UNIVERSITY OF WATERLOO
MIDTERM EXAMINATION
SPRING TERM 2006

Surname: [redacted]

First Name: [redacted]

Id.#: [redacted]

Course Number	MATHEMATICS 239
Course Title	Introduction to Combinatorics
Instructor	☑ P. Schellenberg 9:30 MWF LEC 001 ☐ U. Celmins 1:30 MWF LEC 002
Date of Exam	Tuesday , June 27, 2006
Time Period	4:30 – 6:30 p.m.
Number of Exam Pages (including this cover sheet)	10
Exam Type	Closed Book
Additional Materials Allowed	None
Additional Instructions	1. Write your answers in the space provided. If you require more space, use the back of the *previous* page. 2. Please indicate your professor and section above.

Problem	Value	Mark Awarded	Problem	Value	Mark Awarded
1	13	13	4b	5	5
2	9	5	5	8	8
3	9	5	6	12	12
4a	4	3	Total	60	51

$\times$ 1. (a) **[4 marks]** Determine $[x^{10}]\frac{x^3}{(1-x)^5}$.

$$x^3(1-x)^{-5} = x^3 \sum_{k=0} \binom{5+k-1}{5-1} x^k$$

$$\phi(x) = x^3 \sum_{k=0} \binom{4+k}{4} x^k$$

$$[x^{10}]\phi(x) = [x^{10-3}] \sum_{k=0} \binom{4+k}{4} x^k$$

4

$$= [x^7]\phi(x) = \binom{4+7}{4} = 39$$

$\times$ (b) **[4 marks]** Determine $[x^{13}]\frac{(1-2x^2)^6}{(1-x)^4}$. Express your answer as a sum of products of binomial coefficients.

$$\phi(x) = \frac{(1-2x^2)^6}{(1-x)^4} = (1-2x^2)^6(1-x)^{-4}$$

4

$$= \sum_{k=0}^{\infty} \binom{6}{k}(-2x^2)^k \sum_{j=0}^{\infty} \binom{4+j-1}{4-1} x^j$$

$$= \sum_{k=0}^{\infty} \binom{6}{k}(-2)^k x^{2k} \sum_{j=0}^{\infty} \binom{3+j}{3} x^j$$

$$[x^{13}]\phi(x) = \sum_{2k+j=13} \binom{6}{k}\binom{3+j}{3}(-2)^k$$

$j = 13-2k$

$k = \left\lfloor \frac{13-j}{2} \right\rfloor$

$$= \sum_{k=0}^{6} \binom{6}{k}\binom{3+13-2k}{3}(-2)^k$$

$$= \sum_{k=0}^{6} \binom{6}{k}\binom{16-2k}{3}(-2)^k \checkmark$$

(c) **[5 marks]** Use the Binomial Theorem to prove that

$$\sum_{r+s=t} (-1)^r \binom{n+r-1}{r} \binom{m}{s} = \binom{m-n}{t}.$$

$$= [x^t] \sum_{r=0} \binom{n+r-1}{r} (-1)^r x^r \sum_{s=0} \binom{m}{s} x^s$$

$$= [x^t] \, (1+x)^{-n} (1+x)^m$$

$$= [x^t] \, (1+x)^{(m-n)}$$

$$= [x^t] \sum_{k=0} \binom{m-n}{k} x^k$$

$$= \binom{m-n}{t}$$

5

2. (a) **[4 marks]** List all the 3-part compositions of 5. (Recall that compositions have no zeros.)

(1, 1, 3)
(1, 2, 2)
(1, 3, 1)
~~(3, 1, 1)~~
(2, 1, 2)
(2, 2, 1)

1/4

4 other compositions order matters.

