MATH 239 Tutorial 4 Solution Outline

1. Prove that $\{01, 011, 101\}^*$ is an ambiguous expression.

   Solution. The simplest way to prove an expression is ambiguous is to demonstrate a string that can be generated by the expression in more than one way. For example, $01101 = 01/101 = 011/01$.

2. Prove that $\{101, 110\}^*$ is an unambiguous expression.

   Solution. We will do a proof by induction on the length of any string s generated by the expression. We note that the length of any string generated by the expression must be divisible by 3.

   Base Case: The length of s is 3. In this case, $s = 101$ or $s = 110$, so s is uniquely generated.

   Induction Hypothesis: For positive number $k > 3$ and k divisible by 3, assume that any binary string of length $< k$ generated by the expression is uniquely generated.

   Induction Conclusion: Prove that s of length k as defined above is a string generated by our expression. We wish to prove that it is uniquely generated. Define string r to be the first 3 bits of string s, and string t to be the rest of string s, i.e. $s = rt$.

   Since every block of our regular expression is of length 3, to generate s we must have first picked r, and then constructed t. Since r is only three bits long, it is either 101 or 110, so r is uniquely generated. Since t has length $k - 3$, by induction it is also uniquely generated. Thus r and t were both uniquely generated by our expression, and since s was generated by picking two unique strings, it must be uniquely generated as well.

3. Determine an unambiguous decomposition for each of the following sets of strings. Then determine the generating series of each set with respect to the length of the string.

   (a) The set of binary strings that begin and ends with the same bit.

   Solution. The string either has length at least 2, or length less than 2. If its length is at least 2, then it either begins and ends with 0, or it begins and ends with 1. Thus,

   $$\{0\}\{0,1\}^*\{0\} \cup \{1\}\{0,1\}^*\{1\} \cup \{\varepsilon, 1, 0\}$$

   And so the generating series is

   $$\frac{x^2}{1-2x} + \frac{x^2}{1-2x} + (1 + 2x) = \frac{1-2x^2}{1-2x}.$$

   (b) The set of binary strings where the length of each block of 0's is not divisible by 3.

Solution. Since each block of 0s is not divisible by 3, the length of the block must be congruent to either 1 modulo 3 or 2 modulo 3. We modify the standard block decomposition formula so that the blocks of length 0 reflect this property.

$$\{\varepsilon, \{0, 00\}\{000\}^*\}(\{1\}\{1\}^*\{0, 00\}\{000\}^*)^*\{1\}^*$$

$$\left(1 + (x + x^2)\frac{1}{1 - x^3}\right)\frac{1}{1 - x\frac{1}{1-x}\frac{x+x^2}{1-x^3}}\frac{1}{1 - x} = \frac{1}{1 - x - x^2 - x^3}.$$

(c) The set of binary strings where every block of 1's of even length cannot be followed by at least 5 0's.

Solution. There are two cases: An odd block of 1s can be followed by any number of 0s, and an even block of 1s can be followed by at most 4 0s.

$$\{0\}^*(\{1\}\{11\}^*\{0\}\{0\}^* \cup \{11\}\{11\}^*\{0, 00, 000, 0000\})^*\{1\}^*$$

$$\left(\frac{1}{1 - x}\right)^2 \frac{1}{1 - \left(\frac{x}{1-x^2}\frac{x}{1-x} + \frac{x^2}{1-x^2}\frac{x-x^5}{1-x}\right)}$$

(d) The set of binary strings that begins with a 1, and other than the last 2 bits, the i-th bit is different from the $(i + 2)$-th bit.

Solution. There are three cases: Either the binary string is the single digit 1, or it begins with 11, or it begins with 10. If the first two bits are 11, then the next two must be 00, and then the next two must be 11, etc. The reasoning for the 10 case is similar.

$$\{1\} \cup \{11\}\{0011\}^*\{\varepsilon, 0, 00, 001\} \cup \{10\}\{0110\}^*\{\varepsilon, 0, 01, 011\}$$

$$x + 2\frac{x^2(1 + x + x^2 + x^3)}{1 - x^4} = \frac{x + 2x^2 + 2x^3 + 2x^4 + x^5}{1 - x^4}$$

4. Let k be a fixed positive integer. Let S be the set of binary strings with no k consecutive 1's, and let b_n be the number of strings in S of length n. Prove that for $n \geq k$,

$$b_n = \sum_{i=1}^{k} b_{n-i}.$$

Give a combinatorial proof of this recurrence.

Solution. Let $M = \bigcup_{i=0}^{k-1} 1^i$. Then the decomposition is

$$M(\{0\}M)^*.$$

Generating series is

$$(1 + x + \cdots + x^{k-1})\frac{1}{1 - x(1 + x + \cdots + x^{k-1})} = \frac{1 + x + \cdots + x^{k-1}}{1 - x - x^2 - \cdots - x^k}.$$

The denominator gives the recurrence.

Combinatorial proof (too hard?): Let S_n be the number of binary strings of length n with no k consecutive 1′s. Partition S_n according to the number of 1′s following the last 0: Let T_i be those in S_n where the last 0 is followed by i 1′s. Then

$$S_n = T_0 \cup T_1 \cup \cdots T_{k-1}$$

(we stop at $k-1$ because there cannot be k consecutive 1′s). Each T_i is a bijection with S_{n-i+1} by removing the last $i+1$ bits. The inverse is adding 0 followed by i 1′s.

Additional exercises

1. Determine the generating series for the set of binary strings where every block of 0′s cannot be followed by a block of 1′s of equal or greater length.

   Solution. Let $M = \{01, 0011, 000111, \ldots\}$. Then

   $$S = \{1\}^*(\{0\}\{0\}^*M)^*\{0\}^*.$$

   $$\Phi_M(x) = \frac{x^2}{1-x^2}$$

   $$\Phi_S(x) = \frac{1}{(1-x)^2}\frac{1}{1-\frac{x}{1-x}\frac{x^2}{1-x^2}} = \frac{1+x}{1-x-x^2}$$

2. Prove that $\{00, 101, 11\}^*$ is an unambiguous expression.

   Solution. The block of 0′s either have length 1 (decompose using 101) or it has even length (decompose using 00). The remaining pieces must be from 11′s.

3. Question 4 above suggests that if S is the set of all binary strings with no 2 consecutive 1′s, then the number of strings of length n satisfies the Fibonacci recurrence $b_n = b_{n-1} + b_{n-2}$ with initial conditions $b_0 = 1, b_1 = 2, b_2 = 3$. From class, the number a_n of compositions of n where each part is odd also satisfies the same recurrence, with different initial conditions $a_0 = a_1 = a_2 = 1$. By comparing the two sequences, we can then conclude that for $n \geq 0$, the number of binary strings in S with length n is equal to the number of compositions of $n+2$ where each part is odd. Find a bijection between these two sets of objects.

   Solution. I have no idea.