1. (a) **[2 marks]** List all compositions of 2, 3 and 4 where each part is a positive integer not equal to 2.

   (b) **[2 marks]** Prove that the generating function for the number of compositions of a positive integer n into k parts, such that each part is a positive integer which is not equal to two, is

$$\Phi(x) = \left(\frac{x - x^2 + x^3}{1 - x} \right)^k .$$

   (c) **[2 marks]** Deteremine the generating function for the number, a_n, of compositions of n with no restriction on the number of parts in the composition. As in parts (a) and (b), each part is a positive integer which is not equal to 2.

(d) **[3 marks]** Let $\{b_n\}$ be the sequence whose generating function is defined by

$$\sum_{n\geq 0} b_n x^n = \frac{1+x^4}{1-x-2x^3}.$$

Determine a recurrence relation, together with sufficient initial conditions, to uniquely determine the sequence of b_n's.

(e) **[3 marks]** The recurrence relation $c_n = 5c_{n-1} - 6c_{n-2}$ for all integers $n \geq 2$ with initial conditions $c_0 = -3$ and $c_1 = -4$, defines the terms of a sequence $\{c_n\}$. Solve this recurrence relation.

2. (a) For each of the following sets, write down a decomposition that uniquely creates the elements of that set.

   i. **[2 marks]** The set of $\{0,1\}$-strings where all blocks have even length.

   ii. **[2 marks]** The $\{0,1\}$-strings that have no blocks of 1s with length 2, and no substrings of 0s of length 3.

   iii. **[2 marks]** The set of $\{0,1\}$-strings in which the substring 11000 does not occur.

   (b) **[3 marks]** Let $S = \{1\}^*(\{0\}\{00\}^*\{11\}\{1\}^*)^*\{0\}^*$. Find the generating function for S (for full marks, simplify your answer to be of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials).

3. (a) **[3 marks]** Let G_1 denote the graph with vertex set V_1 equal to the subsets of $\{1, 2, \ldots, 100\}$ of size 5. Any two vertices of V_1, say a and b, are joined if and only if $|a \cap b| = 2$. What is the degree of each vertex? Explain.

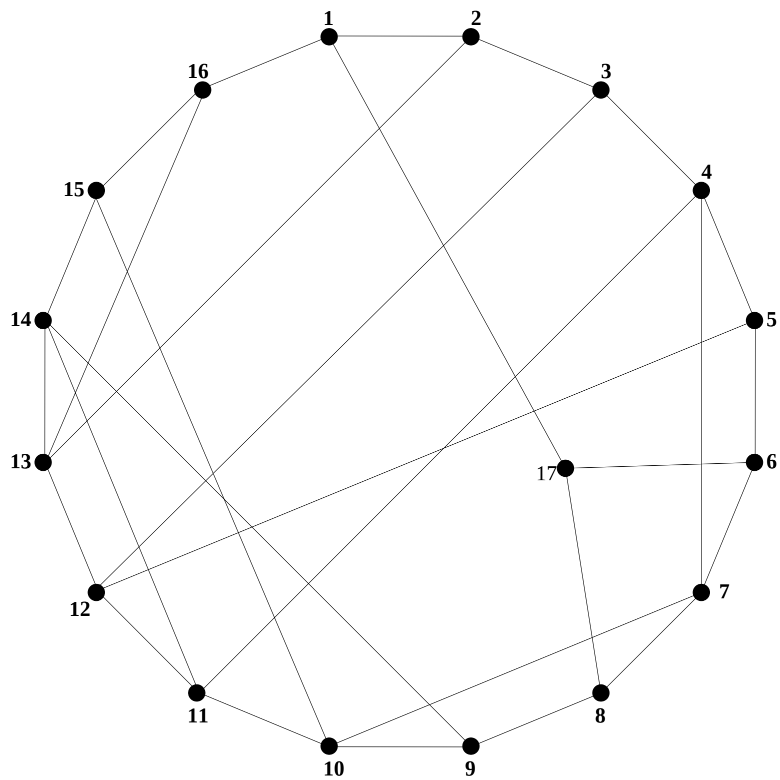
   (b) **[3 marks]** Is G_1 bipartite? Justify your answer.

