COMBINATORICS
& OPTIMIZATION

University of Waterloo

Introduction to Combinatorics

Lecture 2

http://info.iqc.ca/mmosca/2014math239

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Last class we noticed the following correspondences

Subsets of {1,2,3}	Terms in $f(y_1,y_2,y_3)$ $=(1+y_1)(1+y_2)(1+y_3)$	Terms in f(x,x,x) $=(1+x)(1+x)(1+x)$
$\{\}$	$=1$	$=1$
$\{1\}$	$+y_1$	$+x$
$\{2\}$	$+y_2$	$+x$
$\{3\}$	$+y_3$	$+x$
$\{1,2\}$	$+y_1y_2$	$+x^2$
$\{2,3\}$	$+y_2y_3$	$+x^2$
$\{1,3\}$	$+y_1y_3$	$+x^2$
$\{1,2,3\}$	$+y_1y_2y_3$	$+x^3$

More basic counting examples

How many subsets does the set {1,2,…,n} have?

We can apply the idea from last class to see that the total number of subsets is $2 \times 2 \times \cdots \times 2 = 2^n$

"include 1 or not"

"include 2 or not"

"include n or not"

Notice the natural 1-1 correspondence with the subset problem

How many terms does the expansion of

$$f(y_1, y_2, \cdots, y_n) = (1 + y_1)(1 + y_2)\cdots(1 + y_n)$$

have?

don't include y_1

include y_1

don't include y_2

include y_2

don't include y_n

include y_n

$$= 1 + y_1 + y_2 + y_3 + y_1 y_2 + y_2 y_3 + \cdots + y_1 y_2 y_3 \cdots y_n$$

$\{\}$ $\{1\}$ $\{2\}$ $\{3\}$ $\{1,2\}$ $\{2,3\}$ $\{1,2,3,\cdots,n\}$

Exploiting this correspondence

If we "plug in" $y_1 = y_2 = \cdots = y_n = 1$ into

$$f(y_1, y_2, \cdots, y_n)$$

then we are counting the subsets.

$$f(1,1,\cdots,1) = 1 + 1 + 1 + 1 + 1 \cdot 1 + 1 \cdot 1 + \cdots + 1 \cdot 1 \cdots 1 = 2^n$$

(this computation corresponds to counting the subsets one by one)

$$f(1,1,\cdots,1) = (1+1) \cdot (1+1) \cdots (1+1) = 2^n$$

(this computation corresponds to the recursive counting method we used earlier)

More basic counting examples

How many subsets of size k does the set {1,2,…,n} have?

Easier question: How many ordered lists of k distinct numbers from {1,2,3,…,n} are there?

(NB: the elements of a set are not ordered; but order matters for an ordered list)

Even easier question

Even easier question: Let $n_1, n_2, \dots, n_k$ be distinct objects. How many ways can we order the elements $\{n_1, n_2, \dots, n_k\}$?

Answer: $k! = k(k-1)(k-2)\dots(2)(1)$. " k–factorial"

Proof (sketch): (by induction) True for $k=1$.

Suppose it's true for $k=j-1$. Consider the case of ordering j distinct objects.
There are j choices for the first element. We now need to order the remaining $j-1$ distinct elements, which can be done in $(j-1)!$ ways (by induction hypothesis). Thus the total number of ways of order j distinct objects is $j\cdot(j-1)! = j!$.
Induction follows.

Back to the easier question

How many ordered lists of k distinct numbers from $\{1,2,\ldots,n\}$ are there?

Answer: $n(n-1)(n-2)\ldots(n-k+1)$

Proof (sketch): (almost the same as previous proof)

Back to the original question

How many subsets of size k does the set {1,2,…,n} have?

Let $S_{n,k}$ equal the set of all such subsets.

Let $L_{n,k}$ equal the set of ordered lists of k elements from {1,2,…,n}.
We know $|L_{n,k}| = n(n-1)(n-2)\ldots(n-k+1)$.

