MATH 239 - Fall 2013

Assignment 2

Due date : Friday, September 27th, 2013, at noon (sharp)

Submission Guidelines:

- Total number of marks in this assignment is 46.
- Use a cover page to submit your solutions (available on the course webpage).
- Keep a copy of your manuscript before your submission.
- Assignments submissions are exclusively accepted in the following dropboxes

[Section 001]	Dropbox next to the St Jerome's library, 2nd floor of STJ
[Section 002]	Math DropBox #18; Slot #1 A-J, Slot #2 K-S, Slot #3 T-Z
[Section 003]	Math DropBox #18; Slot #4 A-J, Slot #5 K-S, Slot #6 T-Z
[Section 004]	Math DropBox #18; Slot #7 A-J, Slot #8 K-S, Slot #9 T-Z

- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- You are not allowed to post this manuscript (or parts of it) online, nor share it (or parts of it) with anyone not enrolled in this course.

Assignment policies: While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. For example, copying or paraphrasing a solution from some fellow student or old solutions from previous offerings of related courses qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. All students found to be cheating will automatically be given a mark of 0 on the assignment. In addition, there will be a 10/100 penalty to their final mark, as well as all academic offenses will be reported to the Associate Dean for Undergraduate Studies and recorded in the student's file (this may lead to further, more severe consequences).

If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that if you see any marking error, then you should return your assignment paper to the TA of your section within one week and with written notes on all the marking errors; please write the notes on a new sheet and attach it to your assignment paper.

Question 1 [Marks 14=5+5+4]
Suppose $n \geq 2$, and let S be the set of all subsets of $\{1, ..., n\}$ that have the property that at most one of 1 or 2 can be in the subset. Let the weight of a subset be its size.

(a) Determine the number of elements of S with weight k, for $k = 0, ..., n$.
Hint: You may need to treat the $k = 0$ and $k = 1$ cases differently than the $k \geq 2$ case.

Solution. *First there is 1 element of S with weight 0 (the empty set), and n elements of S with weight 1 (the sets $\{i\}$ for $i = 1, ..., n$).*

Next suppose $k \geq 2$. There are a total of $\binom{n}{k}$ subsets of $\{1, ..., n\}$ of size k. Of these, the ones that are not in S are those that contain both 1 and 2. There are $\binom{n-2}{k-2}$ k-element subsets of $\{1, ..., n\}$ which contain both 1 and 2. Therefore there are $\binom{n}{k} - \binom{n-2}{k-2}$ elements in S with weight k when $k \geq 2$.

(Another correct answer for $k \geq 2$ is $\binom{n-2}{k} + 2\binom{n-2}{k-1}$. This is obtained by looking at subsets that contain neither 1 nor 2, subsets that contain 1 and not 2, and subsets that contain 2 and not 1.)

(b) Find $\Phi_S(x)$. Use the binomial theorem to express your final answer without a sum.

Solution. *In part (a) we calculated the coefficients of $\Phi_S(x)$. Putting them into a power series we get:*

$$\begin{aligned}
\Phi_S(x) &= 1 + nx + \sum_{k=2}^{n} \left(\binom{n}{k} - \binom{n-2}{k-2} \right) x^k \\
&= 1 + nx + \sum_{k=2}^{n} \binom{n}{k} x^k - x^2 \sum_{k=2}^{n} \binom{n-2}{k-2} x^{k-2} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k - x^2 \sum_{k=0}^{n-2} \binom{n-2}{k} x^k \\
&= (1+x)^n - x^2(1+x)^{n-2} \qquad \textit{(by the Binomial Theorem)} \\
&= (1+2x)(1+x)^{n-2}
\end{aligned}$$

(c) What is the average size of a subset in S?

