1. Proof required.

2.

$$\left(\frac{x^{10}}{1-x^{10}}\right)\left(\frac{x^{20}}{1-x^{20}}\right)\left(\frac{x^{50}}{1-x^{50}}\right)$$

3. (a) $\{0\}^*\left(\{1\}\{1\}^*\{0\}\{0\}^* \setminus \{1\}\{11\}^*\{0\}\{00\}^*\right)^*\{1\}^*$

   (b) $a_n = a_{n-1} + a_{n-2}, \quad n \geq 2.$

   $a_0 = 1,\ a_1 = 2.$

4. $b_n = (2-n)2^n$.

5. (a) a connected graph with no cycles.

   (b) Let n_i be the number of vertices of degree i. Then using sum of degrees $= 2q$ (Handshake theorem), and also $q = p - 1$ for trees, we can see that

   $$n_1 - n_3 - 2n_4 - 3n_5 - \cdots = -2.$$

   This is true for all trees. Substituting $n_1 = 7$ and $n_4 = 2$ we get

   $$7 - n_3 - 2 \times 4 - 3n_5 - \cdots = -2.$$

   Thus

   $$1 = n_3 + 3n_5 + 4n_6 + \cdots .$$

   Since the n_i are nonnegative integers, the only possibility is $n_3 = 1$ (and $n_5 = n_6 = \cdots = 0$).

   (c) Should be easy.

6. (a) The vertices at levels are as follows (if these are correct, you probably have the right tree):

   level 0: a

   level 1: b e s

   2: c i f o r t

   3: g j d p n u

   4: h k q

   5: m

   (b) The graph is not bipartite because there is an edge between t wo vertices at the same level (gu). Tracing back up the tree we find the cycle dfguqmhd.

7. Use the handshake theorem: sum of vertex degrees is $5p = 2q$.

   Also the analogue for faces (‘face-shaking’): sum of face degrees is $3s = 2q$ (assuming s denotes the number of faces).

   So $p = 2q/5$ and $s = 2q/3$.

   Plug these into Euler’s theorem: since the embedding is connected,

   $$p - q + s = 2$$

   and this gives $q = 30$.

8. (a) is planar. (Draw it.)

   (b) is not planar. It contains an edge subdivision of $K_{3,3}$.

9. Requires a proof. Similar to proof of 6 colour theorem.

10. (a) state Hall's theorem (Bipartite graph with bipartition A, B has matching saturating all vertices in A iff $|N(D)| \geq |D|$ for all $D \subseteq A$.)

    (b) Let $D \subseteq A$. If $D = \emptyset$ then $|N(D)| \geq |D|$ immediately. Next, every vertex has at least 10 neighbours. So every nonempty set D of vertices has $|N(D)| \geq 10$. Thus $|N(D)| \geq |D|$ if $|D| \leq 10$. Now suppose $|D| \geq 11$ for $D \subseteq A$. We show that $|N(D)| = 24$. If not, then there is a vertex $b \in B$ such that $b \notin N(D)$. Then b is adjacent to no vertex in D. That is, $N(b)$ is disjoint from D. But the graph is bipartite and b has degree at least 10, so at least 10 vertices in A are not in D (i.e. the vertices in $N(b)$). Since $|A| = 20$ this means $|D|$ must be at most 10. (Contradiction). So $|N(D)| = 24$ if $|D| \geq 10$. Since $|D| \leq 20$ this implies $|N(D) \geq |D|$, as required.

    Finally, since $|N(D) \geq |D|$ for all $D \subseteq A$, the graph has a matching saturating all vertices in A by Hall's theorem.

11. (a) Vertices added in the following order:

    To X: 3,5

    To Y: 2,4,6,10

    To X: 1,9,7,15

    To Y: 14, 18, 20

    To X: 13, 17,19

    To Y: 8, 12, 16

    Stop: 12 is unsaturated.

    Augmenting path is 5, 10, 15, 14 13, 12.

    New matching has edges 1 2, 6 7, 4 9, 5 10, 15 14, 13 12, 8 11, 17 18, 19 20.

    (b) Minimum cover 2,4,7.