MATH 239 Spring 2012: Assignment 1

Solutions

1. {6 marks} We start with a few counting problems. You do not need to show your work for this question only. Simplify your answer as much as possible.

   (a) How many $\{0,1\}$-strings are there with exactly n 0's and m 1's?

   Solution. These are strings of length $m+n$, and we pick n spots for 0's, so there are $\binom{m+n}{n}$ (or $\binom{m+n}{m}$) of them.

   (b) For some $n \geq 2$, how many subsets of $[2n]$ contain at most two odd numbers?

   Solution. There are 2^n possible subsets of even numbers. Any of them could have 0, 1 or 2 odd numbers, of which there are $1, n, \binom{n}{2}$ choices respectively. So in total there are $2^n(1+n+\binom{n}{2})$.

   (c) A permutation of $[n]$ is a bijection $\sigma : [n] \to [n]$ (in other words, it is any rearrangement of $[n]$). How many permutations σ of $[n]$ satisfy the property that $\sigma(1) \neq 1$?

   Solution. The number of permutations where $\sigma(1) = 1$ is $(n-1)!$. So the number of permutations with $\sigma(1) \neq 1$ is $n! - (n-1)!$.

2. {4 marks} For some $0 \leq r \leq k \leq n$, how many subsets of $[n]$ have r elements in common with the set $\{1, \ldots, k\}$? Describe two sets S and T such that the answer to our question is the cardinality of the cartesian product $S \times T$, then determine what is this answer.

   Solution. Let S be the set of all subsets of $\{1, \ldots, k\}$ of size r, and let T be the set of all subsets of $\{k+1, \ldots, n\}$ (of any size). Then any of our subset of interest is the disjoint union of an element of S and an element of T. Since $|S| = \binom{k}{r}$ and $|T| = 2^{n-k}$, we have $|S \times T| = \binom{k}{r}2^{n-k}$.

3. {8 marks} Let $n \geq 1$. Let S be the set of all subsets of $[n]$ that contains the element 1, and let T be the set of all subsets of $[n]$ that contains the element n. Define a bijection between S and T, and prove that it is a bijection.

   Solution. Define $f : S \to T$ where for each $A \in S$,

$$f(A) = \begin{cases} A & n \in A \\ A \setminus \{1\} \cup \{n\} & n \notin A \end{cases}$$

   (If A contains n, then we keep it the same. If A does not contain n, then we swap 1 and n.)

   To prove that f is a bijection, we provide its inverse mapping. Define $g : T \to S$ where for each $B \in T$,

$$g(B) = \begin{cases} B & 1 \in B \\ B \setminus \{n\} \cup \{1\} & 1 \notin B \end{cases}$$

   For each $A \in S$, if A contains n, then A contains both 1 and n. So $g(f(A)) = A$. If A does not contain n, then

$$g(f(A)) = g(A \setminus \{1\} \cup \{n\}) = (A \setminus \{1\} \cup \{n\}) \setminus \{n\} \cup \{1\} = A.$$

   The case for $B \in T$ is similar. Therefore, f is a bijection.

Alternate solution. Let $A \in S$, and suppose $A = \{1, a_1, \ldots, a_k\}$ where $a_1, \ldots, a_k > 1$. Then we define $f : S \to T$ by

$$f(A) = \{a_1 - 1, \ldots, a_k - 1, n\}.$$

As each element of $f(A)$ is in $[n]$ including n, $f(A) \in B$, so this mapping is well-defined.

To prove that f is a bijection, we provide its inverse mapping. Let $B \in T$, and suppose $B = \{b_1, \ldots, b_l, n\}$. Then we define $g : T \to S$ by

$$g(B) = \{1, b_1 + 1, \ldots, b_l + 1\}.$$

This is also a well-defined function. Then

$$g \circ f(A) = g(\{a_1 - 1, \ldots, a_k - 1, n\}) = \{1, a_1 - 1 + 1, \ldots, a_k - 1 + 1\} = A$$

and

$$f \circ g(B) = f(\{1, b_1 + 1, \ldots, b_l + 1\}) = \{b_1 - 1 + 1, \ldots, b_l - 1 + 1, n\} = B.$$

So g is the inverse of f, hence f is a bijection.