Solution. *From theorem 1.4.3, the average size of a subset in S is $\frac{\Phi_S'(1)}{\Phi_S(1)}$. We have that $\Phi_S'(x) = 2(1+x)^{n-2} + (1+2x)(n-2)(1+x)^{n-3}$, so the desired average is*

$$\begin{aligned}
\frac{\Phi_S'(1)}{\Phi_S(1)} &= \frac{2^{n-1} + 3(n-2)2^{n-3}}{3 \cdot 2^{n-2}} \\
&= \frac{4 + 3(n-2)}{6}.
\end{aligned}$$

Question 2 [Marks 10=5+5]

(a) Let S be a set of configurations, and let w be a weight function on S. Recall that the number of elements in S with weight exactly n is just $[x^n]\Phi_S(x)$. Prove that the number of elements in S with weight *at most* n is $[x^n]\frac{\Phi_S(x)}{1-x}$.

Solution. *Let* $\Phi_S(x) = \sum_{i\geq 0} a_i x^i$. *Then*

$$\begin{aligned} [x^n]\frac{\Phi_S(x)}{1-x} &= [x^n]\left(\sum_{i\geq 0} a_i x^i\right)\left(\sum_{j\geq 0} x^j\right) \\ &= [x^n]\sum_{i\geq 0}\sum_{j\geq 0} a_i x^{i+j} \\ &= [x^n]\sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i\right) x^n \qquad \textit{(setting } j = n-i\textit{)} \\ &= \sum_{i=0}^{n} a_i. \end{aligned}$$

This is exactly the number of elements of S *with weight* n *or less, as required.*

(b) Use part (a) to determine the number of k-tuples $(a_1, ..., a_k)$ of non-negative integers that satisfy the inequality $a_1 + ... + a_k \leq n$.
Hint: Consider Problem 1.6.4 in the notes.

Solution. *Let* S *be the set of all* k*-tuples* $(a_1, ..., a_k)$ *of non-negative integers. Let* $w(a_1, ..., a_k) = a_1 + ... + a_k$. *Then* $S = N_{\geq 0}^k$ *and by the product lemma we have:*

$$\Phi_S(x) = (1 + x + x^2 + ...)^k = \left(\frac{1}{1-x}\right)^k.$$

Now $[x^n]\Phi_S(x)$ *is the number of* k*-tuples* $(a_1, ..., a_k)$ *with weight exactly* n*, meaning they satisfy* $a_1 + ... + a_k = n$. *By part a), the number of elements of* S *with weight at most* n *is*

$$\begin{aligned} [x^n]\frac{\Phi_S(x)}{1-x} &= [x^n]\left(\frac{1}{(1-x)^k}\right)\left(\frac{1}{1-x}\right) \\ &= [x^n](1-x)^{-(k+1)} \\ &= \binom{n + (k+1) - 1}{(k+1) - 1} \qquad \textit{(by Theorem 1.6.5)} \\ &= \binom{n+k}{k}. \end{aligned}$$

Since the number of elements $(a_1, ..., a_k)$ *of* S *with weight at most* n *are precisely the solutions to* $a_1 + ... + a_k \leq n$*, we conclude that there are* $\binom{n+k}{k}$ *of them.*

Question 3 [Marks 10=2+2+2+2+2]
Consider the following four formal power series:

$$A(x) = \frac{1}{1-x^2}, \quad B(x) = \frac{x}{(1-2x)(1+3x)}, \quad C(x) = \sum_{k\geq 1} \frac{x^k}{k}, \quad D(x) = \sum_{k\geq 0}(1+k)x^k.$$

For each of the following, state whether the expression is a well-defined formal power series. You must fully justify your answers.

(a) $A(x) - D(x)$

(b) $\frac{C(x)}{B(x)-D(x)}$

(c) $\frac{C(x)}{A(x)-D(x)+C(x)}$

(d) $B(A(x))$

(e) $B(A(x) - D(x))$

Solution. *(a) Yes. The sum of two formal power series is always a formal power series.*

(b) Yes. $[x^0](B(x) - D(x)) = -1$, *so* $B(x) - D(x)$ *is invertible, and* $C(x)(B(x) - D(x))^{-1}$ *is a formal power series.*

(c) No. $[x^0](A(x) - D(x) + C(x)) = 0$, *so* $A(x) - D(x) + C(x)$ *is not invertible.*

