UNIVERSITY OF WATERLOO
FINAL EXAMINATION
WINTER TERM 2010

Last Name: ______________________

First Name: ______________________

Id.#: ______________________

Indicate your instructor:

P. Haxell	
M. Mosca	

Course Number	MATH 239
Course Title	Introduction to Combinatorics
Date of Exam	Saturday 17 April 2010
Time Period	4:00pm-6:30pm
Number of Exam Pages (including this cover sheet)	10
Exam Type	Closed Book
Additional Materials Allowed	NONE
Additional Instructions	Write your answers in the space provided. If the space is insufficient, use the back of the page and indicate clearly where your solution continues. Show all your work.

Problem	Value	Mark Awarded	Problem	Value	Mark Awarded
1	10		6	4	
2	3		7	6	
3	5		8	14	
4	10		9	10	
5	10		10	8	
			Total	80	

1. **[10 marks]**

(a) Let S be the set of compositions of n with at least one part such that each part is odd and not equal to 3. (Note then that the number of parts is not fixed, and we do not consider the empty composition.) Express S as a union of Cartesian products of certain sets of integers.

(b) Show that the generating function for S is

$$\Phi_S(x) = \frac{(1-x^2)x + x^5}{(1-x)(1-x^2) - x^5}.$$

(c) Compute a linear recurrence for the coefficients of the generating function for S, and determine enough initial values to uniquely specify these coefficients.

2. **[3 marks]**

   Prove that, for any positive integer n the number of binary strings of length n with an even number of 1s equals the number of binary strings with an odd number of 1s.

3. **[5 marks]**

   Let $a_0, a_1, a_2, \ldots$ be the sequence of integers defined by the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$ and the initial conditions $a_0 = 3, a_1 = 1$. Find a_n as an explicit function of n, for all $n \geq 0$.

4. **[10 marks]**

For each set S and weight function in the second column, the generating function $\Phi_S(x)$ is one of the following power series (one of the power series is the answer for two questions):

$$(1)\quad \frac{1}{1-2x^4} \qquad (2)\quad \frac{1-x^5}{1-x} \qquad (3)\quad \frac{x^4}{(1-x)^4} \qquad (4)\quad \frac{1-x^5}{1-2x+x^6}$$

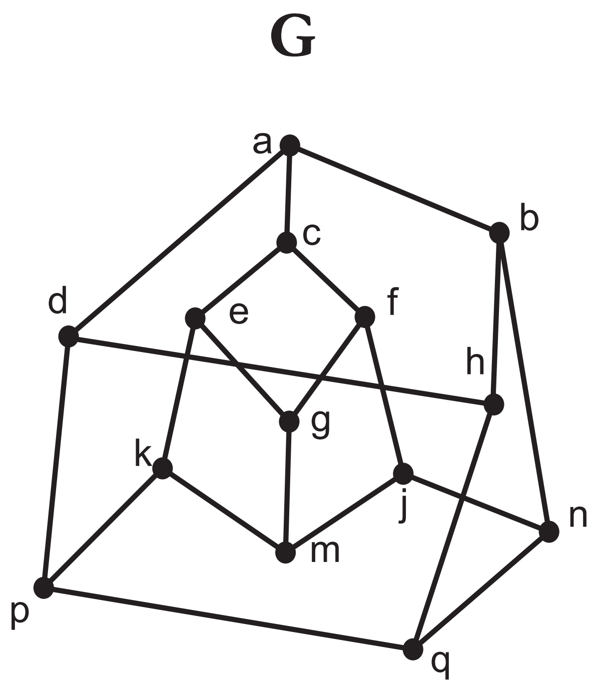
Find the correct generating function for each set. Please place your answer for each set S in the box to the left of the set S.

