MATH 239 Tutorial 10

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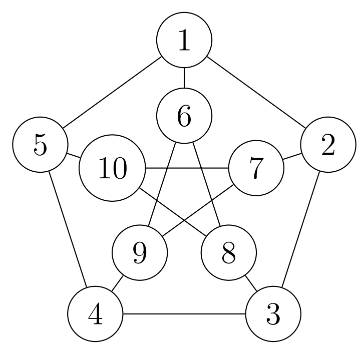
PROBLEMS

1. (Problem set 7.6, #9) Prove that every planar, bipartite graph has a vertex of degree at most 3.
2. (Problem set 7.6, $\#3, 4$) Recall the Petersen graph, which has $p = 10$ vertices and $q = 15$ edges.



   (a) Prove without using Kuratowski's Theorem that the Petersen graph is non-planar.
   (b) Prove using Kuratowski's Theorem that the Petersen graph is non-planar.
   (c) Find two edges of the Petersen graph whose deletion gives a planar graph.

3. (Problem set 7.8, #4) Let G be a connected, planar graph. Suppose G has an embedding in which every face has even degree. Prove that G is 2-colourable.
4. (Problem set 7.4, #1) A graph is platonic if it has an embedding where all of the vertices have the same degree d (at least 3) and all of the faces have the same degree d^* (also at least 3). Show that there is a unique planar embedding of a platonic graph where each vertex has degree $d = 4$ and each face has degree $d^* = 3$.
5. (Problem set 4.6, #7) A Hamilton cycle is one that spans every vertex of the graph. The n-cube is the graph on 2^n vertices (the n-bit strings) where two vertices are adjacent iff their corresponding strings differ in exactly 1 position. Prove that the n-cube has a Hamilton cycle for $n \geq 2$.

SOLUTIONS

1. By way of contradiction, suppose that there exists a planar, bipartite graph with every vertex having degree at least 4. Let's try to determine how many edges q it has.

   Since the graph is bipartite, it can have no odd cycle. Therefore, its girth k is at least 4. (Recall that the girth of a graph is the minimum length of a cycle. If a graph has no cycles, then its girth is infinite.) Since it is planar, we know that $q \leq \frac{k}{k-2}(p-2)$. Since $k \geq 4$, we know that $\frac{k}{k-2} = 1 + \frac{2}{k-2} \leq 1 + 1$. So, $q \leq 2(p-2) = 2p - 4$. On the other hand, we know by the Handshaking Lemma that $2q \geq 4p$. Combining these two inequalities, we have $2p \leq q \leq 2p - 4$, an obvious contradiction. Therefore, any planar, bipartite must have a vertex of degree at most 3.

2. (a) We know that if a graph is planar and connected, then $q \leq 3p - 6$. Therefore, if $q > 3p - 6$, we'll know it's non-planar. The Petersen graph has $q = 15$ and $3p - 6 = 30 - 6 = 24$, which tells us nothing.

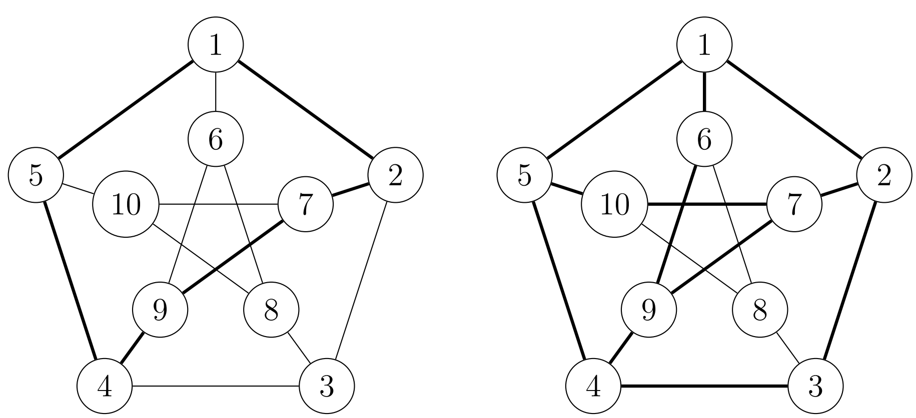
   We also know that if a graph is planar, it must have a vertex of degree at most 5. Therefore, if every vertex of a graph has degree at least 6, then it is non-planar. The Petersen graph has vertices of degree 3, which tells us nothing.

