COMBINATORICS & OPTIMIZATION

University of Waterloo

Introduction to Combinatorics

Lecture 5

http://info.iqc.ca/mmosca/2014math239

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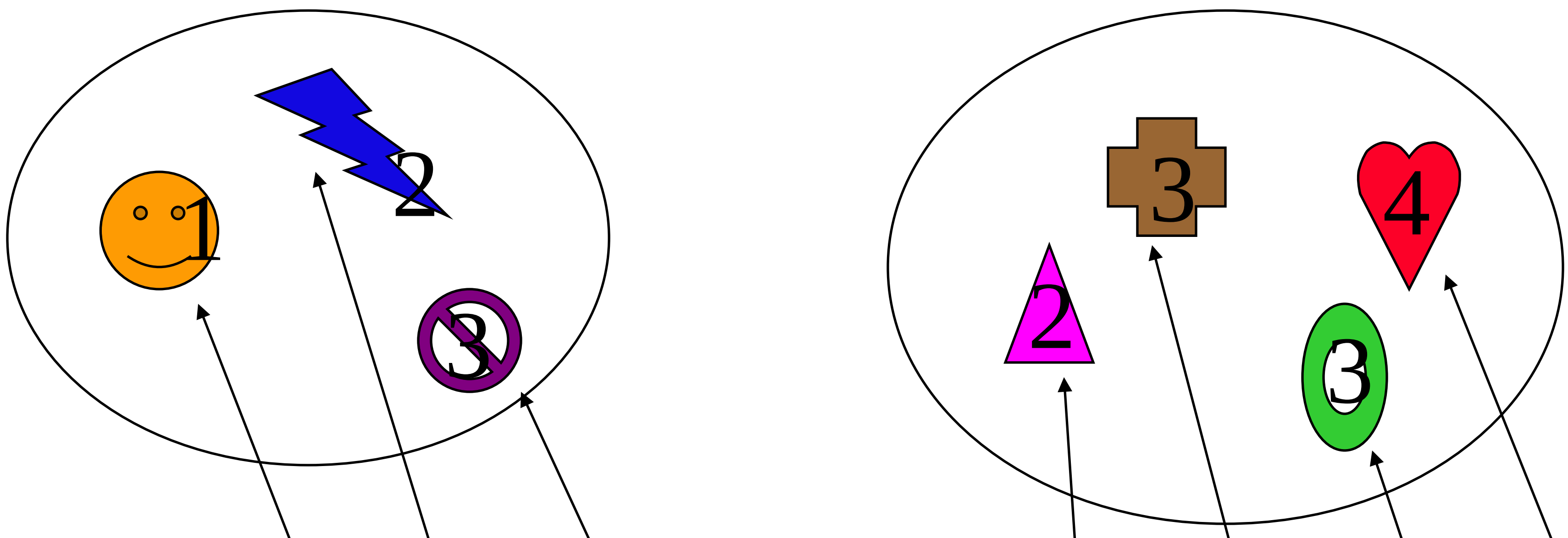
Consider the generating functions of A and B

