Bijections - definitions, and an example of a proof

Definition A function f is **one-to-one** (often written 1-1) if and only if it never takes on the same value twice; that is, $f(a_1) \neq f(a_2)$ whenever $a_1 \neq a_2$.

Definition A function $f : A \to B$ is **onto** if it takes on each value in B at least once; that is, for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

Definition A function f is a bijection if f is both 1-1 and onto.

Bijections are often useful in combinatorics, because if $f : A \to B$ is a bijection, then $|A| = |B|$. (To prove this statement, show that "f is 1-1" implies $|A| \leq |B|$, and that "F is onto" implies $|B| \leq |A|$.)

An example (from Sec 3 tutorial, Jan 16)

Problem. Let E be the set of all subsets of $\{1, 2, .., n\}$ with even size, and let O be the set of all subsets of $\{1, 2, ..., n\}$ with odd size. Let $f : E \to O$, such that for each $A \in E$

$$f(A) = \begin{cases} A \backslash \{1\}, & \text{if } 1 \in A, \\ A \cup \{1\}, & \text{if } 1 \notin A. \end{cases}$$

Show that f is a bijection.

Solution:

***f* is 1-1 (a different proof than in tutorial):**
Suppose $A_1, A_2 \in A$, and $f(A_1) = f(A_2)$. (We must show that $A_1 = A_2$.)
Suppose $f(A_1), f(A_2)$ contain 1. Then (by definition of f)

$$A_1 = f(A_1) \backslash \{1\} = f(A_2) \backslash \{1\} = A_2.$$

Now suppose $f(A_1), f(A_2)$ do not contain 1. Then

$$A_1 = f(A_1) \cup \{1\} = f(A_2) \cup \{1\} = A_2.$$

This proves that f is 1-1.

***f* is onto:**
Let $B \in O$. Suppose $1 \in B$. Then $B \backslash \{1\}$ has even size (i.e. it is in E) and does not contain 1. So

$$f(B \backslash \{1\}) = (B \backslash \{1\}) \cup \{1\} = B.$$

Now suppose $1 \notin B$. Then $B \cup \{1\}$ is in E and contains 1. So

$$f(B \cup \{1\}) = (B \cup \{1\}) \backslash \{1\} = B.$$

We have shown that for every $B \in O$, there exists $A \in E$ such that $f(A) = B$. This proves that f is onto.

Since f is both 1-1 and onto, f is a bijection (as required).