

Introduction to Combinatorics

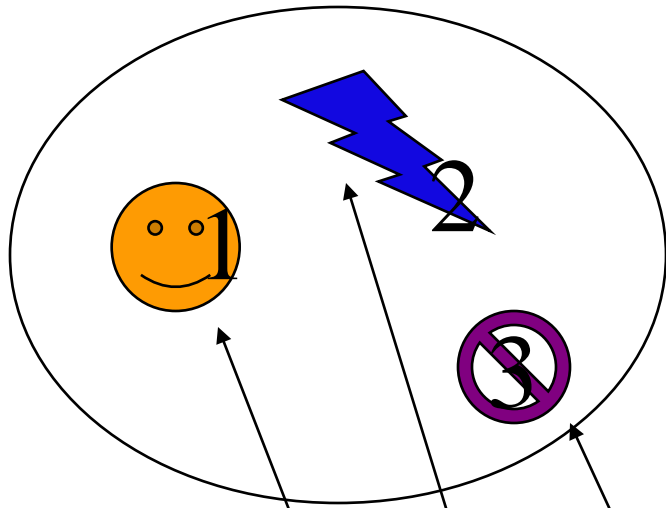
Lecture 5

<http://info.iqc.ca/mmosca/2014math239>

Michele Mosca

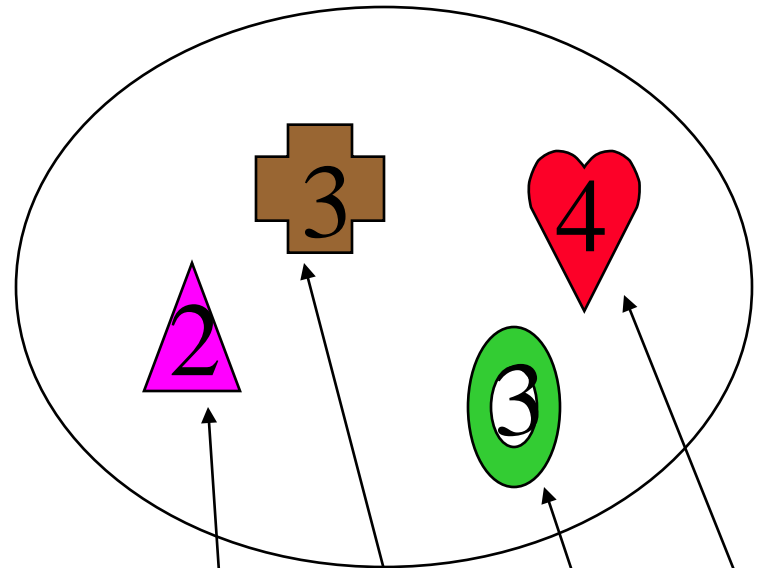
Consider the generating functions of A and B

