COMBINATORICS & OPTIMIZATION

University of Waterloo

Introduction to Combinatorics

Lecture 6

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Recap of Problem 1.6.2

Problem 1.6.2: Let k,n be fixed non-negative integers. How many solutions are there to

$t_1 + \ldots + t_n = k$, where $t_1, \ldots, t_n \in \{0,1\}$?

Answer: The number of solutions to $t_1 + \ldots + t_n = k$ is

$$\left[x^k\right]\Phi_S(x) = \binom{n}{k}$$

Three equivalent problems

Let k and n be fixed non-negative integers.

- How many k-subsets of a set of size n are there?
- How many 0/1 strings of length n containing k ones and n-k zeroes are there?
- How many solutions are there to

$$t_1 + t_2 + t_3 + \cdots + t_n = k$$

where for each j

$$t_j \in \{0,1\}$$

?

Three equivalent problems

(e.g. with n=4, k=2)

$$\{1,2\},\ \{1,3\},\ \{1,4\},\ \{2,3\},\ \{2,4\},\ \{3,4\},$$

$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$

$$1100, 1010, 1001, 0110, 0101, 0011$$

$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$

$$\left\{\begin{matrix} t_1 = 1, \\ t_2 = 1, \\ t_3 = 0, \\ t_4 = 0 \end{matrix}\right\} \quad \left\{\begin{matrix} t_1 = 1, \\ t_2 = 0, \\ t_3 = 1, \\ t_4 = 0 \end{matrix}\right\} \quad \left\{\begin{matrix} t_1 = 1, \\ t_2 = 0, \\ t_3 = 0, \\ t_4 = 1 \end{matrix}\right\} \quad \left\{\begin{matrix} t_1 = 0, \\ t_2 = 1, \\ t_3 = 1, \\ t_4 = 0 \end{matrix}\right\} \quad \left\{\begin{matrix} t_1 = 0, \\ t_2 = 1, \\ t_3 = 0, \\ t_4 = 1 \end{matrix}\right\} \quad \left\{\begin{matrix} t_1 = 0, \\ t_2 = 0, \\ t_3 = 1, \\ t_4 = 1 \end{matrix}\right\}$$

The Product Lemma Generalized

Let $A_1, A_2, \cdots, A_k$ be sets of objects with weight functions $\omega_1, \omega_2, \cdots, \omega_k$ respectively. Let $S = A_1 \times A_2 \times \cdots \times A_k$ and suppose $\omega(\sigma) = \omega_1(a_1) + \omega_2(a_2) + \cdots + \omega_k(a_k)$ for each

$$\sigma = (a_1, a_2, \cdots, a_k) \in A_1 \times A_2 \times \cdots \times A_k$$

Then:

$$\Phi_S^{\omega}(x) = \Phi_{A_1}^{\omega_1}(x) \times \Phi_{A_2}^{\omega_2}(x) \times \cdots \times \Phi_{A_k}^{\omega_k}(x)$$

The Product Lemma Generalized

e.g. suppose $A = A_1 = A_2 = \cdots = A_k$ and $\omega_1 = \omega_2 = \cdots = \omega_k$

Then:

$$\Phi_S^{\omega}(x) = \Phi_A^{\omega_1}(x) \times \Phi_A^{\omega_1}(x) \times \cdots \times \Phi_A^{\omega_1}(x)$$

$$= \left(\Phi_A^{\omega_1}(x)\right)^k$$

Recap of Problem 1.6.4

Problem 1.6.4: Let S be the set of all k-tuples $(a_1, a_2, \cdots, a_k)$ where each $a_1 \in Z_{\geq 0}$. Define the weight of a k-tuple $\sigma = (a_1, a_2, \cdots, a_k)$ by $\omega(\sigma) = a_1 + a_2 + \cdots + a_k$ (implicitly $\omega_j(a_j) = a_j$ for all integers j).

By the Product Lemma, we showed that

$$\Phi_S(x) = (\Phi_A(x))^k = \left(\frac{1}{1-x}\right)^k = (1-x)^{-k}$$

Proof of Theorem 1.6.5

In Theorem 1.6.4, we showed the following:

Let k and n be fixed non-negative integers. The number of solutions to the equation $t_1 + t_2 + t_3 + \cdots + t_k = n$ where $t_1, t_2, \cdots, t_k \in Z_{\geq 0}$ is

$$\left[x^j\right]\left(1-x\right)^{-k}$$

If we can show that the number of solutions is $\binom{k+j-1}{k-1}$ then this proves THM 1.6.5.

