DO NOT REMOVE FROM OFFICE

UNIVERSITY OF WATERLOO
FINAL EXAMINATION
SPRING TERM 2003

Surname: ______________________________

First Name: ______________________________

Name:(Signature) ______________________________

Id.#: ______________________________

Course Number	MATH 239
Course Title	Introduction to Combinatorics
Instructor	001 Mark Bauer 002 Paul Schellenberg
Date of Exam	August 11, 2003
Time Period	9:00 am - 12:00 noon
Number of Exam Pages	14 pages (including this cover sheet)
Exam Type	Closed Book
Additional Materials	calculators with no data storage

NOTES:
1. Please write your name and student identification number in the blanks above.
2. Please cirle the name of your instructor.
3. Write your answers in the space provided. If you require additional space, use the back of the previous page and indicate clearly where your solution continues.

Problem	Value	Mark Awarded
1	8	
2	8	
3	8	
4	10	
5	7	
6	8	
7	8	
8	7	
9	8	
10	10	
11	8	
Total	90	

[marks] 1. Determine the following coefficients explicitly in terms of binomial coefficients.

(a)

$$[x^n]\frac{x^6}{(1+3x)^{10}}$$

[3]

(b)

$$[x^n]\frac{(1-x^2)^3}{(1-2x^3)^7}$$

[5]

[marks] 2.(a) Determine the number of compositions of n into k parts.
[3]

(b) Determine the number of solutions to the equation

$$|x_1| + |x_2| + \cdots + |x_k| = n$$

where the x_i are non-zero integers, and k is a fixed integer greater than
[2] or equal to 1.

(c) Determine the number of solutions to the above equation where k is al-
[3] lowed to vary over all integers greater than or equal to 1.

[marks] 3.(a) Write down a decomposition that uniquely generates the set of binary strings in which every block of 0's is followed by a block of 1's that has
[4] greater length. (For example, 101 is not in the set, but 1011 is.)

(b) Let

$$S = \{11\}^*(\{0\}\{00\}^*\{1\}\{1\}^*)^*\{0\}^*.$$

[4] Determine the generating function $\Phi_S(x)$.

[marks]4.(a) Let T be the set of binary strings in which every block of 0's has length at least 2 and every block of 1's has length at least 3. The generating function $\Phi_T(x)$ is

$$\Phi_T(x) = \sum_{n \geq 0} a_n x^n = \frac{(1-x+x^2)(1-x+x^3)}{1-2x+x^2-x^5}.$$

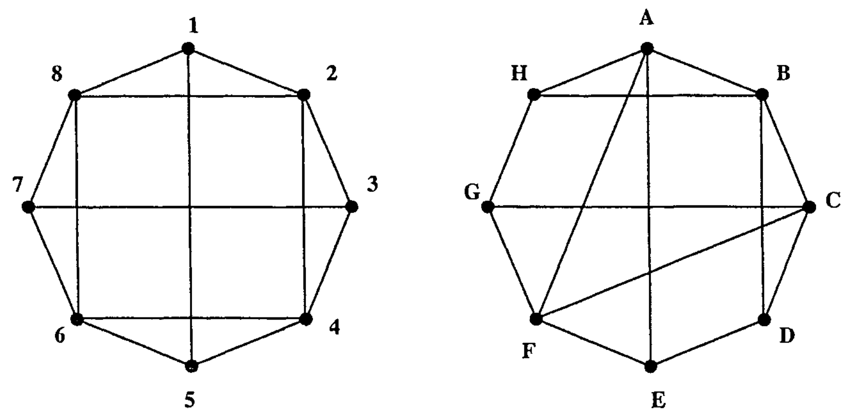
[5] Determine a recurrence relation and sufficient initial conditions to uniquely define the sequence $\{a_n\}$.

