MATH 239 Spring 2012: Assignment 3

Solutions

1. {8 marks} At an intergalactic yard sale, there are three distinct planets costing 5, 7 and 9 gold coins respectively, one comet costing 12 gold coins, 120 identical stars selling for 3 gold coins each, and an unlimited supply of star bits selling for 2 gold coins each. For a positive integer n, how many ways can one spend n gold coins in this sale?

   Solution. We reprsent the cost of each way of buying as $(a_1, \ldots, a_6)$ where a_1, a_2, a_3 represent the cost of the planets with the price of 5, 7, 9 coins respectively, a_4 represent the cost of the comet, a_5 represent the cost of the stars, and a_6 represent the cost of the star bits. The set we are enumerating is

   $$S = \{0, 5\} \times \{0, 7\} \times \{0, 9\} \times \{0, 12\} \times \{0, 3, 6, \ldots, 360\} \times \{0, 2, 4, 6, \ldots\}.$$

   The weight is

   $$w(a_1, \ldots, a_6) = a_1 + \cdots + a_6.$$

   Using the product lemma, the generating series is

   $$\Phi_S(x) = \frac{(1 + x^5)(1 + x^7)(1 + x^9)(1 + x^12)(1 - x^{363})}{(1 - x^3)(1 - x^2)}.$$

   The answer is the coefficient of x^n in this series.

   Alternate solution. We can enumerate the set as

   $$S = \{0, 1\}^4 \times \{0, 1, 2 \ldots, 120\} \times \mathbb{N}_0$$

   and use the weight function

   $$w(a_1, \ldots, a_6) = 5a_1 + 7a_2 + 9a_3 + 12a_4 + 3a_5 + 2a_6.$$

   This will lead to the same generating series.

2. {8 marks} Let k be a fixed integer. How many compositions of n with k parts are there where each part is congruent to 1 modulo 5? Determine an explicit formula.

   Solution. Let $P = \{1, 6, 11, 16, \ldots\}$ be the set of all integers congruent to 1 modulo 5. Then the set we are enumerating is P^k. The generating series is

   $$\Phi_{P^k}(x) = (\Phi_P(x))^k = \left(\frac{x}{1 - x^5}\right)^k.$$

   The answer is

   $$[x^n]\left(\frac{x}{1 - x^5}\right)^k = [x^{n-k}]\frac{1}{(1 - x^5)^k} = \begin{cases} 0 & \text{if } 5 \nmid n - k \\ \binom{\frac{n-k}{5}+k-1}{k-1} & \text{if } 5 \mid n - k \end{cases}$$

3. {8 marks} Let m be a fixed integer. How many compositions of n are there where the largest part is at most m?

Solution. We are looking at compositions (of any number of parts) where each part is at most m. So the set we are enumerating is $S = \bigcup_{k\geq 0}[m]^k$. Using the sum lemma, we get

$$\begin{aligned}\Phi_S(x) &= \sum_{k\geq 0}\Phi_{[m]^k}(x)\\ &= \sum_{k\geq 0}\left(\frac{x(1-x^m)}{1-x}\right)^k\\ &= \frac{1}{1-\frac{x-x^{m+1}}{1-x}}\\ &= \frac{1-x}{1-2x+x^{m+1}}\end{aligned}$$

So the answer is $[x^n]\frac{1-x}{1-2x+x^{m+1}}$.

4. {Extra credit: 5 marks} How many compositions of n into three parts are there where $n = a_1+a_2+a_3$ and $a_1 < a_2 < a_3$?

