# MATH 239 - Fall 2013

# Assignment 8

## Due date : Friday, November 15, 2013, at noon (sharp)

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**Submission Guidelines:**

- Total number of marks in this assignment is 35+1 bonus.
- Use a cover page to submit your solutions (available on the course webpage).
- Keep a copy of your manuscript before your submission.
- Assignments submissions are exclusively accepted in the following dropboxes

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| [Section 001] | Dropbox next to the St Jerome's library, 2nd floor of STJ | | | |
| [Section 002] | Math DropBox #18; | Slot #1 A-J, | Slot #2 K-S, | Slot #3 T-Z |
| [Section 003] | Math DropBox #18; | Slot #4 A-J, | Slot #5 K-S, | Slot #6 T-Z |
| [Section 004] | Math DropBox #18; | Slot #7 A-J, | Slot #8 K-S, | Slot #9 T-Z |

- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- You are not allowed to post this manuscript (or parts of it) online, nor share it (or parts of it) with anyone not enrolled in this course.

**Assignment policies:** While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. For example, copying or paraphrasing a solution from some fellow student or old solutions from previous offerings of related courses qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. All students found to be cheating will automatically be given a mark of 0 on the assignment. In addition, there will be a 10/100 penalty to their final mark, as well as all academic offenses will be reported to the Associate Dean for Undergraduate Studies and recorded in the student's file (this may lead to further, more severe consequences).

If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that if you see any marking error, then you should return your assignment paper to the TA of your section within one week and with written notes on all the marking errors; please write the notes on a new sheet and attach it to your assignment paper.

**Question 1** [Marks 5]
Let $G$ be a connected graph in which every vertex has even degree. Prove that $G$ cannot have a bridge.

**Solution.** *Suppose of the contrary that edge $e = uv$ of $G$ is a bridge. Then by definition of bridge the graph $G - e$ is disconnected. Let $G_u$ denote the component of $G - e$ that contains $u$. Then by Lemma 4.9.2 the vertex $v$ is not in $G_u$. Since every vertex of $G$ had even degree, in $G_u$ every vertex has even degree except $u$, which has degree $deg_G(u) - 1$. But then $G_u$ is a graph which has exactly one vertex of odd degree, which contradicts the handshake theorem. Therefore $G$ cannot have a bridge.*

**Question 2** [Marks 7]
Let $T$ be a tree with two vertices of degree 3, one vertex of degree 5 and two vertices of degree 7. Find the smallest possible number $p$ of vertices $T$ could have. Give an example of a tree with these properties with exactly the minimum number $p$ of vertices.

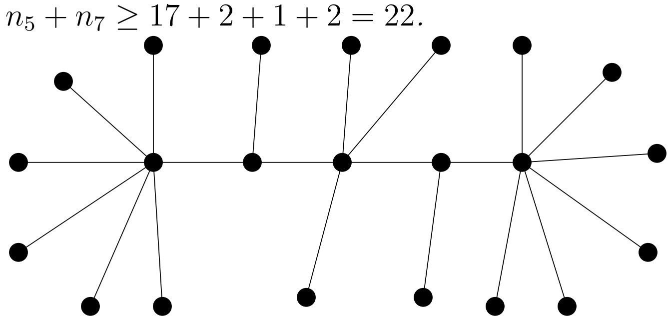
**Solution.** *For each $r$ let $n_r$ denote the number of vertices of $T$ of degree $r$. Then by considering the total number $p = \sum_{r=1}^{p} n_r$ of vertices in $T$, and the fact that $2|E(T)| = 2(p-1) = \sum_{r=1}^{p} rn_r$, as in the alternate proof of Theorem 5.1.4 we find that*