Note that for each element in $S_{n,k}$ there are k! elements in $L_{n,k}$.
This means $|L_{n,k}| = |S_{n,k}|\, k!$.

This means $|S_{n,k}| = |L_{n,k}| / k! = n(n-1)(n-2)\ldots(n-k+1)/k! = \binom{n}{k}$

(“n choose k”)

See Theorem 1.3.1.

Exploiting this correspondence

If we "plug in" $y_1 = y_2 = y_3 = \cdots = y_n = x$ into $f(y_1, y_2, \cdots, y_n)$

$$= 1 + y_1 + y_2 + y_3 + y_1 y_2 + y_2 y_3 + \cdots + y_1 y_2 y_3 \cdots y_n$$

then we get

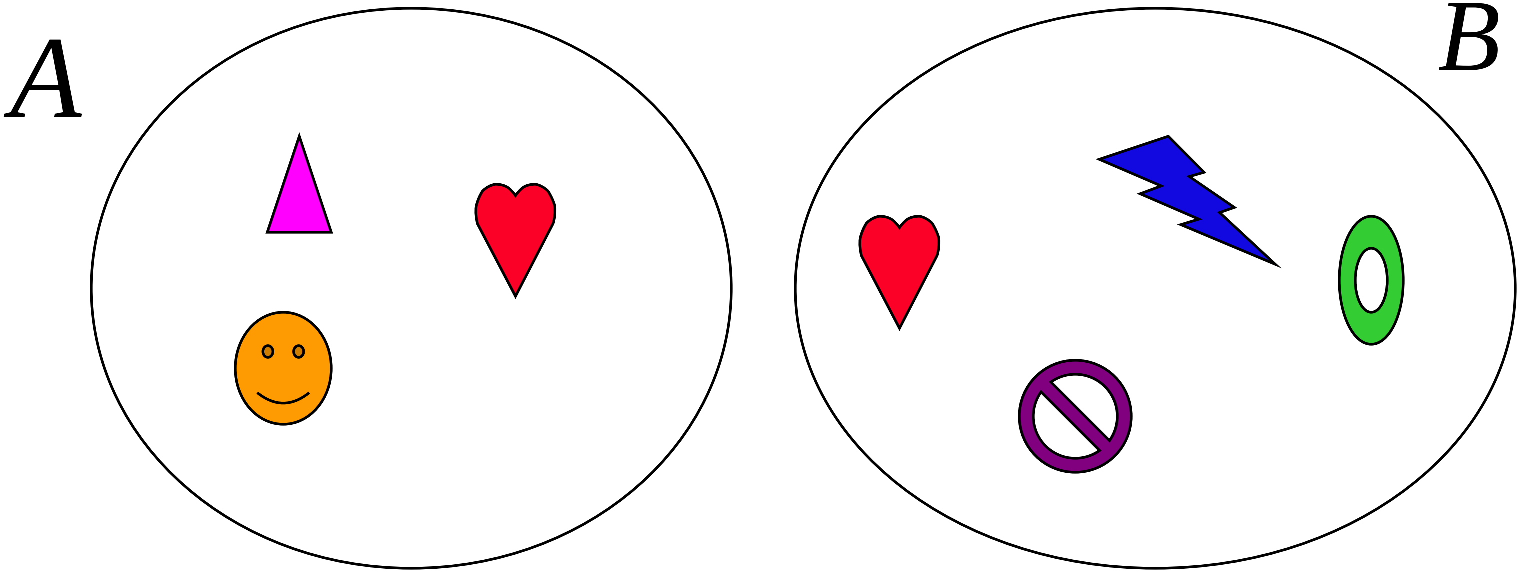
$$f(x, x, \ldots, x) = 1 + \binom{n}{1} x^1 + \binom{n}{2} x^2 + \cdots + \binom{n}{k} x^k + \cdots + x^n = (1+x)^n$$

number of subsets of size k

See THM 1.3.2.

