COMBINATORICS
& OPTIMIZATION

University of Waterloo

Introduction to Combinatorics

Lecture 3

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Notation

Recall (for non-negative integers n and r)

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

We define $0! = 1$

Note that if n<r, then $\binom{n}{r} = 0$

$\binom{n}{r}$ is the number of r-element subsets of a set with n elements.

More combinatorial identities

Prove $$\binom{m+n}{k} = \sum_{i=0}^{m} \binom{m}{i}\binom{n}{k-i}$$

Proof 1(combinatorial):

- Let S be the set of all k-elements subsets of a set of m+n elements. To prove this identity, we will count this set in two different ways.
  - Firstly, we know the number of such subsets is $\binom{m+n}{k}$ which gives us the LHS

More combinatorial identities

Proof 1 (continued):

- Now let's count those same subsets in a different way. First let's separate the m+n objects into two sets, set A of size m and set B of size n.
- For each i between 0 and m, let S_i denote the set of subsets with i elements from A and k-i elements from B.
- Every subset will belong to a unique S_i. So

$$S = S_0 \bigcup S_1 \bigcup \cdots \bigcup S_m = \bigcup_{i=0}^{m} S_i$$

(and this is a disjoint union)

More combinatorial identities

Proof 1(continued).

- $$S = S_0 \bigcup S_1 \bigcup \cdots \bigcup S_m = \bigcup_{i=0}^{m} S_i$$ where the union is disjoint so
$$|S| = |S_0| + |S_1| + \cdots + |S_m| = \sum_{i=0}^{m} |S_i|$$
- We know that $|S_i| = \binom{m}{i}\binom{n}{k-i}$ since there are $\binom{m}{i}$ ways of choosing i elements from A and $\binom{n}{k-i}$ ways of choosing k-i elements from B.

More combinatorial identities

Proof 1(continued).

- We have $$|S| = |S_0| + |S_1| + \cdots + |S_m| = \sum_{i=1}^{m} |S_i|$$

  and $$|S_i| = \binom{m}{i}\binom{n}{k-i}$$

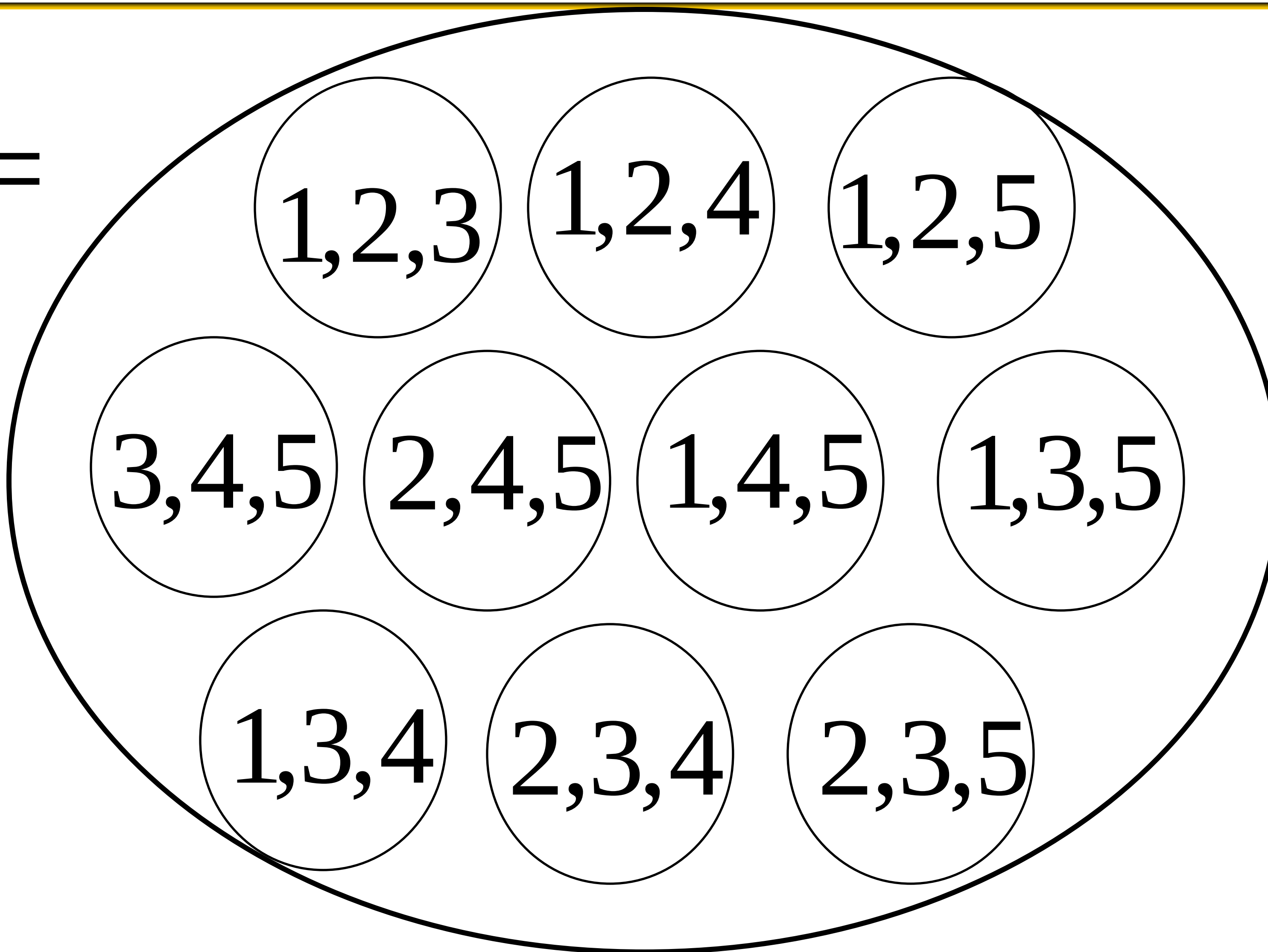
- Therefore the total number of subsets is

