SOS Spring 2012 MATH 239 Final Review Package

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This package includes a list of theorems and definitions that you need to know for your final, as well as several examples.

Keep calm and ace that final.

Topics Covered

(M) - Covered on the midterm, see midterm review package: https://docs.google.com/document/d/1A0XuuHAoif8S6X-ruWhsMpwhvFVvGtBf94oqDBEnlZs/edit

Enumeration:
Basic set definitions **(M)**
Binomial coefficients **(M)**
Combinatorial proofs **(M)**
Bijections **(M)**
Formal power series **(M)**
Manipulating generating series using sum and product lemmas **(M)**
Compositions **(M)**
Recurrence relations **(M)**
Binary strings **(M)**
Solving recurrence relations **(M)**
Asymptotics **(M)**

Graph Theory:
Graphs (edges, degrees) **(M)**
Isomorphisms **(M)**
Bipartite graphs **(M)**
Paths and cycles **(M)**
Connectivity (including bridges) **(M)**
Eulerian circuits
Trees (including minimum spanning trees)
Bipartite characterization
Planarity, Euler's formula, and Kuratowski's theorem
Platonic solids
Colouring
Planar duals
Matchings, König's theorem, Hall's theorem

Eulerian Circuits

Def: A trail is a walk in which all edges are distinct.

Def: A circuit is a closed trail.

Def: An Eulerian trail, or Eulerian path, is a trail in a graph which visits every edge exactly once.

Def: A Eulerian circuit is an Eulerian trail that starts and ends on the same vertex.

Theorem: Let G be a graph with no isolated vertices. Then G has an Eulerian circuit iff G is connected and every vertex of G has even degree.

(Proof of ⇐ on Supplementary 4 page)

Proof of ⇒: Suppose the graph G has an Eulerian circuit. Then every vertex that has an edge is in the circuit, and is therefore connected with all other vertices in the circuit. So G is connected. For each vertex in the circuit except the first one, the walk leaves it just after entering. Thus, every time we use two edges incident to that vertex (i.e., an even number of edges). Since the Eulerian tour uses each edge exactly once, every vertex has even degree. Also, the first vertex v has even degree because the walk leaves it in the beginning but returns to v at the end.

Trees

Def: A tree is a connected graph with no cycles.

Lemma 5.1.2: There is a unique path between every pair of vertices u and v in a tree T.

(Proof on page 116 of course notes)

Lemma 5.1.3: Every edge of a tree T is a bridge.

(Proof on page 116 of course notes)

Theorem 5.1.4: A tree with at least two vertices has at least two vertices of degree one.
(Proof on page 116 of course notes)

Theorem 5.1.5: If T is a tree, then $|E(T)| = |V(T)| - 1$.
(Proof on page 116 of course notes)

Useful fact: Every tree is bipartite.
Proof: A tree contains no cycles, so it does not contain any odd cycles. Then by Theorem 5.3.2, every tree is bipartite.

Spanning Trees

Def: A spanning subgraph that is also a tree is a spanning tree.

Theorem 5.2.1: A graph G is connected if and only if it has a spanning tree.
(Proof on page 118 of course notes)

Corollary 5.2.2: If G is connected, with p vertices and q = p - 1 edges, then G is a tree.
(Proof on page 119 of course notes)

Minimum Spanning Trees

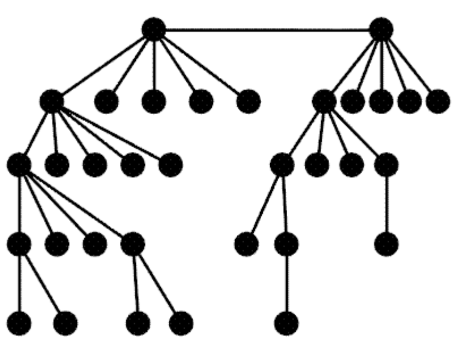
Def: If each edge is given a weight (usually a positive integer) then a minimum spanning tree is a spanning tree with a weight that is less than or equal to the weight of every other spanning tree.

A "greedy" algorithm (Prim-Jarnik algorithm) for finding a minimum spanning tree in graph G:
Let P be the set of vertices already in the tree.
Pick any vertex in G and put it in P.
Pick an edge of minimum weight uv such that u is a vertex in P and v is not in P.
Rinse and repeat until you have all the vertices in G placed in P.