A combinatorial proof

Let k and n be fixed non-negative integers. How many solutions are there to the equation

$t_1 + t_2 + t_3 + \cdots + t_k = n$ where $t_1, t_2, \cdots, t_k \in Z_{\geq 0}$?

Equivalently:

Given n indistinguishable balls to put into k rooms, Room1, Room2, …,Roomk. How many different ways can you distribute the balls?

A direct combinatorial proof of the same result

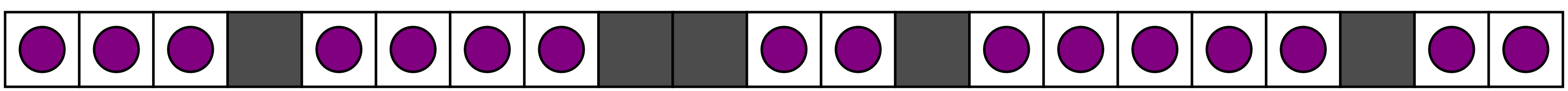
Given n indistinguishable balls to put into k rooms, Room1, Room2, …,Roomk. How many different ways can you distribute the balls? i.e. we put t_1 balls into Room1, t_2 balls into Room2, and so on; how many solutions are there to

$t_1 + t_2 + t_3 + \cdots + t_k = n$ where each t_j is a non-negative integer?

Solution

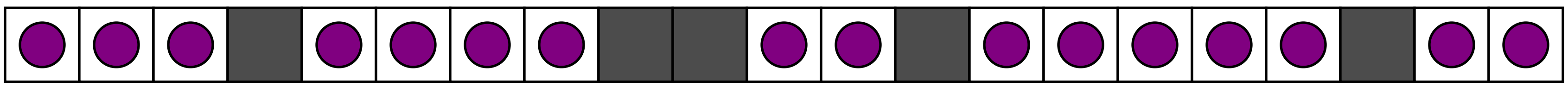
- Line up n+k-1 boxes. Fill k-1 of the boxes with cement; these are the “dividers”, Divider1, Divider2, …, Divider(k-1). (e.g. with n=16, k=6)

- Put the balls in the remaining n boxes.

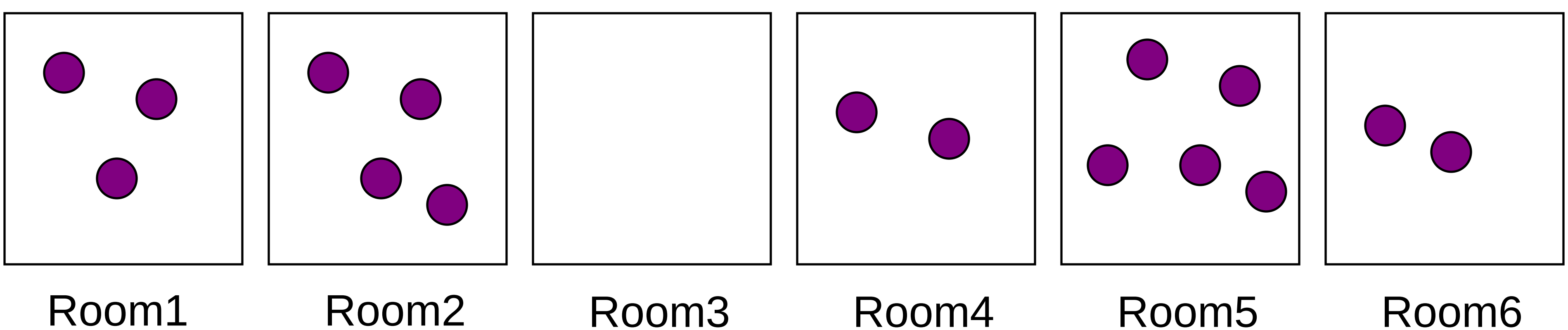


Solution

- (e.g. with n=16, k=6)



- Put the balls preceding Divider1 into Room1, put the balls between Divider1 and Divider2 into Room2, and so on, and finally put the balls after Divider(k-1) into Room(k).



