1. **[10 marks]**

(a) Let S be the set of compositions of n with at least one part such that each part is odd and not equal to 3. (Note then that the number of parts is not fixed, and we do not consider the empty composition.) Express S as a union of Cartesian products of certain sets of integers.

$N_{odd} = \{1, 3, 5, 7, \ldots\}$ $N_{odd} \setminus 3 = \{1, 5, 7, \ldots\}$ - $k \geq 1$

$$S = \bigcup_{k\geq 1} \{1, 5, 7, 9, \ldots\}^k = \bigcup_{k\geq 1} (\{1\} \cup \{5, 7, 9, 11, \ldots\})^k$$

$(1-x)(1-x^2)$
$= 1 - x + x^3 - x^2$

(b) Show that the generating function for S is

$$\Phi_S(x) = \frac{(1-x^2)x + x^5}{(1-x)(1-x^2) - x^5}.$$

$$\Phi_S(x) = \sum_{k\geq 1} \left(\Phi_{\{1\}\cup\{5,7,9,\ldots\}}(x)\right)^k \quad \text{(by Sum and Product Lemmas)}$$

$$= \sum_{k\geq 1} \left(x + \sum_{i\geq 0} x^{2i+5}\right)^k \quad \text{(by Sum Lemma)}$$

$$= \sum_{k\geq 1} \left(x + x^5\left(\frac{1}{1-x^2}\right)\right)^k \quad \text{(by Geometric Series)}$$

$$= \sum_{k\geq 0} \left(\frac{x(1-x^2) + x^5}{1-x^2}\right)^k - 1 \quad \text{(by change of index)}$$

$$= \frac{1}{1 - \frac{(1-x^2)x + x^5}{1-x^2}} - 1 \quad \text{(by Geometric Series)}$$

$$= \frac{1-x^2}{1-x^2 - (1-x^2)x - x^5} - 1 = \frac{x(1-x^2) + x^5}{(1-x)(1-x^2) - x^5} \quad \text{as required}$$

(c) Compute a linear recurrence for the coefficients of the generating function for S, and determine enough initial values to uniquely specify these coefficients.

$a_n := [x^n]\,\Phi_S(x)$ then

$$\sum_{n\geq 0} a_n x^n = \frac{x - x^3 + x^5}{1 - x - x^2 + x^3 - x^5}$$

$$(1 - x - x^2 + x^3 - x^5)\sum_{n\geq 0} a_n x^n = x - x^3 + x^5$$

$$\sum_{n\geq 0} a_n x^n - \sum_{n\geq 0} a_n x^{n+1} - \sum_{n\geq 0} a_n x^{n+2} + \sum_{n\geq 0} a_n x^{n+3} - \sum_{n\geq 0} a_n x^{n+5} = x - x^3 + x^5$$

Thus $\boxed{a_n = a_{n-1} + a_{n-2} - a_{n-3} + a_{n-5}}$ for $n \geq 5$

$\boxed{a_0 = 0}$

$a_1 - a_0 = 1 \Rightarrow \boxed{a_1 = 1}$

$a_2 - a_1 - a_0 = 0 \Rightarrow \boxed{a_2 = 1}$

$a_3 - a_2 - a_1 + a_0 = -1 \Rightarrow \boxed{a_3 = 1}$

$a_4 - a_3 - a_2 + a_1 = 0 \Rightarrow \boxed{a_4 = 1}$

$a_5 - a_4 - a_3 + a_2 - a_0 = 1 \Rightarrow \boxed{a_5 = 2}$

2. **[3 marks]**
Prove that, for any positive integer n the number of binary strings of length n with an even number of 1s equals the number of binary strings with an odd number of 1s.

For any binary string of length n, there exists a bijection between all strings containing an even number of 1's and strings containing an odd number of ones defined as:

$$f(s) \begin{cases} \text{If } |s| = n \text{ is even, flip the first bit in } s \\ \text{If } |s| = n \text{ is odd, flip every bit in } s \end{cases}$$

Since f is a bijection the number of strings of length n with an even number of 1's must equal the number of strings of length n with an odd number of 1's.

