MATH 239 Tutorial 1

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PROBLEMS

1. (Problem set 1.3, question 1) Consider the identity $\Sigma_{i\geq 0}\binom{n}{2i} = \Sigma_{i\geq 0}\binom{n}{2i+1}$.
   (a) Give a combinatorial proof.
   (b) Give an algebraic proof.
2. (Problem set 1.3, question 4) Consider the identity $\Sigma_{i=0}^{n}\binom{n}{i}i = n2^{n-1}$.
   (a) Give a combinatorial proof.
   (b) Give an algebraic proof.
3. (Problem set 1.4, question 5) Let $S = \{1, 2, \ldots\}$. Find the generating series $\Phi_S(x)$ if:
   (a) $w(i) = i$ for all $i \in S$.
   (b) $w(i) = i$ if i is even, or $w(i) = i - 1$ if i is odd.
4. (Problem set 1.4, question 7) Let S be the set of all subsets of $\{1, 2, 3, 4, 5\}$. For a subset A, its weight $w(A)$ is defined to be the number of pairs of consecutive integers in A.
   (a) Find the generating series $\Phi_S(x)$ of S with respect to w.
   (b) Calculate $\Phi_S(1)$.

SOLUTIONS

1. (a) Let $S = \{1, \ldots, n\}$. The left side of the identity counts the number of even subsets of S; the right side counts the number of odd subsets. Let S_E be the set of even subsets of S and let S_O be the set of odd subsets of S. If we can find a bijection between S_E and S_O, then $|S_E| = |S_O|$ so the equation will be true.

   Consider the function f from S_E to S_O that maps a subset $A \in S_E$ to $A \setminus \{1\}$ if $1 \in A$, or $A \cup \{1\}$ if $1 \notin A$.

   - It is onto: Consider any odd subset $B \in S_O$. If it contains 1, its pre-image is the even subset $B \setminus \{1\}$. If it doesn't contain 1, then its pre-image is $B \cup \{1\}$.

- It is one-to-one: Suppose A and A' are distinct even subsets of S. Then, A' contains at least one element, say k, not in A. Consider $f(A)$ and $f(A')$. If $k = 1$, then $f(A) = A \cup \{1\}$ and $f(A') = A \setminus \{1\}$, so the images under f are distinct. If k is not 1, then $f(A)$ doesn't contain k and $f(A')$ still does, so again the images are distinct.

There is a bijection from S_E to S_O, so $|S_E| = |S_O|$.

(b) For an algebraic proof, we try to find a way to apply the Binomial Theorem. We can re-arrange the identity's terms:

$$\begin{aligned}\Sigma_{i\geq 0}\binom{n}{2i} &= \Sigma_{i\geq 0}\binom{n}{2i+1}\\ \Leftrightarrow 0 &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n}\\ \Leftrightarrow 0 &= \Sigma_{i=0}^{n}\binom{n}{i}(-1)^i\end{aligned}$$

Apply the Binomial Theorem with $x = -1$:

$$\begin{aligned}(1+(-1))^n &= \Sigma_{i=0}^{n}\binom{n}{i}(-1)^i\\ 0 &= \Sigma_{i=0}^{n}\binom{n}{i}(-1)^i\end{aligned}$$

2. (a) (Proof 1) Let $S = \{1, \ldots, n\}$. Consider the sum of the sizes of all subsets of S.

There are $\binom{n}{k}$ subsets of size k, so the sum of the sizes of all subsets of S is $\Sigma_{i=0}^{n}\binom{n}{i}i$.

Another way to count the sets is to partition them into subsets A_1 and A_{NO-1} according to whether they contain or do not contain the element 1. Consider the function f from A_1 to A_{NO-1} that maps a subset of S to its complement:

- It is onto: Consider any subset $A \in A_{NO-1}$. Its pre-image is also its complement, $S \setminus A$.
- It is one-to-one: Suppose A and A' are distinct subsets of S that each contain 1. Then, A' contains at least one element, say k, not in A. The image $f(A)$ of A will contain k, but $f(A')$ will not, so the images are distinct.

So there is a bijection from A_1 to A_{NO-1}. We know that $|A_1| = |A_{NO-1}| = 2^{n-1}$. Consider pairing all subsets of S with their complements. The sum of the size of a subset and the size of its complement is $|S| = n$, and there are 2^{n-1} such pairs. Hence, the sum of the sizes of all subsets of S is $n2^{n-1}$.

(Proof 2) Again, let $S = \{1, \ldots, n\}$. Consider the number of ways we can pick a subset of S with a distinguished element.

There are $\binom{n}{k}$ subsets of size k, and k ways to pick a distinguished element from each such subset. So, the number of ways to pick a subset of S with a distinguished element is $\Sigma_{i=0}^{n}\binom{n}{i}i$.