A

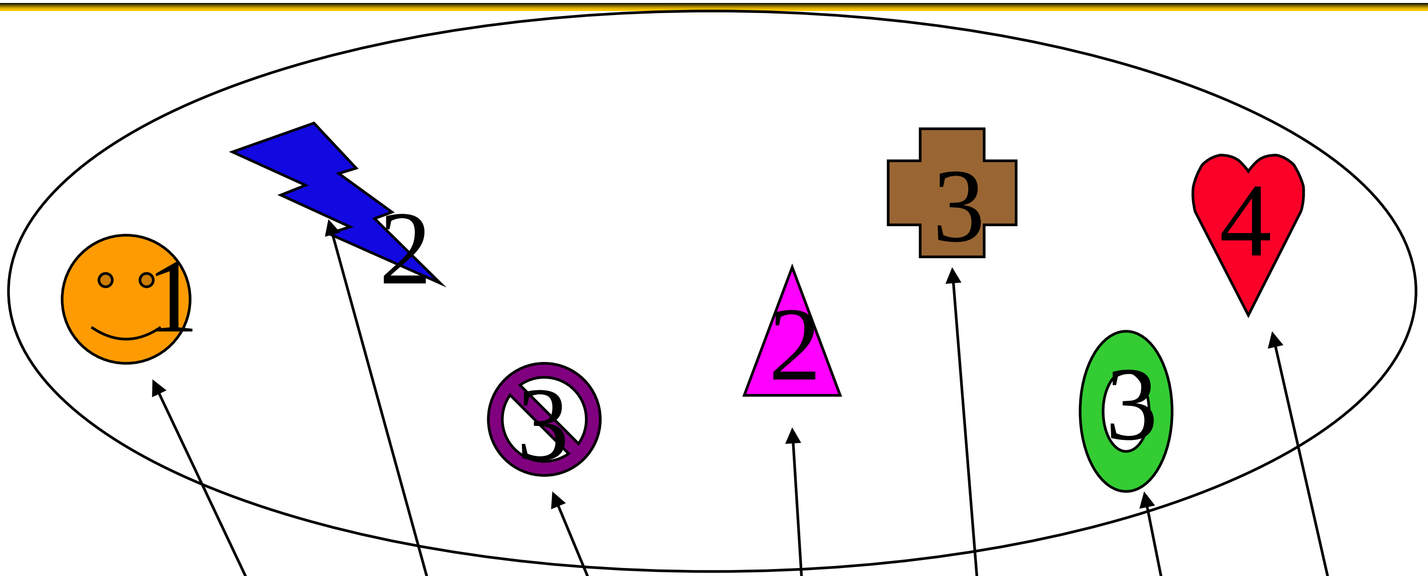
B



$$\Phi_A(x) = x + x^2 + x^3 \qquad \Phi_B(x) = x^2 + x^3 + x^3 + x^4$$

Consider the generating function of

$$S = A \bigcup B$$



$$\Phi_S(x) = x + x^2 + x^3 + x^2 + x^3 + x^3 + x^4$$

$$= \left(x + x^2 + x^3\right) + \left(x^2 + x^3 + x^3 + x^4\right)$$

