MATH 239 Assignment 10

This assignment is for practice only, and is not to be handed in.

1. Find a maximum matching and a minimum cover in the graph in Figure 1.

   Solution:

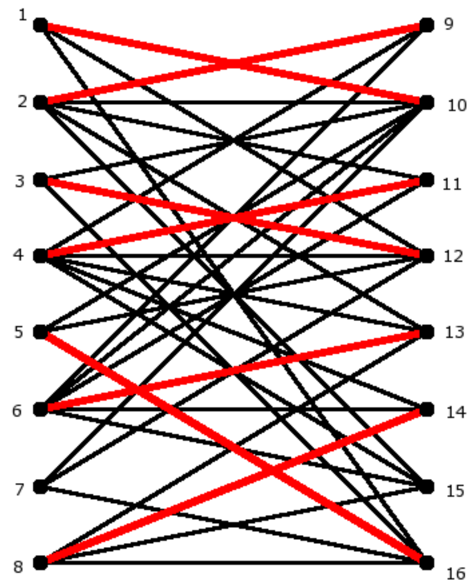

Figure 1: A maximum matching

We claim the matching shown in bold in Figure 1 is maximum. To show this, we find a cover of the same size. Following the bipartite matching algorithm in the course notes, let M be the matching above, and $V = (A, B)$ with A being the vertices $\{1, 2, \ldots, 8\}$.

Step 1. Set $\hat{X} = \{7\}$, $\hat{Y} = \emptyset$.

Step 2. Let $\hat{Y} = \{10, 12, 16\}$ and set $pr(10) = pr(12) = pr(16) = 7$.

Step 3. Step 2 added some vertices to $\hat{Y}$, continue.

Step 4. No unsaturated vertices in $\hat{Y}$.

Step 5. Add $\{1, 3, 5\}$ to $\hat{X}$, and set $pr(1) = 10, pr(3) = 12$ and $pr(5) = 16$. Now $\hat{X} = \{1, 3, 5, 7\}$. Go to Step 2.

Step 2. No new vertices added to $\hat{Y}$.

Step 3. M is a maximum matching and the cover $C = \hat{Y} \cup (A \backslash \hat{X}) = \{2, 4, 6, 8, 10, 12, 16\}$ is minimum.

Indeed the maximum matching above and the minimum cover $C = \{2, 4, 6, 8, 10, 12, 16\}$ have the same size (7).

2. Find a subset D of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ such that $|N(D)| < D$.

   Solution: A suitable choice can be found from the proof of Hall's Theorem as $D = \hat{X}$: so the set $D = \{1, 3, 5, 7\}$ works. Its neighbourhood is $\{10, 12, 16\}$.

3. Let k be a positive integer and suppose G is a bipartite graph in which every vertex has degree precisely k. Prove that G has k perfect matchings, no two having an edge in common.

   Solution: We use induction on k. If $k = 1$ then a 1-regular graph is exactly a perfect matching, so the claim holds.

   Suppose $k \geq 2$ and the claim holds for smaller values of k. From class we know that G has a perfect matching M. Let G' be the graph obtained by removing the edges of M from G. Since M is a perfect matching, the degree of every vertex goes down by exactly one. So every vertex of G' has degree exactly $k - 1$. By induction, G' has $k - 1$ perfect matchings, no two of which share an edge. Then these together with M form k perfect matchings of G, no two of which share an edge.

4. For each positive integer $n \geq 24$, find an example of a bipartite graph with n vertices on each side, with minimum degree at least three, and with no matching of size larger than $n/4$.

   Solution: Let $n \geq 24$ be given. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. The graph G is formed as follows: each $a \in A \setminus \{a_1, a_2, a_3\}$ is adjacent to $\{b_1, b_2, b_3\}$, and each $a \in \{a_1, a_2, a_3\}$ is adjacent to $B \setminus \{b_1, b_2, b_3\}$. Then each $a \in A$ and $b \in B$ has degree at least 3. Moreover, this graph has a cover of size 6, as every edge is incident to a vertex in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. Therefore by König's Theorem, the maximum size of a matching is at most 6. But if $n \geq 24$ then $6 \leq n/4$ so the claim holds.

5. Let G be a graph with $2n$ vertices such that every vertex has degree at least n. Prove that G has a perfect matching.