Example: Let T be a tree having at least two vertices of degree 2, at least three vertices of degree 3, exactly two vertices of degree 5, and at least three vertices of degree 6. Show that T has at least 33 vertices and find an example of such a tree with precisely 33 vertices.
Solution: By "alternate proof of 5.1.4" on page 117 of course notes, we have the following fact:

$$n_1 = 2 + \sum_{r \geq 3} (r-2) n_r$$

Where n_r is the number of vertices of degree r in T. Then we have $n_2 \geq 2$, $n_3 \geq 3$, $n_5 = 2$, $n_6 \geq 3$.
So $n_1 \geq 2 + (3-2)3 + (5-2)2 + (6-2)3 = 23$
Then the total number of vertices must be greater than $23 + 2 + 3 + 2 + 3 = 33$.
An example of such a tree is below.



Example (Similar to Tutorial 8 Question 1): A forest is a graph with no cycles. Prove that a forest with p vertices and q edges has p-q components.

Solution: Since a forest has no cycles, its components must be trees. For each tree component, there is 1 more vertex than there are edges by Theorem 5.1.5. There are p-q more vertices than edges in total, so there must be p-q components.

Example: Find a MST in the following graph.

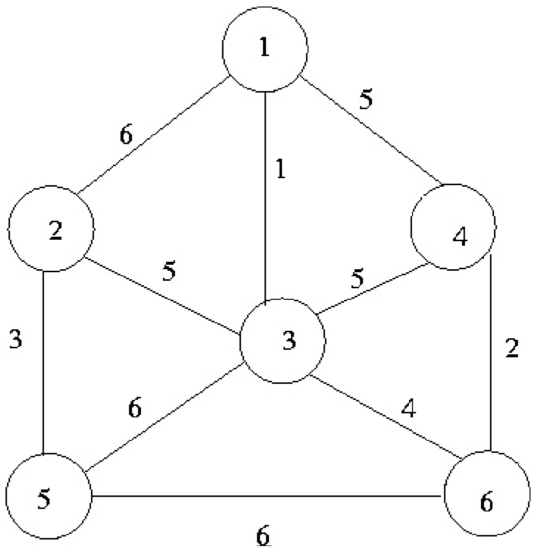


Image source: http://lcm.csa.iisc.ernet.in/dsa/node182.html

Solution: The edges in the MSP are {1,3}, {3,6}, {4,6}, {2,3}, {2,5}.

Bipartite Characterization

Def: An odd cycle is a cycle on an odd number of vertices.

Lemma 5.3.1: An odd cycle is not bipartite.
(Proof on page 120 of course notes)

Theorem 5.3.2: A graph is bipartite if and only if it has no odd cycles.
(Proof on page 120 of course notes)

Lemma 5.3.3: Let T be a spanning tree in a connected graph G. If G is not bipartite, then there is an odd cycle in G that uses exactly one edge not from T.
(Proof on page 121 of course notes)

Planarity, Euler's Formula, and Kuratowski's Theorem

Def: A graph G is planar if it has a drawing in the plane so that its edges intersect only at their ends, and so that no two vertices coincide. The actual drawing is called a planar embedding of G, or a planar map.

Def: A planar embedding partitions the plane into connected regions called faces; one of these regions, called the outer face, is unbounded.

Def: The subgraph formed by the vertices and edges in a face is called the boundary of the face.

Def: Two faces are adjacent if they are incident with a common edge.

Def: A boundary walk of a face f is a closed walk around the perimeter of f.

Def: The degree of a face f is the length of the boundary walk of f.

Useful fact: A bridge contributes 2 to the degree of the face with which it is incident. Every edge in a tree is a bridge, so a planar embedding of a tree T has a single face of degree $2|E(T)|$.

Theorem 7.1.2: If we have a planar embedding of a connected graph G with faces $f_1, f_2, \ldots, f_s$, then $\sum_{i=1}^{s} deg(f_i) = 2|E(G)|$.
(Proof on page 145 of course notes)

Corollary 7.1.3: If the connected graph G has a planar embedding with f faces, the average degree of a face in the embedding is $\frac{2|E(G)|}{f}$.

