MATH 239 Tutorial 1 Problems

1. Let E be the set of all subsets of $[n]$ of even size, and let O be the set of all subsets of $[n]$ of odd size. Prove that

$$\sum_{\substack{i \text{ is even} \\ 0 \le i \le n}} \binom{n}{i} = \sum_{\substack{j \text{ is odd} \\ 0 \le j \le n}} \binom{n}{j}$$

by finding a bijection between E and O. Illustrate your bijection by matching up the E and O for $[4]$.

2. Give a combinatorial proof of the following identity:

$$\binom{2n}{n} = 2\binom{2n-1}{n-1}.$$

3. Give a combinatorial proof and an algebraic proof of the following identity:

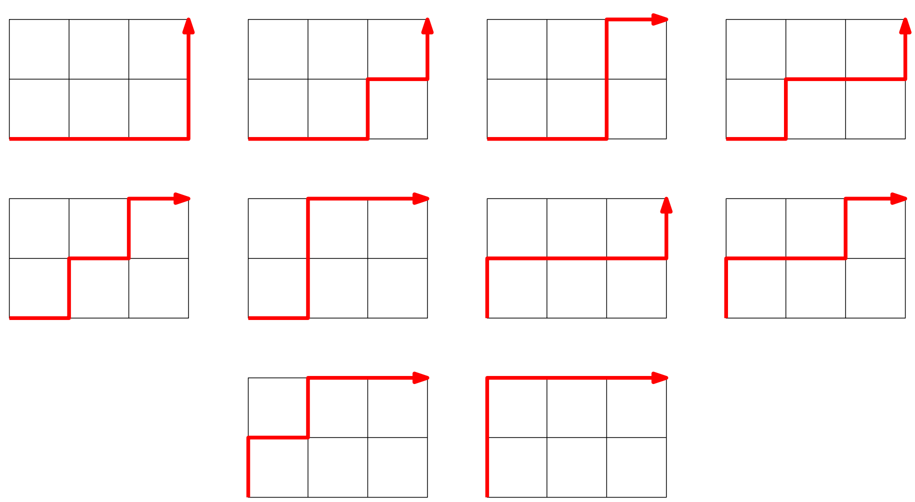
$$\sum_{i=0}^{n} \binom{n}{i} i = n2^{n-1}.$$

4. Determine $[x^{314}]x^3(1+x)^2(1+5x^{10})^{-1}$.

5. Determine $[x^n](1+x^2)^{-5}(1-3x)^{20}$

Additional exercises

1. Consider a road network that resembles an $m \times n$ grid. We wish to walk from the SW corner to the NE corner of the grid so that we only walk in E or N direction. Let $W_{m,n}$ be the set of all such walks. For example, the set of all such walks on a 2×3 grid is illustrated below:



Define $S_{m,n}$ to be the set of all subsets of $[m+n] = \{1, 2, \ldots, m+n\}$ of size m.

Find a bijection between $W_{m,n}$ and $S_{m,n}$, and determine the number of walks in $W_{m,n}$.

2. Consider the k-tuples $(T_1, \ldots, T_k)$ where each $T_i \subseteq [n]$. In other words, if P is the set of all subsets of $[n]$, then such a k-tuple is in the cartesian product P^k. We define the following two subsets of P^k:

   (a) S is all such k-tuples where $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k$.

   (b) T is all such k-tuples that are mutually disjoint, i.e. $T_i \cap T_j = \emptyset$ for any $i \neq j$.

   Find a bijection between S and T, which proves that $|S| = |T|$. What is this cardinality?

3. Prove the following identity (using any correct method):

$$2^{n-r}\binom{n}{r} = \sum_{k=r}^{n} \binom{n}{k}\binom{k}{r}$$