Math 239 - Tutorial 5

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1. Let
$$A(x) = \frac{x^2 - 13x}{(x-1)^2(x+3)}.$$
Find an expression for $[x^n]A(x)$ as a function of n where $A(x)$ is to be considered a formal power series.

Solution.
We write
$$\frac{x^2 - 13x}{(x-1)^2(x+3)} = \frac{Ax+B}{(x-1)^2} + \frac{C}{x+3} \quad \text{(partial fraction expansion).}$$
Clear denominators to get
$$\begin{aligned} x^2 - 13x &= (Ax+B)(x+3) + C(x-1)^2 \\ &= x^2(A+C) + x(3A+B-2C) + (3B-C). \end{aligned}$$
Equate coefficients:
$$\begin{aligned} 1 &= A + C \\ -13 &= 3A + B - 2C \\ 0 &= 3B - C. \end{aligned}$$
Solving the system gives $A = -2$, $B = -1$, $C = 3$. So we can rewrite $A(x)$ as
$$A(x) = \frac{-2x-1}{(x-1)^2} + \frac{3}{x+3}.$$
We wish to write $\frac{-2x-1}{(x-1)^2} = \frac{D}{(x-1)^2} + \frac{E}{x-1}$, where D and E are constants. Solving as before:
$$\begin{aligned} -2x - 1 &= D + E(x-1) \\ \implies E &= -2 \\ D &= E - 1 = -3. \end{aligned}$$

So

$$\begin{aligned}
A(x) &= \frac{-3}{(x-1)^2} + \frac{-2}{x-1} + \frac{3}{x+3} \\
&= \frac{-3}{(1-x)^2} + \frac{2}{1-x} + \frac{1}{1-\frac{-x}{3}}.
\end{aligned}$$

We know that

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \sum_{n\geq 0}(n+1)x^n \\
\frac{1}{1-x} &= \sum_{n\geq 0} x^n \\
\frac{1}{1-\frac{-x}{3}} &= \sum_{n\geq 0}\left(\frac{-1}{3}\right)^n x^n.
\end{aligned}$$

So we have

$$\begin{aligned}
A(x) &= \frac{-3}{(1-x)^2} + \frac{2}{1-x} + \frac{1}{1-\frac{-x}{3}} \\
&= -3\sum_{n\geq 0}(n+1)x^n + 2\sum_{n\geq 0} x^n + \sum_{n\geq 0}\left(\frac{-1}{3}\right)^n x^n.
\end{aligned}$$

Consequently,

$$[x^n]A(x) = -3(n+1) + 2 + (\frac{-1}{3})^n.$$

2. Let $\{a_n\}$ be the sequence that satisfies the recurrence

$$a_n - 3a_{n-2} + 2a_{n-3} = 0$$

for $n \geq 3$, with initial conditions $a_0 = 4$, $a_1 = -1$, $a_2 = 3$. Determine an explicit formula for a_n.

Solution.

The characteristic polynomial is $x^3 - 3x + 2 = (x-1)^2(x+2)$. So by Theorem 3.2.2 the solution is of the form

$$\begin{aligned}
a_n &= P_1(n)(1)^n + P_2(n)(-2)^n, \text{ where } \deg(P_1) \leq 1 \text{ and } \deg(P_2) = 0 \\
&= An + B + C(-2)^n.
\end{aligned}$$

Use initial conditions to set up a system of linear equations:

$$\begin{aligned}
4 &= B + C \\
-1 &= A + B - 2C \\
3 &= 2A + B + 4C
\end{aligned}$$

Solving gives $A = -2$, $B = 3$, $C = 1$. So (substituting these values into our formula above)

$$a_n = -2n + 3 + (-2)^n.$$

3. Let $\{b_n\}$ be the sequence that satisfies for $n \geq 3$ the recurrence

$$b_n - 3b_{n-2} + 2b_{n-3} = 12$$

with initial conditions $b_0 = 0$, $b_1 = 8$, $b_2 = 2$. Determine an explicit formula for b_n.

