MATH 239 - Fall 2013

Assignment 1

Due date : Friday, September 20th, 2013, at noon (sharp)

Submission Guidelines:

- Total number of marks in this assignment is 33.
- Use a cover page to submit your solutions (available on the course webpage).
- Keep a copy of your manuscript before your submission.
- Assignments submissions are exclusively accepted in the following dropboxes

[Section 001] Dropbox next to the St Jerome's library, 2nd floor of STJ
[Section 002] Math DropBox #18; Slot #1 A-J, Slot #2 K-S, Slot #3 T-Z
[Section 003] Math DropBox #18; Slot #4 A-J, Slot #5 K-S, Slot #6 T-Z
[Section 004] Math DropBox #18; Slot #7 A-J, Slot #8 K-S, Slot #9 T-Z

- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- You are not allowed to post this manuscript (or parts of it) online, nor share it (or parts of it) with anyone not enrolled in this course.

Assignment policies: While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. For example, copying or paraphrasing a solution from some fellow student or old solutions from previous offerings of related courses qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. All students found to be cheating will automatically be given a mark of 0 on the assignment. In addition, there will be a 10/100 penalty to their final mark, as well as all academic offenses will be reported to the Associate Dean for Undergraduate Studies and recorded in the student's file (this may lead to further, more severe consequences).

If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that if you see any marking error, then you should return your assignment paper to the TA of your section within one week and with written notes on all the marking errors; please write the notes on a new sheet and attach it to your assignment paper.

Question 1 [Marks 10=5+5]
Prove the following identities using the binomial theorem.

(a) For all integers $n \geq 1$,

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} 2^{n-k} = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

Solution. *The binomial theorem says that*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Setting $x = -1/2$ we get that

$$\frac{1}{2^n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{2^k}$$

and after we multiply by 2^n we obtain

$$1 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k}.$$

By isolating all but the last summands we have

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} 2^{n-k} = 1 - (-1)^n \binom{n}{n} 2^{n-n} = 1 - (-1)^n.$$

Clearly, if n is even, the last expression is 0, and it is 2 otherwise.

(b) For every $x \in \mathbb{R}$, and for all integers $n \geq 0$,

$$\sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k+j} \binom{n}{k} \binom{k}{j} x^j = x^n.$$

Hint: Use that $x = (x-1) + 1$.

Solution. *Below we utilize that $x = (x-1)+1$.*

$$
\begin{aligned}
x^n &= ((x-1)+1)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} (x-1)^k \qquad \textit{(by the Binomial Theorem)} \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+(-x))^k \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-x)^j \qquad \textit{(by the Binomial Theorem)} \\
&= \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k+j} \binom{n}{k} \binom{k}{j} x^j
\end{aligned}
$$

Question 2 [Marks 5]

n balls with distinct colours have to be placed into k distinct boxes (labeled by 1 up to k), such that the size of box i is exactly r_i (i.e. $\sum_{i=1}^{k} r_i = n$). In how many ways can this be done? Simplify your final answer as much as possible.

Hint: Follow the reasoning that counts the size of set $\mathcal{L}$ in the proof of Theorem 1.3.1.

Solution. *There are $\binom{n}{r_1}$ ways to form the first set. For every such option, we can form the second set in $\binom{n-r_1}{r_2}$ different ways. Similarly, the set t can be formed in $\binom{n-\sum_{j=1}^{t-1} r_j}{r_t}$ many ways. So the different partitions are*

$$
\prod_{t=1}^{k} \binom{n-\sum_{j=1}^{t-1} r_j}{r_t} = \prod_{t=1}^{k} \frac{\left(n-\sum_{j=1}^{t-1} r_j\right)!}{r_t! \, \left(n-\sum_{j=1}^{t} r_j\right)!} = \frac{n!}{\prod_{i=1}^{k} r_i!}
$$

Question 3 [Marks 10=5+5]

Using combinatorial arguments, prove the following identities

(a) For every integer $n \geq 0$,

$$
\sum_{t=0}^{n} \binom{n}{t}^2 = \binom{2n}{n}
$$

Solution. *The right-hand-side counts in how many ways we can choose a set of n objects out $2n$ many.*

