Graph Theory (1)

→ degree thm:

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

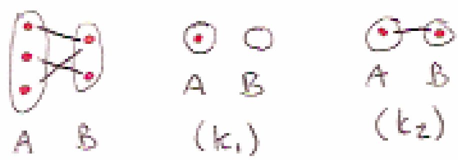
→ handshake thm:

# of verticies of odd degree is even.

→ bipartite graph:

if $V(G) = A \cup B$ and $A \cap B = \varnothing$ and $\forall$ edges of G have a vertex in both A and B.

e.g.



→ Trees:

↳ Every tree with p verticies has $p-1$ ~~verticies~~ edges.

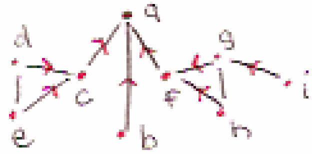
↳ Breadth-First Search Trees (BFST)

- A spanning tree
- Algorithm:

  given: a Connected Graph G
  gives: a Tree T

  1. Select an arbitrary vertex r of G. Set $T := \{r\}$ Set parent$(r) = \varnothing$ this is level $\varnothing$.
  2. IF there are no ~~[illegible]~~ neighbour vertexs out of T to those in T STOP
  3. select a neighbour outside of T. Add the edge and vertex to T, denote the parent ($>$). All neighbours the same distance get added to the level set of distance.

e.g.



lvl 0 = $\{a\}$
1 = $\{b, c, f\}$
2 = $\{d, e, g, h\}$ ⇒ G not bipartite.
3 = $\{i\}$

- Bipartite Check:

  given a BFST, T, on G if no edge of edge joins vertices on the same level in T, G is bipartite.

Graph Theory (2)

→ planar graph

drawn in a way that no edges cross.

↳ quick checks for planarity:

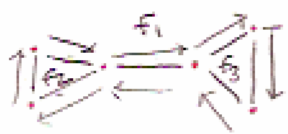
1. if $v \geq 3$ then $e \leq 3v-6$
2. if $v > 3$ and there are no cycles of length 3 then $e \leq 2v-4$

these can only be used to prove non-planarity (ie. one of these fails. both holding true means nothing)

→ faces

degree of a face F is the value of it's walk.

e.g.



$\deg(f_1) = 8$

$\deg(f_2) = \deg(f_3) = 3$.

→ Faceshake thm:

Given $\tilde{G}$, a planar embedding of graph G. $F(\tilde{G})$ is the set of faces.

$$\sum_{f \in F(\tilde{G})} \deg(F) = 2|E(G)|$$

→ Euler's Formula:

Given: G is a connected graph

$\tilde{G}$ is a planar embedding of G.

we let: $p = |V(G)|$

$q = |E(G)|$

$s = |F(\tilde{G})|$

then: $p - q + s = 2$

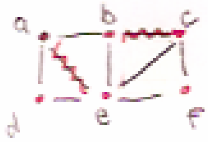
$(|V(G)| - |E(G)| + |F(\tilde{G})| = 2)$

Graph Theory (3)

→ matching:

a matching M in a graph is a

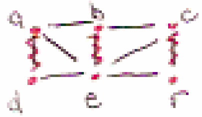
set of edges st no edges in M share a common vertex.

e.g. 

$M = \{ae, bc\}$

→ maximum matching:

a matching with largest possible size.



$M = \{ad, be, cf\}$ ← e.g

→ perfect matching:

a matching with $p/2$ size.

($p = |V(G)|$ & $|V(G)|$ must be even)

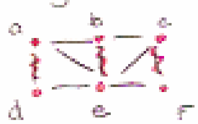
→ alterating path:

Given: a graph G

a matching on G, M

An alterating path P in G wrt M is a path st.

every second edge of P is in M



{d~a−b~e−f~c} is alternating

{d~a−e−f} is not augmenting.

→ augmenting alternating path:

an alterating path, of length ≥1, where both the 1st and last verticies are not saturated by M.

$M = \{ae, bc\}$

{f−e~a−b~c−d} is augmenting alternating.

↳ switching the matching on an augmenting path M to create M' will guarentee

$|M| < |M'|$ (M' is always bigger)

→ Hall's thm:

Given: a bipartite graph G with vertex classes A and B,

a set $D \subseteq A$

the function $N(D) = \{b \in B : ab \in E(G) \text{ for some } a \in D\}$

Then G has a matching that saturates A

$|N(D)| \geq |D|$ iff

simply - the only obstruction to a perfect matching in bipartite graphs is a bad subset

e.g. 

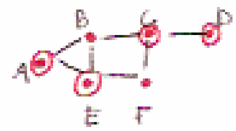
Graph Theory (4)

→ cover:

a <u>cover</u> of graph G is
a set of verticies, C, st $\forall e \in E(G)$ is incident to at least on vertex in C.

e.g. 

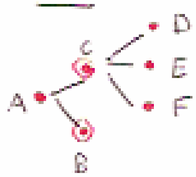
$C = \{A, C, F\}$



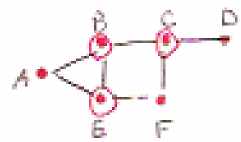
$C = \{A, E, C, D\}$

→ minimum cover:

a <u>cover</u> of the smallest possible size.



$C = \{B, C\}$



$C = \{B, C, E\}$

→ König's thm:

Given a bipartite graph G
a maximal matching M on G.

Then there must exist a cover C in G st.
$|C| = |M|$

→ Bipartite Matching Algorithm:

Given a bipartite graph $G = (V = \{X, Y\}, E)$

The algorithm finds a maximal matching by:
1. finding an augmenting path from each $x \in X$ to Y
2. adding it to the matching if it exists.

More Formally:

Given a bipartite graph $G = (V = \{X, Y\}, E)$
a matching M_0 in G

Gives a maximum matching M_1 in G
a minimum cover C in G

1. let $\hat{X} = \{v \in X : v \text{ not saturated by } M\}$
   $\hat{Y} = \phi$
   $M = M_0$
2. If $\exists v \in Y - \hat{Y}$ st $uv \in E(G)$ for some $u \in \hat{X}$
   then set $\hat{Y} = \hat{Y} \cup \{v\}$ and parent(v)
3. If no such v exists
   then $M_1 = M$
   $C = \hat{Y} \cup (X \setminus \hat{X})$.
   <u>STOP</u>!
4. If v is <u>not saturated</u> by M
   then augment the augmenting path given by parent(v).
   <u>Goto</u> 1.
5. If $\exists w \in X \setminus \hat{X}$ st $uw \in E(G)$ for some $u \in \hat{Y}$
   then set $\hat{X} = \hat{X} \cup \{w\}$ and parent(w)
   <u>Goto</u> 2.