MATH 239 - Fall 2013

Assignment 9

Due date : Friday, November 22, 2013, at noon (sharp)

Submission Guidelines:

- Total number of marks in this assignment is 55.
- Use a cover page to submit your solutions (available on the course webpage).
- Keep a copy of your manuscript before your submission.
- Assignments submissions are exclusively accepted in the following dropboxes

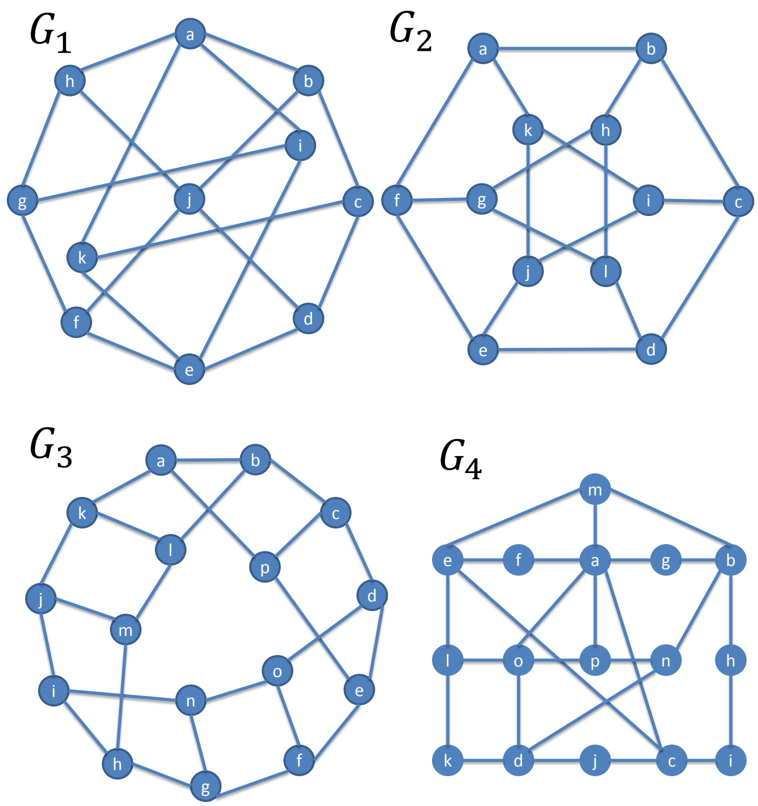
[Section 001]	Dropbox next to the St Jerome's library, 2nd floor of STJ			
[Section 002]	Math DropBox #18;	Slot #1 A-J,	Slot #2 K-S,	Slot #3 T-Z
[Section 003]	Math DropBox #18;	Slot #4 A-J,	Slot #5 K-S,	Slot #6 T-Z
[Section 004]	Math DropBox #18;	Slot #7 A-J,	Slot #8 K-S,	Slot #9 T-Z

- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- You are not allowed to post this manuscript (or parts of it) online, nor share it (or parts of it) with anyone not enrolled in this course.

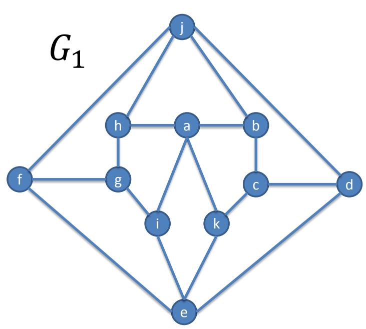
Assignment policies: While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. For example, copying or paraphrasing a solution from some fellow student or old solutions from previous offerings of related courses qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. All students found to be cheating will automatically be given a mark of 0 on the assignment. In addition, there will be a 10/100 penalty to their final mark, as well as all academic offenses will be reported to the Associate Dean for Undergraduate Studies and recorded in the student's file (this may lead to further, more severe consequences).

If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that if you see any marking error, then you should return your assignment paper to the TA of your section within one week and with written notes on all the marking errors; please write the notes on a new sheet and attach it to your assignment paper.

Question 1 [Marks 14=3+3+4+4]
For each of the graphs G_1, G_2, G_3, G_4 below, either provide a planar embedding or use Kuratowksi's Theorem to prove that they are not planar.

