MATH 239 Winter 2013
Assignment 8

Due Friday, March 22, 10am

TOTAL: 50 POINTS

For a graph G, let p be the number of vertices and q be the number of edges.

1. Let G be a connected graph that has a single cycle of length n. Prove that G has exactly n distinct spanning trees.

2. Let G be a connected graph with spanning tree T. Pick a vertex w of T. For any vertex v of G, let $d(v)$ be the length of the unique path in T from v to w. Suppose that for any edge $e = uv$ of G that is not in T, $d(u) - d(v)$ is an odd number. Show that G is bipartite.

3. This problem generalizes Platonic graphs. Let G be a connected planar graph where every vertex has degree at least 3. We say that G is *special* if there is a positive integer k, positive distinct integers $d_1^*, \dots, d_k^* \geq 3$, and positive integers $m_1, \dots, m_k$ such that every vertex v is of degree $m_1 + \cdots + m_k$ and that faces containing v are exactly m_1 faces of degree d_1^*, m_2 faces of degree d_2^*, and so on, up to m_k faces of degree d_k^*. The Platonic graphs are the cases where $k = 1$: a cube has $k = 1$, $m_1 = 3$, $d_1^* = 4$; a dodecahedron has $k = 1$, $m_1 = 3$, $d_1^* = 5$. The edge graph of a soccer ball has $k = 2$, $m_1 = 2, d_1^* = 6, m_2 = 1, d_2^* = 5$. In other words at every vertex of a soccer ball, there are two hexagons and one pentagon.

   (a) Show that
   $$(m_1 + m_2 + \cdots + m_k)p = 2q.$$

   (b) Show that the number of faces of degree d_i^* is
   $$s_i = \frac{m_i p}{d_i^*}.$$

   (c) Prove by using Euler's formula that
   $$\frac{1}{m_1 + m_2 + \cdots + m_k}\left(1 + \frac{m_1}{d_1^*} + \frac{m_2}{d_2^*} + \cdots + \frac{m_k}{d_k^*}\right) = \frac{1}{2} + \frac{1}{q}.$$

4. Let G be a connected planar graph in which every face has degree exactly 3 and every vertex has degree at least 4. Prove that G has at least 12 edges.

5. Show for $n \geq 3$ the following graphs are planar by describing a planar embedding. Please give an example of your explanation for $n = 4$.

   (a) Let G be the graph with $2n$ vertices labelled $v_1, v_2, \dots, v_n$ and $w_1, w_2, \dots, w_n$ with $3n$ edges of the form $v_i v_{i+1}$, $w_i w_{i+1}$, and $v_i w_i$ for $i = 1, \dots, n$. Note: we use the convention that $v_{n+1} = v_1$.

   (b) Let G be the graph with $2n$ vertices labelled $v_1, v_2, \dots, v_n$ and $w_1, w_2, \dots, w_n$ such that there are $4n$ edges of the form: $v_i v_{i+1}$, $w_i w_{i+1}$, $v_i w_i$, and $v_i w_{i+1}$ for $i = 1, \dots, n$.