MATH 239 Tutorial 10 Problems

1. Prove that the 4-cube is not planar. (Find two different proofs of this statement.)

   Solution. The 4-cube has 16 vertices and 32 edges, but any planar bipartite graph with 16 vertices has at most $2 \cdot 16 - 4 = 28$ edges. Also, we can find a $K_{3,3}$ edge subdivision.

2. Let G be a 3-regular planar bipartite graph with 14 vertices. How many faces does G have? Assuming that G has faces of degrees 4 or 6, how many faces of degree 4 does G have? Draw one such graph.

   Solution. There are 21 edges, so using Euler's formula, there should be 9 faces. The total degree of the faces is 42, so there must be 3 faces of degree 6 and 6 faces of degree 4.

3. Suppose G has a planar embedding where every face boundary is an even cycle. Prove that G is bipartite. (Hint: Pick an odd cycle that has the fewest number of faces inside of it.)

   Solution. By way of contradiction, suppose C is an odd cycle that contains the fewest number of faces inside of it. Pick a face C_1 adjacent to a boundary edge inside C. So C_1 is an even cycle. Let the remaining faces inside C form the cycle C_2. Then $|E(C)| = |E(C_1)| + |E(C_2)| - 2|E(C_1) \cap E(C_2)|$. Since C is an odd cycle, this means that C_2 is an odd cycle which contains fewer number of faces inside of it. Contradiction.

4. (a) Let G be a planar embedding where every face has degree 3. Suppose G is 3-colourable. Prove that the dual of G is 3-edge-colourable (meaning there is a 3-colouring of the edges such that all edges joining the same vertex receive different colours).

   Solution. Consider a 3-colouring of G using the colours 1, 2 and 3. We classify each edge in G according to the two colours of its endpoints. Let A, B, C be those that have 12, 13, 23 as their two colours. In the dual, every vertex has degree 3. These three edges around a vertex correspond to edges in G in different classes, one in each of A, B, C. We give edges colours according to their corresponding classification, and this gives a 3-edge-colouring of G^*.

   (b) Find a 3-colouring of the octahedron. The dual of the octahedron is the cube. Find a 3-edge-colouring of the cube.

5. Suppose G is a graph (not necessarily planar) such that the edges of G can be partitioned into two bipartite subgraphs (i.e. $E(G) = A \cup B$ where $A \cap B = \emptyset$, and both $(V(G), A)$ and $(V(G), B)$ are bipartite graphs). Prove that G is 4-colourable.

   Solution. Let G_1, G_2 be the two bipartite subgraphs. They are both 2-colourable, and we use the colours 1, 2 to colour both of them. To find a 4-colouring of G, we use the 4 colours $(1,1), (1,2), (2,1), (2,2)$. For each vertex v, we give it the colour (a, b) where a is the colour of v in G_1 and b is the colour of v in G_2. Notice that for each edge uv in the graph, it is either in G_1 or in G_2. So one coordinate of the colours between u and v must be different. Hence this is a 4-colouring of G.