MATH 239 — Fall 2011 Assignment 8

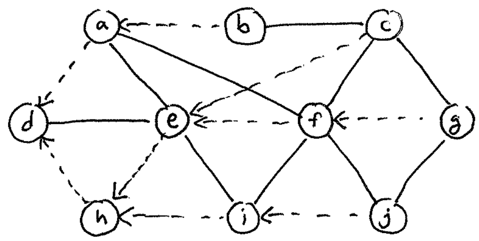
DUE: NOON Friday 25 November 2011 in the drop boxes opposite the Math Tutorial Centre MC 4067 or next to the St. Jerome's library for the St. Jerome's section.

1. Let G be the n-cube. Given a breadth-first search tree for G rooted at $0 \cdots 0$, the function level defined on $V(G)$ can be used as a weight function. Compute $\Phi_{V(G)}(x)$.

   SOLUTION. At level 1, we have all the binary strings of length n that have only 1 copy of 1. At level 2, we have all the binary strings of length n that have only 2 copies of 1. At level k, we have all the binary strings of length n that have exactly k copies of 1. So $|\text{level}^{-1}(k)| = \binom{n}{k}$ and so

$$\Phi_{V(G)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

2. (a) Explain why the search tree, rooted at d and indicated by dotted lines in the graph below is not a breadth-first search tree.



   SOLUTION. Note that vertices level$(a) = 1$ while level$(f) = 3$. If this drawing represented a breadth-first search tree, these two vertices would be at most one level apart since they are adjacent.

   (b) Can you remove only two edges of that graph to make this search tree a breadth-first search tree? If yes, which edges and what is the sequence of the vertices as they are added to the tree (starting with d); if no, why?

   SOLUTION. Yes! Just remove edges $\{d, e\}$ and $\{a, f\}$. The vertices are added to the tree in the following order:

$$d, h, a, e, i, b, f, c, j, g.$$

3. (a) Prove that a bipartite planar graph has a vertex of degree at most 3.

   SOLUTION. Suppose G is a bipartite planar graph. If it is a forest, we know it has a vertex of degree 1. Otherwise, it has a cycle. Let $q = |E(G)|$ and $p = |V(G)|$. Then we know that $q \leq 2p - 4$. Suppose that all vertices have degree at least 4. Then

$$2q = \sum_{v \in V(G)} \deg(v) \geq 4p$$

   hence $q \geq 2p > 2p - 4$. Since it is not possible for q to be both less than or equal and bigger than $2p - 4$.

   (b) Deduce that the n-cube is not planar for $n > 3$.

   SOLUTION. For each string $s \in \{0,1\}^n$, there are n positions where one can modify the string. Hence the n-cube is n-regular . The n-cube is bipartite , with bipartition $A = \{s \in \{0,1\}^n \mid s \text{ contains an odd number of } 1\}$, and $B = \{s \in \{0,1\}^n \mid s \text{ contains an even number of } 1\}$. Therefore the result from part a) applies. This result therefore tells us that if the n-cube is planar, we must have $n \leq 3$.

4. Suppose that G is a non-empty connected graph with an embedding on the sphere where all the faces are hexagons. Let p be the number of vertices, q the number of edges, and s the number of faces of the embedding.

   (a) Prove that $q = 3s$ and $p = 2 + 2s$.

   SOLUTION. Since faces are hexagon, their degree is always 6. So

   $$2q = \sum_{\text{face}} \deg(\text{face}) = 6s.$$

   So $q = 3s$, as desired. Now using Euler's formula , we have $2 = p - q + s = p - 3s + s = p - 2s$, hence $p = 2 + 2s$, as desired.

   (b) Prove that G is bipartite.

   SOLUTION. Every cycle C must contain an integral number of faces, say $f_1, \dots, f_k$. Let q_{in} be the number of edges in the interior of $f_1 \cup \dots \cup f_k$ and q_{out} be the number of edges in the boundary walk of $f_1 \cup \dots \cup f_k$, hence on the cycle C. Every interior edge is in two faces, while every outside edge is on one face only. Hence $6k = 2q_{in} + q_{out}$. That proves that the length of the cycle C is $q_{out} = 2(3k - q_{in})$, so is even. Since that is true for every cycle, G has no odd cycle and is thus bipartite.

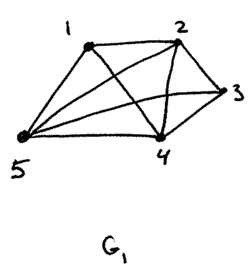
   (c) Deduce that if G is regular, this embedding of G has exactly two faces.

