University of Waterloo

MATH 239

Midterm Test

Thursday, November 10, 2005

4:30 - 6:30 p.m.

Surname: ______________________

Initials: ______________________

Id.#: ______________________

Name:(Signature) ______________________

Please check your lecture section:

☐	Professor Schellenberg	2:30 MWF	Section 001. 004
☐	Professor Wormald	12:30 MWF	Section 002
☐	Professor Haxell	1:30 MWF	Section 003

1. Please check that you have all 8 pages (including this cover page) of this test.

2. Questions are to be answered in the space provided. Show all your work. If you require more space than is provided, please use the back of the *previous* page.

3. No calculators.

Problem	Value	Mark Awarded	Problem	Value	Mark Awarded
1	10		5	11	
2	10		6	9	
3	11				
4	9		Total	60	

[3] 1. (a) Determine $[x^{10}]\frac{x^3}{(1+3x)^6}$. (You may leave your answer as a product of integers, without further simplification.)

[7] (b) Use the Binomial Theorem to prove that

$$\sum_{m=k}^{n}\binom{m}{k}=\binom{n+1}{k+1}.$$

(Hint: expand $(1-x)^{-(k+1)}(1-x)^{-1}=(1-x)^{-(k+2)}$.)

[5] 2. (a) Find the number of compositions of n with 8 parts, in which the first and last parts are even, and all the other parts are odd.

[5] (b) Find the generating function for the number of compositions of n in which every part is an even number greater than or equal to 4.

[4] 3. (a) Write down a decomposition that uniquely creates all binary strings in which each odd-length block of 0's is immediately followed by a non-empty even-length block of 1's. (Blocks of 1's that follow even-length blocks of 0's are unrestricted.)

[3] (b) Write down a decomposition that uniquely creates all binary strings that do not contain the substring 11100. (For example, 1111100000 contains the substring 11100, but 10110000 does not.)

[4] (c) Write down the generating function (with weight being length) for the binary strings in

$$\{0\}^*(\{11\}\{11\}^*\{00, 000\}\{000\}^*)^*\{1\}$$

[4] 4. (a) Consider the generating function $\Phi(x) = \frac{1-x}{1-5x+6x^2}$. Find a linear recurrence with initial conditions for the sequence a_n of coefficients of $\Phi(x)$.

[5] (b) Let the sequence b_n be defined by $b_0 = 11$, $b_1 = 29$, $b_2 = 67$ and $b_n = 11b_{n-1} - 40b_{n-2} + 48b_{n-3}$ for $n \geq 3$. Solve this recurrence to obtain a closed form expression for b_n.

[3] 5. (a) Make a list of all trees on 6 vertices up to isomorphism; that is, each tree on 6 vertices should be isomorphic to exactly one of the trees on your list.

(b) Let A_n be the graph whose vertices are all binary strings of length n, where two vertices (strings) are adjacent if and only if they differ in exactly two consecutive bits. (For example, the vertices 001100 and 001010 are adjacent, but 001100 and 001001 are not.)

[2] i. Draw graphs A_2 and A_3.

[3] ii. (Continued from page 6) Determine the number of edges in A_n.

[3] iii. For $n \geq 2$, is A_n connected? Prove your claim.

[4] 6. (a) A cubic tree is a tree in which every vertex has degree 1 or 3. Prove that a cubic tree with exactly k vertices of degree 1 has $p = 2(k-1)$ vertices.

[5] (b) Let G be a graph with no cycles, p vertices and c components. Determine the exact number of edges in G (in terms of p and c).