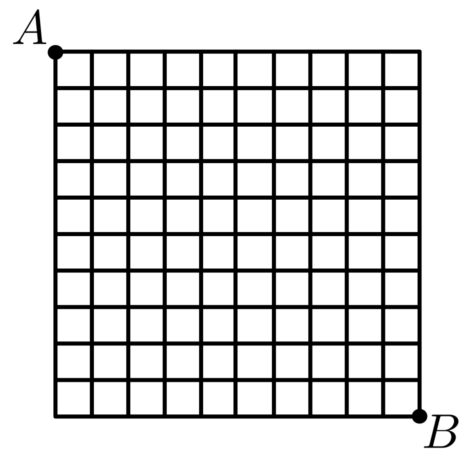
MATH 239 Spring 2012: Midterm
Solutions

1. Combinations and permutations

   (a) {5 marks} Ying lives at location A and works at location B as indicated on the following map. Note that Ying needs to travel at least 20 blocks to go to work, namely 10 block south and 10 block east. Find a formula for the number of ways Ying can get from home to work (under the condition that she only travels 20 blocks).



   Solution. Ying needs to travel exactly 20 blocks, 10 of them have to go south, 10 of them have to go east. Once we have chosen which 10 of the 20 steps are in the southernly direction, we have determined the whole path. So there are $\binom{20}{10}$ ways to do this in total.

   (b) {5 marks} The n children of the Von Trapp family all have different ages. Whenever they sing, they will all line up so that the oldest child is to the right of the youngest child (it does not have to be directly to the right, for example, the oldest child could be 3 to the right of the youngest child). How many ways can they line up?

   Solution. We could formulate the problem as the set S of all permutations σ of $[n]$ where $\sigma(1) < \sigma(n)$. Let T be the set of all permutations where $\sigma(1) > \sigma(n)$. Then there is a bijection between S and T by switching $\sigma(1)$ and $\sigma(n)$. Since $S \cup T$ is the set of all permutations and $|S| = |T|$, $|S| = n!/2$.

2. Identities

   Consider the following identity:

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}.$$

   (a) {5 marks} Give a combinatorial proof of this identity.

   Solution. Let S be the set of all n-subsets of $[2n]$. Partition S into $S_0, S_1, \ldots, S_n$ where S_i consists of all n-subsets of $[2n]$ which has exactly i elements from $[n]$. Any subset in S_i has i elements from $[n]$ and $n-i$ elements from $[2n]\setminus[n]$. Therefore, $|S_i| = \binom{n}{i}\binom{n}{n-i}$. Since $S = \bigcup_{i=0}^{n} S_i$ is a disjoint union, $|S| = \sum_{i=0}^{n} |S_i|$, and the identity holds.

   (b) {5 marks} Give an algebraic proof of this identity. (You may assume the Binomial theorem.)

Solution. We see that $\binom{2n}{n} = [x^n](1+x)^{2n}$. Also,

$$\begin{aligned}[x^n](1+x)^{2n} &= [x^n](1+x)^n(1+x)^n \\ &= [x^n]\sum_{k=0}^{n}\binom{n}{k}x^k\sum_{i=0}^{n}\binom{n}{i}x^i \qquad \text{set } i = n-k \\ &= \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}.\end{aligned}$$

Hence the identity holds.

3. Generating series and compositions

(a) {5 marks} The coffee shop, Tom Hirtons, has a special offer, 10 donuts for 12 dollars. Cathy decides to take advantage of the special offer and eat at Tom Hirtons once a day, for as long as she does not have to eat the same collection of 10 donuts more than once. The donuts come in five different flavours: chocolate, maple glazed, lemon, walnut, and boston cream. How many times will Cathy eat at Tom Hirtons, or equivalently how many ways are there to select 10 donuts with 5 different flavours? Note that it is not necessary to include all five flavours in one selection. Write your answer as the coefficient of x^n (for some n) of a power series in rational form.

