Instructions

1. **EXPLANATIONS ARE ALWAYS REQUIRED.**
   SHOW ALL YOUR WORK.
   STATE ANY THEOREMS YOU ARE USING.
2. No aids of any kind are permitted.
3. Please write your solutions in the space provided. If you need more space, please use the back of a page. Clearly indicate where your solution continues.

Question	Mark	Question	Mark
1 8 points		5 8 points	
2 6 points		6 8 points	
3 4 points		7 10 points	
4 9 points		8 7 points	
		Total 60 points	

Question 1. Consider the following formal power series:

$$\Phi(x) = \frac{1}{1-x} \qquad \Psi(x) = \frac{x}{(1-x)(1+2x)}$$

$$\Lambda(x) = 1 + 10x + 100x^2 + \cdots \qquad \Upsilon(x) = x + 2x^2 + 3x^3 + \cdots .$$

For each of the following, does the expression represent a well-defined formal power series? Be sure to justify your answers. Answers given without justification will receive a mark of zero.

(a) $\Phi(x) + \Psi(x)$

(b) $A(x)$ satisfying $\Lambda(x)A(x) = \Upsilon(x)$

(c) $\Upsilon(x)^{-1}$

(d) $\Phi(\Psi(x))$

Question 2. Let n be a positive integer. Give a combinatorial proof that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

Question 3. Let A and B be sets, and let w be a weight function defined on $A \cup B$. Suppose $A \cap B \neq \emptyset$. Give an expression for the generating series $\Phi_{A \cup B}(x)$ with respect to w, in terms of generating series for smaller sets. Prove your expression is correct.

Question 4. Let k be a fixed positive integer. Let S denote the set of compositions into k parts, where each part is even and at least 6.

(a) Express S as a Cartesian product of suitable sets.

(b) Find the generating function for S, with respect to the usual weight function (the weight of a composition is the sum of its parts). Be sure to state all theorems you use. Express your solution as a rational function.

(c) Let n be a positive integer. Find the number of compositions of n into k parts, where each part is even and at least 6. Express your solution as a closed-form expression in terms of n and k.

Question 5. Let S be the set of all binary strings which start with a 1 and do not contain the substring 00111. Show that the generating series of S is equal to

$$\frac{x}{1-2x+x^5},$$

where the weight of a string is its length.

Question 6. Suppose the sequence $\{a_n\}_{n\geq 0}$ satisfies the homogeneous recurrence relation

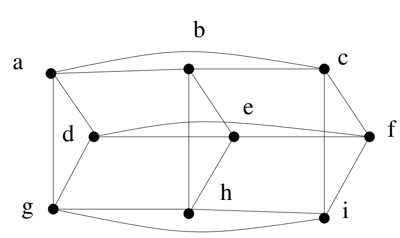
$$a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0 \quad (n \geq 3),$$

along with $a_0 = 3, a_1 = 10, a_2 = 52$. Solve this recurrence to find an explicit formula for a_n in terms of n, for all $n \geq 0$.

Question 7. Let n be a positive integer. Let T_n denote the set of all *ternary* strings of length n. Here a ternary string of length n is a string $\sigma = a_1 a_2 \cdots a_n$ where each $a_i \in \{0, 1, 2\}$. We define a graph H_n as follows. The vertex set of H_n is T_n. Two vertices σ and σ' are joined by an edge of H_n if and only if σ and σ' differ in exactly one position. (For example 012 is adjacent to 022 in H_3 but not to 121.)

(a) Find the number of vertices and the number of edges of H_n.

(b) Prove that H_2 is isomorphic to the graph shown below.



(c) Prove that H_n is connected for every n.

Question 8. Let G be a graph, and let P be a path of maximum length in G. Let x be the first vertex of P.

(a) Prove that all neighbours of x are vertices of P.

(b) Prove that if $k \geq 2$ and all vertices in a graph G have degree at least k, then G contains a cycle of length at least $k + 1$.

(c) For each $k \geq 2$, give an example of a graph in which all vertices have degree at least k and all cycles have length at most $k + 1$.