$$n_1 = 2 + \sum_{r=3}^{p}(r-2)n_r.$$

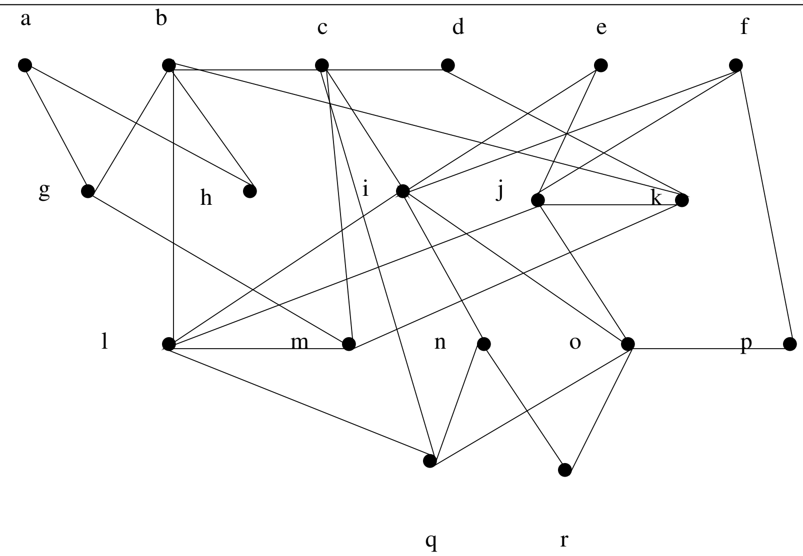
*We are given that $n_3 = 2$, $n_5 = 1$ and $n_7 = 2$. Therefore we find that*

$$n_1 \geq 2 + (1)(2) + (3)(1) + (5)(2) = 17.$$

*Hence $p \geq n_1 + n_3 + n_5 + n_7 \geq 17 + 2 + 1 + 2 = 22$.*



**Question 3** [Marks 10=6+3+1]
Let $G$ be the graph shown.




(a) Find a breadth-first search tree $G$, rooted at the vertex $a$. When considering the vertices adjacent to the vertex being examined, take them in increasing alphabetic order of their labels. List the vertices at each level of your tree. For each edge $uv$ of the tree indicate the predecessor function $pr(v) = u$ by an arrow on $uv$ from $v$ to $u$.

**Solution.** *Level 0:* $a$

*Level 1:* $g, h$

*Level 2:* $b, m$

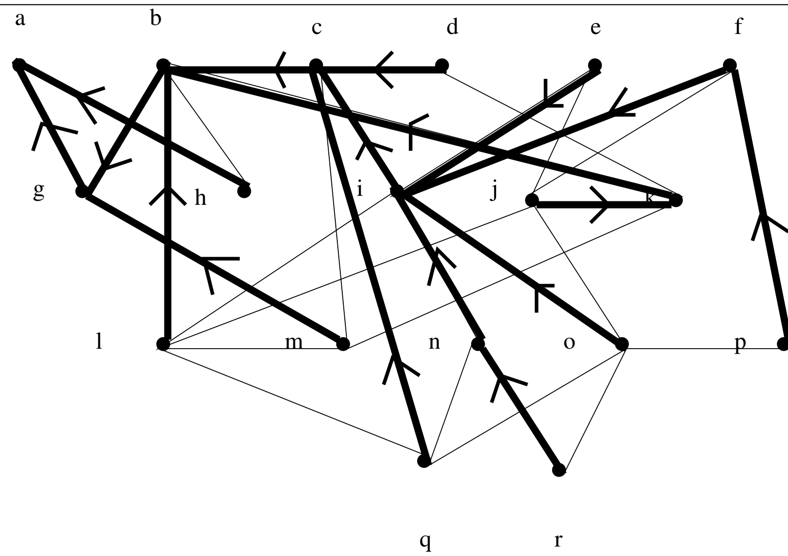
*Level 3:* $c, k, l$

*Level 4:* $d, i, q, j$

*Level 5:* $e, f, n, o$

*Level 6:* $p, r$

*The BFS tree* $T$ *is shown in the diagram.*




(b) Determine whether $G$ is bipartite. If it is bipartite, list the vertices in the two parts $A$ and $B$ of the bipartition. If it is not, list the vertices of an odd cycle.