Solution

- There is a one-to-one correspondence between distributions of the balls, and (k-1)-subsets of the n+k-1 boxes.
- Since there are $\binom{n+k-1}{k-1}$ (k-1)-subsets of the n+k-1 boxes, there must also be $\binom{n+k-1}{k-1}$ ways of distributing the balls, and therefore there are $\binom{n+k-1}{k-1}$ solutions to $t_1 + t_2 + t_3 + \cdots + t_k = n$ for $t_1, t_2, \cdots, t_k \in Z_{\geq 0}$
- Therefore $\left[x^j\right]\left(1-x\right)^{-k} = \binom{n+k-1}{k-1}$

"Compositions"

Definition 2.1.1: A *composition* of n with k parts is an ordered list $(c_1, c_2, \cdots, c_k)$ of positive integers such that $c_1 + c_2 + \cdots + c_k = n$

The positive integers $c_1, c_2, \cdots$ are called the *parts* of the composition.

We also allow a single ("empty") composition with 0 parts, which is a composition of 0.

(one way of thinking about this is that the value of a composition is equal to 0 plus the value of all the parts of the composition; so if the composition has no parts, the sum is just 0).

Problem 2.1.2

How many compositions of n are there with k parts, for $n \geq k \geq 1$?

We can mathematically reformulate this problem as follows:

Let $S = Z_{\geq 1} \times Z_{\geq 1} \times \cdots \times Z_{\geq 1} = \left(Z_{\geq 1}\right)^k$.

Let $\omega((t_1, t_2, \ldots, t_k)) = t_1 + t_2 + \cdots + t_k$.

How many elements of S have weight n?

Problem 2.1.2 solution

We compute

$$\Phi_{Z_{\geq 1}}(x) = x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

By the Product Lemma, we obtain

$$\Phi_S(x) = \left(\Phi_{Z_{\geq 1}}(x)\right)^k = x^k (1-x)^{-k}$$

So for any integer n, the number of compositions of n into k parts （ $n \geq k \geq 1$ ）is:

$$[x^n]\Phi_S(x) = [x^n] x^k (1-x)^{-k}$$

Problem 2.1.2 solution

So for any integer n, the number of compositions of n into k parts $(n \geq k \geq 1)$ is:

$$\left[x^n\right]\Phi_S(x) = \left[x^n\right]x^k(1-x)^{-k}$$

Aside

Note that for $(n \geq m \geq 1)$

$$\left[x^n\right]x^m F(x)$$

$$= \left[x^n\right]x^m\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-m}x^{n-m} + a_{n-m+1}x^{n-m+1} + \cdots\right)$$

$$= \left[x^{n-m}\right]\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-m}x^{n-m} + a_{n-m+1}x^{n-m+1} + \cdots\right)$$

$$= \left[x^{n-m}\right]F(x)$$

(What if n<m??)

Problem 2.1.2 solution

So for any integer n, the number of compositions of n into k parts $(n \geq k \geq 1)$ is:

$$[x^n]\Phi_S(x) = [x^n]x^k(1-x)^{-k}$$

$$= [x^{n-k}](1-x)^{-k}$$

$$= \binom{k+(n-k)-1}{n-k} = \binom{n-1}{n-k} = \binom{n-1}{k-1}$$

More general formulation of Problem 2.1.2

Let k and n be fixed non-negative integers. How many solutions are there to the equation

$t_1 + t_2 + t_3 + \cdots + t_k = n$ where

$t_1 \in A_1, t_2 \in A_2, \ldots, t_k \in A_k$?

In Problem 2.1.2, we had $A_1 = A_2 = \cdots = A_k = Z_{\geq 0}$

In general, we let $S = A_1 \times A_2 \times \cdots \times A_k$.

Let $\omega\left(\left(t_1, t_2, \ldots, t_k\right)\right) = t_1 + t_2 + \cdots + t_k$.

How many elements of S have weight n?

General solution

We compute

$$\Phi_{A_1}(x), \Phi_{A_2}(x), \cdots, \Phi_{A_k}(x)$$

By the Product Lemma, we obtain

$$\Phi_S(x) = \Phi_{A_1}(x)\Phi_{A_2}(x)\cdots\Phi_{A_k}(x)$$

So for any integer n, the number of solutions to
$t_1 + t_2 + t_3 + \cdots + t_k = n$ with $t_j \in A_j$ is

$$[x^n]\Phi_S(x)$$

Problem 2.1.3

How many compositions of n are there with k parts, where each part is odd, for $n \geq k \geq 1$?

We can mathematically reformulate this problem as follows:

Let $S = Z_{odd} \times Z_{odd} \times \cdots \times Z_{odd} = \left(Z_{odd}\right)^k$.

Let $\omega\left(\left(t_1, t_2, \ldots, t_k\right)\right) = t_1 + t_2 + \cdots + t_k$.

How many elements of S have weight n?