Solution. Procedure: we first want to guess some solution that fulfils the recurrence relation (ignoring the initial conditions), and then use Theorem 3.3.1 to find the general solution. Using the general solution, we then find the solution that satisfies the starting conditions.

Trying to find a specific solution c_n, neither $c_n = \alpha$ nor $c_n = \alpha n$ for some α work. So we try $c_n = \alpha n^2$:

$$\begin{aligned} 12 = b_n - 3b_{n-2} + 2b_{n-3} &= \alpha n^2 - 3\alpha(n-2)^2 + 2\alpha(n-3)^2 \\ &= \alpha(n^2 - 3n^2 + 12n - 12 + 2n^2 - 12n + 18) \\ &= \alpha \cdot 6 \end{aligned}$$

So we set $\alpha = 2$ to get a solution $c_n = 2n^2$.

Now by Theorem 3.3.1, we get that the general solution b_n for the above recurrence relation is the sum of a particular solution c_n (like the one we found above) and the general solution to the homogenous system a_n, i.e. the one where the RHS of the recurrence relation is 0, which we found in problem 2. So we have

$$b_n = a_n + c_n = An + B + C(-2)^n + 2n^2$$

The initial conditions now give us:

$$\begin{aligned} 0 &= B + C \\ 8 &= A + B - 2C + 2 \\ 2 &= 2A + B + 4C + 8 \end{aligned}$$

So we get $A = 0, B = 2, C = -2$ to get the solution $b_n = 2 + (-2)^{n+1} + 2n^2$.

4. Let $\{a_n\}$ be the sequence given by

$$a_n = -2n + 3 + (-2)^n.$$

Derive a third order homogeneous recurrence relation that a_n satisfies.

Solution.
Note that the solution above is of the form

$$P_1(n)(1)^n + P_2(n)(-2)^n,$$

where $\deg(P_1) = 1$ and $\deg(P_2) = 0$. Working backwards, we see (By Theorem 3.2.2) that the characteristic polynomial should have 1 as a root with multiplicity two and -2 as a root with multiplicity 1. So the characteristic polynomial is

$$E(x) = (x-1)^2(x+2) = (x^2 - 2x + 1)(x+2) = x^3 - 3x + 2.$$

It follows that a_n satisfies the homogeneous recurrence relation

$$a_n - 3a_{n-2} + 2a_{n-3} = 0.$$

5. Let $\{a_n\}$ be the sequence that satisfies the recurrence

$$a_n - 3a_{n-2} + 2a_{n-3} = 0$$

for $n \geq 3$ with initial conditions $a_0 = 4$, $a_1 = -1$, $a_2 = 3$. Write

$$A(x) = \sum_{n \geq 0} a_n x^n$$

as a rational function, i.e. a quotient of two polynomials.

Solution. The characteristic polynomial is $g(x) = x^3 - 3x + 2$.

Let $Q(x) = x^k g(x^{-1}) = 1 - 3x^2 + 2x^3$. By Theorem 3.2.1, there is a polynomial $P(x)$ of degree less than the degree of $Q(x)$ such that

$$A(x) = \frac{P(x)}{Q(x)}.$$

Let $P(x) = p_2 x^2 + p_1 x + p_0$ satisfying the above. Using the inversion formula we get:

$$A(x) = \frac{P(x)}{Q(x)} = \frac{p_2 x^2 + p_1 x + p_0}{1 - 3x^2 + 2x^3} = (p_2 x^2 + p_1 x + p_0) \sum_{n \geq 0} (3x^2 - 2x^3)^n$$

Consider the constant coefficient of this expression: $p_0 = a_0 = 4$

Consider the linear coefficient: $p_1 = a_1 = -1$

Consider the quadratic term of this expression: $p_2 + 3p_0 = a_2 = 3$ and therefore $p_2 = -9$

So we get $P(x) = -9x^2 - x + 4$ and overall:

$$A(x) = \frac{-9x^2 - x + 4}{1 - 3x^2 + 2x^3}$$