$\square$

3. **[5 marks]**
Let $a_0, a_1, a_2, \ldots$ be the sequence of integers defined by the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$ and the initial conditions $a_0 = 3, a_1 = 1$. Find a_n as an explicit function of n, for all $n \geq 0$.

$a_n - 2a_{n-1} - 3a_{n-2} = 0$ Characteristic polynomial $= x^2 - 2x - 3$
$= (x+1)(x-3)$

So a_n is of the form

$a_n = A(-1)^n + B(3)^n$

$\boxed{\text{So } a_n = 2(-1)^n + 3^n}$

$a_0 = 3 = A + B \Rightarrow A = 3 - B$ ①

$a_1 = 1 = -A + 3B$ ②

① into ②: $1 = -(3-B) + 3B$
$4 = B + 3B$
$B = 1$
Thus $A = 2$

Check: $a_2 = 2a_1 + 3a_0$
$= 2(1) + 3(3)$
$= 11$

$2(-1)^2 + 3^2$
$= 2 + 9$
$= 11$

ok.

4. **[10 marks]**

For each set S and weight function in the second column, the generating function $\Phi_S(x)$ is one of the following power series (one of the power series is the answer for two questions):

$$(1)\ \frac{1}{1-2x^4} \qquad (2)\ \frac{1-x^5}{1-x} \qquad (3)\ \frac{x^4}{(1-x)^4} \qquad (4)\ \frac{1-x^5}{1-2x+x^6}$$

Find the correct generating function for each set. Please place your answer for each set S in the box to the left of the set S.

	Your Answer	The set S
a)	(2)	$S=\{0,1,2,3,4\}$ The weight of i is i
b	(3)	S is the set of all sets of 4 positive integers The weight of a set is its largest element
c)	(1)	$S=\{0000\}^*(\{1111\}\{0000\}^*)^*$ The weight of a string is its length
d)	(4)	$S=\{\epsilon,1,11,111,1111\}(\{0\}\{\epsilon,1,11,111,1111\})^*$ The weight of a string is its length
e)	(3)	S is the set of compositions with exactly 4 parts The weight of a composition is the sum of its parts

$$\boxed{\Phi_S(x)=\sum_{\sigma\in S}x^{w(\sigma)}}$$

$$\boxed{(1-x)^{-k}=\sum_{n\geq 0}\binom{n+k-1}{k-1}x^n}$$

a) $\Phi_S(x) = 1+x+x^2+x^3+x^4$

$= \dfrac{1-x^5}{1-x}$ (by Geometric Series)

c) $\Phi_S(x) = \left(\dfrac{1}{1-x^4}\right)\left(\dfrac{1}{1-\left(\frac{x^4}{1-x^4}\right)}\right)$ (by Sum, Product Star rules)

$=\left(\dfrac{1}{1-x^4}\right)\left(\dfrac{1-x^4}{(1-x^4)-x^4}\right)$

$= \dfrac{1}{1-2x^4}$

d) $\Phi_S(x) = (1+x+x^2+x^3+x^4)\left(\dfrac{1}{1-(1+x+x^2+x^3+x^4)(x)}\right)$

$= \left(\dfrac{1-x^5}{1-x}\right)\left(\dfrac{1}{1-x\frac{(1-x^5)}{1-x}}\right)$ (See (a))

$= \dfrac{1-x^5}{1-x-x(1-x^5)}$

$= \dfrac{1-x^5}{1-2x+x^5}$

b) $\Phi_S(x) = \binom{3}{3}x^4+\binom{4}{3}x^5+\binom{5}{3}x^6+\cdots$

$= \sum_{n\geq 0}\binom{n+3}{3}x^{n+4}$

$= x^4\sum_{n\geq 0}\binom{n+3}{3}x^n$

$= x^4(1-x)^{-4}$ (Binomial Thm)

