MATH 239 Assignment 6

- This assignment is due on Friday, March 1.

1. For $n \geq r \geq 1$, define the graph $G_{n,r}$ as follows: The vertices of $G_{n,r}$ are r-element subsets of $\{1, ..., n\}$. Two vertices U and V are adjacent if and only if $U \cap V \neq \emptyset$.

   (a) Draw $G_{4,1}$ and $G_{4,2}$.

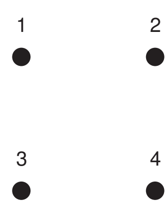
   (b) Determine how many vertices $G_{n,r}$ has.

   (c) Prove that $G_{n,r}$ is k-regular for each $n \geq r \geq 1$, and determine k in terms of n and r.

   (d) Determine how many edges $G_{n,r}$ has.

   Solution:

   (a) $G_{4,1}$ consists of four vertices of degree 0. $G_{4,2}$ is the octahedron:



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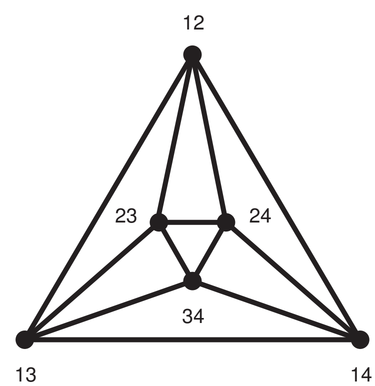

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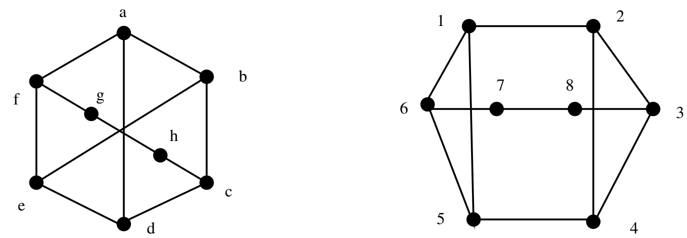
   (b) By definition $G_{n,r}$ has $|V(G_{n,r})| = \binom{n}{r}$ vertices.

   (c) Let U be a vertex of G, so U is an r-subset of $\{1, ..., n\}$. Then U is adjacent to every vertex V of G except itself and those V for which $U \cap V = \emptyset$. The number of such V is the number of r-subsets of $\{1, ..., n\} \setminus U$, which as we know is $\binom{n-r}{r}$, since $|U| = r$. Therefore the degree of U is $|V(G_{n,r})| - 1 - \binom{n-r}{r} = \binom{n}{r} - \binom{n-r}{r} - 1$.
   Since U was an arbitrary vertex, we conclude that $G_{n,r}$ is k-regular where $k = \binom{n}{r} - \binom{n-r}{r} - 1$.

   (d) We know that the number of edges of a graph G is $\frac{1}{2}\sum_{v \in V(G)} deg(v)$. By the previous part we know that $deg(v) = \binom{n}{r} - \binom{n-r}{r} - 1$ for each vertex v of $G_{n,r}$. Therefore the number of edges of $G_{n,r}$ is

$$\frac{1}{2}|V(G_{n,r})|(\binom{n}{r} - \binom{n-r}{r} - 1) = \binom{n}{r}(\binom{n}{r} - \binom{n-r}{r} - 1).$$

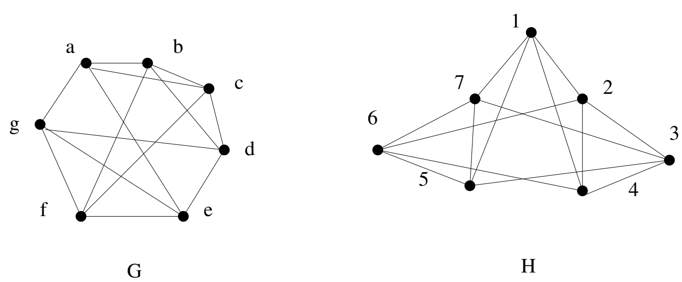
2. For the two graphs shown below, determine whether they are isomorphic. If they are isomorphic, find an explicit isomorphism and verify that it is an isomorphism. If they are not isomorphic, give a proof that they are not.



