MATH 239 Assignment 4

DUE: NOON Friday 21 October 2011 in the drop boxes opposite the Math Tutorial Centre MC 4067 or next to the St. Jerome's library for the St. Jerome's section.

1. For each of the following sets A, determine whether A^* is uniquely created. Prove your assertion in each case. (Hint: to prove a set A^* is uniquely created, try using induction on the length.)

   (a) $A = \{1, 0010, 000110, 011100\}$

   (b) $A = \{0, 0110, 001100, 0001100\}$.

   SOLUTION. In the first case, we show that the elements of A^* are uniquely created. To see this, let σ be a string in A^*. We proceed by induction on the length ℓ of σ, the base case $\ell = 0$ being easy, since the only way to get the empty string is to use none of the elements of A .

   For the induction step, $\ell > 0$. There is some way to express $\sigma = a_1 a_2 \cdots a_k$, with each $a_i \in A$. Consider now the first entry in σ. If it is 1, then a_1 must be 1 . Since $a_2 \cdots a_k \in A^*$ has length $< \ell$, the inductive assumption implies it is uniquely created and, therefore, σ is uniquely created.

   On the other hand, the first entry in σ might be 0. It follows that, in this case, a_1 is one of 0010, 000110, and 011100. In all cases, a_1 is determined by the number of leading 0's in σ (there can only be one, two, or three leading 0's in σ and each determines which of 0010, 000110, and 011100 is equal to a_1). Again, $a_2 \cdots a_k$ is uniquely created by induction, so we see that σ is uniquely created, as claimed.

   For the second case, the elements of A^* are not uniquely created. For example, 001100001100 can be expressed as (0)(0110)(0001100) and as (001100)(001100).

2. For each of the following sets of binary strings S, write a decomposition that precisely describes S, in which the elements are uniquely created. Justify why each has the uniquely created property.

   (a) S is the set of binary strings in which each occurrence of 1 must be immediately followed by a string of at least two 0's.

   SOLUTION. In the 1-decomposition, we have $\{0\}^*(\{1\}\{0\}^*)^*$. In the strings under consideration for this question, the next two elements after a 1 must both be 0's. Thus, the following is the desired decomposition: $\{0\}^*(\{1\}\{00\}\{0\}^*)^*$.

   We notice that any decomposition of a string σ into $\{0\}^*(\{1\}\{00\}\{0\}^*)^*$ is also a decomposition into the 1-decomposition $\{0\}^*(\{1\}\{0\}^*)^*$ (anything in $\{00\}\{0\}^*$ is in $\{0\}^*$). Since the 1-decomposition uniquely creates strings, it follows that so does the new decomposition.

   (b) S is the set of binary strings in which each block of 1's has even length and is followed by a block of 0's whose length is precisely one half that of the length of the block of 1's.

SOLUTION. If the block of 1's has length a, then the succeeding block of 0's must have length $a/2$. The, the possibilities are: 110, 111100, 111111000, ... The standard block decomposition is $\{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^*$. A block of 1's at the end of a string cannot occur in this context, so all our strings are contained in $\{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*$. However, we only want some of these. Instead of all the possibilities $\{1\}\{1\}^*\{0\}\{0\}^*$, we only want $\{110, 111100, 111111000, \dots\}$. Thus, the desired decomposition is $\{0\}^*(\{110, 111100, 111111000, \dots\})^*$.

Every way of expressing a string using $\{0\}^*(\{110, 111100, 111111000, \dots\})^*$ is a way of expressing the string using the standard block decomposition. Since the block decomposition uniquely creates all strings, so does our decomposition.

(c) S is the set of binary strings in which each block of 1's has even length and is followed by a block of 0's whose length is at least one half that of the length of the block of 1's.

SOLUTION. Let S_h denote the set $\{110, 111100, 111111000, \dots\}$ of strings from the solution to (b). The set of strings consisting of a block of 1's of even length followed by a block of 0's of length at least half that of the block of 1's is $S_h\{0\}^*$. Thus, the desired decomposition is $\{0\}^*(S_h\{0\}^*)^*$.

As $S_h\{0\}^*$ is a subset of $\{1\}\{1\}^*\{0\}\{0\}^*$, our decomposition is again a special case of the standard block decomposition. Therefore the elements of $\{0\}^*(S_h\{0\}^*)^*$ are uniquely created.

3. The set S_e of binary strings in which every block of 1's has even length has decomposition $\{11\}^*(\{0\}\{0\}^*\{11\}\{11\}^*)^*\{0\}^*$. (You may assume that this is a correct decomposition and that strings are uniquely created by this decomposition.)

(a) Find the generating function for S_e with weight function being the length of the string.

SOLUTION. Because length adds over the parts of a decomposition, the Product Lemma implies that the generating function for S_e is the product of the generating functions for $\{11\}^*$, $(\{0\}\{0\}^*\{11\}\{11\}^*)^*$, and $\{0\}^*$.

