MATH 239 - Fall 2013

Assignment 6

Due date : Friday, October 25th, 2013, at noon (sharp)

Submission Guidelines:

- Total number of marks in this assignment is 30.
- Use a cover page to submit your solutions (available on the course webpage).
- Keep a copy of your manuscript before your submission.
- Assignments submissions are exclusively accepted in the following dropboxes

[Section 001]	Dropbox next to the St Jerome's library, 2nd floor of STJ
[Section 002]	Math DropBox #18; Slot #1 A-J, Slot #2 K-S, Slot #3 T-Z
[Section 003]	Math DropBox #18; Slot #4 A-J, Slot #5 K-S, Slot #6 T-Z
[Section 004]	Math DropBox #18; Slot #7 A-J, Slot #8 K-S, Slot #9 T-Z

- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- You are not allowed to post this manuscript (or parts of it) online, nor share it (or parts of it) with anyone not enrolled in this course.

Assignment policies: While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. For example, copying or paraphrasing a solution from some fellow student or old solutions from previous offerings of related courses qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. All students found to be cheating will automatically be given a mark of 0 on the assignment. In addition, there will be a 10/100 penalty to their final mark, as well as all academic offenses will be reported to the Associate Dean for Undergraduate Studies and recorded in the student's file (this may lead to further, more severe consequences).

If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that if you see any marking error, then you should return your assignment paper to the TA of your section within one week and with written notes on all the marking errors; please write the notes on a new sheet and attach it to your assignment paper.

Question 1 [Marks 8=4+4]
For $p \geq 1$, define $\mathcal{G}_p$ to be the set of all graphs with vertex set $\{1, ..., p\}$.

(a) Consider the weight function defined by $w(G) = |E(G)|$ for $G \in \mathcal{G}_p$. Determine $\Phi_{\mathcal{G}_p}$ with respect to w. Your final answer should not include a summation.
Hint: You'll need to determine the possible numbers of edges a graph on p vertices can have.

Solution. *Let* $\Phi_{\mathcal{G}_p} = \sum_{q\geq 0} a_q x^q$, *where* a_q *is the number of graphs with vertex set* $\{1, ..., p\}$ *and* q *edges. There are* $\binom{p}{2}$ *possible edges on a graph with* p *vertices. Choosing any* q *from* $q = 0$ *to* $q = \binom{p}{2}$ *of them gives a graph in* $\mathcal{G}_p$ *with* q *edges, so*

$$a_q = \binom{\binom{p}{2}}{q} = \binom{\frac{1}{2}p(p-1)}{q} \quad \textit{for } q = 0, ..., \frac{1}{2}p(p-1),$$

and $a_q = 0$ *otherwise. Therefore, using the binomial theorem, we have*

$$\begin{aligned} \Phi_{\mathcal{G}_p} &= \sum_{q\geq 0} a_q x^q \\ &= \sum_{q=0}^{\frac{1}{2}p(p-1)} \binom{\frac{1}{2}p(p-1)}{q} x^q \\ &= (1+x)^{\frac{1}{2}p(p-1)}. \end{aligned}$$

(b) Next consider the weight function defined by $w'(G) = |V(G)|$ for $G \in \mathcal{G}_p$. Define $\mathcal{G} = \cup_{p\geq 1}\mathcal{G}_p$. Determine $\Phi_{\mathcal{G}}$ with respect to w'.

Solution. *By the sum lemma,* $\Phi_{\mathcal{G}} = \sum_{p\geq 1} \Phi_{\mathcal{G}_p}$, *so we need to find* $\Phi_{\mathcal{G}_p}$ *with respect to* w'. $[x^n]\Phi_{\mathcal{G}_p}$ *is the number of graphs on vertex set* $\{1, ..., p\}$ *with* n *vertices, which is clearly going to be* 0 *if* $n \neq p$. *Otherwise if* $n = p$, *we just need the number of graphs on vertex set* $\{1, ..., p\}$. *As mentioned in part a), there are* $\binom{p}{2} = \frac{1}{2}p(p-1)$ *possible edges for a graph on vertices* $\{1, ..., p\}$. *Any subset of these edges gives a graph, and there are* $2^{\frac{1}{2}p(p-1)}$ *possible subsets. Therefore* $\Phi_{\mathcal{G}_p} = 2^{\frac{1}{2}p(p-1)}x^p$ *and*

$$\Phi_{\mathcal{G}} = \sum_{p\geq 1} \Phi_{\mathcal{G}_p} = \sum_{p\geq 1} 2^{\frac{1}{2}p(p-1)}x^p.$$

Question 2 [Marks 4]
Let G be a graph with 6 or more vertices. Prove that G has either

- a subset of 3 vertices with all 3 edges between them, or
- a subset of 3 vertices with no edges between them.

