MATH 239 Winter 2013
Assignment 8
Solutions

TOTAL: 50 POINTS

For a graph G, let p be the number of vertices and q be the number of edges.

1. Let G be a connected graph that has a single cycle of length n. Prove that G has exactly n distinct spanning trees.

   Solution. We first note that for this graph has the same number of vertices as edges. In fact, let e be any edge in the cycle. Because e is not a bridge, $G-e$ is connected. Since $G-e$ has no cycles, it is a tree and we have $p(G-e) = q(G-e)+1$. Consequently, $p(G) = q(G)$.

   Since a spanning tree is a tree containing all the vertices of G, it has $p(G)-1$ edges. Therefore, a spanning tree is of the form $G-e$ for some edge e. If e were not in the cycle, then e would be a bridge and $G-e$ would be disconnected. Consequently, e must be in the cycle. As noted above, for any e in the cycle, $G-e$ is a tree. Therefore, we may remove any edge in the cycle (of which there are n) to obtain a spanning tree.

2. Let G be a connected graph with spanning tree T. Pick a vertex w of T. For any vertex v of G, let $d(v)$ be the length of the unique path in T from v to w. Suppose that for any edge $e = uv$ of G that is not in T, $d(u) - d(v)$ is an odd number. Show that G is bipartite.

   Solution. We first note that the tree T is bipartite. Colour the vertex w red. Now, colour any vertex v red if $d(v)$ is even or blue if $d(v)$ is odd. This is indeed a bipartition because if $e = uv$ is an edge in T then either the path in T from w to u or the path in T from w to v contains e. Suppose by possibly interchanging u and v that it is the path from w to v. Then the path from w to u is $e_1e_2 \dots e_k$ while the path from w to v is $e_1e_2 \dots e_ke$. Consequently, $d(v) = d(u)+1$ and v and u are given different colours.

   Now suppose that $e = uv$ is an edge not in T. Because $d(u) - d(v)$ is odd, $d(u)$ and $d(v)$ have different parities. Consequently u and v have been given different colours.

3. This problem generalizes Platonic graphs. Let G be a connected planar graph where every vertex has degree at least 3. We say that G is *special* if there is a positive integer k, positive distinct integers $d_1^*, \dots, d_k^* \geq 3$, and positive integers $m_1, \dots, m_k$ such that every vertex v is of degree $m_1 + \cdots + m_k$ and that faces containing v are exactly m_1 faces of degree d_1^*, m_2 faces of degree d_2^*, and so on, up to m_k faces of degree d_k^*. The Platonic graphs are the cases where $k = 1$: a cube has $k = 1$, $m_1 = 3$, $d_1^* = 4$; a dodecahedron has $k = 1$, $m_1 = 3$, $d_1^* = 5$. The edge graph of a soccer ball has $k = 2$, $m_1 = 2, d_1^* = 6, m_2 = 1, d_2^* = 5$. In other words at every vertex of a soccer ball, there are two hexagons and one pentagon.

   (a) Show that
   $$(m_1 + m_2 + \cdots + m_k)p = 2q.$$

   (b) Show that the number of faces of degree d_i^* is
   $$s_i = \frac{m_i p}{d_i^*}.$$

(c) Prove by using Euler's formula that

$$\frac{1}{m_1+m_2+\cdots+m_k}\left(1+\frac{m_1}{d_1^*}+\frac{m_2}{d_2^*}+\cdots+\frac{m_k}{d_k^*}\right)=\frac{1}{2}+\frac{1}{q}.$$

Solution.

(a) By the handshake lemma for vertices,

$$\sum \deg(v) = 2q$$

but since each vertex is of degree $m_1+m_2+\cdots+m_k$, we have the desired equality.

(b) Let us count pairs (v, f) where v is a vertex and f is a face of degree d_i^* containing it. For each vertex v of G, there are m_i faces of degree d_i^* containing it. Consequently, there are $m_i p$ pairs. On the other hand, each face of degree d_i^* contains d_i^* vertices. Consequently, if s_i is the number of faces of degree d_i^*, there are $d_i^* s_i$ pairs. We get the desired equality from

$$d_i^* s_i = m_i p.$$

(c) Euler's formula is

$$p - q + s = 2.$$

By breaking s into contributions from faces of different degrees, we have

$$\begin{aligned} s &= \sum s_i \\ &= \sum \frac{m_i p}{d_i^*}. \end{aligned}$$

where the last equality follows from (b). Consequently, we have

$$\begin{aligned} 2 &= p - q + \sum \frac{m_i p}{d_i^*} \\ &= -q + p(1 + \sum \frac{m_i p}{d_i^*}) \\ &= -q + \frac{2q}{m_1+m_2+\cdots+m_k}(1 + \sum \frac{m_i p}{d_i^*}) \end{aligned}$$

where the last equality follows from (a). We divides both sides of the above equation by $2q$ and rearrange to get the desired equality.

4. Let G be a connected planar graph in which every face has degree exactly 3 and every vertex has degree at least 4. Prove that G has at least 12 edges.

Solution. By the handshake lemma for vertices, $3p = 2q$ hence $p = \frac{2q}{3}$.

By the handshake lemma for faces,

$$4s \le \sum \deg(f_i) = 2q$$

where s is the number of faces. Consequently, $s \le \frac{q}{2}$. Euler's formula gives

$$2 = p - q + s = \frac{2q}{3} - q + s \le \frac{2q}{3} - q + \frac{q}{2} = \frac{q}{6}.$$

Consequently, $q \ge 12$.

5. Show for $n \geq 3$ the following graphs are planar by describing a planar embedding. Please give an example of your explanation for $n = 4$.

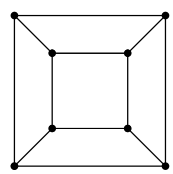
   (a) Let G be the graph with $2n$ vertices labelled $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ with $3n$ edges of the form $v_i v_{i+1}$, $w_i w_{i+1}$, and $v_i w_i$ for $i = 1, \ldots, n$. Note: we use the convention that $v_{n+1} = v_1$.

   (b) Let G be the graph with $2n$ vertices labelled $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ such that there are $4n$ edges of the form: $v_i v_{i+1}$, $w_i w_{i+1}$, $v_i w_i$, and $v_i w_{i+1}$ for $i = 1, \ldots, n$.

Solution.

(a) These graphs are prisms. They can be drawn by putting one n-gon in another n-gon and connecting the respective edges. Here the vertices in the outer n-gon are $v_1, \ldots, v_n$ and the vertices in the inner n-gon are $w_1, \ldots, w_n$

The case for $n = 4$ is illustrated:



(b) These graphs are anti-prisms. They can be drawn by putting one n-gon in another where the inner n-gon is rotated through half an edge. The vertices in the vertices in the outer n-gon are $v_1, \ldots, v_n$ and the vertices in the inner n-gon are $w_1, \ldots, w_n$. Each outer vertex is connected to two adjacent inner vertices.

The case for $n = 4$ is illustrated: