MATH 239 Winter 2013
Assignment 2
Solutions

The term “generating function” in this assignment is called a “generating series” in the notes.

1. Let S be a finite set.

   (a) Let $w : S \to \mathbb{N}_{\geq 0}$ be a function. Let $\Phi_{S,w}(x)$ be the generating function of S with respect to weight function w. Define a new function w' on S by
   $$w'(\sigma) = 2w(\sigma) + 3$$
   for all $\sigma \in S$. Show that $\Phi_{S,w'}$, the generating function of S with respect to w', satisfies
   $$\Phi_{S,w'}(x) = x^3 \Phi_{S,w}(x^2).$$

   (b) Let $w_1, w_2 : S \to \mathbb{N}_{\geq 0}$ be weight functions on S. Let $\Phi_{S,w_1}, \Phi_{S,w_2}$ be their respective generating functions. Let $w_3 : S \to \mathbb{N}_{\geq 0}$ be the function defined by
   $$w_3(\sigma) = w_1(\sigma) + w_2(\sigma).$$
   Is it true that $\Phi_{S,w_3}(x) = \Phi_{S,w_1}(x)\Phi_{S,w_2}(x)$? If true, give a proof. If false, give a set S and weight functions w_1, w_2 for which it is not true.

Solution.

(a) Note
$$\Phi_{S,w}(x) = \sum_{\sigma \in S} x^{w(\sigma)}.$$
Then,
$$\Phi_{S,w'}(x) = \sum_{\sigma \in S} x^{w'(\sigma)} = \sum_{\sigma \in S} x^{3+2w(\sigma)} = x^3 \sum_{\sigma \in S} (x^2)^{w(\sigma)} = \Phi_{S,w}(x^2).$$

(b) This is false. Practically any example is a counterexample. For example, let $S = \{0, 1\}$ with $w_1(0) = w_2(0) = 0$, $w(1) = w_2(1) = 1$. Then
$$\begin{aligned} \Phi_{S,w_1} &= 1 + x \\ \Phi_{S,w_2} &= 1 + x \\ \Phi_{S,w_3} &= 1 + x^2 \end{aligned}$$
while
$$\Phi_{S,w_1}\Phi_{S,w_1} = 1 + 2x + x^2$$

2. For a positive integer n, let N_n denote the set $\{1, 2, \ldots, n\}$, and let $\mathcal{P}_n$ be the set of all subsets of N_n. Let $w : \mathcal{P}_n \to \mathbb{N}_{\geq 0}$ be the weight function taking a subset to its number of elements, that is, for $A \subseteq N_n$,
   $$w(A) = |A|.$$
   For a pair of positive integers m, n with $m < n$, let $\mathcal{T}_{m,n}$ be the set of all subsets A of N_n such that $\max(A) > m$.

(a) Give a formula for $\Phi_{\mathcal{P}_n}(x)$.

(b) Explain why $\mathcal{T}_{m,n} = \mathcal{P}_n \setminus \mathcal{P}_m$, that is, the set of elements of $\mathcal{P}_n$ that are not elements of $\mathcal{P}_m$.

(c) Give a formula for $\Phi_{\mathcal{T}_{m,n}}(x)$ in terms of binomial coefficients.

Solution.

(a) We know that

$$\Phi_{\mathcal{P}_n}(x) = \sum_{A \subseteq N_n} x^{|A|} = \sum_{k=0}^{n} \left(\sum_{\substack{A \subseteq N_n \\ |A|=k}} 1 \right) x^k = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

(b) Let A be a subset of N_m. Since $N_m \subset N_n$, A is a subset of N_n. Note that the largest element of A is less than or equal to m. Conversely, a subset A of N_n whose largest is less than or equal to m is a subset of N_m. Consequently, $A \in \mathcal{P}_n \setminus \mathcal{P}_m$ is equivalent to $A \in \mathcal{P}_n$ and $\max(A) > m$.