$\frac{1}{(1-x^2)}$

$\sum x^{2i}$

(b) **[5 marks]** Determine the generating function for the number of compositions of n in which every part is an even positive integer.

each part : $(x^2 + x^4 + \cdots) = \frac{x^2}{1-x^2}$

for k parts : $\left(\frac{x^2}{1-x^2}\right)^k$

for $k = 1, 2, \ldots$

$$\sum_{k=1}^{\infty}\left(\frac{x^2}{1-x^2}\right)^k = \frac{1}{1-\frac{x^2}{1-x^2}} - 1 = \frac{1-x^2}{1-x^2-x^2} - 1$$

$$= \frac{1-x^2-(1-2x^2)}{1-2x^2} = \frac{x^2}{1-2x^2}$$

$\Phi(x) = \frac{x^2}{1-2x^2}$

4/5

3. (a) **[4 marks]** Write down a decomposition that uniquely creates all binary strings in which each block of 1's has length at least 3.

$\{1\}^*(\{0\}\{1\}^*)^*$

$(\epsilon \cup \{111\}\{1\}^*)(\{0\}(\epsilon \cup \{111\}\{1\}^*))^*$

X

2/4

(b) **[5 marks]** Let A be the set of binary strings defined as follows:

$$A = \{01, 0011, 000111, 00001111, \ldots\}.$$

2 4 6 8

Determine the generating function for the binary strings in the set

$$S = \{1\}^*(\{0\}^*A)^*\{0\}^*.$$

As usual, the weight of a string is its length.

$\Phi_A(x) = x^2 + x^4 + \ldots = \dfrac{x^2}{1-x^2}$

$(\{0\}^*A)^* : \dfrac{1}{1 - \frac{1}{1-x}\frac{x^2}{1-x^2}}$

$\Phi_S(x) = \dfrac{1}{1-x}\left(\dfrac{1}{1-\frac{1}{1-x}\frac{x^2}{1-x^2}}\right)\dfrac{1}{1-x}$

3/5

$= \dfrac{1}{(1-x)^2 - \frac{x^2}{(1-x)(1+x)}} = \dfrac{1}{\frac{(1-x)^3 - x^2}{1-x}}$

$= \dfrac{1-x}{(1-2x+x^2)(1-x) - x^2} = \dfrac{1-x}{1-x-2x+2x^2 \cancel{+x^2} - x^3 \cancel{-x^2}}$

$= \dfrac{1-x}{1-3x+2x^2-x^3}$

4. (a) [**4 marks**] The generating function for a_n, the number of binary strings of length n that do not contain the substring 01110, is

$$\sum_{i\geq 0} a_i x^i = \frac{1+x^4}{1-2x+x^4-x^5}$$

Find a linear recurrence relation satisfied by the a_n's, together with sufficient initial conditions to uniquely determine the sequence $\{a_n\}$.

$(1-2x+x^4-x^5)\,\phi(x) = 1+x^4$

$[x^0]:\quad a_0 = 1 \qquad a_0 = 1$

$[x^1]:\quad a_1 - 2a_0 = 0 \qquad a_1 = 2$

$[x^2]:\quad a_2 - 2a_1 = 0 \qquad a_2 = 4$

$[x^3]:\quad a_3 - 2a_2 = 0 \qquad a_3 = 8$

$[x^4]:\quad a_4 - 2a_3 + a_0 = 1 \qquad a_4 = 15 \quad a_4 = 16$

$[x^n]:\quad a_n - 2a_{n-1} + a_{n-4} - a_{n-5} = 0 \qquad (\text{for } n \geq 5)$

$a_n = 2a_{n-1} - a_{n-4} + a_{n-5}$ ✓

with initial conditions

$a_0 = 1,\ a_1 = 2,\ a_2 = 4,\ a_3 = 8,\ a_4 = 15$

3/4

(b) **[5 marks]** Let the sequence b_n be defined by $b_0 = 3$, $b_1 = 4$, $b_2 = 12$ and

$$b_n - 6b_{n-1} + 12b_{n-2} - 8b_{n-3} = 0 \ \forall n \geq 3.$$

Solve this recurrence relation to obtain a closed form expression for b_n.

