# MATH 239 Tutorial 1 Solution Outline

1. Let $E$ be the set of all subsets of $[n]$ of even size, and let $O$ be the set of all subsets of $[n]$ of odd size. Prove that

$$\sum_{\substack{i \text{ is even} \\ 0 \leq i \leq n}} \binom{n}{i} = \sum_{\substack{j \text{ is odd} \\ 0 \leq j \leq n}} \binom{n}{j}$$

   by finding a bijection between $E$ and $O$. Illustrate your bijection by matching up the $E$ and $O$ for $[4]$.

   **Solution.** One way is $f : E \to O$ where for any $A \in E$,

$$f(A) = \begin{cases} A \setminus \{1\} & \text{when } 1 \in A \\ A \cup \{1\} & \text{when } 1 \notin A \end{cases}$$

   In other words, if the even subset has 1 as an element, remove it. Otherwise, add 1 to the subset. Since we started with an even subset and are either removing or adding an element, we now have an odd subset.

   The inverse function $g$ is exactly the same, except it is defined on the domain $g : O \to E$. To prove that it is the inverse of $f$, we take any even subset $A$ and show that $g(f(A)) = A$.

   Case: $1 \in A$. Then $g(f(A)) = g(A \setminus \{1\}) = A$, since $A \setminus \{1\}$ does not include 1. The other case is similar.

2. Give a combinatorial proof of the following identity:

$$\binom{2n}{n} = 2\binom{2n-1}{n-1}.$$

   **Solution.** For clarity we will first do a bit of algebra on the RHS:

$$2\binom{2n-1}{n-1} = \binom{2n-1}{n-1} + \binom{2n-1}{n-1} = \binom{2n-1}{n-1} + \binom{2n-1}{n}$$

   Now consider what the LHS counts: It is the number of ways to pick an $n$-subset of $[2n]$. For any $n$-subset of $[2n]$, it either contains the element 1, or it doesn't. If we know a subset does contain the element 1, then we still need to pick the other $n-1$ elements from the remaining $2n-1$ elements, so there are $\binom{2n-1}{n-1}$ ways to do this. If the subset does not contain the element 1, then we need to pick the full $n$-subset from $2n-1$ elements (since we can't pick the element 1). There are $\binom{2n-1}{n}$ ways to do this.

3. Give a combinatorial proof and an algebraic proof of the following identity:

$$\sum_{i=0}^{n} \binom{n}{i} i = n2^{n-1}.$$

**Solution.** Combinatorial: Count $\{(A, x) | A \subseteq [n], x \in A\}$ (sort of like picking a committee from $n$ people, and picking a chair for the committee).

How can we build such a committee? On the RHS, we first pick the committee chair ($n$ ways to do this), and then pick the rest of the committee ($2^{n-1}$ ways to do this). On the LHS, we first pick our committee ($\binom{n}{i}$ ways to do this for committee size $i$ ranging from 0 to $n$), and then pick our chair ($i$ ways to do this, if we picked a committee of size $i$).

Algebraic: Differentiate $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$, and then let $x = 1$.

4. Determine $[x^{314}]x^3(1+x)^2(1+5x^{10})^{-1}$.

   **Solution.** This is $[x^{311}](1+x)^2(1+5x^{10})^{-1}$.

   The first term only has powers ranging from 0 to 2, and the second term only has powers that are multiples of 10. Since we only care of the coefficient of $x^311$, this can only occur when the second term contributes $x^{310}$, and the first term contributes $x^1$.

   When the first term contributes $x^1$, its coefficient is 2. When the second term contributes $x^310$, its coefficient is $(-5)^{31}$. Multiplying these together, we get $2 \cdot (-5)^{31}$.

