MATH 239 Spring 2012: Assignment 9
Solutions

1. {10 marks} Let G be a connected graph, and let T be a spanning tree of G. Let x be a vertex in G. For any vertex v in G, define $d(v)$ to be the length of the unique x,v-path in T. Suppose that all the edges in G that are not in T join two vertices whose d-values have the same parity.

   (a) Prove that if uv is an edge in T, then $|d(u) - d(v)| = 1$.

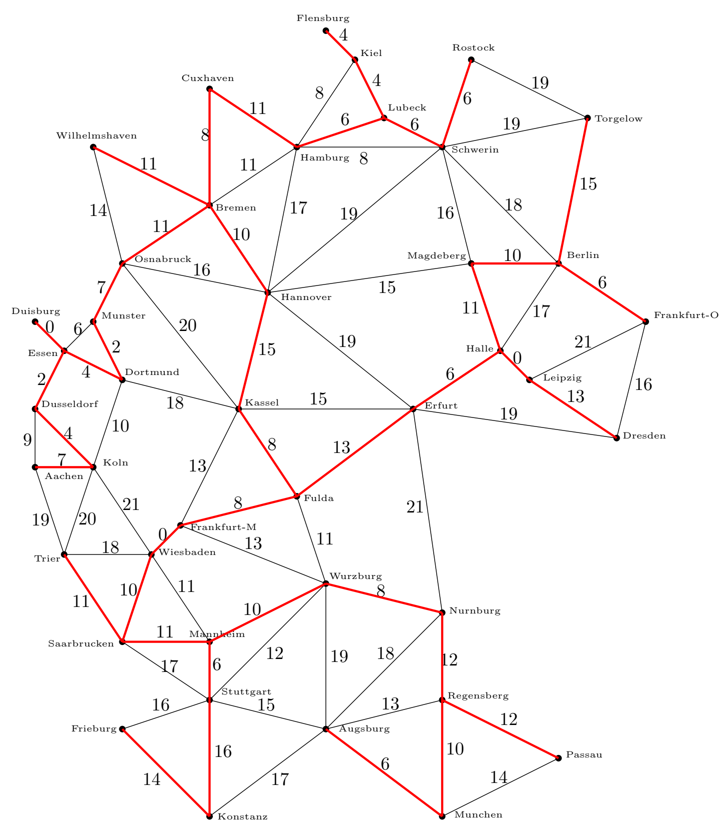
   Solution. Let P be the unique x,u-path in T. If v is not on this path, then $P + uv$ is an x,v-path (in fact the only x,v-path in T), hence $d(v) = d(u) + 1$. Otherwise, v must be on this path, which means the last edge must be uv (for otherwise we get two different u,v-paths in T). So $P - uv$ is an x,v-path, hence $d(v) = d(u) - 1$. In either case, $|d(u) - d(v)| = 1$.

   (b) Prove that any cycle of G contains an even number of edges from T.

   Solution. Let $v_1, v_2, \ldots, v_k, v_1$ be any cycle in G. Notice that if $v_i v_{i+1}$ is an edge in T, then by part (a), $|d(v_i) - d(v_{i+1})| = 1$. In particular, $d(v_i)$ and $d(v_{i+1})$ have different parities. If $v_i v_{i+1}$ is not an edge in T, then by assumption, $d(v_i)$ and $d(v_{i+1})$ have the same parity. So the number of tree edges in the cycle is equal to the number of times the parity of the d-values changes along this cycle. Since we start and end at the same vertex (hence with the same d-value), we must have changed parities an even number of times. Therefore, there is an even number of edges from T in this cycle.

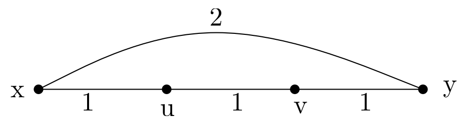
2. {7 marks} Produce a minimum spanning tree of the following graph. You do not need to show your work. (Source: The map of Germany from the board game Power Grid.)

   Solution. This is one possible minimum spanning tree.



3. {5 marks} Let T be a minimum spanning tree of a weighted graph G. For any two vertices u, v in G, is it true that the unique u, v-path in T is a path of minimum weight among all u, v-paths in G? Give a proof or a counterexample.

Solution. This is false. In the following diagram, a minimum spanning tree is the path x, u, v, y. The length of the x, y-path is 3. However, the shortest x, y-path is the edge xy itself which has weight 2.