   We know that if a graph is planar, then $q \leq \frac{k}{k-2}(p-2)$, where k is the girth of the graph. Therefore, if $q > \frac{k}{k-2}(p-2)$, we'll know it's non-planar. The Petersen graph has girth 5, $q = 15$ and $\frac{k}{k-2}(p-2) = \frac{5}{3}(8) = 13\frac{1}{3}$. Thus, it is non-planar.

   (b) Kuratowski's Theorem says that a graph is non-planar if and only if it has a subgraph that is an edge subdivision of $K_{3,3}$ or K_5.

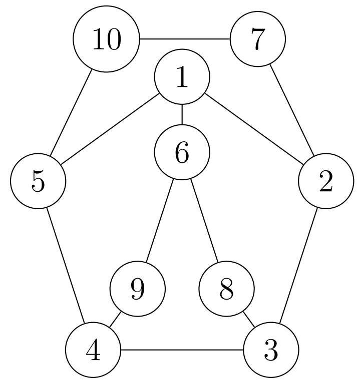
   Does the Petersen graph have an edge subdivision of K_5? No. K_5 has five vertices of degree 4, and the maximum degree of any vertex in the Petersen graph is 3. Replacing edges by paths in K_5 will not change the degrees of the five original ("branch") vertices in K_5.

   Instead, we look for an edge subdivision of $K_{3,3}$, which requires six vertices of degree 3. Two different ways to picture $K_{3,3}$ are as a 6-cycle with three chords connecting vertices opposite each other, and as two groups of three vertices, each connected to all vertices in the other group. We look for the former pattern. Consider the 6-cycle (1,2,7,9,4,5). With the chords (1,6,9), (2,3,4), and (5,10,7), we see that it is an edge subdivision of $K_{3,3}$. (Vertices 1,7,4 are in one partition and vertices 2,9,5 are in the other partition.)



(c) We must find two edges of the Petersen graph whose deletion make the resulting graph planar. From part (b), we see that the edge subdivision of $K_{3,3}$ leaves out the three edges incident with vertex 8, so we shouldn't pick two of these edges.

One possibility is to remove edges $\{8, 10\}$ and $\{7, 9\}$.



3. Recall that a graph is k-colourable if its vertices can be labelled with a set of k or fewer colours so that no two adjacent vertices have the same colour. Labelling a graph's vertices with two colours is equivalent to partitioning the vertices into two classes; a graph is 2-colourable iff it is bipartite.

   Suppose that there exists a planar, connected graph G that is not bipartite but has an embedding where every face has even degree. Then, G must have an odd cycle. Let C be an odd cycle in this planar embedding of G that has the fewest number of faces "in" it (i.e., within its boundary). Let C_0 be the boundary of one of the faces inside C that shares at least one edge with C. C_0 must have an even number of edges since every face has even degree.

   Now, let C' be the boundary of the other faces in C. C' is a set of edge-disjoint cycles. The number of cycles in C' depends on the number of paths in $C \cap C_0$, i.e., how many "times" C_0 intersects the edges of C. Each of the cycles in C' is the boundary of one or more faces contained in C. (Every face in C is either contained in one of the cycles of C', or it is the face C_0.) Every edge of C belongs either to C_0 or to one of the cycles in C', but not to both:

$$E(C) = E(C_0) \triangle E(C')$$

   From the above equation, we know that

$$\begin{aligned} |E(C)| &= |E(C_0) \setminus E(C')| + |E(C') \setminus E(C_0)| \\ &= |E(C_0)| + |E(C')| - 2|E(C_0 \cap C')| \end{aligned}$$

   Since $|E(C)|$ is odd, $|E(C_0)| + |E(C')|$ must also be odd. $|E(C_0)|$ is even since C_0 is the boundary of a face, so $|E(C')|$ must be odd. C' consists of edge-disjoint cycles, so one of them must be an odd cycle. However, it contains fewer faces than C, which was chosen to contain the fewest number of faces. This contradiction implies that G cannot have any odd cycles and hence is 2-colourable.