Properties of Planar Graphs

Euler's Formula (Theorem 7.2.1): Let G be a connected graph with p vertices and q edges. If G has a planar embedding with f faces, then p - q + f = 2.
(Proof on page 145 of course notes)

Lemma 7.5.1: If G is is connected and not a tree, then in a planar embedding of G, the boundary of each face contains a cycle.
(Proof on page 153 of course notes)

Lemma 7.5.2: Let G be a planar embedding with p vertices and q edges. If each face of G has degree at least d*, then $(d^*-2)q \leq d^*(p-2)$.
(Proof on page 154 of course notes)

Theorem 7.5.3: In a connected planar graph with $p \geq 3$ vertices and q edges, we have $q \leq 3p-6$.
(Proof on page 154 of course notes)

Corollary 7.5.4: K_5 is not planar.
(Proof on page 155 of course notes)

Corollary 7.5.5: A planar graph has a vertex of degree at most five.
(Proof on page 156 of course notes)

Lemma 7.5.6: $K_{3,3}$ is not planar.
(Proof on page 157 of course notes)

Def: An edge subdivision of G is created by replacing each edge with a path of length 1 or more.

Kuratowski's Theorem (Theorem 7.6.1): A graph is not planar if and only if it has a subgraph that is an edge subdivision of K_5 or $K_{3,3}$.
(Proof on page 159 of course notes)

Example (Problem Set 7.6 Question 9 from course notes): Prove that a bipartite planar graph has a vertex of degree at most 3.
Solution: Suppose G is a bipartite planar graph. If it is a forest, we know it has a vertex of degree 1. Otherwise, it has a cycle. Let $q=|E(G)|$ and $p=|V(G)|$. Since G is bipartite, then the minimum cycle length is at least 4 ie the minimum face degree is at least 4. Then from Lemma 7.5.2 we have that

$(d^*-2)q \leq d^*(p-2)$

$q \leq \frac{d^*}{d^*-2}(p-2)$

Note that when d*=4, $\frac{d^*}{d^*-2}=2$ and as d* gets larger, $\frac{d^*}{d^*-2}$ approaches 1.
So $q \leq 2(p-2)=2p-4$.

Now suppose every vertex in G has degree at least 4. Then by handshaking, $2q \geq 4p \Rightarrow q \geq 2p$, which is a contradiction. So a bipartite planar graph has a vertex of degree at most 3.

Example: Suppose that G is a non-empty connected graph with an embedding on the sphere where all the faces are hexagons. Let p be the number of vertices, q the number of edges, and s the number of faces of the embedding.

(a) Prove that q = 3s and p = 2 + 2s.

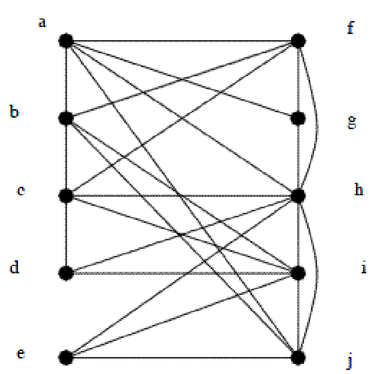
(b) Prove that G is bipartite.

Solution: a) Since faces are hexagon, their degree is always 6. So by faceshaking, 2q=6s which gives us q=3s as desired. Now using Euler's formula , we have 2 = p - q + s = p - 3s + s = p - 2s, hence p = 2 + 2s, as desired.

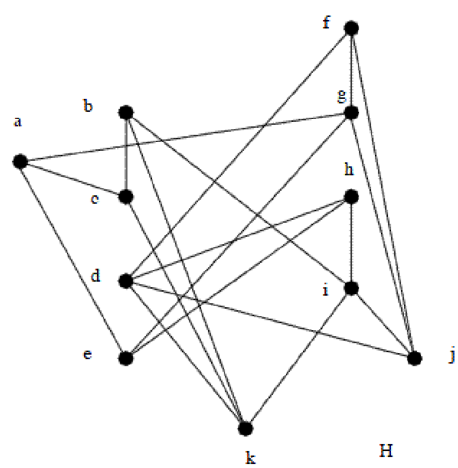
b) Every cycle C must contain an integral number of faces, say $f_1,...,f_k$.

