MATH 239 Supplementary 2: Formal power series

You have learned about power series in calculus 2. A power series has the form

$$f(x) = \sum_{n \geq 0} a_n x^n$$

where each a_n is a number. In combinatorics, we will be using *formal* power series, which are still power series except we treat x as some literal that do not take on any value. We also do not care about convergence or divergence (a big relief to most of you). We are using the coefficients of power series to keep track of answers to our counting problems.

We start with a simple definition. If $f(x)$ is a formal power series, then $[x^n]f(x)$ denotes the coefficient of x^n in $f(x)$. One trick that we often use with this definition is that if $k \leq n$, then

$$[x^n]x^k f(x) = [x^{n-k}]f(x).$$

If $k > n$, then $[x^n]x^k f(x) = 0$.

(Note: This document summarizes the main tools that we will use in this course. For a formal treatment of this subject, you may read section 1.5 from the course notes.)

1 Power series as rational expression

We often represent a formal power series as a rational expression, that is, $\frac{p(x)}{q(x)}$ where $p(x), q(x)$ are polynomials. (Note: Polynomials have finite number of terms, while formal power series may have infinitely many terms.) Here are a few examples:

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \tag{1}$$

$$\frac{1}{1-x} = \sum_{i \geq 0} x^i = 1 + x + x^2 + x^3 + \cdots \tag{2}$$

$$\frac{1-x^{k+1}}{1-x} = \sum_{i=0}^{k} x^i = 1 + x + x^2 + \cdots + x^k \tag{3}$$

$$\frac{1}{(1-x)^k} = \sum_{i \geq 0} \binom{i+k-1}{k-1} x^i \tag{4}$$

Equation (1) is an instance of the Binomial Theorem that you have learned in Math 135.

Equation (2) is a geometric series. This can be explained as follows:

$$(1-x)(1+x+x^2+\cdots) = (1+x+x^2+\cdots) - (x+x^2+\cdots) = 1.$$

Divide both sides by $1-x$ gives us equation (2). You may ask “what happens if $x = 1$?” The answer is we are dealing with formal power series, so x does not take on any value.

Equation (3) can be established using a similar argument.

Equation (4) is somewhat tricky, and will be proved in class using a combinatorial argument.

2 Compositions

For all four equations, you can replace x by any formal power series, *as long as its constant term is 0*. For example, we can replace x with $3x^2$ in equation (2) to get

$$\frac{1}{1-3x^2} = 1 + 3x^2 + 9x^4 + 27x^6 + \cdots = \sum_{i\geq 0} 3^i x^{2i}.$$

This is described as a composition of power series: let $f(x) = 1/(1-x)$ and $g(x) = 3x^2$, then what we have above is $f(g(x))$. Using the notation we have for coefficients, we can say that

$$[x^n]\frac{1}{1-3x^2} = \begin{cases} 3^{n/2} & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}$$

We can be more adventurous and use $g(x) = x/(1-x^2)$ instead, and get

$$f(g(x)) = \frac{1}{1-\frac{x}{1-x^2}} = \frac{1-x^2}{1-x-x^2} = \sum_{i\geq 0}\left(\frac{x}{1-x^2}\right)^i.$$

As to what is the coefficient of x^n in this series, that is harder to determine (exercise). Sometimes it is good enough to give a rational expression as an answer.

We mentioned that we want $g(x)$ in our composition to have constant term 0. When this is the case, the resulting composition will always produce a valid power series (the course notes have a formal proof of why this is). Here is an example of something that goes wrong when the constant term is not 0. If $f(x) = 1/(1-x)$ and $g(x) = 1+x$, then $f(g(x))$ becomes

$$1 + (1+x) + (1+x)^2 + (1+x)^3 + \cdots$$

Notice that in each term in the sum, there is a constant term 1. So in this series, the constant term is infinite, which is not possible.

3 Operations on power series

There are two main operations you can perform on power series: addition and multiplication. Let $f(x) = \sum_{i\geq 0} a_i x^i$ and $g(x) = \sum_{j\geq 0} b_j x^j$.

