Question 2. Suppose the numbers a_n satisfy the non-homogeneous recurrence relation

$$\begin{aligned} a_n - 6a_{n-1} + 9a_{n-2} &= 4, \\ a_0 &= 2, \\ a_1 &= 10, \end{aligned}$$

find an explicit formula for a_n in terms of n, for all $n \geq 0$.

Non-homogeneous part : b_n

let $b_n = An + B$ ... $b_n = 6a_{n-1} - 9a_{n-2} + 4$

$An + B = 6(A(n-1) + B) - 9(A(n-2) + B) + 4$

$= 6An - 6A + 6B - 9An + 18A - 9B + 4$

$= n(6A - 9A) + (-6A + 6B + 18A - 9B + 4)$

$A = -3A \Rightarrow A = 0$

$B = -3B + 4$

$4B = 4$

$B = 1$

So $b_n = 1$

Homogeneous part: c_n

ch. poly $= x^2 - 6x + 9$

$= (x-3)^2$

so $c_n = ((Cn + D)(3^n))$

$a_n = b_n + c_n$

$= (Cn + B) 3^n + 1$

$a_0 = 2 = B + 1 \rightarrow B = 1$

$a_1 = 10 = (C + B)3 + 1$

$9 = 3(C+1)$

$3 = C + 1$

$C = 2$

So $a_n = (2n+1)(3^n) + 1$

test $n = 2$

$a_2 = 6(10) - 9(2) + 4$

$= 60 - 18 + 4$

$= 46$

$5(3^2) + 1 = 5(9) + 1$

$= 45 + 1 = 46$

(c) Consider the power series

$$p(x) = \sum_{n\geq 0} \frac{x^n}{n!}.$$

By actually multiplying out the power series, compute $p(x)p(x)$.

(d) Using **(b)** and **(c)** or otherwise, show that $p(x)p(x) = p(2x)$.

$(p(x))^2 =$

c) $\left(\sum_{n\geq 0} \frac{x^n}{n!}\right)\left(\sum_{m\geq 0} \frac{x^m}{m!}\right) = \sum_{n\geq 0}\sum_{m\geq 0} \frac{x^{m+n}}{m!\,n!}$

let $l = m+n$

$\cancel{n = l-m}$

$m = l-n$

$= \sum_{l\geq 0}\sum_{n=0}^{l} \frac{x^l}{n!\,(l-n)!} = \sum_{l\geq 0}\left(\sum_{n=0}^{l}\frac{1}{n!\,(l-n)!}\right)x^l$

d) $p(2x) = \sum_{n\geq 0} \frac{(2x)^n}{n!}$

$= \sum_{n\geq 0} \frac{2^n x^n}{n!}$

$= \sum_{n\geq 0} \frac{2^n}{n!} x^n$

$= \sum_{n\geq 0}\left(\sum_{i=0}^{n} \frac{1}{i!\,(n-i)!}\right) x^n$

$= (p(x))^2$

Question 3. **(a)** Let $g(x) = \sum_{n\geq 0} a_n x^n$ be a generating function and suppose

$$g(x) = \frac{1+2x}{(1-2x)(1-3x)}.$$

Find a homogeneous recurrence relation satisfied by the sequence a_n, together with enough initial conditions to determine the a_n for all $n \geq 0$.

(b) Determine a_3.

a) $(1-2x)(1-3x) = 1 - 3x - 2x + 6x^2$

$= 1 - 5x + 6x^2$ is the characterist polynomial

So

$a_n - 5a_{n-1} + 6a_{n-2} = 0$ for $n \geq 2$

need a_0 and a_1:

$[x^n] \frac{1+2x}{(1-2x)(1-3x)} = [x^n]\left(\frac{a}{1-2x} + \frac{b}{1-3x}\right)$

