UNIVERSITY OF WATERLOO
FINAL EXAMINATION
WINTER TERM 2009

Surname: ______________________

First Name: ______________________

Id.#: ______________________

MATH 239	
Course Title	Introduction to Combinatorics
Please check your lecture section	☐ Professor Mosca 10:30 MWF LEC 001 ☐ Professor Wormald 1:30 MWF LEC 002
Date of Exam	April 23, 2009
Time Period	9:00 - 11.30 a.m.
Number of Exam Pages (including this cover sheet)	10
Exam Type	Closed Book
Additional Materials Allowed	None
Additional Instructions	Write your answers in the space provided. Give reasons for your answers and show all your work. You may use any theorems proved in lectures. State clearly which theorems you are using. No calculators or other aids permitted.

Problem	Value	Mark Awarded	Problem	Value	Mark Awarded
1	5		7	11	
2	4		8	7	
3	4		9	6	
4	3		10	6	
5	12		11	6	
6	12		12	14	
			TOTAL	90	

1. **[5 marks]** Give a combinatorial proof that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ where $n \geq k \geq 1$ are integers. (In particular, you may not use any exact formula for $\binom{n}{k}$.)

2. **[4 marks]** Let n be an even positive integer, and m a positive integer. Find an expression for the following coefficient, involving no summation notation. Justify your answer.

$$[x^n](1 - 3x^2)^{-m}(1 + x)^2.$$

3. [**4 marks**] Find the generating function for the number of compositions of n into $2k$ parts, where each of the first k parts is an even number, and each of the other parts is at least 2.

4. [**3 marks**] Find a decomposition that uniquely creates the elements of the set of $\{0,1\}$-strings that have no substring of 1s with length 2, and no substrings of 0s of length 3. Justify your answer.

5. [**12 marks**] Let $S = \{11\}^*\left(\{0\}\{00\}^*\{111\}\{11\}^* \cup \{00\}\{00\}^*\{11\}\{11\}^*\right)^*\{00\}^*$.

   (a) List all the elements of S of weight 4.

   (b) How many elements of S have weight 5?

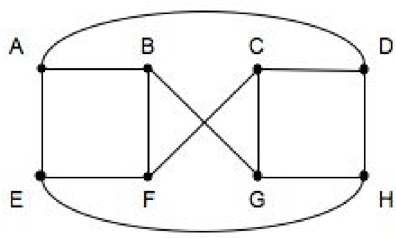
(c) (Recall $S = \{11\}^*\big(\{0\}\{00\}^*\{111\}\{11\}^* \cup \{00\}\{00\}^*\{11\}\{11\}^*\big)^*\{00\}^*$.) Show that the generating function for S is $\dfrac{1}{1-2x^2-x^4}$. (You may assume without proof that this decomposition creates the elements of S uniquely.)

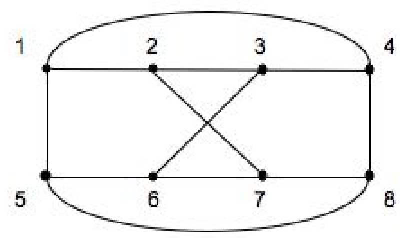
(d) Let b_n equal the number of strings in S with length n. Establish a recurrence relation (and state the values of n for which it holds) with sufficient initial conditions to determine all the values b_n.

(e) Determine from your recurrence relation the value of b_6.

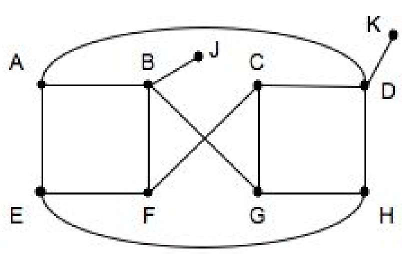
6. [**12 marks**]

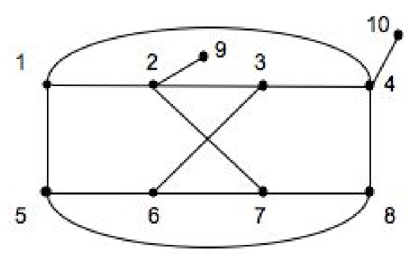
(a) Prove that the following two graphs are isomorphic.





(b) Explain why the following two graphs are *not* isomorphic.

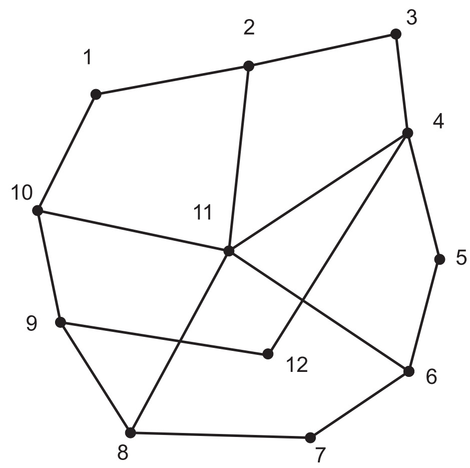




(c) Prove that the first graph in (a) is not planar.

7. **[11 marks]**

(a) Find a breadth-first search tree for the following graph, rooted at vertex 1. When there is a choice for which vertex should enter the BFST next, choose the vertex with the numerically smaller label. Keep a "vertex list" which tracks the order in which vertices entered the BFST, and indicate the parent of every non-root vertex (by an arrow pointing from child to parent).



(b) Prove that the graph given in (a) is not bipartite.

(c) Find a cover of the graph given in (a) consisting of the vertices 4, 9, and 11, together with exactly four other vertices. What, if anything, does this imply about a maximum matching in that graph?