$$= \Phi_A(x) + \Phi_B(x)$$

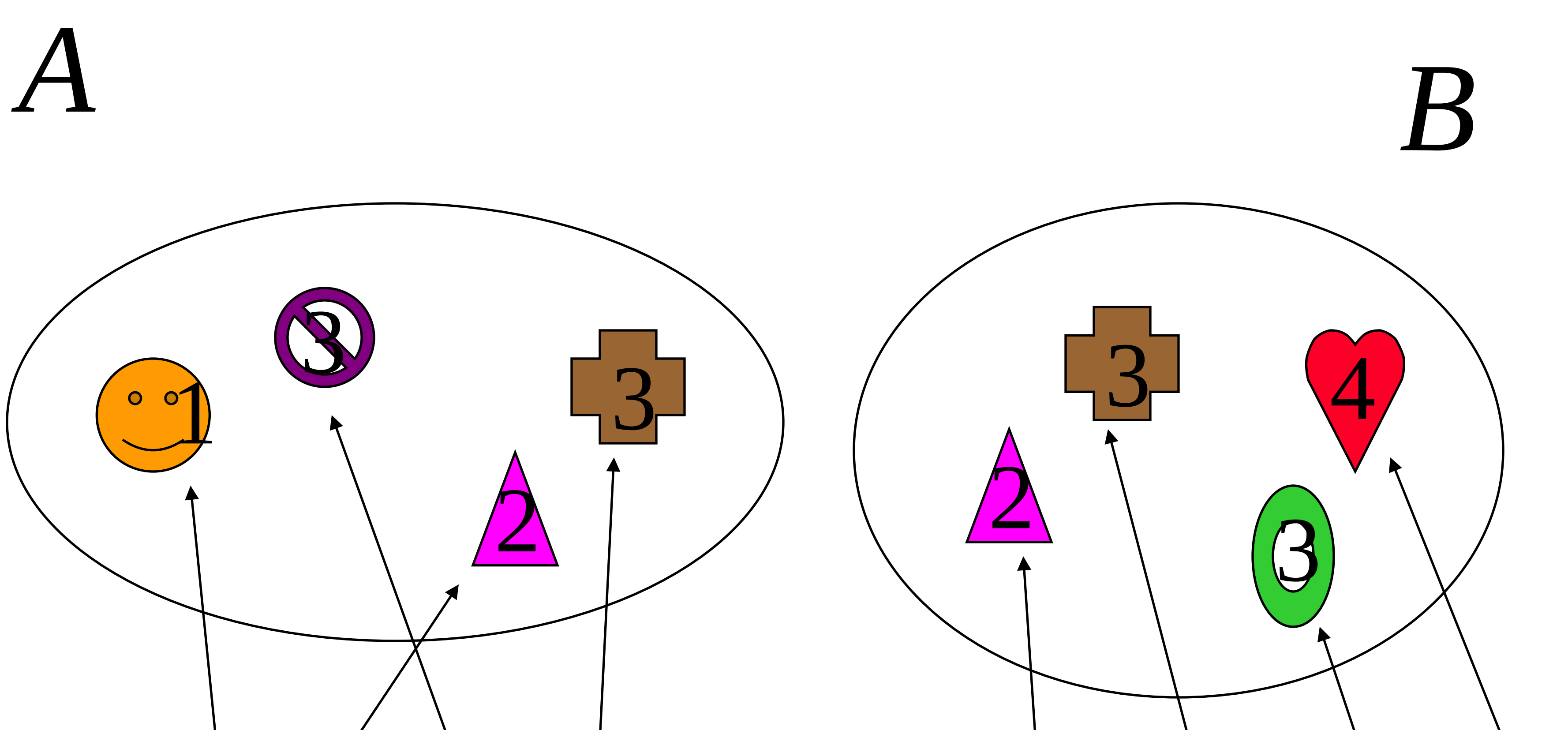
The Sum Lemma

THM1.6.1: Let (A,B) be a partition of a set S (i.e. $S = A \dot{\bigcup} B$, the disjoint union of A and B).

Then:

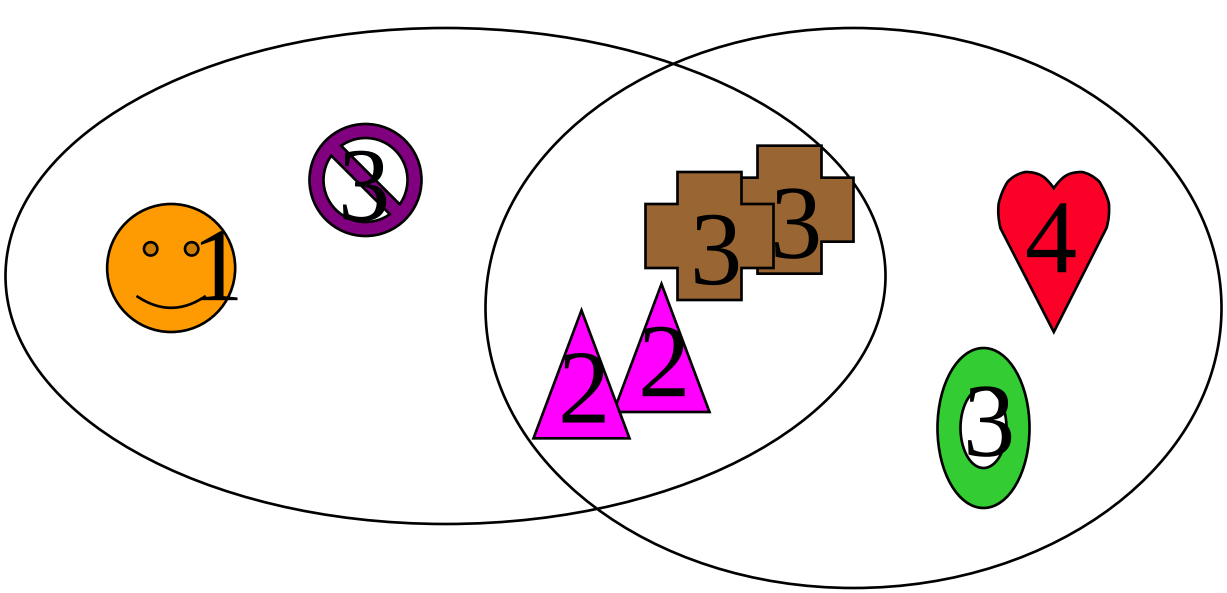
$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$$

