MATH 239 Spring 2012: Assignment 10
Solutions

1. {7 marks} Let G be a simple planar graph with at least 4 vertices where every vertex has degree at least 3.

   (a) Prove that G has at least 4 vertices of degree at most 5. (For {Extra credit: 2 marks}, prove the case without the assumption that "every vertex has degree at least 3.")

   Solution. Suppose G has n vertices, m edges, and at most 3 vertices of degree at most 5. We know that $m \leq 3n-6$. Now we have $n-3$ vertices of degree at least 6, and 3 vertices of degree at least 3. By Handshaking lemma,

$$2m \geq 6(n-3)+3(3) = 6n-9.$$

   This means that

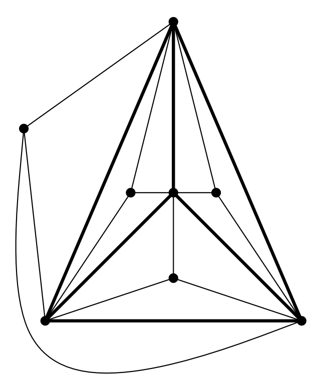
$$m \geq 3n - 4.5.$$

   But this is greater than $3n-6$, which is a contradiction. Hence there are at least 4 vertices of degree at most 5.

   For extra credit: We need to use induction. When all vertices of degree at most 5 have degree at least 3, then we use the argument above. Otherwise, we may assume there is a vertex v of degree 2, 1 or 0. When $deg(v) = 0$, then $G - v$ has at least 4 vertices of degree at most 5 by induction, and they still have degree at most 5 in G. When $deg(v) = 1$, then again, $G-v$ has at least 4 vertices of degree at most 5, but G could have only 3 such vertices as v may be adjacent to one. Then v is another vertex of degree at most 5, giving us 4 such vertices in G. When $deg(v) = 2$, we need to contract an edge e incident with v. Then G/e has at least 4 vertices of degree at most 5, and at least 3 such vertices remain in G. Again, include v to get 4 vertices of degree at most 5.

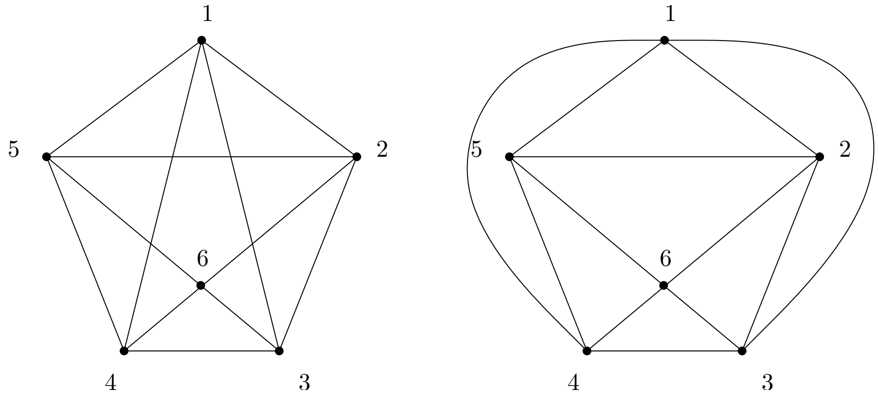
   (b) Give a planar embedding of a graph with 8 vertices which has exactly 4 vertices of degree at most 5.

   Solution. Start with a K_4, add a vertex to each face, and join each vertex to the 3 vertices in that face.

