MATH 239 Supplementary : Eulerian circuits

(An example is provided to illustrate the ideas of this proof.)

Theorem 1. *Let G be a graph with no isolated vertices. Then G has an Eulerian circuit if and only if G is connected and every vertex of G has even degree.*

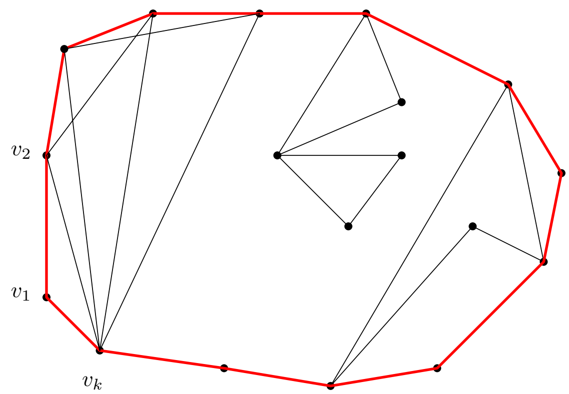
Proof. ($\Rightarrow$) Exercise.

($\Leftarrow$) We will prove by induction on the number of cycles in G.

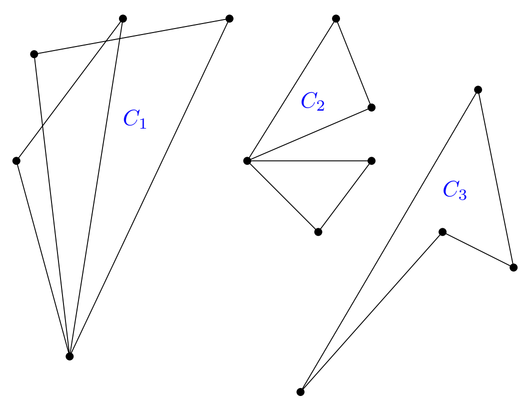
Base case: When G has 1 cycle, it is already an Eulerian circuit.

Induction hypothesis: Assume that any connected graph with fewer cycles than G where every vertex has even degree has an Eulerian circuit.

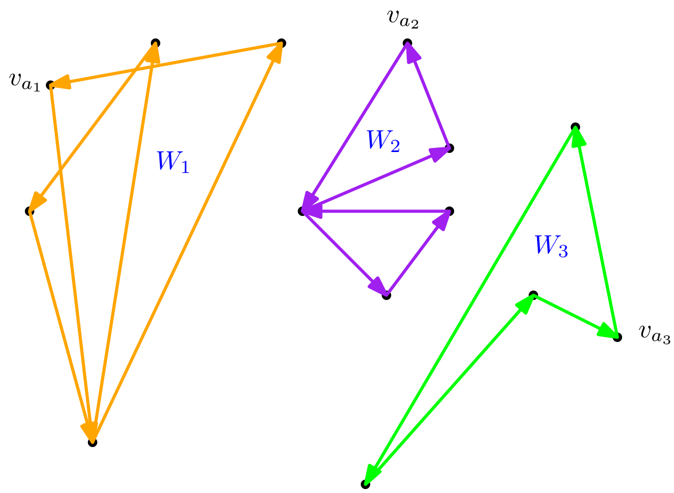
Induction step: Let G be a connected graph where every vertex has even degree. Since there is no isolated vertex, every vertex has degree at least 2. Therefore, there exists a cycle C in G, say the vertices on the cycle in order are $v_1, v_2, \ldots, v_k, v_1$.



Remove edges of C from G and remove isolated vertices to obtain G'. Since every vertex is incident with even number of edges in C, every vertex in G' still has even degree. Now G' consists of components $C_1, \ldots, C_l$, each containing fewer cycles than G.



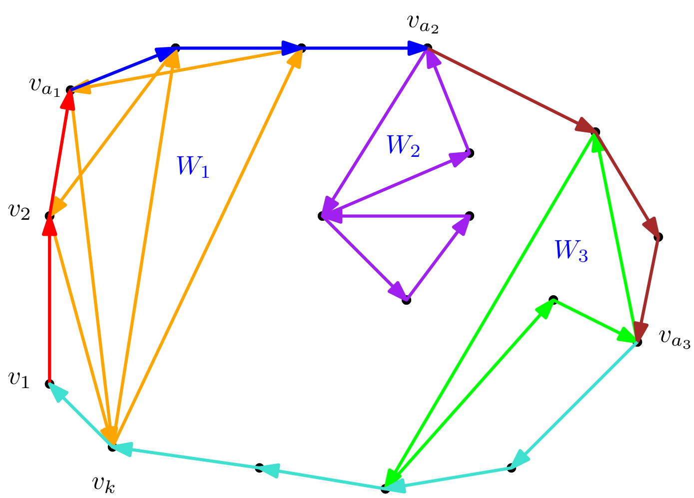
By induction hypothesis, each component C_i has an Eulerian circuit W_i. Moreover, each component must have a common vertex with C, for otherwise G is disconnected.



Let v_{a_i} be one vertex of C_i. Rearrange the components so that $a_1 < a_2 < \cdots < a_l$, and let W_i start and end at v_{a_i}.

Then we can construct an Eulerian circuit for G by walking along C and making detours W_i as we hit v_{a_i}:

$$v_1, \ldots, v_{a_1} - 1, W_1, v_{a_1} + 1, \ldots, v_{a_2} - 1, W_2, v_{a_2} + 1, \ldots\ldots\ldots, v_{a_l} - 1, W_l, v_{a_l} + 1, \ldots, v_k, v_1.$$



$\square$