MATH 239 Tutorial 2: Question 4 (Section 3)

Typed by: Marie-Sarah Lacharité

September 26, 2012

4. Determine the number of compositions of n into k parts, for any positive integer n and some fixed positive integer k, where each part is a multiple of 2.

   We want to find a generating series $\Phi(x)$ such that $[x^n]\Phi(x)$ is the number of compositions of n into k even, positive parts. Let T_k be the set of all k-tuples of even, positive integers:

   $$T_k = \underbrace{\{2,4,6,\ldots\} \times \cdots \times \{2,4,6,\ldots\}}_{k \text{ times}}$$

   We can define a weight function on these k-tuples:

   $$w_{T_k}(t_1, t_2, \ldots, t_k) = t_1 + t_2 + \ldots + t_k$$

   To find $\Phi_{T_k}(x)$, we can use the Product Lemma:

   - We already have T_k as a Cartesian product of sets $T = \{2, 4, 6, \ldots\}$.
   - We can re-write the weight function on T_k as a sum of weight functions on each T:

     $$w_{T_k}(t_1, t_2, \ldots, t_k) = \sum_{i=1}^{k} w_T(t_i)$$

     where we define $w_T(t_i) = t_i$.

   Then, by the Product Lemma,

   $$\Phi_{T_k}(x) = (\Phi_T(x))^k$$

   What is the generating series $\Phi_T(x)$ with respect to w_T? The exponents are the possible weights of elements in T — in this case, the elements of T themselves. The coefficient of each term is the number of elements of T with that particular weight. In this case, there is only one element that has each weight, since the weight is defined as the value.

   Therefore, the generating series of T is:

   $$\Phi_T(x) = \sum_{i\geq 1} x^{2i} = \sum_{i \geq 0} x^{2(i+1)} = x^2 \sum_{i\geq 0} x^{2i} = \frac{x^2}{1-x^2}$$

Returning to the generating series for T_k, we get:

$$\begin{aligned}
\Phi_{T_k}(x) &= \left(\frac{x^2}{1-x^2}\right)^k \\
&= x^{2k}\left(\sum_{m\geq 0}\binom{m+k-1}{k-1}x^{2m}\right) \qquad \text{(since } (1-x)^{-k} = \sum_{m\geq 0}\binom{m+k-1}{k-1}x^m\text{)} \\
&= \sum_{m\geq 0}\binom{m+k-1}{k-1}x^{2(m+k)}
\end{aligned}$$

We designed this generating series so that the coefficient of x^n would be the number of compositions of n into k even, positive parts. The values of n that have non-zero coefficients are restricted. If $n = 2(m+k)$ for some non-negative integer m, then its coefficient is $\binom{m+k-1}{k-1} = \binom{\frac{n}{2}-1}{k-1}$. Therefore, the number of compositions of n into k even, positive parts is:

$$[x^n]\Phi_{T_k}(x) = \begin{cases} \binom{\frac{n}{2}-1}{k-1} & \text{if } n \text{ is even and greater than or equal to } 2k \\ 0 & \text{if } n \text{ is odd or less than } 2k \end{cases}$$