Another way to pick subsets of S that have distinguished elements is to first pick the distinguished element, then choose a subset of S containing it. There are n possible distinguished elements, and we can choose any of the other $n-1$ elements to be in a subset with it. There are 2^{n-1} subsets of S that contain the chosen distinguished element. Hence, there are $n2^{n-1}$ ways to choose a subset of S with a distinguished element.

(b) (Proof 1) Expand the left side of the identity:

$$
\begin{aligned}
\Sigma_{i=0}^{n}\binom{n}{i}i &= \binom{n}{0}0+\binom{n}{1}1+\binom{n}{2}2+\ldots+\binom{n}{n}n \\
&= \frac{n!}{(n-1)!\cdot 1!}1+\frac{n!}{(n-2)!\cdot 2!}2+\ldots+\frac{n!}{(n-n)!\cdot n!}n \\
&= n\left(\frac{(n-1)!}{(n-1)!\cdot(1-1)!}+\frac{(n-1)!}{(n-2)!\cdot(2-1)!}+\ldots+\frac{(n-1)!}{(n-n)!\cdot(n-1)!}\right) \\
&= n\left(\frac{(n-1)!}{(n-1)!\cdot 0!}+\frac{(n-1)!}{(n-2)!\cdot 1!}+\ldots+\frac{(n-1)!}{0!\cdot(n-1)!}\right) \\
&= n\left(\Sigma_{i=0}^{n-1}\binom{n-1}{i}\right) \\
\Sigma_{i=0}^{n}\binom{n}{i}i &= n2^{n-1}
\end{aligned}
$$

(Proof 2) Consider the function $f(x)=(1+x)^n$. Its derivative is $f'(x)=n(1+x)^{n-1}$. By the Binomial Theorem, $f(x)=\Sigma_{i=0}^{n}\binom{n}{i}x^i$, so its derivative can also be expressed as $f'(x)=\Sigma_{i=0}^{n}i\binom{n}{i}x^{i-1}$.
Now evaluate the derivative at $x=1$. On one hand, it's equal to $n(1+1)^{n-1}=n2^{n-1}$. On the other hand, it's equal to $\Sigma_{i=0}^{n}i\binom{n}{i}$.

3. (a) The exponents of the terms in the generating series will be the weights: $1,2,3,\ldots$. The coefficients will be the numbers of elements with a particular exponent as their weight.

$$
\begin{aligned}
\Phi_S(x) &= 0x^0+1x^1+1x^2+1x^3+\ldots \\
\Phi_S(x) &= x+x^2+x^3+\ldots
\end{aligned}
$$

(b) The exponents of the terms in the generating series will be the weights: $0,2,4,6,\ldots$. The coefficients will be the numbers of elements with a particular exponent as their weight.

$$
\begin{aligned}
\Phi_S(x) &= x^0+2x^2+2x^4+2x^6+\ldots \\
\Phi_S(x) &= 1+2x^2+2x^4+2x^6+\ldots
\end{aligned}
$$

4. (a) The exponents of the terms in the generating series will be the weights: $0, 1, 2, 3, 4$. The coefficients will be the number of elements of S (i.e. subsets of $\{1, 2, 3, 4, 5\}$) with that particular exponent as their weight.

weight (k)	frequency ($[x^k]$)	frequency by subset size 0	1	2	3	4	5	subsets
0	13	1	5	6	1	0	0	$\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$, $\{3,5\}$, $\{1,3,5\}$
1	10	0	0	4	6	0	0	$\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{4,5\}$, $\{1,3,4\}$, $\{1,4,5\}$, $\{2,4,5\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{2,3,5\}$
2	6	0	0	0	3	3	0	$\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,5\}$, $\{1,3,4,5\}$, $\{1,2,3,5\}$, $\{1,2,4,5\}$
3	2	0	0	0	0	2	0	$\{1,2,3,4\}$, $\{2,3,4,5\}$
4	1	0	0	0	0	0	1	$\{1,2,3,4,5\}$

So, the generating series is $\Phi_S(x) = 13 + 10x + 6x^2 + 2x^3 + x^4$.

(b) We don't need to compute the sum of the coefficients to determine $\Phi_S(1)$. Every element of S (subset of $\{1, 2, 3, 4, 5\}$) contributes 1 to some coefficient, so $\Phi_S(1)$ will always equal $|S|$. In this case, S is the set of all subsets of $\{1, 2, 3, 4, 5\}$, so $\Phi_S(1) = 2^5 = 32$.

Note that we could have considered binary strings of length 5 instead of subsets of $\{1, 2, 3, 4, 5\}$. Consecutive integers in a subset correspond to adjacent 1s in a bitstring.