A



$$\Phi_A(x) = x + x^2 + x^3$$

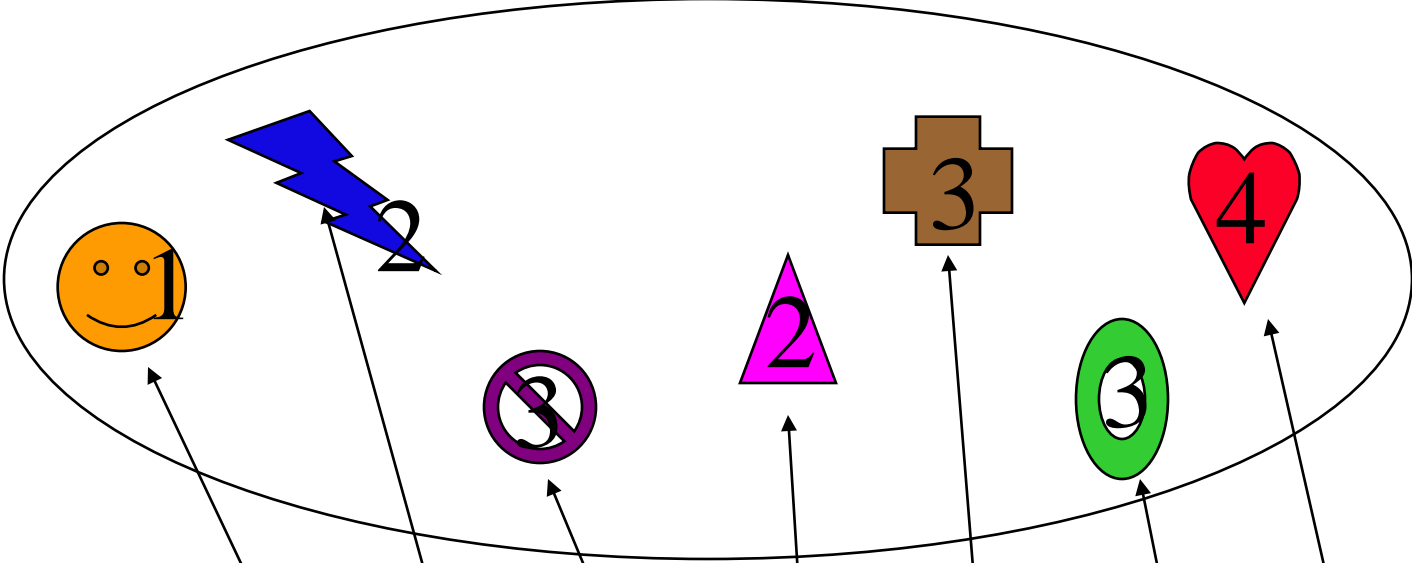
B



$$\Phi_B(x) = x^2 + x^3 + x^3 + x^4$$

Consider the generating function of

$$S = A \cup B$$



The diagram shows a large oval representing the set S . Inside the oval are eight elements, each with a weight indicated by a number: an orange smiley face with '1', a blue lightning bolt with '2', a purple circle with a crossed-out '3', a pink triangle with '2', a brown cross with '3', a green oval with '3', a red heart with '4', and a purple circle with a crossed-out '3'. Arrows point from each element to its corresponding term in the generating function below.

$$\begin{aligned}\Phi_S(x) &= x + x^2 + x^3 + x^2 + x^3 + x^3 + x^4 \\ &= (x + x^2 + x^3) + (x^2 + x^3 + x^3 + x^4) \\ &= \Phi_A(x) + \Phi_B(x)\end{aligned}$$

The Sum Lemma

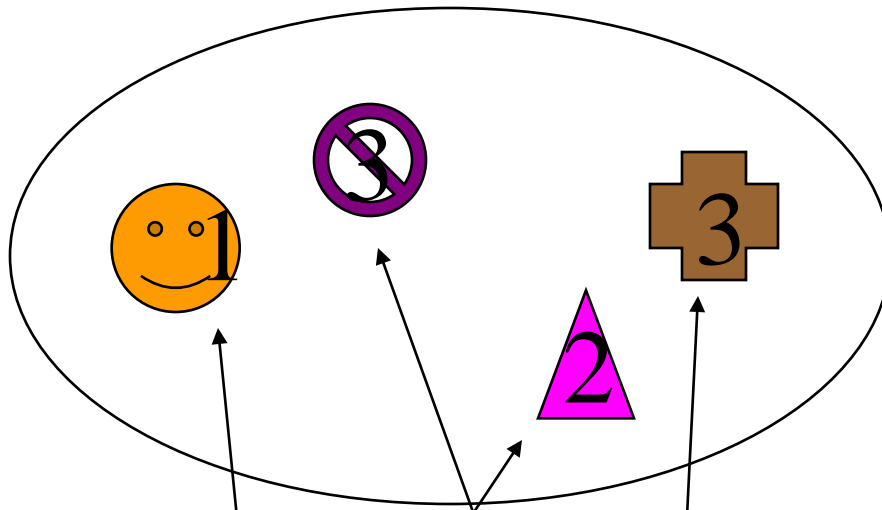
THM1.6.1: Let (A,B) be a partition of a set S (i.e. $S = A \dot{\cup} B$, the disjoint union of A and B).

