MATH 239 Assignment 1

- This assignment is due on Friday, September 21, 2012, at 10am in the drop boxes in St. Jerome's (section 1) or outside MC 4067 (the other two sections).
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.

1. Let $2 \leq k \leq n$ be integers. Consider the identity
$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$$
   (a) Give a proof of this identity using the binomial theorem.
   (b) Give a combinatorial proof of this identity. (Hint: consider the set of all k-subsets of $\{1, 2, \dots, n\}$ and classify them according to whether or not they contain the element 1 and/or the element 2.)

2. (a) Prove that for every positive even integer $n = 2m$,
$$\sum_{i=0}^{m} \binom{n}{2i} 2^{2i} = \sum_{i=0}^{m-1} \binom{n}{2i+1} 2^{2i+1} + 1.$$
   (b) State and prove a similar identity for odd positive integers $n = 2m + 1$.

3. Let $S = \{0, 1, \dots, 15\}$.
   (a) For $0 \leq i \leq 4$ let S_i denote the subset of S consisting of those integers whose binary (i.e. base 2) representation has exactly i ones. Find S_i explicitly for $0 \leq i \leq 4$.
   (b) Find the generating series for S with respect to the weight function $w(\sigma) =$(the number of ones in the binary representation of σ).
   (c) Let $r \geq 1$ be an integer, and let $S(r) = \{0, 1, \dots, 2^r - 1\}$. Find the generating series for $S(r)$ with respect to the weight function w. (Hint: the coefficients will be of the form $\binom{n}{k}$ for some n and k.) Prove your answer is correct.

4. Give a combinatorial proof of the identity
$$\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n.$$
(Hint: the ideas in Question 3 may help you.)