UNIVERSITY OF WATERLOO
FINAL EXAMINATION
FALL TERM 2003

Surname: ____________________

First Name: ____________________

Id.#: ____________________

Course Number	MATH 239
Course Title	Introduction to Combinatorics
Instructor	Professor Goulden 1:30 ☐ Professor Mosca 2:30 ☐ Professor Schellenberg 1:30 ☐ Professor Verstraete 12:30 ☐
Date of Exam	December 13, 2003
Time Period	9 a.m. – 12 Noon
Number of Exam Pages (including this cover sheet)	12 pages
Exam Type	Closed Book

ADDITIONAL INSTRUCTIONS:

1. Write your name and Id.# in the blanks above. Put a check mark in the box next to your instructor's name and lecture time.
2. There are 12 pages to this exam including the cover page. Please be sure you have all 12 pages.
3. Answer each of the problems in the space provided; use the back of the previous page for additional space.

4. You may only use a non-programmable calculator. Show the reasoning used in any calculation.

Problem	Value	Mark Awarded	Problem	Value	Mark Awarded
1	12		6	15	
2	15		7	13	
3	10		8	14	
4	13				
5	8		TOTAL	100	

[5] **1(a)** For fixed positive integers n, k, determine the number of integer solutions to the equation $t_1 + \ldots + t_k = n$, where $t_1 \geq 2, \ldots, t_k \geq 2$.

[7] **(b)** For fixed positive integers n, k, show that the number of integer solutions to the equation $t_1 + \ldots + t_k = n$, where $8 \geq t_1 \geq 2, \ldots, 8 \geq t_k \geq 2$ is given by

$$\sum_{i=0}^{\lfloor \frac{n-2k}{7} \rfloor} \binom{k}{i} (-1)^i \binom{n-k-7i-1}{k-1}.$$

[4] **2(a)** Let p_n, $n \geq 0$, be the number of $\{0,1\}$–strings of length n in which all blocks have odd length. Prove that

$$\sum_{i \geq 0} p_i x^i = \frac{1 + x - x^2}{1 - x - x^2}.$$

[8] **2(b)** Let s_n, $n \geq 0$, be the number of $\{0,1\}$–strings of length n in which a block of odd length is never immediately followed by a block of odd length, and a block of even (positive) length is never immediately followed by a block of even (positive) length (e.g., for $n = 12$, some of the strings of this type are 111111111111, 100001111100, 000010011100, 000111101111, 000000000000; for $n = 3$, the strings of this type are 000, 001, 011, 111, 110, 100). Prove that

$$\sum_{i \geq 0} s_i x^i = \frac{1 + 2x + x^3 - x^4}{1 - 2x^2 - x^3 + x^4}.$$

[3] **(c)** From part (b), deduce a linear recurrence equation for s_n, with initial conditions that uniquely determine $\{s_n\}_{n \geq 0}$.

3. Let $a_n = 5^n - 2^{n+1},\ n \geq 0.$

[4] **(a)** Find a linear, homogeneous recurrence equation for a_n, with initial conditions that uniquely determine $\{a_n\}_{n\geq 0}$.

[6] **(b)** Let $b_n = a_n^2$, $n \geq 0$. Find a linear, homogeneous recurrence equation for b_n, with initial conditions that uniquely determine $\{b_n\}_{n\geq 0}$.

4. Let Q_n, $n \geq 0$, denote the graph whose vertex set is the set of all $\{0, 1\}$– strings of length n, and in which two vertices are joined by an edge if and only if they differ in exactly one position.

[2] **(a)** Draw a planar embedding of Q_3.

[2] **(b)** Determine the number of vertices and edges in Q_n, for $n \geq 0$.

[4] **(c)** Prove that Q_n is bipartite, for $n \geq 0$.

[5] **(d)** Find a minimum cover of Q_n, and prove that it is a minimum cover, for $n \geq 0$.

[4] **5(a)** Draw four non-isomorphic trees with 7 vertices.

[4] **(b)** Are the following two graphs isomorphic? Justify your answer.

[5] **6(a)** Construct a breadth-first search tree for the graph G below, using the vertex labelled 1 as the root vertex. When considering the vertices adjacent to the active vertex, add them to the tree in increasing order of label. Give a list of the vertices in the order that they join the tree.

[3] **(b)** Use the breadth-first search tree from (a) to determine the distance in G from vertex 1 to vertex v for each vertex $v = 2, \ldots, 14$. (Recall that the distance from vertex 1 to v is the length of the shortest path from 1 to v.)

[3] **6(c)** Is G 2–colourable ? Justify your answer. Is G 3–colourable ? Justify your answer.

[4] **(d)** Apply Kuratowski's Theorem to prove that G is not planar.

[2] **7(a)** Prove that every 3–regular graph has an even number of vertices.

[4] **(b)** Prove that every 3–regular graph has at least one cycle.

[3] **(c)** Find an example of a 3–regular connected graph with a bridge.

[4] **(d)** Let G be a connected 3–regular graph with at least one bridge, and a perfect matching M. Prove that every bridge is in M.

[2] **8(a)** State König's Theorem for matchings.

[4] **(b)** Apply the bipartite matching algorithm to construct the sets X and Y for the following bipartite graph G, in which $A = \{2, 4, 6, 8, 10, 12, 14\}$, $B = \{1, 3, 5, 7, 9, 11, 13\}$, and the matching M consists of the thick edges.

[5] **8(c)** From your construction in part (b), find an augmenting path, and use it to find a larger matching M' for the graph G. Is M' a maximum matching ? Justify your answer. (You may find the extra drawing of G, below, helpful.)

[3] **(d)** If a graph G has a matching and a cover of equal size, then must G be bipartite? Justify your answer.