Consider the generating functions of A and B

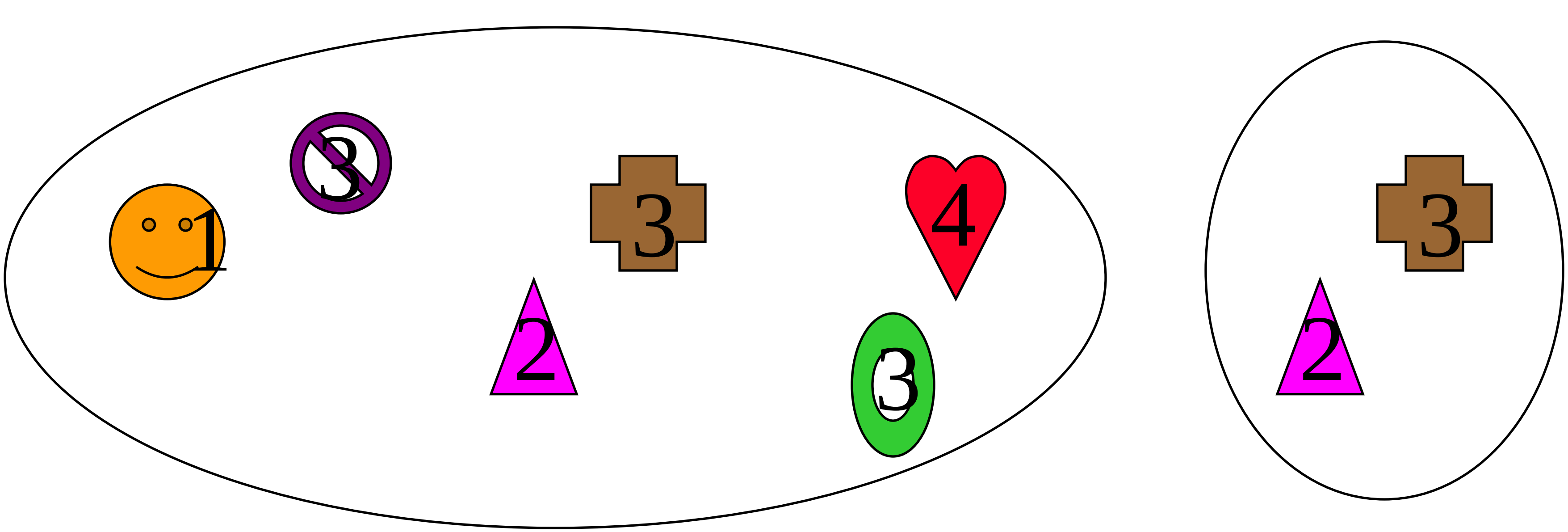


$$\Phi_A(x) = x + x^2 + x^3 + x^3 \qquad \Phi_B(x) = x^2 + x^3 + x^3 + x^4$$

Consider $S = A \bigcup B$ and $T = A \bigcap B$

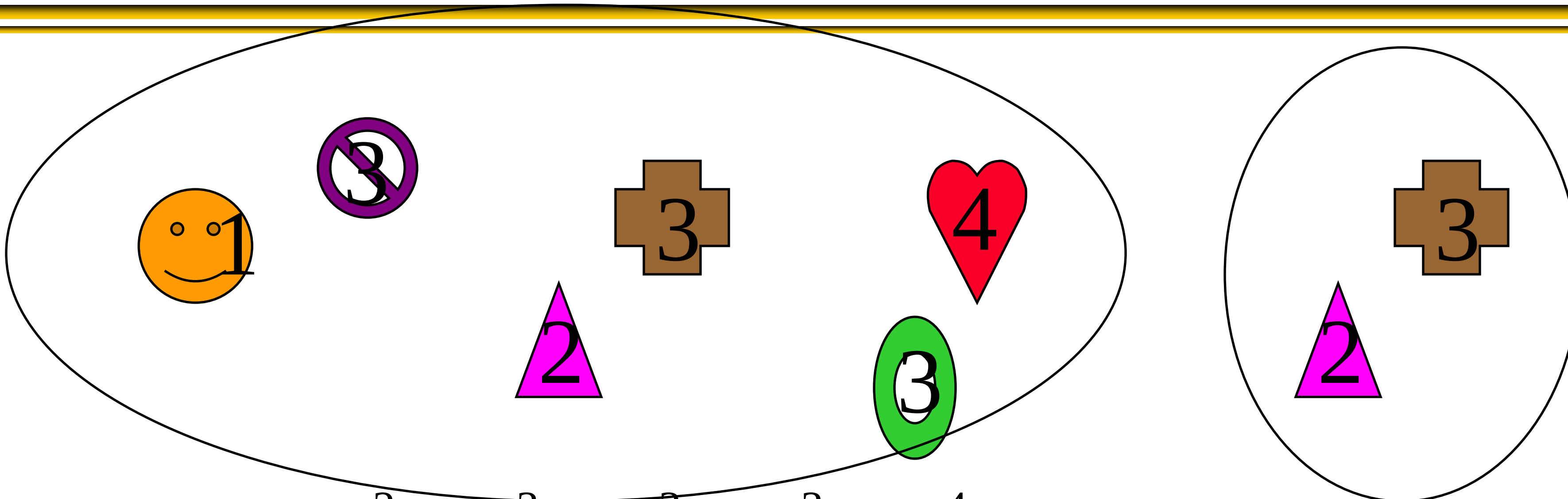


Consider $S = A \bigcup B$ and $T = A \bigcap B$



$S = A \bigcup B$ $\qquad$ $T = A \bigcap B$

$$S = A \bigcup B \qquad T = A \bigcap B$$



$$\Phi_S(x) = x + x^2 + x^3 + x^3 + x^3 + x^4$$

$$\Phi_T(x) = x^2 + x^3$$

