MATH 239 - Tutorial 1

1. Let E be the set of all subsets of $\{1, \ldots, n\}$ of even size, and let O be the set of all subsets of $\{1, \ldots, n\}$ of odd size. Prove that

$$\sum_{i \text{ is even}} \binom{n}{i} = \sum_{i \text{ is odd}} \binom{n}{i}$$

by finding a bijection between E and O.

Solution: Define $f\colon E \to O$ with

$$f(A) = A - \{1\} \text{ if } 1 \in A$$
$$f(A) = A \cup \{1\} \text{ if } 1 \notin A$$

2. Let $S = \{1, 2, ..., n\}$ and let $\mathcal{O}$ be the set of subsets of S such that the sum of its elements is odd (define the sum of the elements of the empty set as 0). Find $|\mathcal{O}|$.

Solution. Let S_{odd} be the set of odd integers in S. Define $k = |S_{odd}|$ and $S_{even} = S \setminus S_{odd}$. Any subset A of S belongs to $\mathcal{O}$ if and only if A has an odd number of odd integers. Hence any set in $\mathcal{O}$ is formed by choosing an odd number of elements in S_{odd} and joining this set to any subset of S_{even}, using Problem 1 we can observe that half of the subsets of a given set has an odd number of elements, hence there are $\frac{2^k}{2} = 2^{k-1}$ sets in S_{odd} of odd cardinality. Hence $|\mathcal{O}| = 2^{k-1}2^{n-k} = 2^{n-1}$.

3. Give a combinatorial and an algebraic proof for the following identity:

$$\sum_{i=0}^{n} \binom{n}{i} i = n2^{n-1}$$

Solution:

Combinatorial: Count $\{(A, x)\colon A \subseteq \{1, \ldots, n\}, x \in A\}$.

Left: There are $\binom{n}{i}$ subsets of size i, and for each of them there are i choices for x.

Right: Pick an x (n choices), and for each of the remaining $n-1$ elements, we have the choice whether to include it in A or not.

Algebraic: Differentiate

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

which gives

$$n(1+x)^{n-1} = \sum_{i=0}^{n} \binom{n}{i} i x^{i-1}$$

Plug in $x = 1$.

4. You are given an infinite number of boxes, which come in sizes 1 through n, and are asked to build a tower of $2n$ boxes. The bottom box must have size n, and each other box must be either the same size as or one size smaller than the box below it. How many different towers can you build?

   Solution: The size difference between each pair of consecutive boxes is either 0 or 1, so we may write the size differences between pairs of consecutive boxes (in order) to obtain a 0,1 string of length $2n-1$. Since the boxes come in sizes 1 through n, and since the box sizes are non-increasing as we move up the tower, this 0,1 string contains at most $n-1$ 1's. So then, an equivalent version of our problem is to count the number of length $n-1$ 0,1 strings having at most $n-1$ 1's. Since the number of length $2n-1$ 0,1 strings with k 1's is $\binom{2n-1}{k}$, we can sum over the possible values of k for our string (i.e. from $k=0$ to $k=n-1$) to see that there are

$$\sum_{k=0}^{n-1}\binom{2n-1}{k}$$

   ways to build the tower. Recall that $\sum_{k=0}^{2n-1}\binom{2n-1}{k}=2^{2n-1}$. This gives

$$\begin{aligned}
2^{2n-1} &= \sum_{k=0}^{2n-1}\binom{2n-1}{k}\\
&= \sum_{k=0}^{n-1}\binom{2n-1}{k}+\sum_{k=n}^{2n-1}\binom{2n-1}{k}\\
&= \sum_{k=0}^{n-1}\binom{2n-1}{k}+\sum_{k=n}^{2n-1}\binom{2n-1}{2n-1-k}\\
&= \sum_{k=0}^{n-1}\binom{2n-1}{k}+\sum_{k=0}^{n-1}\binom{2n-1}{k}\\
&= 2\sum_{k=0}^{n-1}\binom{2n-1}{k}.
\end{aligned}$$

   So the number of ways to build the tower is given by

$$\sum_{k=1}^{n-1}\binom{2n-1}{k}=\frac{1}{2}\cdot 2^{2n-1}=2^{2n-2}.$$

5. The n children of the Von Trapp family all have different ages. Whenever they sing, they stand, shoulder to shoulder, on a line. Indicate in how many ways they can line up under the condition that the youngest child always stands next to the oldest.

   Solution. Let P_1 be the set of permutations keeping the youngest child next to the oldest. Let us partition P_1 into two sets P_1^r and P_1^l where P_1^r is the set of permutations where the youngest child is in the right side of the oldest child, and P_1^l is the set of permutations where the youngest child is in the left side of the oldest child. Consider P_2 the set of permutations of the $n-1$ older children; there is a natural bijection between P_1^r and P_2 just by removing the youngest child from the line in any permutation of P_1^r, hence we have that $|P_1^r|=|P_2|=(n-1)!$. Similarly $|P_1^l|=(n-1)!$, therefore $|P_1|=2(n-1)!$

6. Give a combinatorial and an algebraic proof for the following identity

$$\sum_{i=1}^{n}\binom{n}{i}2^i=3^n.$$

   Combinatorial: Let $S=\{0,1,...,3^n-1\}$. We count S in two different ways. Clearly, $|S|=3^n$. On the other hand, we may count the elements of S base on their base 3 representations, as follows. For each

i, $0 \le i \le n$, let S_i denote the subset of S consisting of those elements of S that have exactly i zeroes in their base 3 representation. Then $3^n = |S| = \sum_{i=0}^{n} |S_i|$.
To find the size of S_i for each i, we have exactly i positions to place a zero, out of the n possible positions, giving a total of $\binom{n}{i}$ choices. For each of these, the remaining $n - i$ positions must each be a 1 or 2, and all choices are possible, giving 2^{n-i} choices. Therefore $|S_i| = \binom{n}{i}2^{n-i}$. This tells us

$$3^n = |S| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}.$$

Then using the fact that $\binom{n}{i} = \binom{n}{n-i}$ and changing the summation index gives us the required statement

$$\sum_{i=1}^{n} \binom{n}{i} 2^i = 3^n.$$

Algebraic:

$$\begin{aligned} 3^n &= (2+1)^n \\ &= \sum_{i=0}^{n} \binom{n}{i} 2^i, \text{ by Binomial Theorem.} \end{aligned}$$

7. Let $S = \{1, 2, 3\}$, and let 2^S be the set of all subsets of S. Let the weight $w(\sigma)$ of $\sigma \in 2^S$ be defined as 1 if $2 \in \sigma$, and be 0 otherwise. Find the generating series $\Phi_s(\sigma)$.

   Solution: First, note that $|2^S| = 2^|S| = 2^3 = 8$. The number of subsets of S that contain 2 is 2^2: adding 2 to any subset of $\{1, 3\}$ to gives a subset of weight 1, and this is a bijection between weight 1 subsets of S and subsets of $\{1, 3\}$. Since the weight 1 subsets and weight 0 subsets partition S, there are 8 - 4 = 4 weight 0 subsets. It follows that the generating series is

$$\Phi_S(\sigma) = 4x^0 + 4x^1 = 4 + 4x.$$