(G could have both conditions satisfied. You only need to show at least one will be true.)
Hint: Pick an arbitrary vertex v and split into two cases: $deg(v) \geq 3$ or $deg(v) \leq 2$.

Solution. *As per the hint, we pick an arbitrary vertex v and split into two cases:*

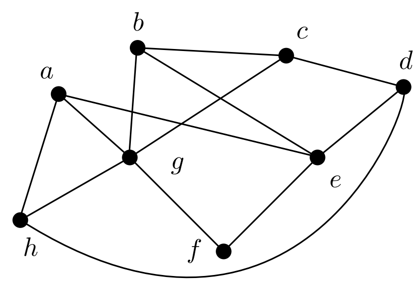
Case 1), $deg(v) \geq 3$. Then v has at least 3 neighbours, say a, b, c are any 3 of them. If there is an edge between any 2 of a, b, c, say a and b, then $\{a, b, v\}$ is a set of 3 vertices with all 3 edges between them. Otherwise $\{a, b, c\}$ is a set of 3 vertices with no edges between them.

Case 2), $deg(v) \leq 2$. Since there are at least 6 vertices in G and v has at most 2 neighbours, there are 3 vertices, say x, y, z that are not neighbours of v. If there is no edge between any 2 of x, y, z, say x and y, then $\{x, y, v\}$ is a set of 3 vertices with no edges between them. Otherwise $\{x, y, z\}$ is a set of 3 vertices with all 3 edges between them.

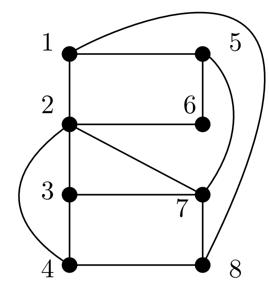
Therefore one of the two properties is always true.

Question 3 [Marks 4=2+2]
In each of the following, determine if G and H are isomorphic. If they are, give an isomorphism. If they are not, justify why no isomorphism exists.

(a)



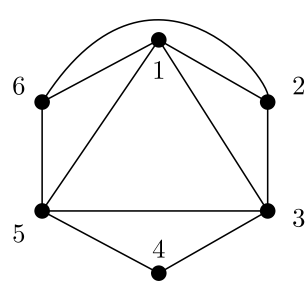
G



H

Solution. *Notice that both G and H have unique vertices of degrees $5, 2$ and 4. If f were an isomorphism from $V(G)$ to $V(H)$, we would need $f(g) = 2, f(f) = 6$, and $f(e) = 7$. However this is not adjacency preserving, since (for example) $\{e, f\} \in E(G)$ but $\{f(e), f(f)\} = \{6, 7\} \notin E(H)$. Thus there is no such isomorphism, and G and H are not isomorphic.*

(b)



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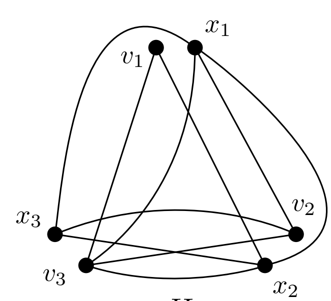


H

Solution. *G and H are isomorphic. A possible isomorphism is $f : V(G) \to V(H)$, where f is defined by*

$$\begin{array}{c|cccccc} a & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline f(a) & x_1 & v_2 & v_3 & v_1 & x_2 & x_3 \end{array}.$$

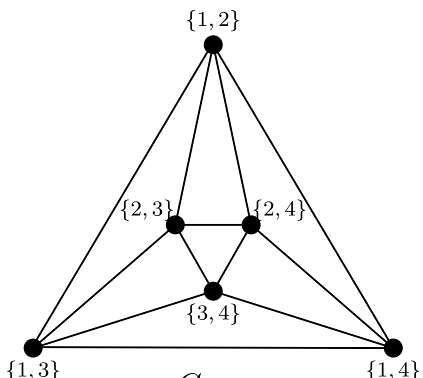
It is straightforward to verify that this preserves adjacency.

