Midterm Solutions - Math 239 Spring 2005

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1. [3] *Find a closed form expression for each of the following formal power series. Your answer should be expressed in the simplest possible form.*
   Solutions.

   (a) Geometric series:

$$1 - x^2 + x^4 - x^6 + x^8 - \cdots = \sum_{i \geq 0} (-x^2)^i = \frac{1}{1+x^2}$$

   (b) Binomial Theorem:

$$1 + nx^2 + \binom{n}{2} x^4 + \binom{n}{3} x^6 + \cdots + \binom{n}{n} x^{2n} = \sum_{i=0}^{n} \binom{n}{i} x^{2i} = (1+x^2)^n$$

   (c) Geometric series:

$$\begin{aligned} &\left(\frac{x}{1-x}\right) + \left(\frac{x}{1-x}\right)^3 + \left(\frac{x}{1-x}\right)^5 + \left(\frac{x}{1-x}\right)^7 + \cdots \\ &= \left(\frac{x}{1-x}\right) \sum_{i \geq 0} \left(\frac{x}{1-x}\right)^{2i} = \frac{\left(\frac{x}{1-x}\right)}{1 - \left(\frac{x}{1-x}\right)^2} = \frac{\left(\frac{x}{1-x}\right)}{\left(\frac{(1-x)^2 - x^2}{(1-x)^2}\right)} \\ &= \frac{x(1-x)}{1-2x} \end{aligned}$$

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2. [3] *Let* $\Phi(x) = \sum_{i \geq 0} a_i x^i$. *Define the formal power series*

$$f(x) = \frac{\Phi(x)}{1-x}.$$

   Determine $[x^n]f(x)$ *in terms of the coefficients* a_i.
   Solution. Using $(1-x)^{-1} = \sum_{j \geq 0} x^j$,

$$f(x) = \frac{\Phi(x)}{1-x} = \left(\sum_{i \geq 0} a_i x^i\right) \left(\sum_{j \geq 0} x^j\right) = \sum_{i \geq 0} \sum_{j \geq 0} a_i x^{i+j}$$

   Thus,

$$[x^n]f(x) = [x^n] \sum_{i \geq 0} \sum_{j \geq 0} a_i x^{i+j} = \sum_{\substack{i \geq 0 \; j \geq 0 \\ i+j=n}} a_i = \sum_{i=0}^{n} a_i$$

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3. [4] *Let* $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ *denote the set of nonnegative integers. Define the weight of integer* n *to be*

$$w(n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ \dfrac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Determine the generating function $\Phi_{\mathbb{N}}(x)$ *as a rational function* $\frac{p(x)}{q(x)}$ *where* $p(x)$ *and* $q(x)$ *are polynomials.*

Solution. Let $\mathbb{N}_{\text{even}}$ denote the set of even natural numbers. Let $\mathbb{N}_{\text{odd}}$ denote the set of odd natural numbers. Note that $\mathbb{N} = \mathbb{N}_{\text{even}} \cup \mathbb{N}_{\text{odd}}$, and that $\mathbb{N}_{\text{even}} \cap \mathbb{N}_{\text{odd}} = \emptyset$. Thus, by the Sum Lemma,

$$\begin{aligned} \Phi_{\mathbb{N}}(x) &= \Phi_{\mathbb{N}_{\text{even}}}(x) + \Phi_{\mathbb{N}_{\text{odd}}}(x) \\ &= \sum_{i \in \mathbb{N}_{\text{even}}} x^{w(i)} + \sum_{j \in \mathbb{N}_{\text{odd}}} x^{w(j)} \\ &= \sum_{i \geq 0} x^i + \sum_{j \geq 0} x^{2j+1} \\ &= \frac{1}{1-x} + \frac{x}{1-x^2} \quad = \quad \frac{1+2x}{1-x^2}. \end{aligned}$$

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4. [8] *Let* n *be a positive integer and let* b_n *denote the number of compositions of* n *into* k *parts, where each part is one or two. For example,* $(1, 2, 1, 2, 1)$ *and* $(2, 2, 1, 1, 1)$ *are two compositions of* $n = 7$ *into* $k = 5$ *parts.*

