1. [10 marks]

Find the generating function with respect to length for the set of nonempty binary strings in which every block of 0's has odd length and the first digit in the string is 1. Express your answer as a rational function.

Consider $\{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$ for all binary strings and restrict it.

Let S be the decompostion of such a string

$$S = \{1\}\{1\}^*(\{0\}\{00\}^*\{1\}\{1\}^*)^*(\{0\}\{00\}^* \cup \{\varepsilon\})$$

Then by the sum and product lemmas

$$\Phi_S(x) = (\Phi_{\{1\}}(x))(\Phi_{\{1\}^*}(x))(\Phi_{(\{0\}\{00\}^*\{1\}\{1\}^*)^*}(x))((\Phi_{\{0\}}(x))(\Phi_{\{00\}^*}(x)) + \Phi_{\{\varepsilon\}}(x))$$

by the star rule

$$= (x)\left(\frac{1}{1-x}\right)\left(\frac{1}{1-(\Phi_{\{0\}\{00\}^*\{1\}\{1\}^*}(x))}\right)\left((x)\left(\frac{1}{1-x^2}\right)+1\right)$$

by product lemma and star rule

$$= \left(\frac{x}{1-x}\right)\left(\frac{1}{1-(x)\left(\frac{1}{1-x^2}\right)(x)\left(\frac{1}{1-x}\right)}\right)\left(\frac{x}{1-x^2}+1\right)$$

$$= \left(\frac{x}{1-x}\right)\left(\frac{1}{1-\frac{x^2}{(1-x)^2(1+x)}}\right)\left(\frac{x+1-x^2}{(1-x)(1+x)}\right)$$

$$= \left(\frac{(x)(1+x-x^2)}{\cancel{(1-x)^2(1+x)}}\right)\left(\frac{\cancel{(1-x)^2(1+x)}}{(1-x)^2(1+x)-x^2}\right)$$

$$= \frac{(x)(1+x-x^2)}{1-x-x^2+x^3-x^2}$$

$$= \frac{x+x^2-x^3}{1-x-2x^2+x^3}$$

2. [12 marks]

(a) [3 marks]
Let $\Phi(x) = \sum_{n\geq 0} a_n x^n$ be a formal power series. Set $b_n = \sum_{i=0}^{n} a_i$. Prove that

$$\sum_{n\geq 0} b_n x^n = \frac{\Phi(x)}{1-x}.$$

$$\sum_{n\geq 0} b_n x^n = \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i\right)x^n = a_0 x^0 + (a_0+a_1)x^1 + (a_0+a_1+a_2)x^2 + \cdots$$

$$= a_0\left(\sum_{n\geq 0} x^n\right) + a_1\left(\sum_{n\geq 1} x^n\right) + a_2\left(\sum_{n\geq 2} x^n\right) + \cdots$$

$$= a_0\left(\sum_{n\geq 0} x^n\right) + a_1 x\left(\sum_{n\geq 0} x^n\right) + a_2 x^2\left(\sum_{n\geq 0} x^n\right) + \cdots$$

$$= \left(\sum_{n\geq 0} x^n\right)\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$$

$$= \left(\sum_{n\geq 0} x^n\right)\left(\sum_{n\geq 0} a_n x^n\right)$$

$$= \left(\frac{1}{1-x}\right)\left(\Phi(x)\right)$$

$$= \frac{\Phi(x)}{1-x}$$

$\square$

(b) [4 marks]
The Fibonacci numbers f_n are given by the recurrence

$$f_0 = 0,$$
$$f_1 = 1,$$
$$f_n = f_{n-1} + f_{n-2}, \quad \text{for all } n \geq 2.$$

Prove that

$$\sum_{n\geq 0} f_n x^n = \frac{x}{1-x-x^2}.$$

Let $g_n := [x^n]\,\frac{x}{1-x-x^2}$

then $\sum_{n\geq 0} g_n x^n = \frac{x}{1-x-x^2}$

$(1-x-x^2)\sum_{n\geq 0} g_n x^n = x$

$\sum_{n\geq 0} g_n x^n - x\sum_{n\geq 0} g_n x^n - x^2\sum_{n\geq 0} g_n x^n = x$

$\sum_{n\geq 0} g_n x^n - \sum_{n\geq 0} g_n x^{n+1} - \sum_{n\geq 0} g_n x^{n+2} = x$

