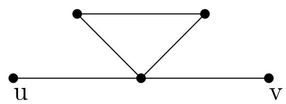
MATH 239 Spring 2012: Assignment 7

Solutions

1. {16 marks} For the following statements, determine whether they are true or false, and give justifications through a proof or counterexample.

   (a) If there is a u, v-walk of odd length, then there is a u, v-path of odd length.

   Solution. False. For example, in the following graph, there exist a u, v-walk of length 5, but the only u, v-path has length 2.

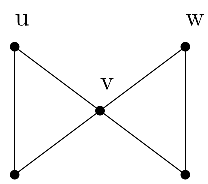


   (b) If G is a bipartite graph, then any closed walk has even length.

   Solution. True. Let (A, B) be a bipartition of the vertices of G. Let $v_0, v_1, \ldots, v_k = v_0$ be a closed walk in G, which has length k. Without loss of generality, we may assume that v_0 is in A. Since v_0v_1 is an edge, v_1 must be in B. Since v_1v_2 is an edge, v_2 must be in A. Following the same argument, we see that $v_0, v_2, v_4, \ldots$ are in A and $v_1, v_3, v_5, \ldots$ are in B. Since the last vertex $v_k = v_0$ which must be in A, k must be even.

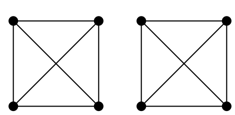
   (c) If there is a cycle containing u and v and another cycle containing v and w, then there is a cycle containing u and w.

   Solution. False.



   (d) Any 3-regular graph must have a Hamilton cycle.

   Solution. False. The following graph (considered as one graph) is not connected, and hence has no Hamilton cycle.



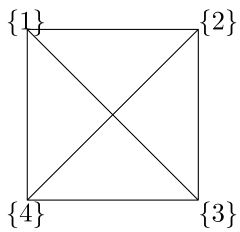
2. {8 marks} Prove that if G is a graph where every vertex has degree at least k, then G contains a path of length at least k.

   Solution. Let $v_0, v_1, \ldots, v_m$ be a path of the longest length in G. Note that neighbours of v_m must all be on the path, for otherwise we may extend the path to get a longer one, contradicting our original choice. Since v_m has degree at least k, it is adjacent to at least k distinct vertices on the path. This implies that $m \geq k$, hence this path has length at least k.

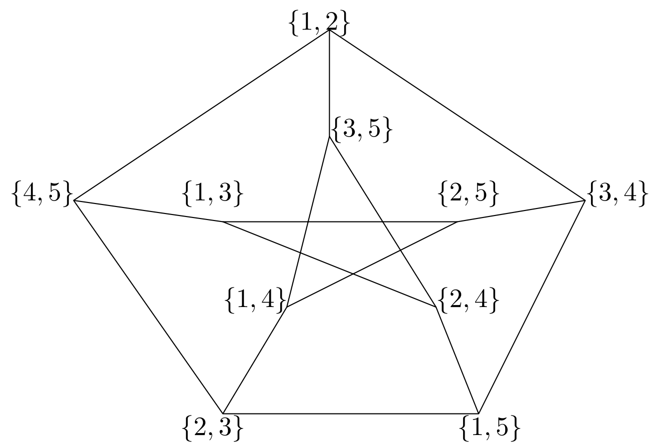
3. {16 marks} The Kneser graph $G_{n,k}$ is the graph where the vertices are all k-subsets of $[n]$, and two vertices are adjacent if and only if their corresponding sets are disjoint.

   (a) Draw $G_{4,1}$ and $G_{5,2}$.

   Solution. $G_{4,1}$ is a K_4, $G_{5,2}$ is the Petersen graph.



$G_{4,1}$



$G_{5,2}$

   (b) How many vertices and edges are there in $G_{n,k}$?

   Solution. There are $\binom{n}{k}$ vertices. Each vertex is adjacent to $\binom{n-k}{k}$ other vertices (these are the k-subsets of the $n-k$ element not in our set). Using the handshaking lemma and we see that the number of edges is $\binom{n}{k}\binom{n-k}{k}/2$.

   (c) Prove that if $n \geq 3k-1$, then $G_{n,k}$ is connected.

   Solution. We will show that there is a path between $S = \{1, \ldots, k\}$ and any other vertices. Let T be any k-subset of $[n]$. If T is disjoint from S, then there is an edge between S and T which forms an S, T-path. Otherwise, $|S \cap T| \geq 1$, and so

$$|S \cup T| = |S| + |T| - |S \cap T| \leq 2k - 1.$$

   Since $n \geq 3k-1$, we see that

$$|[n] \setminus (S \cup T)| = n - |S \cup T| \geq 3k - 1 - (2k-1) = k.$$

   This means that there are at least k elements of $[n]$ that are not in $S \cup T$. Let U be a k-subset of those elements. Then U is disjoint with both S and T, and S, U, T forms an S, T-path.

   (d) Prove that if $n \geq 3k$, then $G_{n,k}$ is not bipartite. As {Extra credit: 4 marks}, prove this statement for $n \geq 2k+1$.

