1. (a) **[3 marks]** Write each of the following rational functions $f(x)$ in form of a formal power series $f(x) = \sum_{i\geq 0} a_i x^i$.

$$\frac{1}{(1-2x)^3}, \qquad \frac{x^3}{(1+4x^2)^5}.$$

Solution.

$$\frac{1}{(1-2x)^3} = \sum_{n\geq 0}\binom{n+2}{n}(2x)^n = \sum_{n\geq 0}\binom{n+2}{n}2^n x^n = \sum_{n\geq 0}\binom{n+2}{2}2^n x^n.$$

$$\frac{x^3}{(1+4x^2)^5} = x^3\sum_{n\geq 0}\binom{n+4}{n}(-4x^2)^n = \sum_{n\geq 0}\binom{n+4}{n}(-4)^n x^{2n+3} = \sum_{n\geq 0}\binom{n+4}{4}(-4)^n x^{2n+3}.$$

(b) **[1 mark]** Determine the coefficient $[x^5]\frac{x^2}{(1+x)^7}$.

Solution.

$$[x^5]\frac{x^2}{(1+x)^7} = [x^5]x^2\sum_{i\geq 0}\binom{i+6}{i}(-x)^i = [x^3]\sum_{i\geq 0}\binom{i+6}{i}(-1)^i x^i = -\binom{9}{3}.$$

(c) **[3 marks]** Write the following formal power series in closed form:

$$\sum_{i\geq 0}\left(-\frac{1}{2}\right)^i x^{3i}, \qquad \sum_{i\geq 0}\binom{i+2}{i}5^i x^i.$$

Solution.

$$\sum_{i\geq 0}\left(-\frac{1}{2}\right)^i x^{3i} = \sum_{i\geq 0}\left(\frac{-x^3}{2}\right)^i = \frac{1}{1+\frac{x^3}{2}} = \frac{2}{2+x^3}.$$

$$\sum_{i\geq 0}\binom{i+2}{i}5^i x^i = \sum_{i\geq 0}\binom{i+2}{i}(5x)^i = \frac{1}{(1-5x)^3}.$$

2. (a) [**2 marks**] Use the Binomial Theorem to prove that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

Solution.
By the Binomial Theorem,
$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$
Letting $x = 1$, we get
$$2^n = \sum_{i=0}^{n} \binom{n}{i}.$$

(b) [**3 marks**] Prove that
$$\sum_{r=0}^{n} \sum_{s=0}^{r} \binom{n}{r} \binom{r}{s} = 3^n.$$
(Hint: Use the Binomial Theorem to expand $(1+2)^n$.)

Solution.
By part (a) and the Binomial Theorem,
$$3^n = (1+2)^n = \sum_{r=0}^{n} \binom{n}{r} 2^r = \sum_{r=0}^{n} \binom{n}{r} \left(\sum_{s=0}^{r} \binom{r}{s} \right) = \sum_{r=0}^{n} \sum_{s=0}^{r} \binom{n}{r} \binom{r}{s}$$
as required.

3. An even composition of n is a composition of n into an even number of parts such that each part is a positive even number. For example, the even compositions of 8 are $(2,6),(4,4),(6,2),(2,2,2,2)$.

   (a) [**4 marks**] Determine the generating function for the number of even compositions of n having $2k$ parts. Use this generating function to determine the number of $2k$-part even compositions of n.
   Solution. Let $E=\{2,4,6,\ldots\}$. By definition,
$$\Phi_E(x)=x^2+x^4+x^6+\cdots=\frac{x^2}{1-x^2}.$$
   The set of all even compositions with $2k$ parts is $S=E^{2k}$ (the Cartesian product of E with itself $2k$ times). By the Product Lemma,
$$\Phi_S(x)=(\Phi_E(x))^{2k}=\left(\frac{x^2}{1-x^2}\right)^{2k}=\frac{x^{4k}}{(1-x^2)^{2k}}.$$
   The number of such even compositions of n is
$$[x^n]\Phi_S(x)=[x^n]\frac{x^{4k}}{(1-x^2)^{2k}}=[x^{n-4k}]\sum_{i\geq 0}\binom{2k+i-1}{i}x^{2i}=\binom{\frac{n}{2}-1}{\frac{n}{2}-2k}.$$
   (Remark: this binomial coefficient is zero unless n is even and $n\geq 4k$.)