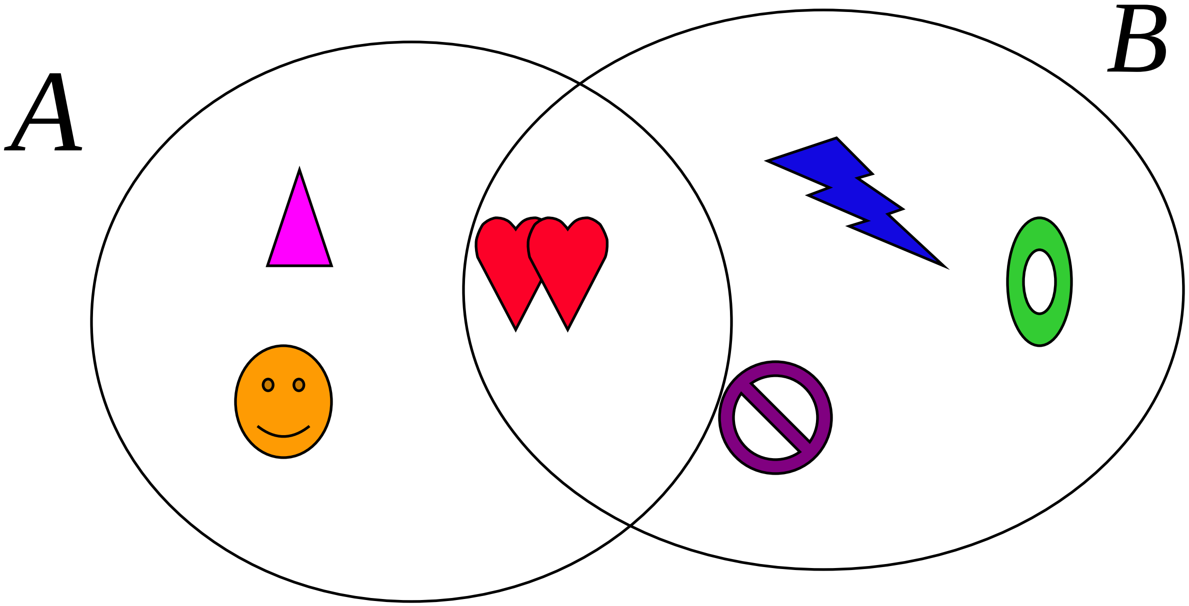
Aside

$A \bigcup B$ means the union of the sets A and B.