$= \dfrac{x^4}{(1-x)^4}$

e) $\Phi_S(x) = \Phi_{N_{\geq 1}^4}(x)$

$= (\Phi_{N_{\geq 1}}(x))^4$ (product Lemma)

$= \left(\sum_{i\geq 1}x^i\right)^4$

$= (x+x^2+x^3\ldots)^4$

$= \left(\dfrac{x}{1-x}\right)^4$ (geometric series)

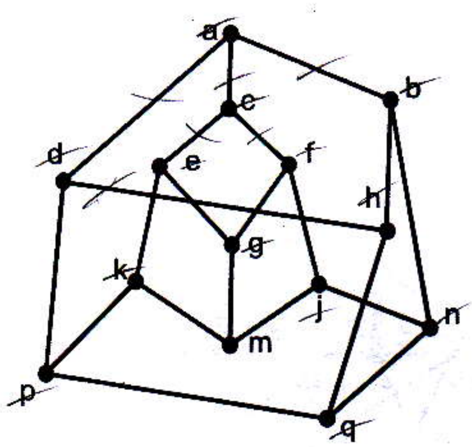
$= \dfrac{x^4}{(1-x)^4}$

5. (a) **[10 marks]**

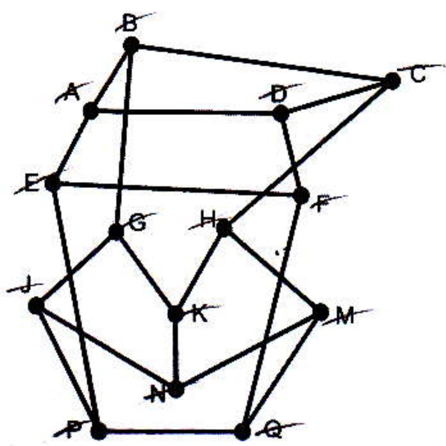
Find a breadth-first search tree for each of the following two graphs rooted at vertex a for graph **G**, and rooted at vertex A for graph **H**.

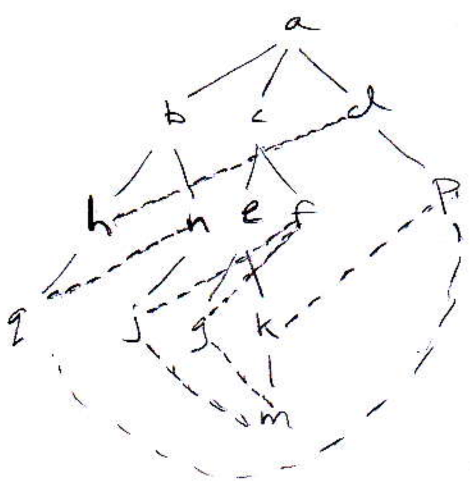
When considering the vertices adjacent to the vertex being examined, take them in alphabetically increasing order of their labels. *List the vertices at each level of your tree.*

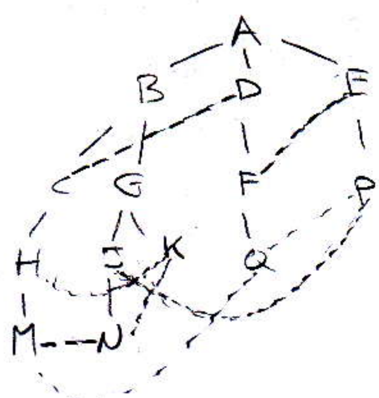
G



H







(b) Determine whether the two graphs are isomorphic. Prove your answer is correct.

They are not isomorphic since G is bipartite since no non-tree edge connects vertices on the same level but H is not-bipartite by similar reasoning.

6. [4 marks]

Let G' be an edge subdivision of a bipartite graph G. Prove that G' is 3-colourable.