the characteristic polynomial is

$P(x) = x^3 - 6x^2 + 12x - 8$

lets discover roots:

$P(1) = 1 - 6 + 12 - 8$ ✗

$P(2) = 8 - 24 + 24 - 8 = 0$ ✓ $x - 2$ is a factor

$$\begin{array}{r|l} & x^2 - 4x + 4 \\ x-2 & x^3 - 6x^2 + 12x - 8 \\ & x^3 - 2x^2 \\ & -4x^2 + 12x \\ & -4x^2 + 8x \\ & 4x - 8 \\ & 4x - 8 \\ & 0 \end{array}$$

$P(x) = (x-2)(x^2 - 4x + 4)$
$= (x-2)^3$

one root, 2, with multiplicity 3.

$b_n = (A + Bn + Cn^2) 2^n$ general soln.

$b_0 = 3 = A \cdot 2^0 = A$

$b_1 = 4 = (A + B + C)2 = 2A + 2B + 2C$

$b_2 = 12 = (A + 2B + 4C)2^2 = 4A + 8B + 16C$

$A = 3$

b_1: $2B = 4 - 2A - 2C$; $B = -1 - C$

b_2: $16C = 12 - 4A - 8B$; $C = \frac{3}{4} - \frac{1}{4}A - \frac{1}{2}B$

$C = \frac{3}{4} - \frac{3}{4} - \frac{1}{2}(-1 - C) = \frac{1}{2} + \frac{1}{2}C$

$C - \frac{1}{2}C = \frac{1}{2}$

$C = 1$

$A = 3$

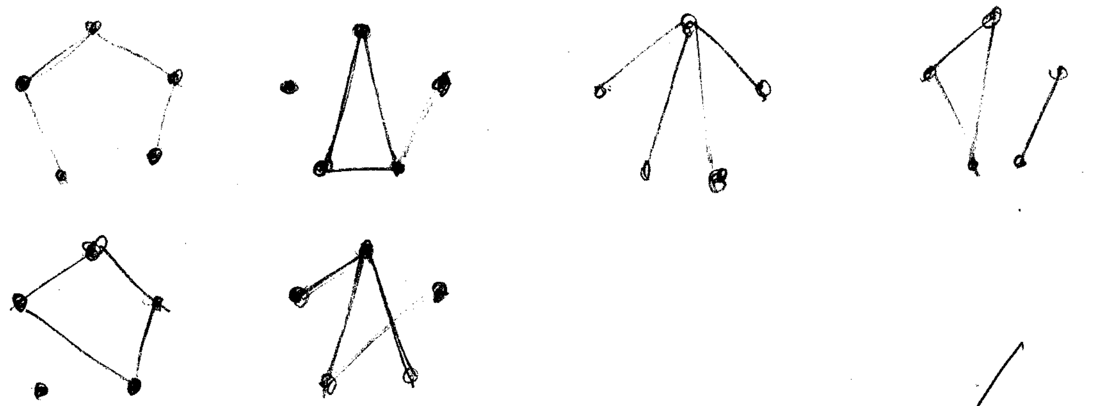
$B = -1 - C = -2$

$\rightarrow b_n = (3 - 2n + n^2)2^n$ ✓

n^x

5. (a) **[4 marks]** Make a list of all graphs, up to isomorphism, having 5 vertices and 4 edges; that is, each graph with 5 vertices and 4 edges should be isomorphic to exactly one of the graphs on your list. (Hint: There are more than 4 and fewer than 8 such graphs.)

$\binom{5}{2}$ possible edges $\frac{5!}{2!\,3!} = \frac{5\cdot 4}{2} = 10$



4X

(b) **[4 marks]** Show that if every vertex of a graph G has degree at least k, then G has a path of length at least k. (Hint: Let $v_0, v_1, v_2, \ldots, v_n$ be a longest path in G. Show that $n \geq k$.)

