MATH 239 Assignment 7

- This assignment is due on Friday, November 2, 2012, at 10am in the drop boxes in St. Jerome's (section 1) or outside MC 4067 (the other two sections).
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.

1. Let G be a bipartite graph with vertex classes A and B. Suppose that every vertex in A has degree k and every vertex in B has degree ℓ. Prove that $|B| = \frac{k}{\ell}|A|$.

   Solution:

   Since G is bipartite, every edge of G has exactly one vertex in A and exactly one vertex in B. Therefore we can count the edges of G in two ways: we get

   $$|E(G)| = \sum_{v \in A} deg(v) = k|A|$$

   and

   $$|E(G)| = \sum_{v \in B} deg(v) = \ell|B|.$$

   Therefore $k|A| = |E(G)| = \ell|B|$, which implies the given statement.

2. Let n be a positive integer. We define a graph G_n as follows. The vertex set of G_n is the set of all permutations of $\{1, 2, \ldots, n\}$. (Recall that a *permutation* of $\{1, 2, \ldots, n\}$ is just an ordering of the elements of $\{1, 2, \ldots, n\}$. Thus in particular $|V(G_n)| = n!$.) Two permutations σ and σ' are joined by an edge of G_n if and only if σ' can be obtained from σ by interchanging two positions. (For example, 3241 and 1243 are adjacent in G_4.)

   (a) Draw G_3 and label the vertices.

   (b) Prove that G_n is bipartite for every n. (Hint: consider partitioning the vertex set according to the function T, where for $\sigma = a_1 a_2 \ldots a_n$, the value of $T(\sigma)$ is the number of pairs $s < t$ such that $a_s > a_t$.)

   Solution:

   (b) Let A denote the set of vertices $\sigma \in V(G_n)$ for which $T(\sigma)$ is even, and let B be the set of vertices for which $T(\sigma)$ is odd. We claim that G_n is bipartite with vertex classes A and B.

   To verify this claim we will show that if σ and σ' are joined by an edge in G then $T(\sigma) - T(\sigma') \equiv 1 \pmod 2$. Let $\sigma = a_1 a_2 \ldots a_n$.

   Suppose σ' is obtained from σ by interchanging a_i and a_j, so

   $$\sigma' = a_1 a_2 a_{i-1} a_j a_{i+1} \ldots a_{j-1} a_i a_{j+1} \ldots a_n.$$

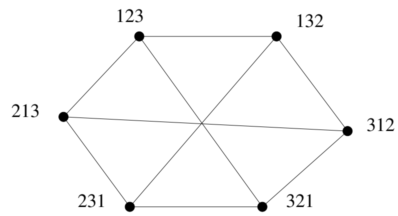
(a) 

Figure 1:

Consider the effect of interchanging a_i and a_j on the value of T. The interchange itself changes T by 1. Let $x = j - i - 1$, the number of elements in $X = \{i+1, \ldots, j-1\}$. Suppose there are y indices q in X with $a_q < a_i$, and therefore $x - y$ indices in X with $a_i < a_q$. Then moving a_i from position i in σ to position j in σ' causes T to go down by y and up by $x - y$, for a total change of $x - 2y$. Similarly (letting z be the number of indices q in X with $a_q > a_j$), moving a_j from position j in σ to position i causes T to go down by z and up by $x - z$, for a total change of $x - 2z$. So the total change to T coming from pairs of indices with one index in $\{i, j\}$ and the other in X is $2x - 2y - 2z$, which is even. Therefore, overall $T(\sigma) - T(\sigma') \equiv 1 \pmod 2$. Since σ and σ' were arbitrary vertices, we find that G_n is bipartite with vertex classes A and B.

3. Let G be a graph that has 20 vertices of degree 25 and 300 vertices of degree 5, and no other vertices. Prove that for every vertex x of degree 25 there exists a path in G from x to a vertex of degree 5.