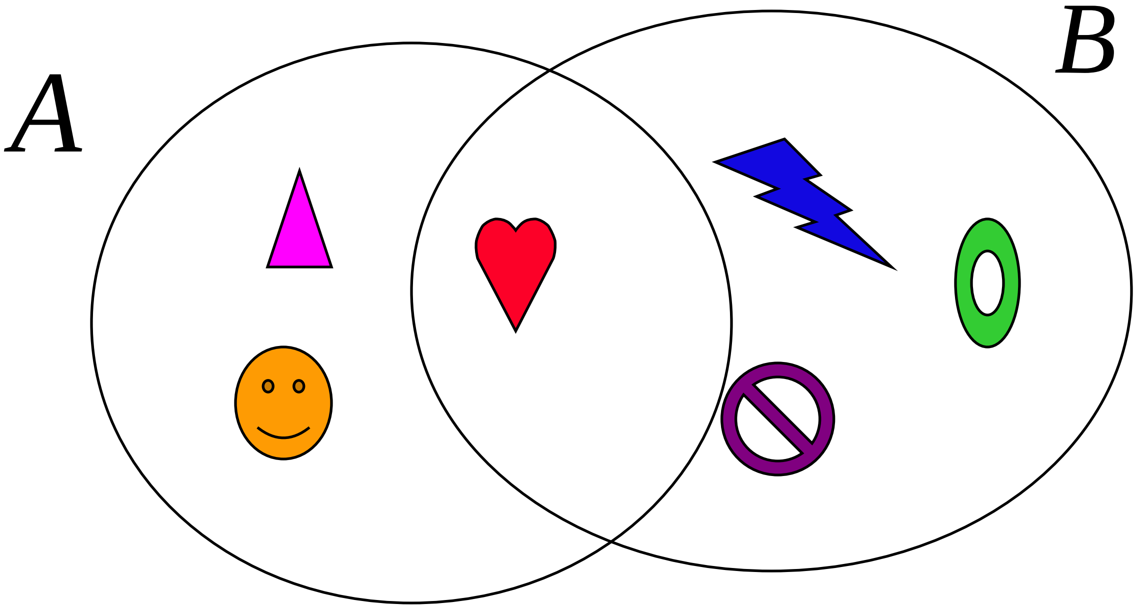
Aside

The union is the result of combining all the elements of A and B into a new set.



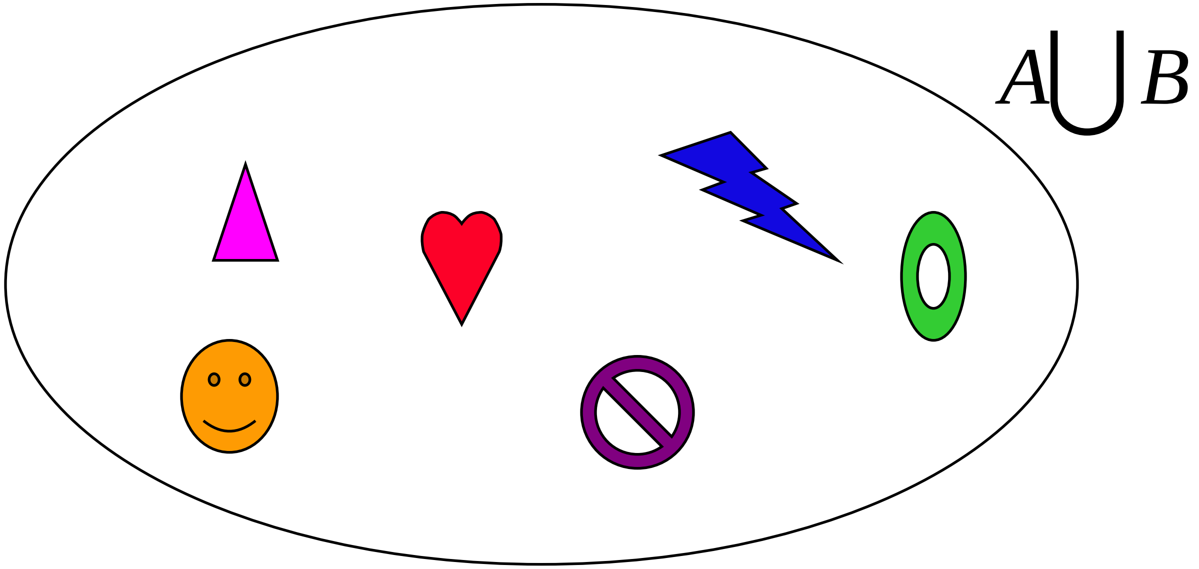
Aside

Only keep one copy of every element.



Aside

The resulting set is the union of A and B.

