Instructions

1. **EXPLANATIONS ARE ALWAYS REQUIRED.**
   SHOW ALL YOUR WORK.
   STATE ANY THEOREMS YOU ARE USING.
2. No aids of any kind are permitted.
3. Please write your solutions in the space provided. If you need more space, please use the back of a page. Clearly indicate where your solution continues.

Question	Mark	Question	Mark
1 7 points		5 12 points	
2 9 points		6 5 points	
3 8 points		7 5 points	
4 6 points		8 8 points	
		Total 60 points	

Question 1. **(a)** Explain (either with the binomial theorem or with a combinatorial argument) why, for any non-negative integer k,

$$2^k = \sum_{i=0}^{k} \binom{k}{i}.$$

(b) Recall that

$$\binom{k}{i} = \frac{k!}{i!(k-i)!}.$$

Using this fact and **(a)** or otherwise, show that

$$\sum_{i=0}^{k} \frac{1}{i!} \frac{1}{(k-i)!} = \frac{2^k}{k!}.$$

(c) Consider the power series

$$p(x) = \sum_{n \geq 0} \frac{x^n}{n!}\,.$$

By actually multiplying out the power series, compute $p(x)p(x)$.

(d) Using **(b)** and **(c)** or otherwise, show that $p(x)p(x) = p(2x)$.

Question 2. Suppose the numbers a_n satisfy the non-homogeneous recurrence relation

$$\begin{aligned} a_n - 6a_{n-1} + 9a_{n-2} &= 4, \\ a_0 &= 2, \\ a_1 &= 10, \end{aligned}$$

find an explicit formula for a_n in terms of n, for all $n \geq 0$.

Question 3. **(a)** Let $g(x) = \sum_{n \geq 0} a_n x^n$ be a generating function and suppose

$$g(x) = \frac{1+2x}{(1-2x)(1-3x)}.$$

Find a homogeneous recurrence relation satisfied by the sequence a_n, together with enough initial conditions to determine the a_n for all $n \geq 0$.

(b) Determine a_3.

Question 4. Let a_n be the number of binary strings in which every 0 that has a 1 somewhere to its right is in a block of at most two 0's. Find the generating function for the sequence a_n; please express your answer as a quotient of two polynomials. *(Examples of such strings are: 1110110010110000 and 11110011001, while 01110001100 is not such a string.)*

Question 5. **(a)** Let p, q, and d be integers with $p > 0$, $q > 0$, and $d \geq 0$. Explain why

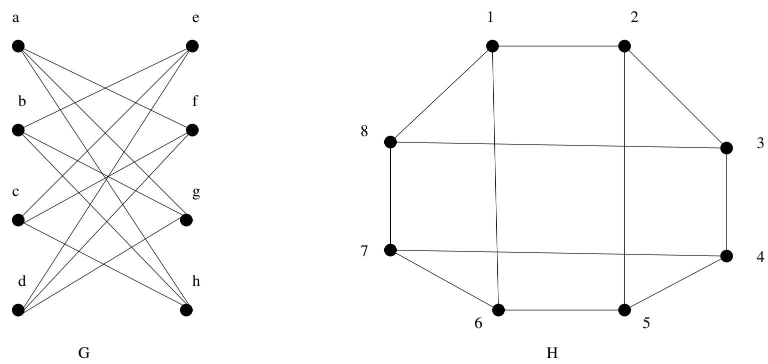
$$x^d + x^{d+q} + x^{d+2q} + \cdots + x^{d+pq} = x^d \left(\frac{1 - x^{(p+1)q}}{1 - x^q} \right).$$

(b) Let n and k be positive integers and let $a_{n,k}$ denote the number of compositions of n with precisely k parts, in which each part is an element of the set $U = \{1, 5, 9, 13, \ldots, 89\}$. Find $a_{n,k}$ in terms of n and k. Express your answer as a sum of products of binomial coefficients.

(c) Prove that the generating function for all compositions with an even number of parts, in which each part is an element of U, is

$$\frac{1 - 2x^4 + x^8}{1 - x^2 - 2x^4 + x^8 + 2x^{94} - x^{186}}.$$

Question 6. Determine whether the two graphs shown are isomorphic. Prove your answer is correct.



Question 7. Let G be a 6-regular connected graph. Prove that G does not have a bridge.

Question 8. Which of the following three sequences are the degree sequences of a graph on seven vertices? If it is, give an example of such a graph; if not, explain why not.

(a) 3,3,3,3,3,3,3

(b) 4,4,4,3,3,3,3

(c) 6,6,3,2,2,2,1

(d) 6,5,5,5,5,5,5