Addition. We have

$$f(x) + g(x) = \sum_{i\geq 0}(a_i + b_i)x^i.$$

Or,

$$[x^n](f(x) + g(x)) = [x^n]f(x) + [x^n]g(x).$$

Multiplication. We have

$$f(x)g(x) = \left(\sum_{i\geq 0} a_i x^i\right)\left(\sum_{j\geq 0} b_j x^j\right).$$

We wish to determine the coefficient of x^n in this product. We can multiply a_0 from $f(x)$ and $b_n x^n$ from $g(x)$ to get $a_0 b_n x^n$. Similarly, we can multiply $a_1 x$ from $f(x)$ with $b_{n-1}x^{n-1}$ from $g(x)$ to get $a_1 b_{n-1} x^n$. Continuing this process, we see that

$$[x^n]f(x)g(x) = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{i=0}^{n} a_i b_{n-i}.$$

Alternatively, if you understand the rules of double sums, we can write

$$f(x)g(x) = \sum_{i\geq 0}\sum_{j\geq 0} a_i b_j x^{i+j}.$$

When looking for the coefficient of x^n, we need $i+j=n$. Certainly i has to be between 0 and n. So for each $i=0,1,\ldots,n$, we need $j=n-i$. So

$$f(x)g(x) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)x^n.$$

Example 1. Let $f(x) = (1-x)^{-5} = \sum_{i\geq 0}\binom{i+4}{4}x^i$ and $g(x) = (1-2x)^{-1} = \sum_{j\geq 0} 2^j x^j$. Then

$$f(x)+g(x) = \sum_{n\geq 0}\left(\binom{n+4}{4} + 2^n\right)x^n.$$

And

$$f(x)g(x) = \sum_{n\geq 0}\left(\sum_{i=0}^{n}\binom{i+4}{4}2^{n-i}\right)x^n.$$

So we could find coefficients like

$$\begin{aligned}[x^3](f(x)+g(x)) &= \binom{7}{4} + 2^3\\ [x^3]f(x)g(x) &= \binom{4}{4}2^3 + \binom{5}{4}2^2 + \binom{6}{4}2 + \binom{7}{4}\end{aligned}$$

Example 2. Now let's keep the same $g(x)$, and change $f(x)$ to $f(x) = (1-x^2)^{-5} = \sum_{i\geq 0}\binom{i+4}{4}x^{2i}$. Because only even powers appear in $f(x)$, the coefficient of $f(x)+g(x)$ has to be dealt in two cases.

$$[x^n](f(x)+g(x)) = \begin{cases}\binom{\frac{n}{2}+4}{4} + 2^n & n \text{ is even}\\ 2^n & n \text{ is odd}\end{cases}$$

For the product, we have

$$f(x)g(x) = \left(\sum_{i\geq 0}\binom{i+4}{4}x^{2i}\right)\left(\sum_{j\geq 0}2^j x^j\right) = \sum_{i\geq 0}\sum_{j\geq 0}\binom{i+4}{4}2^j x^{2i+j}.$$

It is a bit more tricky to find the coefficient of x^n. We need values of i and j such that $2i+j=n$, but now the values of i range from 0 to $\lfloor \frac{n}{2}\rfloor$ (this is the floor function, i.e. the closest integer that is smaller than or equal to $\frac{n}{2}$). For each such i, we need $j = n - 2i$. Therefore,

$$[x^n]f(x)g(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{i+4}{4} 2^{n-2i}.$$

We can illustrate this with an example. For the coefficient of x^5, we can get it from

- constant term of $f(x)$ and x^5 term of $g(x)$;
- x^2 term of $f(x)$ and x^3 term of $g(x)$; and
- x^4 term of $f(x)$ and x term of $g(x)$.

Because there are no x, x^3 terms in $f(x)$, they are ignored. So the coefficient of x^5 here is then

$$[x^5]f(x)g(x) = \binom{4}{4}2^5 + \binom{5}{4}2^3 + \binom{6}{4}2.$$