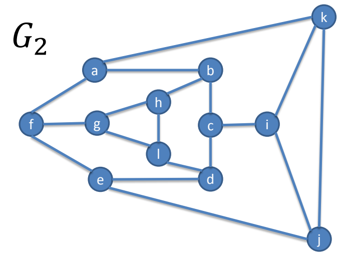


Solution. *(a) G_1 is planar and a planar embedding can be seen in the next drawing.*

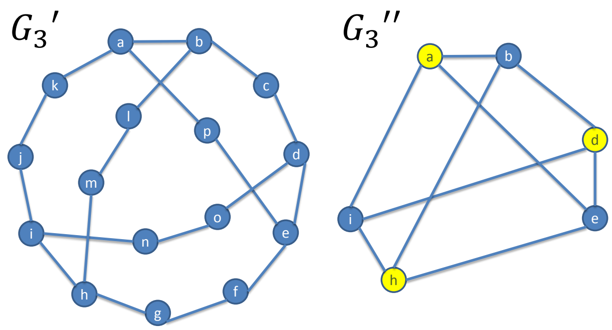


(b) G_2 is planar and a planar embedding can be seen in the next drawing.

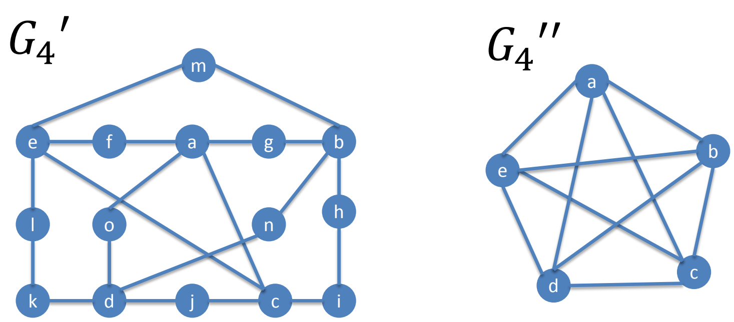
(c) G_3 is not planar. To prove this we identify a subgraph G_3' that is an edge subdivision of $K_{3,3}$ (that is called G_3'' in the figure below). The bipartition of G_3'' is $\{a, d, h\}, \{b, i, e\}$,



and in the figure below the sets of vertices are depicted with colours yellow and blue respectively.



(d) *G_4 is not planar. To prove this we identify a subgraph G_4' that is an edge subdivision of K_5 (that is called G_4'' in the figure below).*



Question 2 [Marks 16=9+7]
Let $G = (E, V)$ be a connected planar graph that does not have a cycle of length less than 6.

(a) Prove that G has at least one vertex of degree at most 2.
Hint: Distinguish the cases where G is or is not a tree. For the latter case attempt a proof by contradiction.

Solution. *if G is a tree, then we know it has at least two vertices of degree 1, if it has at least 2 vertices. If $|V| = 1$, the claim is again true since the degree of the only vertex would be 0 hence the claim follows.*

Next we examine the case where G is not a tree. Denote by p, q the number of vertices and edges in G. Suppose, for the sake of contradiction, that every vertex has degree at least 3. Then, by the hand-shaking lemma we have that

$$2q = \sum_{u \in V} deg(u) \geq 3p.$$

At the same time, G is planar, and its smallest cycle is at least 6. Fix any planar embedding of G. Every face of that embedding contains a cycle, and therefore it has degree at least $d^* = 6$. We conclude (by Lemma 7.5.2) that

$$4q \leq 6(p-2).$$

The two inequalities above derive a contradiction, since in particular the second one says that $2q \leq 3(p-2) < 3p$.

(b) Prove by induction on $|V|$ that G is 3 colourable.

Solution. *Note that if $|V| \leq 5$ then the graph is 3 colourable, since it does not have a cycle, hence it is bipartite and therefore 2-colourable as well. That would complete also the base case of the inductive argument.*

In the inductive step, we assume that every planar connected graph with no cycle of length less than 6, and with $n \geq 5$ vertices is 3-colourable. Now fix some arbitrary planar connected graph G with no cycle of length less than 6, and with $n+1 \geq 6$ vertices.

By part (a) we know that G has a vertex v of degree at most 2. We consider a new graph G' which is the same as G only that v together with all edges incident to v are deleted. Note that G' might be disconnected. However, all of its connected components (maybe still 1 component) are planar and connected graphs with no cycles of length less than 6. Moreover, each of the components has at most n vertices, and therefore is 3-colourable, by the inductive hypothesis. Fix such a 3-colouring that effectively assigns one colour to each vertex of G, except from vertex v. Since v has at most two neighbors, there is always a colour available to colour v that is different from the colours of its (at most 2) neighbors, showing that G is 3-colourable.

Question 3 [Marks 7]

Prove that every planar bipartite graph has at least one vertex of degree at most 3.