Problem 2.1.3 solution

$$Z_{odd} = \{1,3,5,7,\cdots\}$$

We compute

$$\Phi_{Z_{odd}}(x) = x + x^3 + x^5 + \cdots = \frac{x}{1-x^2}$$

By the Product Lemma, we obtain

$$\Phi_S(x) = \left(\Phi_{Z_{odd}}(x)\right)^k = x^k\left(1-x^2\right)^{-k}$$

So for any integer n, the number of compositions of n into k parts （ $n \geq k \geq 1$ ）is:

$$\left[x^n\right]\Phi_S(x) = \left[x^n\right]x^k\left(1-x^2\right)^{-k}$$

Problem 2.1.3 solution

So for any integer n, the number of compositions of n into k odd parts $(n \geq k \geq 1)$ is:

$$[x^n]\Phi_S(x) = [x^n]x^k(1-x^2)^{-k}$$

$$= [x^{n-k}](1-x^2)^{-k}$$

$$= [x^{n-k}]\sum_{i=0}^{\infty}\binom{k+i-1}{i}x^{2i}$$

Problem 2.1.3 solution

$$= \left[x^{n-k}\right] \sum_{i=0}^{\infty} \binom{k+i-1}{i} x^{2i}$$

(need 2i = n-k)

$$= \begin{cases} 0 & \text{if n-k is odd} \\ \dbinom{k+\frac{n-k}{2}-1}{\frac{n-k}{2}} & \text{if n-k is even} \end{cases}$$

$$= \binom{\frac{n+k-2}{2}}{\frac{n-k}{2}}$$

Problem 2.1.4

How many compositions of n are there with k parts, where each part at most 5, for $n \geq k \geq 1$?

We can mathematically reformulate this problem as follows:

Let $S = N_5 \times N_5 \times \cdots \times N_5 = N_5{}^k$ $\qquad N_5 = \{1,2,3,4,5\}$

Let $\omega((t_1, t_2, \ldots, t_k)) = t_1 + t_2 + \cdots + t_k$.

How many elements of S have weight n?

Problem 2.1.5 solution

We compute

$$\Phi_{N_5}(x) = x + x^2 + x^3 + x^4 + x^5$$

$$= x\left(1 + x + x^2 + x^3 + x^4\right) = \frac{x\left(1 - x^5\right)}{1 - x}$$

By the Product Lemma, we obtain

$$\Phi_S(x) = \left(\Phi_{N_5}(x)\right)^k = x^k\left(1 - x^5\right)^k\left(1 - x\right)^{-k}$$

Problem 2.1.5 solution

So for any integer n, the number of compositions of n into k parts of size at most 5 $(n \geq k \geq 1)$ is:

$$\left[x^n\right]\Phi_S\left(x\right) = [x^{n-k}](1-x^5)^k(1-x)^{-k}$$

$$= [x^{n-k}]\left(\sum_{i=0}^{\infty}\binom{k}{i}(-x^5)^i\right)\left(\sum_{j=0}^{\infty}\binom{k+j-1}{j}x^j\right)$$

$$= [x^{n-k}]\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^i\binom{k+j-1}{j}\binom{k}{i}x^{5i+j}\right)$$

Problem 2.1.5 solution

So for any integer n, the number of compositions of n into k parts of size at most 5 $(n \geq k \geq 1)$ is:

$$\left[x^n\right]\Phi_S(x) = \qquad \cdots \qquad =$$

$$= \sum_{i=0}^{\left\lfloor \frac{n-k}{5} \right\rfloor} (-1)^i \binom{k}{i} \binom{n-5i-1}{n-k-5i}$$

Problem 2.1.5

We can also count numbers of compositions without specifying the number of parts.

Let n be a non-negative integer. How many compositions of n are there?

We can mathematically reformulate this problem as follows:

Let $S = \{()\} \bigcup Z_{\geq 1} \bigcup Z_{\geq 1} \times Z_{\geq 1} \bigcup \cdots \bigcup (Z_{\geq 1})^k \bigcup \cdots$

(empty composition)

$$= \bigcup_{k=0}^{\infty} (Z_{\geq 1})^k$$

The Sum Lemma

THM1.6.1: Let (A,B) be a partition of a set S (i.e. $S = A \dot{\bigcup} B$, the disjoint union of A and B).

