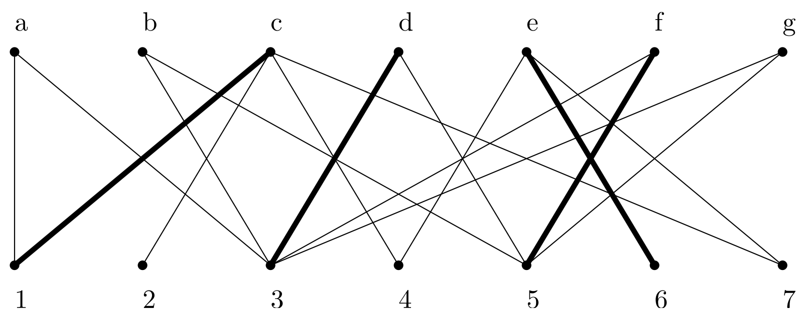
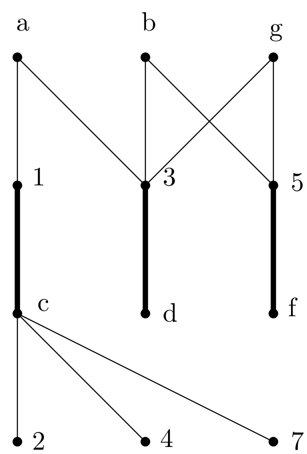
MATH 239 Spring 2012: Assignment 11

Solutions

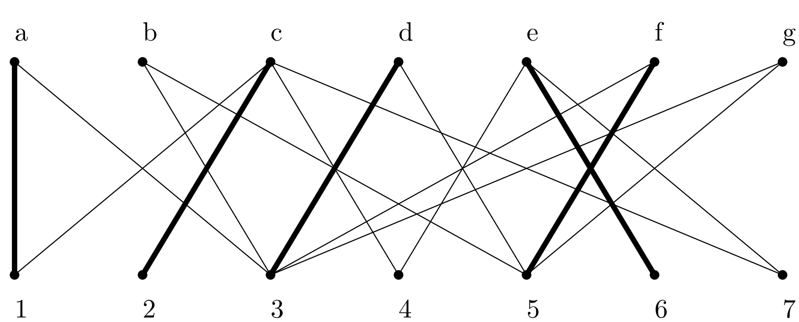
1. For the following bipartite graph with bipartition $A = \{a, b, c, d, e, f, g\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$, perform the maximum matching algorithm using XY-construction. At the end of the algorithm, produce a maximum matching, a minimum cover, and the sets X and Y from the algorithm. Prove that there is no matching that saturates every vertex in A by giving a set $D \subseteq A$ such that $|N(D)| < |D|$.



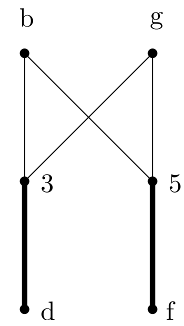
Solution. In the first iteration, we have $X_0 = \{a, b, g\}$, and construct $X = \{a, b, g, c, d, f\}$, $Y = \{1, 3, 5, 2, 4, 7\}$. We also find several augmenting paths, one of which is $a, 1, c, 2$.



Augmenting on this path, we get the following new matching.



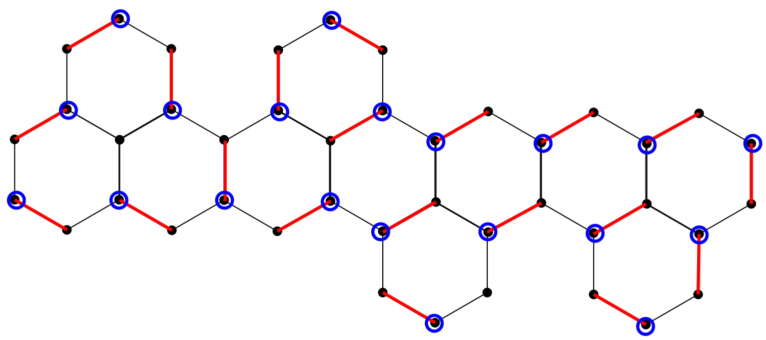
In the second iteration, we have $X_0 = \{b, g\}$, and construct $X = \{b, g, d, f\}$ and $Y = \{3, 5\}$. We cannot find any augmenting paths, so this is a maximum matching with a minimum cover $Y \cup (A \backslash X) = \{3, 5, a, c, e\}$.



To use Hall's Theorem, we can look at the set $X = \{b, g, d, f\}$ and notice that $N(X) = Y$. Here $|X| = 4 > 2 = |N(X)|$, hence X is a set that violates Hall's condition.

