MATH 239 - Tutorial 3

Jan 23, 2013

1. Let $S = \{(a,b,c)\colon a,b \in \mathbb{N}_0, c \in \{0,1\}\}$. Let the weight w of $(a,b,c) \in S$ be given by $w(a,b,c) = a+b+c$. Find a formula for $[x^n]\Phi_S(x)$.

   Solution
   We will begin by finding the generating series of S. Define a weight function w_1 on $\mathbb{N}_0$, with $w_1(a) = a$ for all $a \in \mathbb{N}_0$, and a weight function w_2 on $\{0,1\}$ with $w_2(c) = c$. Then the conditions of the product lemma apply, and

$$\begin{aligned}
\Phi_S(x) &= \Phi_{\mathbb{N}_0^2 \times \{0,1\}}(x) \\
&= \left[\Phi_{\mathbb{N}_0}(x)\right]^2 \cdot \Phi_{\{0,1\}}(x) \\
&= \left[\sum_{i \geq 0} x^i\right]^2 (1+x) \\
&= (1-x)^{-2}(1+x) \\
&= (1+x)\sum_{n \geq 0}\binom{n+1}{1}x^n, \text{ by Thm 1.6.5,} \\
&= \sum_{n \geq 0}(n+1)x^n + \sum_{n \geq 0}(n+1)x^{n+1} \\
&= \sum_{n \geq 0}(n+1)x^n + \sum_{n \geq 1} nx^n, \text{ by reindexing,} \\
&= 1 + \sum_{n \geq 1}(2n+1)x^n.
\end{aligned}$$

   So

$$[x^n]\Phi_S(x) = \begin{cases} 1, & \text{if } n = 0, \\ 2n+1, & \text{if } x \geq 1. \end{cases}$$

   .

2. Let k be a fixed positive integer. Let a_n denote the number of compositions of n with k parts, such that no part is divisible by 3.

   (a) Find a set S and a weight function ω defined on S such that a_n is equal to the number of elements σ of S with $\omega(\sigma) = n$.

   (b) Find a generating series $\Phi_S(x)$ with respect to the weight function ω.

   (c) For $k = 1$, Determine a recurrence relation that $\{a_n\}$ satisfies, together with sufficient initial conditions.

   (d) Find a_n explicitly in terms of n and k.

   Solution.

(a) Let $P := \{1, 2, 4, 5, 7, 8, 10, 11, ...\}$ be the set of all integers not divisible by 3. Let $S = P^k$, for each $\sigma = (s_1, s_2, ..., s_k) \in S$ define the weight function $\omega(\sigma) = \sum_{i=1}^{k} s_i$. Then a_n is equal to the number of elements in S with weight n.

(b) For P, define the weight function $\alpha(i) = i$. Let $P_1 = \{r \in P : r \equiv 1 \mod 3\}$ and $P_2 = \{r \in P : r \equiv 2 \mod 3\}$. Now we have that

$$\Phi_{P_1}(x) = x^1 + x^4 + x^7 + x^{10} + ... = \sum_{i\geq 0} x^{3i+1} = x \sum_{i\geq 0} x^{3i} = x\frac{1}{1-x^3}$$

$$\Phi_{P_2}(x) = x^2 + x^5 + x^8 + x^{11} + ... = \sum_{i\geq 0} x^{3i+2} = x^2 \sum_{i\geq 0} x^{3i} = x^2\frac{1}{1-x^3}$$

As $P_1 \cup P_2$ is a partition of P, by the sum lemma, we have that

$$\Phi_P(x) = x\frac{1}{1-x^3} + x^2\frac{1}{1-x^3} = \frac{x+x^2}{1-x^3}$$

Now using the product lemma (recall that $S = P^k$ and that $\omega(\sigma) = \sum_{i=1}^{k} \alpha(s_i)$), we obtain that

$$\Phi_S(x) = \Phi_{P^k}(x) = (\Phi_P(x))^k = \left(\frac{x+x^2}{1-x^3}\right)^k$$

(c) We have that

$$\begin{aligned}
\sum_{i\geq 0} a_i x^i &= \frac{x+x^2}{1-x^3} \\
(1-x^3)\sum_{i\geq 0} a_i x^i &= x + x^2 \\
\sum_{i\geq 0} a_i x^i - \sum_{i\geq 0} a_i x^{i+3} &= x + x^2 \\
\sum_{i\geq 0} a_i x^i - \sum_{i\geq 3} a_{i-3} x^i &= x + x^2 \\
a_0 + a_1 x + a_2 x^2 + \sum_{i\geq 3}(a_i - a_{i-3})x^i &= x + x^2
\end{aligned}$$

We know that two formal power series are the same if the coefficients for all x^i are the same. So we get that $a_0 = 0$, $a_1 = 1$, $a_2 = 1$, and for $i \geq 3$ that $a_i - a_{i-3} = 0$. Thus we get that $a_i = a_i - 3$ for $i \geq 3$.
This means that $a_{3j} = 0$ while $a_{3j+1} = a_{3j+2} = 1$ and so our power series looks like

$$x + x^2 + x^4 + x^5 + x^7 + x^8 + \cdots = x\sum_{i\geq 0} x^{3i} + x^2 \sum_{i\geq 0} x^{3i}$$

as expected.