(b) The sequence $\{b_n\}$ satisfies the recurrence relation

$$b_n = -b_{n-1} + 8b_{n-2} + 12b_{n-3} \quad \forall n \geq 3$$

[5] with initial conditions $b_0 = 2,\ b_1 = 8,\ b_2 = 10$. Solve for b_n as a function of n.

[marks] 5. Give an isomorphism for the following two graphs or show that they are not isomorphic.

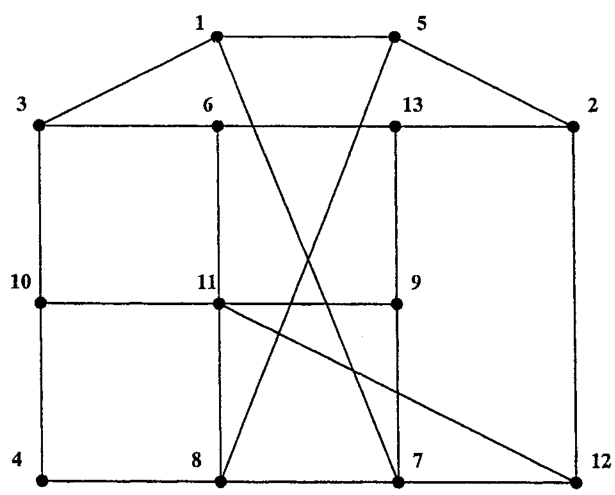


1

[7]

[marks] 6.(a) Construct a breadth-first search tree for the following graph rooted at the vertex labelled 1. When considering the vertices adjacent to the active vertex, take them in increasing order of their labels. Redraw the tree so that the root vertex is at the top, level 1 vertices below it, level 2 vertices below level 1 vertices and so on. In each level, arrange the vertices in the order they join the tree, with the oldest vertex on the left side.

[6]



1

[2] (b) Is the above graph bipartite? Justify your answer.

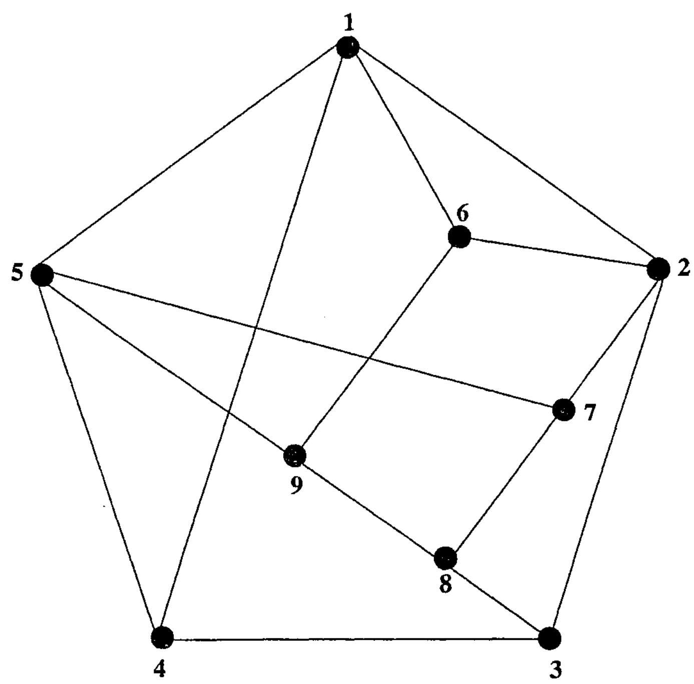
[marks] 7. Let G be a (simple) graph on p vertices such that each vertex has degree at least $\frac{p}{d}$. Prove that G has at most $d-1$ components.
[8]

[marks] 8.(a) Are the strings of the set $\{1, 01, 0110\}^*$ uniquely generated? Justify your answer.
[3]

[4] (b) If every face of a connected planar embedding P has degree 3, show that it has $q = 3p - 6$ edges.

[marks] 9.(a) State Kuratowski's Theorem.
[3]

[5] (b) Is the following graph planar? Justify your answer.



