MATH 239 Assignment 2

- This assignment is due on Friday, September 28, 2012, at 10 am in the drop boxes in St. Jerome's (section 1) or outside MC 4067 (the other two sections).
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.

1. (a) Let S be the set of all binary strings with the same number of zeros and ones, and let the weight of a string be the number of ones it contains. Write $\Phi_S(x)$ as an infinite sum. Be sure to justify your answer.

   (b) It can be shown that $\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n$ for any non-negative integer n. (For a challenge, you can try to prove this identity, but this is not part of the assignment.) Using this fact, show that $\Phi_S(x) = \frac{1}{\sqrt{1-4x}}$.

   (Here if $\Phi(x)$ and $\Psi(x)$ are formal power series, we write $\Phi(x) = \sqrt{\Psi(x)}$ if $(\Phi(x))^2 = \Psi(x)$.)

   Solution:

   (a) Since there are $\binom{2n}{n}$ binary strings of length $2n$ with n ones, we have

   $$\Phi_S(x) = \sum_{n\geq 0}\binom{2n}{n}x^n.$$

   (b) Squaring the above expression, we find

   $$\begin{aligned}(\Phi_S(x))^2 &= \sum_{n,m\geq 0}\binom{2n}{n}\binom{2m}{m}x^{n+m}\\ &= \sum_{k\geq 0}\sum_{n=0}^{k}\binom{2n}{n}\binom{2(n-k)}{n-k}x^k\\ &= \sum_{k\geq 0}(4x)^k\\ &= \frac{1}{1-4x}.\end{aligned}$$

2. Let

   $$P(x) := \sum_{n\geq r} p_n x^n \quad \text{and} \quad Q(x) := \sum_{n\geq s} q_n x^n$$

   be formal power series, where $p_r \neq 0$ and $q_s \neq 0$ (i.e., x^r is the lowest-order nonzero term of $P(x)$ and x^s is the lowest-order nonzero term of $Q(x)$). Recall that by Corollary 1.5.3, $s = 0$ is a sufficient condition for the equation $Q(x)A(x) = P(x)$ to have a solution. Give a *necessary and sufficient* condition for this equation to have a solution, and prove that your answer is correct. When a solution exists, is it unique?

Solution: The equation $Q(x)A(x) = P(x)$ has a solution if and only if $r \geq s$.

If $r \geq s$, then factoring out x^s from both sides of $Q(x)A(x) = P(x)$ shows that $\bar{Q}(x)A(x) = \bar{P}(x)$, where

$$\begin{aligned} \bar{P}(x) &:= \sum_{n \geq r} p_n x^{n-s} = \sum_{n \geq r-s} p_{n+s} x^n \\ \bar{Q}(x) &:= \sum_{n \geq s} q_n x^{n-s} = \sum_{n \geq 0} q_{n+s} x^n. \end{aligned}$$

Since the constant term of $\bar{Q}(x)$ is $q_s \neq 0$, this equation has a solution (by Corollary 1.5.3).

If $r < s$, then factoring out x^r from both sides of $Q(x)A(x) = P(x)$ shows that $\tilde{Q}(x)A(x) = \tilde{P}(x)$, where

$$\begin{aligned} \tilde{P}(x) &:= \sum_{n \geq r} p_n x^{n-r} = \sum_{n \geq 0} p_{n+r} x^n \\ \tilde{Q}(x) &:= \sum_{n \geq s} q_n x^{n-r} = \sum_{n \geq s-r} q_{n+r} x^n = q_s x^{s-r} + q_{s+1} x^{s-r+1} + \cdots . \end{aligned}$$

Since $s > r$, the constant term of $\tilde{Q}(x)$ is 0. Thus the constant term of the left-hand side is $\tilde{Q}(0)\tilde{A}(0) = 0$. But the constant term of the right-hand side is $p_r \neq 0$. Thus the equation cannot be satisfied.

Corollary 1.5.3 also shows that the solution is unique when it exists.

3. Let n be a non-negative integer. Compute $[x^n]\frac{1}{(1-2x)(1+3x^2)}$.

   (Give a closed-form expression: your answer should not involve any sum with a number of terms that depends on n.)

   Solution: We have

$$\frac{1}{1-2x} = \sum_{i \geq 0} (2x)^i$$

and

$$\frac{1}{1+3x^2} = \sum_{j \geq 0} (-3x^2)^j,$$

so

$$\begin{aligned} \frac{1}{(1-2x)(1+3x^2)} &= \sum_{i,j \geq 0} (2x)^i (-3x^2)^j \\ &= \sum_{i,j \geq 0} 2^i (-3)^j x^{i+2j}. \end{aligned}$$

Letting $n = i + 2j$ and summing over n and j gives

$$\frac{1}{(1-2x)(1+3x^2)} = \sum_{n \geq 0} \sum_{j=0}^{\lfloor n/2 \rfloor} 2^{n-2j} (-3)^j x^n,$$

so we have

$$\begin{aligned}
[x^n]\frac{1}{(1-2x)(1+3x^2)} &= \sum_{j=0}^{\lfloor n/2\rfloor} 2^{n-2j}(-3)^j \\
&= 2^n \sum_{j=0}^{\lfloor n/2\rfloor} (-3/4)^j.
\end{aligned}$$

This is a geometric sum, so

$$[x^n]\frac{1}{(1-2x)(1+3x^2)} = 2^n\frac{1-(-3/4)^{\lfloor n/2\rfloor+1}}{1+(3/4)}.$$

4. Fix positive integers k and t, where $t \le k$. In this problem you will determine the number of compositions of a positive integer n into k parts, where exactly t of the parts are multiples of 3.