(d) No. $[x^0]A(x) \neq 0$, *and* $B(x)$ *has infinitely many non-zero terms, so the composition does not give a formal power series.*

(e) Yes. $[x^0](A(x) - D(x)) = 0$, *so the composition does give a formal power series.*

Question 4 [Marks 12=6+6]
Calculate the following coefficients for all $n \geq 0$:

(a) $[x^n](3 + x^2 - x^4)^r$, where $r \geq 0$ is a constant. Your final answer may include a sum. *Hint: You'll need to use the binomial theorem.*

Solution. *We have that*

$$\begin{aligned}(3+x^2-x^4)^r &= 3^r\left(1+\frac{x^2}{3}-\frac{x^4}{3}\right)^r \\ &= 3^r\sum_{i=0}^{r}\binom{r}{i}\left(\frac{x^2}{3}-\frac{x^4}{3}\right)^i \qquad \textit{(by Binomial Theorem)} \\ &= 3^r\sum_{i=0}^{r}\binom{r}{i}\frac{x^{2i}}{3^i}\left(1-x^2\right)^i \\ &= 3^r\sum_{i=0}^{r}\binom{r}{i}\frac{x^{2i}}{3^i}\left(\sum_{j=0}^{i}\binom{i}{j}(-x^2)^j\right) \qquad \textit{(Binomial Theorem again)} \\ &= 3^r\sum_{i=0}^{r}\sum_{j=0}^{i}\binom{r}{i}\binom{i}{j}3^{-i}(-1)^j x^{2i+2j}.\end{aligned}$$

Notice that only even powers of x appear in this sum. Therefore $[x^n](3+x^2-x^4)^r = 0$ if n is odd. On the other hand, suppose n is even. Then,

$$\begin{aligned}[x^n](3+x^2-x^4)^r &= [x^n]3^r\sum_{i=0}^{r}\sum_{j=0}^{i}\binom{r}{i}\binom{i}{j}3^{-i}(-1)^j x^{2i+2j} \\ &= 3^r\sum_{i=0}^{\frac{n}{2}}\binom{r}{i}\binom{i}{\frac{n}{2}-i}3^{-i}(-1)^{\frac{n}{2}-i} \qquad \textit{(setting } j=\frac{n}{2}-i\textit{).}\end{aligned}$$

Therefore,

$$[x^n](3+x^2-x^4)^r = \begin{cases} 3^r\sum_{i=0}^{\frac{n}{2}}\binom{r}{i}\binom{i}{\frac{n}{2}-i}3^{-i}(-1)^{\frac{n}{2}-i} & \textit{if } n \textit{ is even,} \\ 0 & \textit{if } n \textit{ is odd.}\end{cases}$$

(b) $[x^n]\frac{1}{(1-2x)(1+4x^2)}$. Express your final answer without a sum.
Hint: Try to get your answer into the form of a geometric series.

Solution. *First we have that*

$$\begin{aligned}\frac{1}{(1-2x)(1+4x^2)} &= \left(\frac{1}{1-2x}\right)\left(\frac{1}{1+4x^2}\right) \\ &= \left(\sum_{i\geq 0}2^i x^i\right)\left(\sum_{j\geq 0}(-4)^j x^{2j}\right) \\ &= \sum_{i\geq 0}\sum_{j\geq 0}2^i(-4)^j x^{i+2j} \\ &= \sum_{n\geq 0}\left(\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}2^{n-2j}(-4)^j\right)x^n \qquad \textit{(setting } i=n-2j\textit{).}\end{aligned}$$

And therefore,

$$
\begin{aligned}
[x^n]\frac{1}{(1-2x)(1+4x^2)} &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2j}(-4)^j \\
&= 2^n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{-4}{4} \right)^j \\
&= 2^n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \\
&= 2^n \left(\frac{1-(-1)^{\lfloor \frac{n}{2} \rfloor + 1}}{1-(-1)} \right) \\
&= 2^{n-1} \left(1 + (-1)^{\lfloor \frac{n}{2} \rfloor} \right)
\end{aligned}
$$