Math 239 Tutorial 5

1. Consider the formal power series

$$A(x) := \frac{x^2 - 13x}{(x-1)^2(x+3)},$$

and let $a_n := [x^n]A(x)$.

(a) Determine a homogeneous recurrence relation that the sequence $\{a_n\}$ satisfies, along with sufficient initial conditions.

(b) Express a_n as a function of n using the solution of (a).

(c) Express a_n as a function of n without using the solution of (a).

Solution. (a) Rewriting the expression of $A(x)$ yields

$$(x-1)^2(x+3)A(x) = x^2 - 13x,$$

which simplifies to

$$(x^3 + x^2 - 5x + 3)A(x) = x^2 - 13x. \tag{1}$$

Since $A(x) = \sum_{n\geq 0} a_n x^n$, we have

$$\begin{aligned}(x^3 + x^2 - 5x + 3)A(x) &= (x^3 + x^2 - 5x + 3)\sum_{n\geq 0} a_n x^n \\ &= \sum_{n\geq 0} a_n x^{n+3} + \sum_{n\geq 0} a_n x^{n+2} - \sum_{n\geq 0} 5a_n x^{n+1} + \sum_{n\geq 0} 3a_n x^n \\ &= \sum_{n\geq 3} a_{n-3} x^n + \sum_{n\geq 2} a_{n-2} x^n - \sum_{n\geq 1} 5a_{n-1} x^n + \sum_{n\geq 0} 3a_n x^n.\end{aligned}$$

The last line of the expression can be re-written as

$$3a_0 + (3a_1 - 5a_0)x + (3a_2 - 5a_1 + a_0)x^2 + \sum_{n\geq 3}(3a_n - 5a_{n-1} + a_{n-2} + a_{n-3})x^n.$$

This expression is equal to $x^2 - 13x$ by (1), so we have

$$3a_0 = 0, \quad 3a_1 - 5a_0 = -13, \quad 3a_2 - 5a_1 + a_0 = 1, \tag{2}$$

and

$$3a_n - 5a_{n-1} + a_{n-2} + a_{n-3} = 0, \text{ for all } n \geq 3.$$

Solving the equations in (2), we get the initial conditions

$$a_0 = 0, \quad a_1 = -\frac{13}{3}, \quad a_2 = -\frac{62}{9}.$$

(b) The characteristic polynomial of the recurrence relation found in (a) is

$$P(x) = 3x^3 - 5x^2 + x + 1.$$

Clearly $P(1) = 0$, so $x - 1$ is a factor of $P(x)$. By long division, we get

$$P(x) = (x-1)(3x^2 - 2x - 1).$$

Again, $x = 1$ is clearly a root of $3x^2 - 2x - 1$, and long division yields

$$3x^2 - 2x - 1 = (x-1)(3x+1).$$

Hence

$$P(x) = (x-1)^2(3x+1).$$

By Theorem 3.2.2 in the course note, the general solution to the recurrence relation is

$$a_n = (An + B)1^n + C(-\frac{1}{3})^n,$$

where A, B and C are determined by a_0, a_1 and a_2. Using the initial conditions found in (a), we have

$$\begin{cases} 0 = a_0 = B + C \\ -13/3 = a_1 = A + B - (C/3) \\ -62/9 = a_2 = 2A + B + C/9 \end{cases}.$$

Solving the system of equations yields

$$A = -3, \quad B = -1, \quad C = 1.$$

Therefore, for all $n \geq 0$,

$$a_n = -3n - 1 + (-\frac{1}{3})^n.$$

Remark: In the course notes, the characteristic polynomial is only defined when the recurrence has coefficient 1 for the leading term. This is not the case in our question, as the leading coefficient is 3. However, the characteristic polynomial can still be defined similarly in general and the approach for solving recurrence in the notes will still work. This is because if the leading coefficient K for the recurrence is not equal to 1, we could divide the the recurrence by K so that the new recurrence has leading coefficient being 1. Now the new characteristic polynomial would be $(1/K)$ times the old one, and these two polynomial have exactly the same roots and multiplicities. The remaining steps for solving the recurrence thus stay unchanged and we would get the same answer.

