MATH 239 – Tutorial 5

1.

Let $\{a_n\}$ be the sequence that satisfies the recurrence

$$a_n - 3a_{n-2} + 2a_{n-3} = 0$$

for $n \geq 3$, with initial conditions $a_0 = 4$, $a_1 = -1$, $a_2 = 3$. Determine an explicit formula for a_n.

Solution
The characteristic polynomial is $x^3 - 3x + 2 = (x-1)^2(x+2)$, which has roots $x = 1$ with multiplicity 2 and $x = -2$ with multiplicity 1. So

$$a_n = An + B + C(-2)^n.$$

for some constants A, B and C. From the initial conditions, we have $A = -2$, $B = 3$, and $C = 1$. So

$$a_n = -2n + 3 + (-2)^n.$$

2.

Let $\{b_n\}$ be the sequence that satisfies the recurrence

$$b_n - 3b_{n-2} + 2b_{n-3} = 12$$

for $n \geq 3$, with initial conditions $b_0 = 0$, $b_1 = 8$, and $b_2 = 2$. Determine an explicit formula for b_n.

Solution
For a specific solution c_n, it can be seen that $c_n = \alpha$ or $c_n = \alpha n$ does not work (show). So we check $c_n = \alpha n^2$, which gives

$$c_n - 3c_{n-2} + 2c_{n-3} = 6\alpha.$$

So $\alpha = 2$, and $2n^2$ is a specific solution. Then

$$b_n = 2n^2 + An + B + C(-2)^n$$

(from Q1) for some constants A, B and C. Use the initial conditions to get $A = 0$, $B = 2$, and $C = -2$. So

$$b_n = 2n^2 + 2 + (-2)^{n+1}.$$

3.

(a) Find a and b such that

$$\frac{11-2x}{(2-x)(3+2x)} = \frac{a}{2-x} + \frac{b}{3+2x}.$$

(b) Find a closed-form expression for

$$[x^n]\frac{11-2x}{(2-x)(3+2x)}.$$

Solution

(a) Expanding the right-hand side gives

$$\frac{a}{2-x} + \frac{b}{3+2x} = \frac{a(3+2x)+b(2-x)}{(2-x)(3+2x)} = \frac{(3a+2b)+(2a-b)x}{(2-x)(3+2x)},$$

so we have $3a+2b = 11$ and $2a-b=-2$, which has the solution $a=1$, $b=4$.

(b) We have

$$\frac{1}{2-x} = \left(\frac{1}{2}\right)\left(\frac{1}{1-\frac{x}{2}}\right) = \frac{1}{2}\sum_{n\geq 0}\left(\frac{x}{2}\right)^n$$

and

$$\frac{4}{3+2x} = \left(\frac{4}{3}\right)\left(\frac{1}{1+\frac{2}{3}x}\right) = \frac{4}{3}\sum_{n\geq 0}\left(-\frac{2}{3}x\right)^n,$$

so

$$\begin{aligned}
[x^n]\frac{11-2x}{(2-x)(3+2x)} &= [x^n]\left(\frac{1}{2-x}+\frac{4}{3+2x}\right)\\
&= \frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{4}{3}\left(-\frac{2}{3}\right)^n\\
&= \frac{1}{2^{n+1}} + \frac{(-1)^n 2^{n+2}}{3^{n+1}}.
\end{aligned}$$

4. (Problem Set 2.8)

Find the generating series (with respect to length) for the set of binary strings that do not contain the substring 0110.

Solution

Let L denote the set of strings that do not contain the substring 0110, Let M denote the set of strings that contain the substring 0110 exactly once, as a suffix. Claim:

$$\begin{aligned}
\{\epsilon\}\cup L\{0,1\} &= L\cup M\\
L\{0110\} &= M\cup M\{110\}
\end{aligned}$$

(See Problem 2.8.2) For the first equation, if we append a bit to ϵ or a string in L, the resulting string is either in L or in M, which shows that $\{\epsilon\} \cup L\{0,1\} \subseteq L \cup M$. To show the other inclusion, let $\alpha \in L \cup M$. If $\alpha \in L$, then α is either the empty string, or we can remove the last bit to obtain another string in L, so $\alpha \in L\{0,1\}$. If $\alpha \in M$, then by removing the last bit from α, it destroys the only copy of 0110 in α. The remaining string is in L, so $\alpha \in L\{0,1\}$ as well. This shows that $\{\epsilon\} \cup L\{0,1\} \supseteq L \cup M$, hence equality holds.
For the second equation, if $a \in L$, then $b = a0110$ clearly contains 0110 as a substring, but it might contain multiple copies. If a ends in 011 then b contains two copies of 0110, and dropping the final 110 from b gives us a string in M. Otherwise, b contains only one copy of 0110, so $b \in M$. Therefore $L\{0110\} \subseteq M \cup M\{110\}$. On the other hand, it is clear that $M \subseteq L\{0110\}$, and $M\{110\} \subseteq L\{0110\}$, since strings in M and $M\{110\}$ can all be formed by appending 0110 to a string with no occurrence of 0110. Therefore $M \cup M\{0,1\} \subseteq L\{0110\}$, and the second equation holds.

By the sum and composition rules, we have

$$\begin{array}{lcl} 1 + 2x\Phi_L(x) & = & \Phi_L(x) + \Phi_M(x) \\ x^4\Phi_L(x) & = & (1 + x^3)\Phi_M(x). \end{array}$$

From the second equation, $\Phi_M(x) = \frac{x^4}{1+x^3}\Phi_L(x)$. Using this in the first equation gives

$$1 = \left[(1 - 2x) + \frac{x^4}{1 + x^3}\right] \Phi_L(x),$$

so

$$\Phi_L(x) = \left[(1 - 2x) + \frac{x^4}{1 + x^3}\right]^{-1}.$$

This can be simplified to the rational form, giving

$$\frac{1 + x^3}{1 - 2x + x^3 - x^4}$$

for $n \geq 0$.