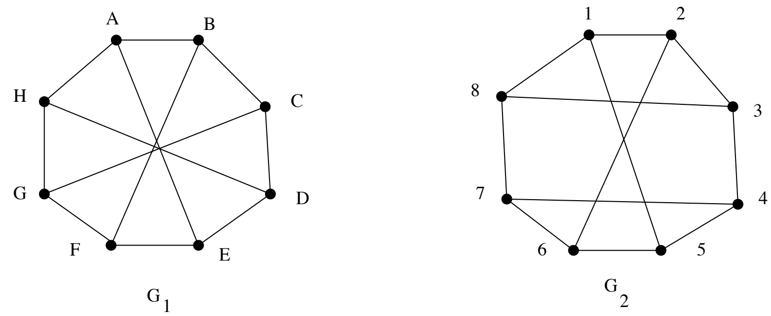
MATH 239 Assignment 6

DUE: NOON Friday 4 November 2011 in the drop boxes opposite the Math Tutorial Centre MC 4067 or next to the St. Jerome's library for the St. Jerome's section.

1. (a) Show that the following two graphs are isomorphic.



SOLUTION. For any $x \in \{A, B, C, D, E, F, G, H\}$, define $f(x) \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ as shown below.

x	A	B	C	D	E	F	G	H
$f(x)$	1	5	4	3	2	6	7	8

If we re-label vertex x in graph G_1 with the label $f(x)$, it is easy to confirm that the vertices 1,2,3,4,5,6,7,8,1 in that order determine a cycle of length 8. By inspection, we see that G_1 also has edges $(1, 5), (2, 6), (3, 8), (4, 7)$. Therefore, graphs G_1 and G_2 are isomorphic. (Note: this isomorphism is not unique; there are many others.)

(b) Show that the following two graphs are isomorphic.

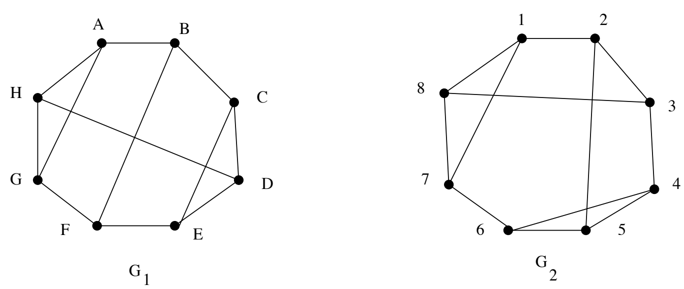


Figure 1: **Two 3-regular graphs on 8 vertices**

SOLUTION. For any $x \in \{A, B, C, D, E, F, G, H\}$, define $f(x) \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ as shown below.

x	A	B	C	D	E	F	G	H
$f(x)$	8	3	4	6	5	2	1	7

If we re-label vertex x in graph G_1 with the label $f(x)$, it is easy to confirm that the vertices 1,2,3,4,5,6,7,8,1 in that order determine a cycle of length 8. By inspection, we see that G_1 also has edges $(1,7),(2,5),(3,8),(4,6)$. Therefore, graphs G_1 and G_2 are isomorphic. (Note: this isomorphism is not unique; there are many others.)

(c) Make a list of all 3-regular graphs with 8 vertices, up to isomorphism. In other words, each 3-regular graph with 8 vertices should be isomorphic to exactly one of the graphs on your list. Briefly explain why no two are isomorphic. (Hint: there are exactly 6 graphs in the list.)

SOLUTION.

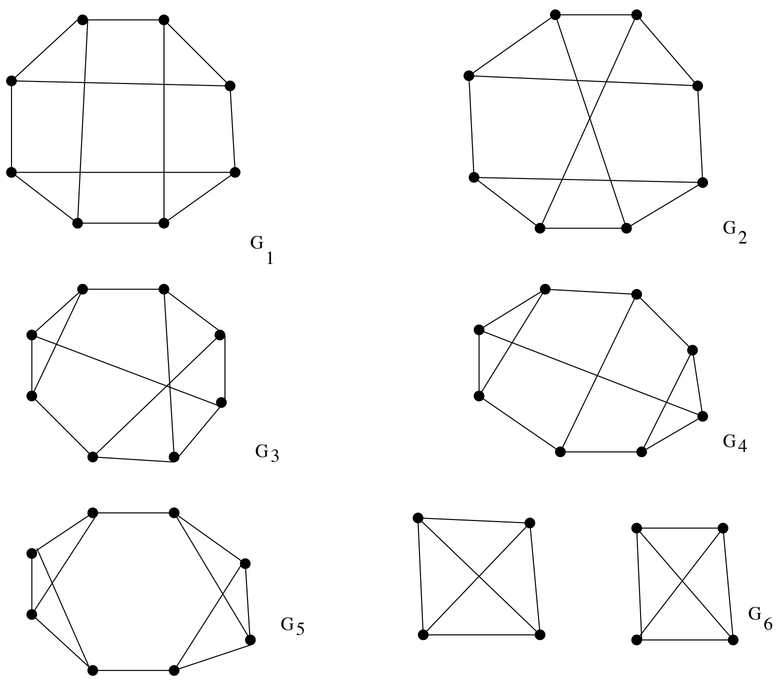


Figure 2: **Nonisomorphic 3-regular graphs with 8 vertices**

Graph G_1 has no cycles of length 3 nor 5; G_2 has no cycles of length 3, but has cycles of length 5; G_3 has one cycle of length 3; G_4 has 2 cycles of length 3; and G_5 has 4 cycles of length 3. These 5 graphs are all connected; G_6 has 2 components. Therefore, no two of these graphs are isomorphic.