(c) Note that the numerator of $A(x)$ has degree smaller than that of the denominator. Since $(x-1)^2$ and $x+3$ are coprime, by Lemma 3.1.2 (partial fractions), there exist constants A, B and C such that

$$A(x) = \frac{Ax+B}{(x-1)^2} + \frac{C}{x+3}.$$

Since

$$\frac{Ax+B}{(x-1)^2} + \frac{C}{x+3} = \frac{(A+C)x^2 + (3A+B-2C)x + (3B+C)}{(x-1)^2(x+3)},$$

we have $(A+C)x^2 + (3A+B-2C)x + (3B+C) = x^2 - 13x$ and so

$$\begin{cases} A + C = 1 \\ 3A + B - 2C = -13 \\ 3B + C = 0 \end{cases}.$$

Solving this system of equations yields

$$A = -2, \quad B = -1, \quad C = 3,$$

so

$$A(x) = \frac{-2x-1}{(x-1)^2} + \frac{3}{x+3}. \tag{3}$$

Now,

$$\begin{aligned}
\frac{-2x-1}{(x-1)^2} &= \frac{-2x-1}{(1-x)^2} \\
&= (-2x-1)\sum_{i\geq 0}\binom{i+2-1}{2-1}x^i \qquad \textit{(by Theorem 1.6.5)} \\
&= (-2x-1)\sum_{i\geq 0}(i+1)x^i \\
&= \sum_{i\geq 0} -2(i+1)x^{i+1} - \sum_{i\geq 0}(i+1)x^i \\
&= \sum_{i\geq 1} -2ix^i - \sum_{i\geq 0}(i+1)x^i \qquad \textit{(change of index)} \\
&= \sum_{i\geq 0} -2ix^i - \sum_{i\geq 0}(i+1)x^i \\
&= \sum_{i\geq 0}(-3i-1)x^i,
\end{aligned}$$

so

$$[x^n]\frac{-2x-1}{(x-1)^2} = -3n-1.$$

On the other hand,

$$\frac{3}{x+3} = \frac{1}{(x/3)+1} = \frac{1}{1-(-x/3)} = \sum_{i\geq 0}(-\frac{x}{3})^i = \sum_{i\geq 0}(-\frac{1}{3})^i x^i,$$

so

$$[x^n]\frac{3}{x+3} = (-\frac{1}{3})^n.$$

Therefore, by expression (3),

$$a_n = [x^n]A(x) = [x^n]\frac{-2x-1}{(x-1)^2} + [x^n]\frac{3}{x+3} = -3n - 1 + (-\frac{1}{3})^n.$$

□

2. Consider the sequence $\{b_n\}$ that satisfies the nonhomogeneous recurrence relation

$$b_n - 4b_{n-1} + 3b_{n-2} = 2^n \tag{4}$$

for all $n \geq 2$, with $b_0 = 0$ and $b_1 = 2$. Find a formula of b_n in terms of n.

Solution. We first find a particular solution to the recurrence relation (4). Based on the right-hand side of the recurrence relation, we suspect that $a_n := k2^n$ is a particular solution for some constant k. By substituting $k2^n$ into (4) for b_n, we get

$$k2^n - 4k2^{n-1} + 3k2^{n-2} = 2^n.$$

By dividing both sides by 2^{n-2}, we get

$$4k - 8k + 3k = 4,$$

which implies $k = -4$. So $a_n = -4(2^n) = -2^{n+2}$ is a particular solution to (4).

Now we consider the underlying homogeneous recurrence relation

$$b_n - 4b_{n-1} + 3b_{n-2} = 0. \tag{5}$$

The characteristic polynomial is $x^2 - 4x + 3 = (x-1)(x-3)$. By Theorem 3.2.2, the general solution of (5) has the form

$$A \cdot 1^n + B \cdot 3^n = A + B \cdot 3^n$$

for some constants A and B. The solution to the nonhomogeneous recurrence relation (4) is

$$b_n = a_n + A + B \cdot 3^n = -2^{n+2} + A + B \cdot 3^n,$$

where A and B are determined by the initial conditions. Now

$$\begin{cases} 0 = b_0 = -2^2 + A + B = -4 + A + B \\ 2 = b_1 = -2^3 + A + 3B = -8 + A + 3B. \end{cases}$$

Solving the system yields $A = 1$ and $B = 3$. So the solution is

$$b_n = -2^{n+2} + 3 \cdot 3^n + 1 = -2^{n+2} + 3^{n+1} + 1.$$

□

3. Consider the sequence $\{a_n\}$, where $a_n := n^2 \cdot 3^{n+1} - 4$ for all integers $n \geq 0$. Determine a homogeneous recurrence relation which a_n satisfies, along with sufficient initial conditions.

Solution. We first seek for the characteristic polynomial $P(x)$ of the desired recurrence. Note that a_n can be re-written as

$$3n^2 \cdot 3^n - 4 \cdot 1^n.$$

With this form, we can tell that the polynomial $P(x)$ has roots 3, 3, 3 and 1, i.e.,

$$P(x) = (x-3)^3(x-1) = x^4 - 10x^3 + 36x^2 - 54x + 27.$$

The recurrence relation is then

$$a_n - 10a_{n-1} + 36a_{n-2} - 54a_{n-3} + 27a_{n-4} = 0$$

for all $n \geq 4$. The initial conditions can be computed using the expression $a_n = n^2 \cdot 3^{n+1} - 4$:

$$a_0 = -4, \quad a_1 = 5, \quad a_2 = 104, \quad a_3 = 725.$$

□