   Solution. Let $S = \{(a_1, a_2, a_3) | 1 \leq a_1 < a_2 < a_3\}$. We define the weight on S as $w(a_1, a_2, a_3) = a_1 + a_2 + a_3$. We find a bijection between S and $\mathbb{N}^3$ as follows: Define $f : S \to \mathbb{N}^3$ by

   $$f(a_1, a_2, a_3) = (a_1, a_2 - a_1, a_3 - a_2).$$

   This has an inverse $g : \mathbb{N}^3 \to S$ where

   $$g(b_1, b_2, b_3) = (b_1, b_1 + b_2, b_1 + b_2 + b_3).$$

   It is clear that both functions are well-defined. Note that for each $(a_1, a_2, a_3) \in S$,

   $$a_1 + a_2 + a_3 = 3a_1 + 2(a_2 - a_1) + (a_3 - a_2).$$

   So if we define the weight on $\mathbb{N}^3$ to be $w'(b_1, b_2, b_3) = 3b_1 + 2b_2 + b_3$, then

   $$w(a_1, a_2, a_3) = w'(f(a_1, a_2, a_3)).$$

   In particular, the generating series for S with respect to w is the same as the generating series of $\mathbb{N}^3$ with respect to w', which is

   $$\left(\frac{x^3}{1-x^3}\right)\left(\frac{x^2}{1-x^2}\right)\left(\frac{x}{1-x}\right) = \frac{x^6}{(1-x^3)(1-x^2)(1-x)}.$$

   The answer is then the coefficient of x^n in this series.

5. {8 marks} For any $n \in \mathbb{N}_0$, let E_n be the set of all compositions of n with even number of parts, and let O_n be the set of all compositions of n with odd number of parts. Prove that for $n \geq 2$, $|E_n| = |O_n|$.

   Solution. We can enumerate the set of all compositions with odd number of parts by $O = \bigcup_{k\geq 0}\mathbb{N}^{2k+1}$. Also, we can enumerate the set of all compositions with even number of parts by

$E = \bigcup_{k\geq 0} \mathbb{N}^{2k}$. If we look at the difference between the generating series for the two sets, we get

$$\begin{aligned}
\Phi_O(x) - \Phi_E(x) &= \sum_{k\geq 0} \left(\frac{x}{1-x}\right)^{2k+1} - \sum_{k\geq 0} \left(\frac{x}{1-x}\right)^{2k} \\
&= \left(\sum_{k\geq 0} \left(\frac{x}{1-x}\right)^{2k}\right) \left(\frac{x}{1-x} - 1\right) \\
&= \frac{1}{1 - \frac{x^2}{(1-x)^2}} \frac{-1+2x}{1-x} \\
&= \frac{(1-x)^2}{1-2x} \frac{-1+2x}{1-x} \\
&= x - 1
\end{aligned}$$

For $n \geq 2$, since the coefficient is 0, it means that $[x^n]\Phi_O(x) = [x^n]\Phi_E(x)$. Hence $|O_n| = |E_n|$.

Alternate solution. We can find a bijection between E_n and O_n. For a composition $(a_1, \ldots, a_k) \in E_n$, define $f : E_n \to O_n$ by

$$f(a_1, \ldots, a_k) = \begin{cases} (a_1 + a_2, a_3, \ldots, a_k) & \text{if } a_1 = 1 \\ (1, a_1 - 1, a_2, \ldots, a_k) & \text{if } a_1 > 1 \end{cases}$$

Note that this bijection only works if $n \geq 2$. Because we change the number of parts by 1 (added one part in the first case, subtracted one part in the second case), we have mapped a composition in E_n to a composition in O_n. The inverse is the same function, and it is easy to show that it is the inverse. Since f is a bijection, $|E_n| = |O_n|$.

6. {8 marks} Let $\{a_n\}$ be the sequence where

$$\sum_{n\geq 0} a_n x^n = \frac{1+2x^3}{1-2x+x^3-3x^4}.$$

Determine a recurrence relation that $\{a_n\}$ satisfies, with sufficient initial conditions to uniquely specify $\{a_n\}$.

