MATH 239 Assignment 6

- This assignment is due on Friday, October 26th, 2012, at 10 am in the drop boxes in St. Jerome's (section 1) or outside MC 4067 (the other two sections).
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.

1. For $n \geq r \geq 1$, define the graph $G_{n,r}$ as follows: The vertices of $G_{n,r}$ are r-element subsets of $\{1, ..., n\}$. Two vertices U and V are adjacent if and only if $|U \cap V| = 1$.

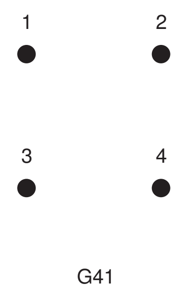
   (a) Draw $G_{4,1}$ and $G_{4,2}$.

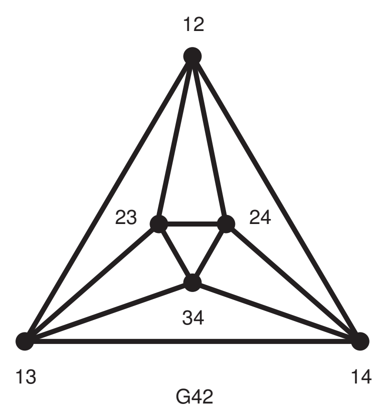
   (b) Prove that for any $n \geq r \geq 1$, $G_{n,r}$ is k-regular and determine k.

   (c) Determine how many vertices and edges $G_{n,r}$ has.

   Solution:

   (a) $G_{4,1}$ consists of four vertices of degree 0. $G_{4,2}$ is the octahedron:





   (b) If $r = 1$ then $G_{n,r}$ consists of n vertices, each of degree 0. Therefore $G_{n,1}$ is 0-regular for all $n \geq 1$.

   Now suppose $r \geq 2$ and fix a vertex $U = \{u_1, ..., u_r\}$ of $G_{n,r}$. Then another vertex $V = \{v_1, ..., v_r\}$ is adjacent to U if and only if $|U \cap V| = 1$. There are r ways to choose which element of U will be the intersection, and $\binom{n-r}{r-1}$ ways to choose $r-1$ remaining elements that are not in U. Thus there are exactly $r\binom{n-r}{r-1}$ vertices adjacent to U.

   Since U was an arbitrary vertex, we conclude that $G_{n,r}$ is k-regular where $k = r\binom{n-r}{r-1}$, for $r \geq 2$.

   (c) Clearly $G_{n,r}$ has $|V(G_{n,r})| = \binom{n}{r}$ vertices. If $r = 1$ then $|E(G_{n,r})| = 0$ for all $n \geq 1$, since $G_{n,1}$ is 0-regular. For $r \geq 2$, we use Theorem 4.3.1 and part b:

$$
\begin{aligned}
|E(G_{n,r})| &= \frac{1}{2}\sum_{V \in V(G_{n,r})} \deg(V) \\
&= \frac{1}{2}\sum_{V \in V(G_{n,r})} r\binom{n-r}{r-1} \\
&= \frac{1}{2} r \binom{n-r}{r-1}\binom{n}{r}.
\end{aligned}
$$

2. Define another graph, $H_{n,r}$, for $n \geq r \geq 1$ as follows: The vertices of $H_{n,r}$ are $\{0,1\}$-strings of length n which have exactly r zeros (and therefore $n-r$ ones). Two vertices $x_1 \cdots x_n$ and $y_1 \cdots y_n$ are adjacent if and only if

$$|\{i : x_i = 0 = y_i\}| = 1.$$

Prove that $H_{n,r}$ is isomorphic to $G_{n,r}$ from the previous question by defining and justifying an isomorphism between the two.

