MATH 239 Assignment 10

This assignment is for practice only, and is not to be handed in.

1. Find a maximum matching and a minimum cover in the graph in Figure 1.

   Solution:

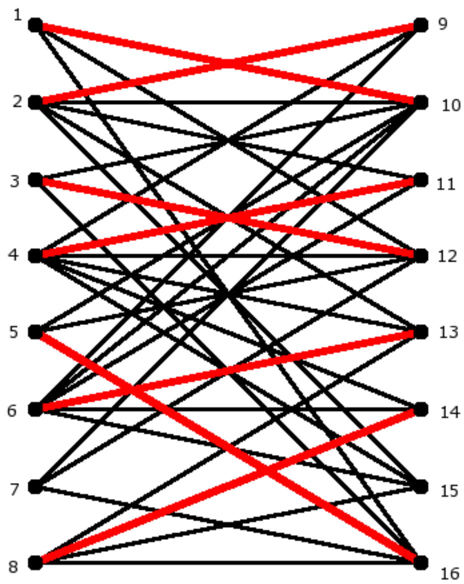

Figure 1: A maximum matching

We claim the matching shown in bold in Figure 1 is maximum. To show this, we find a cover of the same size. Following the bipartite matching algorithm in the course notes, let M be the matching above, and $V = (A, B)$ with A being the vertices $\{1, 2, \ldots, 8\}$.

Step 1. Set $\hat{X} = \{7\}$, $\hat{Y} = \emptyset$.

Step 2. Let $\hat{Y} = \{10, 12, 16\}$ and set $pr(10) = pr(12) = pr(16) = 7$.

Step 3. Step 2 added some vertices to $\hat{Y}$, continue.

Step 4. No unsaturated vertices in $\hat{Y}$.

Step 5. Add $\{1, 3, 5\}$ to $\hat{X}$, and set $pr(1) = 10, pr(3) = 12$ and $pr(5) = 16$. Now $\hat{X} = \{1, 3, 5, 7\}$. Go to Step 2.

Step 2. No new vertices added to $\hat{Y}$.

Step 3. M is a maximum matching and the cover $C = \hat{Y} \cup (A \backslash \hat{X}) = \{2, 4, 6, 8, 10, 12, 16\}$ is minimum.

Indeed the maximum matching above and the minimum cover $C = \{2, 4, 6, 8, 10, 12, 16\}$ have the same size (7).

2. Find a subset D of $\{1,2,3,4,5,6,7,8\}$ such that $|N(D)| < D$.

   Solution: A suitable choice can be found from the proof of Hall's Theorem as $D = \hat{X}$: so the set $D = \{1,3,5,7\}$ works. Its neighbourhood is $\{10,12,16\}$.

3. Let k be a positive integer and suppose G is a bipartite graph in which every vertex has degree precisely k. Show:

   (a) any bipartition (A,B) of G has $|A| = |B|$

   Solution: Let (A,B) be a bipartition of G. Since every vertex has degree exactly k, by counting edges of G by their vertex in A we know that $|E(G)| = k|A|$. But similarly $|E(G)| = k|B|$. Therefore $k|A| = k|B|$ which implies $|A| = |B|$ (since $k \neq 0$).

   (b) G has a perfect matching

   Solution: We apply Hall's Theorem. Let D be a subset of A. Then the set E_D of edges incident to D has size exactly $k|D|$. Since $N(D)$ is by definition the set of vertices in B that are incident to an edge of E_D, and every vertex in B has degree exactly k, we know that $|E_D| \leq k|N(D)|$. Therefore

   $$k|D| = |E_D| \leq k|N(D)|,$$

   which implies $|D| \leq |N(D)|$. Therefore $|N(D)| \geq |D|$. Since this is true for every subset D of A, Hall's Condition holds and therefore G has a perfect matching.

   (c) G has k perfect matchings, no two having an edge in common.

   Solution: We use induction on k. If $k = 1$ then a 1-regular graph is exactly a perfect matching, so the claim holds.

