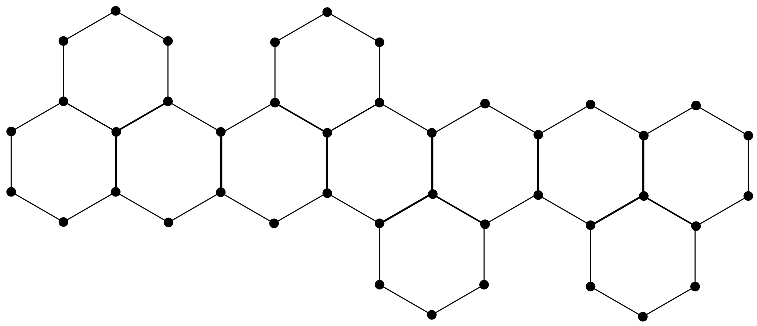
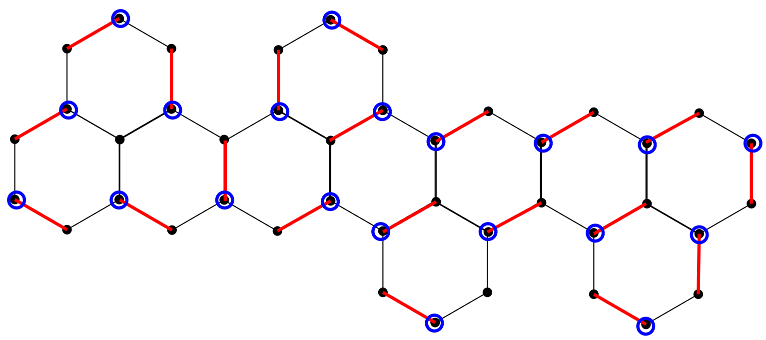
Note: Do not hand in this assignment. Solutions will be posted on Monday April 8th.

Exercise 1. Find a maximum matching M and a minimum cover C in the following graph (using whatever mean you wish). Then find a simple argument proving that M is indeed a maximum matching and that C is indeed a minimum cover.

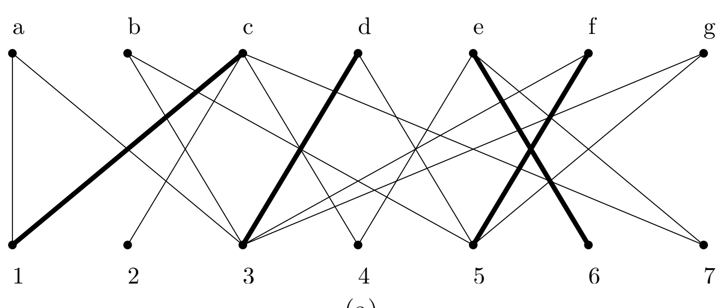


SOLUTION. The following shows a matching M of size 20 and a cover C of size 20.



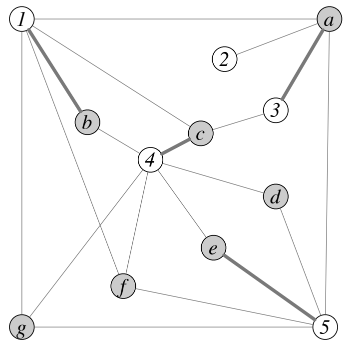
Since the cardinality of any matching is always smaller than the cardinality of any cover, the cover C implies that no matching has cardinality greater than 20. As $|M| = 20$, M is a maximum matching. Similarly, the matching M implies that no cover has cardinality smaller than 20. As $C = 20$, C is a minimum cover.

Exercise 2. Consider the following bipartite graphs,



(a)

and



(b)

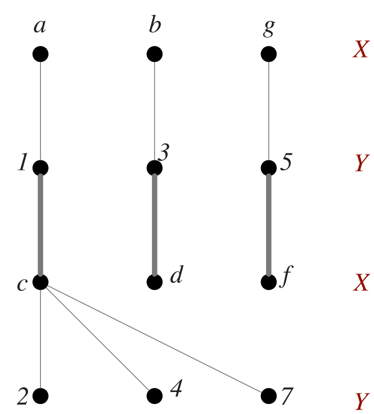
For (a) the bipartition is given by $A = \{a, b, c, d, e, f, g\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$. For (b) the bipartition is given by $A = \{a, b, c, d, e, f, g\}$ and $B = \{1, 2, 3, 4, 5\}$.

Run the maximum matching algorithm starting from the matching indicated by the bold edges for both (a) and (b). At each iteration,

(i) determine the sets $X \subseteq A$ and $Y \subseteq B$, and
(ii) either,
- find a larger matching, or
- prove the matching is maximum by exhibiting a vertex cover.

Explain each iteration of the procedure. In particular, explain how you find a larger matching, explain how you find a vertex cover.