$= [x^n]\left(\frac{-4}{1-2x} + \frac{5}{1-3x}\right)$

$= -4[x^n]\frac{1}{1-2x} + 5[x^n]\frac{1}{1-3x}$

$= -4(2^n) + 5(3^n)$

$a(1-3x) + b(1-2x) = 1+2x$

$a - 3ax + b - 2bx = 1+2x$

$(a+b) + (-3a-2b)x = 1+2x$

$a+b = 1 \rightarrow b = 1-a$

$-3a - 2b = 2$

$-3a - 2(1-a) = 2$

$-3a - 2 + 2a = 2$

$-a = 4$

$a = -4$

$b = 5$

$a_0 = [x^0]g(x) = 1$

$a_1 = [x^1]g(x) = -4(2) + 5(3)$

$= 15 - 8$

$= 7$

$a_3 = [x^3]g(x) = -4(2^3) + 5(3^3)$

$= -4(8) + 5(27)$

$= 135 - 32 = 103$

$\boxed{\text{So } a_n = 5a_{n-1} - 6a_{n-2} \quad n \geq 2 \text{ with } a_0 = 1 \text{ \& } a_1 = 7}$

then $a_2 = 5(a_1) - 6a_0 = 5(7) - 6 = 35 - 6 = 29$

b) $a_3 = 5a_2 - 6a_1 = 5(29) - 6(7) = 145 - 42 = 103$

agrees

Question 4. Let a_n be the number of binary strings in which every 0 that has a 1 somewhere to its right is in a block of at most two 0's. Find the generating function for the sequence a_n; please express your answer as a quotient of two polynomials. *(Examples of such strings are: 1110110010110000 and 11110011001, while 01110001100 is not such a string.)*

$$A = \{1\}^* \left(\{01, 001\}\{1\}^*\right)^* \{0\}^*$$

$$\Phi_A(x) = \left(\frac{1}{1-x}\right)^2 \left(\frac{1}{1 - \left(\frac{x^2+x^3}{1-x}\right)}\right)$$

$$= \frac{1}{(1-x)^2}\left(\frac{\cancel{(1-x)}}{(1-x)-(x^2+x^3)}\right)$$

$$= \frac{1}{(1-x)(1-x-x^2-x^3)}$$

$$= \frac{1}{1-x\cancel{-x^2}\cancel{-x^3}-x\cancel{+x^2}\cancel{+x^3}+x^4}$$

$$= \frac{1}{1-2x+x^4}$$

5a)

$$S = x^d(1 + x^a + x^{2a} + \dots + x^{pa})$$

$$x^a S = x^d(x^a + x^{2a} + \dots + x^{pa} + x^{(p+1)a})$$

$$(1 - x^a)S = x^d(1 - x^{(p+1)a})$$

$$S = x^d\left(\frac{1 - x^{(p+1)a}}{1 - x^a}\right)$$

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b) Let $n, k \in \mathbb{Z}^+$ and $a_{n,k} = \#$ of compositions of n with k parts. each part is in

$U = \{1, 5, 9, \dots, 89\}$

Find $a_{n,k}$ in terms of n and k

Express answer as a sum of products of binomial coefficients

$$\Phi_U(x) = x + x^5 + x^9 + \dots + x^{89} = x\left(\frac{1 - x^{92}}{1 - x^4}\right)$$

$$[x^n]\Phi_{U^k}(x) \overset{[x^n]}{=} (\Phi_U(x))^k = [x^n]x^k\left(\frac{1 - x^{92}}{1 - x^4}\right)^k$$

$$= [x^{n-k}]\ (1 - x^{92})^k (1 - x^4)^{-k}$$

$92i + 4j = n - k$

$j = \frac{n-k}{4} - 23i$

$$= [x^{n-k}]\left(\sum_{i=0}\binom{k}{i}(-x)^{92i}\right)\left(\sum_{j\geq 0}\binom{k+j-1}{k-1}x^{4j}\right)$$

$$= [x^{n-k}]\sum_{i\geq 0}\sum_{j\geq 0}\binom{k}{i}\binom{k+j-1}{k-1}x^{92i+4j}(-1)^i$$

$$= \begin{cases} \displaystyle\sum_{i=0}^{\lfloor\frac{n-k}{92}\rfloor}\binom{k}{i}\binom{k + \frac{n-k}{4} - 23i}{k-1}(-1)^i & \text{if } n-k \equiv 0 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

c) Prove the generating function of all compositions w even # of parts, each part in U is.