   (a) *Determine the generating function for* b_n*.*

   Solution. Let $S = \{1, 2\}^k$ be the set of compositions with k parts, each of which is either 1 or 2. Then, applying the Product Lemma,

$$\begin{aligned} \Phi_S(x) &= (\Phi_{\{1,2\}}(x))^k \\ &= (x + x^2)^k \end{aligned}$$

   (b) *Prove that* $b_n = \binom{k}{n-k}$ *for* $k \leq n \leq 2k$ *and* $b_n = 0$ *otherwise.*

   Solution. Using the binomial series theorem, we find

$$\begin{aligned} b_n &= [x^n]\Phi_S(x) \\ &= [x^n]x^k(1+x)^k \\ &= [x^{n-k}](1+x)^k \\ &= \binom{k}{n-k}. \end{aligned}$$

   Note that if $n < k$, then $n - k < 0$, so $b_n = \binom{k}{n-k} = 0$. If $n > 2k$, then $n - k > k$, so $b_n = \binom{k}{n-k} = 0$.

Alternate solution: A composition of n with k parts will have i parts equal to 2, for some $0 \leq i \leq k$. Note that the number of parts equal to 1 is $k-i$, and $n=(k-i)+2i$, giving $i=n-k$. There are k positions for the $n-k$ parts equal to 2, so there are a total of $\binom{k}{n-k}$ compositions of n into k parts, each either 1 or 2. Note that $0 \leq i \leq k$ implies $k \leq n \leq 2k$, and the number of compositions is zero otherwise.

(c) *Determine the generating function for c_n, the number of compositions of n into any number of parts, each of which is one or two.*
Solution. Let $T = \cup_{k\geq 0}\{1,2\}^k$ be the set of compositions in which the parts are either 1 or 2. Then, applying the Sum and Product Lemma,

$$\begin{aligned}\Phi_T(x) &= \sum_{k\geq 0}(\Phi_{\{1,2\}}(x))^k \\ &= \sum_{k\geq 0}(x+x^2)^k \\ &= \frac{1}{1-x-x^2}.\end{aligned}$$

(d) *Find a recurrence equation for c_n with appropriate initial conditions.*
Solution. Using a theorem from class (namely, Theorem 1.26) and the generating series in part (c) or directly comparing coefficients of x^n in the generating series from part (c), we obtain the recurrence

$$c_n = c_{n-1} + c_{n-2}.$$

We require two initial conditions. Note that $c_0 = 1$ since there is one composition of 0 (the empty composition), and $c_1 = 1$, since there is one composition of 1 (the composition with one part equal to 1).

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5. [4]

(a) *Are the strings of $\{101,\ 010,\ 01010\}^*$ uniquely created? Justify your answer.*
Solution. No. For example, 01010101010 decomposes as 010,101,01010 or 01010,101,010.

(b) *Are the strings of $\{110,\ 011,\ 01010\}^*$ uniquely created? Justify your answer.*
Solution. Yes. Any block of 1's of length one must come from 01010. This implies we can always identify these occurences. Any string must have the form

$$b_1(01010)b_2\dots(01010)b_{n-1}(01010)b_n(01010)b_{n+1}$$

where b_i is a substring generated by $\{110, 011\}^*$. Since both 110 and 011 have length three, we can divide each b_i into substrings of length three and uniquely identify how each was generated.

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6. [4] *For each of the following sets, write down a decomposition that uniquely creates the elements of that set.*

   (a) *Binary strings that do not contain the substring 1111.*
   Solution. From the block decomposition we restrict each block of 1's to have length less than 4:

$$\{\epsilon, 1, 11, 111\}(\{0\}\{0\}^*\{1, 11, 111\})^*\{0\}^*$$

   or

$$\{0\}^*(\{1, 11, 111\}\{0\}\{0\}^*)^*\{\epsilon, 1, 11, 111\}.$$

   Note that as a restriction of the block decomposition this uniquely creates the elements of this set.

