Math 239 Tutorial 1

1. Let n and k be nonnegative integers. How many subsets of $\{1, 2, \ldots, 2n\}$ have exactly k odd integers?

Solution. There are n odd integers and n even integers in the set $\{1, 2, \ldots, 2n\}$. To construct a required set, we first choose exactly k odd integers to put in it, and then add to it any number of even integers. There are $\binom{n}{k}$ ways to choose the odd integers, and there are 2^n ways to pick a collection of even integers (including the empty collection). So the number of required subsets is $2^n\binom{n}{k}$. $\square$

2. Let n be a positive integer. Prove that

$$\sum_{i \text{ is even}} \binom{n}{i} = \sum_{i \text{ is odd}} \binom{n}{i}.$$

Solution. Let $\mathcal{E}$ be the set of all subsets of $\{1, 2, \ldots, n\}$ that have even sizes, and let $\mathcal{O}$ be the set of those that have odd sizes. Note that the left-hand side and the right-hand side of the desired identity are the sizes of $\mathcal{E}$ and $\mathcal{O}$, respectively, so it suffices to show that $|\mathcal{E}| = |\mathcal{O}|$. We do so by providing a bijection between $\mathcal{E}$ and $\mathcal{O}$. A way to do this is to provide a function $f : \mathcal{E} \to \mathcal{O}$ and a function $g : \mathcal{O} \to \mathcal{E}$ such that $g(f(A)) = A$ for all $A \in \mathcal{E}$ and $f(g(B)) = B$ for all $B \in \mathcal{O}$, in which case f and g are called *inverses* of each other. Consider the function $f : \mathcal{E} \to \mathcal{O}$ defined by

$$f(A) = \begin{cases} A \setminus \{1\}, & \text{if } 1 \in A \\ A \cup \{1\}, & \text{if } 1 \notin A. \end{cases}$$

Since adding an element to an even-sized set or removing an element from it yields an odd-sized set, f is well-defined. Similarly, the function $g : \mathcal{O} \to \mathcal{E}$ defined by

$$g(B) = \begin{cases} B \cup \{1\}, & \text{if } 1 \notin B \\ B \setminus \{1\}, & \text{if } 1 \in B \end{cases}$$

is well-defined. From the definition of f and g, it follows that f and g are inverses of each other. Therefore $|\mathcal{E}| = |\mathcal{O}|$, proving the identity. $\square$

3. Let $\mathcal{S}$ be the set of all subsets of $\{1, 2, \ldots, n\}$ whose elements sum up to an odd integer, where $n \geq 1$. Find $|\mathcal{S}|$.

Solution. Let O be the set of odd integers in $\{1, 2, \ldots, n\}$ and let E be the set of even integers in it. Note that a set in $\mathcal{S}$ has an odd number of odd integers and any number of even integers, from which it follows that $|\mathcal{S}| = |\mathcal{A} \times \mathcal{B}|$, where $\mathcal{A}$ is the set of all odd-sized subsets of O, and $\mathcal{B}$ is the set of all subsets of E. Problem 2 implies that among the subsets of a finite set, half of them have odd sizes. In particular, if $|O| = k$ then $|\mathcal{A}| = 2^k/2 = 2^{k-1}$. Clearly $\mathcal{B} = 2^{n-k}$, so

$$|\mathcal{S}| = |\mathcal{A} \times \mathcal{B}| = |\mathcal{A}| \cdot |\mathcal{B}| = 2^{k-1}2^{n-k} = 2^{n-1}.$$

□

4. For any integers n, k and r with $n \geq k \geq r \geq 0$, prove that

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$$

using a combinatorial argument.

Solution. Consider forming a sport team of k players with r leaders out of n persons. We can first pick k players out the n persons, and then select the r leaders out of the k players chosen. From this it follows that there are $\binom{n}{k}\binom{k}{r}$ ways to form the team. Another approach is to first select the r leaders directly from the n persons, and then choose the $k-r$ non-leaders among the remaining $n-r$ persons. From this it follows that there are $\binom{n}{r}\binom{n-r}{k-r}$ ways to form the team. Hence the identity follows. (Symbolically, both sides of the identity count the number of ordered pairs (A, B), where A is a k-subset of $\{1, 2, \ldots, n\}$ and B is a r-subset of A.) □

5. Give a combinatorial proof and an algebraic proof of the following identity:

$$3^n = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}.$$

Solution. We first prove the identity algebraically. By the binomial theorem, we have

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j.$$

By change of index and the fact that $\binom{n}{n-i} = \binom{n}{i}$, we have

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{n-i} x^{n-i} = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}.$$

Substituting $x = 2$ into the equation above yields

$$\sum_{i=0}^{n} \binom{n}{i} 2^{n-i} = (1+2)^n = 3^n,$$

as desired. Combinatorially, let S be the set of strings of length n with alphabets in $\{1, 2, 3\}$, and let S_i be the set of strings in S containing exactly i 3's ($0 \leq i \leq n$). Clearly

$$|S| = 3^n \quad \text{and} \quad S = \bigcup_{i=0}^{n} S_i.$$

For a string in S_i, there are $\binom{n}{i}$ ways to assign the 3's, and upon assigning the 3's there are 2 possible choices for each of the remaining $n - i$ positions, so $|S_i| = \binom{n}{i} 2^{n-i}$. Therefore

$$3^n = |S| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i},$$

as desired. $\square$