Math 239 - Tutorial 3 – Fall 2013

Problem 1:

Find the number of 4-digit positive integers m such that the digits are from the set $\{1,2,3,4,5\}$ and the sum of them is equal to 12.

Solution: We are looking for 4-tuples (t_1,t_2,t_3,t_4) by elements from the set $N=\{1,2,3,4,5\}$. Let us define $S=N^4$, then we are looking for the number of elements of S with weight 12, with the weight function $w(t_1,t_2,t_3,t_4):=t_1+t_2+t_3+t_4$. The generating function of N is $\Phi_N(x)=x+x^2+x^3+x^4+x^5=x\frac{1-x^5}{1-x}$, then:

$$\Phi_S(x)=(\Phi_N(x))^4=x^4(\frac{1-x^5}{1-x})^4$$

The coefficient of x^{12} is:

$$\begin{aligned}
[x^{12}]\Phi_S(x) &= [x^{12}]x^4(1-x^5)^4(1-x)^{-4} \\
&= [x^8](1-x^5)^4(1-x)^{-4} \\
&= [x^8]\left[\sum_{i=0}^{4}(-1)^i\binom{4}{i}x^{5i}\right]\times\left[\sum_{j\geq 0}\binom{4+j-1}{4-1}x^j\right] \\
&= \binom{4+8-1}{4-1}-\binom{4}{1}\binom{4+3-1}{4-1} \\
&= \binom{11}{3}-4\binom{6}{3} \\
&= 85.
\end{aligned}$$

Problem 2:

For a fixed integer k, let a_n be the the number of compositions of n with k parts, where each part is divisible by 3. Find a_n in terms of n and k.

Solution: Let $N:=\{3,6,9,\cdots\}$ and $S:=N^k$, then a_n is the number of elements of S with weight k, where $w(t_1,\cdots,t_k)=t_1+\cdots+t_k$. We have:

$$\Phi_S(x)=(\Phi_N(x))^k=(x^3+x^6+\cdots)^k=(\frac{x^3}{1-x^3})^k.$$

Then a_n is given by:

$$\begin{aligned}
[x^n]\Phi_S(x) &= [x^n]x^{3k}(1-x^3)^{-k}=[x^{n-3k}](1-x^3)^{-k} \\
&= [x^{n-3k}]\sum_{i\geq 0}\binom{k+i-1}{k-1}x^{3i} \\
&= \begin{cases} 0 & \text{if } n \text{ is not divisible by 3,} \\ \binom{\frac{n}{3}-1}{k-1} & \text{if } n \text{ is divisible by 3.} \end{cases}
\end{aligned}$$

Note that by Problem 2.1.2 of the course notes, if n is divisible by 3, the number of desired compositions is equal to the number of compositions of $\frac{n}{3}$ with k parts, when there is no restriction on the parts.

Problem 3:

Let a_n be the number of compositions of n with 3, 4, or 5 parts such that no part is divisible by 3. Find the generating series $\sum_{n\geq 0} a_n x^n$ as a rational function.

Solution: Let us define N as the set of positive integers not divisible by 3; $N = \{1, 2, 4, 5, 7, 8, ...\}$. a_n counts the number of elements of $S = N^3 \cup N^4 \cup N^5$ of weight n, with the weight function $w(t_1, \cdots, t_r) := t_1 + \cdots + t_r$. For the generating function of N we have:

$$\begin{aligned}
\Phi_N(x) &= (x + x^2 + x^3 + x^4 + x^5 + \cdots) - (x^3 + x^6 + \cdots) \\
&= x\left(\sum_{i\geq 0} x^i\right) - x^3\left(\sum_{i\geq 0} x^{3i}\right) \\
&= \frac{x}{1-x} - \frac{x^3}{1-x^3} = \frac{x+x^2}{1-x^3}
\end{aligned}$$

Now we can write:

$$\begin{aligned}
\Phi_S(x) &= \sum_{k=3}^{5} \Phi_{N^k}(x), \quad \text{by Sum Lemma} \\
&= \sum_{k=3}^{5} (\Phi_N(x))^k, \quad \text{by Product Lemma} \\
&= \sum_{k=3}^{5} (\frac{x+x^2}{1-x^3})^k \\
&= \left(\frac{x+x^2}{1-x^3}\right)^3 + \left(\frac{x+x^2}{1-x^3}\right)^4 + \left(\frac{x+x^2}{1-x^3}\right)^5 \\
&= \left(\frac{x+x^2}{1-x^3}\right)^3 \left(\frac{1+x+2x^2-x^5+x^6}{(1-x^3)^2}\right).
\end{aligned}$$

Problem 4:

Let a_n be the number of compositions of $n \geq 0$ with exactly 2 parts, i.e., (t_1, t_2), such that $t_1 < t_2$. Find the generating series $\sum_{n\geq 0} a_n x^n$ as a rational function.

Solution: For each i, let $N_i = \{i, i+1, \cdots\}$. Then a_i is the number of elements of $S = \bigcup_{i\geq 1}\{i\} \times N_{i+1}$ that have weight n, where we have $w(t_1, t_2) = t_1 + t_2$.

$$\begin{aligned}
\Phi_S(x) &= \sum_{i\geq 1} \Phi_{\{i\}\times N_{i+1}}(x), \quad \text{by Sum Lemma} \\
&= \sum_{i\geq 1} x^i \Phi_{N_{i+1}}(x), \quad \text{by Product Lemma} \\
&= \sum_{i\geq 1} x^i (x^{i+1} + x^{i+2} + \cdots) \\
&= \sum_{i\geq 1} x^i \frac{x^{i+1}}{1-x} \\
&= \frac{x^3}{1-x} \sum_{i\geq 0} x^{2i} \\
&= \frac{x^3}{(1-x)(1-x^2)}.
\end{aligned}$$

Problem 5:

Let $S = \{(a,b,c)|a,b \in \{0,1,2,\cdots\}, c \in \{0,1\}\}$. Let the weight w of $(a,b,c) \in S$ be given by $w(a,b,c) = a+b+c$. Find a formula for $[x^n]\Phi_S(x)$.

Solution: Let us define $N = \{0,1,2,\cdots\}$, then we have:

$$\begin{aligned}
\Phi_S(x) &= \Phi_{N^2\times\{0,1\}}(x) \\
&= [\Phi_N(x)]^2 \cdot \Phi_{\{0,1\}}(x), \quad \text{by Product Lemma} \\
&= \left[\sum_{i\geq 0} x^i\right]^2 (1+x) \\
&= (1-x)^{-2}(1+x) \\
&= (1+x)\sum_{n\geq 0}\binom{n+1}{1}x^n, \\
&= \sum_{n\geq 0}(n+1)x^n + \sum_{n\geq 0}(n+1)x^{n+1} \\
&= \sum_{n\geq 0}(n+1)x^n + \sum_{n\geq 1} nx^n, \text{ by reindexing,} \\
&= 1 + \sum_{n\geq 1}(2n+1)x^n.
\end{aligned}$$

So

$$[x^n]\Phi_S(x) = \begin{cases} 1, \text{ if } n = 0, \\ 2n+1, \text{ if } n \geq 1. \end{cases}$$

.