1. **[3 marks]** Show that the number of k-subsets of $\{1, \ldots, n\}$ that contain 1 but do not contain n is $\binom{n-2}{k-1}$.

2. **[7 marks]** Let a_n be the number of compositions of n into three parts, where the first part is a multiple of 10, the second part is a multiple of 20 and the third part is a multiple of 50. Find an expression for the generating function $\sum_{n\geq 0} a_n x^n$ and show that it is equal to

$$\frac{x^{80}}{(1-x^{10})^2(1+x^{10})(1-x^{50})}.$$

3. **[8 marks]**

   (a) Write down a decomposition that uniquely creates the set of binary strings in which an odd-length block of 1's is never followed by an odd-length block of 0's.

   (b) There is a set S of strings with generating function

$$\Phi_S(x) = \frac{1+x}{1-x-x^2}.$$

   Let a_n be the coefficient of x^n in $\Phi_S(x)$. Find a recurrence relation for a_n, with enough initial conditions to uniquely determine the sequence of coefficients.

4. **[12 marks]** Solve the following homogeneous recurrence equation with initial conditions:

$$b_n - 5b_{n-1} + 8b_{n-2} - 4b_{n-3} = 0 \qquad (n \geq 3); \qquad b_0 = 2,\ b_1 = 2,\ b_2 = 0.$$

5. **[10 marks]**

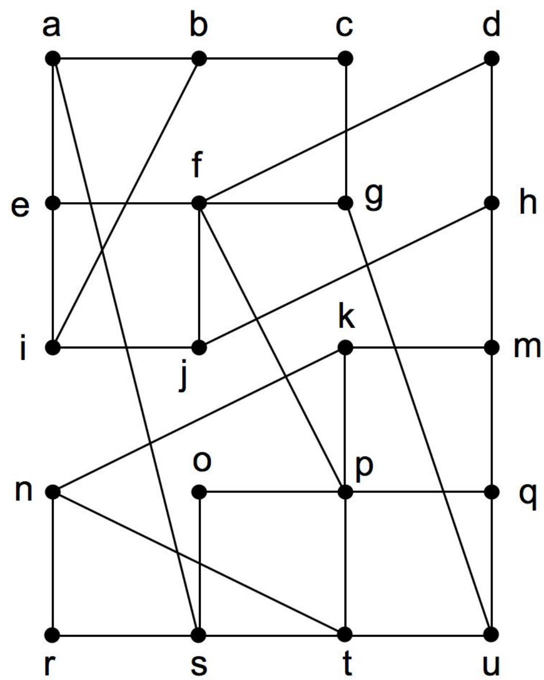
   (a) State the definition of a *tree*.

   (b) Prove that every tree T with exactly 7 vertices of degree 1 and exactly 2 vertices of degree 4 must have exactly 1 vertex of degree 3.

   (c) Draw a tree with 11 vertices in total, with at least 7 vertices of degree 1 and at least 2 vertices of degree 4.

6. **[12 marks]**

   (a) Find a breadth first search tree in the graph shown, rooted at the vertex a. When considering the new vertices adjacent to a vertex in the tree, take them in increasing alphabetical order.

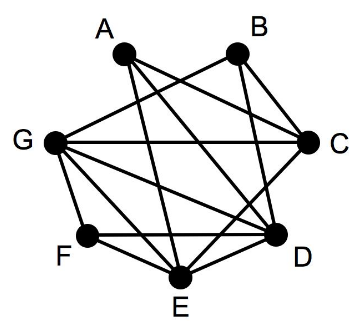


(b) Determine if the graph in (a) is bipartite. Give reasons for your answer. If it is bipartite, find a partition of the vertex set that shows it is bipartite. If not, find an odd cycle.

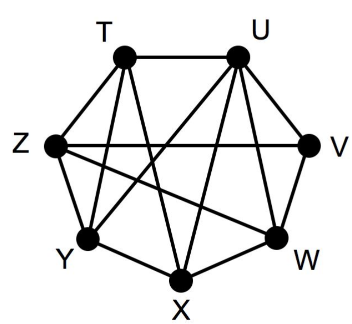
7. **[8 marks]** Prove that if a connected planar embedding has all faces of degree 3 and all vertices of degree 5, then it must have exactly 30 edges.