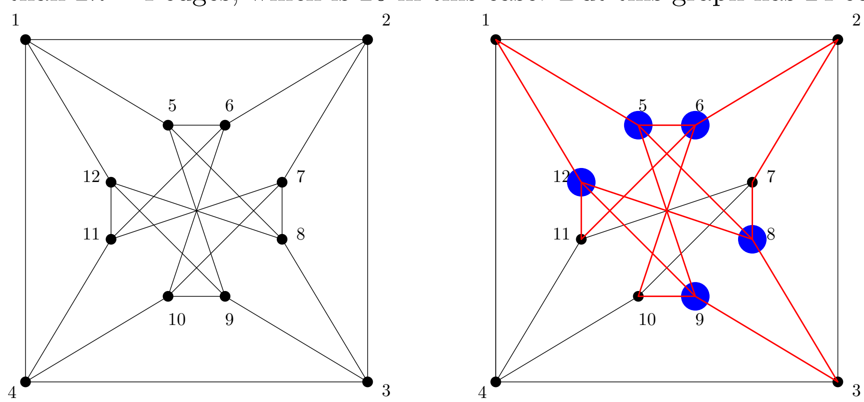


2. {16 marks} For each of the following graphs, determine whether it is planar or not. Prove your assertion. **Solution.**

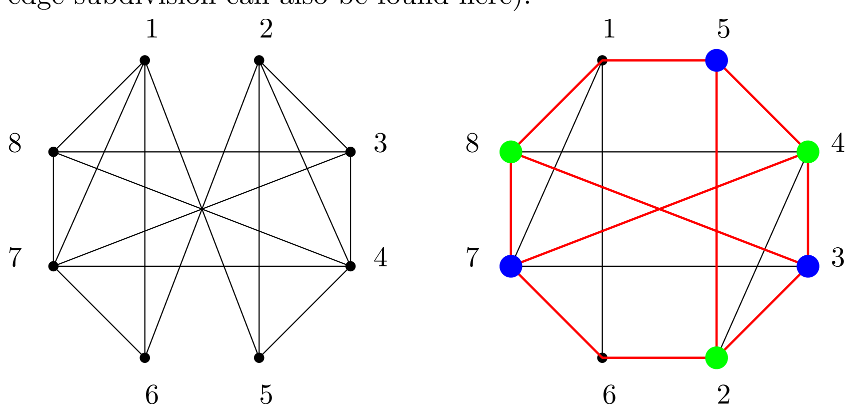
   (a) This graph is planar. We give a planar embedding.



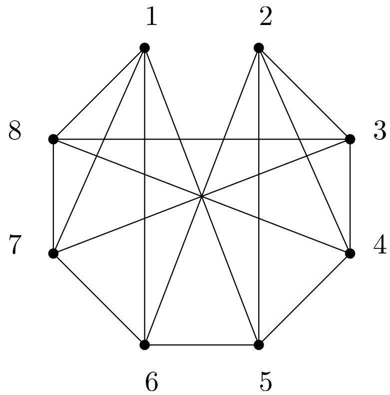
   (b) This graph is not planar. We find an edge subdivision of K_5 here. (Edge subdivisions of $K_{3,3}$ can also be found.) Alternatively, this graph has no cycles of length 3, so it cannot have more than $2n - 4$ edges, which is 20 in this case. But this graph has 24 edges, so it cannot be planar.

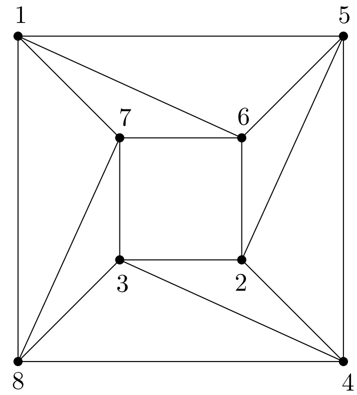


   (c) This graph is not planar. We redraw this graph and find an edge subdivision of $K_{3,3}$ here (a K_5 edge subdivision can also be found here).



(d) This graph is planar. We give a planar embedding.

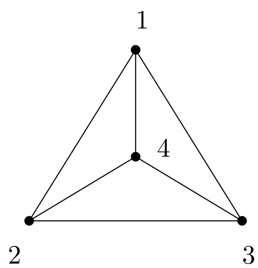




3. {12 marks} For each of the following description of a graph, draw a planar embedding for it and briefly explain why the colouring requirements are met.

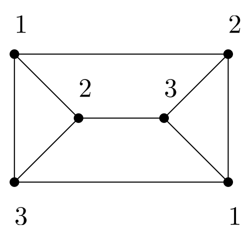
(a) A planar 3-regular graph that is not 3-colourable.