   (a) Let $T := \{3, 6, 9, \ldots\}$ be the set of positive multiples of 3 and let $U := \{1, 2, 4, 5, \ldots\}$ be the set of positive integers that are not multiples of 3. Let the weight of a number be its value. Find $\Phi_T(x)$ and $\Phi_U(x)$.

   (b) Let S be the set of compositions with k parts, where exactly t of the parts are multiples of 3. Express S as a union of cartesian products of the sets T and U.

   (c) Find the generating series $\Phi_S(x)$, where the weight of a composition of n is n.

   (d) Express $[x^n]\Phi_S(x)$ as a finite sum involving binomial coefficients.

   (e) How many compositions of $n = 40$ are there with $k = 4$ parts, where exactly $t = 2$ parts are multiples of 3?

   Solution:

   (a) We have

   $$\Phi_T(x) = \sum_{n\ge 1} x^{3n} = \frac{x^3}{1-x^3}$$

   and

   $$\begin{aligned}
   \Phi_U(x) &= \frac{x}{1-x} - \frac{x^3}{1-x^3} \\
   &= \frac{x(1-x^3) - x^3(1-x)}{(1-x)(1-x^3)} \\
   &= \frac{x - x^4 - x^3 + x^4}{(1-x)(1-x^3)} \\
   &= \frac{x(1-x^2)}{(1-x)(1-x^3)} \\
   &= \frac{x(1-x)(1+x)}{(1-x)(1-x^3)} \\
   &= \frac{x(1+x)}{1-x^3}.
   \end{aligned}$$

(b) We have

$$S = (\underbrace{T \times \cdots \times T}_{t} \times \underbrace{U \times \cdots \times U}_{k-t}) \cup (\underbrace{T \times \cdots \times T}_{t-1} \times U \times T \times \underbrace{U \cdots \times U}_{k-t-1}) \cup \cdots$$
$$\cup (\underbrace{U \times \cdots \times U}_{k-t} \times \underbrace{T \times \cdots \times T}_{t})$$

where we take the union over all of the $\binom{k}{t}$ ways of deciding which t out of k parts are multiples of 3.

(c) By the Product Lemma, the generating function for any one particular choice of which parts are multiples of 3 is

$$\begin{aligned}
\Phi_{T^t \times U^{k-t}}(x) &= (\Phi_T(x))^t (\Phi_U(x))^{k-t} \\
&= \left(\frac{x^3}{1-x^3}\right)^t \left(\frac{x(1+x)}{1-x^3}\right)^{k-t} \\
&= \frac{x^{2t+k}(1+x)^{k-t}}{(1-x^3)^k}.
\end{aligned}$$

Since S is a union of $\binom{k}{t}$ disjoint sets, each with the same generating function, the Sum Lemma shows that the desired generating function is

$$\Phi_S(x) = \binom{k}{t} \Phi_{T^k \times U^{k-t}}(x) = \binom{k}{t} \frac{x^{2t+k}(1+x)^{k-t}}{(1-x^3)^k}.$$

(d) The desired value is

$$[x^n]\Phi_S(x) = \binom{k}{t} [x^{n-2t-k}] \frac{(1+x)^{k-t}}{(1-x^3)^k}.$$

Now

$$\begin{aligned}
\frac{(1+x)^{k-t}}{(1-x^3)^k} &= \sum_{i=0}^{k-t} \binom{k-t}{i} x^i \sum_{j \geq 0} \binom{j+k-1}{j} x^{3j} \\
&= \sum_{i=0}^{k-t} \sum_{j \geq 0} \binom{k-t}{i} \binom{j+k-1}{j} x^{i+3j} \\
&= \sum_{n \geq 0} \sum_{j=0}^{\lfloor n/3 \rfloor} \binom{k-t}{n-3j} \binom{j+k-1}{j} x^n,
\end{aligned}$$

so

$$[x^n]\Phi_s(x) = \binom{k}{t} \sum_{j=0}^{\lfloor (n-2t-k)/3 \rfloor} \binom{k-t}{n-2t-k-3j} \binom{j+k-1}{j}.$$

(e) Plugging in the given values to the above formula, we find that the number of compositions is

$$\begin{aligned}\binom{4}{2}\sum_{j=0}^{\lfloor 32/3\rfloor}\binom{2}{32-3j}\binom{j+3}{j} &= 6\sum_{j=0}^{10}\binom{2}{32-3j}\binom{j+3}{3}\\ &= 6\binom{2}{2}\binom{13}{3}\\ &= 1716.\end{aligned}$$