COMBINATORICS
& OPTIMIZATION

University of Waterloo

Introduction to Combinatorics

Lecture 4

http://info.iqc.ca/mmosca/2014math239

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Definition of a generating function

Given a set S with weight function ω we define the *generating function* (or *generating series*) of S with respect to ω to be

(we usually omit the ω if it is implicit)

$$\Phi_S^{\omega}(x) = \sum_{\sigma \in S} x^{\omega(\sigma)}$$

(tells us what to sum over)

An example

$$\omega(\sigma) = cardinality(\sigma) = \#\sigma$$

$$S = \{\{\,\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

$$\Phi_S^{\omega}(x) = x^{\omega(\{\})} + x^{\omega(\{1\})} + x^{\omega(\{2\})} + x^{\omega(\{3\})}$$

$$+ x^{\omega(\{1,2\})} + x^{\omega(\{1,3\})} + x^{\omega(\{2,3\})} + x^{\omega(\{1,2,3\})}$$

$$= x^0 + x^1 + x^1 + x^1 + x^2 + x^2 + x^2 + x^3$$

$$= 1 + 3x + 3x^2 + x^3$$

The first part of this course

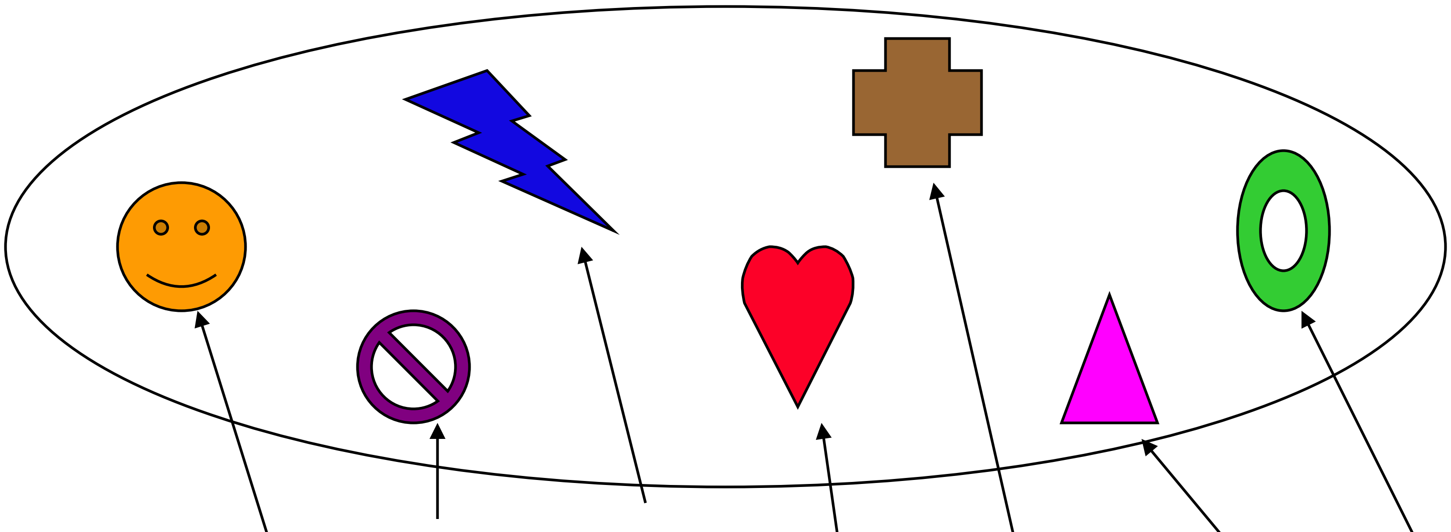
Our general strategy for counting is:

- Start with a counting problem. Define a set S and a weight function ω so that the solution equals the number of elements in S with weight k .
- Find $\Phi_S(x)$
- Extract $[x^k]\Phi_S(x)$ (i.e. the answer)

Other uses of a generating function

IF S is finite, then we can set x=1, and this equals the number of elements in S.