   (c) **[4 marks]** Let G_2 denote the graph with vertex set V_2 equal to the $\{0, 1\}$-strings of length 100 that have exactly 95 zeros and 5 ones. Any two vertices of V_2, say x and y, are joined if and only if x and y differ in exactly 2 positions.

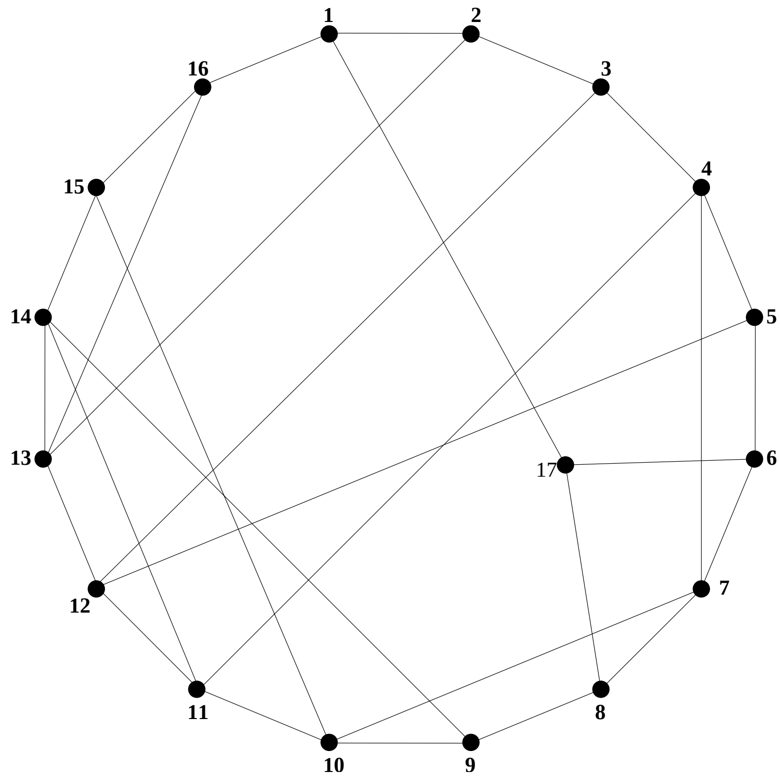
   i. What is the degree of each vertex? Explain.

   ii. Are G_1 and G_2 isomorphic? Justify your answer.

4. (a) **[4 marks]** Find a breadth-first search tree for the following graph G, rooted at 11. Whenever you have a choice of which vertex to have join the tree, choose the vertex whose label is numerically smallest. Draw the breadth-first search tree in a separate diagram, and make a list of the vertices showing the order in which they entered the tree. In your drawing, place the root vertex at the top, and the vertices at level $i+1$ below those at level i for $i = 0, 1, 2, 3, \ldots$.



(b) **[4 marks]** Is this graph planar? Justify your answer.



5. The following statements are all false. Give a counter example for each of them!

   (a) **[1 mark]** Every bipartite graph is planar.

   (b) **[1 mark]** Every planar graph is bipartite.

   (c) **[1 mark]** In every graph, the size of a maximum matching is equal to the size of a minimum cover.

   (d) **[2 marks]** If a connected bipartite graph G has a bipartition (A, B) for which $|A| = |B|$, then G has a perfect matching.

(e) (Continued from page 8.) [**1 mark**] Every 4-colourable graph is planar.

(f) [**1 mark**] Every graph has a spanning tree.

(g) [**1 mark**] Every tree has a perfect matching.

(h) [**2 marks**] If a graph has a perfect matching M, then it has a minimum cover C such that $|C| = |M|$.

6. (a) **[2 marks]** State Euler's Formula for a connected planar embedding with s faces, p vertices and q edges.

   (b) **[2 marks]** State a formula for the sum of the degrees of the faces in a planar embedding with q edges and faces $f_1, f_2, ..., f_s$.