Let q_{in} be the number of edges in the interior of $f_1 \cup ... \cup f_k$ and q_{out} be the number of edges in the boundary walk of $f_1,...,f_k$, hence on the cycle C. Every interior edge is in two faces, while every outside edge is on one face only. Hence $6k = 2q_{in} + q_{out}$. So the length of the cycle C is $q_{out} = 2(3k - q_{in})$, which is even. Since that is true for every cycle, then G has no odd cycle and is thus bipartite.

Example: For each of the graphs shown, determine whether it is planar and prove your assertion.



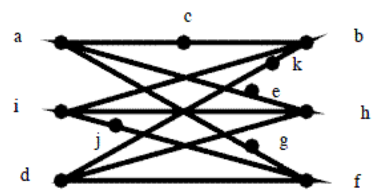
G



Solution: G is planar. To prove G is planar, we exhibit a planar embedding. (Shown at session)

Tip for finding a planar embedding: look for the longest cycle in the graph G, draw that cycle first and then add in the other edges.

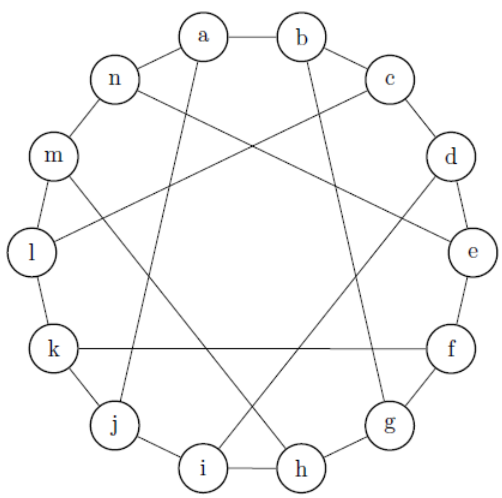
H is not planar by Kuratowski's Theorem as it contains a subdivision of $K_{3,3}$ as shown below.



Example (Problem Set 7.6 Question 10 from course notes): Let G denote the graph below (assume graph G has girth 6).

(a) Let H be any graph obtained from G by deleting two edges.Prove that H is not planar.

(b) Prove there exist 3 edges that can be deleted from G so that the resulting graph is planar.



Solution: a) First note that G has 14 vertices and 21 edges. Then H has 14 vertices and 19 edges. Suppose H is planar. Since the girth of G is 6, then removing edges will not decrease the girth, so the girth of H is 6 or higher. Hence, any face in H must have degree 6 or higher. Let p be the number of vertices in H, q be the number of edges in H, and d* be the minimum face degree of H. Then by Lemma 7.5.2, we have

$$(d^*-2)q \leq d^*(p-2)$$

$$q \leq \frac{d^*}{d^*-2}(p-2)$$

Note that when d*=6, $\frac{d^*}{d^*-2} = \frac{6}{4}$ and as d* gets larger, $\frac{d^*}{d^*-2}$ approaches 1.

Then we have that $q \leq \frac{6}{4}(14-2) = 18$. This is a contradiction since H has 19 edges. So H is not planar.
b) Remove edges {a,j}, {b,g}, and {k,f}, move edges {c,l}, {d,i} outside of the cycle and you have a planar embedding of the resulting graph. So these 3 edges fulfill the requirement.

Platonic Solids

Def: Polyhedra with congruent faces and vertices are platonic solids.

Def: A graph is platonic if admits a planar embedding in which each vertex has the same degree $d \geq 3$ and each face has the same degree $d^* \geq 3$.

Theorem 7.4.1 There are exactly five platonic graphs.
(Proof on page 150 of course notes)

Lemma 7.4.2: Let G be a planar embedding with p vertices, q edges and s faces, in which each vertex has degree $d \geq 3$ and each face has degree $d^* \geq 3$. Then (d,d*) is one of the five pairs {(3,3), (3,4), (4,3), (3,5), (5,3)}.
(Proof on page 150 of course notes)

Lemma 7.4.3: If G is a platonic graph with p vertices, q edges and f faces,where each vertex has degree d and each face degree d*, then

$$q = \frac{2dd^*}{2d + 2d^* - dd^*}$$

and $p = 2q/d$ and $f = 2q/d^*$.
(Proof on page 152 of course notes)

See page 151 on course notes for planar embedding of each platonic solid.

Colouring

Def: A k-colouring of a graph G is a function from V(G) to a set of size k (whose elements are called colours), so that adjacent vertices always have different colours. A graph with a k-colouring is called a k-colourable graph.

