MATH 239 Assignment 9

- This assignment is due on Friday, November 23rd, 2012, at 10 am in the drop boxes in St. Jerome's (section 1) or outside MC 4067 (the other two sections).
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.

1. Suppose G is a 4-regular connected graph with a planar embedding such that every face has degree 3 or 4, and further that any 2 adjacent faces have different degrees.

   (a) Prove that G has no bridges and hence that every edge in G is on the boundary of 2 distinct faces.

   (b) Determine precisely the number of vertices, edges, faces of degree 3, and faces of degree 4 in G.

   (c) Draw a planar embedding of a graph having these properties.

   Solution:

   (a) If G has a bridge $e = \{x, y\}$, and G' is the connected component of $G - e$ containing x, then G' has exactly one vertex of degree 3 (which is x), and all others having degree 4. This is not possible, by the Handshaking Theorem. Therefore G has no bridges.

   Since G has no bridges, it follows immediately that every edge is on the boundary of 2 distinct faces, since any edge that is on the boundary of only a single face must be a bridge.

   (b) Since every edge in G is on the boundary of 2 distinct faces, and any 2 adjacent faces must have different degrees, we conclude that every edge in G is on the boundary of exactly one face with degree 3, and exactly one face of degree 4. Let f_3 and f_4 be the number of faces of degree 3 and 4 respectively in G. Then by summing over the face degrees of the f_3 faces of degree 3, (and likewise the face degrees of the f_4 faces of degree 4), we have

   $$q = 3f_3 = 4f_4.$$

   (Note that the reasoning is the same as the justification for the degree-sum formula for faces). Next the total number of faces in G is $s = f_3 + f_4$. Also since G is 4-regular, the degree-sum formula for vertices gives

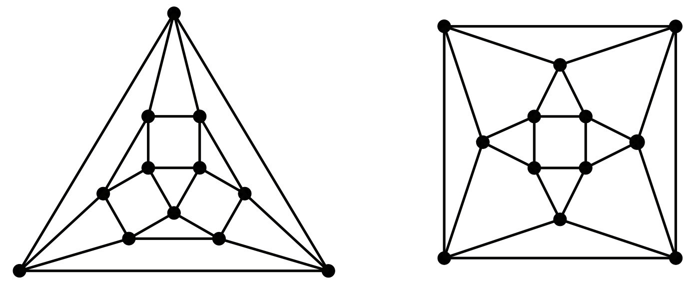
   $$2q = 4p.$$

   Now G is planar and connected, so Euler's Formula applies. Substituting everything we've found above, we have

   $$\begin{aligned} 2 = p - q + s &= \frac{1}{2}q - q + f_3 + f_4 \\ &= \frac{1}{2}q - q + \frac{1}{3}q + \frac{1}{4}q \\ &= \frac{1}{12}q. \end{aligned}$$

Hence $q = 24$, $p = 12$, $f_3 = 8$ and $f_4 = 6$.

(c) The following are 2 different drawings satisfying these properties :



(Note these are the same graph, and G is actually unique. We haven't proved uniqueness, however.)

2. Suppose G is a connected 3-regular planar graph which has a planar embedding such that every face has degree either 5 or 6. Prove that G has precisely 12 faces of degree 5.

Solution: Let f_5 and f_6 be the number of faces of degree 5 and 6 respectively in the planar embedding of G. Then the total number of faces is just $s = f_5 + f_6$, and the sum of the face degrees is $5f_5 + 6f_6$. Therefore by the degree-sum formula for faces we have

$$\begin{aligned} 2q &= \sum_f \deg f = 5f_5 + 6f_6 \\ &= 5f_5 + 6(s - f_5) \\ &= 6s - f_5. \end{aligned}$$

Similarly since G is 3-regular the degree-sum formula for vertices gives

$$2q = \sum_v \deg v = 3p.$$

Finally since G is connected and planar, Euler's Formula applies. Substituting the above two expressions, we have

$$\begin{gathered} p - q + s = 2 \\ \frac{2}{3}q - q + \frac{2}{6}q + \frac{1}{6}f_5 = 2 \\ 0 + \frac{1}{6}f_5 = 2 \end{gathered}$$

and hence G has $f_5 = 12$ faces of degree 5.

3. Recall from assignment 6 the definition of graph complement: If G is a graph, the complement graph of G, denoted $\overline{G}$, is a graph with $V(\overline{G}) = V(G)$, and $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$. Suppose G and $\overline{G}$ are both connected and have $p \geq 11$ vertices. Prove that at least one of G or $\overline{G}$ is not planar.

