) [3 marks]
Give an example of G, T_1 and T_2 in which every vertex of $H = T_1 \cup T_2$ has degree at least 3.

(c) [4 marks] Prove that if $E(T_1) \cap E(T_2) = \emptyset$, and T_1 and T_2 each have a vertex of degree more than $p/2$, then H has a vertex of degree exactly 2.

8. [13 marks]

Let G be the following graph:



and let $M = \{\{1,B\},\{3,E\},\{4,C\},\{6,D\},\{7,F\},\{8,G\}\}$ be the matching indicated in the above figure with bold edges.

(a) [6 marks]

Apply the XY construction to M, and:

- Determine the sets X_0, X, and Y.
- Indicate the order in which vertices are added to X and Y.
- Determine the set U of unsaturated vertices in Y.
- Indicate the covering C produced by the XY construction satisfying $|M| = |C| - |U|$.

(For students in Section 001: The sets X_0, X, Y were called U_A, R_A, and R_B in class.)

(b) [2 marks]
Find an augmenting path for M, or prove that no such path exists.

(c) [3 marks]
Find a maximum matching and a minimum covering for G.

(d) [2 marks] Find a set $D \subset \{1, 2, 3, 4, 5, 6, 7, 8\}$ such that $|D| > |N(D)|$, or prove that no such set exists.

9. [8 marks]

Let k be a positive integer and let G be a bipartite graph with vertex classes A and B. Suppose every vertex in A has degree at least k, and every vertex in B has degree at most k. Prove that G has a matching of size $|A|$.