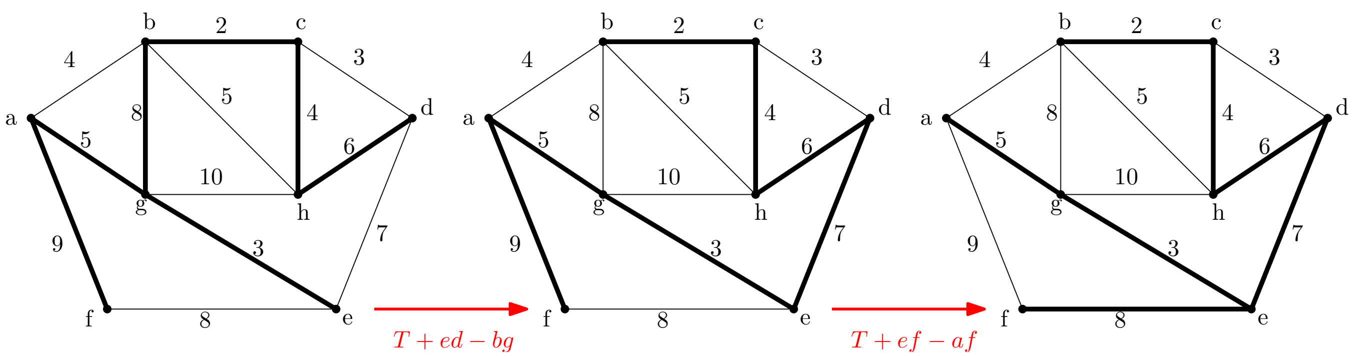
4. {14 marks} We propose another algorithm for finding a minimum spanning tree of a connected graph G. Let $w(e)$ be the weight of an edge. Start with any spanning tree T.

   Find a pair of edges (e, e') such that $e \in E(G) \setminus E(T)$, e' is in the unique cycle of $T + e$, and $w(e) < w(e')$. Replace T by $T + e - e'$.

   The algorithm repeats this process, and it terminates when no such pair of edges can be found.

   (a) Perform 2 iterations of this algorithm on the graph below, using the bolded edges as the starting tree. Indicate which pair of edges you are choosing.

   Solution. One possible solution.



   (b) Prove that when the algorithm terminates, it produces a spanning tree.

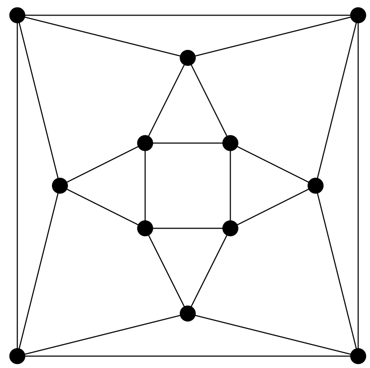
   Solution. We started with a spanning tree. At each step of the algorithm, when we pick (e, e'), $T + e$ creates exactly one cycle, and is still connected. Since e' is on this cycle, it is not a bridge. Hence $T + e - e'$ is still connected. This is a tree since removing e' destroys the only cycle in the graph (or, since it has the same number of edges as T).

   (c) Prove that when the algorithm terminates, it produces a minimum spanning tree. (You may start this way if you wish: Let T be the tree produced by the algorithm, and let T^* be a minimum spanning tree that has the most number of edges in common with T.)

   Solution. Let T and T^* be as above. If $T = T^*$, then T is a minimum spanning tree. Otherwise, let $e \in E(T^*) \setminus E(T)$. Then $T + e$ has a unique cycle C. In $T^* - e$, there are two components, and some edge e' in $C - e$ must join vertices from different components. This edge e' is in T but not in T^*. Since the algorithm terminated, $w(e) \geq w(e')$. On the other hand, $T^* - e + e'$ is also a spanning tree. The weight of this tree is $w(T^*) - w(e) + w(e')$ which must be at least $w(T^*)$, since T^* is a minimum spanning tree.. This means that $w(e) \leq w(e')$, so $w(e) = w(e')$. Then $T^* - e + e'$ has the same weight as T^*, meaning it is also a minimum spanning tree. However, $T^* - e + e'$ has one more edge in common with T than T^*, and that is a contradiction.

5. {8 marks} Let G be a 4-regular connected planar graph with an embedding where every face has degree 3 or 4, and adjacent faces have different face degrees. Determine the number of vertices, edges, faces of degree 3, and faces of degree 4 in G. Draw a planar embedding of G.

   Solution. Suppose G has n vertices, m edges, s_3 faces of degree 3 and s_4 faces of degree 4. From the handshaking lemma, we get $2m = 4n$. From the handshaking lemma for faces, we get $2m = 3s_3 + 4s_4$. From Euler's formula, we get $n - m + s_3 + s_4 = 2$. Since each edge is adjacent to one face of degree 3 and one face of degree 4, the total number of edges is the sum of the degrees of all faces of degree 3, or all faces of degree 4. Therefore, $m = 3s_3 = 4s_4$. Solving these four equations, we get $n = 12, m = 24, s_3 = 8, s_4 = 6$. A planar embedding of G is



   (This is called the rhombicuboctahedron.)

6. {6 marks} Is it true that any planar embedding of any simple connected planar graph has either a vertex of degree at most 3 or a face of degree at most 3? Give a proof or a counterexample.

   Solution. This is true. Suppose by way of contradiction that there is a planar embedding of G where every vertex has degree at least 4 and every face has degree at least 4. Suppose G has n vertices, m edges and s faces. By the handshaking lemma, $2m \geq 4n$, so $n \leq m/2$. By the handshaking lemma for faces, $2m \geq 4s$, so $s \leq m/2$. By Euler's formula,

$$2 = n - m + s \leq m/2 - m + m/2 = 0.$$

   This is a contradiction.