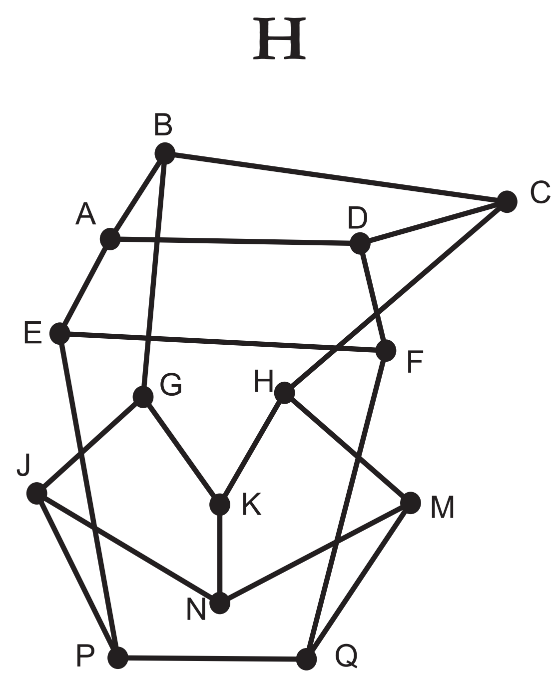
Your Answer	The set S
	$S = \{0,1,2,3,4\}$ The weight of i is i
	S is the set of all sets of 4 positive integers The weight of a set is its largest element
	$S = \{0000\}^* \left(\{1111\}\{0000\}^*\right)^*$ The weight of a string is its length
	$S = \{\epsilon, 1, 11, 111, 1111\} \left(\{0\}\{\epsilon, 1, 11, 111, 1111\}\right)^*$ The weight of a string is its length
	S is the set of compositions with exactly 4 parts The weight of a composition is the sum of its parts

5. (a) **[10 marks]**

Find a breadth-first search tree for each of the following two graphs rooted at vertex a for graph **G**, and rooted at vertex A for graph **H**.

When considering the vertices adjacent to the vertex being examined, take them in alphabetically increasing order of their labels. *List the vertices at each level of your tree.*





(b) Determine whether the two graphs are isomorphic. Prove your answer is correct.

6. [**4 marks**]

   Let G' be an edge subdivision of a bipartite graph G. Prove that G' is 3-colourable.

7. [**6 marks**]

   Let G be a connected planar embedding with exactly 20 faces, each of which has degree 3. How many edges and vertices must G have?

8. **[14 marks]**

The graph G shown below is bipartite. The vertex set A is labelled with small letters, and B is labelled with capital letters.

(a) The matching M in the graph G is indicated by the bold edges. Find an M-augmenting path P in G starting at the vertex y.