   Suppose $k \geq 2$ and the claim holds for smaller values of k. By (b) we know G has a perfect matching M. Let G' be the graph obtained by removing the edges of M from G. Since M is a perfect matching, the degree of every vertex goes down by exactly one. So every vertex of G' has degree exactly $k-1$. By induction, G' has $k-1$ perfect matchings, no two of which share an edge. Then these together with M form k perfect matchings of G, no two of which share an edge.

4. Give an example of a 3-regular graph that does not have a pefect matching. (Note that such a graph cannot be bipartite.)

   Solution: The graph shown in the figure is 3-regular. To see that it has no perfect matching, suppose on the contrary that it does. Then the vertex a must be incident to some matching edge, say without loss of generality ab is in the matching. But then the five vertices g,h,i,j,k cannot all be incident to matching edges.

5. Let G be a bipartite graph with vertex classes A and B, where $|A| = |B| = 2n$. Suppose that $|N(X)| \geq |X|$ for all subsets $X \subset A$ with $|X| \leq n$, and $|N(X)| \geq |X|$ for all subsets $X \subset B$ with $|X| \leq n$. Prove that G has a perfect matching.

   Solution: We verify the condition for Hall's Theorem in G. We are given that $|N(X)| \geq |X|$ for all subsets $X \subset A$ with $|X| \leq n$, so we just need to check that $|N(X)| \geq |X|$ for all subsets $X \subset A$ with $|X| > n$. Let X be such a subset. Since X contains a subset S of size exactly n, we know that $|N(X)| \geq |N(S)| \geq |S| = n$. Suppose on the contrary that $|N(X)| < |X|$. Let $Y = B \setminus N(X)$. Then by definition of neighbourhood, there are no edges of G joining X to

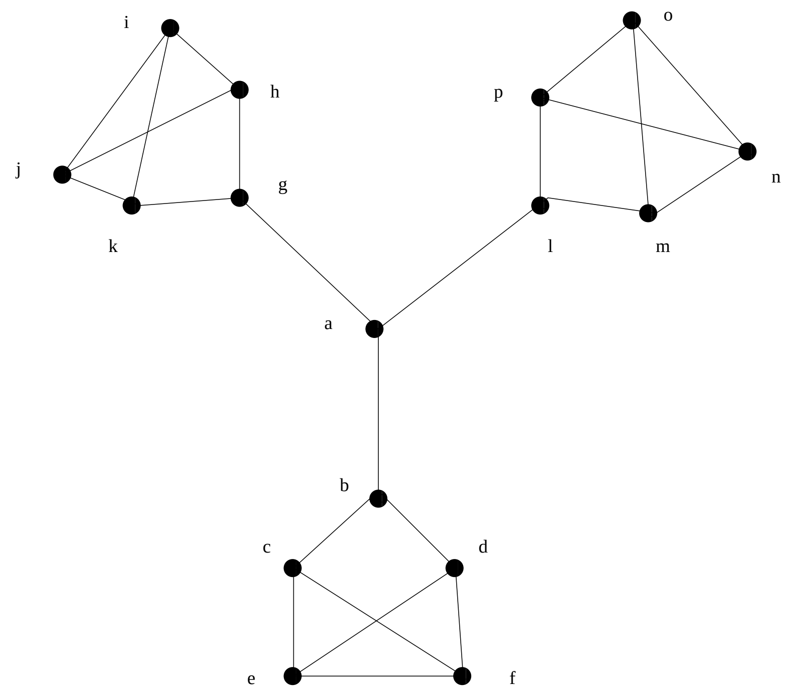

Figure 2:

Y. This implies that $N(Y) \subseteq A \setminus X$. But since $|N(X)| \geq n$ we know $|Y| = |B \setminus N(X)| \leq n$, and so by the given property we know $|N(Y)| \geq |Y|$. Therefore

$$|A| = |N(Y)| + |X| > |Y| + |N(X)| = |B|,$$

contradicting the given fact that $|A| = |B|$. Therefore we must have $|N(X)| \geq |X|$ for every $X \subseteq A$, which by Hall's Theorem implies that G has a perfect matching.