UNIVERSITY OF WATERLOO
MIDTERM EXAMINATION
Winter TERM 2013

Surname: Solutions

First Name: ____________________

Signature: ____________________

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✓ **boxes:** ☐ Section 1 (10:30)
☐ Section 2 (1:30)
☐ Section 3 (2:30)

Course Number	Math 239
Course Title	Introduction to Combinatorics
Instructors	B. Guenin, P. Haxell, E. Katz
Date of Exam	March 7th
Time Period	4:30–6:20PM
Number of Exam Pages (including cover page)	12 pages
Exam Type	Closed Book
Additional Materials Allowed	**none**

Write answers in the space provided.
If you need more space, use the back of the previous page
No calculators are allowed.

Problem	Value	Mark Awarded
1	16	
2	11	
3	10	
4	11	
5	14	
6	10	
7	16	
8	12	
Total	100	

Question 1 (16 marks)**.**

The WaterTech company has n employees. A *committee* is a subset of the employees. A *delegation* is a committee with an elected chair (who is a member of the committee). For instance if WaterTech has $n = 5$, employees, the following are some of the possible different delegations,

- committee $\{1, 3, 5\}$ with chair 3,
- committee $\{1, 3, 5\}$ with chair 5,
- committee $\{2, 3\}$ with chair 2.

(a) Find the number of distinct delegations with exactly i employees (counting the chair).

IMPORTANT: Justify your answer.
No marks will be given for answers without justification.

$\binom{n}{i}$ = # i subsets of $[n]$
= # of commitees with i employees.

We have i choices for the chair.

$\longrightarrow \binom{n}{i} i$ = # of delegations with i employees.

(b) Find the total number of distinct delegations (with arbitrary number of employees). Your answer should NOT involve a sum.

IMPORTANT: Justify your answer.
No marks will be given for answers without justification.

Choose chair $k \in [n]$: n choices

Choose set $S \subseteq [n] \setminus \{k\}$: 2^{n-1} choices.

Any delegation is of the form $\{k\} \cup S$

$\longrightarrow n \times 2^{n-1}$ delegations

(c) Using part (a) and part (b) find a combinatorial proof for the following identity,

$$\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}.$$

Let S = set of all delegations with at most n employees

S_i = set " " " " exactly i " .

Then $S_1, S_2, \ldots, S_n$ is a partition of $S \rightarrow$

$$|S| = |S_1| + |S_2| + \cdots + |S_n|$$

By (a) : $|S_i| = \binom{n}{i} i$

By (b) : $|S| = n \times 2^{n-1}$

Thus $n \times 2^{n-1} = \sum_{i=1}^{n} i \binom{n}{i} = \sum_{i=0} i \binom{n}{i}$

(d) State the binomial theorem (do NOT prove it).

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

(e) Using part (d) find an algebraic proof for the identity in part (c).

Taking the derivative with respect to x :

$$n(1+x)^{n-1} = \sum_{i=0}^{n} \binom{n}{i} i\, x^{i-1}.$$

By setting $x=1$, we get the identity in (c).

Question 2 (11 marks).

State and prove the **Sum Lemma.**

Let S be set with partition A, B.

Spse every $\sigma \in S$ has weight $w(\sigma)$.

Let $\Phi_A(x)$ = generating series for A with weight w

$\Phi_B(x)$ = " " " B " " w

$\Phi_S(x)$ = " " " S " " ".

Then $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$

Pf:

$$\Phi_S(x) \underset{\text{def}}{=} \sum_{\sigma \in S} x^{w(\sigma)}$$

$$\underset{A,B \text{ partition of } S}{=} \sum_{\sigma \in A} x^{w(\sigma)} + \sum_{\sigma \in B} x^{w(\sigma)}$$

$$\underset{\text{def}}{=} \Phi_A(x) + \Phi_B(x)$$

Question 3 (10 marks).

Find the following coefficients:

(a) $[x^{12}]\frac{x^2}{(1-3x)^4}$.

$$\frac{1}{(1-3x)^4} = \sum_{n \geq 0} \binom{n+4-1}{4-1} (3x)^n = \sum_{n \geq 0} \binom{n+3}{3} 3^n x^n$$

$$[x^{12}] \frac{x^2}{(1-3x)^4} = [x^{10}] \frac{1}{(1-3x)^4} = \binom{13}{3} 3^{10}$$

(b) $[x^{12}]\frac{x^3+x^5}{1+x}$.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 + \cdots$$

$$[x^{12}] \frac{x^3+x^5}{1+x} = [x^{12}] \frac{x^3}{1+x} + [x^{12}] \frac{x^5}{1+x}$$

$$= [x^9] \frac{1}{1+x} + [x^7] \frac{1}{1+x} = (-1)^9 + (-1)^7$$

$$= -2$$

Question 4 (11 marks).

