Math 239 - Tutorial 7 – Fall 2013

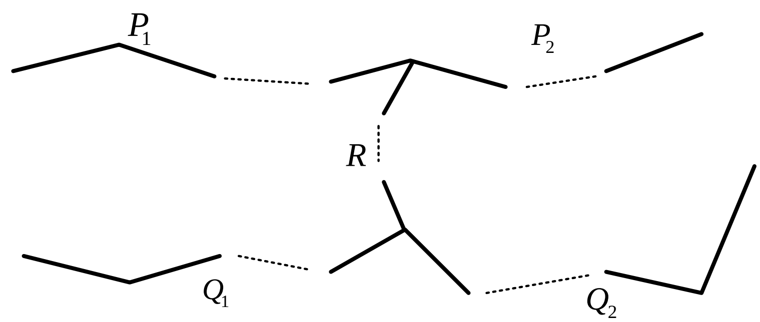
Problem 1:

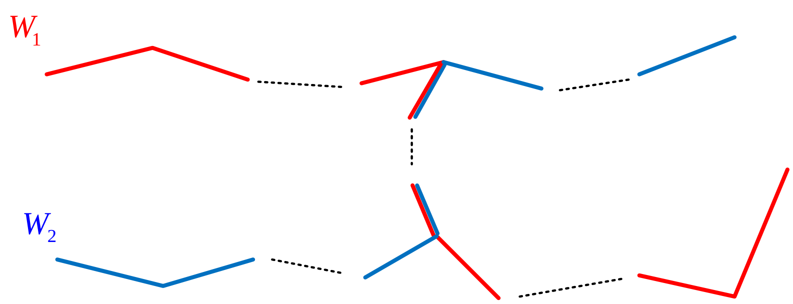
Suppose that P and Q are two paths of maximum length (the length of a path is the number edges of that path) in a connected graph G. Prove that there is at least one vertex that is in both P and Q.

Solution: Look at the following figure. Assume that P and Q have no vertex in common. As G is connected, there is a path from $V(P)$ to $V(Q)$, let R be such a path with the shortest length. Assume that R divides P into P_1 and P_2, and Q into Q_1 and Q_2. As P and Q are two paths of maximum length, we have $|P_1| + |P_2| = |Q_1| + |Q_2| = L$, where L is the length of a maximum length path. Now look at the paths $W_1 := (P_1, R, Q_2)$ and $W_2 := (Q_1, R, P_2)$. We have

$$|W_1| + |W_2| = |P_1| + |P_2| + |Q_1| + |Q_2| + 2|R| = 2L + 2|R| > 2L,$$

where the last inequality is from the fact that $|R| \geq 1$. This means at least one of W_1 and W_2 has length greater than L, which is a contradiction.



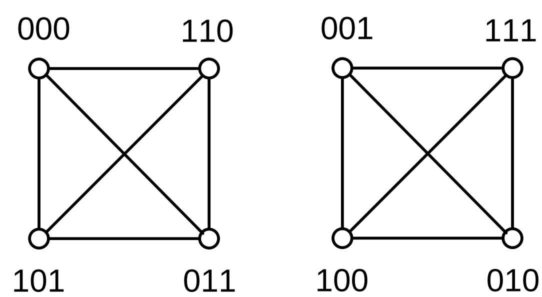


Problem 2:

Define the graph $G_{n,d}$ as follows: the vertices of $G_{n,d}$ are the $\{0, 1\}$-strings of length n. Two strings are adjacent if and only if they differ in exactly d positions.

(a) Draw $G_{3,2}$.

Solution:



(b) Prove that $G_{n,k}$ is k-regular by finding the value of k.

Solution: For any string v, we can choose any d entries and change them to get a neighbour of v. We can do that in $k = \binom{n}{d}$ ways. Note that k does not depend on the chosen vertex, hence the graph is regular.

(c) Show that when d is odd, $G_{n,d}$ is bipartite?

