MATH 239 Assignment 8

- This assignment is due on Friday, November 16, 2012, at 10 am in the drop boxes in St. Jerome's (section 1) or outside MC 4067 (the other two sections).
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.

1. For each positive integer n, what is the largest number of bridges in any n-vertex graph? Prove that your answer is correct.

   Solution:

   The largest possible number of bridges is $n-1$.

   Any tree with n vertices has $n-1$ edges, each of which is a bridge, so the largest possible number is at least this large.

   It remains to show that no n-vertex graph G has more than $n-1$ bridges. If G is connected, consider a spanning tree T of G. We claim that an edge of G that is not part of T cannot be a bridge. If $\{u, v\}$ is an edge of G but not of T, then since there is a path from u to v in T, the edge $\{u, v\}$ is part of a cycle in G, and hence is not a bridge. If G is not connected, with $k > 1$ components, then we can use this result on each component to see that there are at most $n - k < n - 1$ bridges.

2. Suppose there is a unique path between any pair of vertices in G. Prove that G is a tree.

   Solution:

   Since there is a path between every pair of vertices, G is connected.

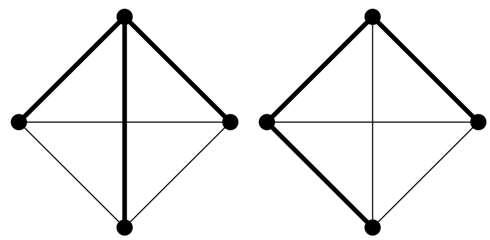
   It remains to show that G has no cycles. For a contradiction, suppose that G has a cycle $v_0v_1v_2\dots v_{n-1}v_0$. Then there are two paths from v_0 to v_{n-1}, namely $v_0v_1v_2\dots v_{n-1}$ and v_0v_{n-1}. But this contradicts the fact that there is a unique path between any pair of vertices; hence G has no cycles.

   Since G is connected and has no cycles, it follows by definition that G is a tree.

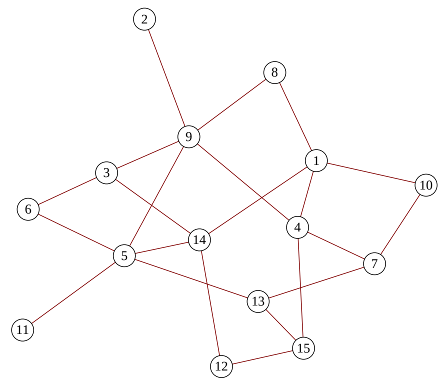
3. Prove or disprove the following statement: Given a graph G, if T and U are spanning trees of G, then T and U are isomorphic.

   Solution:

   This statement is false. For example, consider K_4, the complete graph on 4 vertices. Below, we show two spanning trees of K_4, with the edges of the spanning tree denoted by thicker lines. These two spanning trees are not isomorphic, because the first has a vertex of degree 3, and the latter does not.



4. Let G be the following graph:

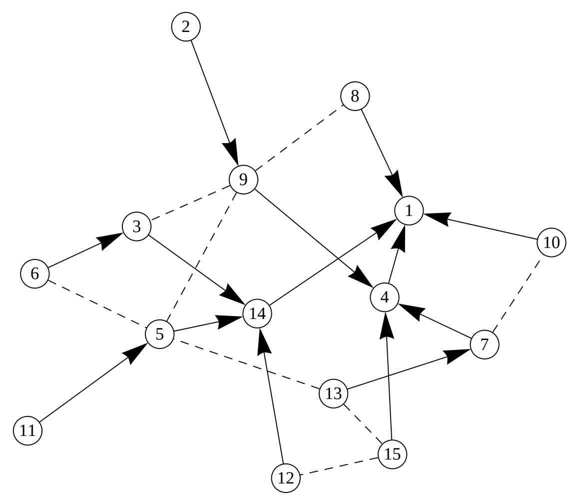


(a) Construct a breadth-first search tree for G. Use vertex 1 as the root, and add vertices to the tree using the smallest possible label first. Denote each tree edge by an arrow from a vertex to its parent, and indicate each non-tree edge by a dashed line.

(b) Is G bipartite? Use a breadth-first search tree to justify your answer.

(c) Give a shortest path between vertices 1 and 6. Use a breadth-first search tree to prove that your path is as short as possible.

Solution:

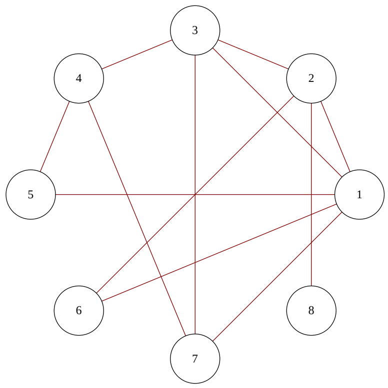


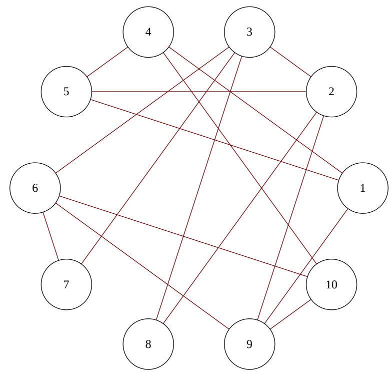
(a)

(b) G is not bipartite, because adjacent vertices 12 and 15 are both at the same level of the breadth-first search tree (level 2).

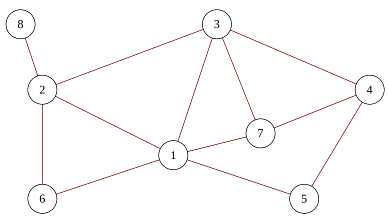
(c) Since vertex 6 is at level 3 of the breadth-first search tree rooted at 1, the length of the shortest path is 3. An example of a shortest path is $1, 14, 3, 6$.

5. Prove that each of the following graphs is planar by exhibiting a planar embedding. (Be sure to label the vertices in your embeddings.)

(a)


(b)


Solution:

(a)


(b)