Let (A,B) be the vertex set bipartition of G.

Color all vertices in G' that belong to A the color a

Color all vertices in G' that belong to B the color b

For each edge subdivision in G' starting with the vertex in A, color each vertex along the subdivison in alternating colors b and a

If there are an even number of vertices in the edge subdivison, these two colours will suffice.

If there are an odd number, color the final vertex before reaching the vertex in B the color c.

The above algorithm will 3-color G', thus G' is 3-colorable. $\square$

7. [6 marks]

Let G be a connected planar embedding with exactly 20 faces, each of which has degree 3. How many edges and vertices must G have?

Let $f_1, \ldots, f_{20}$ be the faces in G

$$2|E(G)| = \sum_{i=1}^{20} \deg(f_i)$$
$$= 20(3)$$
$$= 60$$

Thus $|E(G)| = 30$

Then by Euler's Thm,

$$2 = |V(G)| - 30 + 20$$
$$|V(G)| = 12$$

So G has 12 vertices, and 30 edges.

$|A| = 13$ $|B| = 12$

8. **[14 marks]**

The graph G shown below is bipartite. The vertex set A is labelled with small letters, and B is labelled with capital letters.

(a) The matching M in the graph G is indicated by the bold edges. Find an M-augmenting path P in G starting at the vertex y.

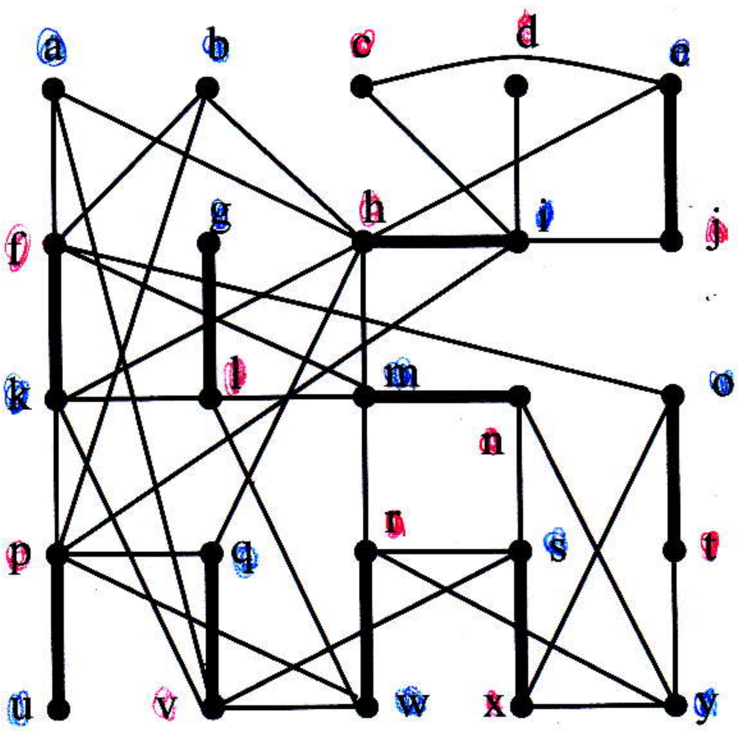


Figure 1: The graph G with matching M

$y\ \{y,n\}\ n\ \underline{\{n,m\}}\ m\ \{m,h\}\ h\ \underline{\{h,i\}}\ i\ \{i,j\}\ j\ \underline{\{j,e\}}\ e\ \{e,c\}\ c$

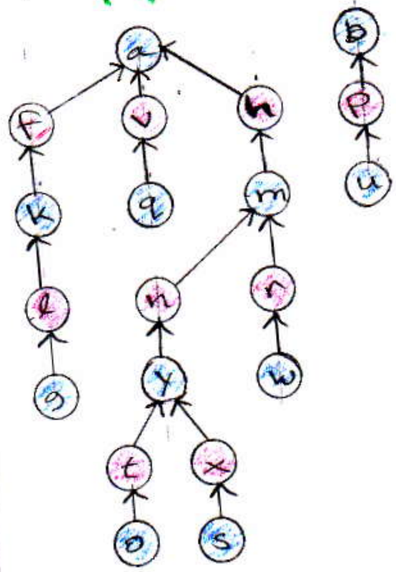
(b) Let M^* be the matching of size $|M|+1$ obtained by switching on the M-augmenting path P you found in the previous part. List the edges of the matching M^*.