   (b) *Binary strings in which every block of 1's of even length is followed by a block of 0's of even length.*
   Solution. A block of 1's either has even or odd length, so the following modification of the block decomposition uniquely generates all binary strings:

$$\{0\}^*(\{11\}\{11\}^*\{0\}\{0\}^* \cup \{1\}\{11\}^*\{0\}\{0\}^*)^*\{1\}^*.$$

   We then restrict this to enforce that every even block of 1's is followed by an even block of 0's (note that this means we cannot end with an even block of ones):

$$\{0\}^*(\{11\}\{11\}^*\{00\}\{00\}^* \cup \{1\}\{11\}^*\{0\}\{0\}^*)^*(\epsilon \cup (\{1\}\{11\}^*).$$

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7. [6] Let S denote the set of strings which do not contain the substring 10001. A combinatorial decomposition of this set of strings is given by

$$S \;=\; \{0\}^* \left(\{1\}\{1\}^*(\{0\}\{0\}^* \backslash \{000\})\right)^* \{1\}^* \{\varepsilon, 1000\}$$

where the strings on the right are uniquely created. Let a_n be the number of binary strings of length n that do not contain the substring 10001.

   (a) *Show that*

$$\sum_{n \geq 0} a_n x^n = \frac{1 + x^4}{1 - 2x + x^4 - x^5}.$$

Solution. By the Product Lemma

$$\begin{aligned}
\Phi_S(x) &= \Phi_{\{0\}^*}(x)\ \Phi_{(\{1\}\{1\}^*(\{0\}\{0\}^*\setminus\{000\}))^*}(x)\ \Phi_{\{1\}^*}(x)\Phi_{\{\varepsilon,1000\}}(x) \\
&= \frac{1}{1-x}\cdot\frac{1}{1-\left(\frac{x}{1-x}\right)\left(\frac{x}{1-x}-x^3\right)}\cdot\frac{1}{1-x}\cdot(1+x^4) \\
&= \frac{1}{(1-x)^2}\cdot\frac{(1-x)^2(1+x^4)}{1-2x+x^2-x(x-x^3+x^4)} \\
&= \frac{1+x^4}{1-2x+x^4-x^5}.
\end{aligned}$$

(b) *Determine a recurrence relation for the sequence of a_n's together with sufficient initial condition to uniquely determine the sequence.*
Solution. By Theorem 1.26, the a_n's satisfy the recurrence relation

$$a_n - 2a_{n-1} + a_{n-4} - a_{n-5} = 0 \quad \forall n \geq 5.$$

The initial conditions, by Theorem 1.26 or directly comparing coefficients of x^0, x^1, x^2, x^3 and x^4 in the generating function, are

$$\begin{aligned}
a_0 &= 1, \\
a_1 - 2a_0 &= 0 \text{ or } a_1 = 2, \\
a_2 - 2a_1 &= 0 \text{ or } a_2 = 4, \\
a_3 - 2a_2 &= 0 \text{ or } a_3 = 8, \\
a_4 - 2a_3 + a_0 &= 1 \text{ or } a_4 = 16.
\end{aligned}$$

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8. [6]

(a) *Solve the homogeneous linear recurrence relation*

$$b_n - 3b_{n-1} + 3b_{n-2} - b_{n-3} = 0 \qquad \forall n \geq 3$$

with initial conditions $b_0 = 1,\ b_1 = 0,\ b_2 = 1.$
Solution. The characteristic polynomial of theis recurrence relation is

$$x^3 - 3x^2 + 3x - 1 = (x-1)^3.$$

By Theorem 1.42, the general solution is of the form

$$b_n = (A + Bn + Cn^2)(1)^n.$$

$$\begin{aligned}
n = 0: &\quad A &= 1, \\
n = 1: &\quad A + B + C &= 0, \\
n = 2: &\quad A + 2B + 4C &= 1.
\end{aligned}$$

Solving this system of 3 equations in 3 unknowns, we get $A = 1,\ B = -2,\ C = 1$. Therefore,

$$b_n = 1 - 2n + n^2 \ \ \forall\, n \geq 0.$$

(b) *Solve the nonhomogeneous linear recurrence relation*

$$a_n - 3a_{n-1} - 4a_{n-2} = 6n - 11 \qquad \forall n \geq 3$$

with initial conditions $a_1 = 5$ *and* $a_2 = 12$.

