# MATH 239 Spring 2012: Assignment 8

Solutions

1. {10 marks} Let $G$ be a connected graph where each vertex has degrees 1 or 3. Let $\mathcal{X}$ be the set of vertices that have degree 1. Suppose there exists a set of edges $\mathcal{E}$ such that after removing $\mathcal{E}$ from $G$, each component of the remaining graph is a tree which contains exactly one vertex from $\mathcal{X}$. Determine $|\mathcal{E}|$ in terms of $|V(G)|$.

   **Solution.** Let $|V(G)| = n$ and $|\mathcal{X}| = k$. So $G$ has $k$ vertices of degree 1 and $n-k$ vertices of degree 3. By Handshaking Lemma,

   $$|E(G)| = \frac{1}{2}(k + 3(n-k)) = \frac{3}{2}n - k.$$

   Suppose that the $k$ components of $G - \mathcal{E}$ are $C_1, \ldots, C_k$. Since each component $C_i$ is a tree, it has $|V(C_i)|$ vertices and $|V(C_i)| - 1$ edges. Therefore, the number of edges in $G - \mathcal{E}$ is

   $$\begin{aligned} |E(G - \mathcal{E})| &= (|V(C_1)| - 1) + \cdots + (|V(C_k)| - 1) \\ &= (|V(C_1)| + \cdots + |V(C_k)|) - k \\ &= |V(G)| - k = n - k. \end{aligned}$$

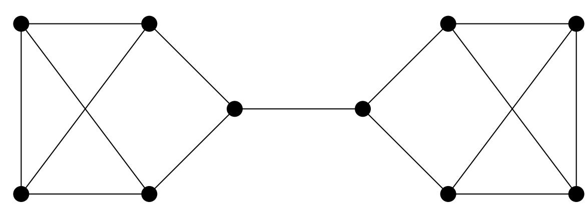
   So the number of edges in $\mathcal{E}$ is

   $$|\mathcal{E}| = |E(G)| - |E(G - \mathcal{E})| = \frac{3}{2}n - k - (n-k) = \frac{1}{2}n = \frac{1}{2}|V(G)|.$$

2. {15 marks}

   (a) Find a 3-regular graph with a bridge.

   **Solution.**

   (b) Prove that if every vertex of $G$ has even degree, then $G$ cannot have a bridge.

   **Solution.** Let $e = xy$ be a bridge in $G$. Let $H$ be the component of $G - e$ containing the vertex $x$. Then every vertex in $H$ has even degree except $x$, which has odd degree (since one edge incident with $x$ is removed). Therefore, $H$ is a graph with exactly 1 odd-degree one vertex, which cannot happen. Therefore, $G$ cannot have a bridge.

   (c) Prove that if $G$ is a $k$-regular bipartite graph where $k \geq 2$, then $G$ cannot have a bridge.

   **Solution.** Let $(A, B)$ be a bipartition of $G$. Let $e = xy$ be a bridge, and let $H$ be the component of $G$ containing the vertex $x$. Let $A' = V(H) \cap A$ and $B' = V(H) \cap B$. Then $H$ is a bipartite graph with bipartition $(A', B')$. Suppose without loss of generality that $x \in A$. In $H$, every vertex has degree $k$ except for $x$, which has degree $k-1$. Since $H$ is a bipartite graph,

   $$\sum_{v \in A'} \deg_H(v) = \sum_{v \in B'} \deg_H(v).$$

So

$$k|A'| - 1 = k|B'|,$$

which means that

$$k(|A'| - |B'|) = 1.$$

Since $k \geq 2$, no integers of $|A'|$ and $|B'|$ could satisfy this equation. Hence $G$ cannot have a bridge.

3. {15 marks} Let $G$ be a connected graph with $2k$ odd-degree vertices, where $k \geq 1$.

   (a) Prove that there exist $k$ walks in $G$ such that each edge of $G$ is used in exactly one walk. (For this question, you may assume that the main theorem about Eulerian circuits is true even for graphs with multiple edges.)

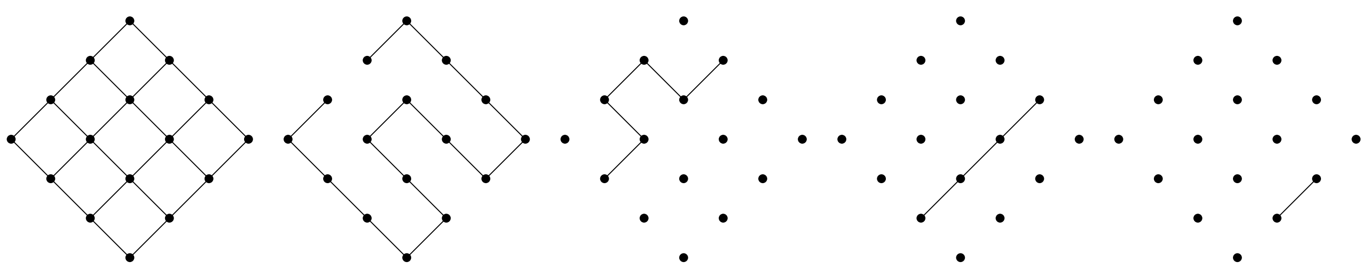
   **Solution.** Let $v_1, v_2, \ldots, v_{2k}$ be the set of all odd-degree vertices in $G$. We obtain $G'$ by adding $k$ edges $v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}$ to $G$. Since we added one to each of these odd-degree vertices, $G'$ is a graph where every vertex has even degree. Therefore, $G'$ contains an Eulerian circuit, i.e. a closed walk containing each edge exactly once. By removing the $k$ edges from the Eulerian circuit, we break it down to $k$ walks where each edge in $G$ is in exactly one of them.

