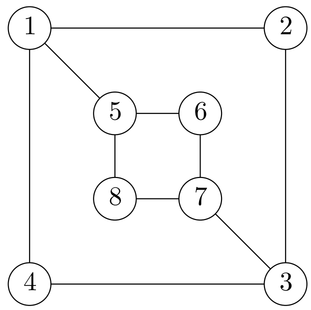
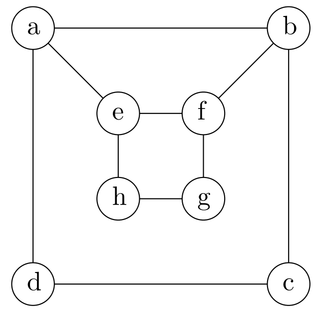
MATH 239 Tutorial 6 Solution Outline

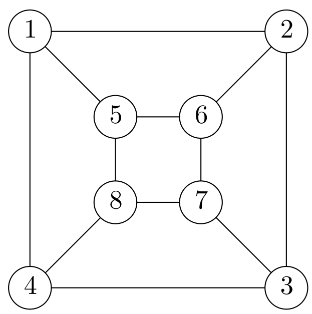
1. Determine if G_1 and G_2 are isomorphic, and if G_3 and G_4 are isomorphic. Prove your claims.



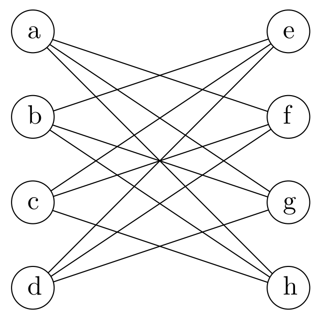
Graph G_1



Graph G_2



Graph G_3



Graph G_4

Solution. G_1 and G_2 are not isomorphic. In G_2, there is a cycle $aefb$ that contains every degree 3 vertex of G_2, whereas in G_1 there is no such cycle. Additionally, every degree 2 vertex of G_2 is adjacent to another vertex of degree 2, whereas the degree 2 vertices of G_1 are only connected to vertices of degree 3.

G_3 is isomorphic to G_4. We note that both G_3 and G_4 possess a great deal of symmetry, which gives us flexibility in determining the isomorphism.

We arbitrarily start our isomorphism $f : V(G_3) \to V(G_4)$ with $f(1) = a$. Since vertex 1 is adjacent to vertices 2, 5, and 4, we try $f(2) = f$, $f(5) = g$, and $f(4) = h$. We now see that vertex 8 is adjacent to vertices 4 and 5 (which correspond to vertices h and g). The only vertices in G_4 which are adjacent to both h and g are a and b, but we have already assigned a, so $f(8) = b$. Continuing this line of deduction and reasoning, our final answer is

$$f(1) = a$$

$$f(2) = f$$

$$f(3) = c$$

$$f(4) = h$$

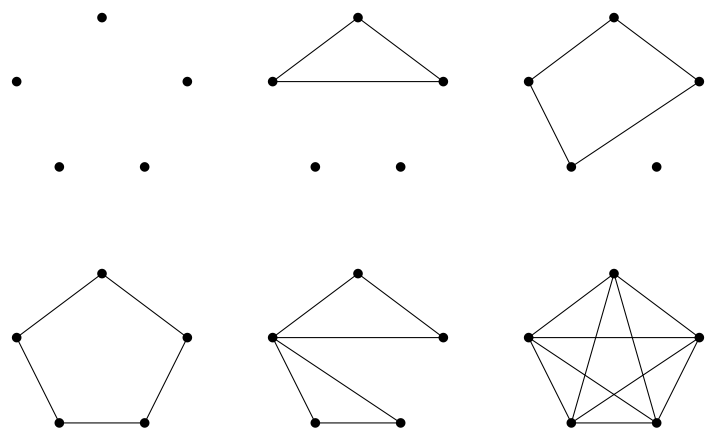
$$f(5) = g$$

$$f(6) = d$$
$$f(7) = e$$
$$f(8) = b$$

Note that this is not the only isomorphism. We could have picked $f(1)$ to be any vertex in G_4, and still have created a valid isomorphism.

2. Draw all non-isomorphic graphs with 5 vertices where the degree of each vertex is even.

   Solution. Since every vertex has even degree, the graphs will be a collection of cycles. Be careful to avoid isomorphism!



3. The *degree sequence* of a graph is the list of vertex degrees, usually written in non-increasing order, as $d_1 \geq \cdots \geq d_p$. Determine whether or not each of the following is the degree sequence of a graph. If so, draw the graph. If not, explain why.

