DUE: NOON Friday 7 October 2011 in the drop boxes opposite the Math Tutorial Centre MC 4067 or next to the St. Jerome's library for the St. Jerome's section.

1. Determine the following coefficients, as summations, where m, n are nonnegative integers:

   (a) $[x^n](1-3x)^m(1-2x^2)^{-3}$

   SOLUTION.

$$\begin{aligned} & (1-3x)^m(1-2x^2)^{-3} \\ = & \left(\sum_{j=0}^{m}\binom{m}{j}(-3)^j x^j\right)\left(\sum_{i\geq 0}\binom{i+2}{2}2^i x^{2i}\right) \\ = & \sum_{j=0}^{m}\sum_{i\geq 0}\binom{m}{j}\binom{i+2}{i}(-1)^j 3^j 2^i x^{j+2i}. \end{aligned}$$

   The term x^n occurs whenever $j+2i=n$, thus we can set $j=n-2i$. Since $j\geq 0$ (and since we need $j=n-2i$), we only need to sum over i ranging from 0 to $\lfloor \frac{n}{2} \rfloor$. Thus we know that $[x^n](1-3x)^m(1-2x^2)^{-3}$ equals

$$\begin{aligned} & [x^n]\sum_{j=0}^{m}\sum_{i\geq 0}\binom{m}{j}\binom{i+2}{2}(-1)^j 3^j 2^i x^{j+2i} \\ = & \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\binom{m}{n-2i}\binom{i+2}{2}(-1)^{n-2i}3^{n-2i}2^i \\ = & (-3)^n\left(\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\binom{m}{n-2i}\binom{i+2}{2}\left(\frac{2}{9}\right)^i\right) \end{aligned}$$

   where the last equality uses the fact that $(-1)^{2i}=1$ for any integer i.

   (b) $[x^n](1+4x+3x^2)^m$

   SOLUTION.
   Note that $1+4x+3x^2=(1+x)(1+3x)$, and thus we are looking for $[x^n](1+x)^m(1+3x)^m$. Applying the binomial theorem we have

$$\begin{aligned} & (1+x)^m(1+3x)^m \\ = & \left(\sum_{k=0}^{m}\binom{m}{k}x^k\right)\left(\sum_{j=0}^{m}\binom{m}{j}3^j x^j\right). \end{aligned}$$

   We get a term of x^n whenever $k+j=n$. So we can set $k=n-j$ and sum over j. We thus get that $[x^n](1+x)^m(1+3x)^m$ equals

$$\begin{aligned} & [x^n]\left(\sum_{k=0}^{m}\binom{m}{k}x^k\right)\left(\sum_{j=0}^{m}\binom{m}{j}3^j x^j\right) \\ = & \sum_{j=0}^{m}3^j\binom{m}{n-j}\binom{m}{j}. \end{aligned}$$

   Alternate solution:

$$\begin{aligned}(1+4x+3x^2)^m &= \sum_{j=0}^{m}\binom{m}{j}(4x+3x^2)^j \\ &= \sum_{j=0}^{m}\binom{m}{j}(4x)^j\left(1+\frac{3}{4}x\right)^j \\ &= \sum_{j=0}^{m}\binom{m}{j}(4x)^j\left(\sum_{k=0}^{j}\binom{j}{k}\left(\frac{3}{4}x\right)^k\right)\end{aligned}$$

Therefore, $[x^n](1+4x+3x^2)$ is (remembering that $0 \le k \le j \le m$)

$$\begin{aligned}&= \sum_{j+k=n}\binom{m}{j}4^j\binom{j}{k}(3/4)^k \\ &= \sum_{k=0}^{m}\binom{m}{n-k}\binom{n-k}{k}3^k4^{(n-2k)}.\end{aligned}$$

2. Let k be a fixed positive integer. Find the number of compositions of n with k parts, in which each part is an even number greater than or equal to 4. Your solution should include a description of the set S of configurations to be counted, and should indicate where theorems from class are applied, e.g. Product Lemma.

SOLUTION. The generating function for the 1-part composition $N_{\geq 4}^{even} = \{4, 6, 8, \ldots\}$ is

$$\begin{aligned}\Phi_{N_{\geq 4}^{even}} &= x^4 + x^6 + x^8 + \ldots \\ &= x^4(1-x^2)^{-1}.\end{aligned}$$

By the Product Lemma , the generating function for the k-part composition $S = N_{\geq 4}^{even} \times \ldots \times N_{\geq 4}^{even} = (N_{\geq 4}^{even})^k$ is

$$\begin{aligned}\Phi_S(x) &= \left(x^4(1-x^2)^{-1}\right)^k \\ &= x^{4k}(1-x^2)^{-k}.\end{aligned}$$

Therefore, the number of configurations of weight n is

$$\begin{aligned}[x^n]\Phi_S(x) &= [x^n]x^{4k}(1-x^2)^{-k} \\ &= [x^{n-4k}](1-x^2)^{-k} \\ &= [x^{n-4k}]\sum_{i\geq 0}\binom{k+i-1}{k-1}x^{2i}.\end{aligned}$$

The coefficient is zero if $n-4k$ is odd, or if $n < 4k$. If $n-4k$ is even then the required coefficient occurs when $i = \frac{n-4k}{2}$ and it is equal to

$$\binom{k+\frac{n-4k}{2}-1}{k-1} = \binom{\frac{n-2k-2}{2}}{k-1}.$$

So the number of k-part compositions of n in which each part is an even number at least 4 is $\binom{\frac{n-2k-2}{2}}{k-1}$ when $n-4k$ is even (in other words, when n is even), and is 0 when n is odd (or when $n < 4k$).