4. First, let's determine how many vertices, edges, and faces this graph and its embedding have. By the Handshaking Lemma, we know that $2q = 4p$. By the similar "Face-shaking" Lemma, we know that $2q = 3s$. Substituting $q = 2p$ and $s = \frac{4}{3}p$ into Euler's formula $p - q + s = 2$, we get $p = 6$. Therefore,

the graph has $p = 6$ vertices (all of degree 4), $q = 12$ edges, and the embedding has $s = 8$ faces (all of degree 3).

Since every face has degree 3, we can draw a 3-cycle (with vertices A, B, C, say) as the boundary of the outer face, and three vertices (D, E, F) in its interior. Since each vertex has degree 4, A must be adjacent to two of the vertices inside the triangle, say D and E.

Now, we consider the other two exterior vertices. Can one of them be adjacent to the same two vertices as A? No. If B were also adjacent to D and E, the triangle's interior would have three faces — one of degree 3 and two of degree 4. By adding the only remaining vertex, F, there would be no way for the graph to remain planar and to reduce the degrees of both degree-4 faces. Therefore, the outer vertices A, B, C are each adjacent to two different inner vertices.

At this point, we have drawn 9 edges. Each of the three outer vertices has degree 4, as required, and each of the three inner vertices has degree 2. There is only one way to connect the inner vertices: each one is adjacent to the other two. These three final edges complete the unique planar embedding of the platonic graph with $d = 4$, $d^* = 3$.

5. To find a Hamilton cycle, we need to find an ordering of the 2^n binary strings of length n so that successive strings differ in exactly one position. We'll use the Gray code of width n, also called the reflected binary code. It is an algorithm for generating binary strings of length n in an order such that successive strings differ in exactly one bit.

   Consider the following algorithm for generating all n-bit strings ($n \geq 1$):

   (a) $A = (a_1, a_2) = (0, 1)$

   (b) $l = 1$

   (c) while $l < n$:

      i. $A = (0a_1, \ldots, 0a_{2^l}, 1a_{2^l}, \ldots, 1a_1)$ (where juxtaposition denotes concatenation)

      ii. $l = l + 1$

   We'll prove by induction on n that the algorithm outputs all binary strings of length n and that successive strings in the n-bit Gray code differ in exactly one bit position.

   Base case ($n = 1$): Clearly, 0 and 1 differ in one bit position and are the only two 1-bit strings.

   Induction hypothesis: Suppose the above algorithm is correct for some $n - 1$, where $n \geq 2$.

   We'll prove that the algorithm is correct for n bits. The algorithm will output

   $A = (0a_1, \ldots, 0a_{2^{n-1}}, 1a_{2^{n-1}}, \ldots, 1a_1)$

   By the induction hypothesis, the first 2^{n-1} strings are distinct and successive strings differ in exactly one position. Similarly, the last 2^{n-1} strings are distinct and successive strings differ in exactly one position. We have generated all 2^n binary strings of length n. We must prove that $0a_{2^{n-1}}$ and $1a_{2^{n-1}}$ differ in exactly one position and that $1a_1$ and $0a_1$ differ in exactly one position. Indeed, both pairs do differ in exactly one position, the first.

   For $n \geq 2$, there will be at least 4 vertices and therefore no repeated edges. Hence, the Gray code gives an ordering of the vertices of the n-cube that forms a Hamilton cycle for $n \geq 2$.

The pattern is more evident when the strings are written vertically, but here are the binary Gray codes for n up to 4:

- $n = 1$: (0,1)
- $n = 2$: (00, 01, 11, 10)
- $n = 3$: (000, 001, 011, 010, 110, 111, 101, 100)
- $n = 4$: (0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1101, 1111, 1110, 1010, 1011, 1001, 1000)