Now we interpret the left-hand side. We partition the $2n$ objects (arbitrarily) into two sets A, B of size n each. We will count again in how many ways we choose n objects out of $A \cup B$. Observe that from set A we will choose either 0, or 1, or $\ldots$, or n objects. If t

elements are to be chosen from A (and this can be done in $\binom{n}{t}$) different ways, we have to choose the remaining $n-t$ from B, and this can be done in $\binom{n}{n-t} = \binom{n}{t}$ different ways. Altogether, the number of different ways to complete this process is exactly $\sum_{t=0}^{n} \binom{n}{t}\binom{n}{t}$.

(b) For every integer $n \geq 0$,

$$\sum_{t=0}^{n} \binom{2n}{t+n}\binom{t+n}{n} = 2^n \binom{2n}{n}$$

Hint: Consider $2n$ distinct items. Colour exactly n of them red, and then colour up to n more items blue.

Solution. *The right-hand-side counts in how many ways we can choose a set of n objects out $2n$ many (and say this results in set A of red items), and then choose any subset of the remaining objects (and call the set of blue items B).*

Now we interpret the left-hand side. Note that set B has size between 0 and n. One way to obtain sets A, B as above is also to first choose $n+t$ elements from the universe of $2n$ elements, and then choose n of them to constitute set A (and the rest will form set B). Since t can be anything between 0 to n, this gives the left-hand side above.

Question 4 [Marks 8]
For integers $n \geq 0$ and $t \geq 1$, consider the sum

$$f_t(n) := \sum_{i_1=0}^{n} \binom{n}{i_1} \sum_{i_2=0}^{n} \binom{i_1}{i_2} \sum_{i_3=0}^{n} \binom{i_2}{i_3} \cdots \sum_{i_t=0}^{n} \binom{i_{t-1}}{i_t}.$$

By induction on t prove that $f_t(n)$ counts all length-n words over an alphabet of $t+1$ letters, and conclude that $f_t(n) = (t+1)^n$.
Hint: For the inductive step verify and use that $f_{t+1}(n) = \sum_{j_1=0}^{n} \binom{n}{j_1} f_t(j_1)$.

Solution. *When $t = 1$, the binomial identity says that $f_1(n) := \sum_{i_1=0}^{n} \binom{n}{i_1} = 2^n$. Note that the left hand-side indeed counts the length-n words over a binary alphabet; just choose i_1 positions to place letter 1 (in $\binom{n}{i_1}$ different ways). Clearly, i_1 can assume any value between 0 and n (and that will be the frequency of letter 1).*

For the inductive hypothesis, assume now that $f_t(n)$ counts all length-n words over an alphabet of $t+1$ letters, for some $t \geq 1$.

Let's count now all length-n words over an alphabet of $t+2$ letters. For this, say that the frequency of letter 1 is $n - j_1$, where j_1 can assume any value between 0 and n. Letter 1 can occupy any of the $\binom{n}{n-j_1} = \binom{n}{j_1}$ positions of the word of length n. Note that we are left with a (sub)word of length j_1 for which we can use $t+1$ letters, and by the inductive hypothesis *there are $f_t(j_1)$ such (sub)words. This shows that the number of all length-n words over an alphabet of $t+1$ letters equals*

$$\sum_{j_1=0}^{n}\binom{n}{j_1}f_t(j_1)=\sum_{j_1=0}^{n}\binom{n}{j_1}\sum_{i_2=0}^{j_1}\binom{j_1}{i_2}\sum_{i_3=0}^{j_1}\binom{i_2}{i_3}\cdots\sum_{i_{t+1}=0}^{j_1}\binom{i_t}{i_{t+1}}$$

Note that when $j_1>n$ we have $\binom{n}{j_1}=0$, so we can change the upper bound of the indices of all summations from j_1 to n. This is by definition $f_{t+1}(n)$ as wanted.

By the principle of mathematical induction, we conclude that for all integers $n\geq 0$ and $t\geq 1$, $f_t(n)$ counts all length-n words over an alphabet of $t+1$ letters. But this is also equal to $(t+1)^n$, since for every position we have $t+1$ many options.