3. Let a_n be the number of compositions of n into an even number of parts, each of which is at least 2. Prove that for $n \geq 0$

$$a_n = [x^n]\frac{1-2x+x^2}{1-2x+x^2-x^4}.$$

As in the previous question, your solution should include a description of the set S of configurations to be counted, and should indicate where theorems from class are applied, e.g. Sum and Product Lemmas.

SOLUTION. The generating function for 1-part composition $N_{\geq 2} = \{2, 3, 4, \ldots\}$ is

$$\begin{aligned} \Phi_{N_{\geq 2}}(x) &= x^2 + x^3 + x^4 + \ldots \\ &= x^2(1-x)^{-1}. \end{aligned}$$

By the Product Lemma, the generating function for a k-part composition $N_{\geq 2} \times \ldots \times N_{\geq 2} = N_{\geq 2}^k$ is

$$\begin{aligned} \Phi_{N_{\geq 2}^k}(x) &= \left(x^2(1-x)^{-1}\right)^k \\ &= x^{2k}(1-x)^{-k}. \end{aligned}$$

By the Sum Lemma, the generating function for all compositions into even number of parts $S = \cup_{k \geq 0, even} N_{\geq 2}^k$ is

$$\begin{aligned} \Phi_S(x) &= \sum_{k\geq 0, even} \Phi_{N_{\geq 2}^k}(x) \\ &= \sum_{k \geq 0, even} x^{2k}(1-x)^{-k} \\ &= \sum_{j \geq 0} \left(x^2(1-x)^{-1}\right)^{2j} \quad \text{setting } k = 2j \\ &= \sum_{j\geq 0} \left(x^4(1-x)^{-2}\right)^j \\ &= \frac{1}{1-x^4(1-x)^{-2}}, \quad \text{by Geometric Series} \\ &= \frac{(1-x)^2}{(1-x)^2 - x^4}, \quad \text{by multiplying top and bottom by } (1-x)^2 \\ &= \frac{1-2x+x^2}{1-2x+x^2-x^4}. \end{aligned}$$

Therefore, the number of compositions a_n of n into even number of parts, each of which is at least 2 equals

$$[x^n]\frac{1-2x+x^2}{1-2x+x^2-x^4}.$$

4. Derive a recurrence relation with initial conditions for the sequence a_n defined in the previous question.

SOLUTION.

$$\begin{aligned}
\sum_{n\geq 0} a_n x^n &= \frac{1-2x+x^2}{1-2x+x^2-x^4} \\
(1-2x+x^2-x^4)\sum_{n\geq 0} a_n x^n &= 1-2x+x^2 \\
\sum_{n\geq 0} a_n x^n - 2\sum_{n\geq 0} a_n x^{n+1} + \sum_{n\geq 0} a_n x^{n+2} - \sum_{n\geq 0} a_n x^{n+4} &= 1-2x+x^2 \\
\sum_{n\geq 0} a_n x^n - 2\sum_{n\geq 1} a_{n-1} x^{n} + \sum_{n\geq 2} a_{n-2} x^{n} - \sum_{n\geq 4} a_{n-4} x^{n} &= 1-2x+x^2.
\end{aligned}$$

Now we compare the coefficient of x^n on both sides for $n = 0, 1, 2, 3$ and general $n \geq 4$ to get:

$$\begin{aligned}
x^0 &: \ a_0 = 1 \\
x^1 &: \ a_1 - 2a_0 = -2 \quad \Longrightarrow \quad a_1 = 2(1) - 2 = 0 \\
x^2 &: \ a_2 - 2a_1 + a_0 = 1 \quad \Longrightarrow \quad a_2 = 2(0) + 1 - 1 = 0 \\
x^3 &: \ a_3 - 2a_2 + a_1 = 0 \quad \Longrightarrow \quad a_3 = 0 + 2(0) - 0 = 0 \\
x^k, k \geq 4 &: \ a_k - 2a_{k-1} + a_{k-2} - a_{k-4} = 0 \quad \Longrightarrow \quad a_k = 2a_{k-1} - a_{k-2} + a_{k-4}.
\end{aligned}$$

Here the last line gives the general recurrence for all $k \geq 4$, and the initial conditions are $a_0 = 1$, $a_1 = a_2 = a_3 = 0$.