MATH 239 Winter 2013
Assignment 5
Solutions

1. *[10 points]* Let

$$A(x) = \frac{2x^2 - 16x + 27}{x^3 - 3x^2 + 4}.$$

Find an expression for $[x^n]A(x)$ as a function of n where $A(x)$ is to be considered a formal power series.

Solution. *[10 points]* We notice that the denominator factors as

$$x^3 - 3x^2 + 4 = (x-2)^2(x-1). \quad 1$$

We write

$$\frac{2x^2 - 16x + 27}{x^3 - 3x^2 + 4} = \frac{Ax + B}{(x-2)^2} + \frac{C}{x+1}. \quad 1$$

We clear denominators to get

$$2x^2 - 16x + 27 = (Ax + B)(x+1) + C(x-2)^2.$$

By equating coefficients, we get $A = -3, B = 7, C = 4$. We can rewrite this as

$$\begin{aligned} \frac{-3x+7}{(x-2)^2} + \frac{4}{x+1} &= \frac{1}{(x-2)^2} - \frac{3}{x-2} + \frac{5}{x+1} \\ &= \frac{1}{4(1-\frac{x}{2})^2} + \frac{3}{2(1-\frac{x}{2})} + \frac{5}{1+x}. \quad 2 \end{aligned}$$

We know

$$\begin{aligned} \frac{1}{(1-\frac{x}{2})^2} &= \sum_{n=0}^{\infty} (n+1)\frac{x^n}{2^n} \\ \frac{1}{1-\frac{x}{2}} &= \sum_{n=0}^{\infty} \frac{x^n}{2^n} \\ \frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n. \quad 2 \end{aligned}$$

So we have

$$\frac{1}{(x-2)^2} - \frac{3}{x-2} + \frac{5}{x+1} = \frac{1}{4}\sum_{n=0}^{\infty}(n+1)\frac{x^n}{2^n} + \frac{3}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n} + 5\sum_{n=0}^{\infty}(-1)^n x^n. \quad 2$$

Consequently,

$$[x^n]A(x) = (n+1)\frac{1}{2^{n+2}} + \frac{3}{2^{n+1}} + 5 \cdot (-1)^n. \quad 2$$

[Marking note: A complete answer must include a derivation for $[x^n]A(x)$ together with an explanation of how it was obtained. This should be done through a partial fraction decomposition. No more than three points should be deducted for linear algebra errors. Students should justify the power series expansions.]

2. *[20 points]* Consider the homogeneous recurrence relation

$$c_n - 13c_{n-2} - 12c_{n-3} = 0 \text{ for } n \geq 3.$$

(a) *[10 points]* Write down a formula for c_n in terms of n, c_0, c_1, and c_2.

(b) *[4 points]* Write down the solution satisfying $c_0 = 0, c_1 = 0, c_2 = 0$.

(c) *[4 points]* Write down the solution satisfying $c_0 = 35, c_1 = 0, c_2 = 0$.

(d) *[6 points]* Write down the solution to the non-homogeneous recurrence relation

$$b_n - 13b_{n-2} - 12b_{n-3} = 24 \text{ for } n \geq 3.$$

satisfying $b_0 = 34, b_1 = -1, b_2 = -1$.

Solution. *[20 points]*

(a) *[10 points]* Note that the characteristic polynomial is

$$x^3 - 13x - 12 = (x+1)(x+3)(x-4)) \; 2$$

so the roots are -1, -3, and 4 with multiplicity 1. Therefore, the general solution is of the form

$$c_n = P_1(n)(-1)^n + P_2(n)(-3)^n + P_3(n)4^n$$

where P_1, P_2 are polynomials with $\deg(P_1) < 2, \deg(P_2) < 1, \deg(P_3)$. In other words, the polynomials are all constant so we may write

$$c_n = A(-1)^n + B(-3)^n + C(4)^n. \; 2$$

for real numbers A, B, C. We get the following linear system by plugging in $n = 1, 2, 3$:

$$\begin{aligned} c_0 &= A + B + C \\ c_1 &= -A - 3B + 4C \\ c_2 &= A + 9B + 16C. \end{aligned}$$