Solution:

We need to define a function which maps the vertices of $G_{n,r}$ to the vertices of $H_{n,r}$. A logical choice is:

$$
\begin{gathered}
f : V(G_{n,r}) \to V(H_{n,r}) \\
f(A) := x_1 \cdots x_n, \text{ where } x_i = \begin{cases} 0 & \text{if } i \in A \\ 1 & \text{if } i \notin A \end{cases}, \quad \text{for } i = 1, ..., n.
\end{gathered}
$$

Notice that if A is an r-element subset of $\{1, ..., n\}$, then $f(A)$ is a $\{0,1\}$-string of length n with exactly r elements equal to zero. Thus f is a well defined map from $V(G_{n,r})$ to $V(H_{n,r})$. We must next verify that f is indeed a bijection, which we can do by verifying that it is both injective and surjective. Alternatively, we can show that f has an inverse. Consider the function:

$$
\begin{gathered}
g : V(H_{n,r}) \to V(G_{n,r}) \\
g(x_1 \cdots x_n) := \{i : x_i = 0\}.
\end{gathered}
$$

Then it is easy to see that for any $A \in V(G_{n,r})$, we have $g(f(A)) = A$. Likewise for any $x_1 \cdots x_n \in V(H_{n,r})$, we have $f(g(x_1 \cdots x_n)) = x_1 \cdots x_n$. Thus $g = f^{-1}$, so f is invertible and hence a bijection, as required.

All that remains to do is verify that f preserves vertex adjacency between $G_{n,r}$ and $H_{n,r}$. Suppose that A and B are two vertices of $G_{n,r}$, and let $f(A) = x_1 \cdots x_n$, $f(B) = y_1 \cdots y_n$. Then

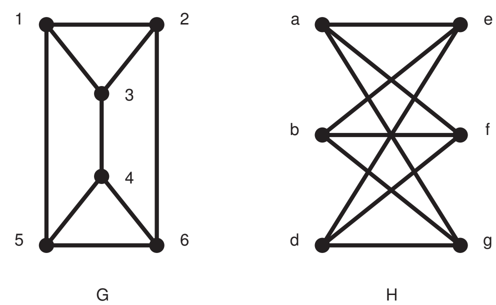
$$
\begin{aligned}
\{A, B\} \in E(G_{n,r}) &\Leftrightarrow |A \cap B| = 1 \\
&\Leftrightarrow A \cap B = \{k\}, \text{ for some } k \in \{1, ..., n\} \\
&\Leftrightarrow x_k = 0 = y_k, \text{ and } x_i = 0 = y_i \text{ only when } i = k. \\
&\Leftrightarrow |\{i : x_i = 0 = y_i\}| = |\{k\}| = 1 \\
&\Leftrightarrow \{f(A), f(B)\} \in E(H_{n,r}).
\end{aligned}
$$

So f preserves adjacency and hence is an isomorphism between $G_{n,r}$ and $H_{n,r}$.

3. Draw two separate graphs which are both 3-regular and have exactly 6 vertices, but are ***not*** isomorphic to each other. Justify that they are non-isomorphic.

   (You can to this by describing some property of one graph which the other graph does not have, but would have to be preserved by an isomorphism).

   Solution: There are, in fact, *only* two non-isomorphic 3-regular graphs on 6 vertices. They look like:



   To see that they are not isomorphic, notice that G has three vertices, $\{1, 2, 3\}$, which are all pairwise adjacent. There are no such three vertices in H, so there is definitely no adjacency preserving map between the two graphs.

4. For a graph G, we define the complement graph of G, denoted $\overline{G}$, with $V(\overline{G}) = V(G)$, and $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$.

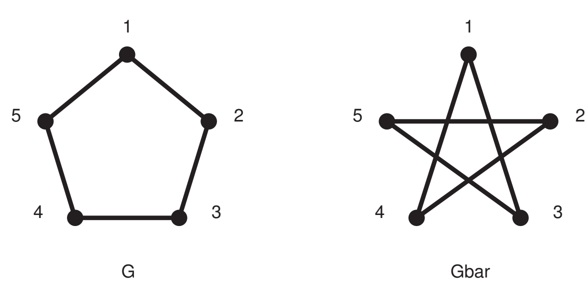
   (a) Define G as $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$. Draw G and $\overline{G}$.