Theorem 7.7.2: A graph is 2-colourable if and only if it is bipartite.

(Proof on page 163 of course notes)

Theorem 7.7.3: K_n is n-colourable, and not k-colourable for any k < n.
(Proof on page 164 of course notes)

Theorem 7.7.4: Every planar graph is 6-colourable.
(Proof on page 164 of course notes)

Def: An edge e = {x, y} of graph G is said to be contracted if it is deleted and vertices x and y are identified. The resulting graph is denoted by G/e.

Note: G/e is planar whenever G is.

Theorem 7.7.6: Every planar graph is 5-colourable.
(Proof on page 166 of course notes)

Theorem 7.7.7: Every planar graph is 4-colourable.
(Proof beyond scope of this course)

Example (Problem Set 7.8 Question 7 from course notes): Show that a graph with 2m vertices and m^2+1 edges is not 2-colourable.
Solution: Note that a bipartite graph on 2m vertices can have at most m^2 edges. So if a graph on 2m vertices has m^2+1 edges, then it is not bipartite. Hence it is not 2-colourable by Theorem 7.7.2.

Example (Problem Set 7.8 Question 6 from course notes): Show that a planar graph with $p>2$ vertices and 2p-3 edges is not 2-colourable.
Solution: Let G be a graph with $p>2$ vertices and 2p-3 edges and suppose on the contrary that G is 2-colourable. Then by Theorem 7.7.2, G is bipartite. Note that since G is bipartite, then G must have girth at least 4, ie the minimum face degree is at least 4. Then from Lemma 7.5.2 we have that

$$(d^*-2)q \leq d^*(p-2)$$

$$q \leq \frac{d^*}{d^*-2}(p-2)$$

Note that when d*=4, $\frac{d^*}{d^*-2}=2$ and as d* gets larger, $\frac{d^*}{d^*-2}$ approaches 1.
So $q \leq 2(p-2)=2p-4$.

However, G has 2p-3 edges, which is a contradiction. Hence, G is not 2-colourable.

Planar Duals

Def: Given a connected planar embedding G, the dual G* is a planar embedding constructed as follows: G* has one vertex for each face of G. Two vertices of G* are joined by an edge whenever the corresponding faces of G have an edge in common (one side for each face), and the edge in G* is drawn to cross this common boundary edge in G.

Notes:

A face of degree k in G becomes a vertex of degree k in G*, and a vertex of degree j in G becomes a face of degree j in G*

(G*)* and G are the same graph.

Note that a bridge in G gives an edge in G* between a vertex and itself (such an edge is called a loop), and more than one edge between two faces in G gives more than one edge between a pair of vertices (these are called, together, a multiple edge). In this case G* is no longer a simple graph.

The Four Colour Theorem for colouring vertices in planar graphs is equivalent to the Four Colour Theorem for colouring faces in planar embeddings, via duality.

Example (Problem Set 7.8 Question 3 from course notes): Let G be a connected planar embedding with p vertices and q edges, and suppose that the dual graph G* is isomorphic to G.

(a) Prove that q=2p-2.

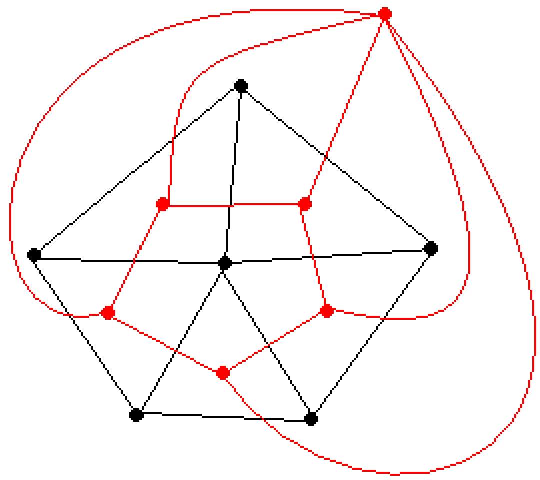
(b) Give an example of such a graph with six vertices.

Solution: a) Each vertex in G becomes a face in G* and vice-versa. Let p be the number of vertices in G, q be the number of edges in G, and s be the number of faces in G. Since the dual graph G* is isomorphic to G, then G* has the same number of vertices as G, ie. p=s. Then by Euler's Formula, we have

p-q+p=2

q=2p-2

b) Example shown below.