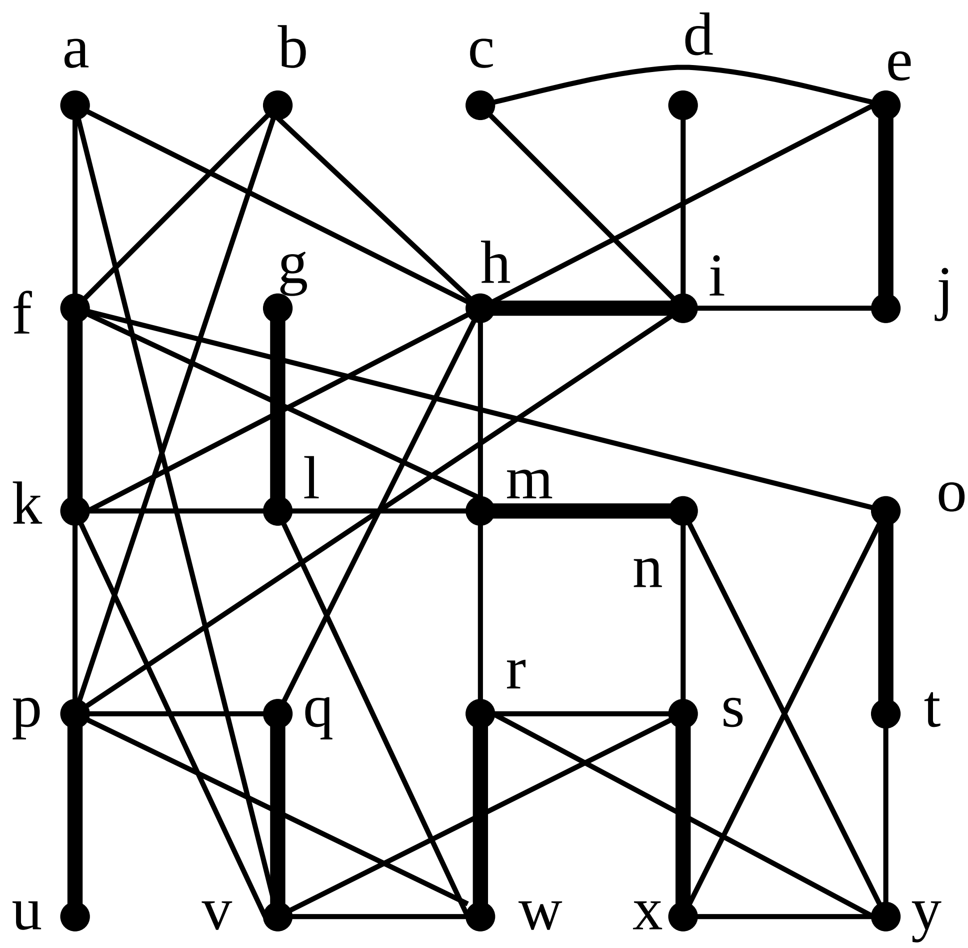


Figure 1: The graph G with matching M

(b) Let M^* be the matching of size $|M| + 1$ obtained by switching on the M-augmenting path P you found in the previous part. List the edges of the matching M^*.

(c) Let X_0 be the M^*-unsaturated vertices in A. Let X be the set of vertices in A that are reachable from a vertex in X_0 by an M^*-alternating path, and let Y be the set of vertices in B that are reachable from a vertex in X_0 by an M^*-alternating path. Use the bipartite matching algorithm to find X_0, X and Y. (Another copy of G is shown here to assist you.)