   (b) [**3 marks**] Determine the generating function for the the number of all even compositions of n.
   Solution. Using part (a), the set of all even compositions is
$$T=E^0\cup E^2\cup E^4\cup E^6\cup\cdots.$$
   By the Sum and Product Lemmas,
$$\begin{aligned}\Phi_T(x) &= (\Phi_E(x))^0\cup(\Phi_E(x))^2\cup(\Phi_E(x))^4\cup(\Phi_E(x))^6\cup\cdots\\ &= \left(\frac{x^2}{1-x^2}\right)^0+\left(\frac{x^2}{1-x^2}\right)^2+\left(\frac{x^2}{1-x^2}\right)^4+\left(\frac{x^2}{1-x^2}\right)^6+\cdots\\ &= \frac{1}{1-\left(\frac{x^2}{1-x^2}\right)^2}=\frac{1-2x^2+x^4}{1-2x^2}=1+\frac{x^4}{1-2x^2}.\end{aligned}$$

   (c) [**3 marks**] Use your answer in part (b) to determine a_n, the number of even compositions of n.
   Solution.
$$\begin{aligned}a_n &= [x^n]\Phi_T(x)\\ &= [x^n]1+x^4\sum_{i\geq 0}2^ix^{2i}\\ &= [x^n]1+\sum_{i\geq 0}2^ix^{2i+4}\\ &= \begin{cases}1 & \text{if } \quad n=0,\\ 2^{\frac{n-4}{2}} & \text{if } n \text{ is even and } \quad n>2,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

4. For each of the following sets, write down a decomposition that uniquely creates the elements of that set.

   (a) [**2 marks**] The $\{0,1\}$-strings that have no blocks of 1s with length 3, and no substrings of 0s of length 2.
   Solution.
   The decomposition for these strings is

$$(\{1\}^* \setminus \{111\}) \quad (\{0\}(\{1\}\{1\}^* \setminus \{111\}))^* \quad \{\varepsilon, 0\}.$$

   (b) [**2 marks**] The set of $\{0,1\}$-strings with no occurrence of the substring 010.
   Solution.
   The decomposition for this set of strings is

$$\{1\}^* \quad (\{0\}\{0\}^*\{11\}\{1\}^*)^* \quad (\{0\}^* \cup \{0\}\{0\}^*\{1\}).$$

5. (a) **[3 marks]** Find the generating function $\Phi_{S_1}(x)$ for the following set of strings:

$$S_1 = \{0\}^* \left(\{1\}\{111\}^*\{00\}\{000\}^*\right)^* \{1\}^*.$$

Solution.
By the Product Lemma, the generating function for S_1 is

$$\begin{aligned}\Phi_{S_1}(x) &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-\left(\frac{x}{1-x^3}\right)\left(\frac{x^2}{1-x^3}\right)}\right)\left(\frac{1}{1-x}\right)\\ &= \frac{(1+x+x^2)^2}{1-3x^3+x^6}.\end{aligned}$$

(b) **[4 marks]** Find the generating function $\Phi_{S_2}(x)$ for the following set of strings:

$$S_2 = \bigcup_{k=0}^{\infty} \{0\}^k \left(\{1\}\{111\}^*\{00\}\{000\}^*\right)^* \{1\}^k.$$

Solution.
Using the Product and Sum Lemmas, we get that

$$\begin{aligned}\Phi_{S_2}(x) &= \sum_{k=0}^{\infty} x^k \left(\frac{1}{1-\left(\frac{x}{1-x^3}\right)\left(\frac{x^2}{1-x^3}\right)}\right) x^k\\ &= \frac{(1-x^3)^2}{1-3x^3+x^6}\sum_{k=0}^{\infty} x^{2k}\\ &= \frac{(1-x^3)^2}{1-3x^3+x^6}\left(\frac{1}{1-x^2}\right) = \frac{1+x+x^2-x^3-x^4-x^5}{1+x-3x^3-3x^4+x^6+x^7}.\end{aligned}$$

(c) **[1 mark]** Find a string that is in S_1 but not in S_2.
Solution.
The string 1 is in S_1 but is not in S_2.

(d) **[1 mark]** Let $a_n = [x^n]\Phi_{S_1}(x)$ and $b_n = [x^n]\Phi_{S_2}(x)$. Without evaluating a_n or b_n, show that $a_n \geq b_n$ for all $n \geq 0$.
Solution.
Clearly every string of S_2 is also in S_1. Therefore $a_n \geq b_n$.

6. Let A be the set of binary strings a such that in each substring of three consecutive positions in a, there is at least one 1 and at least one 0. Let a_n be the number of strings of length n in A.