Then

$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$$

Generalized Sum Lemma

Let the set S be a disjoint union of the sets

$$A_0, A_1, A_2, A_3, \cdots$$

i.e. $S = \bigcup_{k=0}^{\infty} A_k$

Then

$$\Phi_S(x) = \sum_{k=0}^{\infty} \Phi_{A_k}(x)$$

Special Case

Let the set S be a disjoint union of the sets

$$A_0, A_1, A_2, A_3, \cdots$$

where $A_0 = \{()\} = A^0, A_1 = A, A_2 = A \times A, A_k = A^k$

i.e. $S = \bigcup_{k=0}^{\infty} A^k$

(product lemma)

Then

$$\Phi_S(x) = \sum_{k=0}^{\infty} \Phi_{A^k}(x) = \sum_{k=0}^{\infty} \left(\Phi_A(x)\right)^k$$

$$= \frac{1}{1 - \Phi_A(x)}$$

Problem 2.1.8

Let $\omega(()) = 0$.

$$\omega((t_1, t_2, \ldots, t_k)) = t_1 + t_2 + \cdots + t_k, \quad \forall k \geq 1$$

How many elements of S have weight n?

$$[x^n]\Phi_S(x)$$

Problem 2.1.8 solution

We compute

$$\Phi_{Z_{\geq 1}}(x) = x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

and (by the product lemma)

$$\Phi_{(Z_{\geq 1})^k}(x) = \left(\Phi_{Z_{\geq 1}}(x)\right)^k = x^k (1-x)^{-k}$$

So $\Phi_S(x) = \sum_{k=0}^{\infty} \Phi_{(Z_{\geq 1})^k}(x) = \sum_{k=0}^{\infty} \left(x(1-x)^{-1}\right)^k$

$$= \frac{1}{1 - x(1-x)^{-1}}$$

Problem 2.1.8 solution

So $$\Phi_S(x) = \frac{1}{\left(1 - x(1-x)^{-1}\right)} \frac{(1-x)}{(1-x)}$$

$$= \frac{1-x}{1-2x}$$

$$= (1-x)\left(1 + 2x + 4x^2 + \cdots + 2^j x^j + 2^{j+1} x^{j+1} + \cdots\right)$$

$$= 1 + 2x + 4x^2 + \cdots + 2^j x^j + 2^{j+1} x^{j+1} + \cdots$$

$$- x - 2x^2 - \cdots - 2^{j-1} x^j - 2^j x^{j+1} - \cdots$$

Problem 2.1.8 solution

$$\Phi_S(x) = 1 + 2x + 4x^2 + \cdots + 2^j x^j + 2^{j+1} x^{j+1} + \cdots$$
$$- x - 2x^2 - \cdots - 2^{j-1} x^j - 2^j x^{j+1} - \cdots$$

$$= 1 + x + 2x^2 + \cdots + 2^{j-1} x^j + 2^j x^{j+1} + \cdots$$

So the number of elements in S with weight n is:

$$[x^n]\Phi_S(x) = \begin{cases} 1 & \text{if n=0} \\ 2^{n-1} & \text{if n≥1} \end{cases}$$

Another example

Let n be a non-negative integer. How many compositions of n are there in which all parts are at least 3?

We can mathematically reformulate this problem as follows:

Let $S = \{()\} \bigcup Z_{\geq 3} \bigcup Z_{\geq 3} \times Z_{\geq 3} \bigcup \cdots \bigcup (Z_{\geq 3})^k \bigcup \cdots$

$$= \bigcup_{k=0}^{\infty} (Z_{\geq 3})^k$$

(empty composition)

Let $\omega(()) = 0$.

$$\omega((t_1, t_2, \ldots, t_k)) = t_1 + t_2 + \cdots + t_k, \quad \forall k \geq 1$$

How many elements of S have weight n?

$$[x^n]\Phi_S(x)$$

We can compute

$$\Phi_{Z_{\geq 3}}(x) = x^3 + x^4 + x^5 + \cdots = \frac{x^3}{1-x}$$

and (by the product lemma)

$$\Phi_{(Z_{\geq 3})^k}(x) = \left(\Phi_{Z_{\geq 3}}(x)\right)^k = \left(x^3(1-x)^{-1}\right)^k$$

So $\Phi_S(x) = \sum_{k=0}^{\infty} \Phi_{(Z_{\geq 3})^k}(x) = \sum_{k=0}^{\infty} \left(x^3(1-x)^{-1}\right)^k$

$$= \frac{1}{1 - x^3(1-x)^{-1}}$$

So $$\Phi_S(x) = \frac{1}{\left(1 - x^3(1-x)^{-1}\right)} \frac{(1-x)}{(1-x)}$$

$$= \frac{1-x}{1-x-x^3}$$

So the number of elements in S with weight n is:

$$[x^n]\frac{1-x}{1-x-x^3}$$