Question 4 [Marks 9=3+3+3]

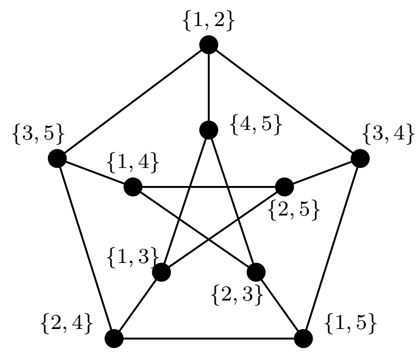
For $n \geq s \geq r \geq 0$, define the graph $G_{n,s,r}$ as follows: The vertices of $G_{n,s,r}$ are s-subsets of $\{1, ..., n\}$. Two vertices U and V are adjacent if and only if $|U \cap V| = r$.

(a) Draw $G_{4,2,1}$ and $G_{5,2,0}$.

Solution. *$G_{4,2,1}$ is the octahedron, and $G_{5,2,0}$ is the Petersen graph:*



$G_{4,2,1}$



$G_{5,2,0}$

(b) Prove that $G_{n,s,r}$ is k-regular and determine k.

Solution. *Fix a vertex $V = \{v_1, ..., v_s\}$ of $G_{n,s,r}$. Another vertex, U is adjacent to v if and only if $|U \cap V| = r$. There are $\binom{s}{r}$ ways to choose the r elements of U in the intersection, and $\binom{n-s}{s-r}$ ways to chose the remaining elements of U that are not in V. Therefore there are exactly $\binom{s}{r}\binom{n-s}{s-r}$ vertices adjacent to V. Since V was an arbitrary vertex, we conclude that $G_{n,s,r}$ is k-regular where $k = \binom{s}{r}\binom{n-s}{s-r}$.*

(c) Determine how many vertices and edges $G_{n,s,r}$ has.

Solution. *Clearly $|V(G_{n,s,r})| = \binom{n}{s}$. To get the number of edges, we use Theorem 4.3.1 and part b:*

$$\begin{aligned} |E(G_{n,s,r})| &= \frac{1}{2} \sum_{V \in V(G_{n,s,r})} deg(V) \\ &= \frac{1}{2} \sum_{V \in V(G_{n,s,r})} \binom{s}{r}\binom{n-s}{s-r} \\ &= \frac{1}{2}\binom{n}{s}\binom{s}{r}\binom{n-s}{s-r}. \end{aligned}$$

Question 5 [Marks 5=2+3]
For a graph G, we define the complement graph of G, denoted $\overline{G}$, with $V(\overline{G}) = V(G)$, and $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$.

(a) Suppose G has $|V(G)| = p$ vertices and $|E(G)| = q$ edges. How many vertices and edges does $\overline{G}$ have? (Express your answers in terms of p and q.)

Solution. *By definition of the complement, $\overline{G}$ clearly has $|V(\overline{G})| = p$ vertices, the same as G. The total number of edges that can exist in a graph with p vertices is $\binom{p}{2}$. Since an edge is in $\overline{G}$ precisely when it is not in G, we conclude that*

$$|E(\overline{G})| = \binom{p}{2} - |E(G)| = \binom{p}{2} - q.$$

(b) Prove that if G is isomorphic to $\overline{G}$, then either $p = 4k$, or $p = 4k + 1$ for some integer k.

Solution. *If G and $\overline{G}$ are isomorphic, then they must have the same number of edges. Therefore by part a, we need*

$$\begin{aligned} q &= \tbinom{p}{2} - q \\ 2q &= \frac{1}{2}p(p-1) \\ q &= \frac{1}{4}p(p-1). \end{aligned}$$

Now $q = \frac{1}{4}p(p-1)$ is an integer, so 4 must divide evenly into $p(p-1)$. But p and $p-1$ share no common factors, so either 4 divides p or 4 divides $p-1$. That is to say, either $p = 4k$ for some integer k, or $p - 1 = 4k$ for some integer k.