MATH 239 Spring 2012: Assignment 5

Solutions

1. {8 marks} A binary string is a palindrome if it reads the same forwards and backwards. Examples of palindromes include 01100110, 11011, 1. Let P be the set of all binary strings that are palindromes. Determine a recursive definition of P, and use it to find the generating series for P.

   Solution. For a palindrome with length at least 2, the first and last bits must be the same, either they are both 0 or both 1, and the remaining string is another palindrome. Palindromes of length 1 or 0 are $\{\varepsilon, 1, 0\}$. This gives us the recursion

   $$P = \{0\}P\{0\} \cup \{1\}P\{1\} \cup \{\varepsilon, 1, 0\}.$$

   This gives us the equation

   $$\Phi_P(x) = x^2\Phi_P(x) + x^2\Phi_P(x) + 1 + 2x.$$

   Solving for $\Phi_P(x)$ and we get

   $$\Phi_P(x) = \frac{1+2x}{1-2x^2}.$$

2. {12 marks} Let S be the set of binary strings which do not contain 01010 as a substring. Let T be the set of binary strings that have exactly one copy of 01010 at the right end. Two (incomplete) recursive relations between the two sets are

   $$\{\varepsilon\} \cup S\{0,1\} = S \cup T$$
   $$S\{01010\} = \dots\dots\dots$$

   (a) Complete the second equation in terms of T and justify this equality.

   Solution. The second equation should be

   $$S\{01010\} = T \cup T\{10\} \cup T\{1010\}.$$

   ($\supseteq$) For any string in $T \cup T\{10\} \cup T\{1010\}$, the last 5 bits are 01010, and removing these 5 bits destroys all copies of 01010, which is in S. So any such string is in $S\{01010\}$.

   ($\subseteq$) Let s be a string in S. Then the string $s01010$ may contain up to 3 copies of 01010. Either

   - The string s ends with 0101, in which case the end of $s01010$ looks like $\dots 010101010 = \dots(01010)1010$. There are 3 copies of 01010 in the string, and the bracketed part is the first copy of 01010, and so $s01010 \in T\{1010\}$.
   - Or, s ends with 01, in which case $s01010$ looks like $\dots 0101010 = \dots(01010)10$. There are 2 copies of 01010 in the string and the bracketed part is the first copy of 01010, and so $s01010 \in T\{10\}$.
   - Otherwise, there is only one copy of 01010, in which case it is in T.

   (b) Determine the generating series for S.

   Solution. The two equations give us

   $$1 + \Phi_S(x)(2x) = \Phi_S(x) + \Phi_T(x)$$
   $$\Phi_S(x)x^5 = \Phi_T(x) + \Phi_T(x)x^2 + \Phi_T(x)x^4$$

   Second equation gives us

   $$\Phi_T(x) = \frac{x^5}{1+x^2+x^4}\Phi_S(x).$$

Substitute into the first equation gives us

$$\Phi_S(x) = \frac{1}{1 - 2x + \frac{x^5}{1+x^2+x^4}}.$$

3. {20 marks}

(a) Let $\{a_n\}$ be the sequence which satsifies

$$a_n - 2a_{n-1} - 7a_{n-2} - 4a_{n-3} = 0$$

for $n \geq 3$ with initial conditions $a_0 = 3, a_1 = 7, a_2 = 8$. Determine an explicit formula for a_n.

Solution. The characteristic polynomial is

$$x^3 - 2x^2 - 7x - 4 = (x+1)^2(x-4).$$

The root -1 has multiplicity 2, and the root 4 has multiplicity 1. So

$$a_n = (A + Bn)(-1)^n + C \cdot 4^n$$

for some constants A, B, C. Plugging in the initial conditions, we get

$$\begin{aligned} 3 &= A + C \\ 7 &= -A - B + 4C \\ 8 &= A + 2B + 16C \end{aligned}$$

Solving this gives us $A = 2, B = -5, C = 1$. So an explicit formula for a_n is

$$a_n = (2 - 5n)(-1)^n + 4^n.$$

(b) Let $\{b_n\}$ be the sequence which satisfies

$$b_n - 2b_{n-1} - 7b_{n-2} - 4b_{n-3} = 3 \cdot (-2)^{n-1}$$

for $n \geq 3$ with initial conditions $b_0 = 1, b_1 = 9, b_2 = 4$. Determine an explicit formula for b_n.