   Solution: Let M be a maximum matching in G, and suppose on the contrary that $|M| \leq n - 1$. Then there exist vertices x and y that are exposed by M, since M saturates at most $2(n-1)$ vertices. Then all neighbours of x and y must be saturated by M, otherwise we could add a new edge to M to get a larger matching. For each $z \in N(x)$, let $u(z)$ denote the vertex such that $zu(z) \in M$. Then the set $U = \{u(z) : z \in N(x)\}$ has size $|N(X)| \geq n$ and every element of U is saturated by M. Since M saturates at most $2(n - 1)$ vertices, there are at most $n - 2$ vertices saturated by M that are not in U. Since every neighbour of y is saturated, this implies that some neighbour w of y is in U. But then $ywu^{-1}(w)x$ is an m-augmenting path, where $u^{-1}(w)$ means the vertex z such that $u(z) = w$. This contradicts the assumption that M is a maximum matching. Therefore G has a perfect matching.

6. Give an example of a 3-regular graph that does not have a perfect matching. (Note that such a graph cannot be bipartite.)

   Solution: The graph shown in the figure is 3-regular. To see that it has no perfect matching, suppose on the contrary that it does. Then the vertex a must be incident to some matching edge, say without loss of generality ab is in the matching. But then the five vertices g, h, i, j, k cannot all be incident to matching edges.

7. Let G be a bipartite graph with vertex classes A and B, where $|A| = |B| = 2n$. Suppose that $|N(X)| \geq |X|$ for all subsets $X \subset A$ with $|X| \leq n$, and $|N(X)| \geq |X|$ for all subsets $X \subset B$ with $|X| \leq n$. Prove that G has a perfect matching.

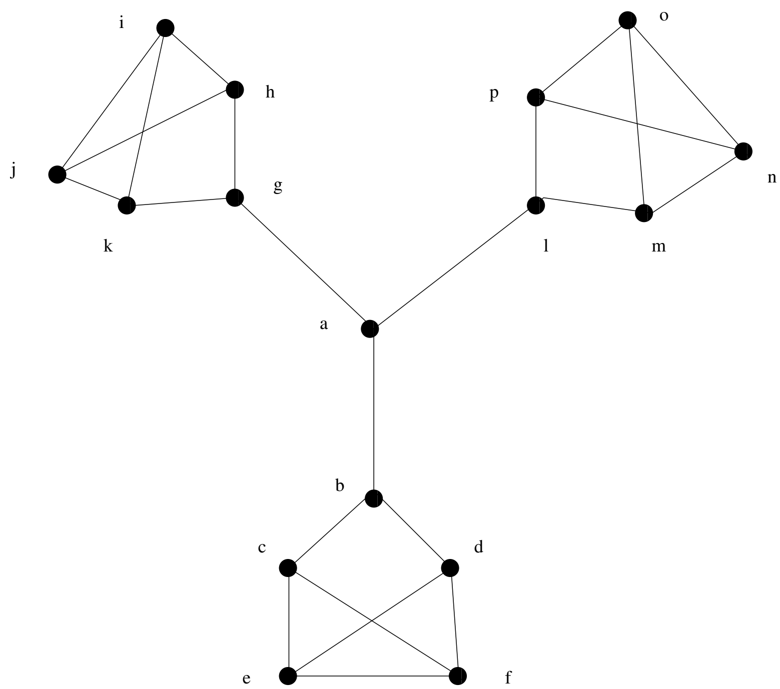

Figure 2:

Solution: We verify the condition for Hall's Theorem in G. We are given that $|N(X)| \geq |X|$ for all subsets $X \subset A$ with $|X| \leq n$, so we just need to check that $|N(X)| \geq |X|$ for all subsets $X \subset A$ with $|X| > n$. Let X be such a subset. Since X contains a subset S of size exactly n, we know that $|N(X)| \geq |N(S)| \geq |S| = n$. Suppose on the contrary that $|N(X)| < |X|$. Let $Y = B \setminus N(X)$. Then by definition of neighbourhood, there are no edges of G joining X to Y. This implies that $N(Y) \subseteq A \setminus X$. But since $|N(X)| \geq n$ we know $|Y| = |B \setminus N(X)| \leq n$, and so by the given property we know $|N(Y)| \geq |Y|$. Therefore

$$|A| = |N(Y)| + |X| > |Y| + |N(X)| = |B|,$$

contradicting the given fact that $|A| = |B|$. Therefore we must have $|N(X)| \geq |X|$ for every $X \subseteq A$, which by Hall's Theorem implies that G has a perfect matching.