Let $k \geq 2$ be a fixed integer. Let a_n denote the number of compositions of n with exactly k parts, in which the first part is greater than 4 and the last part is less than 10.

(a) Find a set $\mathcal{S}$ and a weight function w defined on $\mathcal{S}$ such that a_n is equal to the number of elements σ of $\mathcal{S}$ with $w(\sigma) = n$.

$A = \{5, 6, 7, 8, \ldots\}$

$B = \{1, 2, 3, 4, \ldots\}$

$C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Then $S = A \times B^{k-2} \times C$

For $(a_1, a_2, \ldots, a_k) \in S$,

$$w[(a_1, a_2, \ldots, a_k)] := a_1 + a_2 + \cdots + a_k.$$

(b) Find the generating series $\Phi_{\mathcal{S}}(x)$ with respect to the weight function w.

IMPORTANT: Indicate where theorems from class are applied.

$$\Phi_A(x) = x^5 + x^6 + x^7 + x^8 + \cdots = \frac{x^5}{1-x}$$

$$\Phi_B(x) = x + x^2 + x^3 + x^4 + \cdots = \frac{x}{1-x}$$

$$\Phi_C(x) = x + x^2 + \cdots + x^9 = x\left(\frac{1-x^9}{1-x}\right)$$

product lemma $\longrightarrow$ $$\Phi_S(x) = \frac{x^5}{1-x}\left(\frac{x}{1-x}\right)^{k-2} \times \frac{1-x^9}{1-x}$$

$$= x^{(5+k-2+1)}(1-x^9)\frac{1}{(1-x)^k}$$

$$= x^{k+4}(1-x^9)(1-x)^{-k}$$

Question 5 (14 marks).

(a) Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of binary strings. Let $\Phi_{\mathcal{A}}(x)$ be the generating series for $\mathcal{A}$ and let $\Phi_{\mathcal{B}}(x)$ be the generating series for $\mathcal{B}$. Denote by t_n the number of strings of $\mathcal{AB}$ that have length n. If we do not know whether $\mathcal{AB}$ is unambiguous or not, which of the following statement(s) is guaranteed to be correct,

(i) $[x^n]\phi_{\mathcal{A}}(x)\phi_{\mathcal{B}}(x) \leq t_n$,

(ii) $\boxed{[x^n]\phi_{\mathcal{A}}(x)\phi_{\mathcal{B}}(x) \geq t_n}$,

(iii) $[x^n]\phi_{\mathcal{A}}(x)\phi_{\mathcal{B}}(x) = t_n$.

IMPORTANT: Justify your answer.

product le

$[x^n]\Phi_A(x)\Phi_B(x) =$ # of pairs $(a,b) \in A \times B$ where $\text{length}(a) + \text{length}(b) = n$

$\geq$ # of string $ab \in AB$ where $\text{length}(ab) = n$

(inequality may be strict if AB ambiguous)

(b) Let $\mathcal{S}$ be the set of binary strings that do not have a block that consists of exactly two ones. Find an unambiguous expression for $\mathcal{S}$.

IMPORTANT: Prove that your expression is indeed unambiguous.

$S = \{0\}^* C^* \{\varepsilon, 1, 111, 1111, \ldots\}$

where $C = \{1, 111, 1111, \ldots\}\{0, 00, 000, \ldots\}$.

We can decompose each string in S after each block of 0, as this decomposition rule is unambiguous the corresponding expression above is unambiguous.

(c) Find the generating series for the strings $\mathcal{S}$ defined in (b). Express your answer as a rational function.

IMPORTANT: Indicate where theorems from class are applied.

$$\Phi_C(x) = (x + x^3 + x^4 + x^5 + \cdots)(x + x^2 + x^3 + \cdots) = \left[\frac{x}{1-x} - x^2\right]\frac{x}{1-x}$$

$$\Phi_S(x) = \frac{1}{1-x}\left[1 - \frac{x}{1-x}\left[\frac{x}{1-x} - x^2\right]\right]^{-1}\left[\frac{1}{1-x} - x^2\right]$$

$$= \frac{1 - x^2 + x^3}{1 - 2x + x^3 - x^4}$$

Question 6 (10 marks)**.**

Consider the recurrence relation,

$$a_n - 4a_{n-1} - 4a_{n-2} = 0$$

where

$$a_0 = -3 \qquad \text{and} \qquad a_1 = 2.$$

Solve the recurrence relation, i.e. find a formula for a_n in terms of n for all $n \geq 0$.

Char. polynomial : $p(x) = x^2 - 4x + 4 = (x-2)^2$

General form:

$$a_n = c_1 2^n + c_2 n 2^n$$

$-3 = a_0 = c_1$

$2 = a_1 = 2c_1 + 2c_2 \rightarrow 2 = -6 + 2c_2$

$\rightarrow c_2 = 4$

Thus $\boxed{a_n = -3 \cdot 2^n + 4n2^n}$

Question 7 (16 marks)**.**

Recall that the *complement* $\overline{G}$ of the graph G is the graph with vertex set $V(G)$ and edge set $\{uv : u, v \in V(G), u \neq v, uv \notin E(G)\}$.

