MATH239 Tutorial 2

1. (a) Determine $[x^n]x(1+2x)^{-2}$.

   Solution. Use Theorem 1.6.5 to determine $(1+2x)^{-2}$. The theorem states:

   $$(1-y)^{-k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} y^n$$

   We set $y=-2x$ and $k=2$ to get:

   $$(1+2x)^{-2} = \sum_{n\geq 0} \binom{n+2-1}{2-1}(-2x)^n = \sum_{n\geq 0}(n+1)(-2)^n x^n$$

   So $x(1+2x)^{-2} = \sum_{n\geq 0}(n+1)(-2)^n x^{n+1}$ and thus $[x^n]x(1+2x)^{-2} = n(-2)^{n-1}$.

   (b) Determine $[x^n]\frac{1}{(1+x)(1-x)}$.

   Solution.

   $$\begin{aligned}
   \frac{1}{(1+x)(1-x)} &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1+x}\right) \\
   &= \left(\sum_{i\geq 0} x^i\right)\left(\sum_{j\geq 0}(-1)^j x^j\right) \\
   &= \sum_{i\geq 0}\sum_{j\geq 0}(-1)^j x^{i+j} \\
   &= \sum_{n\geq 0}\left(\sum_{j=0}^{n}(-1)^j\right) x^n, \text{ substituting } i=n-j.
   \end{aligned}$$

   So we have

   $$\begin{aligned}
   [x^n]\frac{1}{(1-x)(1+x)} &= \sum_{j=0}^{n}(-1)^j \\
   &= \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}
   \end{aligned}$$

2. Find the inverse of $Q(x) = 1+2x+3x^2+\cdots = \sum_{n\geq 0}(n+1)x^n$.

   Solution. We want to find a solution $Q(x)$ to $Q(x)A(x)=1$. We will write $q_i = [x^i]Q(x)$; then we have $q_i = i+1$.

   Note that $q_0 = 1$, so we can apply Theorem 1.5.2 which states that the unique solution to $Q(x)A(x) = P(x)$ for given P and Q is $a_n = p_n - q_1 a_{n-1} - q_2 a_{n-2} - \cdots - q_n a_0$. For us, $p_0 = 1$ and $p_n = 0$ for $n \geq 1$. We get $a_0 = p_0 = 1$. Next we get $a_1 = p_1 - q_1 a_0 = 0 - 2\cdot 1 = -2$, and then $a_2 = p_2 - q_1 a_1 - q_2 a_0 = 0+4-3 = 1$.

We will now use induction to show that $a_n = 0$ for $n \geq 3$. Using the induction hypothesis, we get that

$$\begin{aligned} a_n &= p_n - q_1 a_{n-1} - q_2 a_{n-2} - \cdots - q_n a_0 \\ &= 0 - 0 - \cdots - q_{n-2} - q_{n-1}(-2) - q_n \\ &= -(n-1) + 2n - (n+1) = 0. \end{aligned}$$

So $A(x) = 1 - 2x + x^2 = (x-1)^2$ is the inverse of $Q(x)$.

As an exercise, we check that the inverse of $(x-1)^{-2}$ is indeed $Q(x)$. By using the formula for the inverse again, we get that $(x-1)^{-1} = \sum\limits_{n\geq 0} x^n$. Using the product formula, we get that

$$(x-1)^{-2} = \left(\sum_{n\geq 0} x^n\right)^2 = \sum_{n\geq 0}\left(\sum_{i=0}^{n} 1\cdot 1\right)x^n = \sum_{n\geq 0}(n+1)x^n = Q(x)$$

3. Let $A(x) = \frac{1}{1-x}$, $B(x) = \sum\limits_{i\geq 0} ix^i$. State whether the following are well-defined formal power series:

   (a) $\frac{A(x)}{B(x)}$

   (b) $A(B(x))$

   Solution

   (a) Another way to write $B(x)$ is $B(x) = 0x + x + 2x^2 + \cdots$, i.e. the constant term of $B(x)$ is zero. By Thm 1.5.7, $B(x)$ has no inverse. So $\frac{A(x)}{B(x)}$ is not a formal power series.

   (b) Again, the constant term of $B(x)$ is zero. Then Thm 1.5.9 tells us $A(B(x))$ is a formal power series.

4. Let S be the set of all subsets of $\mathbb{N}$. Let the weight w of a set $A \in S$ be the largest element in A. (Ex. If $A = \{1, 4, 5, 9\}$ then $w(A) = 9$.) We define $w(\emptyset) = 0$.

   (a) Determine the number of elements of S with weight k.

   (b) Find $\Phi_S(x)$, and explain why it is a formal power series.

   Solution

   (a) Let A be a subset of $\mathbb{N}$ with weight k. If $k = 0$, then we must have $A = \emptyset$, i.e. there is only one choice for A. Otherwise, $A = \{k\} \cup B$, where B is some subset of $\{1, 2, ..., k-1\}$. Since there are 2^{k-1} choices for B, there are also 2^{k-1} choices for A. So there are 2^{k-1} subsets of $\mathbb{N}$ that have weight $k > 0$.

   (b) Applying the definition of the generating function and using the work from part (a), we get

   $$\Phi_S(x) = 1 + \sum_{k\geq 1} 2^{k-1}x^k.$$

   Since every coefficient $[x^k]\Phi_S(x)$ is finite and rational, $\Phi_S(x)$ is a formal power series.

5. (a) Let S be the set of all subsets of $\{1, 2, ..., n\}$. Let w be the weight function on S such that for each $A \in S$, $w(A)$ is the number of elements of A. Determine the generating series $\Phi_S(x)$ with respect to w.

   (b) Let w' be the weight function on S such that for each $A \in S$, $w'(A)$ is twice the number of elements of A. Determine the generating series $\Phi'_S(x)$ with respect to w'. (Note that here, $\Phi'_S(x)$ is not the derivative of $\Phi_S(x)$.)

(c) What is the relationship between $\Phi_S(x)$ and $\Phi'_S(x)$?

Solution

(a) First, note that the number of subsets of $\{1, 2, ..., n\}$ having weight k is $\binom{n}{k}$. Also, notice that $\binom{n}{k} = 0$ for $k > n$. Then the definition of a generating function gives

$$\Phi_S(x) = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

By the binomial theorem,

$$\Phi_S(x) = (1 + x)^n.$$

(b) With the new weight function, the number of subsets with weight $2k$ is $\binom{n}{k}$, and there are no subsets with odd weight. Then we have

$$\begin{aligned}\Phi'_S(x) &= \sum_{i \geq 0} \binom{n}{k} x^{2k} \\ &= \sum_{i=0}^{n} \binom{n}{k} x^{2k} \\ &= (1 + x^2)^n.\end{aligned}$$

(c) Comparing the expressions we found above, it's easy to see that $\Phi'_S(x) = \Phi_S(x^2)$.