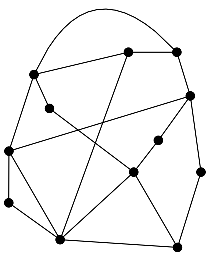
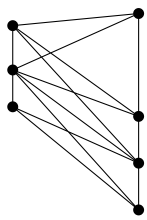
Tutorial 9

March 27, 2013

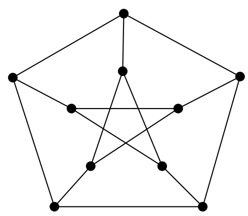
1. For each of the graphs shown, determine whether it is planar. If the graph is planar, exhibit a planar embedding. If the graph is not planar, exhibit a subdivision of K_5 or $K_{3,3}$.



a

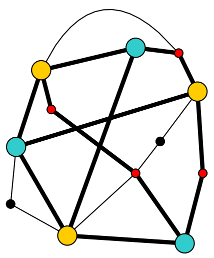


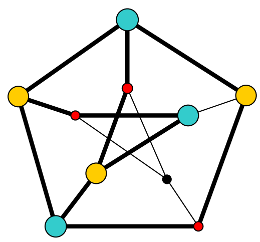
b

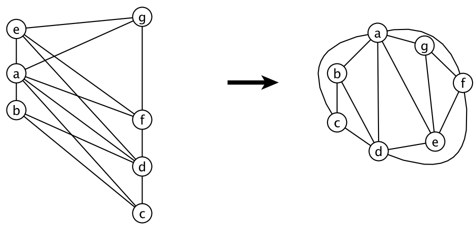


c

Solution. b) is the only planar graph from above. The following picture shows a subdivision of $K_{3,3}$ for a) and c) and a planar embedding for b).







2. Prove that if G is a connected planar graph with girth at least 6, then G is 3-colourable.

   Solution. By Euler's formula, $2 = p - q + f$, where p, q, f denote the number of vertices, edges, and faces of G, respectively. Since the girth of G is at least 6, every face has degree at least 6. So$6f \leq 2q$, and

$$\begin{aligned} 12 &= 6p - 6q + 6f \\ &\leq 6p - 6q + 2q \\ &= 6p - 4q. \end{aligned}$$

   This gives $q = \frac{3p-6}{2}$. Note that the average vertex degree of G is

$$\begin{aligned} \frac{2q}{p} &= \frac{3p-6}{p} \\ &= 3 - \frac{6}{p} \\ &< 3. \end{aligned}$$

   Since the average vertex degree of G is less than 3, G must have a vertex v of degree at most 2.

   We will complete the proof by induction. Suppose that every graph of girth at least 6, with fewer vertices than G, is 3-colourable. (This is clearly true for graphs with one vertex, so the base case holds.)
   It is easy to see that $G - v$ has girth at least 6 (removing a vertex can't decrease the length of the shortest cycle). Since $G - v$ has fewer vertices than G, the inductive hypothesis applies and $G - v$ has a 3-colouring. Now, v has at most two neighbours. So for some colour c used in the colouring of $G - v$, v does not have a neighbour of colour c. It follows that giving v colour c extends our 3-colouring of $G - v$ to a 3-colouring of G.

3. Suppose G has a planar embedding where every face boundary is an even cycle. Prove that G is bipartite.

   Solution. Consider an odd cycle C of length k that has the fewest number of faces inside it. C is not a facial cycle, so there is a path P inside it with endpoints in $V(C)$, let its length be l. Now $C \cup P$ has two cycles inside C, which have the l edges of P in common and share the k edges of C between each other. So one of them gets an odd number of edges from C, and the other one an even number of edges from C as k was odd. But that means that the lengths of these cycles have different parity, so one of these cycles is odd. Now both of these cycles are cycles in G with fewer faces inside them than C. But this contradicts the minimality of C with respect to having fewest facial cycles inside.

4. Let G is a graph on n vertices. Show that if G has two vertices u and v of degree $n - 1$ and $G - u - v$ has a cycle, then G is non-planar.

   Solution. Let C be the cycle in $G - u - v$, and take three consecutive vertices x, y, z along C. Note that u and v are connected to each of these vertices by an edge, and they are also connected to each other, as due to degree $n - 1$ they are connected to all other vertices. But then these three vertices of the cycle together with u and v give a subdivision of K_5, where the edge from x to z is subdivided into a path along C that does not use y.