Solution: Divide $V(G_{n,d})$ into two partitions A and B as follows: A is the set of all strings of length n with even number of 1's and B is the set of all strings of length n with odd number of 1's. Clearly $V(G_{n,d}) = A \cup B$ and $A \cap B = \emptyset$. To prove that $G_{n,d}$ is bipartite, we can show that there is no edge between any two vertices of A and any two vertices of B. Assume that $v \in A$, and we choose d positions to change to get a neighbour of v. If these d positions cover even number of 1's, as d is odd, even number of 1's is removed and odd number of 1's is added, so the parity of 1's is changing. If these d positions cover odd number of 1's, odd number of 1's is removed and even number of 1's is added, so again the parity of 1's is changing. This means that all the neighbours of v is in B and this is true for all the vertices of A. A similar reasoning can be used for set B. So $G_{n,d}$ is bipartite.

(d) Use Theorem 4.8.5 to show that $G_{n,d}$ is not connected for even d.

Solution: Let X be a proper subset of $V(G_{n,d})$. X induce an **empty cut** if there is no edge between X and $V(G_{n,d})\backslash X$. By Theorem 4.8.5, $G_{n,d}$ is not connected if and only if we have an empty cut. Let us define X as the set of all strings that have even number of 1's. Let v be any string of length n, and assume that we choose d positions to change to get a neighbour of v, say $\bar{v}$. If there are even (odd) number of 1's among those d positions, as d is even, there must also be even (odd) number of 0's. It means that the parity of 1's for v and $\bar{v}$ must be the same. Hence, every edge in $G_{n,d}$ either joins two vertices in X or two vertices in $V(G_{n,d})\backslash X$. So X is an empty cut.

(e) Prove by induction on n that if $n > d$ and d is odd, then $G_{n,d}$ is connected.

Solution: The base case of induction is $G_{d+1,d}$. For any string of length $d+1$, there are $d+1$ possible ways to choose d entries to change. Let us define f_i as the operation of choosing all the entries except the ith one and change them, so we have $d+1$ possible operations $f_1, \cdots, f_{d+1}$. We want to prove that there is a path from any string v to the string of all 0, v_0, in $G_{d+1,d}$. If v has even number of 1's in positions $i_1, \cdots, i_k$ (it also has even number of 0's as $d+1$ is even), then if we apply $f_{i_1}, \cdots, f_{i_k}$ to v, we get v_0. Just note that applying these even number of operations changes entries $i_1, \cdots, i_k$ odd number of times and all the other entries even number of times, as we wanted. If v has odd number of 1's, it must have odd number of 0's in positions $i_1, \cdots, i_k$. This time if we apply $f_{i_1}, \cdots, f_{i_k}$ to v, we get v_0. By a similar reasoning, applying these odd number of operations changes entries $i_1, \cdots, i_k$ even number of times and all the other entries odd number of times, as we wanted. Hence, there is a sequence of operations that changes any string v to v_0, which is being translated to a path in $G_{d+1,d}$.

For the body of induction, assume that $G_{n,d}$, $n \geq d+1$, is connected. Now any string of length $n+1$ can be written as $v0$ or $v1$, where v is a string of length n. Define H_j, $j \in \{0, 1\}$, as the subgraph of $G_{n,d}$ induced by vertices of the form vj. We claim that both H_0 and H_1 are isomorphic to $G_{n,d}$. Consider the vertices of H_j, $j \in \{0, 1\}$: define a bijection f between the vertices of H_j and $G_{n,d}$ as $f(vj) = v$. Any two vertex of H_j have the same last entry, so they are adjacent iff they differ in d positions among the first n positions. This means $v_1 j$ and $v_2 j$ are adjacent in H_j if and only if v_1 and v_2 are adjacent in $G_{n,d}$, so H_0 and H_1 are isomorphic to $G_{n,d}$ and both of them are connected by induction hypothesis. We just have to prove that there is an edge between these two subgraphs. Pick any string of the form $v1$, pick any d entries of this string that contains the last entry, and change those entries, we get a string of the form $v0$.

Problem 3:

Prove that if every vertex of a graph G has degree at least 3, then G contains a cycle of even length.

Solution: Let P be a longest path in G and v be the last vertex of the path. We want to use the result of Assignemt 7, Q2-(a) that all the neighbours of v are vertices of the path. Look at the following figure. As v has degree at least 3, let u_1, u_2, and u_3 be any three of its neighbours on P. Also let P_1 be part of P from u_1 to u_2 and P_2 be part of P from u_2 to u_3. If $|P_1|$ is even, then vP_1v is an even cycle. If $|P_2|$ is even, then vP_2v is an even cycle. if both $|P_1|$ and $|P_2|$ are odd, then vP_1P_2v is an even cycle.