   (a) 7, 6, 5, 4, 4, 3, 2, 2, 2, 2
   (b) 3, 3, 3, 2, 2, 1
   (c) 5, 5, 1, 1, 1, 1

   Solution.

   (a) Not possible. There are 3 vertices of odd degree, and by a corollary of Handshaking, every graph must have an even number of odd-degree vertices.
   (b) Possible.
   (c) Not possible. The graph has 6 vertices, and the first vertex of degree 5 is connected to all other vertices. However, when we try to draw the edges for the other vertex of degree 5, we see that the remaining 4 vertices are already degree 1, so they cannot be joined to any more edges.

4. The definition of the complement of a graph is in the assignment.

   (a) Draw the complement of the cube.

(b) Find a graph G with 4 vertices that is isomorphic to $\overline{G}$.

(c) Suppose $|V(G)| = p$. Determine $|E(G)| + |E(\overline{G})|$.

(d) A subset of vertices S of G is an *independent set* if no two vertices in S are adjacent in G. Prove that if a graph's largest independent set is of size n, then $\overline{G}$ contains the subgraph K_n, and there is no $m > n$ such that K_m is a subgraph of $\overline{G}$ (i.e. K_n is the biggest complete subgraph of $\overline{G}$).

Solution.

(a) An ugly drawing. Keep in mind that if uv is an edge of the cube G, then uv is not an edge of its complement $\overline{G}$, and vice-versa. Since each vertex has degree 3 in G, each vertex in $\overline{G}$ should have degree 4.

(b) A path of length 3 is self-complementary.

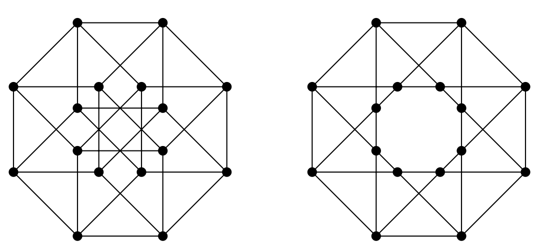
(c) Consider any vertex pair (u, v) of $V(G)$. We note that the edge uv is either in G, OR it's in $\overline{G}$, but not both. Thus the edges of G and $\overline{G}$ is a partition of all possible vertex pairs (u, v). The sets $E(G)$ and $E(\overline{G})$ are disjoint by the definition of complementarity, so $|E(G)| + |E(\overline{G})| = \binom{n}{2}$.

(d) Consider the largest independent set $S \subseteq V(G)$. Since S is independent, for any two $u, v \in S$, the edge uv is not in G. Thus, the edge uv is in $\overline{G}$, by definition of complementary graphs. Thus every pair of vertices in S is joined by an edge in $\overline{G}$, and since $|S| = n$, so the vertices of S correspond to the subgraph K_n in $\overline{G}$.

Now we will show that this K_n is the largest complete subgraph of $\overline{G}$. By way of contradiction, assume there exists $m > n$ such that K_m is a subgraph of $\overline{G}$, with vertex set T. But then the vertices of T all have edges between them in $\overline{G}$, so they must not have edges between them in G. Thus T is an independent set of size m in G. But this is a contradiction, since we stated that n is the largest size of an independent set in G.

Additional exercises

1. Are these two graphs isomorphic?



Solution. No. The left one is the 4-cube. On the right, take any cycle of length 4, and you get to opposite vertices having 3 common neighbours. This is not possible in the 4-cube.

2. For $k \geq 1$, is it true that if G is a k-regular graph, then there exists a spanning subgraph of G that is $(k-1)$-regular?

   Solution. No. Triangle.

3. Married couple Mario and Peach invited 3 other couples to the castle on the mountain for a cake party (and it's no lie). During the party, some handshaking took place with the restriction that a person cannot shake hands with themselves nor with their spouse. After all the shakings were done, Peach went around to ask the 7 others in the party how many people they shook hands with, and she received a different answer from everyone. How many hands did Mario shake? How many hands did Peach shake? What happens if Mario and Peach invited n couples to the party?

   Solution. Both shook hands with 3 people. In general, n shakes.

4. A graph is *self-complementary* if it is isomorphic to its complement.

   (a) Prove that the number of vertices in a self-complementary graph is congruent to 0 or 1 modulo 4.

   Solution. When $\binom{p}{2}$ is even.

   (b) (Hard.) Construct a self-complementary graph with n vertices for every $n \equiv 0, 1 \pmod 4$.

   (Hint: For $n \equiv 0 \pmod 4$, generalize the answer from Q4(b) of the tutorial problems by partitioning the vertices into 4 groups. For $n \equiv 1 \pmod 4$, add a vertex to your construction of $n-1$.)