So $g_0 = 0$

$g_1 - g_0 = 1 \Rightarrow g_1 = 1$

for $n\geq 2$

$g_n - g_{n-1} - g_{n-2} = 0$

So $g_n = g_{n-1} + g_{n-2}$

Thus $g_n = f_n$, so

$\sum_{n\geq 0} f_n x^n = \frac{x}{1-x-x^2}$

(c) [2 marks] Let $b_n = \sum_{i=0}^{n} f_i$. Compute

$$\sum_{n\geq 0} b_n x^n$$

as a rational function. You may use the result of parts (a) and (b), even if you did not solve parts (a) or (b).

$$\sum_{n\geq 0} b_n x^n = \frac{\sum_{n\geq 0} f_i x^n}{1-x} \quad \text{(by part (a))}$$

$$= \frac{\left(\frac{x}{1-x-x^2}\right)}{1-x} \quad \text{(by part (b))}$$

$$= \frac{x}{(1-x-x^2)(1-x)}$$

$$= \frac{x}{1-x-x+x^2-x^2+x^3} = \frac{x}{1+x^3}$$

(d) [3 marks] Let $c_n = f_{n+2} - 1$. Compute

$$\sum_{n>0} c_n x^n$$

as a rational function, and show that $b_n = c_n$ for all n, where b_n is as in part (c).

$$\sum_{n\geq 0} c_n x^n = \cancelto{0}{(f_2 - 1)} + (f_3 - 1)x + (f_4 - 1)x^2 + \cdots$$

$$= (f_2 + f_1 - 1)x + (f_3 + f_2 - 1)x^2 + \cdots$$

$$= f_0 + (f_1 + \cancel{(f_1 - 1)} + f_0)x + (f_2 + f_1 + f_0)x^2 + \cdots$$

$$= \sum_{n\geq 0}\left(\sum_{i=0}^{n} f_i\right)x^n$$

$$= \sum_{n\geq 0} b_n x^n$$

Thus $c_n = b_n$

3. [9 marks]

Let t be a positive integer. Prove that

$$\sum_{i=0}^{t} 2^{2i}\binom{2t}{2i} = 1 + \sum_{i=0}^{t-1} 2^{2i+1}\binom{2t}{2i+1}.$$

Hint: consider $(1+y)^n$ for suitable y and n.

Consider $(1+(-2))^{2t} = ((-1)^2)^t = 1^t = 1$

However

$$(1+(-2))^{2t} = \sum_{k=0}^{2t}\binom{2t}{k}(-2)^k \quad \text{(by Binomial Thm)}$$

$$= \sum_{k \text{ even}}^{2t}\binom{2t}{k}(-2)^k + \sum_{k \text{ odd}}^{2t}\binom{2t}{k}(-2)^k$$

$$= \sum_{i=0}^{t}\binom{2t}{2i}2^{2i} - \sum_{i=0}^{t-1}\binom{2t}{2i+1}2^{2i+1}$$

Thus

$$\sum_{i=0}^{t} 2^{2i}\binom{2t}{2i} - \sum_{i=0}^{t-1} 2^{2i+1}\binom{2t}{2i+1} = 1$$

$$\sum_{i=0}^{t} 2^{2i}\binom{2t}{2i} = 1 + \sum_{i=0}^{t-1} 2^{2i+1}\binom{2t}{2i+1}$$

as required.

$\square$

4. [10 marks]

(a) Find a connected 4-regular graph with a bridge, or prove that no such graph exists.

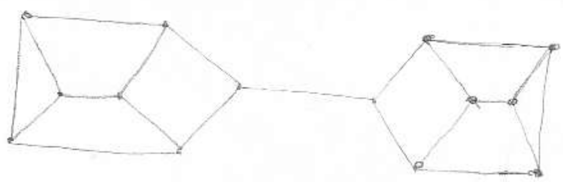
Suppose for contradiction that a 4-regular graph G exists that has a bridge $e = \{u, v\} \in E(G)$

Then consider the component of $G - e$ that contains the vertex u. This is the only vertex in that component with an odd degree (3), so the sum of the degrees of all vertices in that component would be odd, but $\sum_{v \in V(H)} \deg(v) = 2|E(H)|$ for any graph H, thus the component in question cannot be a graph, which is a contradiction. Therefore no such graph exists. $\square$

(b) Find a connected 3-regular graph with a bridge, or prove that ~~no~~ such graph exists.

Such a graph exists;

Example:



5. [8 marks]

Suppose a connected graph G has a cycle C of length n. Prove that in any breadth-first search tree of G, any two vertices in C are at most $\lfloor n/2 \rfloor$ levels apart.