We solve this to obtain

$$\begin{aligned} A &= \frac{6c_0}{5} + \frac{c_1}{10} - \frac{c_2}{10} \\ B &= -\frac{2c_0}{7} - \frac{3c_1}{14} + \frac{c_2}{14} \\ C &= \frac{3c_0}{35} + \frac{4c_1}{35} + \frac{c_2}{35}. \; 4 \end{aligned}$$

So we have

$$c_n = \left(\frac{6c_0}{5} + \frac{c_1}{10} - \frac{c_2}{10}\right)(-1)^n + \left(-\frac{2c_0}{7} - \frac{3c_1}{14} + \frac{c_2}{14}\right)(-3)^n + \left(\frac{3c_0}{35} + \frac{4c_1}{35} + \frac{c_2}{35}\right)(4)^n. \; 2$$

[Marking note: A complete answer must give the general solution for c_n. Some students may solve this problem by first writing the solution as a rational function and then doing a partial fraction decomposition. They should not lose points for doing that. Because the linear system

is rather involved, students should not lose more than three points for linear algebra errors. Some students may solve the linear system by using a computer software package. They should lose no points for doing that.]

(b) *[2 points]* By induction or just setting $c_0 = c_1 = c_2 = 0$ above we get

$$c_n = 0$$

for all n.

[Marking note: Points should be all or nothing at all.]

(c) *[2 points]* Plugging $c_0 = 35, c_1 = 0, c_2 = 0$ into the solution of (a), we get

$$c_n = 42(-1)^n - 10(-3)^n + 3(4)^n.$$

[Marking note: Points should be all or nothing at all.]

(d) *[6 points]* We first find a particular solution to

$$b_n - 13b_{n-2} - 12b_{n-3} = 24 \text{ for } n \geq 3.$$

We guess that a constant $b_n = \alpha$ might work. Substituting in α for c_n, we obtain $-24\alpha = 24$ so $\alpha = -1$. 2 Then, we now know that a general solution of the equation is of the form $-1+c_n$ for c_n a solution to (a). From $b_0 = 34, b_1 = -1, b_2 = -1$, we get $c_0 = 35, c_1 = 0, c_2 = 0$. 2 Coincidentally, those were exactly the initial conditions for (c). Therefore,

$$c_n = 42(-1)^n - 10(-3)^n + 3(4)^n.$$

so

$$c_n = -1 + 42(-1)^n - 10(-3)^n + 3(4)^n. \; 2$$

[Marking note: Students should not lose points for finding the solution by an alternative method, provided they get the right answer.]

3. *[20 points]* Consider the homogeneous recurrence relation

$$c_n + 4c_{n-1} + 3c_{n-2} = 0 \text{ for } n \geq 2.$$

 (a) *[6 points]* Write down a formula for c_n in terms of n, c_0, and c_1.

 (b) *[10 points]* For the solution found in (a), define the formal power series $C(x)$ by

$$C(x) = \sum_{n \geq 0} c_n x^n.$$

 Find polynomials $P(x), Q(x)$ with $\deg Q = 2$ such that

$$C(x) = \frac{P(x)}{Q(x)}.$$

 Express the coefficients of $P(x)$ in terms of c_0, c_1.

Solution.

(a) *[10 points]* The characteristic polynomial is $x^2 + 4x + 3 = (x + 1)(x + 3)$ whose roots are $-1, -3$. 2 Consequently, the general solution is

$$c_n = A(-1)^n + B(-3)^n. \quad 2$$

Now we have the following expressions for the first two values of the general solution:

$$\begin{aligned} c_0 &= A + B \\ c_1 &= -A - 3B \quad 2 \end{aligned}$$

If we solve for A, B in terms of c_0, c_1, we get

$$\begin{aligned} A &= \frac{3c_0 + c_1}{2} \\ B &= -\frac{c_0 + c_1}{2} \quad 2 \end{aligned}$$

so the desired solution is

$$c_n = \frac{3c_0 + c_1}{2}(-1)^n - \frac{c_0 + c_1}{2}(-3)^n. \quad 2$$

[Marking note: Students should both write the general solution and solve for the coefficients in terms of c_0, c_1 to get full credit.]

