MATH 239 Assignment 2

**DUE: NOON Friday 30 September 2011** in the drop boxes opposite the Math Tutorial Centre MC 4067 or next to the St. Jerome's library for the St. Jerome's section.

1. Determine $[x^k]\frac{1}{(1-x)^2(1-3x)}$.

   **SOLUTION.** Since the constant term of $3x$ is zero, we can substituting $3x$ in $\frac{1}{1-x} = \sum_{n\geq 0} x^n$ to get

   $$\frac{1}{1-3x} = 1 + 3x + (3x)^2 + (3x)^3 + \cdots = \sum_{n\geq 0} 3^n x^n.$$

   We also have

   $$\frac{1}{(1-x)^2} = (1 + x + x^2 + \cdots)^2 = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n\geq 0}(n+1)x^n.$$

   Therefore

   $$\begin{aligned}[x^k]\frac{1}{(1-x)^2(1-3x)} &= [x^k]\big(1 + 2x + \cdots + (k+1)x^k\big)\big(1 + 3x + \cdots + 3^k x^k\big) \\ &= 3^k + 2\cdot 3^{k-1} + 3\cdot 3^{k-2} + \cdots + (k+1)3^0 \\ &= \sum_{i=0}^{k}(i+1)3^{k-i}.\end{aligned}$$

2. Let $A(x), B(x)$ be formal power series that are not polynomials. Let $C(x)$ be a polynomial. Suppose that $A(x)$ are invertible, and that $B(C(x))$ exists as a formal power series. Are the following formal power series? (Justify your answers.)

   (a) $B(A(x))$ **SOLUTION. It is not.** For $A(x)$ to be invertible, it must have a non-zero constant term. Therefore we cannot substitute it in another series unless it is a polynomial. But $B(x)$ is not a polynomial.

   (b) $C(x)^{-1}A(x)$ **SOLUTION. It is not.** Since $B(C(x))$ is a formal power series and $B(x)$ is not a polynomial, it must be that $[x^0]C(x) = 0$. Therefore it is not invertible.

   (c) $A(B(x))$ **SOLUTION. It could be either way.** It depends on whether $[x^0]B(x) = 0$ or not. If $[x^0]B(x) = 0$, the formal power series $A(B(x))$ is well defined. Otherwise, it is not a formal power series.

3. Let $S$ be a set of configurations with a certain weight function. Show that, for any non-negative integer $n$,

   $$[x^n]\frac{\Phi_S(x)}{1-x^2}$$

   counts the number of configurations in $S$ whose weight is at most $n$ and has the same parity as $n$. *Two numbers have the same parity if the are either both even or both odd.*

**SOLUTION.** Let's denote the coefficients of $\Phi_S(x)$ by $a_n$, that is $a_n$ is the number of configurations of weight $n$.

We must consider two separate case. Suppose first that $n = 2m$:

$$\begin{aligned}
[x^n]\frac{\Phi_S(x)}{1-x^2} &= [x^n]\left(a_0 + a_1 x + \cdots\right)\left(1 + x^2 + x^4 + \cdots\right) \\
&= [x^{2m}]\left(a_0 + a_1 x + \cdots + a_{2m}x^{2m}\right)(1 + x^2 + x^4 + \cdots + x^{2m}) \\
&= [x^{2m}]\left(\sum_{k=0}^{m} a_{2k}x^{2k}x^{2m-2k}\right) \\
&= \sum_{k=0}^{m} a_{2k} \\
&= \text{number of configurations in } S \text{ whose weight is even and at most } n.
\end{aligned}$$

Suppose now that $n = 2m + 1$:

$$\begin{aligned}
[x^n]\frac{\Phi_S(x)}{1-x^2} &= [x^n]\left(a_0 + a_1 x + \cdots\right)\left(1 + x^2 + x^4 + \cdots\right) \\
&= [x^{2m+1}]\left(a_0 + a_1 x + \cdots + a_{2m+1}x^{2m+1}\right)(1 + x^2 + x^4 + \cdots + x^{2m}) \\
&= [x^{2m+1}]\left(\sum_{k=0}^{m} a_{2k+1}x^{2k+1}x^{2m-2k}\right) \\
&= \sum_{k=0}^{m} a_{2k+1} \\
&= \text{number of configurations in } S \text{ whose weight is odd and at most } n.
\end{aligned}$$

4. Let $S$ be the set of compositions of integers with at most 4 parts and with all parts greater than or equal to 3. Let

$$\begin{aligned}
w\colon S &\to \mathbb{N} \\
(a_1, \ldots, a_k) &\mapsto a_1 + \cdots + a_k
\end{aligned}$$

be the weight function. Find $\Phi_S(x)$ and express it as a quotient of polynomials.

**SOLUTION.** Let $\tilde{\mathbb{N}} = \mathbb{N} \setminus \{0, 1, 2\}$. On $\mathbb{N}$, we consider the identity $\mathbb{N} \to \mathbb{N}$ as the standard weight function, and we restrict that function to $\tilde{\mathbb{N}}$ to define a weight function on $\tilde{\mathbb{N}}$. For $n \geq 3$, there is a unique element in $\tilde{\mathbb{N}}$ of weight $n$. We therefore have

$$\begin{aligned}
\Phi_{\tilde{\mathbb{N}}}(x) &= \sum_{n\geq 3} x^n \\
&= x^3 \sum_{i\geq 0} x^i \\
&= \frac{x^3}{1-x}.
\end{aligned}$$

A composition that has only parts greater than or equal to 3 and has exactly $k$ parts is an element of $\mathbb{N}^k$ with the natural weight function on $\tilde{\mathbb{N}}^k$ is defined to be $(a_1, \ldots, a_k) \mapsto a_1 + \ldots + a_k$. There is thus a natural identification

$$S = \tilde{\mathbb{N}} \cup \tilde{\mathbb{N}}^2 \cup \tilde{\mathbb{N}}^3 \cup \tilde{\mathbb{N}}^4$$

and that identification preserves weights. Therefore

$$\begin{aligned}
\Phi_S(x) &= \Phi_{\tilde{\mathbb{N}}}(x) + \Phi_{\tilde{\mathbb{N}}^2}(x) + \Phi_{\tilde{\mathbb{N}}^3}(x) + \Phi_{\tilde{\mathbb{N}}^4}(x) \quad \text{(by the Sum Lemma)} \\
&= \Phi_{\tilde{\mathbb{N}}}(x) + \Phi_{\tilde{\mathbb{N}}}(x)^2 + \Phi_{\tilde{\mathbb{N}}}(x)^3 + \Phi_{\tilde{\mathbb{N}}}(x)^4 \quad \text{(by the Product Lemma)} \\
&= \frac{x^3}{1-x} + \frac{x^6}{(1-x)^2} + \frac{x^9}{(1-x)^3} + \frac{x^{12}}{(1-x)^4} \\
&= \frac{x^3(1-x)^3 + x^6(1-x)^2 + x^9(1-x) + x^{12}}{(1-x)^4} \\
&= \frac{x^3 - 3x^4 + 3x^5 - x^6 + x^6 - 2x^7 + x^8 + x^9 - x^{10} + x^{12}}{(1-x)^4} \\
&= \frac{x^3 - 3x^4 + 3x^5 - 2x^7 + x^8 + x^9 - x^{10} + x^{12}}{(1-x)^4}.
\end{aligned}$$