Solution. *For the sake of contradiction, assume that all vertices of some planar bipartite graph G have degree at least 4. From the hand-shaking lemma we know that*

$$2|E(G)| = \sum_{u \in V(G)} deg(u) \geq 4|V(G)|.$$

At the same time, we know that G is not a tree, as otherwise, it should have at least one vertex of degree at most 2. Hence, G has a cycle, and as it is bipartite, its shortest cycle (which cannot be odd) is of length at least 4. In any planar embedding of G, any cycle then of G would have degree at least 4. Then, by Lemma 7.5.2 we have

$$(4-2)|E(G)| \leq 4(|V(G)|-2).$$

Together the above inequalities imply that

$$2(|V(G)|-2) \geq |E(G)| \geq 2|V(G)|,$$

a contradiction.

Question 4 [Marks 18=5+8+5]
The complement $\overline{G}$ of the graph G is a graph with vertices $V(\overline{G}) = V(G)$ and edges $E(\overline{G}) = \{\{i,j\} : \{i,j\} \notin E(G)\}$.

(a) Prove that every planar graph G for which $\overline{G}$ is planar too has at most 10 vertices.

Solution. *Denote by p, q the number of vertices and edges in G. Clearly, $\overline{G}$ has $\binom{p}{2} - q$ many edges. By Corollary 7.5.5 and for both $G, \overline{G}$ to be planar we must have*

$$q \leq 3p - 6$$
$$\binom{p}{2} - q \leq 3p - 6.$$

We conclude that for the number of vertices p we must have $\binom{p}{2} - 6p + 12 \leq 0$, which is satisfied for $\frac{1}{2}\left(13 - \sqrt{73}\right) \leq p \leq \frac{1}{2}\left(13 + \sqrt{73}\right)$. Therefore $p \leq 10.7$, and since p is an integer we have $p \leq 10$.

(b) Suppose that for a graph G with 10 vertices, its complement $\overline{G}$ has two connected components of at least 2 vertices each. Prove that G and $\overline{G}$ cannot be both planar.

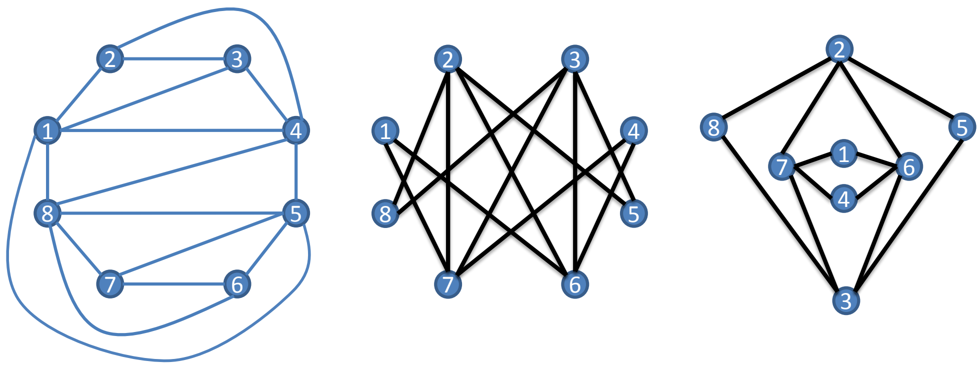
Solution. *Let G be planar. If q denotes the number of its vertices, we have $q \leq 3\cdot 10 - 6 = 24$ (by Corollary 7.5.5). But then, graph $\overline{G}$ has at least $\binom{10}{2} - 24 = 45 - 24 = 21$ edges.*

Next we recall that $\overline{G}$ has two components, say $\overline{G_1}, \overline{G_2}$, both of size at least 2. If the smallest component had size 3, then G would have the following bipartite graph as a subgraph: vertices would be $V(\overline{G_1}) \cup V(\overline{G_2})$, and all edges between $V(\overline{G_1}), V(\overline{G_2})$ would be present. Since both sets have size at least 3, that would induce a $K_{3,3}$ subgraph of G, a contradiction to the planarity of G.

Hence, the components of $\overline{G}$ have size 2 and 8 respectively. The smallest component can have at most 1 edge, which means that the component with the 8 vertices should have at least $21 - 1 = 20$ many edges. But then, for that component we have that $20 \not\leq 3\cdot 8 - 6 = 18$, which by Corollary 7.5.5 means that the component cannot be planar.

(c) Find a graph G on 8 vertices for which both $G, \overline{G}$ are planar. For both graphs, provide the planar embedding.

Solution. *There are different solutions. One is the following.*



On the left is graph G. In the middle we depict $\overline{G}$, while on the right we show a planar embedding of $\overline{G}$.