   (a) [**2 marks**] For $n = 1, 2, 3, 4$, write down all the strings of length n in A.
   Solution.

   $$\begin{array}{ll} n=1 & 0,1 \\ n=2 & 00,01,10,11 \\ n=3 & 001,010,100,011,101,110 \\ n=4 & 0010,0011,0100,0101,0110,1001,1010,1011,1100,1101 \end{array}$$

   (b) [**3 marks**] Prove that the generating function for a_n is given by

   $$\Phi_A(x) = \frac{1+2x+3x^2+2x^3+x^4}{1-x^2-2x^3-x^4}.$$

   Solution.
   A decomposition for the strings of A is

   $$A = \{\varepsilon, 1, 11\}\{01, 011, 001, 0011\}^*\{\varepsilon, 0, 00\}.$$

   By the Sum and Product Lemmas, the generating function for A is

   $$\begin{aligned} \Phi_A(x) &= (1+x+x^2)\left(\frac{1}{1-x^2-2x^3-x^4}\right)(1+x+x^2) \\ &= \frac{1+2x+3x^2+2x^3+x^4}{1-x^2-2x^3-x^4} \end{aligned}$$

   as required.

   (c) [**2 marks**] From part (b), find a recurrence relation for the numbers a_n, together with an appropriate set of initial conditions.
   Solution.
   By Theorem 1.26, a linear homogeneous recurrence relation for a_n's is

   $$a_n - a_{n-2} - 2a_{n-3} - a_{n-4} = 0 \quad \forall n \geq 5.$$

   From part (a), the initial conditions that uniquely determine the sequence $\{a_n\}$ are

   $$a_0 = 1, \quad a_1 = 2, \quad a_2 = 4, \quad a_3 = 6, \quad a_4 = 10.$$

7. (a) **[4 marks]** Solve the linear, homogeneous recurrence equation

$$a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0 \quad \forall \quad n \geq 3$$

with initial conditions $a_0 = 4,\ a_1 = 8,\ a_2 = 2$.

Solution.

The characteristic polynomial of this recurrence equation is

$$x^3 + x^2 - 8x - 12 = (x+2)^2(x-3).$$

By Theorem 1.42, the general solution of this recurrence equation is

$$a_n = (A + Bn)(-2)^n + C3^n.$$

Setting n equal to 0, 1, 2, we get the following 3 linear equations in A, B, C:

$$\begin{aligned} A + C &= 4, \\ -2A - 2B + 3C &= 8, \\ 4A + 8B + 9C &= 2. \end{aligned}$$

The solution to this system of equations is $(A, B, C) = (2, -3, 2)$. Therefore, the solution for a_n is

$$a_n = (2 - 3n)(-2)^n + 2(3^n) \quad \forall n \geq 0.$$

(b) **[1 marks]** What is the asymptotic value of a_n?

Solution.

From part (a) we get that the asymptotic value of a_n is $a_n \sim 2(3^n)$.

(c) **[3 marks]** Find a particular solution of the nonhomogeneous linear recurrence equation

$$a_n - a_{n-1} - 5a_{n-2} - 3a_{n-3} = \frac{15}{8}(-2)^n \quad \forall \quad n \geq 3.$$

Solution.

Substitute $a_n = \alpha(-2)^n$ in this recurrence equation and solve for α.

$$\begin{aligned} \frac{15}{8}(-2)^n &= \alpha(-2)^n - \alpha(-2)^{n-1} - 5\alpha(-2)^{n-2} - 3\alpha(-2)^{n-3} \\ &= \alpha(-2)^{n-3}(-8 - 4 + 10 - 3) \\ &= -5\alpha(-2)^{n-3} = \frac{5}{8}\alpha(-2)^n. \end{aligned}$$

Therefore, $\alpha = 3$, and we obtain the particular solution $a_n = 3(-2)^n$.

8. (a) **[2 marks]** Let G be a 3-regular graph with 10 vertices. (This means every vertex has degree 3.) How many edges does G have? Justify your answer.

Solution.

Let the vertices of G be $v_1, v_2, \ldots, v_{10}$ and let the number of edges be q. By the Handshake Theorem,

$$2q = \sum_{i=1}^{10} \deg(v_i) = \sum_{i=1}^{10} 3 = 30.$$

Therefore, the number of edges is $q = 15$.

(b) **[3 marks]** Let G be a connected graph with 27 edges and exactly one cycle. How many vertices does G have? Justify your answer.

Solution.

Let e be any edge of the cycle in G. Then $G - e$ is connected (because e is not a bridge) and has no cycle (removing edge e destroys the unique cycle in G). Therefore, $G - e$ is a tree with 26 edges. Therefore, by Theorem 2.29, $G - e$ has 27 vertices, as does G.

