MATH 239 Assignment 6

- This assignment is due on Friday, October 26th, 2012, at 10 am in the drop boxes in St. Jerome's (section 1) or outside MC 4067 (the other two sections).
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.

1. For $n \geq r \geq 1$, define the graph $G_{n,r}$ as follows: The vertices of $G_{n,r}$ are r-element subsets of $\{1, ..., n\}$. Two vertices U and V are adjacent if and only if $|U \cap V| = 1$.

    (a) Draw $G_{4,1}$ and $G_{4,2}$.

    (b) Prove that for any $n \geq r \geq 1$, $G_{n,r}$ is k-regular and determine k.

    (c) Determine how many vertices and edges $G_{n,r}$ has.

2. Define another graph, $H_{n,r}$, for $n \geq r \geq 1$ as follows: The vertices of $H_{n,r}$ are $\{0, 1\}$-strings of length n which have exactly r zeros (and therefore $n - r$ ones). Two vertices $x_1 \cdots x_n$ and $y_1 \cdots y_n$ are adjacent if and only if

    $$|\{i : x_i = 0 = y_i\}| = 1.$$

    Prove that $H_{n,r}$ is isomorphic to $G_{n,r}$ from the previous question by defining and justifying an isomorphism between the two.

3. Draw two separate graphs which are both 3-regular and have exactly 6 vertices, but are ***not*** isomorphic to each other. Justify that they are non-isomorphic.

    (You can to this by describing some property of one graph which the other graph does not have, but would have to be preserved by an isomorphism).

4. For a graph G, we define the complement graph of G, denoted $\overline{G}$, with $V(\overline{G}) = V(G)$, and $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$.

    (a) Define G as $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$. Draw G and $\overline{G}$.

    (b) Suppose an arbitrary graph G has $|V(G)| = p$ vertices and $|E(G)| = q$ edges. How many vertices and edges does $\overline{G}$ have? (Express your answers in terms of p and q.)

    (c) Prove that if G is isomorphic to $\overline{G}$, then either $p \equiv 0 \mod 4$, or $p \equiv 1 \mod 4$.

5. Let $\mathcal{G}_p$ be the set of all graphs with vertex set $\{1, ..., p\}$. Let $\mathcal{G} = \cup_{p \geq 0} \mathcal{G}_p$.

    (a) Define a weight function on $\mathcal{G}$ by $w(G) = |V(G)|$ for all $G \in \mathcal{G}$. Determine $\Phi_{\mathcal{G}}(x)$ with respect to w. Your final answer may be in the form of an infinite sum.

    (b) Next consider the weight function $w'(G) = |E(G)|$ for all $G \in \mathcal{G}$. Determine $\Phi_{\mathcal{G}_p}(x)$ with respect to w', where $p \geq 0$. Your final answer should not include a large summation.