This is the midterm examination from Fall 2004. It was both long and challenging. We will try to make our midterm shorter and more routine.

1. (a) **[3 marks]** Write each of the following rational functions $f(x)$ in form of a formal power series $f(x) = \sum_{i\geq 0} a_i x^i$.
$$\frac{1}{(1-2x)^3}, \qquad \frac{x^3}{(1+4x^2)^5}.$$
   (b) **[1 mark]** Determine the coefficient $[x^5]\frac{x^2}{(1+x)^7}$.
   (c) **[3 marks]** Write the following formal power series in closed form:
$$\sum_{i\geq 0}\left(-\frac{1}{2}\right)^i x^{3i}, \qquad \sum_{i\geq 0}\binom{i+2}{i}5^i x^i.$$

2. (a) **[2 marks]** Use the Binomial Theorem to prove that $\sum_{i=0}^{n}\binom{n}{i} = 2^n$.
   (b) **[3 marks]** Prove that
$$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s} = 3^n.$$
   (Hint: Use the Binomial Theorem to expand $(1+2)^n$.)

3. An even composition of n is a composition of n into an even number of parts such that each part is a positive even number. For example, the even compositions of 8 are
$(2,6),(4,4),(6,2),(2,2,2,2)$.
   (a) **[4 marks]** Determine the generating function for the number of even compositions of n having $2k$ parts. Use this generating function to determine the number of $2k$-part even compositions of n.
   (b) **[3 marks]** Determine the generating function for the the number of all even compositions of n.
   (c) **[3 marks]** Use your answer in part (b) to determine a_n, the number of even compositions of n.

4. For each of the following sets, write down a decomposition that uniquely creates the elements of that set.
   (a) **[2 marks]** The $\{0,1\}$-strings that have no blocks of 1s with length 3, and no substrings of 0s of length 2.
   (b) **[2 marks]** The set of $\{0,1\}$-strings with no occurrence of the substring 010.

5. (a) **[3 marks]** Find the generating function $\Phi_{S_1}(x)$ for the following set of strings:
$$S_1 = \{0\}^*\left(\{1\}\{111\}^*\{00\}\{000\}^*\right)^*\{1\}^*.$$
   (b) **[4 marks]** Find the generating function $\Phi_{S_2}(x)$ for the following set of strings:
$$S_2 = \bigcup_{k=0}^{\infty}\{0\}^k\left(\{1\}\{111\}^*\{00\}\{000\}^*\right)^*\{1\}^k.$$

(c) **[1 mark]** Find a string that is in S_1 but not in S_2.

(d) **[1 mark]** Let $a_n = [x^n]\Phi_{S_1}(x)$ and $b_n = [x^n]\Phi_{S_2}(x)$. Without evaluating a_n or b_n, show that $a_n \geq b_n$ for all $n \geq 0$.

6. Let A be the set of binary strings a such that in each substring of three consecutive positions in a, there is at least one 1 and at least one 0. Let a_n be the number of strings of length n in A.

   (a) **[2 marks]** For $n = 1, 2, 3, 4$, write down all the strings of length n in A.

   (b) **[3 marks]** Prove that the generating function for a_n is given by
   $$\Phi_A(x) = \frac{1 + 2x + 3x^2 + 2x^3 + x^4}{1 - x^2 - 2x^3 - x^4}.$$

   (c) **[2 marks]** From part (b), find a recurrence relation for the numbers a_n, together with an appropriate set of initial conditions.

7. (a) **[4 marks]** Solve the linear, homogeneous recurrence equation
   $$a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0 \quad \forall \quad n \geq 3$$
   with initial conditions $a_0 = 4,\ a_1 = 8,\ a_2 = 2$.

   (b) **[1 marks]** What is the asymptotic value of a_n?

   (c) **[3 marks]** Find a particular solution of the nonhomogeneous linear recurrence equation
   $$a_n - a_{n-1} - 5a_{n-2} - 3a_{n-3} = \frac{15}{8}(-2)^n \quad \forall \quad n \geq 3.$$

8. (a) **[2 marks]** Let G be a 3-regular graph with 10 vertices. (This means every vertex has degree 3.) How many edges does G have? Justify your answer.

   (b) **[3 marks]** Let G be a connected graph with 27 edges and exactly one cycle. How many vertices does G have? Justify your answer.

   (c) **[3 marks]** Is there a bipartite graph with 9 vertices and 23 edges? Explain.

9. Let Q_n denote the n-cube graph: the vertex set of Q_n consists of all binary sequences (i.e. binary strings) of length n and the edge set of Q_n consists of all pairs of binary sequences which differ in exactly one position.

   (a) **[3 marks]** Determine the number of vertices and edges in Q_n.

   (b) **[2 marks]** Show that Q_n is bipartite.

   (c) **[3 marks]** Prove that Q_n has no bridges.

   (d) **[4 marks]** Prove that that the graph Q_n does not contain a subgraph isomorphic to the graph drawn below: