# MATH 239 Spring 2012: Assignment 2

Due: 9:29 AM, Friday, May 18 2012 in the dropboxes outside MC 4066

**Note:** We use $\mathbb{N} = \{1, 2, 3, \ldots\}$ to denote the set of all positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Simplify your generating series as much as possible. Remember to always prove/justify that your answers are correct.

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Last Name:

First Name:

I.D. Number:

Section:

Mark (For the marker only): /50

Acknowledgments:

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1. $\{10$ marks$\}$

   (a) Let $S$ be the set of all finite subsets of $\mathbb{N}$. For each $A \in S$, define $w(A)$ to be the largest element in $A$, with $w(\emptyset) = 0$. Determine the generating series for $S$ with respect to $w$.

   (b) Suppose we change the word "largest" to "smallest" in part (a). Explain why the new weight function does not have a generating series.

2. {20 marks} You are playing a game with a deck of 11 cards. There is one card whose value is $i$ for each $i \in \{1, 2, \ldots, 11\}$. You will draw cards one at a time, each time placing the card back into the deck after the draw. Your score is the total value of the cards that you draw. (Note: The order which you draw the cards matters. We would consider drawing a 2 and then a 7 to be different from drawing a 7 and then a 2 in scoring a 9.) For each of the following questions, write down the set you are enumerating, the weight function you are using for this set, and represent your answer as the coefficient of some generating series.

   (a) How many ways can you score 21 after exactly 3 draws?

   (b) How many ways can you score 21 after 2, 3, or 4 draws?

   (c) For some positive integer $n$, how many ways can you score $n$? You are not limited to the number of draws you take.

3. {8 marks} Let $S$ be a set of objects, and suppose $w$ is a weight function on $S$ with generating series $\Phi_S(x)$. Let $w^*$ be a new weight function for $S$ defined by $w^*(a) = 4w(a) + 2$ for all $a \in S$. Determine the generating series $\Phi_S^*(x)$ with respect to the weight function $w^*$ in terms of $\Phi_S(x)$.

4. {Extra credit: 3 marks} Prove the following:

$$(1-x)^{-1} = \prod_{i \geq 0} (1 + x^{2^i}).$$

5. {12 marks} For a permutation $\sigma$ of $[n]$, a pair $(i, j)$ is called an *inversion* of $\sigma$ if $i < j$ and $\sigma(i) > \sigma(j)$. For example, the permutation (32415) on [5] has 4 inversions: $(1, 2), (1, 4), (2, 4), (3, 4)$. Define the weight function $w$ on a permutation $\sigma$ to be the number of inversions in $\sigma$. Let $S_n$ be the set of all permutations of $[n]$.

   (a) Determine the generating series for $S_1, S_2, S_3$ with respect to the weight function $w$.

   (b) Prove that
   $$\Phi_{S_n}(x) = (1 + x + \cdots + x^{n-1})\Phi_{S_{n-1}}(x).$$
   You may use the following (non-standard) notation: If $\sigma$ is a permutation of $[n]$, denote $\sigma'$ to be the permutation of $[n-1]$ obtained from $\sigma$ by removing the element $n$. For example, if $\sigma = (31524)$, then $\sigma' = (3124)$.

   (c) Prove that the number of permutations of $[n]$ with $k$ inversions is
   $$[x^k]\frac{\prod_{i=1}^{n}(1 - x^i)}{(1 - x)^n}.$$