MATH 239 – Midterm Tutorial

1.

Recall the definition of a power series: $\sum_{k\geq 0} a_k x^k$, $a_k \in \mathbb{Q}\ \forall k$.

(a) Compute $[x^n](1-2x+x^2)^{-k}$.
(b) Express the following as rational functions.
(i) $1+2x^2+4x^4+8x^6+16x^8+32x^{10}+\ldots$ and
(ii) $1-x^3+x^6-x^9+x^{12}-x^{15}+\ldots$

Solution
(a) Notice that

$$\begin{aligned}(1-2x+x^2)^{-k} &= ((x-1)^2)^{-k} \\ &= (x-1)^{-2k} \\ &= (-1)^{-2k}(1-x)^{-2k} \\ &= (1-x)^{-2k}.\end{aligned}$$

Therefore

$$\begin{aligned}(1-2x+x^2)^{-k} &= (1-x)^{-2k} \\ &= \sum_{n\geq 0}\binom{n+2k-1}{2k-1}x^n,\end{aligned}$$

and

$$[x^n](1-2x+x^2)^{-k} = \binom{n+2k-1}{2k-1} = \binom{n+2k-1}{n},$$

since $\binom{n}{k} = \binom{n}{n-k}$.

(b) (i) Notice that the exponents are multiples of 2, while the coefficients are twice the previous one (i.e. 2^n). Therefore, the closed form expression is $\sum_{n\geq 0} 2^n x^{2n} = \sum_{n\geq 0}(2x^2)^n$. From this, we see that the formal power series can be expressed as $(1-2x^2)^{-1}$.

(ii) In this case, the exponents are multiples of 3, while the coefficients are alternatively 1 and -1 (i.e. $(-1)^n$). Therefore, the closed form expression is $\sum_{n\geq 0}(-1)^n x^{3n} = \sum_{n\geq 0}(-x^3)^n$. From this, we see that the formal power series can be expressed as $(1+x^3)^{-1}$.

2.

Determine $[x^n](1-x^2)^{-5}(1-3x)^{20}$.

Solution

$$\begin{aligned}(1+x^2)^{-5}(1-3x)^{20} &= \sum_{i\geq 0}\binom{i+4}{4}(x^2)^i\sum_{j=0}^{20}\binom{20}{j}(-3)^j x^j\\ &= \sum_{i\geq 0}\sum_{j=0}^{20}\binom{i+4}{4}\binom{20}{j}(-3)^j x^{2i+j}.\end{aligned}$$

We need $n = 2i+j$, so $j = n-2i$. But $0 \geq j \geq 20$, so $\frac{n-20}{2} \geq i \geq \frac{n}{2}$. So the coefficient of x^n is

$$\sum_{i=\lceil\frac{n-20}{2}\rceil}^{\lfloor\frac{n}{2}\rfloor}\binom{i+4}{4}\binom{20}{n-2i}(-3)^{n-2i}.$$

3.

Let $S = \{0\}^*(\{1\}\{11\}^*\{00\}\{00\}^* \cup \{11\}\{11\}^*\{0\}\{00\}^*)^*$.
(a) What is in S?
(b) Find the generating series for S with respect to length.

Solution
(a) S contains the strings such that a block of 1's of odd length is followed by a block of 0's of even length, and a block of 1's of even length is followed by a block of 0's of odd length.
It can be shown that S is unambiguous since it is a restriction of the block decomposition.