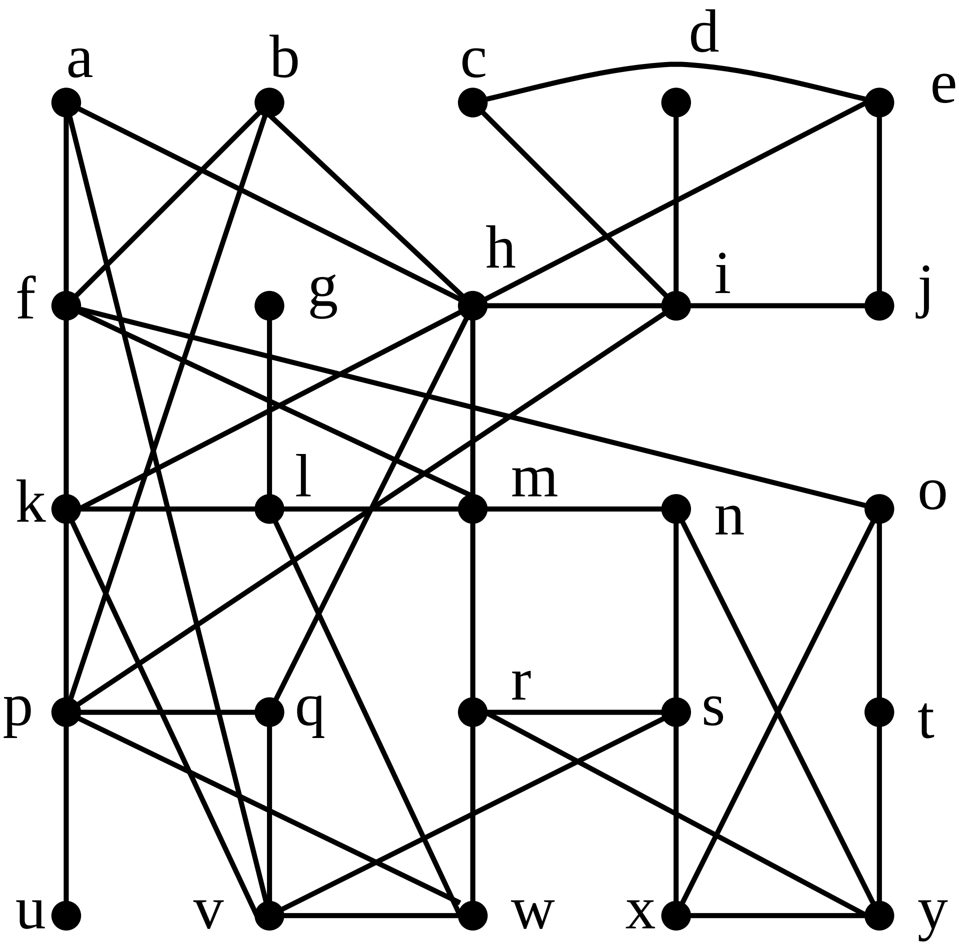


Figure 2: The graph G

(d) Prove that M^* is a maximum matching in G by finding a cover C with $|C| = |M^*|$.

(e) State Hall's Theorem.

(f) Find a subset $D \subseteq B$ such that $|D| > |N(D)|$.

9. **[10 marks]** Let G be a connected planar graph with p vertices, where $p \geq 3$. Let t denote the number of vertices in G with degree less than 6.

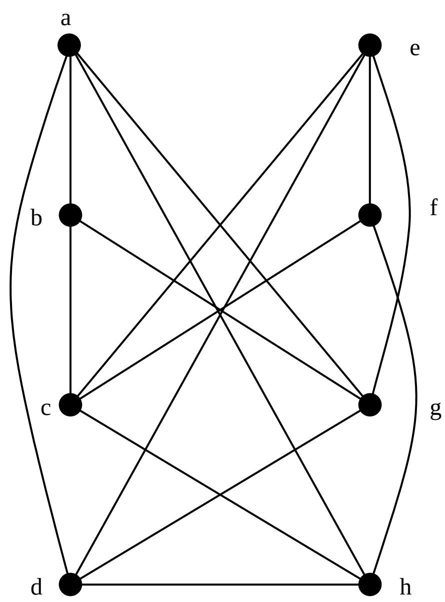
(a) Prove that if G has no cycles then $t \geq \frac{4p+2}{5}$.

(b) Give an example of a graph for which equality ($t = \frac{4p+2}{5}$) holds.

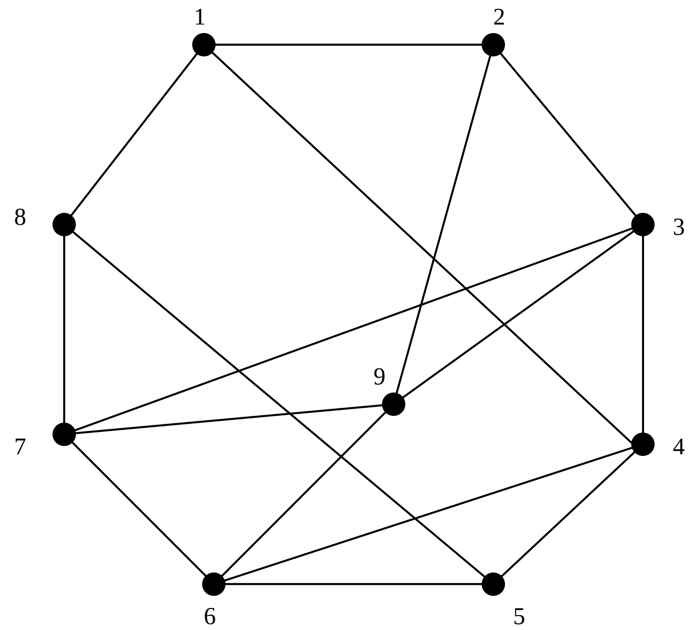
(c) Prove that if G contains a cycle then $t \geq 3$.

(d) Give an example of a graph for which equality ($t = 3$) holds.

10. [**8 marks**] Determine whether each of the graphs shown is planar or nonplanar. Justify your answer in each case.



G



H