SOLUTION (a). Currently our matching is $M = \{c1, d3, e6, f5\}$.
Iteration 1. $\{a, b, g\}$ are unsaturated vertices in A, thus are contained in X. To find the sets X and Y we construct the following set of trees representing alternating paths from $\{a, b, g\}$ to each vertex in X and Y [1].

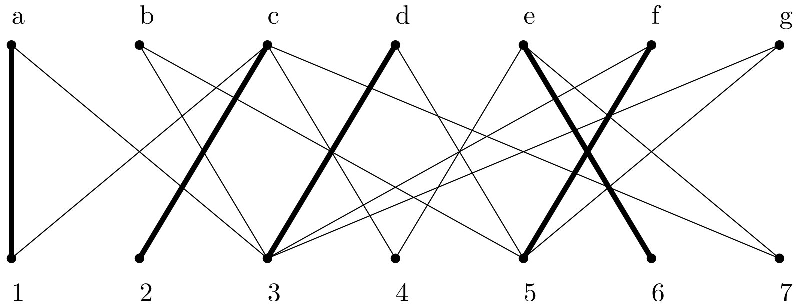


It follows that $X = \{a, b, g, c, d, f\}$, $Y = \{1, 3, 5, 2, 4, 7\}$. Observe that 2 is an unsaturated vertex of Y. We can see in the previous figure that $P = a1, 1c, c2$ is an augmenting path. We define our new matching as,

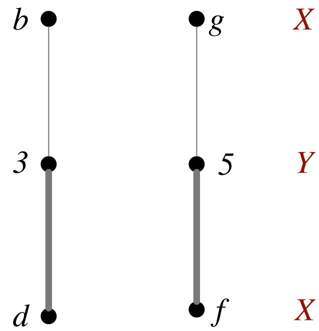
$$M \triangle P = \{c1, d3, e6, f5\} \triangle \{a1, c1, c2\} = \{d3, e6, f5, a1, c2\}.$$

We represent our new matching in the following figure,

[1] Note, that there is more than one possible set of trees.



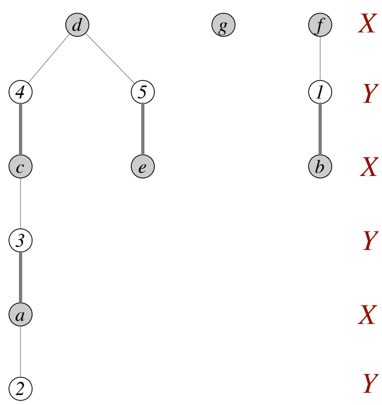
Iteration 2. $\{b, g\}$ are unsaturated vertices in A, thus are contained in X. To find the sets X and Y we construct the following set of trees representing alternating paths from $\{b, g\}$ to each vertex in X and Y.



It follows that $X = \{b, g, d, f\}$, $Y = \{3, 5\}$. Observe that all vertices in Y are saturated, thus our matching must be maximum. We prove that this is indeed the case by constructing a minimum cover C of the same size, we set

$$C = Y \cup (A \setminus X) = \{3, 5, a, c, e\}.$$

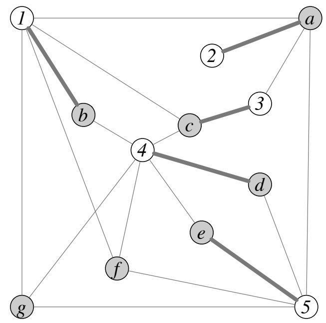
SOLUTION (b). Currently our matching is $M = \{a3, b1, c4, 5e\}$.
Iteration 1. $\{d, g, f\}$ are unsaturated vertices in A, thus are contained in X. To find the sets X and Y we construct the following set of trees representing alternating paths from $\{a, b, g\}$ to each vertex in X and Y.



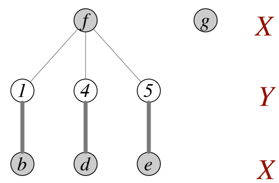
It follows that $X = A$ and $Y = B$. Observe that 2 is an unsaturated vertex of Y. We can see in the previous figure that $P = d4, 4c, c3, 3a, a2$ is an augmenting path. We define our new matching as,

$$M \triangle P = \{a3, b1, c4, 5e\} \triangle \{d4, 4c, c3, 3a, a2\} = \{b1, e5, d4, c3, a2\}.$$

We represent our new matching in the following figure,



<u>Iteration 2.</u> $\{f, g\}$ are unsaturated vertices in A, thus are contained in X. To find the sets X and Y we construct the following set of trees representing alternating paths from $\{f, g\}$ to each vertex in X and Y



It follows that $X = \{f, g, b, d, e\}$ and $Y = \{1, 4, 5\}$. Observe that all vertices in Y are saturated, thus our matching must be maximum. We prove that this is indeed the case by constructing a minimum cover C of the same size, we set

$$C = Y \cup (A \setminus X) = \{a, c, 1, 4, 5\}.$$