MATH 239 Winter 2013
Assignment 2

Due Friday, January 25, 10am

The term "generating function" in this assignment is called a "generating series" in the notes.

1. Let S be a finite set.

   (a) Let $w : S \to \mathbb{N}_{\geq 0}$ be a function. Let $\Phi_{S,w}(x)$ be the generating function of S with respect to weight function w. Define a new function w' on S by
   $$w'(\sigma) = 2w(\sigma) + 3$$
   for all $\sigma \in S$. Show that $\Phi_{S,w'}$, the generating function of S with respect to w', satisfies
   $$\Phi_{S,w'}(x) = x^3 \Phi_{S,w}(x^2).$$

   (b) Let $w_1, w_2 : S \to \mathbb{N}_{\geq 0}$ be weight functions on S. Let $\Phi_{S,w_1}, \Phi_{S,w_2}$ be their respective generating functions. Let $w_3 : S \to \mathbb{N}_{\geq 0}$ be the function defined by
   $$w_3(\sigma) = w_1(\sigma) + w_2(\sigma).$$
   Is it true that $\Phi_{S,w_3}(x) = \Phi_{S,w_1}(x)\Phi_{S,w_2}(x)$? If true, give a proof. If false, give a set S and weight functions w_1, w_2 for which it is not true.

2. For a positive integer n, let N_n denote the set $\{1, 2, \ldots, n\}$, and let $\mathcal{P}_n$ be the set of all subsets of N_n. Let $w : \mathcal{P}_n \to \mathbb{N}_{\geq 0}$ be the weight function taking a subset to its number of elements, that is, for $A \subseteq N_n$,
   $$w(A) = |A|.$$
   For a pair of positive integers m, n with $m < n$, let $\mathcal{T}_{m,n}$ be the set of all subsets A of N_n such that $\max(A) > m$.

   (a) Give a formula for $\Phi_{\mathcal{P}_n}(x)$.

   (b) Explain why $\mathcal{T}_{m,n} = \mathcal{P}_n \setminus \mathcal{P}_m$, that is, the set of elements of $\mathcal{P}_n$ that are not elements of $\mathcal{P}_m$.

   (c) Give a formula for $\Phi_{\mathcal{T}_{m,n}}(x)$ in terms of binomial coefficients.

3. Let N be a non-negative integer. Consider the set of all 3-tuples (n_1, n_2, n_3) of non-negative integers satisfying
   $$n_1 + 5n_2 + 10n_3 = N.$$
   Write down a formal power series $\Phi(x)$ such that $[x^N]\Phi(x)$ is the number of solutions (n_1, n_2, n_3) of the above equation. You do not need to find a closed formula for the number of solutions.

   Hint: Use the product lemma.

4. Let $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$ be the power series that satisfies

$$A(x) = \sum_{n=0}^{\infty} (x + x^2)^n.$$

Prove that $a_0 = 1$, $a_1 = 1$, and for $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$.

Hint: Recall that if $B(x)$ is a power series with no constant term then

$$\frac{1}{1 - B(x)} = \sum_{n=0}^{\infty} B(x)^n.$$

5. Find an expression for each of the coefficients of the following formal power series (your expressions may be sums of binomial coefficients; you may want to break some expressions into cases)

   (a) $[x^n](1-x)^{-2}(1+2x^3)$

   (b) $[x^n](1-2x^2)^{-3}$

   (c) $[x^n](1-x)^{-2}(1-x^3)^2$

6. Let S be a set with weight function $w : S \to \mathbb{N}_{\geq 0}$ Let $\Phi_{S,w}(x)$ be the generating function. Write $c_n = [x^n]\Phi_{S,w}(x)$ so that c_n is equal to the number of elements of S with weight equal to n. Let $A(x)$ be the formal power series given by $A(x) = \frac{\Phi_{S,w}(x)}{1-x^2}$. Write $a_n = [x^n]A(x)$. Prove that

$$a_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n-2k}.$$