10.(a) [marks] Below is a bipatrite graph G with bipatrition (A, B) where $A = \{1, 2, \ldots, 10\}$ and $B = \{a, b, \ldots, j\}$. The 8 thick edges are the edges of the matching M. Determine the set X_0. Then apply the bipartite matching algorithm to construct a larger matching M'. As you apply the algorithm, construct the sets $\hat{X}$ and $\hat{Y}$.

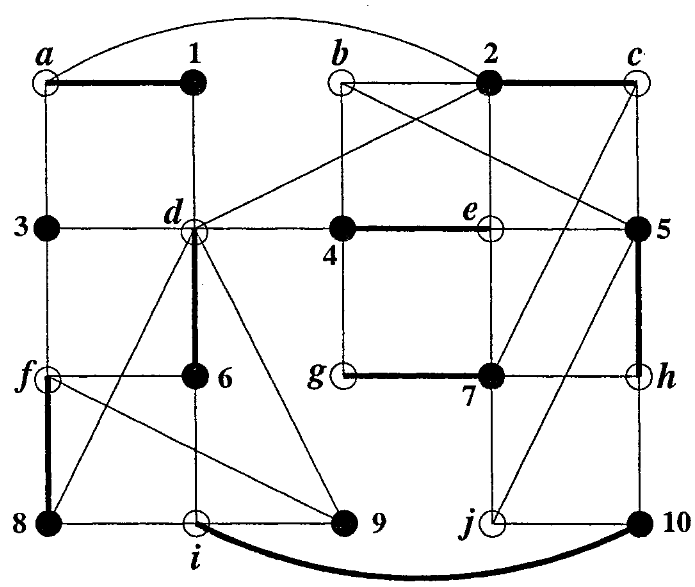


Figure 1: Bipartite graph G

[5]

$X_0 = \{$ ______________________ $\}$.

$\hat{X} = \{$ ____ $\}$.

$\hat{Y} = \{$ ____ $\}$.

[marks] (b) Mark your matching M' from part (a) on the graph below, and continue applying the bipartite matching algorithm. Either construct a larger matching M'' or else prove that M' is a maximum matching.

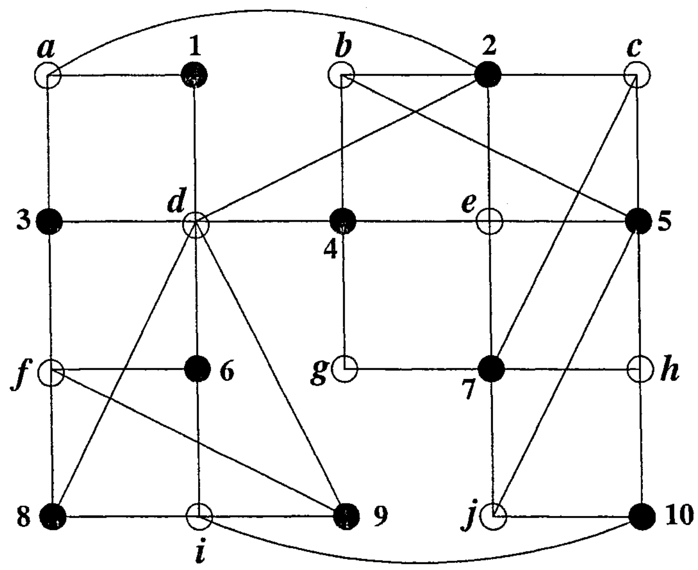


Figure 2: Bipartite graph G

[5]

$X_0 = \{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\}$.

$\hat{X} = \{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\}$.

$\hat{Y} = \{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\}$.

[2] 11.(a) Define $\bar{G}$, the complement of (simple) graph G.
[marks]

[6] (b) Show that if G is a simple planar graph with $p \geq 11$ vertices, then $\bar{G}$ is nonplanar.