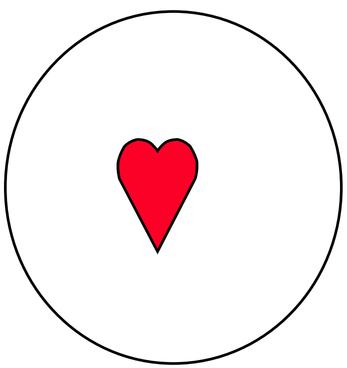


Notice that $\left| A \bigcup B \right| = |A| + |B| - \left| A \bigcap B \right|$

Aside

Notice that $\left|A \bigcup B\right| = |A| + |B| - \left|A \bigcap B\right|$ where

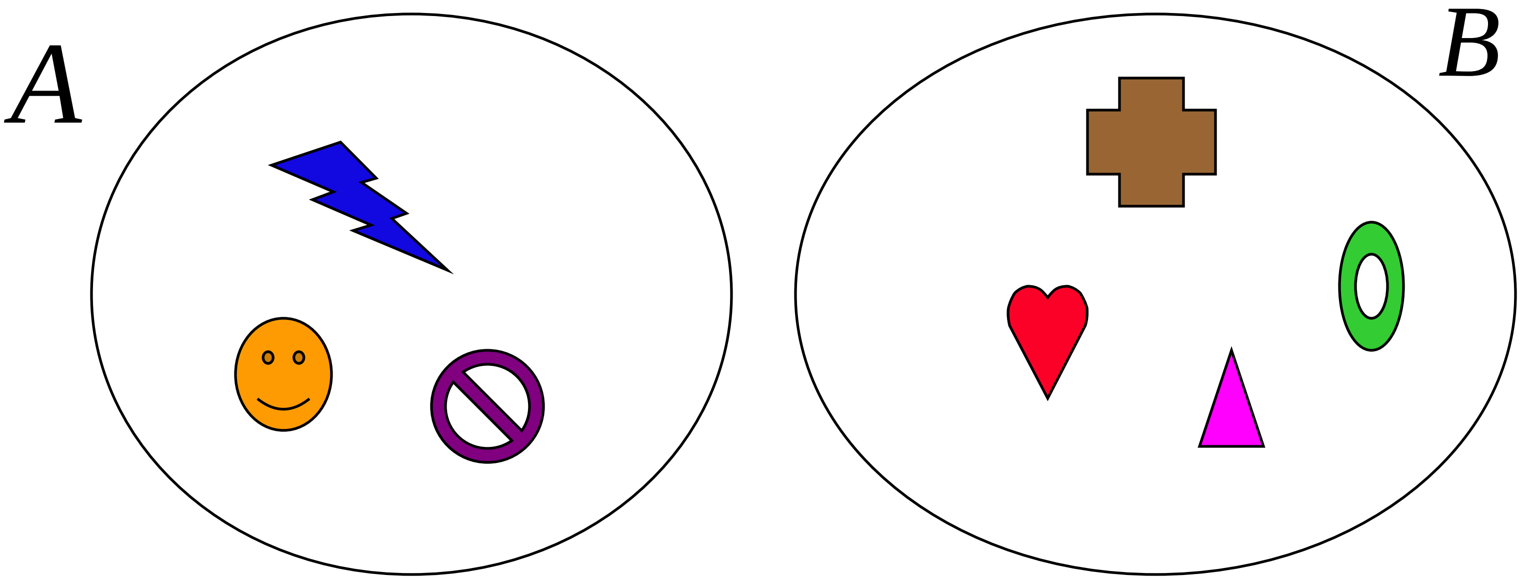
$\left|A \bigcap B\right|$ is the "intersection" of A and B; that is, the elements that they have in common.



