# MATH 239 - Fall 2013

# Assignment 4

## Due date : Friday, October 11, 2013, at noon (sharp)

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**Submission Guidelines:**

- Total number of marks in this assignment is 34.
- Use a cover page to submit your solutions (available on the course webpage).
- Keep a copy of your manuscript before your submission.
- Assignments submissions are exclusively accepted in the following dropboxes

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| [Section 001] Dropbox next to the St Jerome's library, 2nd floor of STJ |
| [Section 002] Math DropBox #18; Slot #1 A-J, Slot #2 K-S, Slot #3 T-Z |
| [Section 003] Math DropBox #18; Slot #4 A-J, Slot #5 K-S, Slot #6 T-Z |
| [Section 004] Math DropBox #18; Slot #7 A-J, Slot #8 K-S, Slot #9 T-Z |

- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- You are not allowed to post this manuscript (or parts of it) online, nor share it (or parts of it) with anyone not enrolled in this course.

**Assignment policies:** While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. For example, copying or paraphrasing a solution from some fellow student or old solutions from previous offerings of related courses qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. All students found to be cheating will automatically be given a mark of 0 on the assignment. In addition, there will be a 10/100 penalty to their final mark, as well as all academic offenses will be reported to the Associate Dean for Undergraduate Studies and recorded in the student's file (this may lead to further, more severe consequences).

If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that if you see any marking error, then you should return your assignment paper to the TA of your section within one week and with written notes on all the marking errors; please write the notes on a new sheet and attach it to your assignment paper.

**Question 1** [Marks l0=5+5]
For each set of strings below, show that the expression is ambiguous. Give a description (in words) of the strings in the set, and explain any reasoning. Then use this description to write down another expression that generates the same set of strings unambiguously.

(a) $X = \{0\}^*\{1\}^*\{0\}^*$

(b) $Y = \{00, 11, 00000, 11111\}^*$

**Solution.**
*(a) To see that the expression is ambiguous, note that* $00 \in X$ *can be written as* $(00)(\epsilon)(\epsilon)$ *or* $(0)(\epsilon)(0)$.

*A string in* $X$ *is of the form* $abc$ *where* $a \in \{0\}^*$, $b \in \{1\}^*$, *and* $c \in \{0\}^*$. *If* $b = \epsilon$, *then the string consists only of zeros. Otherwise, there is a single* 1*-block, and there may be* 0*-blocks on either side of it (depending on whether or not* $a$ *and* $c$ *are empty). More concisely* $X$ *is the set of* $\{0,1\}$*-strings with at most one* 1*-block.*

*The set of strings with no* 1*-blocks is* $\{0\}^*$, *and the set of strings with exactly one* 1*-block is* $\{0\}^*\{1\}\{1\}^*\{0\}^*$. *These are unambiguous and disjoint, so an unambiguous expression for* $X$ *is:*

$$X = \{0\}^* \cup \{0\}^*\{1\}\{1\}^*\{0\}^* .$$

*(b) Consider the string* $0000000000 \in Y$ *(10 zeros). This can be written as* $(00)(00)(00)(00)(00)$ *or as* $(00000)(00000)$, *which shows that the expression is ambiguous.*

*Let's consider what type of blocks a string from* $Y$ *could have. We get a* 0*-block by concatenating strings from* $\{00, 00000\}$ *in any order. By doing so, we can obtain any even number of zeros (by concatenating* 00 *together several times), or any odd number* $\geq 5$ *(by concatenating* 00000 *with several copies of* 00*). Thus a* 0*-block can have any length except* 1 *or* 3*. Similarly a* 1*-block can have any length except* 1 *or* 3*. Hence the strings in* $Y$ *are those which have no blocks of length* 1 *or* 3*.*

*We can express this unambiguously, using the block decomposition:*

$$Y = (\{\epsilon, 00\} \cup \{0000\}\{0\}^*)((\{11\} \cup \{1111\}\{1\}^*)(\{00\} \cup \{0000\}\{0\}^*))^*(\{\epsilon, 11\} \cup \{1111\}\{1\}^*) .$$

**Question 2** [Marks 15=5+5+5]
For each set of strings below, determine the generating function with respect to length.

(a) The set $A$ of $\{0,1\}$-strings in which all blocks have length 1 or 4.