2. Find a maximum matching of the following graph. Prove that your matching is maximum using a vertex cover.

   Solution. The following shows a matching of size 20 and a cover of size 20.



3. An *independent set* of a graph G is a subset of the vertices $S \subseteq V(G)$ such that no two vertices in S are adjacent. Prove that C is a vertex cover of G if and only if $V(G) \setminus C$ is an independent set. If x is the size of a maximum independent set and y is the size of a minimum vertex cover, determine $x + y$.

   Solution. C is a vertex cover if and only if each edge of G has at least one end in C, if and only if no edge joins two vertices of $V(G) \setminus C$, if and only if $V(G) \setminus C$ is an independent set.

   If C is a minimum vertex cover with size x, then $V(G) \setminus C$ is an independent set with size $|V(G)| - x$. If there is any larger independent set I, then $V(G) \setminus I$ is a vertex cover whose size is smaller than C, which is not possible. So the size of a largest independent set is $y = |V(G)| - x$, and so $x + y = |V(G)|$.

4. Suppose that a connected graph G has exactly one maximum matching. Prove that G has a perfect matching.

   Solution. Let M be the only maximum matching in G. Suppose M is not a perfect matching. Then there exists an unsaturated vertex v in G. Since G is connected, v has degree at least 1. Let u be a neighbour of v. Now u must be saturated, for otherwise we could add the edge uv and get a larger matching. Suppose uw is a matching edge in M. Then $M - uw + uv$ is another matching of G, which has the same size as M, so it is another maximum matching in G. This is a contradiction.

5. Prove that the edges of a k-regular bipartite graph can be partitioned into k perfect matchings.

   Solution. We prove our statement by induction on k.

   Base case: When $k = 0$, there are no edges, this is trivially true.

Induction hypothesis: Assume that the edges of any $(k-1)$-regular bipartite graph can be partitioned into $k-1$ perfect matchings.

Induction step: Let G be a k-regular bipartite graph. From class, we know that G has a perfect matching, let M be one of them. Now M is a 1-regular graph, so $G-M$ is a $(k-1)$-regular bipartite graph. By induction hypothesis, the edges of $G-M$ can be partitioned into $k-1$ perfect matchings. Together with M, we partitioned $E(G)$ into k perfect matchings for G.

6. Let G be a bipartite graph with bipartition (A, B) where $|A| = |B| = 2n$. Suppose for each $X \subseteq A$ where $|X| \leq n$, $|N(X)| \geq |X|$, and for each $Y \subseteq B$ where $|Y| \leq n$, $|N(Y)| \geq |Y|$ (i.e. Hall's condition holds for subsets of A and B of size at most n). Prove that G has a perfect matching.

   Solution. To use Hall's Theorem, it suffices to show that for any set $X \subseteq A$ where $|X| > n$, $|N(X)| \geq |X|$. Suppose by way of contradiction that there exists one set $X \subseteq A$ and $|X| > n$ where $|N(X)| < |X|$. Let X' be a subset of X of size exactly n. Then $|N(X')| \geq |X'| = n$ by assumption. But $N(X') \subseteq N(X)$, so $|N(X)| \geq n$. Let $Y = B \setminus N(X)$. So $|Y| = |B| - |N(X)| \leq n$. By assumption, $|N(Y)| \geq |Y|$. Since there is no edge between X and Y, $N(Y) \subseteq A \setminus X$. Therefore,

   $$|A| \geq |X| + |N(Y)| > |N(X)| + |Y| = |B|.$$

   This is a contradiction since $|A| = |B|$.

7. Two people play a game on a graph G by alternately selecting distinct vertices $v_1, v_2, \ldots$ forming a path. The last player able to select a vertex wins. Prove that the second player has a winning strategy if G has a perfect matching, and the first player has a winning strategy if G has no perfect matching.

   Solution. In both cases, the idea is to build alternating paths using a maximum matching. If one player is the first to select a vertex of a matching edge, then the next player is guaranteed to have a move by selecting the other end of the matching edge.

   If G has a perfect matching, then no matter what the first player picks, it's going to be one vertex of a matching edge. So the second player can pick the other end of the matching edge. Regardless of where the first player moves next (if any), it will be part of another matching edge, in which case the second player can pick the other end. Since the second player here never runs out of moves, they will always win.

   If G does not have a perfect matching, then the first player must first pick an unsaturated vertex in a maximum matching. Regardless of the vertex The second player picks, it must be part of a matching edge (for otherwise we can add an edge to our maximum matching). The same process then follows from the case above, and the first player has the winning strategy this time.