   (c) **[4 marks]** Let P be a connected planar embedding in which every vertex has degree 4 and every face has degree 3. Determine the number of vertices p, the number of edges q, and the number of faces s in P. Draw such a planar embedding P.

7. Let G be a planar graph on $p \geq 3$ vertices which does not contain any triangles.

   (a) [**2 marks**] Determine an upper bound for the number of edges, q, in G as a function of p.

   (b) [**3 marks**] Show that G contains a vertex of degree less than four.

   (c) [**6 marks**] Prove, by induction on p, that every planar graph G on p vertices which does not contain any triangles is 4-colorable. [Hint: in the induction step, delete a vertex of smallest degree from G.]

8. (a) Consider the graph G with bipartition $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{a, b, c, d, e, f, g, h, i, j\}$. Let M be the matching determined by the thick edges in this graph.

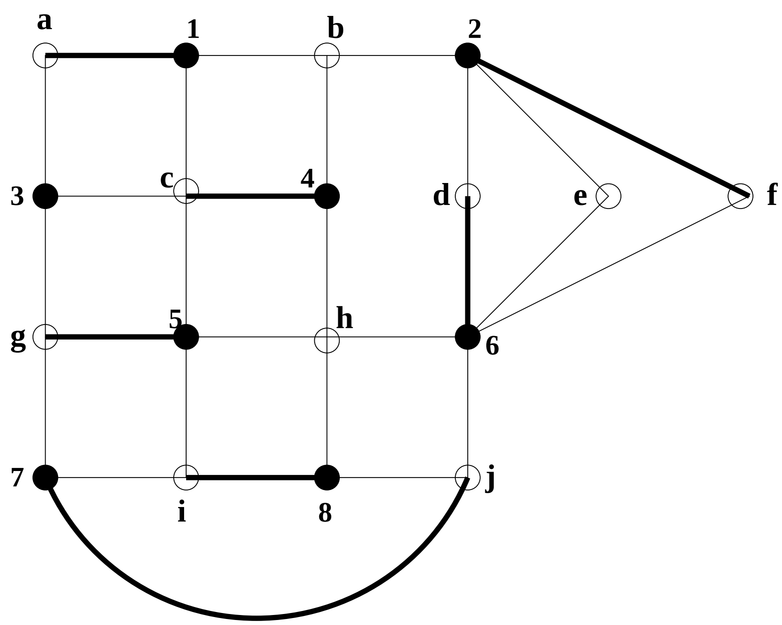


Figure 1: The graph G

i. [**2 marks**] Determine the set X of the XY-construction.

ii. [**2 marks**] Determine the set Y of the XY-construction.

iii. [**2 marks**] Is the matching M a maximum matching? Explain.

(b) Consider the graph H with bipartition $A = \{a, b, c, d, e, f\}$ and $B = \{g, h, i, j, k, \ell\}$ and Let M be the matching determined by the thick edges in this graph.

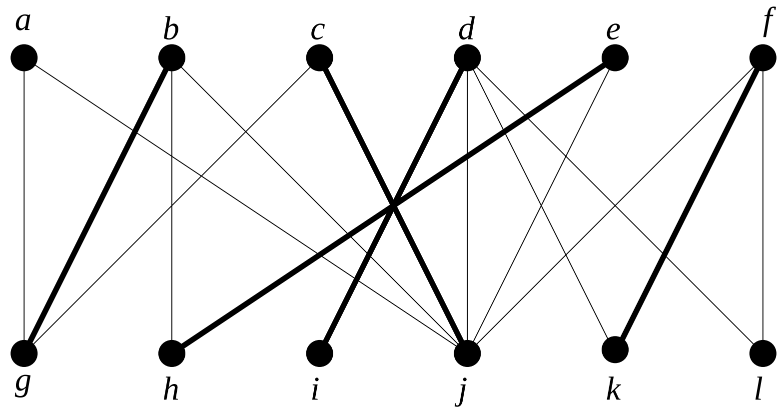


Figure 2: The graph H

i. **[2 marks]** Determine the sets X and Y of the XY-construction.

ii. **[2 marks]** Determine the cover produced by the XY-construction.

iii. **[2 marks]** Is M a maximum matching? Justify your answer.