MATH 239 Tutorial 5 Problems

1. Determine the generating series for the set of binary strings that do not contain 11011 as a substring.

2. Let $\{a_n\}$ be the sequence that satisfies the recurrence
$$a_n - 3a_{n-2} + 2a_{n-3} = 0$$
for $n \geq 3$, with initial conditions $a_0 = 4, a_1 = -1, a_2 = 3$. Determine an explicit formula for a_n.

3. Let $\{b_n\}$ be the sequence that satisfies the recurrence
$$b_n - 3b_{n-2} + 2b_{n-3} = 12$$
for $n \geq 3$, with initial conditions $b_0 = 0, b_1 = 8, b_2 = 2$. Determine an explicit formula for b_n.

4. Let $c_n = (n-2)3^n + (-2)^n$.
   (a) Determine a homogeneous recurrence that c_n satisfies, together with sufficient initial conditions.
   (b) Determine a rational expression for the power series $\sum_{n \geq 0} c_n x^n$.

Additional exercises

1. Consider the two sequences $\{a_n\}, \{b_n\}$ given by
$$a_n = n2^n - 3^n$$
$$b_n = n2^n - 3^n + 4$$
   (a) Determine a homogeneous recurrence that a_n satisfies.
   (b) Determine a non-homogeneous recurrence that b_n satisfies whose left-hand side is the same as part (a).

2. Prove that
$$\frac{(1+\sqrt{3})^n - (1-\sqrt{3})^n}{2\sqrt{3}}$$
is an integer for each nonnegative integer n. (Challenge: Find 3 different proofs.)

3. Let a be a non-zero integer, and let k be any positive integer. Consider the power series $\frac{p(x)}{(1-ax)^k}$ where the degree of $p(x)$ is less than k. Prove that there exist constants $C_1, \ldots, C_k$ such that
$$\frac{p(x)}{(1-ax)^k} = \frac{C_1}{1-ax} + \frac{C_2}{(1-ax)^2} + \cdots + \frac{C_k}{(1-ax)^k}.$$
(Hint: You may consider proving that a certain set of polynomials is a basis for the vector space P_{k-1}, the space of all polynomials of degree at most $k-1$.)