MATH 239 Assignment 1

- This assignment is due on Friday, September 21, 2012, at 10am in the drop boxes in St. Jerome's (section 1) or outside MC 4067 (the other two sections).
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.

1. Let $2 \leq k \leq n$ be integers. Consider the identity

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$$

   (a) Give a proof of this identity using the binomial theorem.

   (b) Give a combinatorial proof of this identity. (Hint: consider the set of all k-subsets of $\{1, 2, \ldots, n\}$ and classify them according to whether or not they contain the element 1 and/or the element 2.)

Solution:

(a) By the binomial theorem we have

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n = (1+x)^2(1+x)^{n-2} = (1+2x+x^2)(1+x)^{n-2}.$$

Using the binomial theorem again we get

$$(1+x)^{n-2} = \sum_{k=0}^{n-2} \binom{n-2}{k} x^k.$$

Therefore

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} x^k &= \sum_{k=0}^{n-2} \binom{n-2}{k} x^k + 2\sum_{k=0}^{n-2} \binom{n-2}{k} x^{k+1} + \sum_{k=0}^{n-2} \binom{n-2}{k} x^{k+2} \\
&= \sum_{k=0}^{n-2} \binom{n-2}{k} x^k + 2\sum_{k=1}^{n-1} \binom{n-2}{k-1} x^k + \sum_{k=2}^{n} \binom{n-2}{k-2} x^k.
\end{aligned}$$

Comparing the coefficient of x^k on both sides for any $k \geq 2$ gives

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$$

(b) Let S be the set of all k-subsets of $\{1, 2, \ldots, n\}$, then we know that $|S| = \binom{n}{k}$. Let S_0 be the subset of S consisting of those sets that do not contain the element 1 or the element 2. Let S_1 be the subset of S consisting of those sets that contain 1 but not 2, and let S_2 be the subset of S consisting of those sets that contain 2 but not 1. Finally let S_3 be the subset of S consisting of those sets that contain both 1 and 2. Then

$$|S| = |S_0| + |S_1| + |S_2| + |S_3|.$$

We find the sizes of the sets S_0, S_1, S_2, and S_3:

- S_0 consists of all k-subsets of $\{3, 4, \ldots, n\}$, so $|S_0| = \binom{n-2}{k}$,
- S_1 consists of all k-subsets made up of 1 together with a $(k-1)$-subset of $\{3, \ldots, n\}$, so $|S_1| = \binom{n-2}{k-1}$,
- S_2 consists of all k-subsets made up of 2 together with a $(k-1)$-subset of $\{3, \ldots, n\}$, so $|S_2| = \binom{n-2}{k-1}$,
- S_3 consists of all k-subsets made up of 1 and 2 together with a $(k-2)$-subset of $\{3, \ldots, n\}$, so $|S_3| = \binom{n-2}{k-2}$.

Therefore $\binom{n}{k} = |S| = |S_0| + |S_1| + |S_2| + |S_3| = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$, as required.

2. (a) Prove that for every positive even integer $n = 2m$,

$$\sum_{i=0}^{m} \binom{n}{2i} 2^{2i} = \sum_{i=0}^{m-1} \binom{n}{2i+1} 2^{2i+1} + 1.$$

(b) State and prove a similar identity for odd positive integers $n = 2m + 1$.

Solution:

(a) Start with the binomial theorem

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j.$$

Substitute $x = -2$ to obtain

$$(-1)^n = \sum_{j=0}^{n} \binom{n}{j} (-2)^j.$$

Since n is even we get $(-1)^n = 1$, and therefore

$$1 = -\sum_{j \text{ odd}} \binom{n}{j} 2^j + \sum_{j \text{ even}} \binom{n}{j} 2^j.$$

Expressing j odd as $j = 2i + 1$ and j even as $j = 2i$ we get

$$1 = -\sum_{i=0}^{m-1} \binom{n}{2i+1} 2^{2i+1} + \sum_{i=0}^{m} \binom{n}{2i} 2^{2i}.$$

Rearranging gives

$$\sum_{i=0}^{m} \binom{n}{2i} 2^{2i} = \sum_{i=0}^{m-1} \binom{n}{2i+1} 2^{2i+1} + 1$$

as required.