$A \bigcap B$

Aside

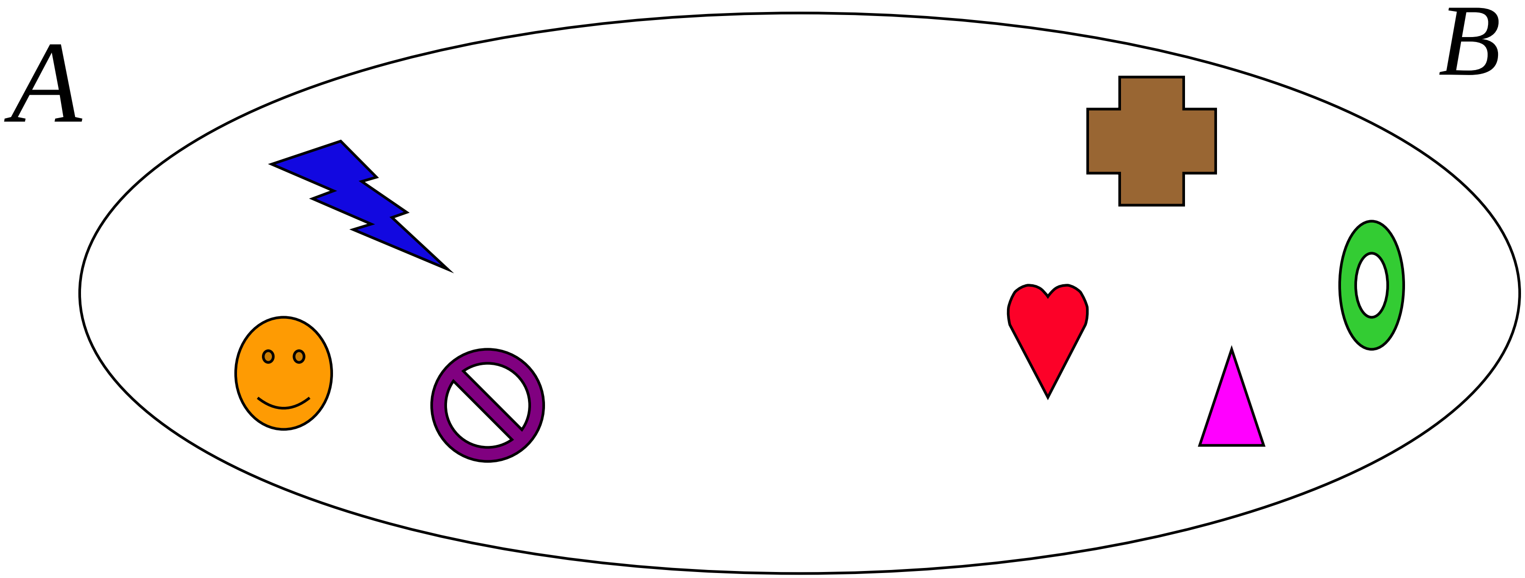
$A \dot{\bigcup} B$ means the disjoint union of the sets A and B
i.e. you can use the dot to denote that $A \bigcap B = \{\}$



Aside

$A \dot{\bigcup} B$ means the union of A and B is disjoint

i.e. you can use the dot to denote that $A \bigcap B = \{\}$



Disjointness implies that $\left| A \bigcup B \right| = |A| + |B|$

A combinatorial equality

THM1.3.3: For non-negative integers n and k

$$\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$$

Proof:

Note that LHS corresponds to the number of subsets of {1,2,…,n+k} of size n.

Let's reorganize those subsets according to their largest element; since every subset has a largest element we will count every subset exactly once.

Counting to prove equalities

THM1.1.3: For non-negative integers n and k

$$\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$$

Proof (continued):

Each subset of size n has a biggest element, equal to n, n+1, n+2,…, or n+k.

So let S_{n+i} be the set of subsets with largest element equal to n+i.

So $$S_{n+k,n} = S_n \dot{\bigcup} S_{n+1} \dot{\bigcup} S_{n+2} \dot{\bigcup} \cdots \dot{\bigcup} S_{n+k}$$

So $$\left|S_{n+k,n}\right| = \left|S_n\right| + \left|S_{n+1}\right| + \cdots + \left|S_{n+k}\right|$$

Counting to prove equalities

What is $|S_{n+i}|$, the size of S_{n+i} ?

In other words, how many subsets of n number from {1,2,3,…,n+k} have n+i as the largest element?

Equivalently, how many subsets of n-1 numbers from {1,2,3,…,n+i-1} are there?

Answer: $$\binom{n+i-1}{n-1}$$

The result now follows.