By the *-Lemma,

$$\Phi_{\{11\}^*}(x) = \frac{1}{1-\Phi_{\{11\}}} = \frac{1}{1-x^2}\,.$$

The same analysis implies

$$\Phi_{\{0\}^*}(x) = \frac{1}{1-x}\,.$$

Likewise

$$\begin{aligned}\Phi_{(\{0\}\{0\}^*\{11\}\{11\}^*)^*}(x) &= \frac{1}{1-\Phi_{\{0\}\{0\}^*\{11\}\{11\}^*}} \\ &= \frac{1}{1-x\left(\frac{1}{1-x}\right)x^2\left(\frac{1}{1-x^2}\right)}\end{aligned}$$

Therefore,

$$\begin{aligned}
\Phi_{S_e}(x) &= \left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x\left(\frac{1}{1-x}\right)x^2\left(\frac{1}{1-x^2}\right)}\right)\left(\frac{1}{1-x}\right) \\
&= \frac{1}{(1-x)(1-x^2)-x^3} \\
&= \frac{1}{1-x-x^2}\,.
\end{aligned}$$

(b) Let $S_{\geq 1\,odd}$ denote the set of the binary strings in which at least one block of 1's has odd length. Find the generating function for $S_{\geq 1\,odd}$.

SOLUTION. Let S_{all} denote the set of all binary strings. Then $S_{all} = S_{\geq 1\,odd} \cup S_e$. By the Sum Lemma, $\Phi_{S_{all}}(x) = \Phi_{S_{\geq 1\,odd}}(x) + \Phi_{S_e}(x)$. Since we know that

$$\Phi_{S_{all}}(x) = \frac{1}{1-2x}\,,$$

and (from (a))

$$\Phi_{S_e}(x) = \frac{1}{1-x-x^2}\,,$$

we deduce that

$$\begin{aligned}
\Phi_{S_{\geq 1\,odd}}(x) &= \Phi_{S_{all}}(x) - \Phi_{S_e}(x) \\
&= \frac{1}{1-2x} - \frac{1}{1-x-x^2} \\
&= \frac{(1-x-x^2)-(1-2x)}{(1-2x)(1-x-x^2)} \\
&= \frac{x-x^2}{1-3x+x^2+2x^3}\,.
\end{aligned}$$

4. The generating function for some set S with weight function w satisfies the equation

$$\Phi_S(x) = \frac{x-x^2}{1-3x+x^2+2x^3}\,.$$

Find a simple closed form expression (not a recurrence relation) for the number of elements of S that have weight n.

SOLUTION. The roots of x^3-3x^2+x+2 are 2 and $(1\pm\sqrt{5})/2$. Therefore,

$$[x^n]\Phi_S(x) = A2^n + B\left(\frac{1+\sqrt{5}}{2}\right)^n + C\left(\frac{1-\sqrt{5}}{2}\right)^n\,.$$

We use the facts that $\Phi_S(x) = \frac{x-x^2}{1-3x+x^2+2x^3}$ and $\Phi_S(x) = \sum_{n\geq 0} a_n x^n$ to determine the first few a_n. We note they satisfy the recurrence $a_n - 3a_{n-1} + a_{n-2} + 2a_{n-3} = 0$ for $n \geq 3$.

We compute a_0, a_1, and a_2 from the equation

$$x - x^2 = (1 - 3x + x^2 + 2x^3)\left(\sum_{n\geq 0} a_n x^n\right).$$

- Comparing the coefficients of x^0 on both sides, we see $0 = a_0$.
- Comparing the coefficients of x^1 on both sides, we see $1 = a_1 - 3a_0$. Since $a_0 = 0$, we conclude $a_1 = 1$.
- Comparing the coefficients of x^2 on both sides, we see $-1 = a_2 - 3a_1 + a_0$. Since $a_0 = 0$ and $a_1 = 1$, we conclude that $a_2 = 2$.

These three values $a_0 = 0$, $a_1 = 1$, and $a_2 = 2$ allow us to compute A, B, and C.

- $n = 0$: $0 = A + B + C$.
- $n = 1$: $1 = 2A + \frac{1+\sqrt{5}}{2}B + \frac{1-\sqrt{5}}{2}C$.
- $n = 2$: $2 = 4A + (\frac{1+\sqrt{5}}{2})^2 B + (\frac{1-\sqrt{5}}{2})^2 C$, so $2 = 4A + \frac{(3+\sqrt{5})}{2}B + \frac{(3-\sqrt{5})}{2}C$.

Rewriting the second and third equations, we get:

$$\begin{aligned} 1 &= 2A + \frac{1}{2}(B + C) + \frac{\sqrt{5}}{2}(B - C) \quad \text{and} \\ 2 &= 4A + \frac{3}{2}(B + C) + \frac{\sqrt{5}}{2}(B - C)\,. \end{aligned}$$

Using $\frac{1}{2}(A + B + C) = 0$ and $\frac{3}{2}(A + B + C) = 0$, these become

$$\begin{aligned} 1 &= \frac{3}{2}A + \frac{\sqrt{5}}{2}(B - C) \quad \text{and} \\ 2 &= \frac{5}{2}A + \frac{\sqrt{5}}{2}(B - C)\,. \end{aligned}$$

Subtracting the first from the second implies $A = 1$. The equations $A + B + C = 0$ and $1 = (3/2)A + (\sqrt{5}/2)(B - C)$ become

$$\begin{aligned} -1 &= B + C \quad \text{and} \\ -\frac{1}{2} &= \frac{\sqrt{5}}{2}(B - C)\,. \end{aligned}$$

The second is the same as $B - C = -\sqrt{5}/5$. Adding this to $B + C = -1$, we conclude $2B = (-5 - \sqrt{5})/5$, or $B = (-5 - \sqrt{5})/10$. Thus, $C = (-5 + \sqrt{5})/10$.

It follows that

$$a_n = 2^n - \left(\frac{5 + \sqrt{5}}{10}\right)\left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{-5 + \sqrt{5}}{10}\right)\left(\frac{1 - \sqrt{5}}{2}\right)^n.$$