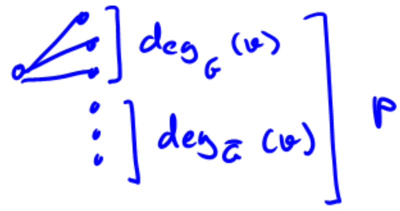
(a) Let G be a graph with p vertices. Prove that

$$\sum_{v \in V(G)} \big(p - 1 - deg(v)\big)$$

is an even number.

$\deg_{\bar{G}}(v) = \#$ of vertices of G not adjacent to v

$= p - 1 - \deg(v)$



$2 \times \#$ edges in $\bar{G} = \sum_{v \in V} \deg_{\bar{G}}(v)$

$= \sum_{v \in V} (p - 1 - \deg(v))$

(b) Give an example of a graph G with $p = 4$ vertices for which both G and $\overline{G}$ are bipartite.

G

(c) Prove that if $p \geq 5$ and G is a bipartite graph then $\overline{G}$ cannot be bipartite.

Spse G is bipartite with partition A, B where $|A| + |B| \geq 5$.

Wma $|A| \geq 3$.

Let $u, v, w \in A$.

Then in $\bar{G}$ u, v, w are pairwise adjacent

$\longrightarrow$ $\bar{G}$ is not bipartite

(d) Prove that if graphs G and H are isomorphic then $\overline{G}$ and $\overline{H}$ are isomorphic.

G, H isomorphic $\to$

$\exists$ bijection $\phi : V(G) \to V(H)$ st

$\forall u, v \in V(G)$,

$uv \in E(G)$ iff $\phi(u)\phi(v) \in E(H)$

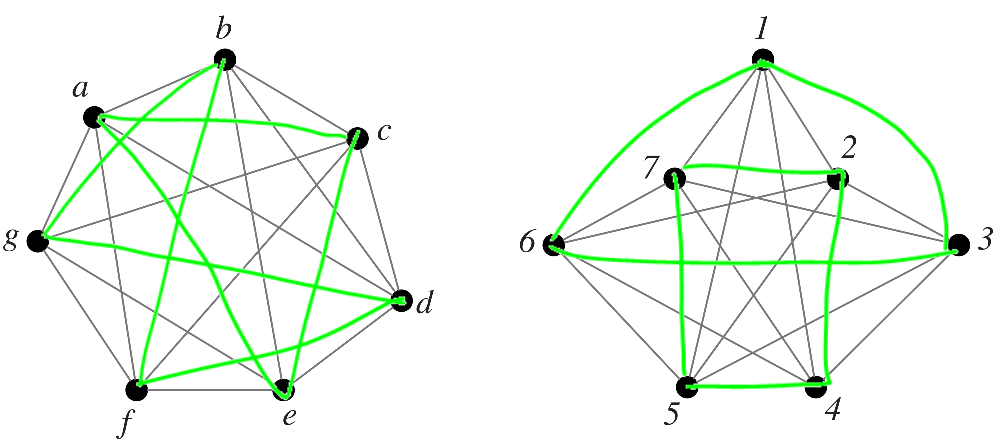
$\Updownarrow \qquad \Updownarrow$

$uv \in E(\overline{G})$ iff $\phi(u)\phi(v) \in E(\overline{H})$.

$\to \overline{G}, \overline{H}$ isomorphic

(e) Determine whether the two graphs shown are isomorphic. Justify your answer.

complement



In both cases the complement graphs can be redrawn as □ △

by (d) the original graphs are isomorphic.

Question 8 (12 marks)**.**

For each of the following questions indicate if the answer is TRUE or FALSE.

You do NOT need to justify your answer.

(a) The following formal power series has an inverse: $x + x^2 + x^3 + x^4 + \dots$.

No

(b) $1, 2, 2, 2, 3, 4, 5, 6$ is the degree sequence of a graph.

No

(c) Let G_1, G_2, G_3 be graphs where G_1 is isomorphic to G_2 and G_2 is isomorphic to G_3. Then G_1 is isomorphic to G_3.

Yes

(d) $\frac{x}{1+2x}$ is the generating series of some set S with respect to some weight function w.

No

(e) Let $\mathcal{S}$ be the set of all binary strings. Then $\mathcal{S} = \{0, 1\}\mathcal{S}$.

No

(f) Let A, B be sets. Then we always have $|A \cup B| = |A| + |B|$.

No

IMPORTANT:

- You will receive 2 points for each correct answer;
- You will receive 0 point for each unanswered question;
- You will lose 2 points for each incorrect answer;
- A negative total will be counted as 0.