(c) Now, we have a disjoint union

$$\mathcal{P}_n = \mathcal{P}_m \cup \mathcal{T}_{m,n}.$$

Therefore, by the sum lemma,

$$\begin{aligned} \Phi_{\mathcal{P}_n}(x) &= \Phi_{\mathcal{P}_m}(x) + \Phi_{\mathcal{T}_{m,n}}(x) \\ \sum_{k=0}^{n} \binom{n}{k} x^k &= \sum_{k=0}^{m} \binom{m}{k} x^k + \Phi_{\mathcal{T}_{m,n}}. \end{aligned}$$

Consequently,

$$\Phi_{\mathcal{T}_{m,n}} = \sum_{k=0}^{n} \left(\binom{n}{k} - \binom{m}{k} \right) x^k$$

where we use the convention that $\binom{n}{k} = 0$ if $k > n$.

3. Let N be a non-negative integer. Consider the set of all 3-tuples (n_1, n_2, n_3) of non-negative integers satisfying

$$n_1 + 5n_2 + 10n_3 = N.$$

Write down a formal power series $\Phi(x)$ such that $[x^N]\Phi(x)$ is the number of solutions (n_1, n_2, n_3) of the above equation. You do not need to find a closed formula for the number of solutions.

Hint: Use the product lemma.

Solution. Consider the set $\mathbb{N}_{\geq 0}$ equipped with functions $w_1, w_2, w_3 : \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ given by

$$\begin{aligned} w_1(n) &= n \\ w_2(n) &= 5n \\ w_3(n) &= 10n \end{aligned}$$

Now, the generating functions are

$$\begin{aligned}
\Phi_{\mathbb{N}_{\geq 0},w_1}(x) &= 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \\
\Phi_{\mathbb{N}_{\geq 0},w_2}(x) &= 1 + x^5 + x^{10} + x^{15} + \cdots = \frac{1}{1-x^5} \\
\Phi_{\mathbb{N}_{\geq 0},w_3}(x) &= 1 + x^{10} + x^{20} + x^{30} + \cdots = \frac{1}{1-x^{10}}
\end{aligned}$$

Now, consider the set $S = \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0}$ with function $w : S \to \mathbb{N}_{\geq 0}$ given by

$$w(n_1, n_2, n_3) = n_1 + 5n_2 + 10n_3.$$

Then the generating function of S with respect to w is given by

$$\Phi_{S,w}(x) = \sum_{(n_1,n_2,n_3)\in S} x^{n_1+5n_2+10n_3}.$$

Consequently, the coefficient of x^N in $\Phi_{S,w}(x)$ is the number of triples (n_1, n_2, n_3) satisfying $n_1 + 5n_2 + 10n_3 = N$. Because $w(n_1, n_2, n_3) = w_1(n_1) + w_2(n_2) + w_3(n_3)$, by the product lemma, we have

$$\Phi_{S,w}(x) = \Phi_{\mathbb{N}_{\geq 0},w_1}(x)\Phi_{\mathbb{N}_{\geq 0},w_2}(x)\Phi_{\mathbb{N}_{\geq 0},w_3}(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})}.$$

4. Let $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$ be the power series that satisfies

$$A(x) = \sum_{n=0}^{\infty} (x + x^2)^n.$$

Prove that $a_0 = 1$, $a_1 = 1$, and for $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$.