Solution:

The graphs are not isomorphic. To see this, notice that the graph on the right has a triangle, i.e. three vertices, $\{2, 3, 4\}$, which are all pairwise adjacent. There are no such three vertices in H, so there is definitely no adjacency preserving map between the two graphs.

3. For the two graphs shown below, determine whether they are isomorphic. If they are isomorphic, find an explicit isomorphism and verify that it is an isomorphism. If they are not isomorphic, give a proof that they are not.



Solution:

The graphs are isomorphic, via the isomorphism $q(a) = 6$, $q(b) = 1$, $q(c) = 3$, $q(d) = 5$, $q(e) = 7$, $q(f) = 2$, $q(g) = 4$. To see this is an isomorphism, imagine a re-drawing of the first graph that fixes the 4-cycle $defg$ and "folds out" the three triangles dea, efb, and fgc to form the same drawing as seen in the second graph.

4. For a graph G, we define the complement graph of G, denoted $\overline{G}$, with $V(\overline{G}) = V(G)$, and $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$.

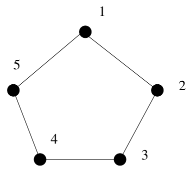
   (a) Let G be the graph with vertex set $\{1, 2, 3, 4, 5\}$ and edge set $\{\{i, j\} : j \equiv i + 1 \mod 5\}$. Draw G and $\overline{G}$.

   (b) For a graph G, the *degree sequence* of G is a list $d_1, \ldots, d_p$ of the degrees of all the vertices of G, written in nondecreasing order. For example, the degree sequence of the graph G in part (a) is $2, 2, 2, 2, 2$. Prove that $d_1, \ldots, d_p$ is the degree sequence of a graph if and only if $p - d_p - 1, \ldots, p - d_1 - 1$ is the degree sequence of a graph.

   (c) Prove that $4, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8$ is not the degree sequence of a graph.

Solution:

(a) Both G and $\overline{G}$ are cycles of length 5:



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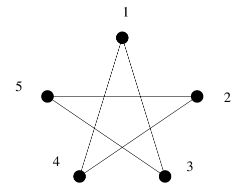

$\overline{G}$

(b) Let $N(v)$ denote the set of vertices in G that are adjacent to v. Since the set of vertices adjacent to v in $\overline{G}$ is precisely the set $V(G) \setminus (\{v\} \cup N(v))$, we see that $deg_{\overline{G}}(v) = p - 1 - deg_G(v)$. Therefore if $d_1, \ldots, d_p$ is the degree sequence of a graph G then $p - d_p - 1, \ldots, p - d_1 - 1$ is the degree sequence of the graph $\overline{G}$. To see that the converse is also true, observe that the complement of the graph $\overline{G}$ is the graph G, and so the same statement implies the converse as well.

(c) Using (b) it suffices to show that the sequence $1, 1, 1, 1, 1, 1, 5, 5, 5, 5$ is not the degree sequence of a graph G. Suppose it is. Let $A = \{a_1, a_2, a_3, a_4\}$ denote the set of vertices of degree 5. Then each $a \in A$ is adjacent to at most 3 vertices in A, so it has at least 2 edges going to the set $B = V(G) \setminus A$. Thus at least $4(2) = 8$ edges of G join vertices in A to vertices in B. But each vertex of B has degree 1, so it can be incident to at most one edge from A. Hence the number of edges joining A to B is at most 6. This contradiction shows that no graph G can have this degree sequence.

5. Let G be a graph with at least two vertices. Prove that G has two vertices of the same degree.

Solution:

Let p denote the number of vertices of G. By definition of degree, the possible values for the degree of a vertex in G are $\{0, 1, \ldots, p-1\}$. Suppose on the contrary that all p vertices of G have distinct degrees. Then since the number of possible degrees is p, there is exactly one vertex of each of the possible degrees $\{0, 1, \ldots, p-1\}$. Then the vertex x of degree $p-1$ must be adjacent to every vertex of $V(G) \setminus \{x\}$. But the vertex y of degree 0 cannot be adjacent to any vertex of $V(G) \setminus \{y\}$, which contradicts the fact it is adjacent to x.

Therefore G must have two vertices of the same degree.