MATH 239 Tutorial 2 Problems

1. (a) Let S be the set of all subsets of $[3]$. Let w be the weight function on S such that for each $A \in S$, $w(A)$ is the sum of all elements of A. Determine the generating series $\Phi_S(x)$ with respect to w.

   (b) Let w' be the weight function on S such that for each $A \in S$, $w'(A)$ is twice the sum of all elements of A. Determine the generating series $\Phi'_S(x)$ with respect to w'.

   (c) What is the relationship between $\Phi_S(x)$ and $\Phi'_S(x)$?

2. Let w be the weight function defined on $\mathbb{N}_0$ as follows: for each $a \in \mathbb{N}_0$,

$$w(a) = \begin{cases} a/2 & a \text{ is even} \\ 2a & a \text{ is odd} \end{cases}$$

   Determine the generating series of $\mathbb{N}_0$ with respect to w.

3. For a binary string s, define its weight $w(s)$ to be the number of 1's in the string plus the length of the string itself. For example, $w(110100001) = 13$.

   (a) Let S_n be the set of all binary strings of length n. Use the product lemma to determine $\Phi_{S_n}(x)$.

   (b) Let T be the set of all binary strings (regardless of length). Determine $\Phi_T(x)$.

4. Let S_n be the set of all subsets of $[n]$, and for each $A \in S_n$, define $w(A)$ to be the sum of the elements in A. Give a combinatorial interpretation of the following:

$$\Phi_{S_n}(x) = (1 + x^n)\Phi_{S_{n-1}}(x).$$

Additional exercises

1. How many ways can you make up n cents using an unlimited supply of pennies, nickels, dimes and quarters? For example, 7 cents can be made up in two ways: 7 pennies, or 2 pennies and 1 nickel. How would this change if you are allowed to use up to 42 nickels? Express your answers as coefficients of generating series.

2. Let S be the set of all finite subsets of $\mathbb{N}$, and suppose the weight of a subset is the sum of all its elements. Find a "nice" expression for the generating series of S.

3. Using mathematical induction on k, prove that

$$(1 - x)^{-k} = \sum_{n \geq 0} \binom{n + k - 1}{k - 1} x^n.$$