$M^* = \{\{u,p\},\{f,k\},\{g,l\},\{q,v\},\{r,w\},\{s,x\},\{o,t\},$
$\{y,n\},\{m,h\},\{i,j\},\{e,c\}\}$

(c) Let X_0 be the M^*-unsaturated vertices in A. Let X be the set of vertices in A that are reachable from a vertex in X_0 by an M^*-alternating path, and let Y be the set of vertices in B that are reachable from a vertex in X_0 by an M^*-alternating path. Use the bipartite matching algorithm to find X_0, X and Y. (Another copy of G is shown here to assist you.)

(A, B)

M^*



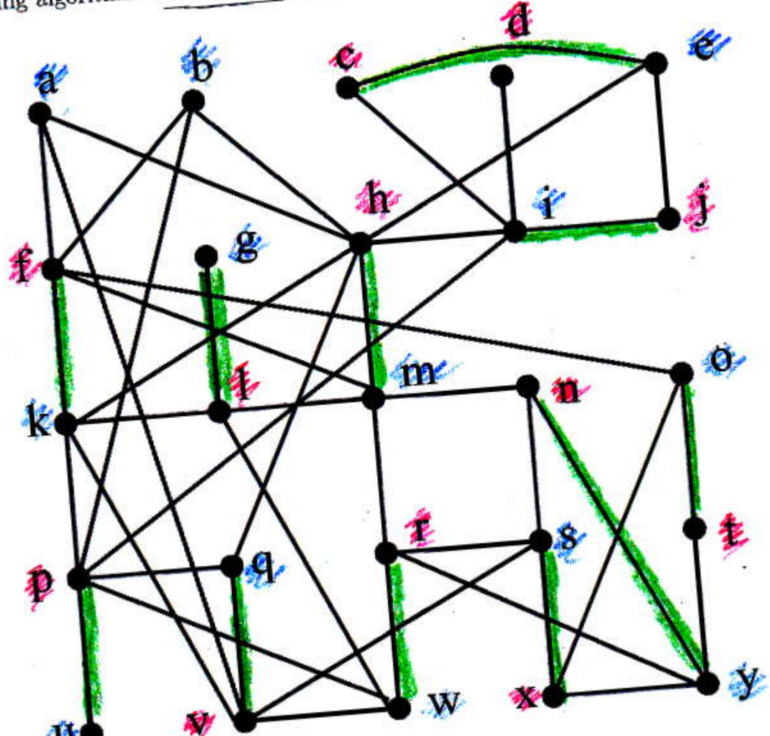


Figure 2: The graph G

$X_0 = \{a, b\}$

$X = \{a, b, k, q, m, u, g, y, w, o, s\}$

$Y = \{f, v, h, p, l, n, r, t, x\}$

(d) Prove that M^* is a maximum matching in G by finding a cover C with $|C| = |M^*|$.

$C = Y \cup A \setminus X = \{f, v, h, p, l, n, r, t, x, e, i\}$

$|C| = |M^*| = 11$

(e) State Hall's Theorem.

If G is bipartite with classes (A, B) then G has a matching of size $|A|$ iff $|N(D)| \geq |D|$ for all $D \subseteq A$

(f) Find a subset $D \subseteq B$ such that $|D| > |N(D)|$.

$D = \{c, d, j\}$ $\quad N(D) = \{i, e\}$

Hint: D should always include at least one unsaturated vertex.