$$\Phi_S(x) + \Phi_T(x)$$

$$= \left(x + x^2 + x^3 + x^3\right) + \left(x^2 + x^3 + x^3 + x^4\right)$$

$$= \Phi_A(x) + \Phi_B(x)$$

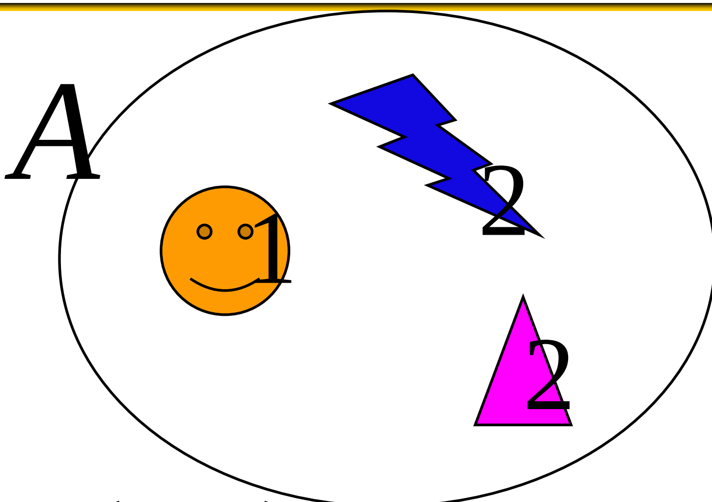
The Sum Lemma

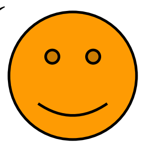
THM1.6.1*: Let S be the union of A and B (i.e. $S = A \bigcup B$).

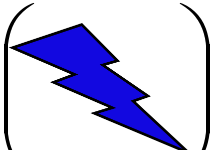
Then:

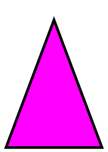
$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x) - \Phi_{A \bigcap B}(x)$$