Solution. Multiplying the denominator gives us

$$\begin{aligned}
1 + 2x^3 &= (1 - 2x + x^3 - 3x^4) \sum_{n\geq 0} a_n x^n \\
&= \sum_{n\geq 0} a_n x^n - 2\sum_{n\geq 0} a_n x^{n+1} + \sum_{n\geq 0} a_n x^{n+3} - 3\sum_{n\geq 0} a_n x^{n+4} \\
&= a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1)x^2 + (a_3 - 2a_2 + a_0)x^3 + \sum_{n\geq 4}(a_n - 2a_{n-1} + a_{n-3} - 3a_{n-4})x^n.
\end{aligned}$$

The constant term is 1, so $a_0 = 1$. The x term is 0, so $a_1 - 2a_0 = 0$, hence $a_1 = 2$. The x^2 term is also 0, so $a_2 - 2a_1 = 0$, hence $a_2 = 4$. The x^3 term has coefficient 2, so $a_3 - 2a_2 + a_0 = 2$, hence $a_3 = 9$. For any higher powers of x, the coefficient is 0. Hence for $n \geq 4$,

$$a_n - 2a_{n-1} + a_{n-3} - 3a_{n-4} = 0,$$

with initial conditions $a_0 = 1, a_1 = 2, a_2 = 4, a_3 = 9$.

7. {10 marks}

(a) Let a_n denote the number of compositions of n. In class, we found out that for $n \geq 1$, $a_n = 2^{n-1}$. This tells us that for $n \geq 2$, a_n satisfies the recurrence $a_n = 2a_{n-1}$. Give a combinatorial interpretation of this recurrence.

Solution. Let C_n be the set of all compositions of n. Partition C_n into two sets A and B where A contains all compositions of n where the last part is 1, and B contains all compositions of n where the last part is greater than 1. We will establish two bijections: one between A and C_{n-1}, and another between B and C_{n-1}. This will prove that $|C_n| = |A| + |B| = 2|C_{n-1}|$, i.e. $a_n = 2a_{n-1}$.

For each composition $(a_1, \ldots, a_{k-1}, 1) \in A$, we define a bijection $f : A \to C_{n-1}$ by

$$f(a_1, \ldots, a_{k-1}, 1) = (a_1, \ldots, a_{k-1}).$$

Clearly $(a_1, \ldots, a_{k-1})$ is a composition of $n-1$, hence it is in C_{n-1}. The inverse is simple: for any $(c_1, \ldots, c_l) \in C_{n-1}$, define $g : C_{n-1} \to A$ by

$$g(c_1, \ldots, c_l) = (c_1, \ldots, c_l, 1).$$

It is easy to see that this is the inverse, hence f is a bijection.

For each composition $(b_1, \ldots, b_k) \in B$ (where we know $b_k \geq 2$), we define another bijection $f' : B \to C_{n-1}$ by

$$f'(b_1, \ldots, b_k) = (b_1, \ldots, b_{k-1}, b_k - 1).$$

The inverse is $g' : C_{n-1} \to B$ where

$$g'(c_1, \ldots, c_l) = (c_1, \ldots, c_{l-1}, c_l + 1).$$

So f' is a bijection.

(b) Let b_n denote the total number of parts of all possible composition of n. For example, compositions of 3 are $(3), (1,2), (2,1), (1,1,1)$, so $b_3 = 1 + 2 + 2 + 3 = 8$. Determine a recurrence relation that $\{b_n\}$ satisfies, with sufficient initial conditions to uniquely specify $\{b_n\}$. Use your recurrence to generate b_5.

Solution. In the two bijections in part (a), each composition in A has one more part than its corresponding composition in C_{n-1}, and each composition in B has the same number of parts as its corresponding composition in C_{n-1}. So the total number of parts in C_n is then 2 times the total number of parts in C_{n-1}, plus the number of compositions in C_{n-1}. So the recurrence is

$$b_n = 2b_{n-1} + a_{n-1} = 2b_{n-1} + 2^{n-2}.$$

This works for $n \geq 2$, so we need the initial conditions $b_0 = 0, b_1 = 1$. This gives us that

$$b_2 = 2 + 1 = 3, b_3 = 6 + 2 = 8, b_4 = 16 + 4 = 20, b_5 = 40 + 8 = 48.$$