Matchings, König's Theorem, Hall's Theorem

Def: A matching in a graph G is a set M of edges of G such that no two edges in M have a common end.

Def: We say that a vertex v of G is saturated by M, or that M saturates v, if v is incident with an edge in M.

Def: A largest matching in G is called a maximum matching of G.

Def: A perfect matching saturates every vertex.

Note: Not every graph has a perfect matching.

Def: An alternating path with respect to a matching M is a path where every other edge is in M.

Def: An augmenting path with respect to M is an alternating path joining two distinct vertices neither of which is saturated by M. (ie. The first and last edges of the path are not in M.)

Lemma 8.1.1: If M has an augmenting path, it is not maximum matching.
(Proof on page 173 of course notes)

Def: A cover of a graph G is a set C of vertices such that every edge of G has at least one end in C.

Lemma 8.2.1: If M is a matching of G and C is a cover of G, then $|M| \leq |C|$.
(Proof on page 174 of course notes)

Lemma 8.2.2: If M is matching and C is a cover and |M| = |C|, then M is a maximum matching and C is a minimum cover.
(Proof on page 174 of course notes)

Konig's Theorem (Theorem 8.3.1): In a bipartite graph the maximum size of a matching is the minimum size of a cover.
(Proof on page 176 of course notes)

Finding a maximum matching in a bipartite graph:

Def: We define an XY-construction as follows:
Let X_0 be the set of vertices in A not saturated by M and let Z denote the set of vertices in G that are joined to a vertex in X_0 by an alternating path. If $v \in Z$, we use P(v) to denote an alternating path that joins v to X_0. Now define:
(a) $X = A \cap Z$.
(b) $Y = B \cap Z$.

Notes: If $v \in X$, then the last edge of P(v) is in M (this is true vacuously if $v \in X_0$).
If $v \in Y$, then the last edge of P(v) is not in M.
If w is a vertex of an alternating path P(v) from X_0 to $v \in Z$, then $w \in Z$.

Lemma 8.3.2: Let M be a matching of bipartite graph G with bipartition A, B ,and let X and Y be as defined above.Then:
(a) There is no edge of G from X to B \ Y;
(b) C = Y U (A \ X) is a cover of G;
(c) There is no edge of M from Y to A \ X;
(d) |M| = |C| - |U| where U is the set of unsaturated vertices in Y;
(e) There is an augmenting path to each vertex in U.
(Proof on page 177 of course notes)

Lemma 8.3.3: Let G be a bipartite graph with bipartition A,B, where |A| = |B| = n.If G has q edges, then G has a matching of size at least q/n.
(Proof on page 178 of course notes)

Algorithm for maximum matching in bipartite graphs:

0. Let M be any matching of G.

1. Set $\widehat{X} = \{v \in A: v\ is\ unsaturated\}$, set $\widehat{Y} = \emptyset$, and set pr(v) be undefined for all $v \in V\{G\}$.

2. For each vertex $v \in B \setminus \widehat{Y}$ such that there is an edge {u, v} with $u \in \widehat{X}$,add v to $\widehat{Y}$ and set pr(v) = u.

3. If step 2 added no vertex to $\widehat{Y}$, return the maximum matching M and the minimum cover $C = \widehat{Y} \cup (A \setminus \widehat{X})$ and stop.

4. If step 2 added an unsaturated vertex v to $\widehat{Y}$, use pr values to trace an augmenting path from v to an unsaturated element of $\widehat{X}$, use the path to produce a larger matching M' ,replace M by M' and go to step 1.

5. For each vertex $v \in A \setminus \widehat{X}$ such that there is an edge $\{u,v\} \in M$ with $u \in \widehat{Y}$, add v to $\widehat{X}$ and set pr(v) = u.Go to step 2.

<u>Def</u>: For any subset D of vertices of a graph G, the <u>neighbour set N(D) of D</u> is $\{v \in V(G): there\ exists\ u \in D\ with\ \{u,v\} \in E(G)\}$.

