MATH 239 Tutorial 3 Solution Outline

1. A house with 9 students have collectively bought n apples. Among the students, Lucas is required to eat an odd number of apples, Ariel is required to eat between 5 to 10 apples, and each of the remaining students is required to eat at least 3 apples. How many ways can all the apples be eaten by the students?

   Solution.
   $$S = \mathbb{N}_{odd} \times \{5,6,7,8,9,10\} \times \mathbb{N}_{\geq 3}^{7}.$$
   The answer is
   $$[x^n]\Phi_S(x) = \frac{x}{(1-x^2)} \frac{x^5(1-x^6)}{1-x} \left(\frac{x^3}{1-x}\right)^7.$$

2. How many compositions of n with k parts are there where no part is divisible by 3?

   Solution. Let $P = \{1,2,4,5,7,8,10,11,\ldots\}$ be all integers not divisible by 3. Then
   $$\Phi_P(x) = \frac{x+x^2}{1-x^3}.$$

   Here are two potential ways to have derived this generating function. We know that P is the disjoint union of $A = \{1,4,7,10,...\}$ and $B = \{2,5,8,11,...\}$. The generating functions for these sets are
   $$\Phi_A(x) = \frac{x}{1-x^3}$$
   and
   $$\Phi_B(x) = \frac{x^2}{1-x^3}$$
   Using the sum lemma, we combine the two to get the generating function for P. Alternatively, we could let $C = \{3,6,9,12,...\}$, and thus $P = \mathbb{N} \setminus C$. Thus
   $$\Phi_P(x) = \Phi_{\mathbb{N}}(x) - \Phi_C(x) = \frac{x}{1-x} - \frac{x^3}{1-x^3} = \frac{x+x^2}{1-x^3}$$

   We are enumerating P^k. So
   $$\Phi_{P^k}(x) = \left(\frac{x+x^2}{1-x^3}\right)^k.$$

3. Let $\{a_n\}$ be the sequence with the corresponding power series
   $$\sum_{n\geq 0} a_n x^n = \frac{1-x+2x^2}{1-x-2x^3}.$$

   Determine a recurrence relation that $\{a_n\}$ satisfies, together with sufficient initial conditions. Use this recurrence to find a_5.

Solution. Multiplying both sides by the denominator $1-x-2x^3$,

$$1-x+2x^2 = (1-x-2x^3)\sum_{n\geq 0} a_n x^n = a_0+(a_1-a_0)x+(a_2-a_1)x^2+\sum_{n\geq 3}(a_n-a_{n-1}-2a_{n-3})x^3$$

So $a_0 = 1$. $a_1 - a_0 = -1$ so $a_1 = 0$. $a_2 - a_1 = 2$, so $a_2 = 2$. For $n \geq 3$,

$$a_n - a_{n-1} - 2a_{n-3} = 0.$$

To find a_5,

$$a_3 = a_2 + 2a_0 = 4, a_4 = a_3 + 2a_1 = 4, a_5 = a_4 + 2a_2 = 8.$$

4. (a) How many compositions of n are there where each part is greater than 1?
   (b) Let a_n be the answer to part (a). Derive a recurrence relation for $\{a_n\}$ with sufficient initial conditions.
   (c) Give a combinatorial interpretation of the recurrence relation from part (b) through a bijection.

   Solution.

   (a) Let S be the set of all compositions where each part is greater than 2.

   $$S = \bigcup_{k\geq 0} \mathbb{N}_{\geq 2}^k.$$

   Thus the answer is

   $$[x^n]\Phi_S(x) = [x^n]\sum_{k\geq 0}\left(\frac{x^2}{1-x}\right)^k = [x^n]\frac{1-x}{1-x-x^2}.$$

   (b) Using similar methods as in Question 3, we multiply both sides by the denominator and get $a_0 = 1, a_1 = 0$, and for $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$.

   (c) Let C_n be the set of all compositions of n where each part is greater than 1. Define a bijection $f : C_n \to C_{n-1} \cup C_{n-2}$ by

   $$f(a_1,\ldots,a_k) = \begin{cases} (a_1,\ldots,a_{k-1},a_k-1) & \text{if } a_k > 2 \\ (a_1,\ldots,a_{k-1}) & \text{if } a_k = 2 \end{cases}$$

   In other words, if the last element in the composition is greater than 2, we will reduce it by 1 (making the total sum $n-1$), and if the last element is exactly 2, we will remove it completely (making the total sum $n-2$). Compositions in C_n either end with 2 or end with a number greater than 2, so these sets are disjoint, and the function is well defined.