Solution. Suppose $c_n = \alpha(-2)^n$ is a specific solution to this recurrence. Then

$$\begin{aligned} c_n - 2c_{n-1} - 7c_{n-2} - 4c_{n-3} &= \alpha(-2)^n - 2\alpha(-2)^{n-1} - 7\alpha(-2)^{n-2} - 4\alpha(-2)^{n-3} \\ &= \alpha(-2)^{n-3}((-2)^3 - 2(-2)^2 - 7(-2) - 4) \\ &= -6\alpha(-2)^{n-3}. \end{aligned}$$

This equals $3 \cdot (-2)^{n-1} = 12 \cdot (-2)^{n-3}$, so $\alpha = -2$. So $c_n = (-2)^{n+1}$ is a specific solution. The homogeneous solution is in the same form as part (a). So there exist constants A, B, C such that

$$b_n = (-2)^{n+1} + (A + Bn)(-1)^n + C \cdot 4^n.$$

Using the initial conditions, we get

$$\begin{aligned} 1 &= -2 + A + C \\ 9 &= 4 - A - B + 4C \\ 4 &= -8 + A + 2B + 16C \end{aligned}$$

Solving this gives $A = 2, B = -3, C = 1$. So an explicit formula for b_n is

$$b_n = (-2)^{n+1} + (2 - 3n)(-1)^n + 4^n.$$

4. {5 marks} Let $\{a_n\}$ be the sequence where $a_n = (n^2-1)(-2)^n + 3^{n+1}$. Determine a homogeneous recurrence that a_n satsifies, together with sufficient initial conditions.

   Solution. The formula for a_n suggests a recurrence whose characteristic polynomial has -2 as a root with multiplicity 3, and 3 as a root with multiplicity 1. One such polynomial is

   $$(x+2)^3(x-3) = x^4 + 3x^3 - 6x^2 - 28x - 24.$$

   So for $n \geq 4$, a_n satisfies the recurrence

   $$a_n + 3a_{n-1} - 6a_{n-2} - 28a_{n-3} - 24a_{n-4} = 0.$$

   We need initial conditions for the first 4 terms, and we can get them by calculating a_n from the given formula.

   $$a_0 = 2, a_1 = 9, a_2 = 39, a_3 = 17.$$

5. {5 marks} In assignment 4, we have an unambiguous decomposition for the set of binary strings that begin with 1 and whose values as binary representations are multiples of 3:

   $$(1(0\{1\}^*0)^*1\{0\}^*)^*$$

   In particular, this set includes no string of length 1 and 1 string of length 2. For $n \geq 1$, determine the number of such binary strings of length n as an explicit formula. (Your answer should make sense in some way.)

   Solution. Let a_n be the number of strings in this set of length n. We can find the generating series for this set of binary strings:

   $$\sum_{n\geq 0} a_n x^n = \frac{1}{1 - x\frac{1}{1-\frac{x^2}{1-x}}x\frac{1}{1-x}} = \frac{1-x-x^2}{1-x-2x^2}$$

   The denominator factors into $(1-2x)(1+x)$, so we know that

   $$a_n = A \cdot 2^n + B \cdot (-1)^n$$

   for some constants A and B. Using $a_1 = 0, a_2 = 1$ as initial conditions, we see that

   $$0 = 2A - B$$
   $$1 = 4A + B$$

   So $A = 1/6$ and $B = 1/3$. So an explicit formula for a_n is

   $$a_n = \tfrac{1}{6} \cdot 2^n + \tfrac{1}{3}(-1)^n = \tfrac{1}{3}(2^{n-1} + (-1)^n).$$

   (This formula makes sense since 2^{n-1} binary strings of length n start with 1, and among them, about a third are multiples of 3.)

   Note. We are a bit careless in the solution above. Notice that our formula doesn't work when $n = 0$, as it produces $a_0 = 1/2$. The reason is that the degree of the numerator is the same as the degree of the denominator. To use partial fractions and results about recurrences from class, we need to first use division algorithm to make this a proper fraction:

   $$\frac{1-x-x^2}{1-x-2x^2} = \frac{1}{2} + \frac{\frac{1}{2} - \frac{1}{2}x}{1-x-x^2}.$$

   Our solution for a_n is actually the coefficient of x^n of the rational expression $\frac{\frac{1}{2}-\frac{1}{2}x}{1-x-x^2}$. Now we see why $a_0 = 1/2$ makes sense: together with the 1/2, the constant term is actually 1, which is what we expected.