Let the cycle C be denoted by $v_0 e_1 v_1 e_2 \cdots e_{n-1} v_{n-1} e_n v_0$ where each v_i and e_i are distinct (hence length $C = n$)

The length of the shortest path in C between any two vertices is at most $\frac{n}{2}$

The primary property of BFS tells us that any non-tree edge in a graph connects vertices at most 1 level apart in the tree.

Also it is clear that tree edges connect vertices exactly 1 level apart in the tree.

Thus every edge in C must represent either a tree edge or non-tree edge, and the difference in levels between any two vertices in C is at most

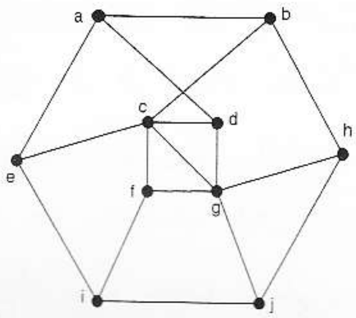
$$1 \cdot \frac{n}{2} = \frac{n}{2}$$

□

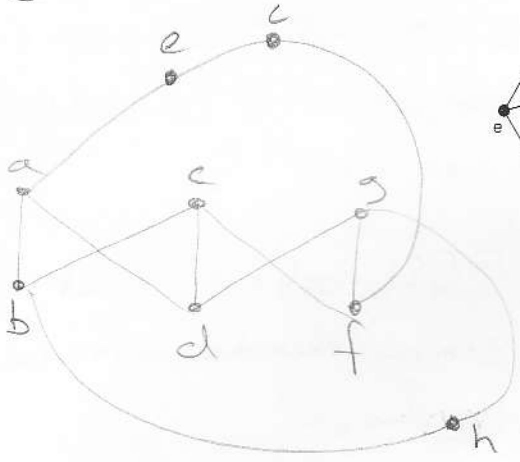
6. [9 marks]

(a) [4 marks]

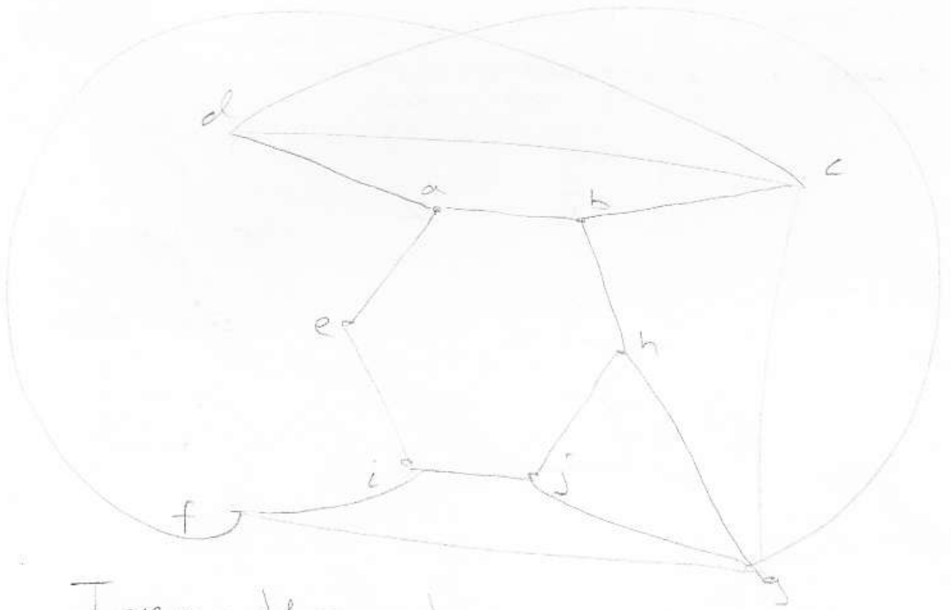
Prove the following graph is nonplanar, or give a planar embedding for it:



$K_{3,3}$ edge subdivision :

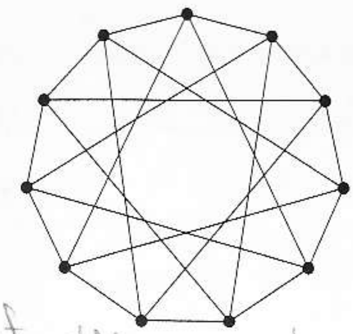


This graph is non-planar since it contains a subgraph which is a $K_{3,3}$ edge subdivision (By Kuratowski's Theorem).



Ignore this; it was just to help better visualize the $K_{3,3}$

(b) [5 marks]
The *Möbius ladder graph* shown below has 11 vertices, 22 edges, and girth 4. Prove that if any three edges are removed, the resulting graph is still nonplanar.