   (b) Suppose an arbitrary graph G has $|V(G)| = p$ vertices and $|E(G)| = q$ edges. How many vertices and edges does $\overline{G}$ have? (Express your answers in terms of p and q.)

   (c) Prove that if G is isomorphic to $\overline{G}$, then either $p \equiv 0 \mod 4$, or $p \equiv 1 \mod 4$.

   Solution:

   (a) Both G and $\overline{G}$ are cycles of length 5:



   (b) By definition of the complement, $\overline{G}$ clearly has $|V(\overline{G})| = p$ vertices, the same as G. To count the number of edges, we observe the total number of edges that can exist in a graph with p vertices is $\binom{p}{2}$, the number of edges in the complete graph K_p. Since an edge is in $\overline{G}$ precisely when it is not in G, we conclude that

$$|E(\overline{G})| = \tbinom{p}{2} - |E(G)| = \tbinom{p}{2} - q.$$

(c) If G and $\overline{G}$ are isomorphic, then they must have the same number of edges. So by the previous part we must have

$$\begin{aligned} |E(G)| &= |E(\overline{G})| \\ q &= \tbinom{p}{2} - q \\ 2q &= \frac{1}{2}p(p-1). \end{aligned}$$

Therefore $q = \frac{1}{4}p(p-1)$. Since q is an integer, we must have that 4 divides $p(p-1)$. Since 2 divides either p or $p-1$ (and not both), we conclude that either 4 divides p or 4 divides $p-1$. The result follows.

5. Let $\mathcal{G}_p$ be the set of all graphs with vertex set $\{1, ..., p\}$. Let $\mathcal{G} = \cup_{p\geq 0}\mathcal{G}_p$.

   (a) Define a weight function on $\mathcal{G}$ by $w(G) = |V(G)|$ for all $G \in \mathcal{G}$. Determine $\Phi_{\mathcal{G}}(x)$ with respect to w. Your final answer may be in the form of an infinite sum.

   (b) Next consider the weight function $w'(G) = |E(G)|$ for all $G \in \mathcal{G}$. Determine $\Phi_{\mathcal{G}_p}(x)$ with respect to w', where $p \geq 0$. Your final answer should not include a large summation.

Solution:

Note: The below solutions assume that $\mathcal{G}_0$ has a single element (called the null graph). Solutions may vary slightly if you assume that $\mathcal{G}_0 = \emptyset$.

(a) By definition of generating functions, $\Phi_{\mathcal{G}}(x) = \sum_{n\geq 0} g_n x^n$, where g_n is the number of graphs on vertex set $\{1, ..., n\}$. To determine what g_n is, we first observe that there are $\binom{n}{2}$ total possible edges in a graph on vertex set $\{1, ..., n\}$. Any subset of these edges gives a unique graph, and there are $2^{\binom{n}{2}} = 2^{\frac{1}{2}n(n-1)}$ possible subsets . Thus,

$$\Phi_{\mathcal{G}}(x) = \sum_{n\geq 0} g_n x^n = \sum_{n\geq 0} 2^{\frac{1}{2}n(n-1)} x^n.$$

(b) Similar to the first part, $\Phi_{\mathcal{G}_p}(x) = \sum_{n\geq 0} p_n x^n$, where p_n is the number of graphs with n edges on vertex set $\{1, ..., p\}$. As we already observed, there are $\binom{p}{2}$ total possible edges in a graph with vertex set $\{1, ..., p\}$. Hence there are

$$\binom{\binom{p}{2}}{n} = \binom{\frac{1}{2}p(p-1)}{n}$$

ways to choose n edges for a graph on vertex set $\{1, ..., p\}$. Each of these choices gives a different graph, so

$$\begin{aligned} \Phi_{\mathcal{G}_p}(x) &= \sum_{n\geq 0} p_n x^n = \sum_{n\geq 0} \binom{\frac{1}{2}p(p-1)}{n} x^n. \\ &= (1+x)^{\frac{1}{2}p(p-1)}. \end{aligned}$$