Then:

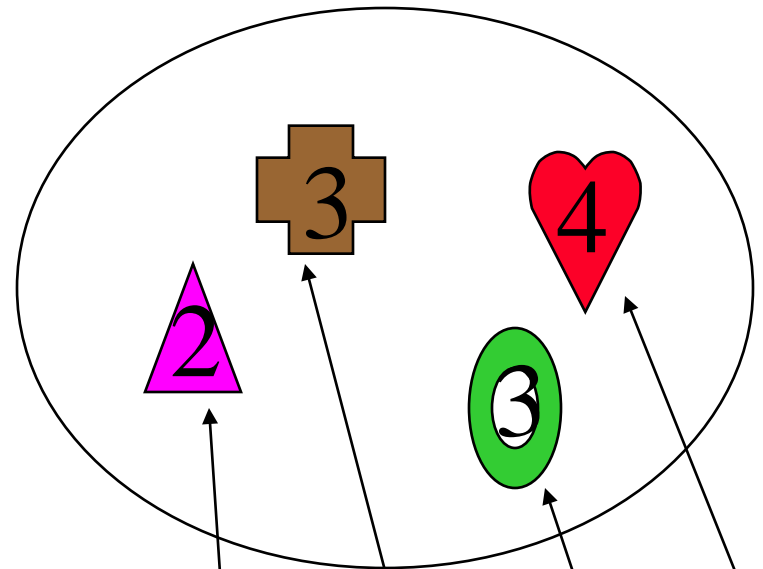
$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$$