S

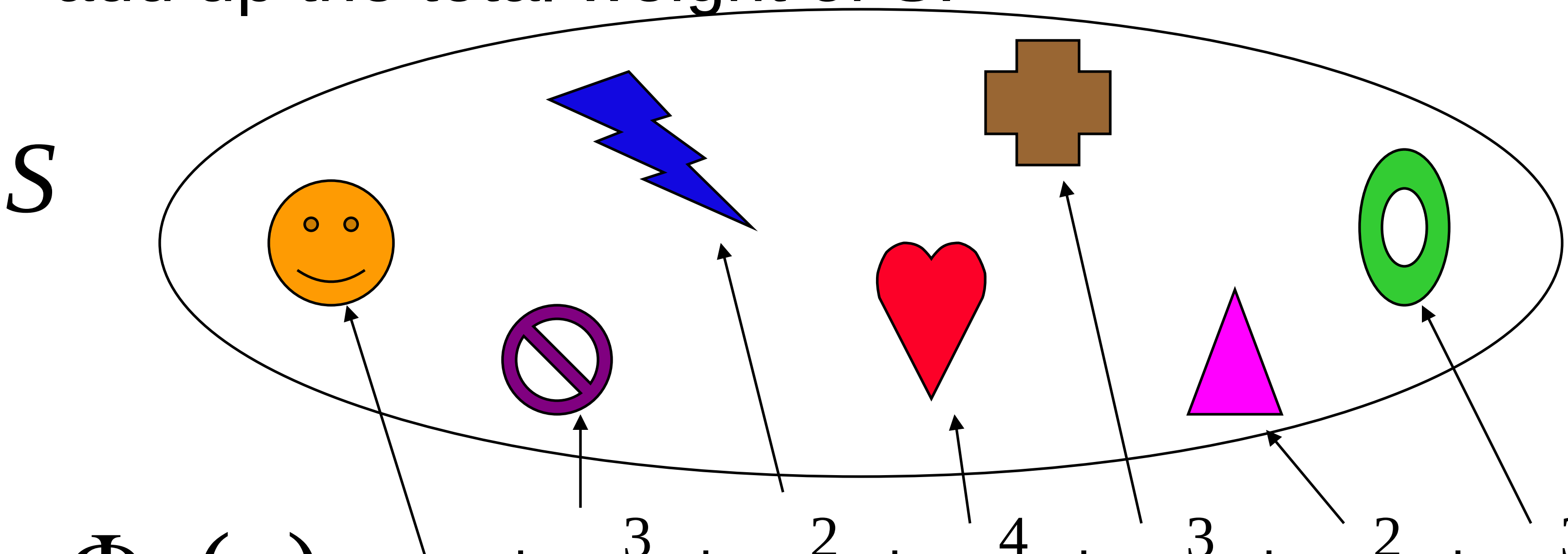


$$\Phi_S(x) = x + x^3 + x^2 + x^4 + x^3 + x^2 + x^3$$

$$\Phi_S(1) = 1+1+1+1+1+1+1 = 7$$

Other uses of a generating function

IF S is finite, we can differentiate, and then set x=1 to add up the total weight of S.



$$\Phi_S(x) = x + x^3 + x^2 + x^4 + x^3 + x^2 + x^3$$

$$\Phi_S'(x) = 1x^0 + 3x^2 + 2x^1 + 4x^3 + 3x^2 + 2x^1 + 3x^2$$

$$\Phi_S'(1) = 1 + 3 + 2 + 4 + 3 + 2 + 3 = 18$$

Other uses of a generating function

IF S is finite, then $\dfrac{\Phi'_S(1)}{\Phi_S(1)}$ gives us the average weight of the elements of S.

On formal power series

The “generating functions” we are dealing with are *formal power series*.

A formal power series

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

is a convenient way of encoding the sequence of rational (or even complex) numbers

$$\left(a_0, a_1, a_2, \cdots\right)$$

On formal power series

If the number of terms is finite, then these are just polynomials. We add and multiply formal power series the same way we add and multiply polynomials.

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_k x^k + \cdots$$

$$A(x) + B(x)$$

$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_k + b_k)x^k + \cdots$$

On formal power series

$$A(x)B(x)$$

$$= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \cdots\right)\left(b_0 + b_1 x + b_2 x^2 + \cdots + b_k x^k \cdots\right)$$

$$= \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \cdots\right) b_0$$

$$+ \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \cdots\right) b_1 x$$

$$+ \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \cdots\right) b_2 x^2$$

$$+ \cdots$$

$$+ \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \cdots\right) b_j x^j$$

$$+ \cdots$$

On formal power series

$$
\begin{aligned}
&= \left(a_0b_0 + a_1b_0x + a_2b_0x^2 + \cdots + a_kb_0x^k \cdots\right) \\
&\qquad + \left(a_0b_1x + a_1b_1x^2 + a_2b_1x^3 + \cdots + a_kb_1x^{k+1} \cdots\right) \\
&\qquad\qquad + \left(a_0b_2x^2 + a_1b_2x^3 + a_2b_2x^4 + \cdots + a_kb_2x^{k+2} \cdots\right) \\
&\qquad\qquad\quad + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&= a_0b_0 + \left(a_1b_0 + a_0b_1\right)x + \left(a_2b_0 + a_1b_1 + a_0b_2\right)x^2 + \cdots \\
&\qquad + \left(a_kb_0 + a_{k-1}b_1 + \cdots + a_{k-j}b_j + \cdots + a_0b_k\right)x^k + \cdots
\end{aligned}
$$

On formal power series

$$= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_{k-j} b_j \right) x^k$$

$$= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$$

Example

Problem 1.5.1: Show that the following equation has a solution and that the solution is unique:
$(1-x -x^2)A(x) =1+x$.

