Combinatorics & Optimization

University of Waterloo

Introduction to Combinatorics

Lecture 1

Michele Mosca

mmosca@uwaterloo.ca

This course:
Introduction to Combinatorics

- We will use D2L: https://learn.uwaterloo.ca/

It is the responsibility of students to check the web page regularly.

- My supplementary notes are available at

http://info.iqc.ca/mmosca/2014math239

This course:
Introduction to Combinatorics

Course Notes:
Introduction to Combinatorics: Course Notes for Math 239

Outline:
The first portion (five weeks) of the course is Combinatorial Analysis.

The second portion (seven weeks) of the course is Graph Theory.

This course:
Introduction to Combinatorics

Tutorial Centre:
Graduate student Teaching Assistants will be in the Tutorial Centre, MC4067 (check course web page for times and dates).

Assignments:
Assignments will be due on Fridays at ***3pm*** in the drop boxes opposite MC 4067. The assignments and solutions will be posted on the course web page.

Late assignments will not be accepted.

This course: Introduction to Combinatorics

Tutorials: Graduate student Teaching Assistants will be responsible for the tutorials. They will present examples, answer student questions, and return marked assignments. They will ***not*** present solutions to assignment problems before the deadline. However, they can do related examples, and can also answer some specific questions related to an assignment problem (without giving away the solution) provided the problem has been seriously attempted.

This course:
Introduction to Combinatorics

Academic Integrity: In order to maintain a culture of academic integrity, members of the University of Waterloo community are expected to promote honesty, trust, fairness, respect and responsibility.

[Check www.uwaterloo.ca/academicintegrity/ for more information.]

Grievance: A student who believes that a decision affecting some aspect of his/her university life has been unfair or unreasonable may have grounds for initiating a grievance. www.adm.uwaterloo.ca/infosec/Policies/policy70.htm. When in doubt please be certain to contact the department/school's administrative assistant who will provide further assistance.

This course: Introduction to Combinatorics

Discipline: A student is expected to know what constitutes academic integrity to avoid committing academic offenses and to take responsibility for his/her actions. A student who is unsure whether an action constitutes an offense, or who needs help in learning how to avoid offenses (e.g. plagiarism, cheating) or about ”rules” for group work/collaboration should seek guidance from the course professor, academic advisor, or the undergraduate associate dean. For information on categories of offenses and types of penalties, students should refer to Policy 71, Student Discipline,

www.adm.uwaterloo.ca/infosec/Policies/policy71.htm.

This course:
Introduction to Combinatorics

Appeals: A decision made or penalty imposed under Policy 70, Student Petitions and Grievances (other than a petition) or Policy 71, Student Discipline may be appealed if there is a ground. A student who believes he/she has a ground for an appeal should refer to Policy 72, Student Appeals,

www.adm.uwaterloo.ca/infosec/Policies/policy72.htm

Note for students with disabilities:
AccessAbility Services is located in Needles Hall, room 1132.

This course:
Introduction to Combinatorics

Final grade:
Final exam 55%, Midterm 30%, Assignments15%.

Midterm date:
Thursday 6 March 4:30-6:20pm

This course:
Introduction to Combinatorics

Policy on Lectures and Assignments:

Students are expected to attend all lectures and to submit all assignments for grading.

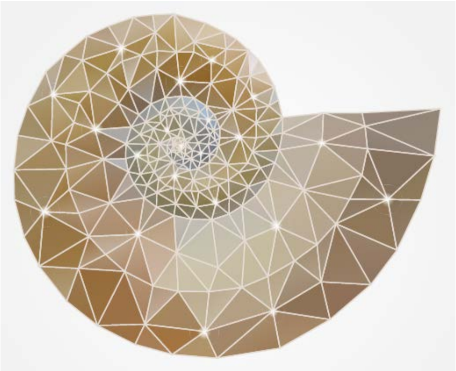
Policy on INC Grades:

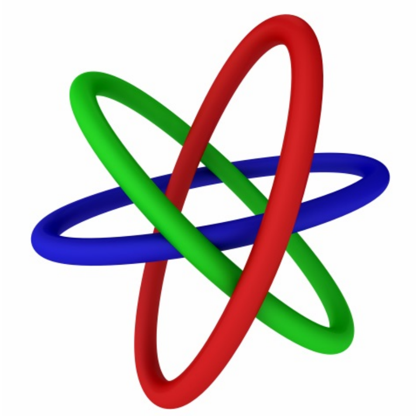
A grade of INC (incomplete) will be *only* awarded to students who cannot write the final exam for reasons acceptable to the instructor, such as a medical certificate by a recognized medical professional. In addition such students need to be in *good standing* prior to the final exam. To be in good standing a student must

- submit and pass 7 out of 10 assignments,
- write and pass the midterm exam, and
- attend classes.

