MATH 239 Assignment 1

- This assignment is due on Friday, January 18th, 2013, at 10am in the drop boxes outside MC 4067. Late assignments will not be graded.
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.

Exercise 1 (10 pts)**.**

(a) In how may ways is it possible to rearrange the letters in "MISSISSAUGA"?

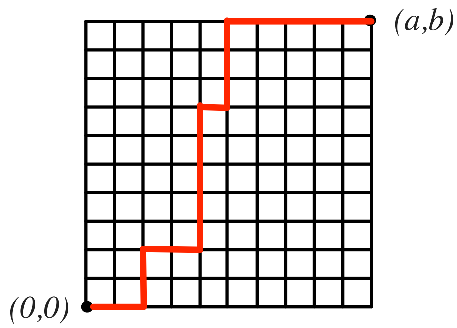
HINT: Consider in how many ways the letter "M" can be placed, and then in how many ways the letters "I" can be placed, etc

(b) Consider a string using only letters $A_1, \ldots, A_n$. Let $\mathcal{S}$ be the family of strings with exactly i_1 letters A_1; i_2 letters A_2; $\ldots$; i_n letters A_n (where $i_1, \ldots, i_n$ are some non-negative integers). Find a formula for the cardinality of $\mathcal{S}$.

HINT: This generalizes the problem in part (a), the proof is similar.

Exercise 2 (15 pts)**.**

Consider a grid where the lower left corner corresponds by the point $(0, 0)$ and the upper right corner to the point (a, b). Thus to reach (a, b) from $(0, 0)$ we can travel up b steps and to the right a steps. Let $\mathcal{P}$ be the set of paths starting at location $(0, 0)$, ending at (a, b) where we only travel either up or to the right. The following figure gives an example where $(a, b) = (10, 10)$ and a member $P \in \mathcal{P}$ is indicated by the thick red path.



(a) Find a formula for the cardinality of $\mathcal{P}$.

HINT: Very little work involved here.

(b) For $i \in \{0, \ldots, a\}$, let $\mathcal{P}_i$ be the set of paths in $\mathcal{P}$ that first reach the top of the grid in position $x = i$. For instance the solid red path $P \in \mathcal{P}$ in the figure is in $\mathcal{P}_5$. Find a formula for the cardinality of $\mathcal{P}_i$.

HINT: The last part of $P \in \mathcal{P}_i$ is going up.

(c) Use the fact that $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_n$ form a partition of $\mathcal{P}$ to give a combinatorial proof that for any non-negative integers a, b the following relation holds,

$$\binom{a+b}{a} = \sum_{i=0}^{a} \binom{i+b-1}{i}.$$

Exercise 3 (10 pts)**.** Consider the relation

$$k^n = \sum_{i=0}^{n} \binom{n}{i} (k-1)^{n-i}, \qquad (\star)$$

where $k \geq 2$ is an integer.

(a) Use the binomial theorem to prove $(\star)$.

HINT: Very little work involved here.

(b) Find a combinatorial proof of $(\star)$.

HINT: Consider strings of length n with k distinct letters.

Exercise 4 (10 pts)**.** Consider the expression

$$(1 + x + y)^{27}.$$

What is the coefficient of $x^7 y^9$?

HINT: The argument is similar to that of the proof of the binomial theorem.

Exercise 5 (10 pts)**.** The n children of the Von Trapp family all have different ages. Whenever they sing, they stand, shoulder to shoulder, on a line. Indicate in how many ways they can line up under the following conditions,

(1) The youngest child is never on the left-most position.

HINT: Look at the complementary case.

(2) The oldest child is to the right of the youngest child.

NOTE: it does not have to be directly to the right, for example, the oldest child could be 3 to the right of the youngest child.

Exercise 6 (10 pts)**.** Let $\mathcal{S}$ be the set of all finite subsets of the positive integers. For each $A \in S$, define the weight $w(A)$ to be the largest element in A (we define $w(\emptyset) = 0$).

(a) Determine the generating series for $\mathcal{S}$ with weights w.

(b) Suppose we change the word "largest" to "smallest" in part (a). Can we still define a generating series for $\mathcal{S}$ with weights w that is a formal power series?