Tutorial 10 - April 3, 2013

April 2, 2013

1. Show that any tree T has at most one perfect matching.

   Solution. Induction on $|V(T)|$. If $|V(T)| = 1$ or 2, then T is either an isolated vertex or an edge, hence in each case T has at most one perfect matching. Let $|V(T)| > 2$ and suppose that any tree T' with less than $|V(T)|$ vertices has at most one perfect matching. If T has no perfect matching we are done. Suppose that T has a perfect matching, let v be a leaf of T. Observe that any perfect matching M must have the unique edge e incident to v. Let M_1, M_2 be perfect matchings in T. Let $T' = T - e$, observe that $M_1 \setminus \{e\}, M_2 \setminus \{e\}$ are perfect matchings in T', therefore by induction hypothesis $M_1 \setminus \{e\} = M_2 \setminus \{e\}$ and hence $M_1 = M_2$. Then T has exactly one perfect matching

2. How many perfect matchings are there in K_n.

   Solution. Observe that if n is odd, there are no perfect matchings. Assume from now that $n > 0$ is even. Let P_n be the number of perfect matchings in K_n. We will show by induction that $P_n = (n-1)(n-3)(n-5)\cdots(5)(3)$. If $n = 2$ then $P_n = 1$. Now let x be a fixed vertex of K_n, observe that any perfect matching in K_n can be obtained by selecting vertex y in $V(K_n) - x$ that is matched with x and choosing a perfect matching in $K_n - xy \cong K_n$. Therefore we obtained the recursive formula $P_n = (n-1)P_{n-2} = (n-1)(n-3)\cdots(5)(3)$ because $P_{n-2} = (n-3)\cdots(5)(3)$ by induction hypothesis.

3. (a) Show that if two opposite corner squares of a chessboard are removed, then the resulting board cannot be covered with 31 dominoes.

   Solution. We may assume that the removed squares are white. Then the chessboard has 30 white squares, and 32 black squares. Construct a bipartite graph G with white vertices corresponding to white squares, black vertices corresponding to black squares, and an edge between a black and white vertex exactly when the corresponding squares share an edge. Then covering each square of the chessboard with 31 dominoes is equivalent to finding a perfect matching in G.

   Notice that every edge in G has a white vertex as an endpoint (since G is bipartite). Then the 30 white vertices are a vertex cover of

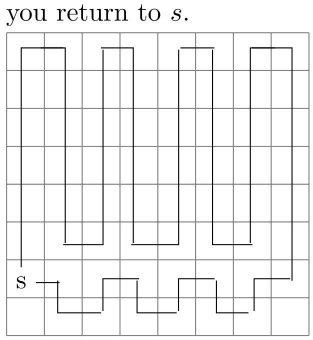
G. Since every matching of G has smaller size than any cover of G, this means G has no matching with 31 edges. So G does not have a perfect matching, and the chessboard cannot be covered with 31 dominoes.

(b) Show that if two adjacent corner squares of a chessboard are removed, then the resulting board can be covered with 31 dominoes.

Solution. (Note – we could just give a covering by dominoes, but instead we'll use results for graphs.)

Consider the chessboard with two adjacent corner squares removed (the bottom left and the bottom right, say). Let s be the square above the bottom left square. We will find a way to "walk around" the chessboard, starting and ending at s, moving between squares that share an edge, without using any square more than once. We will call this path P.

Starting at s, proceed up to the top of the chessboard. Go right one square, then go down 5 squares. Go right one, and up to the top. Repeat until you reach the top right corner. Then go down 6 squares. Move left one, down one, left one, up one; repeat this pattern until you return to s.



We will call this cycle C. In graph G (from part (a)), C is a hamiltonian cycle (visits each vertex of G exactly once. Furthermore, since there are an even number of vertices in G, C has an even number of edges. Let M be a set of alternating edges of C. Since C has an even number of edges, no two edges of M share an endpoint. Also, every vertex of G is the endpoint of some edge of C. It follows that M is a perfect matching of G. This gives the covering by dominoes that we require.

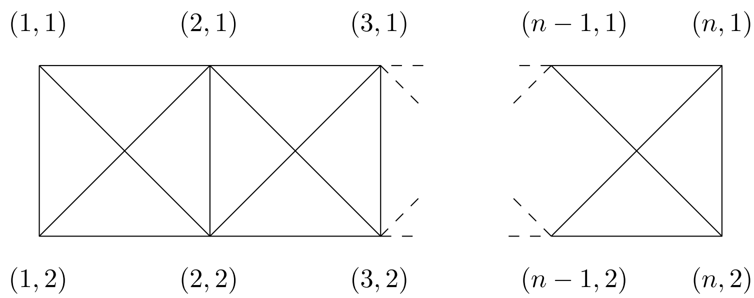
4. Show that it is not always true that there exists a matching M and a cover C of the same size.

   Solution: Consider the triangle K_3. The largest matching is of size 1, but we need at least two vertices in a cover.

5. Show that G is bridgeless if and only if G^* is loopless.

   Solution: Note that a bridge e is an edge whose deletion separates the graph into components C_1 and C_2. But this means that the two faces incident to e must be the same, as otherwise there would have to be a path separating this face connecting C_1 and C_2. But this means that the edge corresponding to e in the dual is a loop. Now consider a loop in G^*. It corresponds to an edge $e = uv$ in G whose two incident faces are the same face f. But this means there cannot be another path from u to v, as that would separate f. So e is a bridge.

6. Let L_n be the graph with vertex set $V = [n] \times [2]$ and edges between a pair (u_1, u_2) and (v_1, v_2) if and only if either $u_2 = v_2$ and $u_1 \neq v_1$, or $|u_1 - v_1| = 1$. Determine the number c_n of perfect matchings in L_n.



Solution: Consider the options for $(1,1)$ to be matched up with. If it is $(1,2)$, then this matching can be completed to a matching of L_n by any other matching remaining graph, which is isomorphic to L_{n-1}, so there are c_{n-1} such matchings. If we match up $(1,1)$ with $(2,1)$ or $(2,2)$, then $(1,2)$ must be matched up with $(2,2)$ or $(2,1)$ respectively. In either case, the remaining graph is isomorphic to L_{n-2}, and thus there are $2c_{n-2}$ such matchings. Thus we get $c_n = c_{n-1} + 2c_{n-2}$. For $n = 1$ and $n = 2$ we get 1 and 3 matchings respectively, therefore the starting conditions are $c_1 = 1$, $c_2 = 3$.

The characteristic polynomial is $x^2 - x - 2 = (x+1)(x-2)$. So by Theorem 3.2.2 the solution is of the form $c_n = A(-1)^n + B2^n$. Use initial conditions to set up a system of linear equations:

$$\begin{aligned} 1 &= -A + 2B \\ 3 &= A + 4B \end{aligned}$$

Solving gives $A = \frac{1}{3}$, $B = \frac{2}{3}$. So (substituting these values into our formula above) $c_n = \frac{1}{3}(-1)^n + \frac{2}{3}2^n$.