<u>Hall's Theorem (Theorem 8.4.1)</u>: A bipartite graph G with bipartition A, B has a matching saturating every vertex in A, if and only if every subset D of A satisfies $|N(D)| \geq |D|$.
(Proof on page 183 of course notes)

Perfect Matchings in Bipartite Graphs

<u>Corollary 8.6.1</u>: A bipartite graph G with bipartition A,B has a perfect matching if and only if $|A|=|B|$ and every subset D of A satisfies $|N(D)| \geq |D|$.
(Proof on page 186 of course notes)

<u>Theorem 8.6.2</u>: If G is a k-regular bipartite graph with $k \geq 1$, then G has a perfect matching.
(Proof on page 186 of course notes)

Example (Problem Set 8.2 Question 12 from course notes): Show that it is not always true that there exist a matching M and a cover C of the same size.

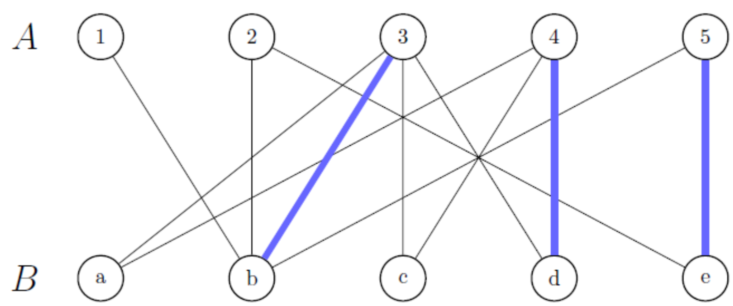
Solution: A triangle graph has a matching M of size 1 and a cover C of size 2.

Example (Problem Set 8.2 Question 12 from course notes): Find a bipartite graph G with bipartition A,B where $|A|=|B|=5$, and having the following properties. Every vertex has degree at least 2, the total number of edges is 16, and G has no perfect matching. Why does your graph not have a perfect matching?

Solution: Solution is presented at session. Hint: By Hall's theorem, a bipartite graph G with bipartition A, B has a matching saturating every vertex in A, if and only if every subset D of A satisfies $|N(D)| \geq |D|$ . .

Try to find a graph that does not satisfy "every subset D of A satisfies $|N(D)| \geq |D|$. ."

Example (Problem Set 8.3 Question 5 from course notes): Find a maximum matching and a minimum cover in the graph below by applying the algorithm of finding the maximum matching.



Solution: We begin with M={{3,b}, {4,d}, {5,c}}

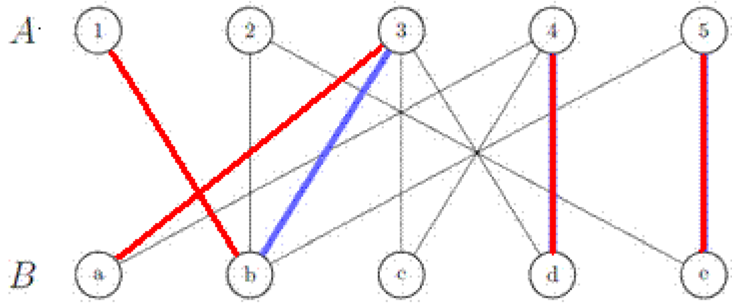
1. Set $\widehat{X} = \{1,2\}$, and $\widehat{Y} = \emptyset$.

2. Set $\widehat{Y} = \{b,e\}$, pr(b)=1, pr(e)=2.

5. Set $\widehat{X} = \{1,2,3,5\}$ and pr(3)=b, pr(5)=e

2. Set $\widehat{Y} = \{a,b,c,d,e\}$ and pr(a)=3, pr(c)=3, pr(d)=3.

4. Note that a is unsaturated. Then an augmenting path is 1b3a. So 1b and 3a would be larger than 3b. Then replace the current matching with M'={{1,b}, {3,a}, {4,d}, {5,c}}. M' is shown in red.



1. Set $\widehat{X} = \{2\}$, and $\widehat{Y} = \emptyset$.

2. Set $\widehat{Y} = \{b,e\}$, pr(b)=2, pr(e)=2.

5. Set $\widehat{X} = \{1,2,5\}$, pr(1)=b, pr(5)=e.

2. Set $\widehat{Y} = \{b,e\}$.

3. Maximum matching is M'={{1,b}, {3,a}, {4,d}, {5,c}} and minimum cover is C={b,e,3,4}