Consider A, B



$\omega_A\Big($  $\Big) = 1$

$\omega_A\Big($  $\Big) = 2$

$\omega_A\Big($  $\Big) = 2$

$\omega_B\big(\text{University of Waterloo}\big) = 2$

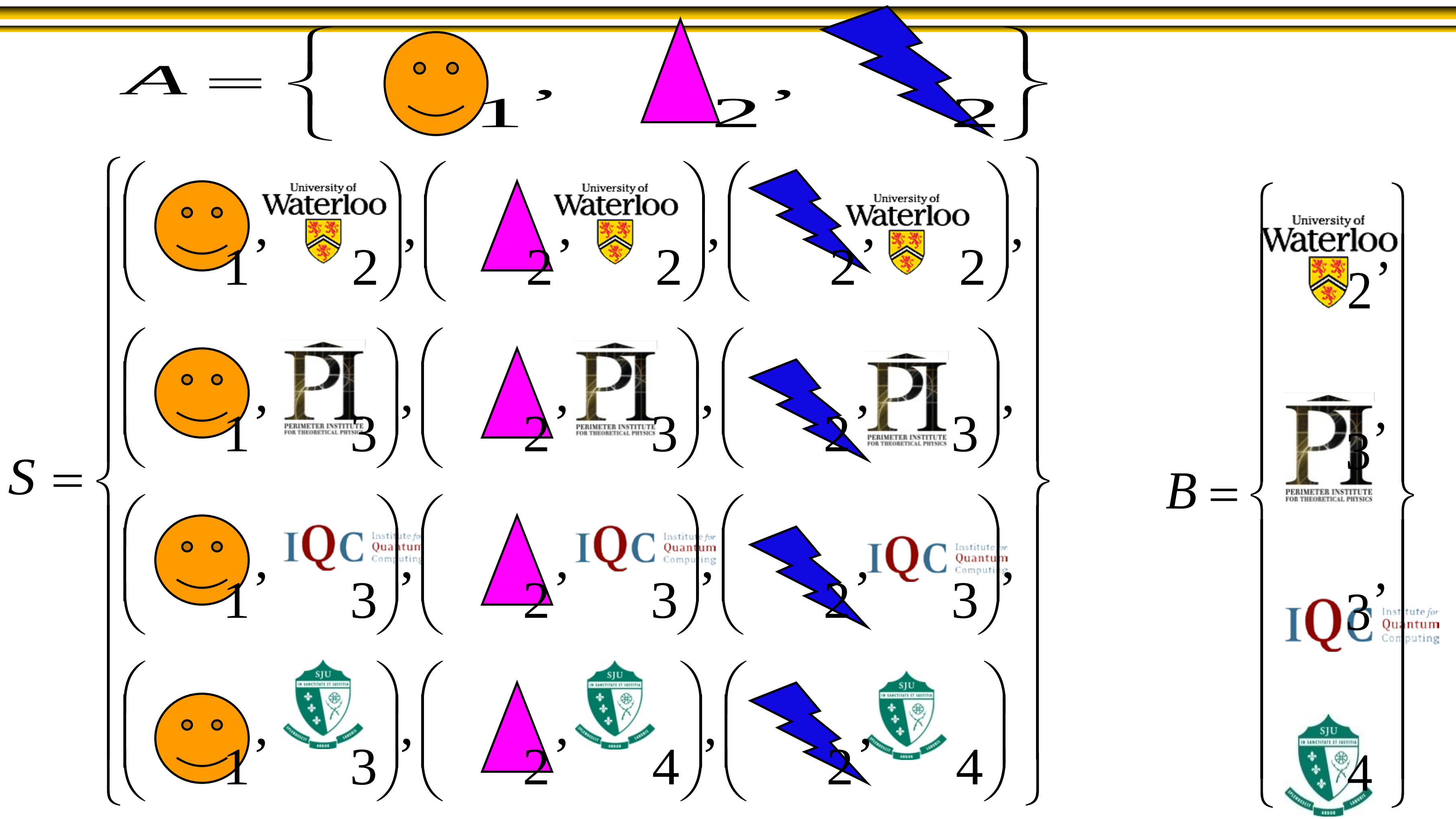
$\omega_B\big(\text{Perimeter Institute for Theoretical Physics}\big) = 3$

$\omega_B\big(\text{IQC Institute for Quantum Computing}\big) = 3$

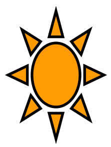
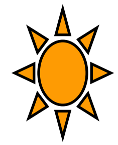
$\omega_B\big(\text{SJU}\big) = 4$

Consider the *Cartesian product*

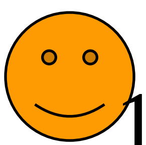
$$S = A \times B$$



Generalizing the weight function

$\omega_S\Big(\big($  $, \;\big)\Big) = \omega_A\big($  $\big) + \omega_B\big(\;\big)$

e.g.

$\omega\Big(\big($  $_1, \;_3\big)\Big) = \omega_A\big($  $_1\big) + \omega_B\big(\;_3\big)$

$$= 1 + 3 = 4$$

Consider $S = A \times B$

$$\Phi_A = x^1 + x^2 + x^2$$

$$\Phi_S(x) = x^{1+2} + x^{2+2} + x^{2+2}$$
$$+ x^{1+3} + x^{2+3} + x^{2+3}$$
$$+ x^{1+3} + x^{2+3} + x^{2+3}$$
$$+ x^{1+4} + x^{2+4} + x^{2+4}$$

$$\Phi_B(x)$$
$$= x^2$$
$$+ x^3$$
$$+ x^3$$
$$+ x^4$$

Consider $S = A \times B$

$$\Phi_A = x^1 + x^2 + x^2$$

$$\Phi_S(x) = x^3 + x^4 + x^4$$

$$+ x^4 + x^5 + x^5$$

$$+ x^4 + x^5 + x^5$$

$$+ x^5 + x^6 + x^6$$

$$= \Phi_A(x) \cdot \Phi_B(x)$$

$$\Phi_B(x)$$

$$= x^2$$

$$+ x^3$$

$$+ x^3$$

$$+ x^4$$

The Product Lemma

THM1.6.2: Let A and B be sets of objects with weight functions ω_A and ω_B respectively. Let $S = A \times B$ and suppose $\omega(\sigma) = \omega_A(a) + \omega_B(b)$ for each $\sigma = (a,b) \in A \times B$