   The inverse g is defined from $g : C_{n-1} \cup C_{n-2} \to C_n$, where we the element 2 to the end if the composition is of $n-2$, and increment the last element by 1 if the composition is of $n-1$.

5. How many k-tuples $(a_1, \ldots, a_k)$ of positive integers satisfy the inequality $a_1 + \cdots + a_k < n$?

   Solution. We can find a bijection between the k-tuples satisfying the inequality with the compositions of $(a_1, ..., a_k, a_{k+1})$ of n. Since the k-tuple adds up to a sum strictly less than n, there is unique, positive a_{k+1} such that the $k+1$-tuple is a composition of n.

   Thus, the answer is the number of compositions of n with $k+1$ parts, i.e.

$$[x^n]\left(\frac{x}{1-x}\right)^{k+1} = \binom{n-1}{k}$$

Additional exercises

1. How many compositions of n are there where the i-th part is congruent to $i \pmod 2$?

   Solution. Partition such compositions into those that have odd parts O and those that have even parts E. Then

$$O = \mathbb{N}_{odd} \times \bigcup_{k\geq 0}(\mathbb{N}_{even} \times \mathbb{N}_{odd})$$
$$E = \bigcup_{k\geq 0}(\mathbb{N}_{odd} \times \mathbb{N}_{even})$$

   So

$$\Phi_O(x) = \frac{x}{1-x^2} \cdot \frac{1}{1 - \frac{x^2}{1-x^2}\frac{x}{1-x^2}} = \frac{x - x^3}{1 - 2x^2 - x^3 + x^4}$$
$$\Phi_E(x) = \frac{1}{1 - \frac{x}{1-x^2}\frac{x^2}{1-x^2}} = \frac{1 - 2x^2 + x^4}{1 - 2x^2 - x^3 + x^4}$$

   By sum lemma, the generating series is

$$\Phi_O(x) + \Phi_E(x) = \frac{1 + x - 2x^2 - x^3 + x^4}{1 - 2x^2 - x^3 + x^4}.$$

2. This question asks you to reverse engineer the process of finding a recurrence relation from a rational function. Suppose a sequence $\{a_n\}$ satisfies $a_0 = 1$, $a_1 = 2$, $a_2 = 2$, and for $n \geq 3$, $a_n = 2a_{n-1} - a_{n-2} + a_{n-3}$. Find a rational function whose power series representation is $\sum_{n\geq 0} a_n x^n$.

   Solution. Denominator is $1 - 2x + x^2 - x^3$. Numerator equals

$$\begin{aligned}(1 - 2x + x^2 - x^3)\sum_{n\geq 0} a_n x^n &= a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1 + a_0)x^2 \\ &\quad + \sum_{n\geq 3}(a_n - 2a_{n-1} + a_{n-2} - a_{n-3})x^n \\ &= 1 - 3x - x^2\end{aligned}$$

3. Prove that for $n \geq 2$, the number of compositions of n with even number of even parts is equal to the number of compositions of n with odd number of even parts.

   Solution. The bijection $f(a_1, \ldots, a_k) = \begin{cases} (a_1, \ldots, a_{k-1} + 1) & a_k = 1 \\ (a_1, \ldots, a_k - 1, 1) & a_k > 1 \end{cases}$ changes the parity of the number of even parts. Inverse is the same function.

4. Let S be the set of all compositions of n. In class, you have learned that for $n \geq 1$, $|S| = 2^{n-1}$, which is also the number of subsets of $[n-1]$. Let T be the set of all subsets of $[n-1]$. Find a bijection between S and T, and provide its inverse. Illustrate your bijection by writing down the subset of [13] that corresponds to the composition $(3, 1, 4, 1, 5)$ of 14.

   Solution. Define bijection $f : S \to T$ by

   $$f(a_1, \ldots, a_k) = \{a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + \cdots + a_{k-1}\}$$

   with inverse

   $$g(\{b_1, \ldots, b_l\}) = (b_1, b_2 - b_1, b_3 - b_2, \ldots, n - b_l).$$

   $f(3, 1, 4, 1, 5) = \{3, 4, 8, 9\}$.

5. Let a_n be the set of all compositions of n. Give a combinatorial proof that for $n \geq 1$,

   $$a_n = a_{n-1} + a_{n-2} + \cdots + a_1 + a_0.$$

   Note: In particular, this proves that

   $$2^{n-1} = 1 + \sum_{i=0}^{n-2} 2^i.$$

   Solution. Let C_n be the set of all compositions of n. Then $C_n = \bigcup_{k=1}^{n} A_k$ where A_k is the set of all compositions of n where the last part is k. Then there is a bijection between A_k and C_k by removing the last part.