4. {12 marks} Consider the following identity:

$$3^n = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}.$$

(a) Give a combinatorial proof of this identity.

Solution. Consider the set $S = \{1, 2, 3\}^n$, which consist of all n-tuples $(a_1, \ldots, a_n)$ where each $a_i \in \{1, 2, 3\}$. Clearly $|S| = 3^n$.

We partition S into $n + 1$ sets $S_0, \ldots, S_n$ where for each $i = 0, \ldots, n$, S_i is the set of elements of S that contains exactly i 1's. We can count S_i by first deciding which i of the n spots are 1's, then fill in the remaining $n - i$ spots with either 2 or 3. There are $\binom{n}{i}$ ways to choose the i spots, and 2^{n-i} ways to fill in the remaining spots. So $|S_i| = \binom{n}{i} 2^{n-i}$.

Since $S = S_0 \cup \cdots \cup S_n$ is a disjoint union,

$$3^n = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}.$$

(b) Give an algebraic proof of this identity.

Solution. By the binomial theorem,

$$(1 + 2x)^n = \sum_{j=0}^{n} \binom{n}{j} (2x)^j = \sum_{j=0}^{n} \binom{n}{n-j} (2x)^j.$$

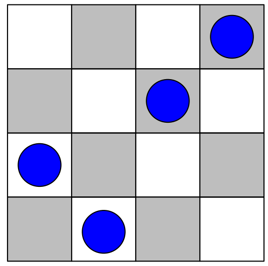
Putting $i = n - j$, we get

$$(1 + 2x)^n = \sum_{i=0}^{n} \binom{n}{i} (2x)^{n-i}.$$

Substitute $x = 1$ to get the identity.

5. {8 marks} We wish to consider the problem of placing n rooks on an $n \times n$ board so that no rook can attack another. This is equivalent to placing them so that no two are on the same column or row. A little bit of thought would hopefully convince you that there are $n!$ ways to do this.

Now let's restrict the problem so that no rook can appear on the main diagonal. Then it is not so obvious how many ways we can do this. Let R_n be the set of all such placements. We can represent a



placement by listing all rook positions in a set. For example, $\{(3,1),(4,2),(2,3),(1,4)\}$ is an element of R_4, representing the following placement:

Let $[n]=\{1,2,\ldots,n\}$. A **derangement** is a bijection $\sigma:[n]\to[n]$ such that $\sigma(x)\neq x$ for each $x\in[n]$. Let S_n be the set of all derangements of $[n]$. We can represent a $\sigma\in S_n$ by $(\sigma(1)\sigma(2)\cdots\sigma(n))$. For example, (3142) is a derangement of $[4]$, but (2431) is not, since $\sigma(3)=3$. There are 2 derangements of $[3]$, 9 derangements of $[4]$, but the general formula for $[n]$ is a bit more involved (not important for this question).

Find a bijection between R_n and S_n, hence proving that they have the same cardinality. You need to prove that it is indeed a bijection, and illustrate your bijection by drawing the placement corresponding to the derangement (73428561).

Solution. Let $\sigma\in S_n$. Then we define $f:S_n\to R_n$ by

$$f(\sigma)=\{(\sigma(1),1),(\sigma(2),2),\ldots,(\sigma(n),n)\}.$$

We first show that f is well-defined by showing that $f(\sigma)\in R_n$. Clearly $f(\sigma)$ has one rook in each column. Since σ is a permutation, $\sigma(i)\neq\sigma(j)$ whenever $i\neq j$, so $f(\sigma)$ has one rook in each row. Now since σ is a derangement, $\sigma(i)\neq i$ for each $i=1,\ldots,n$, so no rook appears on the main diagonal. Hence, $f(\sigma)\in R_n$ and f is well-defined.

