MATH 239 Tutorial 5 Solution Outline

1. Determine the generating series for the set of binary strings that do not contain 11011 as a substring.

   Solution. Let S be the set of binary strings that do not contain 11011. Let T be the set of binary strings that contain exactly one copy of 11011, at the right end. Then

   $$\begin{aligned}\{\varepsilon\} \cup S\{0,1\} &= S \cup T \\ S\{11011\} &= T \cup T\{011\} \cup T\{1011\}\end{aligned}$$

   Generating series

   $$\begin{aligned}1 + \Phi_S(x)(2x) &= \Phi_S(x) + \Phi_T(x) \\ \Phi_S(x)x^5 &= \Phi_T(x) + \Phi_T(x)x^3 + \Phi_T(x)x^4\end{aligned}$$

   So

   $$\Phi_S(x) = \frac{1}{1 - 2x + \frac{x^5}{1+x^3+x^4}}.$$

2. Let $\{a_n\}$ be the sequence that satisfies the recurrence

   $$a_n - 3a_{n-2} + 2a_{n-3} = 0$$

   for $n \geq 3$, with initial conditions $a_0 = 4, a_1 = -1, a_2 = 3$. Determine an explicit formula for a_n.

   Solution. The characteristic polynomial is $x^3 - 3x + 2 = (x-1)^2(x+2)$, implying 1 is a root of multiplicity 2, and -2 is a root of multiplicity 1. So

   $$a_n = (An + B)(1)^n + C(-2)^n$$

   We use the initial conditions for $n = 0, 1, 2$ to solve for our constant, yielding $A = -2, B = 3, C = 1$. Thus

   $$a_n = -2n + 3 + (-2)^n.$$

3. Let $\{b_n\}$ be the sequence that satisfies the recurrence

   $$b_n - 3b_{n-2} + 2b_{n-3} = 12$$

   for $n \geq 3$, with initial conditions $b_0 = 0, b_1 = 8, b_2 = 2$. Determine an explicit formula for b_n.

   Solution. For a specific solution c_n, checking $c_n = \alpha$ or $c_n = \alpha n$ does not work. So we check $c_n = \alpha n^2$, which gives

   $$c_n - 3c_{n-2} + 2c_{n-3} = 6\alpha = 12$$

   So $\alpha = 2$, and $2n^2$ is a specific solution. We notice that the homogeneous recurrence is the same as in Question 2, so we can re-use our solution of $(An + B) + C(-2)^n$ (Notice that we cannot use the final solution from Question 2, since the initial conditions are not the same). The solution to a non-homogeneous recurrence is the sum of a specific solution with the homogeneous solution, and thus

$$b_n = 2n^2 + An + B + C(-2)^n.$$

Use initial conditions, we get $A = 0, B = 2, C = -2$. So

$$b_n = 2n^2 + 2 + (-2)^{n+1}.$$

4. Let $c_n = (n-2)3^n + (-2)^n$.

   (a) Determine a homogeneous recurrence that c_n satisfies, together with sufficient initial conditions.

   Solution. We work in reverse to the methods used in Question 2. Previously, we used the recurrence and found the characteristic polynomial, then the roots, then the general equation. In this case, we are given the equation first, from which we see that the roots are 3 (with multiplicity 2) and -2 (with multiplicity 1). Thus our characteristic polynomial is

   $$(x-3)^2(x+2) = x^3 - 4x^2 - 3x + 18.$$

   So the recurrence is

   $$a_n - 4a_{n-1} - 3a_{n-2} + 18a_{n-3} = 0$$

   for $n \geq 3$. Initial conditions can be obtained by using the given formula.

   $$c_0 = -1, c_1 = -5, c_2 = 4, c_3 = 19.$$

   (b) Determine a rational expression for the power series $\sum_{n\geq 0} c_n x^n$.

   Solution. If $\sum_{n\geq 0} c_n x^n = \frac{p(x)}{q(x)}$ for some polynomials $p(x), q(x)$, then we first multiply both sides by the denominator, getting

   $$q(x) \sum_{n\geq 0} c_n x^n = p(x)$$

   c_n satisfies the recurrence in part (a) for all $n \geq 3$, so we can deduce that $p(x)$ has degree at most 2, and that $q(x)$, when multiplied by $\sum_{n\geq 0} c_n x^n$, will give us the same recurrence as in (a).
   Thus, the denominator is $1 - 4x - 3x^2 + 18x^3$ (notice that this is the characteristic polynomial, but with powers exchanged). The numerator is

   $$\begin{aligned}&(1 - 4x - 3x^2 + 18x^3) \sum_{n\geq 0} c_n x^n \\ =&a_0 + (a_1 - 4a_0)x + (a_2 - 4a_1 - 3a_0)x^2 + \sum_{n\geq 3}(a_n - 4a_{n-1} - 3a_{n-2} + 18a_{n-3})x^n \\ =&-1 - x + 27x^2\end{aligned}$$

   The answer is

   $$\frac{-1 - x + 27x^2}{1 - 4x - 3x^2 + 18x^3}$$

Additional exercises

1. Consider the two sequences $\{a_n\}, \{b_n\}$ given by

$$a_n = n2^n - 3^n$$
$$b_n = n2^n - 3^n + 4$$

   (a) Determine a homogeneous recurrence that a_n satisfies.

   (b) Determine a non-homogeneous recurrence that b_n satisfies whose left-hand side is the same as part (a).

2. Prove that

$$\frac{(1+\sqrt{3})^n - (1-\sqrt{3})^n}{2\sqrt{3}}$$

   is an integer for each nonnegative integer n. (Challenge: Find 3 different proofs.)

   Solution. Proof 1: Induction on n. Proof 2: Expand using binomial theorem. Proof 3: This is the solution of a recurrence.

3. Let a be a non-zero integer, and let k be any positive integer. Consider the power series $\frac{p(x)}{(1-ax)^k}$ where the degree of $p(x)$ is less than k. Prove that there exist constants $C_1, \ldots, C_k$ such that

$$\frac{p(x)}{(1-ax)^k} = \frac{C_1}{1-ax} + \frac{C_2}{(1-ax)^2} + \cdots + \frac{C_k}{(1-ax)^k}.$$

   (Hint: You may consider proving that a certain set of polynomials is a basis for the vector space P_{k-1}, the space of all polynomials of degree at most $k-1$.)

   Solution. $\{1, 1-ax, (1-ax)^2, \ldots, (1-ax)^k\}$ is a basis for P_{k-1}. The numerator of the RHS is a linear combination of these polynomials. So it must cover all possible polynomials of degree at most $k-1$.