$$\sum_{i=0}^{m} \binom{m}{i}\binom{n}{k-i}$$ giving the RHS.

$\underbrace{1,2,}_{m=2}\ \underbrace{3,4,5}_{n=3}$

e.g. with m=2, n=3, k=3

$S =$



$A = \{1,2\}$

$B = \{3,4,5\}$ e.g. with m=2, n=3, k=3

$$|S_0| = \binom{2}{0}\binom{3}{3} = 1 \qquad |S_1| = \binom{2}{1}\binom{3}{2} = 6$$

$$S_0 = 3,4,5$$

$$|S_2| = \binom{2}{2}\binom{3}{1} = 3$$

$$S_1 = \quad 2,4,5 \quad 1,4,5 \quad 1,3,5 \quad 1,3,4 \quad 2,3,4 \quad 2,3,5$$

$$S_2 = \quad 1,2,4 \quad 1,2,3 \quad 1,2,5$$

Alternative proof

Prove $$\binom{m+n}{k} = \sum_{i=0}^{m} \binom{m}{i}\binom{n}{k-i}$$

Proof 2(algebraic):

- Consider the polynomial identity
$$(1+x)^{m+n} = (1+x)^m (1+x)^n$$
- We will compute the coefficient of x^k on both sides.
- From the binomial theorem we know that the coefficient of x^k on the LHS is $\binom{m+n}{k}$

Alternative proof

Proof 2(algebraic):

- The binomial theorem also lets us expand the RHS as $(1+x)^m(1+x)^n$

$$= \left(1 + mx + \binom{m}{2}x^2 + \cdots + \binom{m}{i}x^i + \cdots + x^m\right)$$

$$\times \left(1 + nx + \binom{n}{2}x^2 + \cdots + \binom{n}{j}x^j + \cdots + x^n\right)$$

$$= \left(\sum_{i=0}^{m}\binom{m}{i}x^i\right) \times \left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)$$

Alternative proof

Proof 2(algebraic): $\left(\sum_{i=0}^{m}\binom{m}{i}x^i\right)\times\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)$

- We get an x^k term precisely when i+j=k; so the coefficient of x^k is

$$=\sum_{\substack{(i,j)\\ i+j=k}}\binom{m}{i}\binom{n}{j}$$

$$=1\binom{n}{k}+m\binom{n}{k-1}+\binom{m}{2}\binom{n}{k-2}+\cdots+\binom{m}{i}\binom{n}{k-i}+\cdots+\binom{m}{k}\binom{n}{0}$$

$$=\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$$ giving us the RHS.