Solution. The K_4 is a 3-regular planar graph, but since it is a complete graph, it is not 3-colourable.



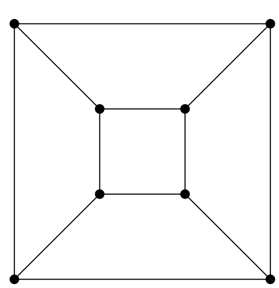
(b) A planar 3-regular graph that is 3-colourable but not 2-colourable.

Solution. The following graph is 3-colourable (as given), but not 2-colourable since it is not bipartite (it contains a triangle).



(c) A planar 3-regular graph that is 2-colourable.

Solution. The cube is 2-colourable since it is bipartite.



4. {10 marks} Let G be a planar graph whose shortest cycle has length at least 6 (if it has any cycle at all).

   (a) Prove that G contains a vertex of degree at most 2.

   Solution. If G is a tree, then certainly it has a vertex of degree at most 2. Otherwise, the degree of each face is at least 6. Suppose G has n vertices, m edges and s faces. Using the handshaking lemma for faces, we have $2m \geq 6s$, or $m \geq 3s$. Using Euler's formula, we have $m \geq 3s = 3(2-n+m)$, so $m \leq \frac{3}{2}n - 3$. If every vertex has degree at least 3, then $m \geq \frac{3}{2}n$, which is a contradiction. Hence there is a vertex of degree at most 2.

   (b) Prove that G is 3-colourable.

   Solution. We prove by induction on the number of vertices n.

   Base case: When $n = 1$, it is just a single vertex, which is 3-colourable.

   Induction hypothesis: Assume that any planar graph whose shortest cycle has length at least 6 is 3-colourable.

   Induction step: Let G be a planar graph on n vertices whose shortest cycle has length at least 6. Let v be a vertex of degree at most 2 (as proved in part (a)). Then $G - v$ is still planar, and since we cannot create new cycles by deleting a vertex, the shortest cycle of $G - v$ still has length at least 6. By induction hypothesis, $G - v$ is 3-colourable. We keep the same colouring for G. Since v has at most 2 neighbours that are coloured, there is one colour that is not used by its neighbours. Assign v with this colour to obtain a 3-colouring of G.

5. {5 marks} Let G be a simple planar graph, and let G^* be the dual of a planar embedding of G. Prove that if G is isomorphic to G^*, then G is not bipartite.

   Solution. Suppose G has n vertices, m edges and s faces. Then G^* has s vertices, m edges and n faces. Since G and G^* are isomorphic, they have the same number of vertices, so $n = s$. Using Euler's formula, $n - m + s = 2n - m = 2$, so $m = 2n - 2$. But any planar bipartite graph has at most $2n - 4$ edges, so G cannot be bipartite.

6. {Extra credit: 5 marks} Let G be a planar graph. Prove that there is a bipartite subgraph of G containing at least $\frac{2}{3}|E(G)|$ edges.

   Solution. Since G is planar, it is 4-colourable. Suppose in a 4-colouring of G, we partition the vertices into 4 sets V_1, V_2, V_3, V_4 where vertices in the same set receive the same colours. Notice that every edge in G must join vertices in different sets. Let E_1 be the set of edges that join V_1 to V_2, and edges that join V_3 to V_4. Let E_2 be the st of edges that join V_1 to V_3, and V_2 to V_4. Let E_3 be the set of edges that join V_1 to V_4, and V_2 to V_3. Notice that E_1, E_2, E_3 partitions the set of edges of G, and moreover, each E_i forms a bipartite subgraph of G (for example, in E_1, we have a bipartition $(V_1 \cup V_3, V_2 \cup V_4)$). Also, any two of these three sets of edges combined would still form a bipartite subgraph of G (for example, in $E_1 \cup E_2$, we have a bipartition $(V_1 \cup V_4, V_2 \cup V_3)$). We discard the set that has the fewest number of edges. The remaining two sets form a bipartite subgraph which must contain at least $\frac{2}{3}|E(G)|$ edges by pigeonhole principle.