MATH 239 Assignment 1

DUE: NOON Friday 23 September 2011 in the drop boxes opposite the Math Tutorial Centre MC 4067 or next to the St. Jerome's library for the St. Jerome's section.

1. It is easy to check that $1 = 1^2$, $1 + 3 = 2^2$, $1 + 3 + 5 = 3^2$. Fill in the details in the following combinatorial outline of the proof that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

   (a) Explain why n^2 is the number of ordered pairs (a, b), with $a, b \in \{1, 2, \ldots, n\}$.
   SOLUTION. There are n choices for each of a and b.

   (b) Explain why, for a given b, the number of ordered pairs (a, b) with $a < b$ is equal to $b - 1$.
   SOLUTION. The value of a can be any of the numbers 1, 2, ..., $b - 1$.

   (c) Explain why the number of ordered pairs (a, b) with $\max\{a, b\} = i$ is equal to $2i - 1$.
   SOLUTION. The number with $b = i$ and $a < i$ is $i - 1$ (from the preceding part). Likewise, the number with $a = i$ and $b < i$ is $i - 1$. And there is also (i, i), for a total of $2i - 1$.

   (d) Deduce that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.
   SOLUTION. Each ordered pair (a, b) has a largest entry, which is obviously one of the numbers 1, 2, ..., n. For each $i = 1, 2, \ldots, n$, there are $2i - 1$ with largest entry i. Therefore, there are $\sum_{i=1}^{n}(2i - 1)$ different ordered pairs , i.e., $\sum_{i=1}^{n}(2i - 1) = n^2$.

2. Give two proofs, one combinatorial and one using the binomial theorem , of

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k .$$

As a hint for the combinatorial proof, consider the strings of length n in which each term is one of 0, 1, or 2. How many of these have a given number of 2's?

SOLUTION. *Combinatorial Proof.* We consider the strings of length n with each term being one of 0, 1, or 2. There are three choices for each position; therefore, there are 3^n such strings.

On the other hand, if we count them by the number of them that have exactly k 2's, for $k = 0, 1, 2, \ldots, n$, there are $\binom{n}{k}$ ways of selecting the positions where the 2's will occur. The remaining $n-k$ positions are filled with 0's and 1's only. For each of these $n - k$ positions, there are two choices to fill the position , so there are 2^{n-k} strings, for each of the $\binom{n}{k}$ choices of where the 2's go . Therefore, there are $\binom{n}{k}2^{n-k}$ strings having exactly k 2's.

Therefore, $3^n = \sum_{k=0}^{n} \binom{n}{k}2^{n-k}$. If we set $j = n - k$, then the sum is the same as $\sum_{j=0}^{n} \binom{n}{n-j}2^j$. Finally, we recall that $\binom{n}{n-j} = \binom{n}{j}$, so that $3^n = \sum_{j=0}^{n} \binom{n}{j}2^j$, as required.

Binomial Theorem Proof. The Binomial Theorem says that, if n is a non-negative integer, then $(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k}x^k$. Plugging in $x = 2$ yields $3^n = \sum_{k=0}^{n} \binom{n}{k}2^k$.

3. Let $S = \{1, 2, 3\}$. Below we describe various weight functions w for each subset T of S. In each case, write down the generating function for the set of all subsets of S with respect to the given weight function.

   (a) $w(T)$ is $|T|$ (that is, the number of elements of T).
   SOLUTION. In this case, the generating function is $x^0 + 3x^1 + 3x^2 + x^3$, because there is one set ($\varnothing$) with weight 0, three ($\{1\}$, $\{2\}$, and $\{3\}$) with weight 1, three ($\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$) with weight 2, and one ($\{1, 2, 3\}$) with weight 3.

   (b) $w(T)$ is the smallest number in T (if $T = \varnothing$, then $w(T) = 4$).
   SOLUTION. $w(\varnothing) = 4$, $w(\{1\}) = 1$, $w(\{2\} = 2$, $w(\{3\}) = 3$, $w(\{1, 2\}) = 1$, $w(\{1, 3\}) = 1$, $w(\{2, 3\}) = 2$, and $w(\{1, 2, 3\}) = 1$. Therefore the generating function in this case is $x^4 + x^1 + x^2 + x^3 + x^1 + x^1 + x^2 + x^1 = 4x^1 + 2x^2 + x^3 + x^4$.

   (c) $w(T)$ is the sum of the elements of T (if $T = \varnothing$, then $w(T) = 0$).
   SOLUTION. $w(\varnothing) = 0$, $w(\{1\}) = 1$, $w(\{2\} = 2$, $w(\{3\}) = 3$, $w(\{1, 2\}) = 3$, $w(\{1, 3\}) = 4$, $w(\{2, 3\}) = 5$, and $w(\{1, 2, 3\}) = 6$. Therefore the generating function in this case is $x^0 + x^1 + x^2 + x^3 + x^3 + x^4 + x^5 + x^6 = x^0 + x^1 + x^2 + 2x^3 + x^4 + x^5 + x^6$.

4. Two 6-sided dice are thrown, one red and one blue. The six numbers 1,2,3,4,5,6 occur, one on each of the six different sides of the die. In each of the two cases below for the weight of a throw, determine the generating function for the set of 36 outcomes.

   (a) The weight of the throw is the sum of the two face up numbers.
   SOLUTION. There are 36 possible outcomes for the throw:

$b\backslash r$	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

   Let S be the set of 36 outcomes (i, j), with the weight of outcome (i, j) being $i + j$. Therefore, the generating function is

$$
\begin{aligned}
\Phi_S(x) &= (x^2 + x^3 + x^4 + x^5 + x^6 + x^7) + (x^3 + \cdots + x^8) + (x^4 + \cdots + x^9) \\
&\quad + (x^5 + \cdots + x^{10}) + (x^6 + \cdots + x^{11}) + (x^7 + \cdots + x^{12}) \\
&= x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12} .
\end{aligned}
$$

(b) The weight of the throw is the larger of the two face up numbers.

SOLUTION. There are 36 possible outcomes for the throw:

$_b\backslash^r$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

Let S be the set of 36 outcomes (i, j), with the weight of outcome (i, j) being $\max\{i, j\}$. Therefore, the generating function is

$$\begin{aligned}
\Phi_S(x) &= (x^1 + x^2 + x^3 + x^4 + x^5 + x^6) + (x^2 + x^2 + x^3 + x^4 + x^5 + x^6) + \\
&\quad (x^3 + x^3 + x^3 + x^4 + x^5 + x^6) + (x^4 + x^4 + x^4 + x^4 + x^5 + x^6) + \\
&\quad (x^5 + x^5 + x^5 + x^5 + x^5 + x^6) + (x^6 + x^6 + x^6 + x^6 + x^6 + x^6) \\
&= x^1 + 3x^2 + 5x^3 + 7x^4 + 9x^5 + 11x^6 \,.
\end{aligned}$$