Notes on disjoint paths

Given a graph $G = (V, E)$, a set of paths from s to t are *edge disjoint* if no two of these paths share an edge. For $S \subseteq V$ we denote by $\delta(S)$ the set $\{uv \in E : u \in S, v \notin S\}$. If $s \in S$ and $t \notin S$, then $\delta(S)$ is an *st-cut*. Two edges of G are in a *series* if they share a common vertex v that has degree 2. The following result by Menger, characterizes when there exists k disjoint paths between a fixed pair of vertices in a graph[1].

Theorem 1. Let $G = (V, E)$ be a graph with distinct vertices s and t. Then

(1) there exists edge disjoint paths $P_1, \dots, P_k$ from s to t if and only if

(2) every st-cut of G contains at least k edges.

Proof. Let us prove (1) implies (2). Consider $S \subset V$ where $s \in S$ and $t \notin S$. For $i \in \{1, \dots, k\}$, let v_i be the last vertex of P_i in S and let w_i be the first vertex of P_i outside S (such vertices exists as $s \in S$ and $t \notin S$). Denote by e_i the edge of P_i with endpoints u_i and w_i. Then $e_i \in \delta(S)$, in particular, $\delta(S) \supseteq \{e_1, \dots, e_k\}$. As $P_1, \dots, P_k$ are all edge disjoint, $e_1, \dots, e_k$ are distinct edges and $|\delta(S)| \geq k$, as required.

Let us prove (2) implies (1). Proceed by induction on $\ell := |V| + |E|$. The base case is $\ell = 2$, where $V = \{s, t\}$ and $E = \emptyset$. Then $k = 0$ in (1) and $|\delta(\{x\})| = 0$ in (2). Thus we may assume that $\ell \geq 3$ and that the result hold for any graph $H = (V', E')$ where $|V'| + |E'| < |V| + |E|$. If every path, say $P_1, \dots, P_r$ of G consists of a single edge, or two series edges then the st-cut that consists of the first edge of each of $P_1, \dots, P_r$ has exactly r edges as required. Otherwise there exists $e_1 \in E$ that is not incident to either s or t. If every st-cut of G that contains e_1 has at least $k + 1$ edges, then for the graph H obtained from G by deleting edge e_1, every st-cut has at least k edges. It follows by induction that H has k edge disjoint paths from s to t and hence so does G. Thus we may assume there exists some st-cut of G that has exactly k edges, including e_1 and say edges $e_2, \dots, e_k$. Construct a graph H_s by identifying all vertices in s to a single vertex $\bar{s}$. [2] Similarly construct a graph H_t by identifying all vertices in t to a single vertex $\bar{t}$. Then $e_1, \dots, e_k$ are the set of edges incident to $\bar{s}$ in H_s and are the set of edges incident to $\bar{t}$ in H_t. Since e_1 is not incident to s, H_s has fewer vertices (and no more edges) than G. As every st-cut of H_s is an st-cut of G, all st-cuts of H_s have at least k edges. Thus by induction there exists k edge-disjoint paths $Q_1, \dots, Q_k$ from $\bar{s}$ to t in H_s. Similarly, there exists k edge-disjoint paths $Q'_1, \dots, Q'_k$ from s to $\bar{t}$ in H_t. We may assume for $i = 1, \dots, k$ that Q_i and Q'_i use edge e_i. Construct a path P_i from s to t in G by appending Q_i at the end of Q'_i (keeping one copy of e_i). Then $P_1, \dots, P_k$ are k edge-disjoint paths from s to t in G. ∎

[1] We allow parallel edges in this statement.

[2] Delete all edges with both endpoints in S and every edge of G with exactly one endpoint in S is incident to $\bar{s}$ in H_s.