Hint: Recall that if $B(x)$ is a power series with no constant term then

$$\frac{1}{1 - B(x)} = \sum_{n=0}^{\infty} B(x)^n.$$

Solution. Observe that by the hint, we have

$$\frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} (x + x^2)^n.$$

We solve for $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$ by writing

$$(1 - x - x^2)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots) = 1.$$

Expanding the left side and grouping like powers, we get

$$a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + \cdots + (a_n - a_{n-1} + a_{n-2})x^n + \cdots + = 1 + 0x + 0x^2 \ldots.$$

Equating coefficients, we get a linear system in infinitely many variables given by

$$\begin{array}{rcl} a_0 &=& 1 \\ a_1 - a_0 &=& 0 \\ a_2 - a_1 - a_0 &=& 0 \\ &\vdots& \\ a_n - a_{n-1} - a_{n-2} &=& 0 \\ &\vdots& \end{array}$$

Consequently $a_0 = 1$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

5. Find an expression for each of the coefficients of the following formal power series (your expressions may be sums of binomial coefficients; you may want to break some expressions into cases)

   (a) $[x^n](1-x)^{-2}(1+2x^3)$

   (b) $[x^n](1-2x^2)^{-3}$

   (c) $[x^n](1-x)^{-2}(1-x^3)^2$

   Solution.

   (a) We know from class that

   $$(1-x)^{-2} = \sum_{n=0}^{\infty} \binom{n+1}{1} x^n = \sum_{n=0}^{\infty} (n+1)x^n.$$

   Consequently,

   $$\begin{array}{rcl} (1-x)^{-2}(1+2x^3) &=& (1+2x^3)\left(\sum_{n=0}^{\infty}(n+1)x^n\right) \\ &=& \sum_{n=0}^{\infty}(n+1)x^n + 2\sum_{n=0}^{\infty}(n+1)x^{n+3} \\ &=& \sum_{n=0}^{\infty}(n+1)x^n + 2\sum_{n=3}^{\infty}(n-2)x^n \end{array}$$

   Therefore,

   $$[x^n](1-x)^{-2}(1+2x^3) = \begin{cases} n+1 \text{ if } 0 \leq n \leq 2 \\ 3n-3 \text{ if } 3 \leq n \end{cases}$$

   (b) We know

   $$(1-x)^{-3} = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n.$$

   Consequently,

   $$(1-2x^2)^{-3} = \sum_{n=0}^{\infty} \binom{n+2}{2} (2x^2)^n = \sum_{n=0}^{\infty} 2^n \binom{n+2}{2} x^{2n}$$

Therefore,

$$[x^n](1-2x^2)^{-3} = \begin{cases} 2^{\frac{n}{2}}\binom{\frac{n}{2}+2}{2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

(c) We have

$$\frac{(1-x^3)^2}{(1-x)^2} = \left(\frac{1-x^3}{1-x}\right)^2 = (1+x+x^2)^2 = 1+2x+3x^2+2x+1$$

The coefficients can be read from this expression.

6. Let S be a set with weight function $w : S \to \mathbb{N}_{\geq 0}$ Let $\Phi_{S,w}(x)$ be the generating function. Write $c_n = [x^n]\Phi_{S,w}(x)$ so that c_n is equal to the number of elements of S with weight equal to n. Let $A(x)$ be the formal power series given by $A(x) = \frac{\Phi_{S,w}(x)}{1-x^2}$. Write $a_n = [x^n]A(x)$. Prove that

$$a_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n-2k}.$$

Solution. Note that

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \ldots.$$

Consequently,

$$\begin{aligned}
[x^n]\frac{\Phi_{S,w}(x)}{1-x^2} &= [x^n]\left((1+x^2+x^4+x^6+\ldots)\Phi_{S,w}(x)\right) \\
&= [x^n]\Phi_{S,w}(x) + [x^n](x^2\Phi_{S,w}(x)) + [x^n](x^4\Phi_{S,w}(x)) + \ldots \\
&= [x^n]\Phi_{S,w}(x) + [x^{n-2}]\Phi_{S,w}(x) + [x^{n-4}]\Phi_{S,w}(x) + \ldots \\
&= c_n + c_{n-2} + c_{n-4} + \ldots
\end{aligned}$$

where we used $[x^n](x^k C(x)) = [x^{n-k}]C(x)$ for the third equality. Translated into Σ-notation, this last quantity is

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n-2k}.$$