Similarly following the multiplication rule, we can show that there is no solution to x B(x) = 1.

We can also prove that for any A(x) with *non-zero* constant term, there exists a B(x) such that A(x)B(x)=1. We call such a B(x) the “inverse” of A(x).

Inverses

We call $B(x)$ the "inverse" of $A(x)$ if $A(x)B(x) = 1$

e.g. the "inverse" of $(1-x)$ is $1 + x + x^2 + \cdots + x^k + \cdots$

$$(1-x)\left(1 + x + x^2 + \cdots + x^k + x^{k+1} + \cdots\right)$$

$$= \left(1 + x + x^2 + \cdots + x^k + x^{k+1} + \cdots\right)$$

$$- x\left(1 + x + x^2 + \cdots + x^k + x^{k+1} + \cdots\right)$$

$$= \left(1 + x + x^2 + \cdots + x^k + x^{k+1} + \cdots\right)$$

$$- x - x^2 - x^3 - \cdots - x^{k+1} - x^{k+2} + \cdots$$

$$= 1$$

Inverses

We call $B(x)$ the “inverse” of $A(x)$ if $A(x)B(x)=1$

We denote this as $B(x)=A(x)^{-1}$ or $B(x)=\dfrac{1}{A(x)}$

e.g. $(1-x)^{-1}=1+x+x^2+\cdots$

Inverses

So $A(x)^{-1}$ or $\dfrac{1}{A(x)}$ is just shorthand for the power series which happens to be the inverse of $A(x)$

Inverses

Note that $\dfrac{1-x^{k+1}}{1-x} = \left(1-x^{k+1}\right)\left(1-x\right)^{-1}$

$$= \left(1-x^{k+1}\right)\left(1+x+x^2+\cdots+x^k+x^{k+1}+\cdots\right)$$

$$= \left(1+x+x^2+\cdots+x^k+x^{k+1}+\cdots\right) - \left(x^{k+1}+x^{k+2}+x^{k+3}+\cdots\right)$$

$$= 1+x+x^2+\cdots+x^k$$

Inverses

THM 1.5.7:

A formal power series has an inverse if and only if it has a non-zero constant term.

If a formal power series has an inverse, then it has a unique inverse.

More generally

Using the multiplication rule, we can prove

THM 1.5.2

Let $A(x) = a_0 + a_1x + a_2x^2 + \ldots$,

$P(X) = p_0 + p_1x + p_2x^2 + \ldots$, and

$Q(x) = 1 - q_1x - q_2x^2 + \ldots$ be formal power series.

Then $Q(x)A(x) = P(x)$ if and only if, for each $n \geq 0$,

$$a_n = p_n + q_1a_{n-1} + q_2a_{n-2} + \ldots + q_{n-1}a_0.$$

Corollary

Corollary 1.5.3:

Let $P(x)$ and $Q(x)$ be formal power series. If the constant term of $Q(x)$ is non-zero, then there is a formal power series $A(x)$ satisfying

$Q(x)A(x) = P(x)$.

Moreover the solution, $A(x)$, is unique.

Inverses

We can show by induction (see THM1.6.5)

$$\left((1-x)^{-1}\right)^{n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^{k}$$

We will denote this as $(1-x)^{-n}$.

This power series is the inverse of $(1-x)^{n}$.

Inverses

Problem 1.5.10: Determine the inverse of 1-x +2x^2.

What can we do?

If we let $y = x - 2x^2$ then we could try to use the fact that

$$(1-y)^{-1} = 1 + y + y^2 + \cdots$$

and claim that

$$\left(1 - x + 2x^2\right)^{-1} = 1 + (x + 2x^2) + (x + 2x^2)^2 + \cdots$$

and simplify.

Can we do that??

Not always! (try finding the inverse of 1-(1-x) this way…)

Composition of power series

Theorem 1.5.9. If $A(x)$ and $B(x)$ are formal power series with the constant term of $B(x)$ equal to zero, then $A(B(x))$ is a formal power series.

Test question

Find a closed form expression for

$$A(x) = \sum_{k=0}^{\infty} x^{5k} = 1 + x^5 + x^{10} + x^{15} + \cdots$$

Hint: let $y=x^5$ ... What is $1 + y + y^2 + y^3 + \cdots$?

$$1 + y + y^2 + y^3 + \cdots = \frac{1}{1-y} = \frac{1}{1-x^5}$$

Composing power series

In other words, we can compose the power series

$$B(y) = \sum_{k=0}^{\infty} y^k = \frac{1}{1-y}$$

with the power series $C(x) = x^5$

to get

$$B(C(x)) = \frac{1}{1-C(x)} = \frac{1}{1-x^5}$$