Solution. Let S_1, S_2, S_3, S_4, S_5 represent possible numbers of donuts that Cathy buys for each of the five flavours. Then $S_i = \{0, 1, 2, \ldots\}$. For each i, $\Phi_{S_i}(x) = 1/(1-x)$. The possible selections of donuts can be represented by the cartesian product $S_1 \times S_2 \times S_3 \times S_4 \times S_5$. Using the product lemma, the generating series for this is $(\frac{1}{1-x})^5$. Since Cathy is buying 10 donuts, the number of ways to get them is

$$[x^{10}]\left(\frac{1}{1-x}\right)^5.$$

(b) {5 marks} Prove that for $n \geq 2$, the number of compositions of n where the first part is 1 is equal to the number of compositions of n where the first part is greater than 1.

Solution. Let S be the set of compositions of n where the first part is 1, let T be the set of compositions of n where the first part is greater than 1. We can find a bijection between S and T as follows: Let $(1, a_1, a_2, \ldots, a_k) \in S$. Define $f : S \to T$ by

$$f(1, a_1, a_2, \ldots, a_k) = (a_1 + 1, a_2, \ldots, a_k).$$

We can easily find its inverse: Let $(a_1, a_2, \ldots, a_k) \in T$. Define $g : T \to S$ by

$$g(a_1, a_2, \ldots, a_k) = (1, a_1 - 1, a_2, \ldots, a_k).$$

Since this is a bijection, $|S| = |T|$.

4. Binary strings

(a) {5 marks} Let S be the set of binary strings with the following decomposition:

$$S = (\{0\}^*\{11\}\{1\}^*)^*$$

Explain why this is an ambiguous expression for S.

Solution. The string 1111 can be expressed in two ways.

$$1111 = (\{0\}^0\{11\}\{1\}^2), \quad (\{0\}^0\{11\}\{1\}^0)(\{0\}^0\{11\}\{1\}^0).$$

Hence this is an ambiguous expression.

(b) {5 marks} Suppose the weight of a string is its length. Let $\Phi_S(x)$ be the generating series for S. Suppose we obtain $\Phi'_S(x)$ by applying the product and sum lemmas for strings using the ambiguous expression in part (a). Which of the following are true?

i. For all $n \geq 1$, $[x^n]\Phi_S(x) > [x^n]\Phi'_S(x)$.
ii. For all $n \geq 1$, $[x^n]\Phi_S(x) < [x^n]\Phi'_S(x)$.
iii. There exists an $n \geq 1$ such that $[x^n]\Phi_S(x) > [x^n]\Phi'_S(x)$.
iv. There exists an $n \geq 1$ such that $[x^n]\Phi_S(x) < [x^n]\Phi'_S(x)$.

Circle all that are true, and briefly justify your answer.

Solution. Only iv is true. For $n = 4$, we saw that the string 1111 contributes at least 2 to the coefficient of x^4 in $\Phi'_S(x)$, however, it only contributes 1 to the same coefficient in $\Phi_S(x)$. Hence $[x^4]\Phi_S(x) < [x^4]\Phi'_S(x)$. (Note: Choice ii is not true since for $n = 1$, the coefficient is 0 for both series.)

(c) {5 marks} Let $\mathcal{A}$ be the set of all binary strings in which every block of 0's has even length, and every block of 1's has length at least two. Determine an unambiguous expression that generates every string in $\mathcal{A}$.

Solution. Derive from the block decomposition: $\{00\}^*(\{11\}\{1\}^*\{00\}\{00\}^*)^*(\varepsilon \cup \{11\}\{1\}^*)$.

5. Recurrences

{10 marks} Solve the following recurrence relation

$$a_n - 4a_{n-2} = -3n + 8$$

for the initial values $a_0 = 2, a_1 = 1$. (Notice that the index of the second term in the recurrence is $n - 2$.)

