Math 239 - Tutorial 6

Feb 26, 2013

1. The **odd graph** O_n is the graph whose vertices are the n-subsets of a $(2n+1)$-set, two such subsets being adjacent if and only if they are disjoint.

   (a) Determine how many vertices O_n has.

   (b) Prove that O_n is k-regular, and determine k in terms of n.

   (c) Determine how many edges O_n has.

   Solution:

   (a) There are $\binom{2n+1}{n}$ n-subsets of a $(2n+1)$-set. So O_n has $\binom{2n+1}{n}$ vertices.

   (b) Let T be a particular n-subset of our $(2n+1)$-set. Then there are

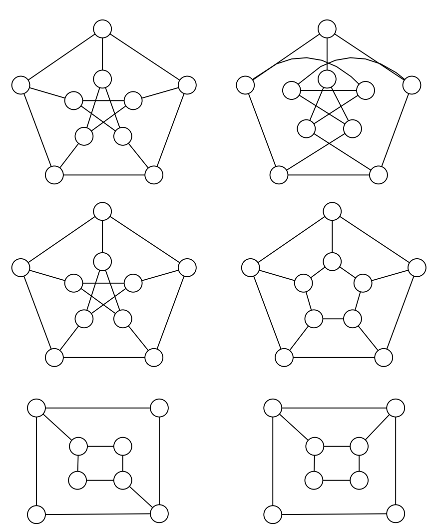
   $$\binom{2n+1-n}{n} = \binom{n+1}{n} = n+1$$

   n-subsets that do not contain any element of T. So each vertex of O_n has degree $n+1$, i.e. O_n is $(n+1)$-regular.

   (c) By Theorem 4.3.1,

   $$\begin{aligned} |E(O_n)| &= \frac{1}{2} \sum_{v \in V(O_n)} \deg(v) \\ &= \frac{1}{2}\binom{2n+1}{n}(n+1) \text{ (since } O_n \text{ is } (n+1)\text{-regular).} \end{aligned}$$

2. Determine which of the following pairs of graphs are isomorphic.



Solution: Pair 1: Isomorphic (just flip horizontally the inside pentagon). Pair 2: Non isomorphic. The second graph has a 4 cycle while the first graph has no 4 cycles. Pair 3: Non isomorphic. The second graph has two adjacent vertices of degree 2, while in the first graph the pairs of vertices of degree 2 are not adjacent.

3. Determine whether or not the following sequences are degree sequences of a graph.

   a) 2,2,2,4,5,5

   b) 3,3,3,4,4,5,5

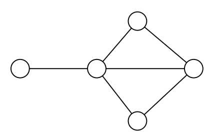
   c) 1,2,2,3,4

   Solution:

   a) There is no graph with this degree sequence. Suppose for the sake of contradiction that there is a graph with degree sequence 2,2,2,4,5,5. Let H be the induced subgraph consisting of the vertices of degree at least 4 in G. H has at most 3 edges, these edges contribute in at most 2 to the sum of the degrees in H, therefore there are at least $4+5+5-3(2)=8$ edges joining vertices in H and vertices in $V(G)\setminus V(H)$. Let H' be the graph induced by the vertices in $V(G)\setminus V(H)$, as any vertex in H' has degree 2 in G, there are at most $2+2+2=6$ edges joining vertices in H and vertices in $V(G)\setminus V(H)$. Contradiction.

   b) This sequence has an odd number of odd degree vertices. By Theorem 4.3.1 this is impossible.

   c) .



4. A subset of vertices S of G is an **independent set** if no two vertices in S are adjacent in G. Prove that the size of a largest independent set in G is equal to the size of a largest complete subgraph of $\bar{G}$ (the complement of G). (note that by "size", we mean the number of vertices)

   Solution:
   Recall that

   - $\bar{G}$ is the graph with $V(\bar{G})=V(G)$, and $\{u,v\}\in E(\bar{(G)})$ if and only if $\{u,v\}\notin E(G)$
   - a complete graph is one in which all pairs of distinct vertices are adjacent.

   Then (using the definitions) an independent set in G will be a complete subgraph of $\bar{G}$. So the size of a largest independent set in G must be the same as the size of a largest complete subgraph of $\bar{G}$.

5. A graph G is called self-complimentary if G is isomorphic to $\bar{G}$. Show that if a graph G is self-complimentary, then the number of vertices n is congruent to 0 or 1 modulo 4.

   Solution: Note that G and $\bar{G}$ have the same number of edges m, and their union is K_n. K_n has $\frac{1}{2}n(n-1)$ edges, and as G has half as many edges as K_n, G has $\frac{1}{4}n(n-1)$ edges. As the number of edges is integer, this means that $n(n-1)$ must be divisible by 4. As either n or $n-1$ is an odd number, this means that the other factor has to be divisible by 4 on its own, meaning that either n or $n-1$ is divisible by 4. But this is equivalent to n being congruent to either 0 or 1 modulo 4.