Technical note

You might notice slight deviations in the range of the summations. Notice that since

$\binom{m}{i} = 0$ if i>m and $\binom{n}{k-i} = 0$ if i>k

then the following are all equal

$$\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i} = \sum_{i=0}^{m}\binom{m}{i}\binom{n}{k-i} = \sum_{i=0}^{\infty}\binom{m}{i}\binom{n}{k-i} = \sum_{i=0}^{\min(k,m)}\binom{m}{i}\binom{n}{k-i}$$

and often just denoted

$$\sum_{i}\binom{m}{i}\binom{n}{k-i}$$

Counting

We will consider problems that can be manipulated into the following form:

- There will be some implicit set S
- There will be a "weight function" $\omega(\sigma)$ on objects $\sigma \in S$ i.e. $\omega : S \to Z_{\geq 0}$
- We will want to know how many elements of S have weight equal to k.

e.g.

This is a very general framework. For example

- "How many 0-1 strings of length n are there?" corresponds to letting S equal the set of all 0-1 strings and ω be the length function, i.e.

$$S = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots.\}$$

$$\omega(\sigma) = length(\sigma)$$

So, e.g., $\omega(010011) = 6$

e.g.

Another example

- “How many subsets of {1,2,…,n} have size k?” corresponds to letting S equal the set of all subsets of {1,2,…,n} and ω give the cardinality of its input i.e.

$$S = \left\{ \{\,\}, \{1\}, \{2\}, \ldots, \{1, 2, \ldots, n\} \right\}$$

$$\omega(\sigma) = cardinality\left(\sigma\right) = \#\,\sigma$$

$$\omega(\{2,5,8\}) = 3$$

Definition of a generating function

Given a set S with weight function ω we define the *generating function* (or *generating series*) of S with respect to ω to be

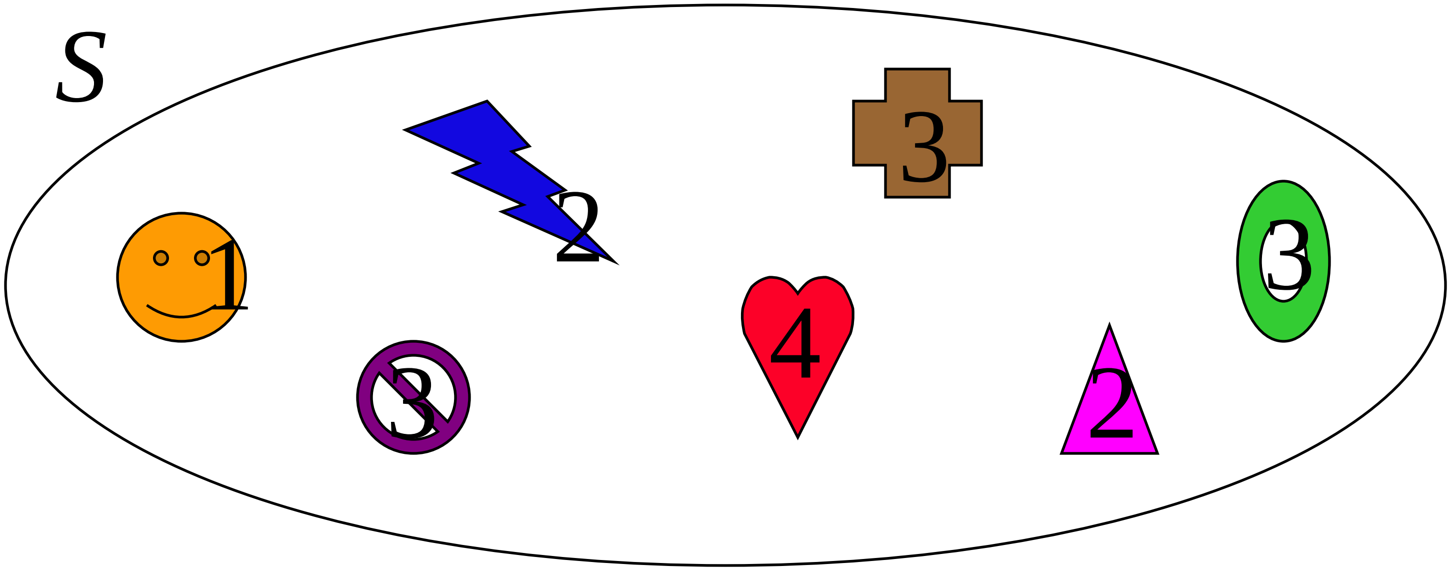
(we usually omit the ω if it is implicit)

$$\Phi_S^{\omega}(x) = \sum_{\sigma \in S} x^{\omega(\sigma)}$$

(tells us what to sum over)