Solution. To find a particular solution of this recurrence relation we try a solution of the form $a_n = \alpha n$ where α is some constant. Substituting into the recurrence relation, we get

$$\begin{aligned} 6n - 11 &= \alpha n - 3\alpha(n-1) - 4\alpha(n-2) \\ &= \alpha n - 3\alpha n + 3\alpha - 4\alpha n + 8\alpha \\ &= -6\alpha n + 11\alpha \\ &= \alpha(-6n + 11). \end{aligned}$$

Therefore, $\alpha = -1$ and a particular solution is $a_n = -n$.

The characteristic polynomial of the corresponding homogeneous linear recurrence relation is

$$x^2 - 3x - 4 = (x-4)(x+1).$$

Therefore, the general solution of the nonhomogeneous recurrence relation is of the form

$$a_n = A(4)^n + B(-1)^n - n.$$

Setting n equal to 1 and 2 respectively, we get

$$\begin{aligned} n = 1: &\quad 4A - B - 1 &= 5, \\ n = 2: &\quad 16A + B - 2 &= 12. \end{aligned}$$

Solving these 2 equations in 2 unknowns, we get $A = 1$ and $B = -2$. Therefore,

$$a_n = 4^n - 2(-1)^n - n \;\; \forall n \geq 0.$$

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9. [3] *Determine which pairs of the following graphs are isomorphic. For those which are not isomorphic, give a reason.*

Solution. We have to check six pairs of graphs for isomorphism. The graphs G_1 and G_4 are isomorphic. Now G_2 is not isomorphic to G_1 or G_3 since G_2 contains no cycle of length four, whereas G_1 and G_3 each contain a cycle of length four. Finally G_1 and G_3 are not isomorphic since G_1 contains five cycles of length four whereas G_3 contains exactly two cycles of length four. Since G_1 and G_4 are isomorphic, G_2 and G_3 are also not isomorphic to G_4. So we have checked all six pairs of graphs.

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10. [9] *Let* $G(n,r)$ *denote the graph whose vertices are subsets of* $\{1,2,\ldots,n\}$ *of size* r*, and whose edges are pairs of subsets of* $\{1,2,\ldots,n\}$ *of size* r *which are disjoint.*

(a) *Draw $G(3,1)$ and $G(4,2)$.*
Solution. The graphs G_3 and G_4 are drawn below:

(b) *Determine the number of vertices in $G(n,r)$.*
Solution. The number of vertices in $G(n,r)$ is the number of subsets of $\{1,2,\ldots,n\}$ of size r, since each vertex is a subset of $\{1,2,\ldots,n\}$ of size r. So there are $\binom{n}{r}$ vertices in $G(n,r)$.

(c) *Determine the degree of every vertex of $G(n,r)$.*
Solution. Fix a vertex S (which is an r-set in $\{1,2,\ldots,n\}$). Then S is adjacent to all vertices T such that $S \cap T = \emptyset$. So the degree of vertex S is the number of choices of an r-set $T \subset \{1,2,\ldots,n\} \backslash S$. Since $|\{1,2,\ldots,n\} \backslash S| = n-r$, there are $\binom{n-r}{r}$ choices of T, so every vertex has degree $\binom{n-r}{r}$.

(d) *Determine the number of edges in $G(n,r)$.*
Solution. By the Handshaking Lemma, the number of edges is $\frac{1}{2}\binom{n}{r}\binom{n-r}{r}$.

(e) *Prove that $G(3r,r)$ is connected for all $r \geq 1$.*
Solution. A graph is connected if and only if each pair of its vertices is joined by a path. In $G(3r,r)$, consider two vertices S and T; we show that there is a path between S and T. Now $|S \cup T| \leq 2r$, since S and T are sets of size r, so there exists a set U of size r in $\{1,2,\ldots,3r\} \backslash (S \cup T)$. By definition of adjacency in $G(3r,r)$, U is adjacent to S and to T, since it is disjoint from both. Therefore SUT is a path (of length two) from S to T. This works for any two vertices S and T in $G(3r,r)$, so we have proved that any two vertices of $G(3r,r)$ is joined by a path of length two. So $G(3r,r)$ is connected.