(b) The set $B$ of $\{0,1\}$-strings with an equal number of 0-blocks and 1-blocks. (NOTE: A 0-block is another name for a block of "0"s, and a 1-block is a block of "1"s.)

(c) The set $C$ of $\{0,1\}$-strings in which each 0-block is followed by a (strictly) longer 1-block. (HINT: Let $D$ be the set of finite strings that consist of two blocks: a 0-block followed a 1-block of equal length. Find an expression that generates $C$ unambiguously in terms of $D$.)

**Solution.**

*(a) Using the block decomposition:*

$$A = \{\epsilon, 0, 0000\}(\{1, 1111\}\{0, 0000\})^*\{\epsilon, 1, 1111\}\,.$$

*Therefore:*

$$\begin{aligned}\Phi_A(x) &= \Phi_{\{\epsilon,0,0000\}}(x)\frac{1}{1-\Phi_{\{1,1111\}\{0,0000\}}(x)}\Phi_{\{\epsilon,1,1111\}}(x)\\ &= (1+x+x^4)\frac{1}{1-(x+x^4)(x+x^4)}(1+x+x^4)\\ &= \frac{(1+x+x^4)^2}{1-(x+x^4)^2}\,.\end{aligned}$$

*(b) Since the blocks of a string alternate between* $0$*-blocks and* $1$*-blocks, a string will have an equal number of each if and only if its first and last digits are different (or the string could be empty). If it starts with* $1$ *it must end with* $0$*, and vice-versa; in the middle we can have an arbitrary string. Thus we have the unambiguous expression*

$$B = \{\epsilon\} \cup \{1\}\{0, 1\}^*\{0\} \cup \{0\}\{0, 1\}^*\{1\}\,,$$

*from which we compute:*

$$\begin{aligned}\Phi_B(x) &= \Phi_{\{\epsilon\}}(x) + \Phi_{\{1\}\{0,1\}^*\{0\}}(x) + \Phi_{\{0\}\{0,1\}^*\{1\}}(x)\\ &= 1 + \Phi_{\{1\}}(x)\frac{1}{1-\Phi_{\{0,1\}}(x)}\Phi_{\{0\}}(x) + \Phi_{\{0\}}(x)\frac{1}{1-\Phi_{\{0,1\}}(x)}\Phi_{\{1\}}(x)\\ &= 1 + x\cdot\frac{1}{1-2x}\cdot x + x\cdot\frac{1}{1-2x}\cdot x\\ &= \frac{1-2x+2x^2}{1-2x}\,.\end{aligned}$$

*(c) As in the hint, let* $D = \{01, 0011, 000111, \ldots\}$*. Then* $D\{1\}\{1\}^*$ *is the set of all strings with two blocks: a* $0$*-block followed by a longer* $1$*-block, and*

$$C = \{1\}^*(D\{1\}\{1\}^*)^*\,.$$

*This is unambiguous, because it is a specialization of the block decomposition.*

*Using* $\Phi_D(x) = \frac{x^2}{1-x^2}$*, we obtain*

$$\begin{aligned}\Phi_C(x) &= \Phi_{\{1\}^*}(x)\frac{1}{1-\Phi_D(x)\Phi_{\{1\}\{1\}^*}(x)}\\ &= \frac{1}{1-x}\cdot\frac{1}{1-\frac{x^2}{1-x^2}\cdot\frac{x}{1-x}}\\ &= \frac{1-x^2}{1-x-x^2}\,.\end{aligned}$$

**Question 3** [Marks 9=6+3]
Let $a_n$ be the number of $\{0,1\}$-strings of length $n$ that do not have 111000 as a substring. Let $b_n$ be the number of $\{0,1\}$-strings of length $n$ that do not have 011111 as a substring.