9. **[10 marks]** Let G be a connected planar graph with p vertices, where $p \geq 3$. Let t denote the number of vertices in G with degree less than 6.

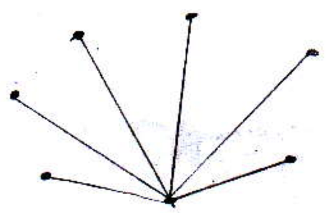
(a) Prove that if G has no cycles then $t \geq \frac{4p+2}{5}$.

If G has no cycles, G is a tree and has $p-1$ edges

So $2e = 2p-2 = \sum_{v \in V(G)} \deg(v) \geq 6(p-t)+t = 6p-5t$ (Since G is connected every vertex has $\deg \geq 1$)

Thus
$$2p-2 \geq 6p-5t$$
$$5t \geq 6p-2p+2$$
$$5t \geq 4p+2$$
$$t \geq \frac{4p+2}{5}$$ as required. □

(b) Give an example of a graph for which equality ($t = \frac{4p+2}{5}$) holds.



$t=6$

$p=7$

check: $\frac{4(7)+2}{5} = \frac{28+2}{5} = 6$

(c) Prove that if G contains a cycle then $t \geq 3$.

$2e = \sum_{v \in V(G)} \deg(v) \geq 6(p-t)+t = 6p-5t$ (Since G is connected, no vertex can have $\deg < 1$)

and since G is planar $e \leq 3p-6 \Rightarrow 2e \leq 6p-12$

Thus
$$6p-5t \leq 6p-12$$
$$-5t \leq -12$$
$$t \geq 12/5 > 2$$

Therefore $t \geq 3$ as required. □

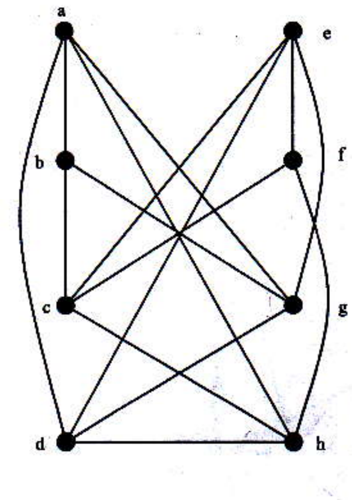
(d) Give an example of a graph for which equality ($t = 3$) holds.

K_3:

$t=3$

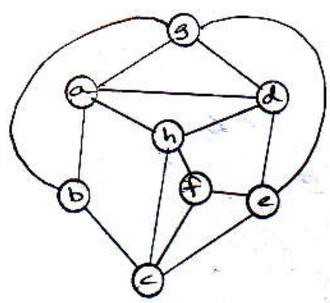
$p=3$

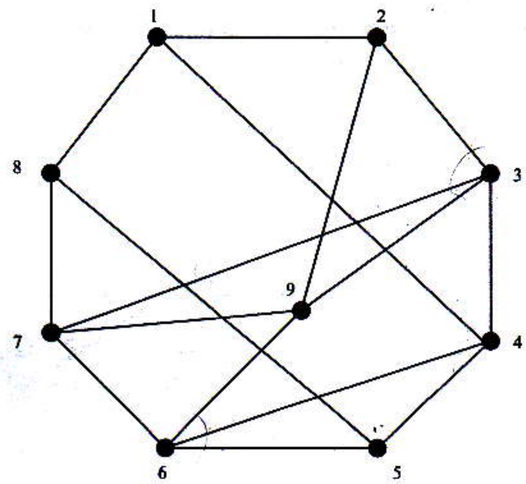
10. **[8 marks]** Determine whether each of the graphs shown is planar or nonplanar. Justify your answer in each case.



G

is planar





H

is not planar because it contains a $K_{3,3}$ edge subdivision subgraph (Kuratowski's Th'm)