Since the girth of this graph is 4, removing any edges cannot make the girth smaller, only possibly larger.

So, suppose for contradiction that the above graph with 3 edges removed is planar. In that graph, every cycle has length ≥ 4, so every face in a planar embedding has degree ≥ 4. Suppose there are f faces in the graph, denoted $f_1, f_2 \ldots f_f$. Then

$$2e = \sum_{i=1}^{f} \deg(f_i) \geq 4f$$, where e is the number of edges.

Thus $4f \leq 38$ (since the graph has $22-3=19$ edges

However, since the graph is planar, Euler's Formula Applies

$$2 = v - e + f$$
$$2 = 11 - 19 + f$$
$$10 = f$$

Then $40 = 4f \leq 38$ which is a contradiction

Therefore the Möbius ladder graph less any 3 edges is still non-planar.

$\square$

7. [11 marks]

Let G be a connected graph with p vertices, and let T_1 and T_2 be two spanning trees of G. Define the spanning subgraph $H = T_1 \cup T_2$ of G to be the spanning subgraph with $E(H) = E(T_1) \cup E(T_2)$.

(a) [4 marks]
Prove that H has a vertex v with $\deg(v) \leq 3$.

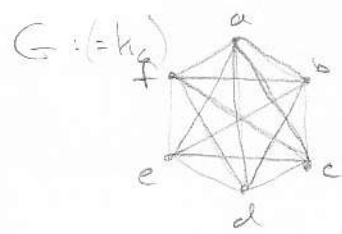
Suppose for contradiction that every vertex in H has deg ≥ 4 (ie deg > 3)

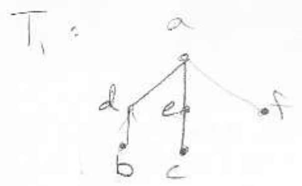
Now $2|E(H)| = \sum_{v \in V(H)} \deg(v) \geq 4p$ so $|E(H)| \geq 2p$

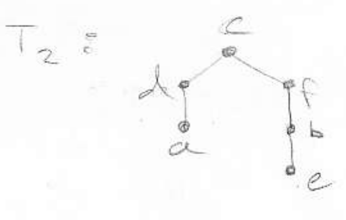
However $|E(H)| \leq |E(T_1)| + |E(T_2)|$
$= (p-1) + (p-1)$ (since T_1 and T_2 are spanning trees)
$= 2p-2$

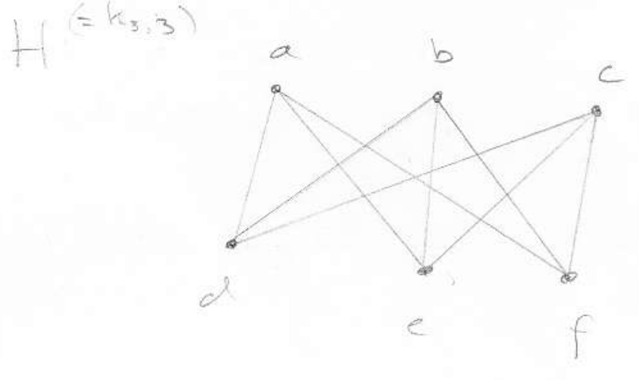
Then $2p \leq |E(H)| \leq 2p-2$
$2p \leq 2p-2$
$0 \leq -2$
which is a contradiction.
Therefore at least one vertex in H must have deg ≤ 3. $\square$

(b) [3 marks]
Give an example of G, T_1 and T_2 in which every vertex of $H = T_1 \cup T_2$ has degree at least 3.









(c) [4 marks] Prove that if $E(T_1) \cap E(T_2) = \emptyset$, and T_1 and T_2 each have a vertex of degree more than $p/2$, then H has a vertex of degree exactly 2.

$\to p = \#$ of vertices

① A tree with p verties has $p-1$ edges. (From a Corallary)

② Since $E(T_1) \cap E(T_2) = \emptyset$ and T_1 and T_2 are spanning trees of G, every vertex in H must have degree ≥ 2

Now, Suppose for contradiction that every vertex in H has degree at least 3.

Then $2|E(H)| = \sum_{v \in V(H)} \deg(v) \geq 3p$ (by ②)

But by ①, $|E(H)| = p-1$

So

$$2(p-1) \geq 3p$$

$$2p-2 \geq 3p$$

$$-2 \geq p$$

Which is a contradiction.