   Solution:

   Consider the 25 neighbours of x. At most 19 of them can be vertices of degree 25, since in total there are only 20 vertices of degree 25. Thus the remaining 6 neighbours of x must be vertices of degree 5. So in fact there is a path of length 1 from x to a vertex of degree 5.

4. Let $W = v_0 e_1 v_1 \ldots e_n v_n$ be a walk in a graph G, such that $v_0 = v_n$ and all the edges $e_1, \ldots, e_n$ are distinct. Prove that there exists a set $\{C_1, \ldots, C_m\}$ of cycles in G such that $\{e_1, \ldots, e_n\} = E(C_1) \cup \ldots \cup E(C_m)$, and $E(C_s) \cap E(C_r) = \emptyset$ for all $s \neq r$. (Hint: use induction on n.)

   Solution:

   We use induction on n. If $n = 0$ then the empty set of cycles satisfies the requirements.

   Induction hypothesis: Assume $n \geq 1$ and that the given statement holds for all walks of length less than n that satisfy the given conditions.

   Let $W = v_0 e_1 v_1 \ldots e_n v_n$ be given. If all the vertices $v_1, \ldots, v_n$ are distinct then by definition W is itself a cycle, and so satisfies the requirements with $m = 1$. If there are repeated vertices, then let $i < j$ be such that $v_i = v_j$, and among all such pairs of indices, choose i and j such that $j - i$ is smallest. We claim that $C = v_i e_{i+1} v_{i+1} \ldots e_j v_j$ is a cycle in G.

   To verify this, note that $v_i = v_j$ by assumption, and the length is $j - i \geq 1$. By the given conditions on W we know the edges $e_{i+1} \ldots e_j$ are all distinct. To verify that $v_{i+1} \ldots e_j v_j$ is a path, note that if on the contrary there were repeated vertices in this sequence then we would have a pair of indices $k < \ell$ such that $v_k = v_\ell$ and $\ell - k < j - i$, contradicting our choice of i and j. Therefore C is a cycle.

Now observe that $W' = v_0 e_1 v_1 \dots v_i e_{j+1} v_{j+1} \dots e_n v_n$ is a walk in G with $v_0 = v_n$ and all edges are distinct, and the length of W' is less than n. Therefore by the induction hypothesis, there exists a set $\{C_1, \dots, C_t\}$ of cycles in G such that $\{e_1, \dots, e_i, e_{j+1} \dots, e_n\} = E(C_1) \cup \dots \cup E(C_t)$ and $E(C_s) \cap E(C_r) = \emptyset$ for all $s \neq r$. Moreover, since C was a segment of the original walk W which is not present in W', we know that $E(C) \cap E(C_s) = \emptyset$ for all $1 \leq s \leq t$ because the edges of W were all distinct. Therefore $\{C_1, \dots, C_t\} \cup \{C\}$ satisfies the requirements for G.

5. Let G be a graph with p vertices. Suppose every vertex in G has degree at least $\frac{p-1}{2}$. Prove that G is connected.

   Solution:

   Let x and y be arbitrary vertices in G. We need to show that there is a path from x to y in G.

   If x and y are joined by an edge of G then we have a path of length 1 joining them.

   If x and y are not joined by an edge then consider the set of neighbours of x. Since $deg(x) \geq \frac{p-1}{2}$, the set W of vertices in G that are not adjacent to x has size at most $\frac{p+1}{2}$. We also know $x \in W$ and $y \in W$. If all neighbours of y were in W then $deg(y) \leq |W| - 2 \leq \frac{p-3}{2}$ (noting that y cannot be adjacent to itself, and we know y is not adjacent to x). This would contradict the assumption on the degrees in G. Therefore y must have a neighbour z not in W, so since $z \notin \{x, y\}$ it must be that z is a neighbour of x. Therefore x and y are joined by the path of length 2 with vertices xzy.

   Thus by definition G is connected.