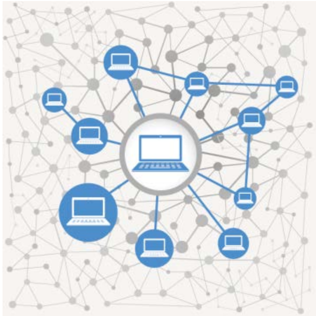
Why should I care about this course?

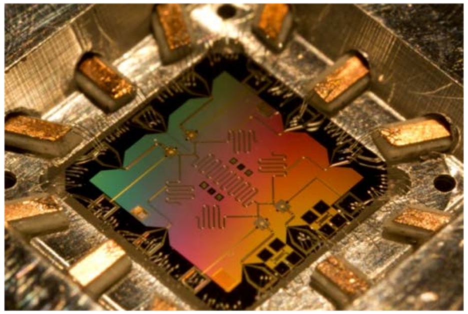
- Combinatorics is a beautiful, elegant and interesting area of mathematics.



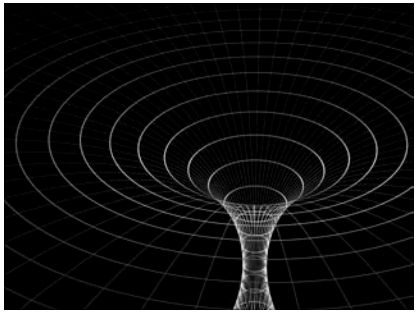


- Combinatorics has many important and interesting applications, from computer science to fundamental physics.



E. Lucero, D. Mariantoni, and M. Mariantoni



How do I get a good mark in this course?

- **Don't try to cram.** It takes time to get used to these ideas. You have to practise math regularly to get good at it.

- **Give the assignments an honest effort.** Their primary purpose is to help you learn the material. Don't just show up to the tutorials or office hours fishing for answers. Try the problems first.

A simple example of a counting problem: counting subsets

A basic counting example

How many subsets does the set $\{1,2,3\}$ have?

Note that the subsets of $\{1,2\}$ are:

$\{\}$,

$\{1\}$,

$\{2\}$,

$\{1,2\}$

(so, 4 in total)

The subsets of $\{1,2,3\}$ are:

$\{\}$, $\{3\}$,

$\{1\}$, $\{1,3\}$,

$\{2\}$, $\{2,3\}$,

$\{2,3\}$, $\{1,2,3\}$

(so, 8 in total)

Some basic counting examples

What's the connection between the number of subsets of {1,2,3} (call this number S_3) and the number of subsets of {1,2} (call this number S_2)?

$S_3 = 2S_2$ because for every subset of {1,2} there are 2 subsets of {1,2,3}: one where we include '3' and one where we don't include '3'

Similarly, $S_2 = 2S_1$ because for every subset of {1} there are 2 subsets of {1,2}: one where we include '2' and one were we don't include '2'.

Lastly, $S_1 = 2$ because we either include '1' (and get the subset {1} or we don't include '1' and get { }).

Some basic counting examples

How many subsets does the set {1,2,3} have?

By the reductions we just gave, we see that the total number of subsets of {1,2,3} is

$$S_3 = 2 \times S_2 = 2 \times 2 \times S_1 = 2 \times 2 \times 2 = 2^3 = 8$$

“include ‘3’ or not”

“include ‘1’ or not”

“include ‘2’ or not”

A related problem

Notice the natural 1-1 correspondence with the subset problem

How many terms does the expansion of

$f(y_1, y_2, y_3) = (1+y_1)(1+y_2)(1+y_3)$ have?

