MATH 239 - Tutorial 2

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1. Determine $[x^n]x(1+2x)^{-2}$.

   Solution 1. We use the formula for the inverse:

   $$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

   Set $y = -2x$ to get

   $$\frac{1}{1+2x} = \sum_{n\geq 0}(-2)^n x^n$$

   To compute

   $$\left(\sum_{n\geq 0}(-2)^n x^n\right)^2$$

   we use the product rule which states

   $$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)x^n$$

   By setting $a_i = b_i = (-2)^i$ we get

   $$\frac{1}{1+2x^2} = \left(\sum_{n\geq 0}(-2)^n x^n\right)^2 = \sum_{n\geq 0}\left(\sum_{i=0}^{n}(-2)^n\right)x^n = \sum_{n\geq 0}(n+1)(-2)^n x^n$$

   So

   $$\frac{x}{1+2x^2} = \sum_{n\geq 0}(n+1)(-2)^n x^{n+1}$$

   and thus

   $$[x^n]\frac{x}{1+2x^2} = n(-2)^{n-1}$$

   Solution 2. Use Theorem 1.6.5 to determine $(1+2x)^{-2}$. The theorem states:

   $$(1-y)^{-k} = \sum_{n\geq 0}\binom{n+k-1}{k-1}y^n$$

   We set $y = -2x$ and $k = 2$ to get:

   $$(1+2x)^{-2} = \sum_{n\geq 0}\binom{n+2-1}{2-1}(-2x)^n = \sum_{n\geq 0}(n+1)(-2)^n x^n$$

   So $x(1+2x)^{-2} = \sum_{n\geq 0}(n+1)(-2)^n x^{n+1}$ and thus $[x^n]x(1+2x)^{-2} = n(-2)^{n-1}$.

2. Find the inverse of $Q(x) = 1 + 2x + 3x^2 + \cdots = \sum_{n\geq 0}(n+1)x^n$.

Solution. We want to find a solution $Q(x)$ to $Q(x)A(x) = 1$. Here $q_i = i + 1$.

Note that $q_0 = 1$, so we can apply Theorem 1.5.2 which states that the unique solution to $Q(x)A(x) = P(x)$ for given P and Q is $a_n = p_n - q_1 a_{n-1} - q_2 a_{n-2} - \cdots - q_n a_0$. For us, $p_0 = 1$ and $p_n = 0$ for $n \geq 1$. We get $a_0 = p_0 = 1$. Next we get $a_1 = p_1 - q_1 a_0 = 0 - 2 \cdot 1 = -2$, and then $a_2 = p_2 - q_1 a_1 - q_2 a_0 = 0 + 4 - 3 = 1$.

We will now use induction to show that $a_n = 0$ for $n \geq 3$. Using the induction hypothesis, we get that $a_n = p_n - q_1 a_{n-1} - q_2 a_{n-2} - \cdots - q_n a_0 = 0 - 0 - \cdots - q_{n-2} 1 - q_{n-1}(-2) - q_n 1 = -(n-1) + 2n - (n+1) = 0$. So $A(x) = 1 - 2x + x^2 = (x-1)^2$ is the inverse of $Q(x)$.

As an exercise, we check that the inverse of $(x-1)^{-2}$ is indeed $Q(x)$. By using the formula for the inverse again, we get that $(x-1)^{-1} = \sum_{n\geq 0} x^n$. Using the product formula, we get that

$$(x-1)^{-2} = \left(\sum_{n\geq 0} x^n\right)^2 = \sum_{n\geq 0}\left(\sum_{i=0}^{n} 1 \cdot 1\right) x^n = \sum_{n\geq 0}(n+1)x^n$$

3. Let w be the weight fuction defined on $\mathbb{N}_0$ as follows. For each $a \in \mathbb{N}_0$,

$$w(a) = \begin{cases} a/2 & a \text{ is even} \\ 2a & a \text{ is odd} \end{cases}$$

Determine the generating series of $\mathbb{N}_0$ with respect to w.

Solution. Let $E = \{0, 2, 4, 6, ...\}$ and $O = \{1, 3, 5, 7, ...\}$. Then $N_0 = E \cup O$ is a disjoint union. Using the sum lemma, we have

$$\Phi_E(x) = 1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

$$\Phi_O(x) = x^2 + x^6 + x^{10} + x^{14} + ... = \frac{x^2}{1-x^4}$$

$$\Phi_{N_0}(x) = \frac{1}{1-x} + \frac{x^2}{1-x^4} = \frac{1 + x^2 - x^3 - x^4}{1 - x - x^4 + x^5}$$

4. Let S be a set of configurations with weight function w. Show that for any non-negative integer n,

$$[x^n]\frac{\Phi_S(x)}{1-x}$$

counts the number of configurations in S with weight at most n.

Solution. Let $\Phi_S(x) = \sum_{k\geq 0} a_k x^k$. Then

$$\frac{\Phi_S(x)}{1-x} = \left(\sum_{k\geq 0} a_k x^k\right)\left(\sum_{k\geq 0} x^k\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i(1)\right) x^n.$$

Hence $[x^n]\frac{\Phi_S(x)}{1-x} = \sum_{i=0}^{n} a_i = \sum_{i\leq n} a_i$.