(c) **[3 marks]** Is there a bipartite graph with 9 vertices and 23 edges? Explain.

Solution.

Let G be a bipartite graph with bipartition A, B, where A and B have a and b vertices respectively. If every vertex of A is adjacent to every vertex of B, then G has ab edges. Clearly, no bipartite graph with bipartition A, B can have more than ab edges.

If G is a bipartite graph with 9 vertices, then (a, b) must be one of $(0, 9), (1, 8), (2, 7), (3, 6)$ or $(4, 5)$. In each case, $ab \leq 20$. Therefore, no bipartite graph on 9 vertices has 23 edges.

9. Let Q_n denote the n-cube graph: the vertex set of Q_n consists of all binary sequences (i.e. binary strings) of length n and the edge set of Q_n consists of all pairs of binary sequences which differ in exactly one position.

   (a) [**3 marks**] Determine the number of vertices and edges in Q_n.

   Solution.

   Since there are 2^n binary strings of length n, Q_n has 2^n vertices.

   Since each binary string has n positions, it is adjacent to n binary strings; that is, each vertex has degree n. By the Handshake Theorem,

$$2q = \sum_{v \in V(Q_n)} \deg(v) = \sum_{v \in V(Q_n)} n = n(2^n).$$

   Therefore, the number of edges in Q_n is $q = n(2^{n-1})$.

   (b) [**2 marks**] Show that Q_n is bipartite.

   Solution.

   Let E be the set of all binary strings that have an even number of 1's, and let O be the set that have an odd number of ones. Let $S = s_1 \dots s_n$ and $T = t_1 \dots t_n$ be adjacent binary strings. They must be identical in $n-1$ positions and different in 1 position, say position i. Without loss of generality, we may assume the $s_i = 1$ and $t_i = 0$. Since S has one more 1 than T has, one of S and T is in E and the other is in O. Since every edge joins a vertex in E to a vertex in O, Q_n is a bipartite graph with bipartition E, O.

   (c) [**3 marks**] Prove that Q_n has no bridges.

   Solution.

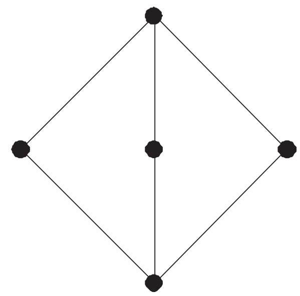
   For $n = 1$, Q_n is the complete graph on 2 vertices, which has precisely one edge. This one edge is a bridge. We now prove that for $n \geq 2$, Q_n has no bridges.

   Let S and T be any pair of adjacent vertices as described in part (b). Let $U = u_1 \dots u_n$ be another vertex adjacent to T. Assume U differs from T in position $j \neq i$. Then $u_j = 1 - t_j = 1 - s_j$ and $u_i = t_i = 0$. Let $V = v_1 \dots v_n$ be the string that has

$$v_i = s_i = 1, \quad v_j = u_j = 1 - t_j = 1 - s_j$$

   and is the same as S, T and U in all other positions. Vertex V is adjacent to both U and S. Hence S, T, U, V, S is a cycle of length 4 in Q_n. Since edge $\{S, T\}$ is in a cycle of length 4, it is not a bridge.

(d) **[4 marks]** (Continued from page 10.) Prove that that the graph Q_n does not contain a subgraph isomorphic to the graph drawn below:



Solution.
Let the top vertex be $S = s_1 \ldots s_n$, let the middle 3 vertices be $A = a_1 \ldots a_n$, $B = b - 1 \ldots b_n$ and $C = c_1 \ldots c_n$, respectively, and let the bottom vertex be $W = w_1 \ldots w_n$. Let S differ from vertex A in position i and be the same as A in all other positions. Similarly, let S differ from vertices B and C in positions j and k respectively. Then $(a_i, a_j, a_k) = (1 - s_i, s_j, s_k)$, $(b_i, b_j, b_k) = (s_i, 1 - s_j, s_k)$ and $(c_i, c_j, c_k) = (s_i, s_j, 1 - s_k)$. Since vertex W is adjacent to both A and B, and $W \neq S$, $(w_i, w_j, w_k) = (1-s_i, 1-s_j, s_k)$ and is identical to vertices S, A, B, C in all other positions. (Note that this solution for W is uniquely determined by S, A and B.) Now W differs from C in each of positions i, j, k. Hence, W is not adjacent to vertex C. Therefore, Q_n does not contain a subgraph isomorphic to the above graph.