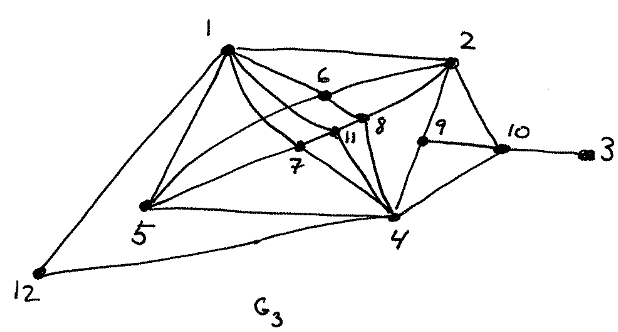
   SOLUTION. From above, since G is bipartite, there is a vertex of degree at most 3. Since G is regular, say k-regular, we have $k \leq 3$. The case $k = 1$ is impossible since G is connected and has at least 6 vertices. Suppose that $k = 3$. Then

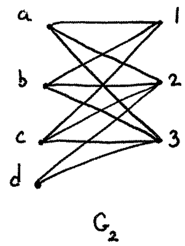
   $$3p = \sum_{v} \sum_{v \in f} 1 = \sum_{f} \sum_{v \in f} 1 = 6s$$

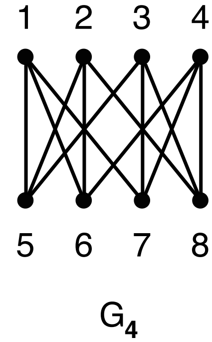
   so $p = 2s$. Since we have from a) that $p = 2 + 2s$, we must have $2 = 0$, that is impossible. Hence the only possibility is $k = 2$, and in this case we must have exactly one hexagon and exactly two faces.

5. Determine if the following graphs G_1, G_2, G_3, G_4 below are planar graphs. If they are, draw a planar embedding, if they are not, explain why.

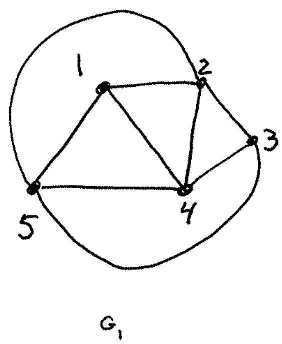






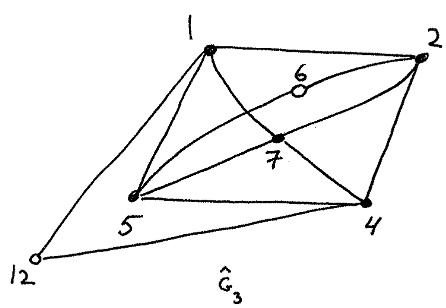


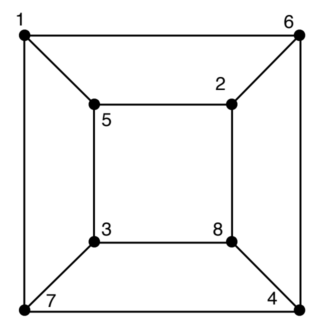
SOLUTION. Graph G_1 is **planar**, as can be seen by moving the edges $\{2,5\}$ and $\{3,5\}$ as in the drawing below.



Obviously, $G_2 - d$ is $K_{3,3}$ so is **non-planar**. Alternative proof: If $q = |E(G_2)|$ and $p = |V(G_2)|$, we have $p = 7$ and $q = 11$. On one hand, $q = 11 > 10 = 2p - 4$. On the other hand, G_2 is bipartite, with bipartition ({1,2,3},{a,b,c,d}), so if it is planar, $q \leq 2p - 4$. Both inequality cannot be true simultaneously, so G_2 is **non-planar**.

The subgraph $\hat{G}_3$ of G_3, depicted below is an edge subdivision of K_5 with $V(K_5) = \{1,2,4,5,7\}$ and the extra vertices 6 and 12 subdividing the edges $\{2,5\}$ and $\{1,4\}$ respectively. The vertices 6 and 12 are drawn in a different style so that the statement is clearer to see. Since G_3 has a subgraph that is an edge subdivision of K_5, Kuratowski's theorem guarantees that G_3 is **non-planar**.





Graph G_4 is **planar**, as can be seen by drawing the graph above