8. **[7 marks]**

   (a) Draw a graph on 5 vertices that is 2-regular, and then draw its complement.

   (b) Prove that if a graph G has 11 vertices then either G is nonplanar or its complement $\overline{G}$ is nonplanar.

9. **[6 marks]**

   (a) Draw a planar embedding of a graph that has exactly four vertices and exactly two faces, with each face of odd degree.

   (b) Prove that in a planar embedding, there cannot be exactly one face of odd degree, the rest being even.

10. [**6 marks**] Prove that if G is a planar graph with no cycles of length less than 6, then G is 3-colourable. You may assume without proof that every planar graph with no cycles of length less than 6 must have a vertex of degree at most 2.

11. [**6 marks**] For any positive integer n, let G_n be the graph whose vertices are the binary vectors of length n, where two vertices are adjacent if they differ in exactly 3 positions.

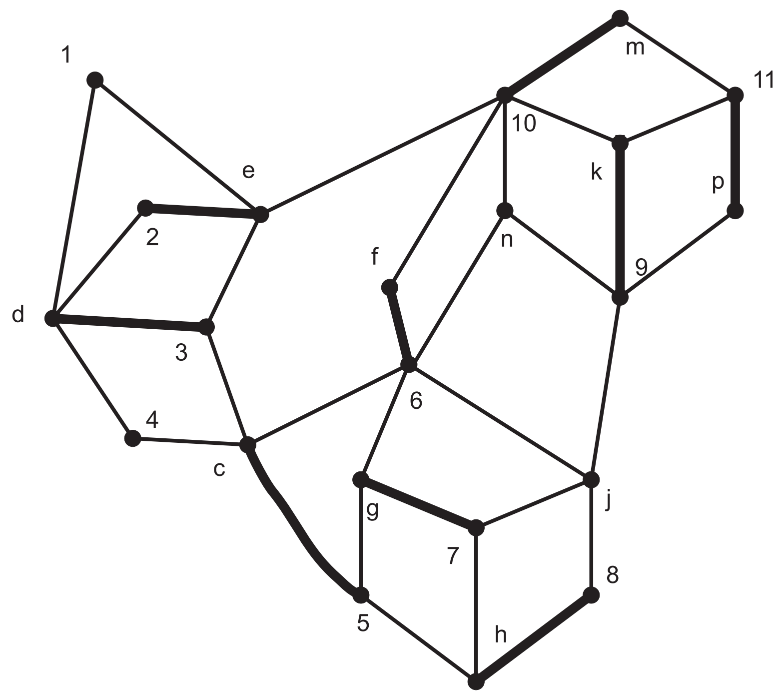
    (a) Draw G_2 and G_3.

    (b) Find the number of vertices and edges of G_n.

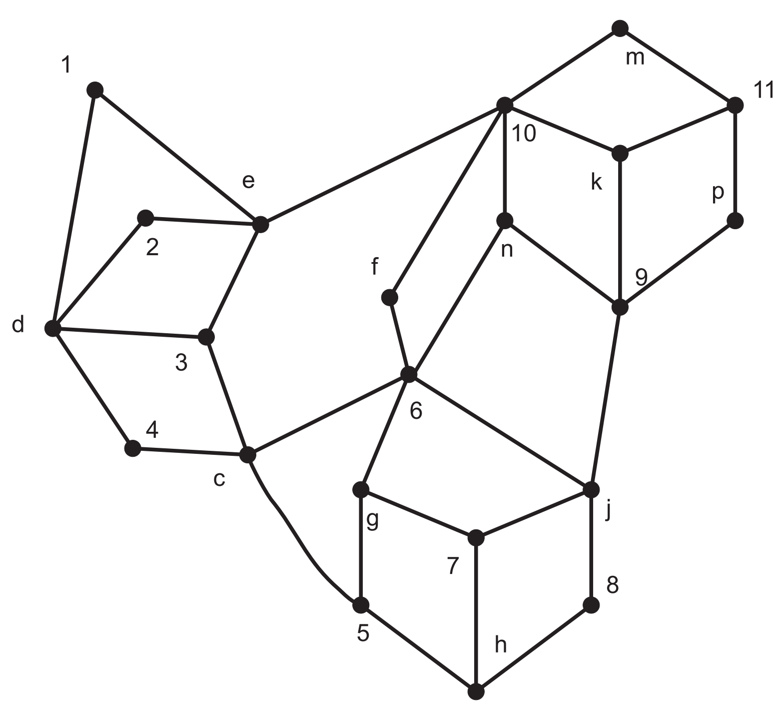
12. **[14 marks]**

In the following bipartite graph, let A correspond to the vertices labelled with integers and B the vertices labelled with letters. Let M denote the edges indicated with thicker lines, i.e. $M = \{\{e, 2\}, \{d, 3\}, \{c, 5\}, \{g, 7\}, \{h, 8\}, \{f, 6\}, \{k, 9\}, \{m, 10\}, \{p, 11\}\}$.

(a) Find the sets $X_0, X \subseteq A$ and $Y \subseteq B$ obtained via the XY-construction with respect to the matching M. In particular, *make it clear in what order the vertices are added to the sets X and Y during the XY-construction*: either list the vertices in order added to X and to Y, or show the trees constructed level by level.



(b) Use the sets X and Y to find an augmenting path, and use it to derive a larger matching for G. Indicate this new matching on the following figure.



(c) Repeat the previous two steps as many times as necessary in order to find a maximum matching for G. Find a minimum cover and prove that the matching is maximum.

[END OF EXAM]