(b) *[10 points]* By general facts about recurrence relations covered in the notes, we know that

$$C(x) = \frac{P(x)}{Q(x)}$$

where $Q(x) = 1 + 4x + 3x^2$ and $\deg(P) < 2$. We can see that by noting that for $n \geq 2$,

$$\begin{aligned} [x^n]((1 + 4x + 3x^2)C(x)) &= [x^n]C(x) + 4[x^n](xC(x)) + 3[x^n](x^2C(x)) \\ &= [x^n]C(x) + 4[x^{n-1}]C(x) + 3[x^{n-2}]C(x) \\ &= c_n + 4c_{n-1} + 3c_{n-2} \\ &= 0. \end{aligned}$$

Consequently, $(1 + 4x + 3x^2)C(x)$ is a polynomial of degree at most 1. 5

We write $P(x) = p_0 + p_1x$ and expand

$$\frac{P(x)}{Q(x)} = \frac{p_0 + p_1x}{1 + 4x + 3x^2} = (p_0+p_1x)(1+(-4x-3x^2)+(-4x-3x^2)^2+\dots) = p_0+(p_1-4p_0)x+\dots$$

where $\dots$ denotes higher order terms. If we equate this expression to $c_0 + c_1x + \dots$, we get

$$\begin{aligned} c_0 &= p_0 \\ c_1 &= p_1 - 4p_0. \quad 2 \end{aligned}$$

Solving for p_0, p_1 in terms of c_0, c_1, we get

$$p_0 = c_0,\ p_1 = c_1 + 4c_0. \quad 2$$

It follows that

$$C(x) = \frac{c_0 + (c_1 + 4c_0)x}{1 + 4x + 3x^2}. \quad (1)$$

[Marking note: Students must identify both the denominator and the numerator (in terms of c_0, c_1) for full credit.]

4. *[10 points]* Consider the two sequences $\{a_n\}, \{b_n\}$ given by

$$\begin{aligned} a_n &= n3^n - 6 \\ b_n &= n3^n - 6 + 4n \end{aligned}$$

   (a) *[7 points]* Derive a third order homogeneous recurrence relation that a_n satisfies.

   (b) *[3 points]* Write down a third order non-homogeneous reoccurrence relation that b_n satisfies whose left-hand side is the same as that found in (a).

Solution. *[10 points]*

(a) *[7 points]* The solution is of the form

$$P_1(n) \cdot 3^n + P_2(n) \cdot 1^n$$

where $\deg(P_1) = 1$, $\deg(P_2) = 0$. Working backwards, we see that the characteristic polynomials should have 3 as a root with multiplicity two and 1 as a root with multiplicity one. It follows that the characteristic polynomial is

$$E(x) = (x-3)^2(x-1) = x^3 - 7x^2 + 15x - 9 \quad (4)$$

It follows that a_n satisfies the homogeneous recurrence relation

$$a_n - 7a_{n-1} + 15a_{n-2} - 9a_{n-3} = 0. \quad (3)$$

[Marking note: Students should get a recurrence relation. It must be third order. Their answer may differ from the given one by a constant multiple.]

(b) *[3 points]* An answer is of the form:

$$b_n - 7b_{n-1} + 15b_{n-2} - 9b_{n-3} = f(n)$$

where we must determine the function $f(n)$. We can find the function $f(n)$ be plugging in $b_n = n3^n - 6 + 4n$ into the left-hand side. Doing so, we get

$$b_n - 7b_{n-1} + 15b_{n-2} - 9b_{n-3} = 4(n - 7(n-1) + 15(n-2) - 9(n-3)) = 16.$$

The desired non-homogeneous recurrence relation is therefore,

$$b_n - 7b_{n-1} + 15b_{n-2} - 9b_{n-3} = 16.$$

[Marking note: Students should substitute in the particular solution to find their right hand side. Their solutions may look different from the given one, but they should get the same answer (up to a constant multiple).]