(d)

$$\left(\frac{x+x^2}{1-x^3}\right)^k = x^k(1+x)^k(1-x^3)^{-k} = x^k \sum_{j\geq 0}\binom{k}{j}x^j \sum_{i\geq 0}\binom{i+k-1}{k-1}x^{3i} =$$

$$= \sum_{r\geq 0}\left(\sum_{i=0}^{\lfloor r/3\rfloor}\binom{k}{r-3i}\binom{i+k-1}{k-1}\right)x^r$$

So $[x^n]\Phi_S(x) = [x^{n-k}]\sum_{r\geq 0}\left(\sum_{i=0}^{\lfloor r/3\rfloor}\binom{k}{r-3i}\binom{i+k-1}{k-1}\right)x^r = \sum_{i=0}^{\lfloor (n-k)/3\rfloor}\binom{k}{n-k-3i}\binom{i+k-1}{k-1}$

3. Let $\{a_n\}$ be the sequence with the corresponding power series

$$\sum_{n\geq 0} a_n x^n = \frac{1-x+2x^2}{1-x-2x^3}.$$

Determine a recurrence relation that $\{a_n\}$ satisfies, together with sufficient initial conditions. Use this recurrence to find a_5.

Solution.

$$1-x+2x^2 = (1-x-2x^3)\sum_{n\geq 0} a_n x^n$$
$$1-x+2x^2 = \sum_{n\geq 0} a_n x^n - \sum_{n\geq 0} a_n x^{n+1} - \sum_{n\geq 0} 2a_n x^{n+3}$$
$$1-x+2x^2 = \sum_{n\geq 0} a_n x^n - \sum_{n\geq 1} a_{n-1} x^n - \sum_{n\geq 3} 2a_{n+3} x^n$$
$$1-x+2x^2 = a_0 + (a_1-a_0)x + (a_2-a_1)x^2 + \sum_{n\geq 3}(a_n - a_{n-1} - 2a_{n-3})x^n$$

So $a_0 = 1$, $a_1 - a_0 = -1$ thus $a_1 = 0$. $a_2 - a_1 = 2$, then $a_2 = 2$. For $n \geq 3$, $a_n - a_{n-1} - 2a_{n-3} = 0$. Therefore the recurrence relation is the following

$$a_n = a_{n-1} + 2a_{n-3}, \text{ for } a_0 = 1, a_1 = -1$$

To find a_5,

$$a_3 = a_2 + 2a_0 = 4, a_4 = a_3 + 2a_2 = 4, a_5 = a_4 + 2a_3 = 8$$

4. Let a_N denote the number of compositions of N with an odd number of parts, in which each part is a positive odd integer. Find a_N.

Solution

We want to find a generating series with a_N as coefficients. First consider the generating series for the odd numbers:

$$x + x^3 + x^5 + x^7 + \cdots = \sum_{n\geq 0} x^{2n+1} = x\sum_{n\geq 0}(x^2)^n = \frac{x}{1-x^2}$$

Consider the number of ways we can write N as the sum of k different odd numbers. Using the product lemma, we get that this is:

$$\left(\frac{x}{1-x^2}\right)^k$$

We are looking for all partitions with an odd number of parts, so we have to consider the union of the partitions with k parts for odd k. By the sum lemma, we sum the above over all odd k, or equivalently over all j writing $k = 2j+1$:

$$
\begin{aligned}
\sum_{k\geq 0 \text{ odd}} \left(\frac{x}{1-x^2}\right)^k &= \sum_{j\geq 0} \left(\frac{x}{1-x^2}\right)^{2j+1} \\
&= \sum_{j\geq 0} \left(x^{2j+1} \left(\frac{1}{1-x^2}\right)^{2j+1} \right) \\
&= \sum_{j\geq 0} \left(x^{2j+1} \sum_{n\geq 0} \binom{n+(2j+1)-1}{(2j+1)-1} x^{2n} \right) \qquad \text{by Thm 1.6.5} \\
&= \sum_{j\geq 0} \sum_{n\geq 0} \binom{n+2j}{2j} x^{2n+2j+1}
\end{aligned}
$$

Looking at this sum, we see that all terms x^k have an odd exponent k. This means that the coefficent for x^N is 0 if N is even, so we cannot write an even number as a sum of an odd amount of odd numbers.

If N is odd, then we want those terms where $2n + 2j + 1 = N$, so we write $j = (N-1)/2 - n$, and the N-th coefficient of this is

$$
\sum_{n\geq 0} \binom{n+2j}{2j} = \sum_{n\geq 0} \binom{N-n-1}{N-2n-1}
$$

Note that this is a finite sum, as for $n \geq N/2$ the binomial coefficient is 0.