(a) By computing generating functions, prove that $a_n = b_n$, for all $n \in \mathbb{N}$.

(b) Given any $\{0,1\}$-string $\alpha$ of length 6, we can do a similar thing: define $c_n$ be the number of $\{0,1\}$-strings of length $n$ that do not have $\alpha$ as a substring. Can you find an example of an $\alpha$ for which it is NOT true that $a_n = c_n$ for all $n \in \mathbb{N}$?

**Solution.** *(a) Let $A$ be the set of binary strings that do not have* 111000 *as a substring. Let $B$ be the set of binary strings that do not have* 011111 *as a substring.*

*We can describe $A$ using the block decomposition: every* 1*-block that is followed by a* 0*-block must either have length $\leq 2$, or the* 0*-block that follows it must have length $\leq 2$. Hence*

$$A = \{0\}^*(\{1,11\}\{0\}\{0\}^* \cup \{111\}\{1\}^*\{0,00\})^*\{1\}^*$$

*We compute that*

$$\begin{aligned}
\Phi_A(x) &= \Phi_{\{0\}^*}(x)\frac{1}{1-(\Phi_{\{1,11\}}(x)\Phi_{\{0\}\{0\}^*}(x) + \Phi_{\{111\}\{1\}^*}\Phi_{\{0,00\}}(x))}\Phi_{\{1\}^*}(x) \\
&= \frac{1}{1-x}\cdot\frac{1}{1-((x+x^2)\frac{x}{1-x} + \frac{x^3}{1-x}(x+x^2))}\cdot\frac{1}{1-x} \\
&= \frac{1}{1-2x+x^6}
\end{aligned}$$

*Similarly $B$ can be described using the block decomposition. Every* 1*-block that follows a* 0*-block must have length at most* 4*. Thus*

$$B = \{1\}^*(\{0\}\{0\}^*\{1,11,111,1111\})^*\{0\}^*$$

*and so*

$$\begin{aligned}
\Phi_B(x) &= \Phi_{\{1\}^*}(x)\frac{1}{1-\Phi_{\{0\}\{0\}^*}(x)\Phi_{\{1,11,111,1111\}}(x)}\Phi_{\{0\}^*}(x) \\
&= \frac{1}{1-x}\cdot\frac{1}{1-\frac{x}{1-x}(x+x^2+x^3+x^4)}\cdot\frac{1}{1-x} \\
&= \frac{1}{1-2x+x^6}
\end{aligned}$$

*Thus we see that $\Phi_A(x) = \Phi_B(x)$. Taking coefficients of $x^n$ on both sides, we have $a_n = [x^n]\Phi_A(x) = [x^n]\Phi_B(x) = b_n$, as required.*

*(b) Consider $\alpha =$ 000000. Let $C$ be the set of $\{0,1\}$-strings that do not have $\alpha$ as a substring. Then $C$ is equivalently described as the set of strings whose* 0*-blocks have length at most* 5*. Using the block decomposition we have:*

$$C = \{\epsilon, 0, 00, 000, 0000, 00000\}(\{1\}\{1\}^*\{0,00,000,0000,00000\})^*\{1\}^*,$$

*and so*

$$\begin{aligned}
\Phi_C(x) &= \Phi_{\{\epsilon,0,00,000,0000,00000\}}(x)\frac{1}{1-\Phi_{\{1\}\{1\}^*}(x)\Phi_{\{0,00,000,0000,00000\}}(x)}\Phi_{\{1\}^*}(x) \\
&= (1+x+x^2+x^3+x^4+x^5)\cdot\frac{1}{1-\frac{x}{1-x}(1+x+x^2+x^3+x^4+x^5)}\cdot\frac{1}{1-x} \\
&= \frac{1+x+x^2+x^3+x^4+x^5}{1-x-x^2-x^3-x^4-x^5-x^6}\,.
\end{aligned}$$

*Since* $\Phi_C(x) \neq \Phi_A(x)$*, it is not true that* $a_n = c_n$ *for all* $n \in \mathbb{N}$*.*