   (b) Prove that it is not possible to find $k-1$ walks in $G$ such that each edge is used in exactly one walk.

   **Solution.** Let $W_1, \ldots, W_{k-1}$ be edge-disjoint walks in $G$. For each $W_i$, if it is a closed walk, then the edges contribute an even degree to every vertex. If it is not a closed walk, then the edges contribute an even degree to every vertex except the two endpoints, which have odd degrees. Over all $k-1$ walks, we have at most $2(k-1)$ vertices of odd degrees, which is not possible since there are $2k$ vertices of odd degrees.

   (c) Partition the edges of the leftmost graph below into as few walks as possible.

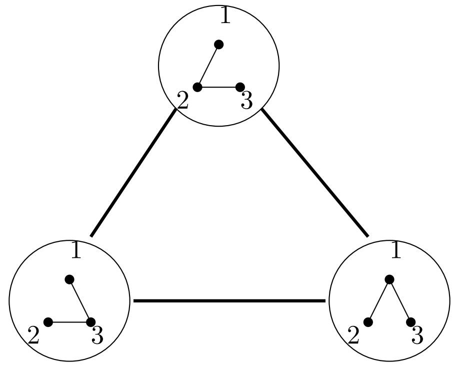
   **Solution.** The idea is to find walks that start and end at odd-degree vertices. There are many solutions, here is one of them.



4. {10 marks} Consider the graph $G_n$ where each vertex is a spanning tree of $K_n$ with vertices labelled with $[n]$, and two trees $T_1$ and $T_2$ are adjacent if and only if $|E(T_1) \setminus E(T_2)| = 1$ (i.e. there is one edge in $T_1$ that is not in $T_2$).

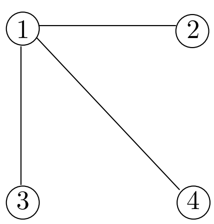
(a) Draw $G_3$.

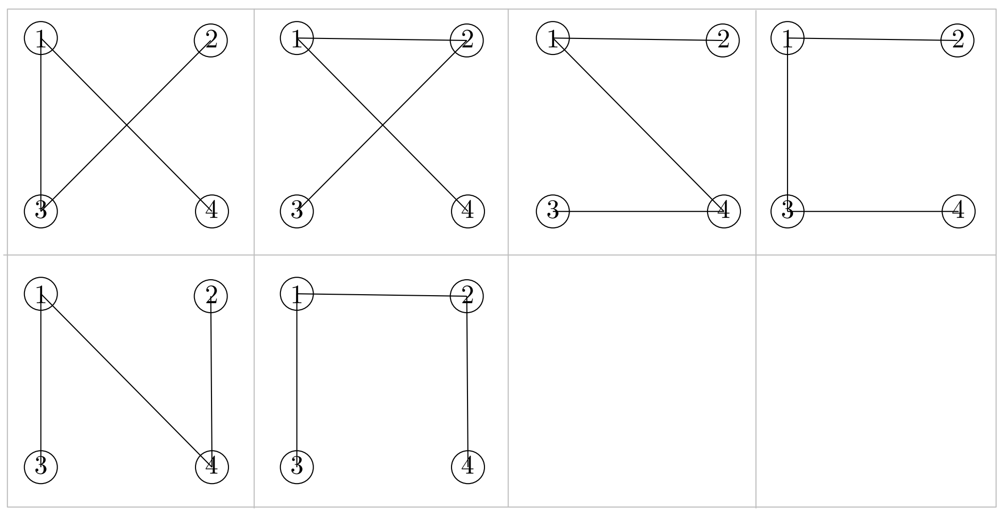
**Solution.** It's a $K_3$.



(b) In $G_4$, what are the neighbours of the tree on the left?

**Solution.**





(c) Prove that $G_n$ is connected. (Hint: Use induction on $|E(T_1) \setminus E(T_2)|$.)

**Solution.** Given two spanning trees $T_1$ and $T_2$, we will prove by induction that there is a path between $T_1$ and $T_2$ whenever $|E(T_1) \setminus E(T_2)| = k$.

Base case: When $k = 0$, $T_1 = T_2$, so such a path exists.

Induction hypothesis: We assume that there is a path between two spanning trees when $|E(T_1) \setminus E(T_2)| = k - 1$.

Induction step: Suppose $|E(T_1) \setminus E(T_2)| = k$. Let $e \in E(T_1) \setminus E(T_2)$. Then $T_1 - e$ consists of two components, let $C$ be one of them. Let $e'$ be an edge in $T_2$ that is in the cut induced by $V(C)$. This edge exists since $T_2$ is connected. Notice that $e'$ cannot be in $T_1$ for otherwise $e$ is not a bridge. Let $T_3 = T_1 - e + e'$. This is a spanning tree in $G_n$ where $|E(T_1) \setminus E(T_3)| = 1$ and $|E(T_3) \setminus E(T_2)| = k - 1$. So $T_1$ and $T_3$ are adjacent in $G_n$, and by induction hypothesis, there is a path between $T_3$ and $T_2$ in $G_n$. Hence there is a path between $T_1$ and $T_2$.