Consider the generating functions of A and B

A



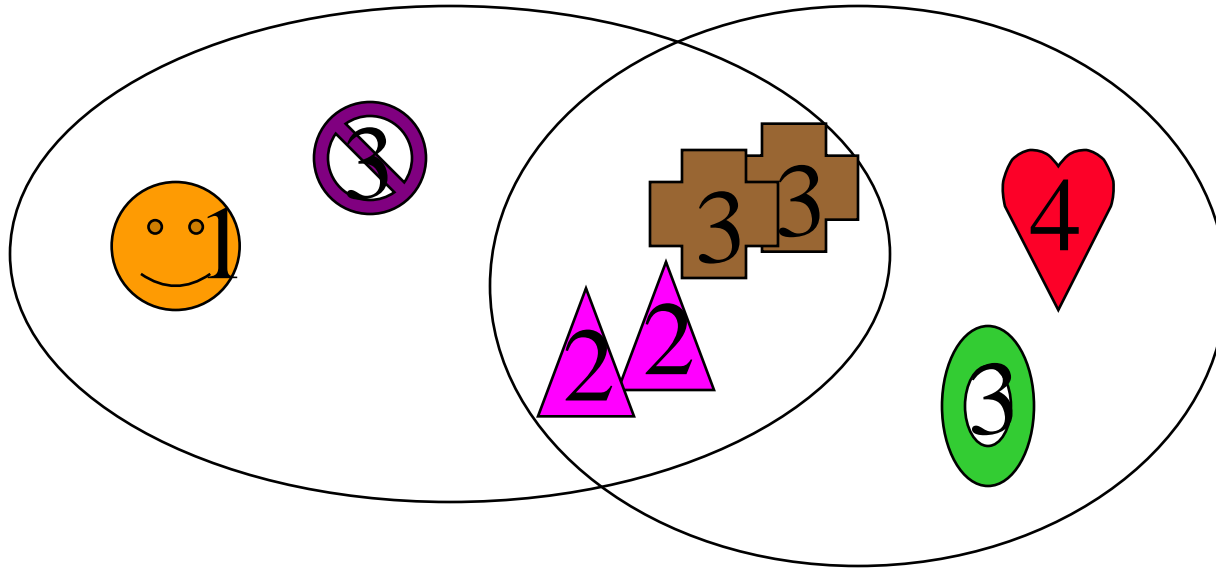
B



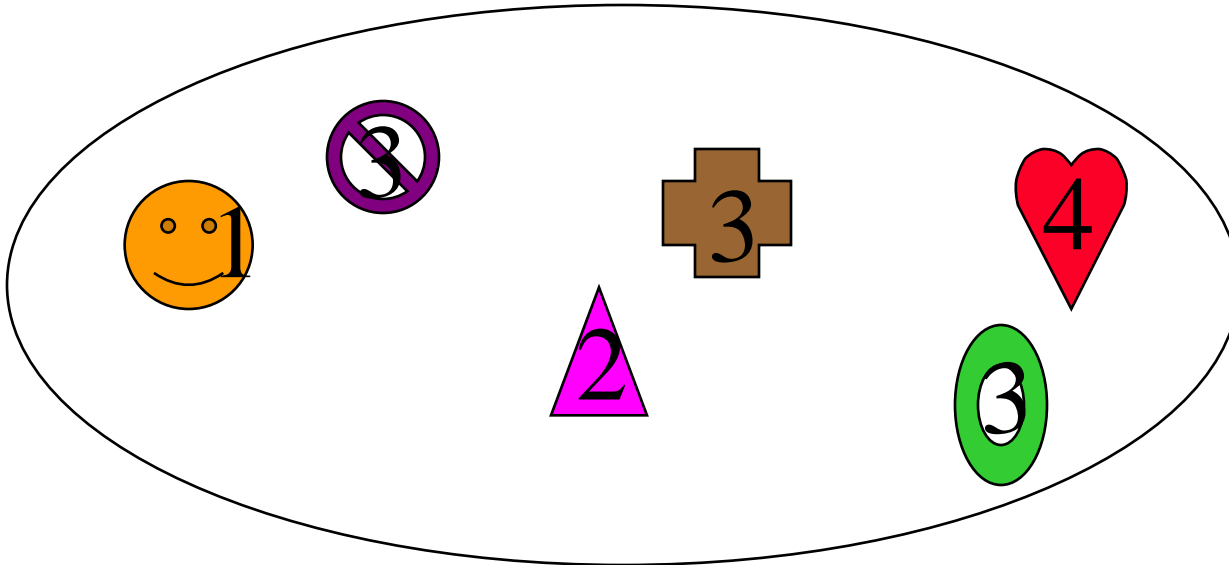
$$\Phi_A(x) = x + x^2 + x^3 + x^3$$

$$\Phi_B(x) = x^2 + x^3 + x^3 + x^4$$

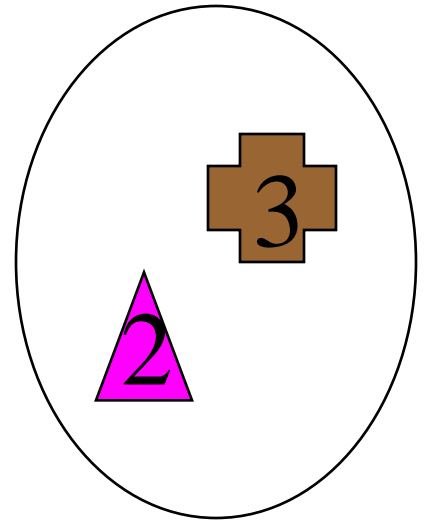
Consider $S = A \cup B$ and $T = A \cap B$