8. **[10 marks]** For each of the two graphs below, determine if it is planar. Prove your answer is correct in each case.

G_1:



G_2:



9. **[5 marks]** Prove that every planar graph with no 3-cycles is 4-colourable, assuming that every planar graph with no 3-cycles has a vertex of degree at most 3.

   You may not assume that the 4-colour theorem is true, and you *do not* need to prove the assumption that there is a vertex of degree at most 3.

10. **[8 marks]**

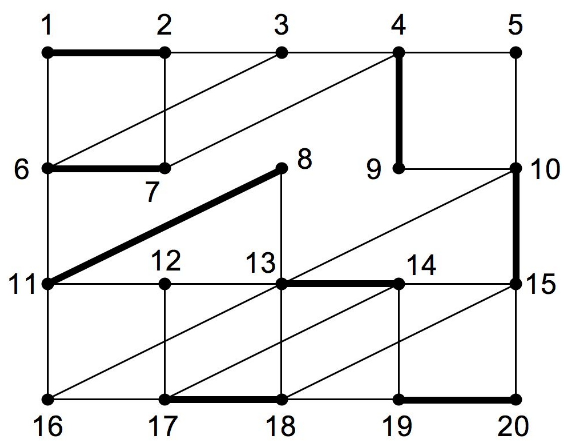
    (a) State Hall's Theorem for when a bipartite graph with bipartition (A, B) has a matching saturating all vertices in A.

    (b) Let G be a bipartite graph with bipartition (A, B) such that $|A| = 20$, $|B| = 24$, and all vertices have degree at least 10. Show that $|N(D)| \geq |D|$ for all $D \subseteq A$ with $|D| \leq 10$. Using this, or otherwise, prove that G has matching saturating all vertices in A.

11. **[12 marks]** Let G be the bipartite graph below, with bipartition (A, B) where A consists of the vertices with odd numbers and B the even numbered ones.

    (a) Beginning with the matching indicated with thick lines, apply the bipartite matching algorithm to find a larger matching.

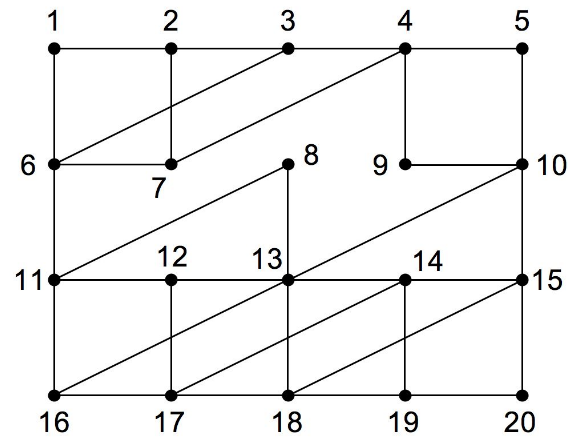
    Show the vertices of X and Y in the order that they are added to these two sets during the course of the algorithm.



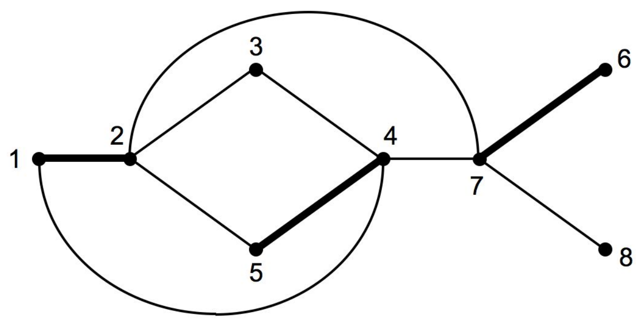
Vertices added to X :

Vertices added to Y :

Indicate the matching found:



(b) Show the sets X and Y that are obtained if the XY construction is used in the following bipartite graph, where $A = \{1, 3, 5, 7\}$ and the starting matching is the one shown with thick lines. Hence give a minimum cover for the graph.



X:

Y:

Minimum cover: