MATH 239

TUTORIAL 7

Question 1:
Let K be a bipartite graph with bipartitions (X, Y):
a) If $K = K_{m,n}$, how many vertices and edges does it have?
b) Suppose K has p vertices, show that K has at most $\left\lfloor \frac{p^2}{4} \right\rfloor$ edges.
c) Suppose K is k-regular with $k > 0$. Show that $|X| = |Y|$.

Answer 1:
a) $K_{m,n}$ has $m + n$ vertices and mn edges. The latter comes from the fact that are m vertices in X, and each of them must join to all n vertices of Y.

b) Suppose $|X| = m$ and $|Y| = n$, where $m + n = p$. Then, K is a subgraph of $K_{m,n}$. That is to say, $K_{m,n}$ contains all vertices and edges of K, and possibly some more. Therefore,

$$\begin{aligned} |E(K)| &\leq |E(K_{m,n})| \\ &\leq m(p-m) \\ &\leq -\left(\frac{p}{2} - m\right)^2 + \frac{p^2}{4} \\ &\leq \frac{p^2}{4} \\ &\leq \left\lfloor \frac{p^2}{4} \right\rfloor \end{aligned}$$

as $|E(K)|$ must be an integer.

c) Notice that each vertex of $|X|$ must join to exactly k vertices of $|Y|$, and vice versa. Therefore, $k|X| = |E(K)| = k|Y|$, and we deduce that $|X| = |Y|$.

Question 2:
Let n be a positive integer. We define a graph G_n as follows. The vertex set of G_n is the set of all permutations of $\{1, 2, \ldots, n\}$. Two permutations σ and σ' are joined by an edge G_n if and only if σ' can be obtained from σ by integerchanging two positions. (This graph is the same as Question 2 of Assignment 7.)
a) Give the neighbours of 13542 of G_5.
b) Determine the number of vertices of G_n.
c) Determine the number of edges of G_n.

Answer 2:
a) The neighbours of 13542 can be achieved by swapping any 2 positions. Therefore, they are 31542, 53142, 43512, 23541, 15342, 14532, 12543, 13452, 13245, 13524.

b) We count the number of permutations of n. We have n choices for the first position of the permutation, $n - 1$ on the second (anything not the same as the first position), $n - 2$ on the third, down to 1 choice for the last position. Therefore, $|V(G)| = n!$.

c) There are $\binom{n}{2}$ ways to can choose two positions and swap them, so each vertex is adjacent $\binom{n}{2}$ edges. Since the total number of edges is twice the sum of the degree, we have $|E(G)| = \frac{1}{2} n! \binom{n}{2}$.

Date: 10/26/12.

Question 3:

Let G be a graph with minimum degree k, where $k \geq 2$. Prove that

a) G contains a path of length at least k.

b) G contains a cycle of length at least k.

Answer 3:

a) Let $P = v_0v_1 \cdots v_n$ be a longest path of G. That is, there does not exist a path P' of length $n+1$. (Notice that P is not necessary unique.) Suppose for contradiction that $n < k$. However, vertex v_n has k neighbours, and $\{v_0, \ldots, v_{n-1}\}$ is a set of $n < k$ vertices. Therefore, there exists a neighbour $u \in V(G)$ that is not in P. This means that $P' = v_0v_1 \cdots v_nu$ is a longer path of G. Contradiction.

b) Again, let $P = v_0v_1 \cdots v_n$ be a longest path of G. By part a), $n \geq k$. Now consider the neighbours of v_0. These vertices must all be in P, as otherwise we can extend P like in a). Furthermore, they cannot all lie in $\{v_1, \ldots, v_{k-1}\}$ as that is a set of $k-1$ vertices. So, v_0 must have a neighbour v_i, where $i \geq k$. Then, the cycle $C = v_0v_1 \cdots v_iv_0$ is a cycle of length $i+1$, which is at least $k+1$ as desired.

Question 4:

Let G be a connected graph with 5 vertices of degree 10, and the rest of the vertices of G have degree 1. Find the minimum and maximum number of vertices possible.

Answer 4:

Let the set of degree 10 vertices be X and the set of other vertices be Y. Let s be the number of edges joining 2 vertices of X together. Since each vertex of Y must join to a vertex of X, we have of $|Y| = 5 \cdot 10 - 2s$ by counting neighbours of X.

Now, in the minimum case, we want to maximize s, so the vertices of X should join to each other to get the complete grah K_5. As K_5 has 10 edges, we have $|Y| = 5 \cdot 10 - 2 \cdot 10 = 30$. Hence, $|V(G)| = 35$.

On the other hand, a path from a vertex of X to another must only use vertices in X, so the subgraph of those 5 vertices are connected. Since adding an edge can degree the number of component by at most 1, there must be at least 4 edges joining those 5 vertices. For example, they can form a path of length 4. This means $|Y| = 5 \cdot 10 - 2 \cdot 4 = 42$. Hence, $|V(G)| = 47$.

Question 5:

Prove that, if G is connected, any two longest paths have a vertex in common.

Answer 5:

Suppose for contradiction $P = v_0v_1 \cdots v_n$ and $P' = u_0u_1 \cdots u_n$ are both longest paths of G. Since G is connected, there exists a path from v_i to u_j for all $0 \leq i, j \leq n$. Let $Q = v_iq_1q_2 \cdots q_{m-1}u_j$ be a shortest path that satisfies this property. Without loss of generality, assume $i \geq j$. Now, none of the internal vertices q_s can be in P or P', as otherwise this contradicts with Q being a shortest path. Then, $v_0v_1 \cdots v_iq_1q_2 \cdots q_{m-1}u_ju_{j+1} \cdots u_n$ is path since v_i and u_j are the only common vertices of P, P', and Q . Furthermore, this path contains at least $(i+1) + (n-j+1) = n + (i-j) + 2 \geq n+2$ vertices, so it is a path of length at least $n+1$