Consider $S = A \cup B$ and $T = A \cap B$



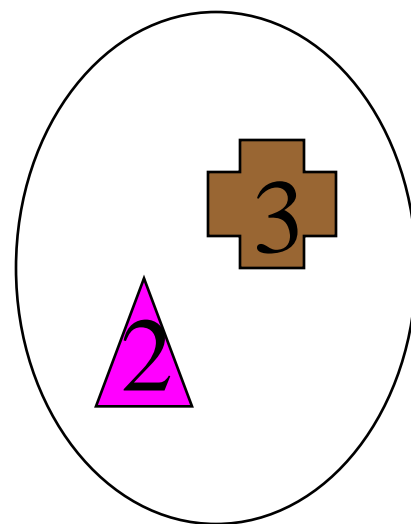
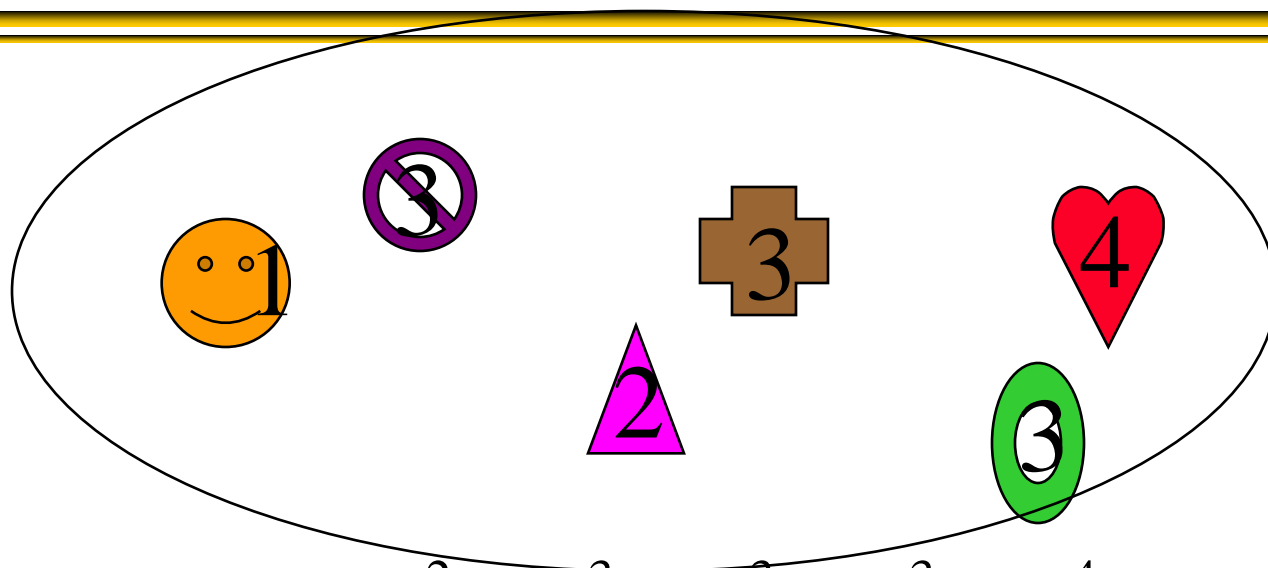
$$S = A \cup B$$



$$T = A \cap B$$

$$S = A \cup B$$

$$T = A \cap B$$



$$\Phi_S(x) = x + x^2 + x^3 + x^3 + x^3 + x^4$$

$$\Phi_T(x) = x^2 + x^3$$

$$\Phi_S(x) + \Phi_T(x)$$

$$= (x + x^2 + x^3 + x^3) + (x^2 + x^3 + x^3 + x^4)$$

$$= \Phi_A(x) + \Phi_B(x)$$

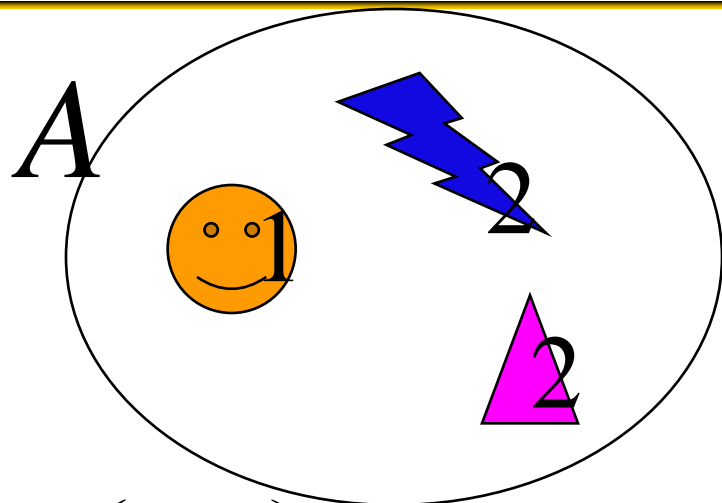
The Sum Lemma

THM1.6.1*: Let S be the union of A and B (i.e. $S = A \cup B$).