(b) For every positive odd integer $n = 2m+1$,

$$\sum_{i=0}^{m} \binom{n}{2i} 2^{2i} = \sum_{i=0}^{m} \binom{n}{2i+1} 2^{2i+1} - 1.$$

To prove this we follow the previous argument: Start with the binomial theorem

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j.$$

Substitute $x = -2$ to obtain

$$(-1)^n = \sum_{j=0}^{n} \binom{n}{j} (-2)^j.$$

Since n is odd we get $(-1)^n = -1$, and therefore

$$-1 = -\sum_{j \text{ odd}} \binom{n}{j} 2^j + \sum_{j \text{ even}} \binom{n}{j} 2^j.$$

Expressing j odd as $j = 2i+1$ and j even as $j = 2i$ we get

$$-1 = -\sum_{i=0}^{m} \binom{n}{2i+1} 2^{2i+1} + \sum_{i=0}^{m} \binom{n}{2i} 2^{2i}.$$

Rearranging gives

$$\sum_{i=0}^{m} \binom{n}{2i} 2^{2i} = \sum_{i=0}^{m} \binom{n}{2i+1} 2^{2i+1} - 1$$

as required.

3. Let $S = \{0, 1, \ldots, 15\}$.

   (a) For $0 \le i \le 4$ let S_i denote the subset of S consisting of those integers whose binary (i.e. base 2) representation has exactly i ones. Find S_i explicitly for $0 \le i \le 4$.

   (b) Find the generating series for S with respect to the weight function $w(\sigma) =$(the number of ones in the binary representation of σ).

   (c) Let $r \ge 1$ be an integer, and let $S(r) = \{0, 1, \ldots, 2^r - 1\}$. Find the generating series for $S(r)$ with respect to the weight function w. (Hint: the coefficients will be of the form $\binom{n}{k}$ for some n and k.) Prove your answer is correct.

Solution:

(a) We have $S_0 = \{0\}$, $S_1 = \{1, 2, 4, 8\}$, $S_2 = \{3, 5, 6, 9, 10, 12\}$, $S_3 = \{7, 11, 13, 14\}$, and $S_1 = \{15\}$.

(b) Using the explicit description of each S_i from above we get

$$\Phi_S(x) = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

(c) We claim that

$$\Phi_{S(r)}(x) = \sum_{i=0}^{r} \binom{r}{i} x^i.$$

To prove this, observe that the elements of $S(r)$ in their binary representations are in one-to-one correspondence with the set of all 01-strings of length r. The number of such strings that have exactly i ones is $\binom{r}{i}$, since we have exactly i positions to choose from the r total positions to put a one. Therefore the coefficient of x^i in $\Phi_{S(r)}(x)$ is $\binom{r}{i}$ for each $0 \leq i \leq r$.

4. Give a combinatorial proof of the identity

$$\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n.$$

(Hint: the ideas in Question 3 may help you.)

Solution: Let $S = \{0, 1, \ldots, 3^n - 1\}$. . We count S in two different ways. Clearly $|S| = 3^n$. On the other hand, we may count the elements of S based on their base 3 representations, as follows. For each i, $0 \leq i \leq n$, let S_i denote the subset of S consisting of those elements of S that have exactly i zeroes in their base 3 representation. Then $3^n = |S| = \sum_{i=0}^{n} |S_i|$.

To find the size of S_i for each i, we have exactly i positions to place a zero, out of the n possible positions, giving a total of $\binom{n}{i}$ choices. For each of these, the remaining $n - i$ positions must each be a 1 or a 2, and all choices are possible, giving 2^{n-i} choices. Therefore $|S_i| = \binom{n}{i} 2^{n-i}$.

This tells us

$$3^n = |S| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}.$$

Then using the fact that $\binom{n}{i} = \binom{n}{n-i}$ and changing the summation index gives us the required statement

$$\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n.$$