2. Determine whether the graphs X and Y shown in Figure 2 below are isomorphic. Prove your claim is correct.

SOLUTION.

The graphs are not isomorphic. To see this, observe that the vertices of X of degree 4 are B, C, E, F, and the subgraph of X consisting of these four vertices and the edges of X joining them has 5 edges. In Y, the vertices of degree 4 are $1, 2, 5, 8$. The corresponding subgraph of Y consisting of just these 4 vertices and their edges has only 3 edges. Thus the two graphs can't be isomorphic.

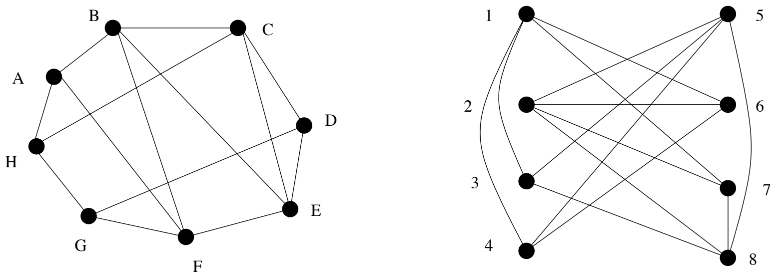


Figure 3: **The graphs X and Y**

3. For n a positive integer, define the *prime graph* P_n as follows: $V(P_n) = \{1, 2, \ldots, n\}$ and

$$E(P_n) = \{\{u, v\} \subseteq \{1, 2, \ldots, n\} \mid u + v \text{ is a prime}\}.$$

(a) Prove that P_n is bipartite for all n.

SOLUTION. If $\{u, v\}$ is an edge of P_n, then $u + v \geq 1 + 2 = 3$; hence, $u + v$ is an odd prime. Therefore, one of u and v is odd, and the other is even. Since every edge joins an odd vertex to an even vertex, P_n is a bipartite graph with bipartition

$$A = \{2i \mid 2i \in \{1, 2, \ldots, n\}\} \text{ and } B = \{2i + 1 \mid 2i + 1 \in \{1, 2, \ldots, n\}\}.$$

(b) Prove that P_n is connected for all n. (You may assume, without proof, that for every integer $k > 1$, there is a prime number r such that $k < r < 2k$.)

SOLUTION. We prove this by induction on the integer n.

(Base Case) The prime graph P_1 has just 1 vertex, and hence is connected.

(Induction Hypothesis) For any integer $n \geq 1$, assume P_n is connected.

(Inductive Step) We now show that P_{n+1} is also connected. Let r be any prime in the range $n + 1 < r < 2n + 2$. Then $r - n - 1 < 2n + 2 - n - 1 = n + 1$; that is, $r - n - 1$ is a vertex of P_n.

Since $(r - n - 1) + (n + 1) = r$, a prime, vertex $n + 1$ is adjacent to vertex $r - n - 1$ in P_{n+1}. Since $r - n - 1$ is a vertex in P_n and, by our Induction Hypothesis, P_n is connected, there is a path from $r - n - 1$ to every vertex in P_n. Therefore, there is a path from $r - n - 1$ to every vertex in P_{n+1}. Therefore P_{n+1} is connected.

Thus by induction, it follows that P_n is connected for every positive integer n.

4. Let G be a graph, and suppose v and w are the only vertices in G of odd degree. Prove that G contains a path from v to w.

SOLUTION. Suppose on the contrary that no path exists between the only two vertices v and w having odd degree. Then v and w are in different components of G, say C_1 contains v and C_2 contains w. But then C_1 is itself a graph that contains exactly one vertex of odd degree. This contradicts the corollary of the Handshake

Lemma which tells us that the number of vertices of odd degree in any graph is even. Thus v and w must be joined by a path in G.

5. Let G be a connected graph and suppose e is a bridge of G. Let x and y be vertices of G that are joined by a path P that contains e. Prove that every path joining x and y must contain e.

   SOLUTION.

   Let $e = uv$, so that the path P is of the form $xz_1 \dots z_t uvw_1 \dots w_s y$. Since e is a bridge, the graph $G - e$ has two components, C_u containing u and C_v containing v. The path $xz_1 \dots z_t u$ shows that $x \in C_u$, and the path $vw_1 \dots w_s y$ shows that $y \in C_v$. Suppose on the contrary that Q is a path in G joining x and y that does not contain e. Then Q is also a path in $G - e$ from the vertex $x \in C_u$ to the vertex $y \in C_v$. This contradicts that fact that C_u and C_v are distinct components of $G - e$. Therefore no such path can exist.