- let $P_1 = v_0 v_1 v_2 \ldots v_n$ be a longest path in G.
- each v_i is distinct.

- SPS. $n < k$.
- each v_i connects to at least k other nodes
- because v_n is connected to at least k other vertices, it is connected to $k-n+1$ vertices not in P_1 (i.e. because there are only $n-1 < k$ other vertices in P_1.
- this implies that there exists some vertex v_{n+1} in G distinct from $v_0 v_1 \ldots v_n$, giving that the length of $P1 \geq n+1$, contradicting the assumption that $P1$ is length n, proving $n \geq k$.

4X

6. For each integer $n \geq 1$, let A_n be the graph whose vertices are all subsets of $\{1, 2, \ldots, 2n\}$ having n or $n+1$ elements, and two distinct vertices (subsets) are adjacent if and only if one contains the other. For example, the vertex $\{1,2,3,4\}$ is adjacent to the vertex $\{1,2,4\}$ in A_3.

   (a) **[4 marks]** Determine the number of edges in A_n.

2n
n

* $\binom{2n}{n} + \binom{2n}{n+1}$ vertices
* vertices adjacent iff one set contains the other.
* vertices of size n can only contain themselves
* each (n+1)-vertex contains (n+1) other vertices.
  * proof: by removing a single element we get a set contained by an (n+1)-vertex. there are n+1 elements in the (n+1)-vertex, and so there are (n+1) ways to do this and (n+1) 'contained' vertices.
* the graph is bipartite — no (n+1)-vertex links to any other (n+1)-vertex, and no n-vertex links to any n-vertex,
* each of the $\binom{2n}{n+1}$ (n+1)-vertices links only to (n+1) other n-vertices so there are $(n+1)\binom{2n}{n+1}$ edges.

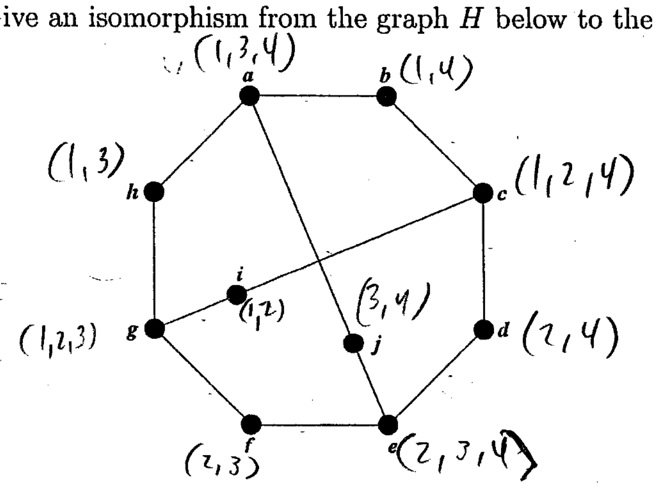
2n

   (b) **[4 marks]** Give an isomorphism from the graph H below to the graph A_2.

{1,2,3,4}

1,2,3
1,2,4
2,3,4
1,3,4

5



4

12

Figure 1: Graph H

x	a	b	c	d	e	f	g	h	i	j
$f(x)$	(1,3,4)	(1,4)	(1,2,4)	(2,4)	(2,3,4)	(2,3)	(1,2,3)	(1,3)	(1,2)	(3,4)

(c) [**4 marks**] Is A_n bipartite? Justify your answer.

An is bipartite.

Partition the set into two subsets — those with n elements and those with n+1 elements.

an n-element set can only contain itself, but we're not considering loops. therefore no n-element set links to any other n-element set.

an (n+1)-element set can contain (n+1) n-element sets, but cannot contain any (n+1)-element sets for the same reason as above. therefore no (n+1) element vertex links to any other.

therefore the given partition is a bipartition.