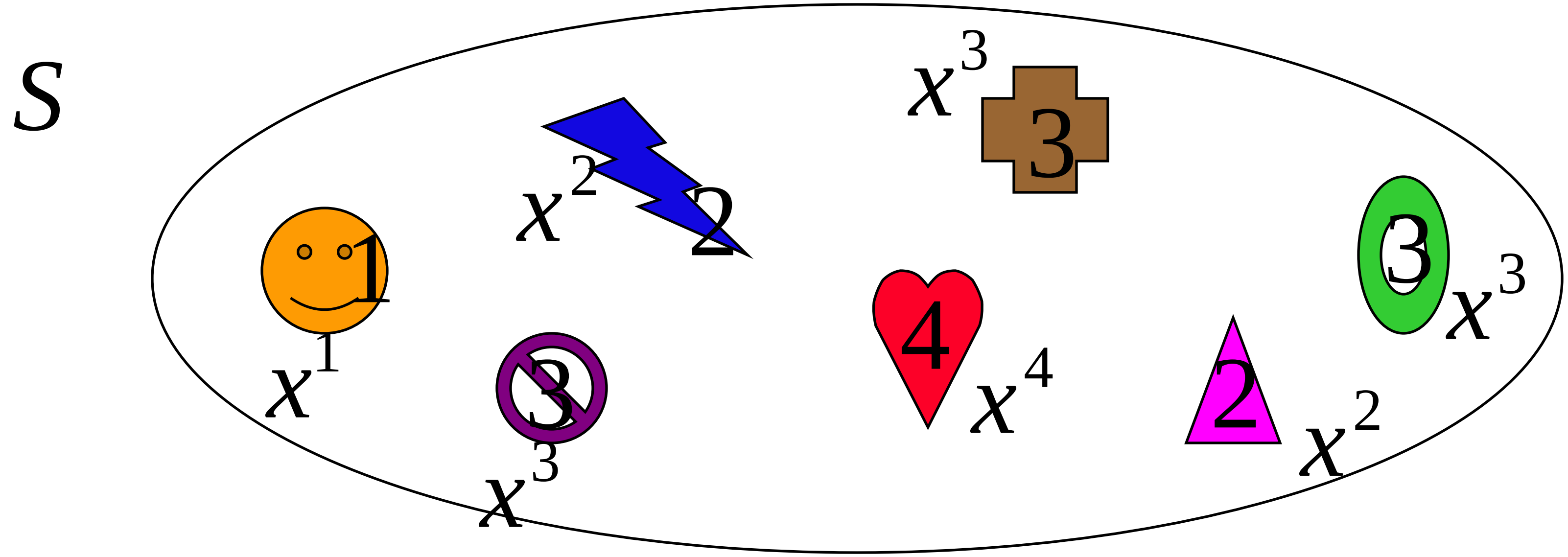
e.g.

$$\Phi_S^{\omega}(x) = \sum_{\sigma \in S} x^{\omega(\sigma)}$$



(here the weights are written on the objects)

e.g. $$\Phi_S^{\omega}(x) = \sum_{\sigma \in S} x^{\omega(\sigma)}$$



e.g. $$\Phi_S^{\omega}(x) = \sum_{\sigma \in S} x^{\omega(\sigma)}$$

x^3

x^2

x^3

x^1

x^4

x^2

x^3

e.g. $$\Phi_S^{\omega}(x) = \sum_{\sigma \in S} x^{\omega(\sigma)}$$

$$\Phi_S(x) = x + x^2 + x^2 + x^3 + x^3 + x^3 + x^4$$

$$= x + \left(x^2 + x^2\right) + \left(x^3 + x^3 + x^3\right) + x^4$$

$$= \sum_k \left(\sum_{\substack{\sigma \in S \\ \omega(\sigma) = k}} x^k \right) = x + 2x^2 + 3x^3 + x^4$$

The coefficient of x^k in $\Phi_S(x)$, denoted $\left[x^k\right]\Phi_S(x)$, gives us the number of elements in S with weight k.

e.g.

$$\omega(\sigma) = cardinality\left(\sigma\right) = \#\,\sigma$$

$$S = \left\{\{\,\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\right\}$$

$$\Phi_S^{\omega}(x) = x^{\omega(\{\})} + x^{\omega(\{1\})} + x^{\omega(\{2\})} + x^{\omega(\{3\})}$$

$$+ x^{\omega(\{1,2\})} + x^{\omega(\{1,3\})} + x^{\omega(\{2,3\})} + x^{\omega(\{1,2,3\})}$$

$$= x^0 + x^1 + x^1 + x^1 + x^2 + x^2 + x^2 + x^3$$

$$= 1 + 3x + 3x^2 + x^3$$