Then:

$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x) - \Phi_{A \cap B}(x)$$

Consider A, B



$$\omega_A \left(\text{smiley face} \right) = 1$$

$$\omega_A \left(\text{lightning bolt} \right) = 2$$

$$\omega_A \left(\text{pink triangle} \right) = 2$$



$$\omega_B \left(\text{University of Waterloo} \right) = 2$$

$$\omega_B \left(\text{Perimeter Institute} \right) = 3$$

$$\omega_B \left(\text{IQC} \right) = 3$$

$$\omega_B \left(\text{SJU} \right) = 4$$

Consider the *Cartesian product*

$$S = A \times B$$

$$A = \left\{ \text{smiley}_1, \text{triangle}_2, \text{lightning}_2 \right\}$$

$$S = \left\{ \begin{array}{l} \left(\text{smiley}_1, \text{University of Waterloo}_2 \right), \left(\text{triangle}_2, \text{University of Waterloo}_2 \right), \left(\text{lightning}_2, \text{University of Waterloo}_2 \right), \\ \left(\text{smiley}_1, \text{PI}_3 \right), \left(\text{triangle}_2, \text{PI}_3 \right), \left(\text{lightning}_2, \text{PI}_3 \right), \\ \left(\text{smiley}_1, \text{IQC}_3 \right), \left(\text{triangle}_2, \text{IQC}_3 \right), \left(\text{lightning}_2, \text{IQC}_3 \right), \\ \left(\text{smiley}_1, \text{SJU}_4 \right), \left(\text{triangle}_2, \text{SJU}_4 \right), \left(\text{lightning}_2, \text{SJU}_4 \right) \end{array} \right\}$$

$$B = \left\{ \text{University of Waterloo}_2, \text{PI}_3, \text{IQC}_3, \text{SJU}_4 \right\}$$

Generalizing the weight function

$$\omega_S \left(\left(\begin{array}{c} \text{Sun} \end{array}, \begin{array}{c} \text{Wing} \end{array} \right) \right) = \omega_A \left(\begin{array}{c} \text{Sun} \end{array} \right) + \omega_B \left(\begin{array}{c} \text{Wing} \end{array} \right)$$

e.g.

$$\omega \left(\left(\begin{array}{c} \text{Smiley}_1, \text{PI}_3 \end{array} \right) \right) = \omega_A \left(\begin{array}{c} \text{Smiley}_1 \end{array} \right) + \omega_B \left(\begin{array}{c} \text{PI}_3 \end{array} \right)$$

$$= 1 + 3 = 4$$

Consider $S = A \times B$

$$\Phi_A = x^1 + x^2 + x^2$$

$$\begin{aligned}\Phi_S(x) = & x^{1+2} + x^{2+2} + x^{2+2} \\ & + x^{1+3} + x^{2+3} + x^{2+3} \\ & + x^{1+3} + x^{2+3} + x^{2+3} \\ & + x^{1+4} + x^{2+4} + x^{2+4}\end{aligned}$$

$$\begin{aligned}\Phi_B(x) &= x^2 \\ &+ x^3 \\ &+ x^3 \\ &+ x^4\end{aligned}$$

Consider $S = A \times B$

$$\Phi_A = x^1 + x^2 + x^2$$

$$\Phi_S(x) = x^3 + x^4 + x^4$$

$$+ x^4 + x^5 + x^5$$

$$+ x^4 + x^5 + x^5$$

$$+ x^5 + x^6 + x^6$$

$$= \Phi_A(x) \cdot \Phi_B(x)$$

$$\Phi_B(x)$$

$$= x^2$$

$$+ x^3$$

$$+ x^3$$

$$+ x^4$$

The Product Lemma

THM1.6.2: Let A and B be sets of objects with weight functions ω_A and ω_B respectively. Let $S = A \times B$ and suppose $\omega(\sigma) = \omega_A(a) + \omega_B(b)$ for each $\sigma = (a, b) \in A \times B$