5. For a binary string x, define its weight $w(s)$ to be the number of 1's in the string plus the length of the string itself. For example, $w(\texttt{110100001}) = 13$.

   (a) Let S_n be the set of all binary strings of length n. Use the product lemma to determine $\Phi_{S_n}(X)$.

   (b) Let T be the set of all binary strings (regardless of length). Determine $\Phi_T(x)$.

Solution 1 to (a)
Let $S = \{\texttt{0}, \texttt{1}\}$ with weight function $w(\texttt{0}) = 1$ and $w(\texttt{1}) = 2$. The generating series is

$$\Phi_S(x) = x + x^2$$

The intuition here is that a `0` in a binary string contributes 1 to the weight (increases the length of 1) and a `1` contributes 2 to the weight.

Note that we can consider n-tuples of `0`s and `1`s as strings of length n. So we can use the product lemma to get the generating series for strings of length n:

$$\Phi_{S^n}(x) = (x + x^2)^n$$

Solution 2 to (a)
To find the generating series, we must find the number a_k of strings in S_n with weight k, for every $k \geq 0$. Let $A \in S_n$. Then $w(A)$ is the number of 1's in A, plus n. It follows that to determine a_k, we need only find the number of length n strings containing $k - n$ 1's.

Then it is easy to see that

$$a_k = \begin{cases} 0, \text{ if } k < n, \\ \binom{n}{k-n}, \text{ otherwise.} \end{cases}$$

So

$$\begin{aligned} \Phi_{S_n}(x) &= \sum_{k \geq n} \binom{n}{k-n} x^k \\ &= x^n \sum_{k \geq 0} \binom{n}{k} x^k \\ &= x^n (1+x)^n \text{ (binomial theorem)} \\ &= (x + x^2)^n. \end{aligned}$$

Solution to (b)
The sets S_n of part (a) form a partition of T. So we can apply the Sum Lemma to get

$$\begin{aligned} \Phi_T(x) &= \sum_{n \geq 0} \Phi_{S_n}(x) \\ &= \sum_{n \geq 0} (x - x^2)^n. \end{aligned}$$

6. Determine a generating series for the number of k-combinations (i.e. collections of k elements, where 2 or more may be alike) of the letters M, A, T, H, in which M and A can appear any number of times but T and H can appear at most once. Which coefficient in this generating series gives the number of 5-collections?

Solution 1
Define the following sets:

- S_1: strings only consisting of Ms, length 0 or 1

- S_2: strings only consisting of As, length 0 or 1
- S_3: strings only consisting of Ts, arbitrarily length
- S_4: strings only consisting of Hs, arbitrarily length

For each, the weight function is the length of the string.

The generating series are $\Phi_{S_1}(x) = \Phi_{S_2}(x) = 1 + x$ and $\Phi_{S_3}(x) = \Phi_{S_4}(x) = 1 + x + x^2 + \ldots$.

Using the product lemma, we get the generating series for $S_1 \times S_2 \times S_3 \times S_4$ where the weight is the sum of the weights. We can intepret such 4-tuples as strings, and their weight is again their length. Note that these strings are sorted (i.e. M before A before T before H), so for each combination of letters M, A, T, H using M and A only once, there is exactly one string in $S_1 \times S_2 \times S_3 \times S_4$.

So the generating series we're interested in is the product of the 4 individual generating series:

$$\Phi_{S_1 \times S_2 \times S_3 \times S_4} = (1+x)^2 \left[\sum_{k \geq 0} x^k \right]^2$$

The coefficient we're interested in is the one corresponding to x^5, as this is the number of strings of length 5 in $S_1 \times S_2 \times S_3 \times S_4$.

Solution 2

Let S be the set of the k-combinations of the letters M,A,T,H, in which T and H appear at most once. First we define a weight function $w(A) = |A|$ for every $A \in S$.

There is exactly one 0-combination, namely $\emptyset$. By inspection, there are exactly 4 1-combinations, M, A, T, and H. For $k \geq 2$, the number of k-combinations can be determined as follows:
Number of k-combinations containing neither T nor H: $k+1$
Number of k-combinations containing exactly one of T and H: k
Number of k-combinations containing both T and H: $k-1$.

Summing these expressions, we see that the number of k-combinations, where $k \geq 2$, is given by

$$(k+1) + 2k + (k-1) = 4k.$$

Then the generating series $\Phi_S(x)$ is given by

$$\begin{aligned}
\Phi_S(x) &= 1 + 4x + \sum_{k \geq 2} 4kx^k \\
&= \sum_{k \geq 0} (k+1)x^k + 2\sum_{k \geq 1} kx^k + \sum_{k \geq 2} (k-1)x^k \\
&= (1 + 2x + x^2) \sum_{k \geq 0} (k+1)x^k \\
&= (1+x)^2 \left[\sum_{k \geq 0} x^k \right]^2.
\end{aligned}$$