$$\frac{1-2x^4+x^8}{1-x^2-2x^4+x^8+2x^{94}-x^{186}}$$

$$\Phi_{U^{2k}}(x) = \left(\Phi_U(x)\right)^{2k}$$
$$= \left(\left(x\left(\frac{1-x^{92}}{1-x^4}\right)\right)^2\right)^k$$

$$S = \bigcup_{k\geq 0} U^{2k}$$

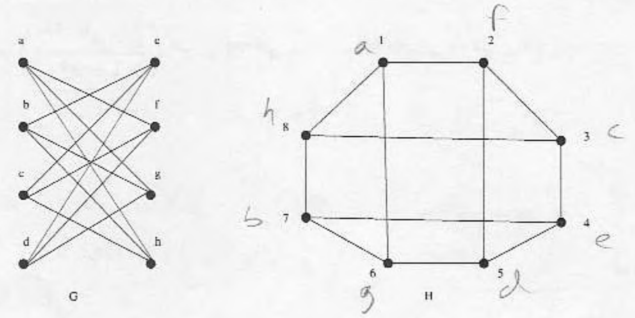
$$\Phi_S(x) = \sum_{k\geq 0}\left(\left(x\,\frac{1-x^{92}}{1-x^4}\right)^2\right)^k$$

$$= \frac{1}{1-\left(\frac{x(1-x^{92})}{1-x^4}\right)^2}$$

$$= \frac{1}{1-\frac{x^2(1-2x^{92}+x^{184})}{1-2x^4+x^8}}$$

$$= \frac{1-2x^4+x^8}{1-2x^4+x^8-x^2+2x^{94}-x^{186}} = \frac{1-2x^4+8}{1-x^2-2x^4+x^8+2x^{94}-x^{186}}$$

Question 6. Determine whether the two graphs shown are isomorphic. Prove your answer is correct.



$f(a) = 1$ $\quad f(e) = 4$

$f(f) = 2$ $\quad f(b) = 7$

$f(g) = 6$ $\quad f(c) = 3$

$f(h) = 8$ $\quad f(d) = 5$

Question 7. Let G be a 6-regular connected graph. Prove that G does not have a bridge.

$\deg(v) = 6$

if G has a bridge then let $\{u, v\}$ be that bridge

then both u and v must have 5 edges that connect elsewere

Suppose G has a bridge $e = \{u, v\}$

then $G - e$ would give 2 connected components

~~Whats~~ lets call ~~them~~ one G_1 ~~and G_2~~

$$\sum_{v \in G_1} \deg(v) = (|V(G_1)| - 1)(6) + (5)$$

$$= 6|V(G_1)| - 6 + 5$$

$$= 6 \cdot |V(G_1)| - 1$$

$$\equiv 1 \mod 2$$

but by handshake thm

$$\sum_{v \in G_1} \deg(v) = 2|E(G_1)|$$

$$\equiv 0 \mod 2$$

$\therefore$ Contradiction

$\therefore$ It cannot have a bridge.

Question 8. Which of the following three sequences are the degree sequences of a graph on seven vertices? If it is, give an example of such a graph; if not, explain why not.

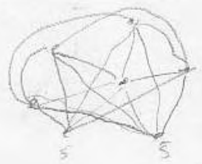
(a) 3,3,3,3,3,3,3 × by handshake

(b) 4,4,4,3,3,3,3 ✓

(c) 6,6,3,2,2,2,1 × deg 1 vertex would need to be adj to both deg 6 vertices

(d) 6,5,5,5,5,5,5 ✓

d)



b)