Math 239 - Tutorial 8

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1. Show that the following graph is planar:

   $V = \{v_0, v_1, \ldots, v_{n-1}, u, w\}$, the edges are $v_i v_{i+1}$ where $i+1$ is taken modulo n, and for all i there are edges uv_i and wv_i.

   Solution. We embed the graph into the sphere: The v_i form a cycle, put them on the equator of the sphere. Put u at the north pole and w at the south pole.

   For an embedding in the plane, embed the cycle first, put u inside and v outside, and connect u and v to the rest. This embedding is in fact equivalent to the stereographic projection of the embedding in the sphere.

2. Let G be a graph with a special vertex u. For $v \in G$ let $d(v)$ be the distance between u and v. Show that

   $$\sum_{v \in V} d(v) \leq \binom{n}{2}$$

   When does equality hold?

   Solution. Let n be the number of vertices. Consider a vertex v with $d(v) = i$ maximal, then all the $i+1$ vertices along the shortest path from u to v (including the endpoints) have distance $0, 1, 2, \ldots, i$ from u respectively. As there are only n vertices in total, there are $n - i - 1$ vertices apart from these on the path, and as $d(v) = i$ was maximal, they all have distance from u at most i. This gives:

   $$\sum_{v \in V} d(v) \leq 0 + 1 + \cdots + i + (n-i-1)i \leq 0 + 1 + \cdots + i + (i+1) + \cdots + (n-1) = \frac{n(n-1)}{2} = \binom{n}{2}$$

   Equality holds if G is a path of length $n-1$ and u is one of its endpoints.

3. Prove the more general version of Euler's formula: If G is a graph with a planar embedding with p vertices, q edges, s faces and c components, then

   $$p - q + s = 1 + c$$

   Solution. Let $G_1, ...G_c$ be the components of G, and for each $i = 1, ..., c$ let p_i and q_i be the number of vertices and edges in G_i. As G is embedded on the plane, also its components are embedded on disjoint regions of the plane, hence each of them satisfy Euler's formula

   $$p_i - q_i + s_i = 2 \text{ for } i = 1, ..., c \tag{1}$$

   Where s_i is the number of faces in the embedding of G_i. Notice that the embeddings of all the components share exactly one face (the unbounded face), hence if when we add up $\sum_{i=1}^{c} s_i$ we are counting the unbounded face c times. Hence adding the equations in (1) we get $p - q + (s + c - 1) = 2c$. Therefore $p - q + s = 1 + c$.

4. Prove that any planar graph in which every face has degree at least 4, has at most $2p-4$ edges.

Solution. Let $f_1, ..., f_s$ be the faces of G, by Theorem 7.1.2 we have that

$$2q = \sum_{i=1}^{s} deg(f_i) \geq \sum_{i=1}^{s} 4 = 4s$$

Now by Euler's formula

$$4 = 2p - 2q + 2s \leq 2p - 2q + q = 2p - q$$

Therefore $q \leq 2p - 4$.

5. Suppose that G is a graph which contains two edge-disjoint spanning trees T_1 and T_2.

   (a) Prove that G does not have any bridge.
   Solution. Recall that an edge e of G is a bridge if $G - e$ has more components than G. Suppose vertices v and w are in different components of $G - e$. Then any path in G from v to w uses edge e. (Since G is connected, we know that such a path exists.) So there are not two edge-disjoint paths from v to w in G.

   (We just proved the statement "bridge $\implies$ there are not two edge disjoint paths between every pair of vertices in G", or equivalently, "two edge-disjoint paths between every pair of vertices of $G \implies$ no bridge".)

   Now, since G contains two edge-disjoint spanning trees, we can find two edge-disjoint paths between any pair of vertices of G – one path in T_1, and another in T_2. From our work above, we see that G has no bridge.

   (b) Let $e \in E(T_1) \setminus E(T_2)$. Prove that there exists $e' \in E(T_2) \setminus E(T_1)$ such that $T_1 - e + e'$ is a spanning tree of G.

   Solution. Since T_1 is a tree, $T_1 - e$ has exactly two components, C_1 and C_2. Since T_2 is a spanning tree of G (i.e. is connected, and contains every vertex of G), T_2 contains some edge e' with one end point in C_1, and the other in C_2. We claim that $T_1 - e + e'$ is a spanning tree of G.

   $T_1 - e + e'$ is a tree: since $T_1 - e + e'$ is connected and $T_1 - e$ is not, e' is a bridge. Then, by Theorem 4.9.3, e' is not contained in a cycle of $T_1 - e + e'$. Since T_1 was a tree, $T_1 - e + e'$ has no cycles that do not contain e'. So $T_1 - e + e'$ contains no cycles and is therefore a tree.

   $T_1 - e + e'$ is clearly spanning (contains every vertex of G), since $T_1 - e$ contains every vertex of G. So $T_1 - e + e'$ is a spanning tree, as required.