Then:

$$\Phi_S^{\omega}(x) = \Phi_A^{\omega_A}(x) \times \Phi_B^{\omega_B}(x)$$

The Product Lemma Generalized

Let $A_1, A_2, \cdots, A_k$ be sets of objects with weight functions $\omega_1, \omega_2, \cdots, \omega_k$ respectively. Let $S = A_1 \times A_2 \times \cdots \times A_k$ and suppose $\omega(\sigma) = \omega_1(a_1) + \omega_2(a_2) + \cdots + \omega_k(a_k)$ for each

$$\sigma = (a_1, a_2, \cdots, a_k) \in A_1 \times A_2 \times \cdots \times A_k$$

Then:

$$\Phi_S^{\omega}(x) = \Phi_{A_1}^{\omega_1}(x) \times \Phi_{A_2}^{\omega_2}(x) \times \cdots \times \Phi_{A_k}^{\omega_k}(x)$$

The Product Lemma Generalized

e.g. suppose $A = A_1 = A_2 = \cdots = A_k$ and $\omega_1 = \omega_2 = \cdots = \omega_k$

Then:

$$\Phi_S^{\omega}(x) = \Phi_A^{\omega_1}(x) \times \Phi_A^{\omega_1}(x) \times \cdots \times \Phi_A^{\omega_1}(x)$$

$$= \left(\Phi_A^{\omega_1}(x)\right)^k$$

Problem 1.6.2: Let k,n be fixed non-negative integers. How many solutions are there to

$t_1 + \ldots + t_n = k$, where $t_1, \ldots, t_n \in \{0,1\}$?

Let S be the set of all k-tuples $(a_1, a_2, \cdots, a_n)$ where each $a_i \in \{0,1\}$.

Define the weight of a k-tuple $\sigma = (a_1, a_2, \cdots, a_n)$ by

$\omega(\sigma) = a_1 + a_2 + \cdots + a_n$

(implicitly $\omega_j(a_j) = a_j$ for all integers j).

We start by determining $\Phi_S(x)$.

Denote $A = \{0,1\}$.

So $S = A \times A \times \cdots \times A = A^n$

Since $$A = \{0,1\}$$

and $\omega(n) = n$, we can easily compute

$$\Phi_A(x) = 1 + x$$

By the Product Lemma, we therefore have

$$\Phi_S(x) = (\Phi_A(x))^n = (1+x)^n$$

The number of solutions to $t_1 + \ldots + t_n = k$ is

$$[x^k]\Phi_S(x) = \binom{n}{k}$$

Problem 1.6.4: Let S be the set of all k-tuples $(a_1, a_2, \cdots, a_k)$ where each $a_1 \in Z_{\geq 0}$. Define the weight of a k-tuple $\sigma = (a_1, a_2, \cdots, a_k)$ by $\omega(\sigma) = a_1 + a_2 + \cdots + a_k$ (implicitly $\omega_j(a_j) = a_j$ for all integers j).

Determine $\Phi_S(x)$.

Solution: Let $A = Z_{\geq 0}$.

Let $S = A \times A \times \cdots \times A = A^k$

Since $A = Z_{\geq 0} = \{0,1,2,3\cdots\}$

and $\omega(n) = n$, we can easily compute

$$\Phi_A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

By the Product Lemma, we therefore have

$$\Phi_S(x) = (\Phi_A(x))^k = \left(\frac{1}{1-x}\right)^k = (1-x)^{-k}$$

Extracting coefficients

We mentioned earlier (Thm 1.6.5) how to extract coefficients from this generating function:

$$\Phi_S(x) = (1-x)^{-k} = \sum_{j=0}^{\infty} \binom{k+j-1}{k-1} x^j$$

So we can now easily determine, for any integer j, the number of k-tuples that sum to j:

$$[x^j]\Phi_S(x) = \binom{k+j-1}{k-1}$$