**Solution.** *Because there is no edge of $G$ joining two vertices of $T$ at the same level, we know that $G$ is bipartite. A bipartition is given by the even and odd levels of $T$, that is* $A = \{a, b, m, d, i, q, j, p, r\}$ *and* $B = \{g, h, c, k, l, e, f, n, o\}$.

(c) Find a shortest path in $G$ from $p$ to $a$.

**Solution.** *A shortest path is given by the path in $T$ from $p$ to $a$, namely $pficbga$.*

**Question 4** [Marks 10=2+5+2+1]
Let $G$ be a 3-regular connected planar graph. Fix a planar drawing of $G$, and suppose that every face in the drawing has degree 5 or 6.

(a) Prove that $5k + 6m = 2q$, where $k$ is the number of faces of degree 5, $m$ is the number of faces of degree 6, and $q$ is the number of edges.

**Solution.**
*Since the sum of all face degrees is $2q$ where $q$ denotes the number of edges of $G$, we get* $2q = 5k + 6m$.

(b) Prove that $k = 12$.

**Solution.** *By Euler's Formula we know that* $p - q + s = 2$, *where* $s = k + m$ *denotes the number of faces. Since* $G$ *is 3-regular, the degree-sum formula tells us that* $2q = \sum_{v \in V(G)} deg(v) = 3p$, *and so* $p = 2q/3$. *Therefore we get*

$$\begin{aligned} p - q + s &= 2 \\ 2q/3 - q + k + m &= 2 \\ -q/3 + k + m &= 2 \\ -2q + 6k + 6m &= 12. \end{aligned}$$

*Part (a) implies that* $-2q + 5k + 6m = 0$. *Subtracting this from the above equation shows that* $k = 12$.

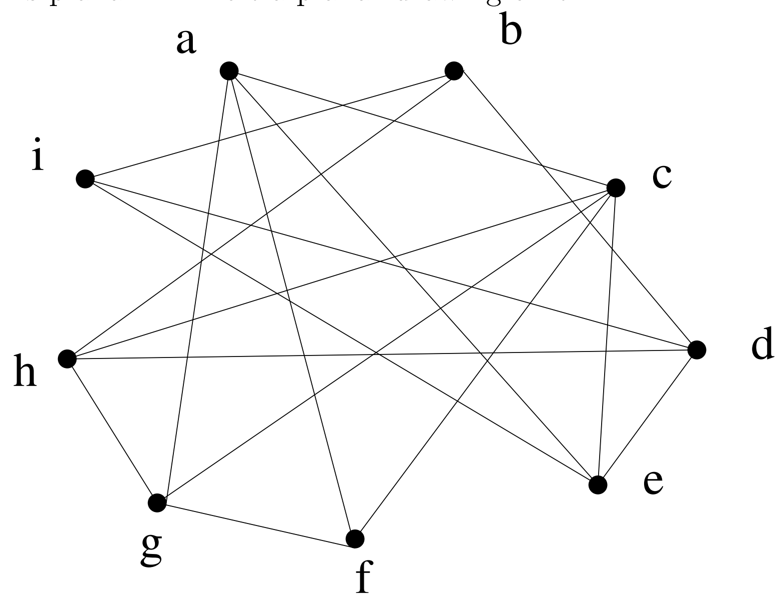
(c) Prove that if $G$ has $p = 60$ vertices, then $m = 20$ and $q = 90$.

**Solution.** *As in (b) we know* $3p = 2q$, *which implies* $q = 90$. *From (a) we have* $5k + 6m = 2q$ *and so* $5(12) + 6m = 2(90)$, *which tells us* $m = 120/6 = 20$.

(d) (BONUS) What everyday object is an example of such a graph $G$ with the parameters in (c)?

**Solution.** *The soccer ball.*

**Question 5** [Marks 4]
The graph shown is planar. Exhibit a planar drawing of it.



**Solution.**