We prove that f is a bijection by providing its inverse mapping. Let $A=\{(r_1,1),(r_2,2),\ldots,(r_n,n)\}\in R_n$. We define $g:R_n\to S_n$ by

$$g(A)=(r_1r_2\ldots r_n).$$

Since $A\in R_n$, $r_i\neq r_j$ whenever $i\neq j$, and $r_i\neq i$. So $g(A)$ is a derangement, and g is well-defined.

Finally, we see that

$$g\circ f(\sigma)=g(\{(\sigma(1),1),(\sigma(2),2),\ldots,(\sigma(n),n)\})=(\sigma(1)\sigma(2)\cdots\sigma(n))=\sigma$$

and

$$f\circ g(A)=f((r_1r_2\ldots r_n))=\{(r_1,1),(r_2,2),\ldots,(r_n,n)\}=A.$$

So g is indeed the inverse of f, hence f is a bijection.

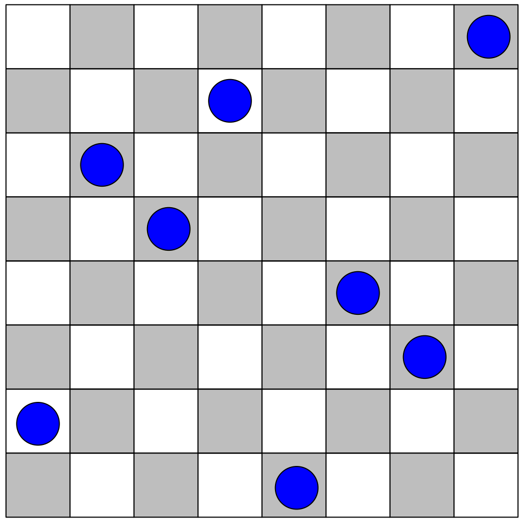
We illustrate $g(73428561)$ below.

6. {12 marks} Read the supplementary file on formal power series. (For a formal treatment of the subject, see Section 1.5 of the course notes.)

   (a) Suppose

$$\begin{aligned}f(x)&=x^5+x^8+x^{11}+x^{14}+\cdots+x^{1337}\\ g(x)&=1+(x+f(x))^2+(x+f(x))^4+(x+f(x))^6+\cdots\end{aligned}$$

   Express $g(x)$ in the form $\frac{p(x)}{q(x)}$ where $p(x),q(x)$ are polynomials.



Solution.

$$f(x) = x^5(1 + x^3 + x^6 + \cdots + x^{1332}) = x^5 \frac{1 - x^{1335}}{1 - x^3}.$$

Using geometric series,

$$g(x) = \frac{1}{1 - (x + f(x))^2} = \frac{1}{1 - (x + \frac{x^5(1-x^{1335})}{1-x^3})^2} = \frac{(1 - x^3)^2}{(1 - x^3)^2 - (x - x^4 + x^5 - x^{1340})^2}.$$

(b) Determine $[x^7](1 + x^2)(1 + 2x)^{-36}$.

Solution. We first note that

$$(1 + 2x)^{-36} = \sum_{i \geq 0} \binom{i + 35}{35} (-2)^i x^i.$$

So

$$\begin{aligned} [x^7](1 + x^2)(1 + 2x)^{-36} &= [x^7](1 + 2x)^{-36} + [x^7]x^2(1 + 2x)^{-36} \\ &= [x^7](1 + 2x)^{-36} + [x^5](1 + 2x)^{-36} \\ &= -\binom{42}{35} 2^7 - \binom{40}{35} 2^5. \end{aligned}$$

(c) For some fixed $k \geq 1$, determine $[x^{3n}](1 - 2x^3)^{-4}(1 - 5x)^{-k}$.