(b) Using the definition of generating series, we can see that

$$\begin{aligned}\Phi_{\{0\}}(x) &= x &= \Phi_{\{1\}},\\ \Phi_{\{00\}^*}(x) &= \frac{1}{1-x^2} &= \Phi_{\{11\}^*},\\ \Phi_{\{00\}}(x) &= x^2 &= \Phi_{\{11\}},\\ \Phi_{\{0\}^*} &= \frac{1}{1-x}.\end{aligned}$$

Therefore by the *-lemma, the Product Lemma and the Sum Lemma we get

$$
\begin{aligned}
\Phi_S(x) &= \left(\frac{1}{1-x}\right)\left[\frac{1}{1-\left(\frac{x}{1-x^2}\cdot\frac{x^2}{1-x^2}+\frac{x^2}{1-x^2}\cdot\frac{x}{1-x^2}\right)}\right], \\
&= \left(\frac{1}{1-x}\right)\left[\frac{1}{1-\frac{2x^3}{(1-x^2)^2}}\right], \\
&= \left(\frac{1}{1-x}\right)\left[\frac{(1-x^2)^2}{1-2x^2-2x^3+x^4}\right], \\
&= \frac{(1-x^2)(1+x)}{1-2x^2-2x^3+x^4}.
\end{aligned}
$$

4.

Consider $\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$.

(a) Give a combinatorial proof.

(b) Give an algebraic proof.

Solution

(a) $\binom{m+n}{k}$ is the number of ways of choosing k elements from a set of $m+n$ elements. On the other hand, $\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$ is the number of ways of picking i elements from a set of m elements, and $k-i$ elements from a set of n elements, for all i from 0 to k.

(b) Notice that $(1+x)^{m+n} = (1+x)^m \cdot (1+x)^n$. Expand both sides using the binomial formula to obtain

$$
\begin{aligned}
\sum_{k=0}^{m+n}\binom{m+n}{k}x^k &= \sum_{i=0}^{m}\binom{m}{i}x^k \cdot \sum_{j=0}^{n}\binom{n}{j}x^k \\
&= \sum_{k\geq 0}\left(\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}\right)x^k.
\end{aligned}
$$

By comparing the kth coefficient we get

$$\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}.$$

5.

Consider the recurrence equation $c_n = c_{n-1} + 2c_{n-2}$, for $n \geq 2$, with $c_0 = c_1 = 1$. Determine c_n explicitly for all non-negatice integers n.

Solution
The characteristic polynomial is

$$x^2 - x - 2 = (x-2)(x+1),$$

which has roots $x = -1$ and $x = 2$, each with multiplicity 1. Therefore

$$c_n = A(-1)^n + B(2)^n$$

for some constants A and B. From the initial conditions, we have

$$\begin{array}{rcl} 1 & = & A + B, \\ 1 & = & -A + 2B. \end{array}$$

This gives $A = \frac{1}{3}$ and $B = \frac{2}{3}$. Therefore

$$c_n = \frac{1}{3}(-1)^n + \frac{2}{3} \cdot 2^n = \frac{(-1)^n + 2^{n+1}}{3}$$

.

6.

How many k-tuples $(a_1, a_2, \ldots, a_k)$ of positive integers satisfy the inequality $a_1 + a_2 + \ldots + a_k < n$?

Solution
There is a bijection between such k-tuples and compositions $(a_1, a_2, \ldots, a_k, a_{k+1})$ of n.
–Given a k-tuple $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \ldots + a_k < n$, let $a_{k+1} = n - (a_1 + a_2 + \ldots + a_k)$. Therefore, $(a_1, a_2, \ldots, a_k, a_{k+1})$ is a composition of n.
–Given a composition $(a_1, a_2, \ldots, a_k, a_{k+1})$ of n, $(a_1, a_2, \ldots, a_k)$ is a k-tuple such that $a_1 + a_2 + \ldots + a_k < n$.

Since there is a bijection, the two sets have the same size. Therefore, the number of k-tuples that satisfy the given property is

$$\begin{array}{rcl} [x^n]\left(\frac{x}{1-x}\right)^{k+1} & = & [x^n]x^{k+1}(1-x)^{-(k+1)} \\ & = & [x^{n-(k+1)}](1-x)^{-(k+1)} \\ & = & \binom{(n-k-1)+(k+1)-1}{(k+1)-1} \\ & = & \binom{n-1}{k}. \end{array}$$