MATH 239 Tutorial 6 Solution Outline

1. Let G be a graph with p vertices and every vertex of G has degree at least $(p-1)/2$. Prove that G is connected.

   Solution. To prove that G is connected, we wish to show that for any two vertices x and y of G, there is a path between them in G.

   Case 1: x and y are adjacent. In this case their is obviously a path from x to y; it's the edge xy.

   Case 2: Suppose x and y are not adjacent. Since each vertex has degree at least $(p-1)/2$, x must have at least $(p-1)/2$ neighbours, as does y. We claim that the neighbours of x and the neighbours of y overlap. If they didn't, then the graph has at least

   $$(p-1)/2 + (p-1)/2 + 2 = p + 1$$

   vertices, which is a contradiction since G has exactly p vertices.

   Thus the neighbours of x and y overlap in at least one vertex, call it z. Thus xzy is a path from x to y. Thus our graph is connected.

2. Let G_n be the graph where the vertices are all binary strings of length n, and two vertices are adjacent if the two strings differ in exactly 2 positions.

   (a) Draw G_2 and G_3.
   Solution. G_2 is a matching of size 2. G_3 is two disjoint copies of K_4.

   (b) How many edges are in G_n?
   Solution. There are 2^n vertices in G_n (the number of binary strings of length n). Two binary strings are adjacent if they differe in exactly 2 positions. The number of ways to swap 2 bits in a binary string of length n is $\binom{n}{2}$.
   Thus, each vertex has degree $\binom{n}{2}$, and there are 2^n vertices. By handshaking,

   $$2|E(G)| = \sum_{v \in V(G)} \deg(v) = 2^n \cdot \binom{n}{2}$$

   so $|E(G)| = 2^{n-1} \cdot \binom{n}{2}$.

   (c) For what values of n is G_n connected?
   Solution. None. Consider any vertex with an even number of 1s. Every time we move along an edge, we swap exactly two bits. Thus the number of 1s can either increase by two, decrease by two, or stay the same. Thus, every time we move along an edge, we get to another vertex with an even number of 1s. Therefore, this vertex cannot be connected to any vertex with an odd number of 1s, so our graph is not connected.

   (d) For what values of n is G_n bipartite?
   Solution. Only for $n \leq 2$. For $n = 3$, consider the three vertices $000, 110, 101$. These vertices are all connected to each other in G_3, forming a cycle of length 3, so G_3 cannot be bipartite. We can extend this to any G_n for $n > 3$ by adding a 0 in front of each string. For example, G_4 has the 3-cycle $0000, 0110, 0101$.

3. Prove that if every vertex of a graph G has degree at least 3, then G contains a cycle of even length.

   Solution. Let P be a longest path in G, $P = v_0v_1v_2...v_k$. Consider the end vertex v_0. All neighbours of v_0 must be in the path P, since otherwise we could extend P to be a longer path, which is a contradiction.

   Case 1: v_0 is adjacent to some v_i, for i odd. Then the subpath $v_0v_1...v_i$ along with edge v_iv_0 forms a cycle of even length.

   Case 2: v_0 is only adjacent to vertices with even index. Let two of them be v_i and v_j, for even i, j, and without loss of generality, let $i < j$. Now consider the edge v_0v_i, subpath $v_iv_{i+1}...v_j$, and edge v_jv_0. These three combine to form a cycle of even length, since subpath $v_iv_{i+1}...v_j$ has even length.

4. How many Hamilton cycles are there in K_n where the vertices are labelled with $1, 2, \dots, n$? We consider two Hamilton cycles to be the same if they use the same set of edges.

   Solution. Any permutation of the n labels corresponds to a Hamilton cycle in K_n. Note, however, that we are double-counting since the permutation can be read forward and backwards, resulting in the same cycle, so we must divide our number by 2. Additionally, any shift of the permutation results in the same Hamilton cycle (for example, for K_4, the permutation 1234 results in the same cycle as 2341, and 3412, and 4123). There are n ways to shift the permutation. Thus our final answer is $(n-1)!/2$.

Additional exercises

1. Let $k \geq 1$. If G is a k-regular bipartite graph with a bipartition (A, B) of the vertices, then $|A| = |B|$.

   Solution. $\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v)$, so $k|A| = k|B|$. Since $k \geq 1$, $|A| = |B|$.

2. Determine (with proof) a bipartite graph with the fewest number of edges such that it is NOT the subgraph of any n-cube.

   Solution. $K_{2,3}$

3. Prove that for $n \geq 2$, the n-cube has a Hamilton cycle.

   Solution. Induction on n. The two copies of the $n-1$-cube have Hamilton cycles by induction. Join them together to get a Hamilton cycle of the n-cube.

4. Suppose that P and Q are two paths of maximum length in a connected graph G. Prove that there is at least one vertex that is in both P and Q.

   Solution. Suppose P and Q don't share any vertices. Since G is connected, there must be a path between them. This path separates each of P and Q into two parts. Pick the longer of the two parts, plus the path between them, this is a longer path, contradiction.

5. Let G_n be the graph whose vertices are all permutations of $[n]$, and two vertices are adjacent if and only if one permutation can be obtained from another by swapping

two entries. For example, in G_4, (1234) is adjacent to (1324) and (1432), but not (3142).

(a) Draw G_2 and G_3.

Solution. G_2 is one edge, G_3 is $K_{3,3}$.

(b) How many vertices and edges are in G_n?

Solution. $n!$ vertices, $\binom{n}{2}$-regular, so $n!\binom{n}{2}/2$ edges.

(c) Prove that G_n is bipartite.

Solution. Take A to be permutations that can be obtained from $(123...n)$ through even number of switches, and B to be those through odd number of switches.

(d) Prove that G_n is connected.

Solution. From any permutation, we can make at most $n-1$ switches to get back to $(123...n)$.