MATH 239 Tutorial 3 Problems

1. A house with 9 students have collectively bought n apples. Among the students, Lucas is required to eat an odd number of apples, Ariel is required to eat between 5 to 10 apples, and each of the remaining students is required to eat at least 3 apples. How many ways can all the apples be eaten by the students?
2. How many compositions of n with k parts are there where no part is divisible by 3?
3. Let $\{a_n\}$ be the sequence with the corresponding power series
$$\sum_{n\geq 0} a_n x^n = \frac{1-x+2x^2}{1-x-2x^3}.$$
Determine a recurrence relation that $\{a_n\}$ satisfies, together with sufficient initial conditions. Use this recurrence to find a_5.
4. (a) How many compositions of n are there where each part is greater than 1?
   (b) Let a_n be the answer to part (a). Derive a recurrence relation for $\{a_n\}$ with sufficient initial conditions.
   (c) Give a combinatorial interpretation of the recurrence relation from part (b) through a bijection.
5. How many k-tuples $(a_1, \ldots, a_k)$ of positive integers satisfy the inequality $a_1 + \cdots + a_k < n$?

Additional exercises

1. How many compositions of n are there where the i-th part is congruent to $i \pmod 2$?
2. This question asks you to reverse engineer the process of finding a recurrence relation from a rational function. Suppose a sequence $\{a_n\}$ satisfies $a_0 = 1$, $a_1 = 2$, $a_2 = 2$, and for $n \geq 3$, $a_n = 2a_{n-1} - a_{n-2} + a_{n-3}$. Find a rational function whose power series representation is $\sum_{n\geq 0} a_n x^n$.
3. Prove that for $n \geq 2$, the number of compositions of n with even number of even parts is equal to the number of compositions of n with odd number of even parts.
4. Let S be the set of all compositions of n. In class, you have learned that for $n \geq 1$, $|S| = 2^{n-1}$, which is also the number of subsets of $[n-1]$. Let T be the set of all subsets of $[n-1]$. Find a bijection between S and T, and provide its inverse. Illustrate your bijection by writing down the subset of $[13]$ that corresponds to the composition $(3, 1, 4, 1, 5)$ of 14.
5. Let a_n be the set of all compositions of n. Give a combinatorial proof that for $n \geq 1$,
$$a_n = a_{n-1} + a_{n-2} + \cdots + a_1 + a_0.$$
Note: In particular, this proves that
$$2^{n-1} = 1 + \sum_{i=0}^{n-2} 2^i.$$