MATH 239 Tutorial 8 Problems

1. What is the fewest number of edges that can exist in a graph with n vertices and k components?

   Solution. For the graph to have as few edges a possible, each component must have as few edges as possible. Thus each component must be a tree, and so our graph is a collection of k trees. Let our components be $C_1, C_2, ..., C_k$. Since each component is a tree, C_i has $V(C_i) - 1$ edges.

   Adding these equations together for all k components, we get

   $$|E(G)| = V(C_1) - 1 + V(C_2) - 1 + ... + V(C_k) - 1 = V(G) - k$$

2. Let G be a connected graph on $n \geq 3$ vertices where removing any edge from G results in a spanning tree. Determine (with proof) all possible graphs that satisfy this condition.

   Solution. Since removing any one edge e of G results in a tree T, we know that $T + e = G$. $T + e$ has a unique cycle C, and we claim that every edge of $T + e$ is in that cycle C. If not, then there is some edge $e' \in T + e$ that is not in C, so e' is a bridge of $T + e$. But $T + e = G$, so e' is a bridge of G.

   G has the property that removing any edge results in a spanning tree, but since e' is a bridge of G, $G - e'$ is disconnected and cannot be a spanning tree. Thus e' does not exist, so all edges of $T + e$ are in the cycle C. Thus $G = T + e = C$, so G can only be the cycle on n vertices.

3. Let T be a tree with n vertices where each vertex has degree either 1 or 4. Determine the number of leaves in T in terms of n.

   Solution. Let x be the number of vertices of degree 1, and y be the number of vertices of degree 4. Thus $x + y = n$. By handshaking,

   $$2E = 1x + 4y$$

   Since T is a tree, we also know that $E = n - 1$. Putting these three equations together and eliminating y and E, we get

   $$x = \frac{2}{3}(n + 1)$$

4. Suppose that G is a connected graph. Prove that an edge e is a bridge if and only if e is in every spanning tree of G.

   Solution. ($\Rightarrow$) Let e be a bridge, and we will prove that it is in every spanning tree of G. Let T be a spanning tree of G, and by way of contradiction assume that it does not use e. Then T is also a spanning tree of $G - e$. But e is a bridge, so $G - e$ is disconnected, and thus $G - e$ cannot have a spanning tree. Therefore T does not exist, a contradiction.

   ($\Leftarrow$) Let every spanning tree of G contain the edge e, and we will prove that e is a bridge. By way of contradiction, assume that e is not a bridge. Thus e is in some cycle C. Consider a spanning tree T. By our assumption, it contains e, and thus e is a bridge of T, so $T - e$ has two components C_1 and C_2. But since e is part of a cycle, there must be another edge of G connecting the vertices of C_1 to the vertices of C_2, call it e'. $e' \notin T$, since otherwise T would have a cycle.

Consider $T' = T - e + e'$. T' is acyclic, connected, and spanning, so it is a spanning tree. But T' does not contain e, a contradiction since every spanning tree of G must contain e. Thus e must be a bridge.

5. Prove that the edges of a graph G can be partitioned into edge-disjoint cycles if and only if every vertex of G has even degree.

   Solution. ($\Rightarrow$) Assume that G can be partitioned into edge-disjoint cycles. Consider any vertex v. Since G is partitioned into edge-disjoint cycles, v has degree either 0 or 2 in each of those cycles. Since these cycles are edge-disjoint, and each edge of G is in exactly one of these cycles, v must have even degree in G.

   ($\Leftarrow$) Assume that every vertex of G has even degree. Each component that is not an isolated vertex has the property that every vertex has degree at least 2. So it contains a cycle. Remove the cycle, the vertices still have even degree in the remaining graph. Use induction to decompose into edge-disjoint cycles.

6. Suppose that G is a graph which is the union of two edge-disjoint spanning trees T_1 and T_2.

   (a) Prove that G does not have any bridge.

   (b) Let $e \in E(T_1) \setminus E(T_2)$. Prove that there exists $e' \in E(T_2) \setminus E(T_1)$ such that $T_1 - e + e'$ is a spanning tree of G.

   (c) Let $X \subseteq V(G)$ be a nonempty subset. What is the maximum number of edges in G that joins two vertices in X? Write this in terms of $|X|$.

   Solution.

   (a) By way of contradiction, assume edge uv is a bridge of G. Since T_1 is a spanning tree, it has a unique uv-path, call it P_1. Similarly, T_2 has a unique uv-path, call it P_2. Since P_1 and P_2 are disjoint, at most one of them can contain the edge uv. Thus in $G - uv$, there is still at least one path from u to v. Thus the edge uv is in a cycle, so it cannot be a bridge.

   (b) All edges of a tree are bridges of that tree, so e is a bridge of T_1. Thus $T_1 - e$ has two components, C_1 and C_2. Since T_2 is a spanning tree, there must be at least one edge e' of T_2 from the vertices of C_1 to the vertices of C_2 (since otherwise, T_2 is not connected).

   This edge e' is not e (since e is not in T_2), and it cannot be in T_1, since otherwise T_1 would contain a cycle. Now consider $T_1 + e'$. $T_1 + e'$ contains a unique cycle C, and this cycle must contain edge e (since it is the only edge of T_1 joining the vertices of C_1 to the vertices of C_2). Thus e is not a bridge of $T_1 + e'$, so $T_1 + e' - e$ is still connected and spanning. Additionally, $T_1 + e'$ had only one cycle, and by deleting e we removed that cycle, so $T_1 + e' - e$ is acyclic, and is thus a spanning tree.

(c) Consider the vertex set X in T_1. Since T_1 is acyclic, the edges between vertices of X in T_1 cannot form cycles, and thus there are at most $|X| - 1$ of them. Similarly, there are at most $|X| - 1$ edges of T_2 between the vertices of X. Since G is the union of T_1 and T_2, and these two trees are edge disjoint, G has at most $2(|X| - 1)$ edges between the vertices of X.