5. Determine $[x^n](1+x^2)^{-5}(1-3x)^{20}$

   **Solution.**

$$(1+x^2)^{-5}(1-3x)^{20} = \sum_{i \geq 0} \binom{i+4}{4} (-x)^{2i} \sum_{j=0}^{20} (-3)^j \binom{20}{j} x^j = \sum_{i \geq 0} \sum_{j=0}^{20} \binom{i+4}{4} (-3)^j \binom{20}{j} x^{2i+j}.$$

   We need $2i + j = n$, so $j = n - 2i$. But $0 \leq j \leq 20$, so $\frac{n-20}{2} \leq i \leq \frac{n}{2}$. So the coefficient of $x^n$ is

$$\sum_{i=\lceil \frac{n-20}{2} \rceil}^{\lfloor \frac{n}{2} \rfloor} \binom{i+4}{4} (-3)^{n-2i} \binom{20}{n-2i}.$$

   (Note: We don't have to be this precise. We can simply put $i \geq 0$, and note that $\binom{20}{n-2i} = 0$ when $n - 2i > 20$ or $n - 2i < 0$.)

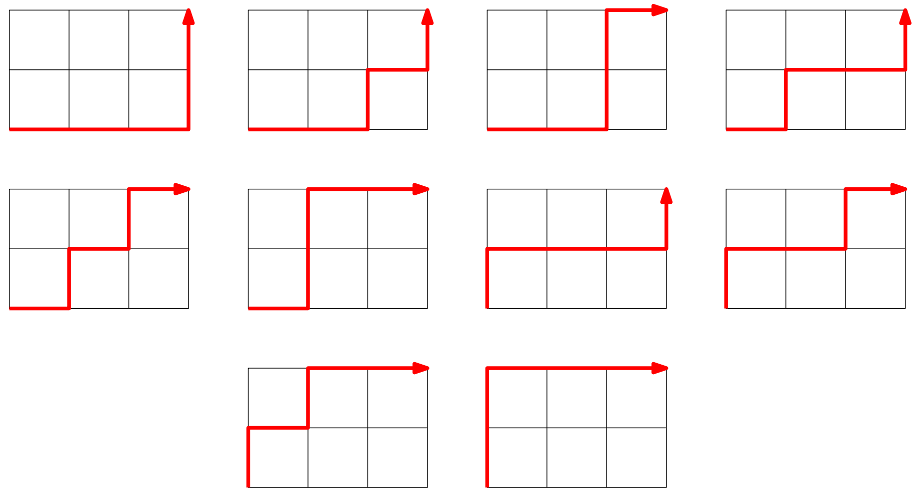
## Additional exercises

1. Consider a road network that resembles an $m \times n$ grid. We wish to walk from the SW corner to the NE corner of the grid so that we only walk in E or N direction. Let $W_{m,n}$ be the set of all such walks. For example, the set of all such walks on a $2 \times 3$ grid is illustrated below:

   Define $S_{m,n}$ to be the set of all subsets of $[m+n] = \{1, 2, \ldots, m+n\}$ of size $m$.

   Find a bijection between $W_{m,n}$ and $S_{m,n}$, and determine the number of walks in $W_{m,n}$.

   **Solution.** Write the walk $W$ as a sequence of E's and N's, map a walk to the subset $\{i | W_i = E\}$. This is reversible. The number is $\binom{m+n}{m}$.



2. Consider the $k$-tuples $(T_1, \ldots, T_k)$ where each $T_i \subseteq [n]$. In other words, if $P$ is the set of all subsets of $[n]$, then such a $k$-tuple is in the cartesian product $P^k$. We define the following two subsets of $P^k$:

   (a) $S$ is all such $k$-tuples where $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k$.

   (b) $T$ is all such $k$-tuples that are mutually disjoint, i.e. $T_i \cap T_j = \emptyset$ for any $i \neq j$.

   Find a bijection between $S$ and $T$, which proves that $|S| = |T|$. What is this cardinality?

   **Solution.** For each $(T_1, \ldots, T_k)$ in $S$, form $(T_1, T_2 \setminus T_1, T_3 \setminus T_2, \ldots, T_k \setminus T_{k-1})$ in $T$. The cardinality is $(k+1)^n$ (for $T$, for each element in $[n]$, it is in either one of the $k$ sets, or not at all. So $k+1$ choices for each of the $n$ elements.)

3. Prove the following identity (using any correct method):

$$2^{n-r}\binom{n}{r} = \sum_{k=r}^{n} \binom{n}{k}\binom{k}{r}$$

   **Solution.** Count the set of objects $\{(A, B) | A \subseteq B \subseteq [n], |A| = r\}$.