Solution. The characteristic polynomial of this recurrence is $x^2 - 4 = (x + 2)(x - 2)$. So its roots are 2 and -2. The general solution to the homogeneous part of this recurrence is $A \cdot 2^n + B \cdot (-2)^n$.

Let b_n be a specific solution to the nonhomogeneous recurrence. Suppose $b_n = \alpha n + \beta$ for some constants α, β. Then

$$\begin{aligned} b_n - 4b_{n-2} &= (\alpha n + \beta) - 4(\alpha(n-2) + \beta) \\ &= (\alpha - 4\alpha)n + (\beta + 8\alpha - 4\beta) \\ &= -3\alpha n + (8\alpha - 3\beta). \end{aligned}$$

Since this equals $-3n + 8$, we get $\alpha = 1$ and $\beta = 0$. Therefore, $b_n = n$.

So the general solution to a_n is

$$a_n = n + A \cdot 2^n + B \cdot (-2)^n.$$

Using the initial conditions $a_0 = 2, a_1 = 1$, we get

$$2 = A + B, \quad 1 = 1 + 2A - 2B.$$

This gives us $A = 1, B = 1$. Hence the solution to the recurrence is

$$a_n = n + 2^n + (-2)^n.$$

6. Graph theory

(a) {5 marks} A group of 1337 software engineers gathered to change a lightbulb. Some of them have 13 friends within the group, some have 33 friends, and the rest have 37 friends. Prove (using graph theory) that this group of software engineers does not exist.

Solution. Let each softie be a vertex, and suppose two vertices are adjacent if their corresponding softies are friends. Each vertex has degree either 13, 33, or 37, so every vertex has odd degree. Also, there are 1337 vertices in this graph, which is also an odd number. So this graph has odd number of odd-degree vertices, which contradicts the corollary to the handshaking lemma. Hence, this group of softies does not exist.

(b) {5 marks} Let G_1, G_2, G_3 be three graphs where G_1 is isomorphic to G_2 and G_2 is isomorphic to G_3. Using the definition of isomorphism, explain why G_1 is isomorphic to G_3.

Solution. Since G_1 is isomorphic to G_2, there exists a bijection $f : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. Similarly, since G_2 is isomorphic to G_3, there exists a bijection $g : V(G_2) \to V(G_3)$ such that $xy \in E(G_2)$ if and only if $g(x)g(y) \in E(G_3)$. The composition of two bijections is a bijection, so $g \circ f : V(G_1) \to V(G_3)$ is a bijection, and $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$ if and only if $g(f(u))g(f(v)) \in E(G_3)$. Hence $g \circ f$ is an isomorphism between G_1 and G_3, and these two graphs are isomorphic.

(c) {5 marks} Let $k \geq 2$. Prove that if G is k-regular, then G contains a cycle with at least $k+1$ edges.

Solution. Let $P = v_1, v_2, \ldots, v_m$ be a path of the longest length in G. The neighbours of v_1 has to be on P, for otherwise we can extend P to obtain a longer path. Since G is k-regular, v_1 has exactly k neighbours, and they are all vertices on P. One such neighbour v_i must be at least k away from v_1, so $i \geq k+1$. Then $v_1, v_2, \ldots, v_i, v_1$ is a cycle that uses at least $k+1$ edges.

(d) {5 marks} Suppose that G is a graph on at least 5 vertices. Prove that G and $\overline{G}$ cannot both be bipartite graphs.

(Recall: $\overline{G}$ is the complement of G, meaning $V(G) = V(\overline{G})$, and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.)

Solution. Suppose G is bipartite. Let (A, B) be a bipartition of G. Since G has at least 5 vertices, at least one of A and B has at least 3 vertices, say it is A. Let x, y, z be three vertices in A. Then xy, yz, xz are not edges in G, hence they are edges in $\overline{G}$, which forms a cycle of length 3, which is not bipartite. Since $\overline{G}$ contains a non-bipartite subgraph, $\overline{G}$ is not bipartite.