UNIVERSITY OF WATERLOO
**MIDTERM EXAMINATION**
**SPRING TERM 2005**

**Surname:** ____________________

**First Name:** ____________________

**Id.#:** ____________________

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|---|---|
| Course Number | MATHEMATICS 239 |
| Course Title | Introduction to Combinatorics |
| Instructor | ☐ P. Schellenberg 9:30 MWF LEC 001<br>☐ J. Verstraete 1:30 MWF LEC 002 |
| Date of Exam | Thursday, June 23, 2005 |
| Time Period | 4:30 – 6:30 p.m. |
| Number of Exam Pages (including this cover sheet) | 12 |
| Exam Type | Closed Book |
| Additional Materials Allowed | None |
| Additional Instructions | Write your answers in the space provided. |

| Problem | Value | Mark Awarded | Problem | Value | Mark Awarded |
|---|---|---|---|---|---|
| 1 | 3 | | 6 | 4 | |
| 2 | 3 | | 7 | 6 | |
| 3 | 4 | | 8 | 6 | |
| 4 | 8 | | 9 | 3 | |
| 5 | 4 | | 10 | 9 | |
| | | | Total | 50 | |

[3] **1.** Find a closed form expression for each of the following formal power series. Your answer should be expressed in the simplest possible form.

(a) $1 - x^2 + x^4 - x^6 + x^8 - \dots$

(b) $1 + nx^2 + \binom{n}{2}x^4 + \binom{n}{3}x^6 + \cdots + \binom{n}{n}x^{2n}$.

(c) $\left(\frac{x}{1-x}\right) + \left(\frac{x}{1-x}\right)^3 + \left(\frac{x}{1-x}\right)^5 + \left(\frac{x}{1-x}\right)^7 + \dots$

[3] **2.** Let $\Phi(x) = \sum_{i\geq 0} a_i x^i$. Define the formal power series

$$f(x) \;=\; \frac{\Phi(x)}{1-x}.$$

Determine $[x^n]f(x)$ in terms of the coefficients $a_i$.

[4] **3.** Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ denote the set of nonnegative integers. Define the weight of integer $n$ to be

$$w(n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ \dfrac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Determine the generating function $\Phi_{\mathbb{N}}(x)$ as a rational function $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

[8] **4.** Let $n$ be a positive integer and let $b_n$ denote the number of compositions of $n$ into $k$ parts, where each part is one or two. For example, $(1,2,1,2,1)$ and $(2,2,1,1,1)$ are two compositions of $n = 7$ into $k = 5$ parts.

(a) Determine the generating function for $b_n$.

(b) Prove that $b_n = \binom{k}{n-k}$ for $k \leq n \leq 2k$ and $b_n = 0$ otherwise.

(c) Determine the generating function for $c_n$, the number of compositions of $n$ into any number of parts, each of which is one or two.

(d) Find a recurrence equation for $c_n$ with appropriate initial conditions.

[4] **5.**

(a) Are the strings of $\{101,\ 010,\ 01010\}^*$ uniquely created? Justify your answer.

(b) Are the strings of $\{110,\ 011,\ 01010\}^*$ uniquely created? Justify your answer.

[4] **6.** For each of the following sets, write down a decomposition that uniquely creates the elements of that set.

(a) Binary strings that do not contain the substring 1111.

(b) Binary strings in which every block of 1's of even length is followed by a block of 0's of even length.

[6] **7.** Let $S$ denote the set of strings which do not contain the substring 10001. A combinatorial decomposition of this set of strings is given by

$$S \;=\; \{0\}^* \left(\{1\}\{1\}^*(\{0\}\{0\}^*\backslash\{000\})\right)^* \{1\}^* \{\varepsilon, 1000\}$$

where the strings on the right are uniquely created. Let $a_n$ be the number of binary strings of length $n$ that do not contain the substring 10001.

(a) Show that

$$\sum_{n\geq 0} a_n x^n = \frac{1+x^4}{1-2x+x^4-x^5}.$$

(b) Determine a recurrence relation for the sequence of $a_n$'s together with sufficient initial condition to uniquely determine the sequence.

[6] **8.** (a) Solve the homogeneous linear recurrence relation

$$b_n - 3b_{n-1} + 3b_{n-2} - b_{n-3} = 0 \qquad \forall n \geq 3$$

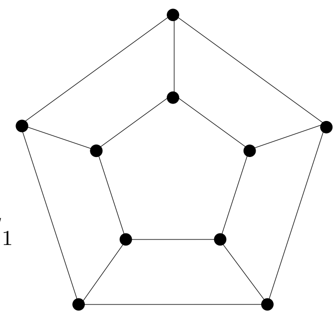
with initial conditions $b_0 = 1,\ b_1 = 0,\ b_2 = 1.$

(b) Solve the nonhomogeneous linear recurrence relation

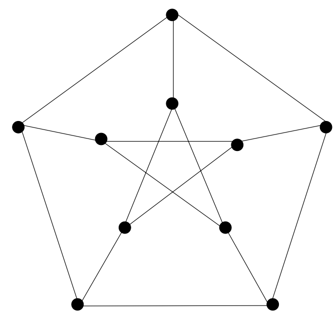
$$a_n - 3a_{n-1} - 4a_{n-2} = 6n - 11 \qquad \forall n \geq 3$$

with initial conditions $a_1 = 5$ and $a_2 = 12$.

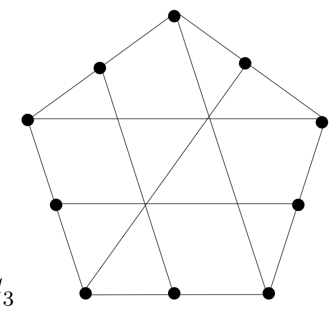
[3] **9.** Determine which pairs of the following graphs are isomorphic. For those which are not isomorphic, give a reason.



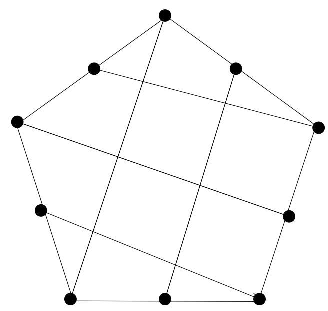
$G_1$



$G_2$



$G_3$



$G_4$

[9] **10.** Let $G(n, r)$ denote the graph whose vertices are subsets of $\{1, 2, \ldots, n\}$ of size $r$, and whose edges are pairs of subsets of $\{1, 2, \ldots, n\}$ of size $r$ which are disjoint.

(a) Draw $G(3, 1)$ and $G(4, 2)$.

(b) Determine the number of vertices in $G(n, r)$.

(c) Determine the degree of every vertex of $G(n, r)$.

(d) Determine the number of edges in $G(n, r)$.

(e) Prove that $G(3r, r)$ is connected for all $r \geq 1$.

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