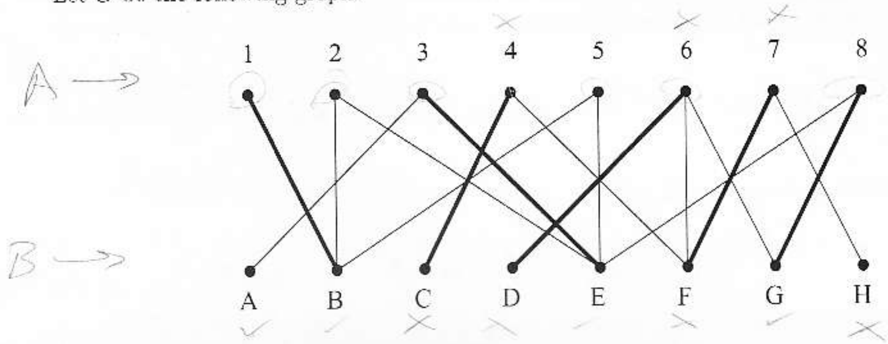
Thus at least one vertex must have degree less than 3, but by ② every vertex must have degree at most 2, thus there exists at least 1 vertex in H with degree exactly 2.

$\square$

8. [13 marks]

B 3 E

Let G be the following graph:



and let $M = \{\{1, B\}, \{3, E\}, \{4, C\}, \{6, D\}, \{7, F\}, \{8, G\}\}$ be the matching indicated in the above figure with bold edges.

(a) [6 marks]

Apply the XY construction to M, and:

- Determine the sets X_0, X, and Y.
- ~~Indicate the order in which vertices are added to X and Y.~~ → Prof Childs agreed this is unclear.
- Determine the set U of unsaturated vertices in Y.
- Indicate the covering C produced by the XY construction satisfying $|M| = |C| - |U|$.

(For students in Section 001: The sets X_0, X, Y were called U_A, R_A, and R_B in class.)

$X_0 = \{2, 5\}$ $Z = \{1, 2, 3, 5, A, B, E\}$

$X = A \cap Z = \{1, 2, 3, 5\}$ $A \setminus X = \{4, 6, 7, 8\}$

$Y = B \cap Z = \{A, B, E\}$ $B \setminus X = \{C, D, F, G, H\}$

$U = \{A\}$

$C = Y \cup A \setminus X = \{4, 6, 7, 8, A, B, E\}$

$|C| = 7$ $|M| = 6$ $|U| = 1$

and $|M| = |C| - |U|$ holds.

(b) [2 marks]
Find an augmenting path for M, or prove that no such path exists.

2 {2, E} E {E, 3} 3 {3, A} A

(c) [3 marks]
Find a maximum matching and a minimum covering for G.

M = { {1, B}, {2, E}, {3, A}, {4, C}, {6, D}, {7, F}, {8, G} }

C = { 3, 4, 6, 7, 8, B, E }

|M| = |C| Therefore M is maximum
and C is minimum
(by König's Thm)

(d) [2 marks] Find a set $D \subset \{1, 2, 3, 4, 5, 6, 7, 8\}$ such that $|D| > |N(D)|$, or prove that no such set exists.

D = {1, 2, 3, 5, 8}

N(D) = {A, B, E, G}

|D| > |N(D)|

or

D = {1, 2, 5}

N(D) = {B, E}

|D| > |N(D)|

9. [8 marks]

Let k be a positive integer and let G be a bipartite graph with vertex classes A and B. Suppose every vertex in A has degree at least k, and every vertex in B has degree at most k. Prove that G has a matching of size $|A|$.

Hall's Thm: A bipartite graph G with bipartition (A,B) has a matching saturating every vertex in A iff for all $D \subseteq A$, $|N(D)| \geq |D|$

Proof: Let $D \subseteq A$.

Then every edge in D has the other end in $N(D)$

so $\sum_{v \in D} \deg(v) \leq \sum_{v \in N(D)} \deg(v)$

and $\sum_{v \in D} \deg(v) \geq k|D|$ since $D \subseteq A$

and $\sum_{v \in N(D)} \deg(v) \leq k|N(D)|$ since $N(D) \subseteq B$

Thus

$$k|D| \leq \sum_{v \in D} \deg(v) \leq \sum_{v \in N(D)} \deg(v) \leq k|N(D)|$$

and

$$k|D| \leq k|N(D)|$$
$$|D| \leq |N(D)| \quad (\text{since } k > 0)$$

Thus by Hall's Thm, G has a matching the size of $|A|$ as required. $\square$