# MATH 239

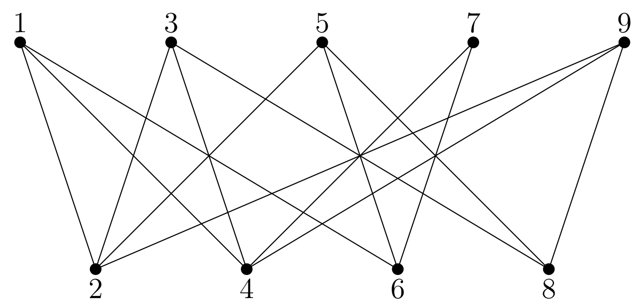
TUTORIAL 6

**Question 1:**
For $n$ a positive integer, define the prime graph $B_n$ to be the graph with vertex set $\{1, \ldots, n\}$, where $uv$ is an edge if and only if $u + v$ is a prime.

a) Draw $B_9$.

b) Prove that $B_n$ is bipartite.

**Answer 1:**
a) The prime numbers up to 18 are 2,3,5,7,11,13,17. So, draw lines between vertices that sum up to those numbers, starting from 1+2=3.




b) As suggested by the picture above, we can form a bipartition $V = (X, Y)$ where $v \in X$ if $v$ is odd, and $v \in Y$ if $v$ is even. To show that this is a bipartiton, we only need to show that there are no edges $uv$ with both $u, v \in X$, or both $u, v \in Y$. If $u, v \in X$, then $u$ and $v$ are both odd, so their sum must be even. Furthermore, since $u$ and $v$ cannot both be 1, the sum must be at least 4. Hence, $u + v$ must be composite. Similarly, if $u, v \in Y$, then $u$ and $v$ are both even, with a sum of at least 6. Hence, there can be no edges between vertices of the same bipartition, so $B_n$ is bipartite.

**Question 2:**
For a graph $G$, we define the complement of graph of $G$, denoted $\overline{G}$, with $V\left(\overline{G}\right) = V(G)$, and $uv \in E\left(\overline{G}\right)$ if and only if $uv \notin G$. Let $G$ and $H$ be two graphs and $\overline{G}$ and $\overline{H}$ be their complements. Show that $G$ and $H$ are isomorphic if and only if $\overline{G}$ and $\overline{H}$ are isomorphic.

**Answer 2:**
Let $V(G) = \{g_1, \ldots, g_n\}$ be the vertices of $G$ and $f\colon V(G) \to V(H)$ be an isomorphism between $G$ and $H$. Now, let $g_i$ and $g_j$ be vertices of $\overline{G}$. Observe the following chain of equivalence:

$$\begin{aligned} g_i \text{and } g_j \text{are adjacent in } \overline{G} &\iff g_i \text{and } g_j \text{are not adjacent in } G \\ &\iff f(g_i) \text{and } f(g_j) \text{are not adjacent in } H \\ &\iff f(g_i) \text{and } f(g_j) \text{are adjacent in } \overline{H} \end{aligned}$$

This means that $f$ is an isomorphism between $\overline{G}$ and $\overline{H}$ as well.

*Date*: 10/24/12.

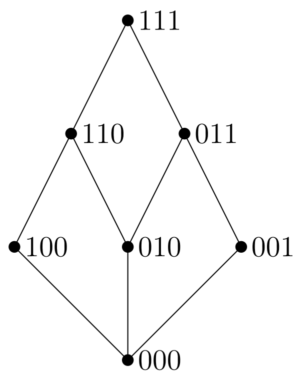
For the reverse direction, we will first show that $\overline{\overline{G}} = G$. This follows since:

$$\begin{aligned} g_i\text{and } g_j\text{are adjacent in } G &\iff g_i\text{and } g_j\text{are not adjacent in } \overline{G} \\ &\iff g_i\text{and } g_j\text{are adjacent in } \overline{\overline{G}} \end{aligned}$$

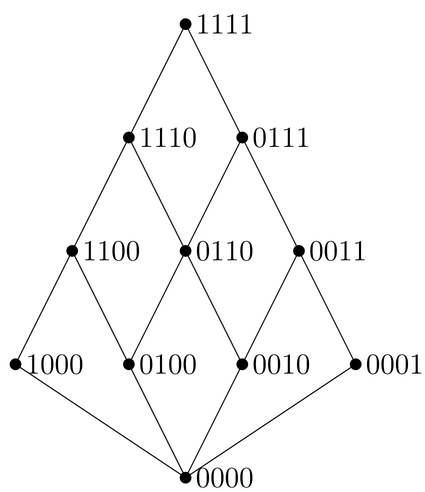
Applying the previous direction, we get that if $\overline{G}$ and $\overline{H}$ are isomorphic, then so must $G = \overline{\overline{G}}$ and $H = \overline{\overline{H}}$.

**Question 3:**
For $n \geq 0$, define the graph $G_n$ as follows: $V(G_n)$ is the set of binary strings of length $n$ having at most one block of 1's. Two vertices are adjacent if they differ in exactly 1 position.
a) Draw $G_3$ and $G_4$.
b) Find $|V(G_n)|$.
c) Find $|E(G_n)|$.

**Answer 3:**
a) $G_3$




$G_4$




b) For $1 \leq k \leq n$, there are exactly $n - k + 1$ strings with one block of 1's of length $k$, where the block of 1's start from position 1 to $n - k + 1$ respectively. There is also one string with no block of 1's. Therefore,

$$\begin{aligned}
|V(G_n)| &= 1+\sum_{k=1}^{n}(n-k+1) \\
&= 1+\frac{n(n+1)}{2} \\
&= \frac{n^2+n+2}{2}
\end{aligned}$$

c) To get the number of edges, we use the fact that the sum of the degree is twice the number of edges. Since $n=1$ and $n=2$ needs to be handled separately, we will assume $n \geq 3$.

First, the string with no 1's is adjacent to all strings with one 1, which means it has degree $n$. Now, let $s$ be a string with $k$ 1's, where $2 \leq k \leq n-1$. If $s$ does not start or end with 1, you can either change a 0 to 1 on both ends, or change a 1 to 0 on both ends. For example, the string 001100 is adjacent to 011100, 001110, 000100, and 001000. Therefore, it is adjacent to 4 other strings. If $s$ does start or ends with 1, you can no longer change one of the 0's to 1, so it is only adjacent to 3 other strings. Of the $n-k+1$ strings with $k$ 1's, $n-k-1$ of them are of the former type, and 2 are of the latter. Therefore, the sum of degree of all strings with $k$ 1's is $4(n-k+1)-2$. The same is true for strings with one 1, except you can only change one 1 to 0, so you have $3n-2$ as the total degree. Finally, the all 1's string is adjacent to only 2 strings. Tallying up, we have

$$\begin{aligned}
2|E(G_n)| &= n+(3n-2)+\sum_{k=2}^{n-1}(4(n-k+1)-2)+2 \\
&= 4n+\frac{4n(n-2)}{2} \\
&= 2n^2 \\
|E(G_n)| &= n^2
\end{aligned}$$

It is easy to check that this formula holds for $n=1,2$ as well.