MATH 239 Spring 2012: Assignment 2

Solutions

1. {10 marks}

   (a) Let S be the set of all finite subsets of $\mathbb{N}$. For each $A \in S$, define $w(A)$ to be the largest element in A, with $w(\emptyset) = 0$. Determine the generating series for S with respect to w.

   Solution. For the generating series of S, the coefficient of x^n represents the number of sets whose largest element is n. For $n \geq 1$, there are 2^{n-1} such sets: any subset of $[n-1]$ union with $\{n\}$. For $n = 0$, there is one such set, which is $\emptyset$. So the generating series is

$$\Phi_S(x) = 1 + \sum_{n \geq 1} 2^{n-1} x^n = 1 + \frac{x}{1-2x} = \frac{1-x}{1-2x}.$$

   (b) Suppose we change the word "largest" to "smallest" in part (a). Explain why the new weight function does not have a generating series.

   Solution. There are infinitely many sets whose smallest element is, say, 1. So we cannot find a coefficient for x.

2. {20 marks} You are playing a game with a deck of 11 cards. There is one card whose value is i for each $i \in \{1, 2, \ldots, 11\}$. You will draw cards one at a time, each time placing the card back into the deck after the draw. Your score is the total value of the cards that you draw. (Note: The order which you draw the cards matters. We would consider drawing a 2 and then a 7 to be different from drawing a 7 and then a 2 in scoring a 9.) For each of the following questions, write down the set you are enumerating, the weight function you are using for this set, and represent your answer as the coefficient of some generating series.

   (a) How many ways can you score 21 after exactly 3 draws?

   Solution. The set of cards we are dealing with has values in $[11]$. Using $w(a) = a$, the generating series for $[11]$ is $\Phi_S(x) = \frac{x(1-x^{11})}{1-x}$. For this question, we are counting the set $[11] \times [11] \times [11]$, and the weight of a 3-card draw is $w(a, b, c) = a + b + c$. Using the product lemma, the answer is

$$[x^{21}] \left(\frac{x(1-x^{11})}{1-x} \right)^3.$$

   (b) How many ways can you score 21 after 2, 3, or 4 draws?

   Solution. The set we are counting here is $[11]^2 \cup [11]^3 \cup [11]^4$. The weight of each element is the sum of its parts. Using the sum and product lemmas, the answer is

$$[x^{21}] \left(\left(\frac{x(1-x^{11})}{1-x} \right)^2 + \left(\frac{x(1-x^{11})}{1-x} \right)^3 + \left(\frac{x(1-x^{11})}{1-x} \right)^4 \right).$$

   (c) For some positive integer n, how many ways can you score n? You are not limited to the number of draws you take.

   Solution. The set we are counting here is $\emptyset \cup [11] \cup [11]^2 \cup [11]^3 \cup \cdots = \bigcup_{i \geq 0} [11]^i$ (the only reason we include $\emptyset$ is to make the generating series slightly simpler, but it is not a necessity). The weight function is the same as before. The answer is

$$[x^n](1 + \Phi_{[11]}(x) + \Phi_{[11]^2}(x) + \cdots) = [x^n] \frac{1}{1 - \Phi_{[11]}(x)} = [x^n] \frac{1}{1 - \frac{x(1-x^{11})}{1-x}} = [x^n] \frac{1-x}{1-2x+x^{12}}.$$

3. {8 marks} Let S be a set of objects, and suppose w is a weight function on S with generating series $\Phi_S(x)$. Let w^* be a new weight function for S defined by $w^*(a) = 4w(a) + 2$ for all $a \in S$. Determine the generating series $\Phi_S^*(x)$ with respect to the weight function w^* in terms of $\Phi_S(x)$.

   Solution. Using the definition of generating series, we have

$$\begin{aligned}\Phi_S^*(x) &= \sum_{\sigma \in S} x^{w^*(\sigma)} \\ &= \sum_{\sigma \in S} x^{4w(\sigma)+2} \\ &= x^2 \sum_{\sigma \in S} (x^4)^{w(\sigma)} \\ &= x^2 \Phi_S(x^4).\end{aligned}$$

4. {Extra credit: 3 marks} Prove the following:

$$(1-x)^{-1} = \prod_{i \geq 0} (1 + x^{2^i}).$$

   Solution. The coefficient of x^n on the LHS is 1. The RHS can be interpreted as the generating series for the set

$$\{0,1\} \times \{0,2\} \times \{0,4\} \times \{0,8\} \times \{0,16\} \times \cdots$$

   whose weight function is the sum of its parts. The coefficient of x^n on the RHS is then the number of ways n can be written as a sum of distinct powers of 2. But there is only one way to do this (which is its binary representation), so the coefficient must be 1. Since the coefficients on both sides of the equation are the same, the identity holds.