Solution. We first note that

$$\begin{aligned} (1 - 2x^3)^{-4} &= \sum_{i \geq 0} \binom{i + 3}{3} 2^i x^{3i} \\ (1 - 5x)^{-k} &= \sum_{j \geq 0} \binom{j + k - 1}{k - 1} 5^j x^j \end{aligned}$$

So

$$(1 - 2x^3)^{-4}(1 - 5x)^{-k} = \sum_{i \geq 0} \sum_{j \geq 0} \binom{i + 3}{3} \binom{j + k - 1}{k - 1} 2^i 5^j x^{3i+j}.$$

Looking at the power x^{3i+j}, in order to generate the x^{3n} term, we only need $i = 0, 1, \ldots, n$. For each i, we need $j = 3n - 3i$. So

$$[x^{3n}](1-2x^3)^{-4}(1-5x)^{-k} = \sum_{i=0}^{n} \binom{i+3}{3}\binom{3n-3i+k-1}{k-1} 2^i 5^{3n-3i}.$$

7. {Extra credit: 5 marks} For $0 \le r \le n$, prove the following identity (using any correct method):

$$\binom{n}{r} 3^{n-r} = \sum_{k=r}^{n} \binom{n}{k}\binom{k}{r} 2^{n-k}.$$

Solution. We give two different proofs of this identity. The main idea of this identity comes from Q2 of this assignment.

Combinatorial proof. Consider the set

$$S = \{(A, B) | A, B \subseteq [n], |A \cap B| = r\}.$$

We split S into subsets $S_r, S_{r+1}, \ldots, S_n$ where for each $k = r, \ldots, n$, S_i consists of those elements (A, B) in S where $|A| = k$ (clearly $|A| \ge r$). To count S_k, we see that there are $\binom{n}{k}$ choices for A. For each choice of A, we can pick B by first picking the r elements in common with A (of which there are $\binom{k}{r}$ choices), and for the $n - r$ elements of $[n] \setminus A$, each could be in B or not in B (so there are 2^{n-r} choices). In total, $|S_k| = \binom{n}{k}\binom{k}{r}2^{n-r}$.

We now count S in a different way. We first pick the r elements of $[n]$ that are in both A and B, of which there are $\binom{n}{r}$ choices. For the remaining $n - r$ elements, each can be either in A only, in B only, or in neither A nor B, so there are 3^{n-r} choices here. Therefore, $|S| = \binom{n}{r}3^{n-r}$.

The result follows from the fact that $S = S_r \cup S_{r+1} \cup \cdots \cup S_n$, which is a disjoint union of sets.

Algebraic proof. The LHS is the coefficient of x^{n-r} in the polynomial $(1+3x)^n$. We now calculate this coefficient in a different way.

$$\begin{aligned}
[x^{n-r}](1+3x)^n &= [x^{n-r}]((1+x)+2x)^n \\
&= [x^{n-r}] \sum_{k=0}^{n} \binom{n}{k} (1+x)^k (2x)^{n-k} \\
&= [x^{n-r}] \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{i=0}^{k} \binom{k}{i} x^i \right) 2^{n-k} x^{n-k} \\
&= [x^{n-r}] \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{n}{k}\binom{k}{i} 2^{n-k} x^{n-k+i} \\
&= \sum_{k=0}^{n} \binom{n}{k}\binom{k}{k-r} 2^{n-k} \qquad \text{since } i = k - r, \text{ so that } n - k + i = n - r \\
&= \sum_{k=r}^{n} \binom{n}{k}\binom{k}{r} 2^{n-k} \qquad \text{since } \binom{k}{r} = 0 \text{ when } k < r.
\end{aligned}$$

Another algebraic proof. Starting from the RHS, recall from (some) class, $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$.

Then

$$\begin{aligned}
\sum_{k=r}^{n}\binom{n}{k}\binom{k}{r}2^{n-k} &= \sum_{k=r}^{n}\binom{n}{r}\binom{n-r}{k-r}2^{n-k} \\
&= \binom{n}{r}\sum_{k=r}^{n}\binom{n-r}{k-r}2^{n-k} \text{ since } n, r \text{ are constants} \\
&= \binom{n}{r}\sum_{i=0}^{n-r}\binom{n-r}{i}2^{n-i} \text{ using } i = k-r \\
&= \binom{n}{r}3^{n-r} \text{ by question 3.}
\end{aligned}$$