$$\begin{aligned} &= (1+y_1)(1+y_2) \\ &+ (1+y_1)(1+y_2)y_3 \end{aligned}$$

$$\begin{aligned} &= (1+y_1) \\ &+ (1+y_1)y_2 \\ &+ (1+y_1)y_3 \\ &+ (1+y_1)y_2y_3 \end{aligned}$$

$$\begin{aligned} &= 1 \\ &+ y_1 \\ &+ y_2 \\ &+ y_1y_2 \\ &+ y_3 \\ &+ y_1y_3 \\ &+ y_2y_3 \\ &+ y_1y_2y_3 \end{aligned}$$

Notice the natural 1-to-1 correspondence with the subset problem

How many terms does the expansion of

$$f(y_1, y_2, y_3) = (1 + y_1)(1 + y_2)(1 + y_3)$$ have?

don't include y_1

include y_1

don't include y_2

include y_2

don't include y_3

include y_3

$$= 1 + y_1 + y_2 + y_3 + y_1y_2 + y_2y_3 + y_1y_3 + y_1y_2y_3$$

$\{\}$ $\{1\}$ $\{2\}$ $\{3\}$ $\{1,2\}$ $\{2,3\}$ $\{1,3\}$ $\{1,2,3\}$

There is a 1-to-1 correspondence between terms of $f(y_1,y_2,y_3)$ and subsets of $\{1,2,3\}$.

Who cares?

Exploiting this correspondence

If we "plug in" $y_1 = y_2 = y_3 = 1$ into

$$1 + y_1 + y_2 + y_3 + y_1 y_2 + y_2 y_3 + y_1 y_3 + y_1 y_2 y_3$$

then we are counting the subsets.

$$f(1,1,1) = 1+1+1+1+1\cdot 1+1\cdot 1+1\cdot 1+1\cdot 1\cdot 1 = 8$$

(this computation corresponds to counting the subsets one by one)

$$f(1,1,1) = (1+1)\cdot(1+1)\cdot(1+1) = 2^3 = 8$$

(this computation corresponds to the recursive counting method we used earlier)

Exploiting this correspondence

If we "plug in" $y_1 = y_2 = y_3 = x$ into

$$1 + y_1 + y_2 + y_3 + y_1 y_2 + y_2 y_3 + y_1 y_3 + y_1 y_2 y_3$$

then we make the terms with the same size indistinguishable:

$$f(x,x,x) = 1 + x + x + x + x \cdot x + x \cdot x + x \cdot x + x \cdot x \cdot x$$

$$= 1 + 3x + 3x^2 + x^3 = (1+x)^3$$

But the information about the sizes of the terms and the number of terms of each size is preserved.

Exploiting this correspondence

But the information about the sizes of the terms and the number of terms of each size is preserved.

(note that $1 = x^0, x = x^1$ and $x^m = 1x^m$ and absence of a term means its coefficient is 0)

$$f(x, x, x) = 1x^0 + 3x^1 + 3x^2 + 1x^3$$

1 subset of size 0

3 subsets of size 1

3 subsets of size 2

1 subsets of size 3

A general lesson on proving two sets have the same size

Proving two sets have the same size

Let A and B denote two finite sets.

A function $f: A \rightarrow B$ maps every element of A to a unique element of B (this is just the definition of a function).

Proving two sets have the same size

The function f is “one-to-one” or “injective” if for every value b in B, there is at most one value a in A such that f(a)=b (another way of saying this is that f(x) = f(y) if and only if x=y).

The function f is “onto” or “surjective” if for every value b in B, there exists at least one value a in A such that f(a)=b.

Proving two sets have the same size

If there is an injective function $f:A \rightarrow B$, then $|A|\leq|B|$.

If there is a surjective function $g:A \rightarrow B$, then $|A|\geq|B|$.

Therefore, one way to prove $|A|=|B|$ is to find an injection $f:A\rightarrow B$, and a surjection $g:A\rightarrow B$.

Alternatively, one can do it in one shot by finding a “one-to-one correspondence”.

Proving two sets have the same size

A function is a "bijection" or "one-to-one correspondence" if it is both injective and surjective.

If $f:A \rightarrow B$ is a bijection, then $|A|=|B|$.

One way to prove a function f is a bijection is to find an inverse function $g=f^{-1}$ that maps $B \rightarrow A$ with the property that for all b in B, $f(g(b))=b$, and for all a in A, $g(f(a))=a$

(verify this; note that the $f(g(b))=b$ part shows that f is onto, and $g(f(a))=a$ part shows it is one to one).

Please read sections 1.1, 1.2, 1.3 and 1.4 this week.