   Solution. It suffices to show this statement for $n = 3k$, as $G_{3k,k}$ is a subgraph of $G_{n,k}$ for any larger n. In $G_{3k,k}$, we consider the three vertices

$$S = \{1, 2, \ldots, k\}, T = \{k+1, \ldots, 2k\}, U = \{2k+1, \ldots, 3k\}.$$

   These three vertices are mutually disjoint, hence they are adjacent to each other. This forms a cycle of length 3 in $G_{3k,k}$, hence this graph is not bipartite.
   For $n \geq 2k+1$, again we only need to show this for $n = 2k+1$. We will find a cycle of length $2k+1$ as follows: Let $S_1 = \{1, 2, \ldots, k\}$, $T_1 = \{k+1, k+2, \ldots, 2k\}$, $S_2 = \{2, 3, \ldots, k, 2k+1\}$, $T_2 = \{1, k+2, k+3, \ldots, 2k\}$. For each $i = 3, \ldots, k$, we define

$$S_i = \{i, \ldots, k\} \cup \{k+1, \ldots, k+i-2\} \cup \{2k+1\}, T_i = \{1, \ldots, i-1\} \cup \{k+i, \ldots, 2k\}.$$

It is easy to check that S_i and T_i are disjoint, and T_i and S_{i+1} are also disjoint. We now define $S_{k+1} = \{k+1, k+2, \ldots, 2k-1, 2k+1\}$, which is disjoint from both S_1 and $T_k = \{1, \ldots, k-1, 2k\}$. This means that

$$S_1, T_1, S_2, T_2, \ldots, S_k, T_k, S_{k+1}$$

is a cycle of length $2k+1$. This implies that $G_{n,k}$ is not bipartite.

Just to illustrate this construction, this cycle in $G_{7,3}$ is $S_1, T_1, S_2, T_2, S_3, T_3, S_4$ where

$$\begin{aligned} S_1 &= \{1,2,3\} \\ T_1 &= \{4,5,6\} \\ S_2 &= \{2,3,7\} \\ T_2 &= \{1,5,6\} \\ S_3 &= \{3,4,7\} \\ T_3 &= \{1,2,6\} \\ S_4 &= \{4,5,7\} \end{aligned}$$

4. {10 marks} Let a_n be the number of cycles of length 4 in an n-cube. So $a_1 = 0, a_2 = 1, a_3 = 6$. For $n \geq 1$, determine an explicit formula for a_n.

   (Hint: There are at least 2 ways to solve this question. One way is to use the recursive construction of the n-cube to generate a recurrence relation, and solve it. Another way is to enumerate certain pairs of binary strings.)

   Solution. Recall that we can construct the n-cube by taking two copies of the $(n-1)$-cube and join corresponding vertices of the two copies with an edge. Let's call the two copies C and C' where a vertex v in C is joined to the vertex v' in C'. Each 4-cycle in C and C' is still a 4-cycle in the n-cube. There are a_{n-1} of those 4-cycles, so this accounts for $2a_{n-1}$ 4-cycles in the n-cube. In addition, for each edge st in C, we have an additional 4-cycle s, t, t', s' in the n-cube. Since there are $2^{n-2}(n-1)$ edges in C, this accounts for an additional $2^{n-2}(n-1)$ 4-cycles in the n-cube. So a_n satisfies the recurrence

   $$a_n = 2a_{n-1} + n2^{n-2} - 2^{n-2}.$$

   The homogeneous part of the solution is $A \cdot 2^n$. For a specific solution, we need to use

   $$b_n = \alpha n^2 2^{n-2} + \beta n 2^{n-2}.$$

   This will give us

   $$\begin{aligned} b_n - 2b_{n-1} &= \alpha n^2 2^{n-2} + \beta n 2^{n-2} - 2\alpha(n-1)^2 2^{n-3} - 2\beta(n-1)2^{n-3} \\ &= 2\alpha n 2^{n-2} - \alpha 2^{n-2} + \beta 2^{n-2} \end{aligned}$$

   This means that $2\alpha = 1$, so $\alpha = 1/2$. And $\beta - \alpha = -1$, so $\beta = -1/2$. Therefore, a specific solution is

   $$b_n = n^2 2^{n-3} - n 2^{n-3}.$$

   The general solution is

   $$a_n = A \cdot 2^n + (n^2 - n)2^{n-3}.$$

   Plugging in $a_1 = 0$, we get that $A = 0$. So the solution for a_n is

   $$a_n = (n^2 - n)2^{n-3}.$$

   Alternate solution. Let A be the set of all pairs (s,t) of binary strings of length n where s and t differ in exactly 2 positions. There are 2^n ways to choose s, and for each s, there are $\binom{n}{2}$ ways to choose t (just pick the two positions to change). So $|A| = 2^n \binom{n}{2}$.

Now for each 4-cycle a, b, c, d in the n-cube, notice that the opposite ends a, c or b, d are binary strings that differ in exactly two positions. For each such cycle, we create 4 pairs in A: $(a, c), (c, a), (b, d), (d, b)$. So then the number of 4-cycles in the n-cube is $|A|/4 = 2^{n-2}\binom{n}{2}$. (Note that this is exactly the same as our previous solution when you expand $\binom{n}{2} = n(n-1)/2$.)