Then:

$$\Phi_S^{\omega}(x) = \Phi_A^{\omega_A}(x) \times \Phi_B^{\omega_B}(x)$$

The Product Lemma Generalized

Let A_1, A_2, \dots, A_k be sets of objects with weight functions $\omega_1, \omega_2, \dots, \omega_k$ respectively. Let $S = A_1 \times A_2 \times \dots \times A_k$ and suppose $\omega(\sigma) = \omega_1(a_1) + \omega_2(a_2) + \dots + \omega_k(a_k)$ for each $\sigma = (a_1, a_2, \dots, a_k) \in A_1 \times A_2 \times \dots \times A_k$

Then:

$$\Phi_S^\omega(x) = \Phi_{A_1}^{\omega_1}(x) \times \Phi_{A_2}^{\omega_2}(x) \times \dots \times \Phi_{A_k}^{\omega_k}(x)$$

The Product Lemma Generalized

e.g. suppose $A = A_1 = A_2 = \cdots = A_k$ and
 $\omega_1 = \omega_2 = \cdots = \omega_k$

Then:

$$\begin{aligned}\Phi_S^\omega(x) &= \Phi_A^{\omega_1}(x) \times \Phi_A^{\omega_1}(x) \times \cdots \times \Phi_A^{\omega_1}(x) \\ &= \left(\Phi_A^{\omega_1}(x)\right)^k\end{aligned}$$

Problem 1.6.2: Let k, n be fixed non-negative integers. How many solutions are there to

$$t_1 + \dots + t_n = k, \text{ where } t_1, \dots, t_n \in \{0, 1\} \text{ ?}$$

Let S be the set of all k -tuples (a_1, a_2, \dots, a_n) where each $a_i \in \{0, 1\}$.

Define the weight of a k -tuple $\sigma = (a_1, a_2, \dots, a_n)$ by

$$\omega(\sigma) = a_1 + a_2 + \dots + a_n$$

(implicitly $\omega_j(a_j) = a_j$ for all integers j).

We start by determining $\Phi_S(x)$.

Denote $A = \{0,1\}$.

So $S = A \times A \times \cdots \times A = A^n$

Since $A = \{0,1\}$
and $\omega(n) = n$, we can easily compute

$$\Phi_A(x) = 1 + x$$

By the Product Lemma, we therefore have

$$\Phi_S(x) = (\Phi_A(x))^n = (1 + x)^n$$

The number of solutions to $t_1 + \dots + t_n = k$ is

$$[x^k] \Phi_S(x) = \binom{n}{k}$$

Problem 1.6.4: Let S be the set of all k -tuples (a_1, a_2, \dots, a_k) where each $a_i \in \mathbb{Z}_{\geq 0}$. Define the weight of a k -tuple $\sigma = (a_1, a_2, \dots, a_k)$ by $\omega(\sigma) = a_1 + a_2 + \dots + a_k$ (implicitly $\omega_j(a_j) = a_j$ for all integers j).

Determine $\Phi_S(x)$.

Solution: Let $A = \mathbb{Z}_{\geq 0}$.

Let $S = A \times A \times \dots \times A = A^k$

Since $A = Z_{\geq 0} = \{0, 1, 2, 3, \dots\}$
and $\omega(n) = n$, we can easily compute

$$\Phi_A(x) = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

By the Product Lemma, we therefore have

$$\Phi_S(x) = (\Phi_A(x))^k = \left(\frac{1}{1-x} \right)^k = (1-x)^{-k}$$

Extracting coefficients

We mentioned earlier (Thm 1.6.5) how to extract coefficients from this generating function:

$$\Phi_S(x) = (1-x)^{-k} = \sum_{j=0}^{\infty} \binom{k+j-1}{k-1} x^j$$

So we can now easily determine, for any integer j , the number of k -tuples that sum to j :

$$[x^j] \Phi_S(x) = \binom{k+j-1}{k-1}$$