5. {12 marks} For a permutation σ of $[n]$, a pair (i, j) is called an *inversion* of σ if $i < j$ and $\sigma(i) > \sigma(j)$. For example, the permutation (32415) on [5] has 4 inversions: $(1, 2), (1, 4), (2, 4), (3, 4)$. Define the weight function w on a permutation σ to be the number of inversions in σ. Let S_n be the set of all permutations of $[n]$.

   (a) Determine the generating series for S_1, S_2, S_3 with respect to the weight function w.

   Solution. $S_1 = \{(1)\}$, which has 0 inversions. So $\Phi_{S_1}(x) = 1$.
   For S_2, (12) has no inversions, but (21) has one inversion. So $\Phi_{S_2}(x) = 1 + x$.
   For S_3, (123) has no inversions, (132), (213) have one inversion, (231), (312) have two inversions, and (321) has three inversions. So $\Phi_{S_3}(x) = 1 + 2x + 2x^2 + x^3$.

   (b) Prove that

$$\Phi_{S_n}(x) = (1 + x + \cdots + x^{n-1})\Phi_{S_{n-1}}(x).$$

   You may use the following (non-standard) notation: If σ is a permutation of $[n]$, denote σ' to be the permutation of $[n-1]$ obtained from σ by removing the element n. For example, if $\sigma = (31524)$, then $\sigma' = (3124)$.

   Solution. We split S_n into $n+1$ sets according to the location of the element n in the permutation. For $i = 1, \ldots, n$, let T_i be the set of all permutations $\sigma \in S_n$ where $\sigma(i) = n$. Then

$$S_n = T_1 \cup T_2 \cup \cdots \cup T_n.$$

   For each T_i, we can form a bijection between T_i and S_{n-1} as follows: $f : T_i \to S_{n-1}$ where $f(\sigma) = \sigma'$. We now compare the number of inversions between σ and σ'. Each inversion in σ'

is still an inversion of σ. However, there are additional inversions introduced by n in σ. Since n is the largest possible element in σ, it creates an inversion with any element after it. Since $\sigma(i) = n$, there are $n - i$ additional inversions, namely $(i, i + 1), (i, i + 2), \ldots, (i, n)$. Therefore, $w(\sigma) = w(\sigma') + (n - i)$. Since we have a bijection between T_i and S_{n-1}, we can say that

$$\Phi_{T_i}(x) = x^{n-i}\Phi_{S_{n-1}}.$$

Using the sum lemma, we get

$$\Phi_{S_n}(x) = \sum_{i=1}^{n} \Phi_{T_i}(x) = \sum_{i=1}^{n} x^{n-i}\Phi_{S_{n-1}}(x) = (1 + x + x^2 + \cdots + x^{n-1})\Phi_{S_{n-1}}(x).$$

(c) Prove that the number of permutations of $[n]$ with k inversions is

$$[x^k]\frac{\prod_{i=1}^{n}(1 - x^i)}{(1 - x)^n}.$$

Solution. We see that $\Phi_{S_1}(x) = 1$ from part (a), so this is satisfied for $n = 1$. Using induction, we see that

$$\begin{aligned}
\Phi_{S_n}(x) &= (1 + x + \cdots + x^{n-1})\Phi_{S_{n-1}}(x) \text{ by part (b)} \\
&= \frac{1 - x^n}{1 - x}\Phi_{S_{n-1}}(x) \\
&= \frac{1 - x^n}{1 - x}\frac{\prod_{i=1}^{n-1}(1 - x^i)}{(1 - x)^{n-1}} \text{ by ind hyp} \\
&= \frac{\prod_{i=1}^{n}(1 - x^i)}{(1 - x)^n}
\end{aligned}$$