Solution: We recall from assignment 6 that if G has p vertices and q edges, then $\overline{G}$ has p vertices and $\binom{p}{2} - q$ edges. Suppose for a contradiction that both G and $\overline{G}$ are planar. Then by theorem 7.5.3, both of the following must be true:

$$q \leq 3p - 6$$
$$\binom{p}{2} - q \leq 3p - 6.$$

Adding these 2 inequalities together:

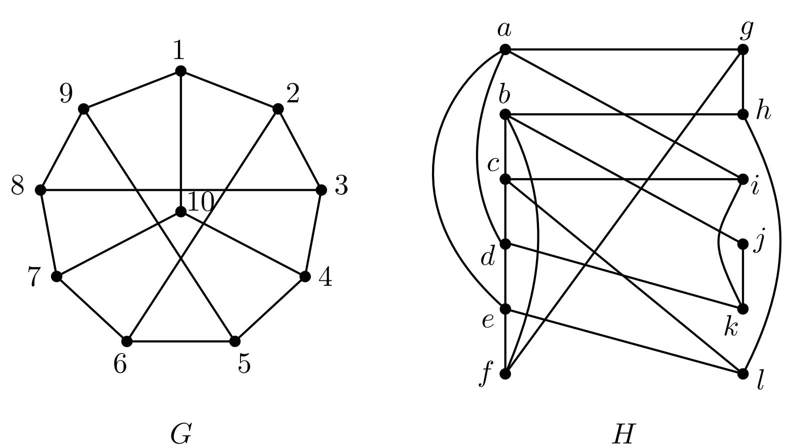
$$\binom{p}{2} \leq 6p - 12$$
$$p^2 - 13p + 24 \leq 0.$$

Finally we show that the above is false whenever $p \geq 11$. It is easy to verify it is false for $p = 11$ and $p = 12$. And when $p \geq 13$ we have

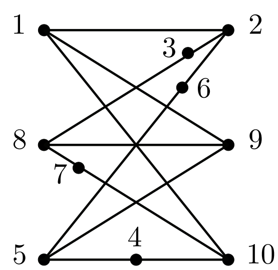
$$p^2 - 13p + 24 \geq 13p - 13p + 24 = 24 > 0.$$

Therefore we have a contradiction and conclude that at least one of G or $\overline{G}$ is not planar. (Note there are many different ways to show the above inequality is false when $p \geq 11$, including using calculus.)

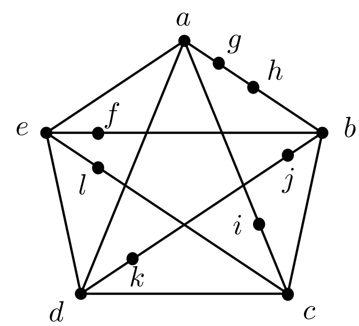
4. For each of G and H below, either give a planar embedding of the graph, or use Kuratowski's Theorem to prove that none exist.



Solution: G has an edge subdivision of $K_{3,3}$ and H has a edge subdivision of K_5. These are verified by examining the following subgraphs:



subgraph of G



subgraph of H

Therefore neither are planar, by Kuratowski's Theorem.

5. (a) Suppose G is a connected planar graph having girth at least 6. Prove that G has at least one vertex with degree at most 2.

   (b) Prove that all connected planar graphs with girth at least 6 are 3-colourable.

Solution:

(a) If G is a tree, then G has a vertex of degree 1, by Theorem 5.1.4. Otherwise assume G is not a tree, and suppose for a contradiction that $\deg(v) \geq 3$ for all vertices v in G. Then by the degree-sum formula,

$$2q = \sum_v \deg(v) \geq 3p.$$

Next consider a face f in a planar embedding of G. Since G is not a tree, Lemma 7.5.1 tells us that the boundary of f contains a cycle. This cycle must have length at least 6, so $\deg(f) \geq 6$. This is true for all faces, so by Lemma 7.5.2, we must have

$$\begin{aligned}(6-2)q &\leq 6(p-2)\\ 2q &\leq 3p-6.\end{aligned}$$

(Alternatively this inequality can be found by directly applying the result of question 2 in Problem Set 7.6, which may have been seen in lecture.)

Combined with the first equation above, this says that $3p \leq 3p-6$, or $0 \leq -6$, a contradiction. Therefore G must have a vertex with degree at most 2.

(b) Again if G is a tree than G is bipartite and the result is immediate (G is 2-colourable). Therefore assume that G is not a tree and proceed by induction on the number of vertices p in G. The smallest p can be is 6, and the only non-tree graph on 6 vertices with girth at least 6 is the 6-cycle, which is indeed 3-colourable (in fact it's bipartite, so 2-colourable).

Now suppose G has $p > 6$ vertices, girth at least 6, and that any planar graph with less than p vertices and girth at least 6 is 3-colourable. By part a), G has some vertex v which has degree at most 2. Let G' be the graph obtained by removing v and all incident edges from G. Then G' is still planar, still has girth at least 6, and has $p-1$ vertices. Hence by the induction hypothesis it is 3-colourable.

Finally the 3-colouring or G' can be extended to a 3-colouring of G by colouring v a different colour than its neighbours, which is possible since v has at most 2 neighbours. The result follows.