MATH 239 Spring 2012: Assignment 6
Solutions

1. {8 marks} In assignment 3, you have discovered that if b_n is the total number of parts over all compositions of n, then b_n satisfies the recurrence

$$b_n = 2b_{n-1} + 2^{n-2}$$

with initial condition $b_1 = 1$. Determine an explicit formula for b_n when $n \geq 1$.

Solution. The characteristic polynomial is $x - 2$, so it has one root 2. So the homogeneous part of the recurrence has the form $A \cdot 2^n$.

To find a specific solution to the nonhomogeneous recurrence, we cannot use $c_n = \alpha 2^n$. So we try $c_n = \alpha n 2^n$. Then

$$c_n - 2c_{n-1} = \alpha(n2^n - 2(n-1)2^{n-1}) = \alpha(n2^n(n-1)2^n) = \alpha 2^n.$$

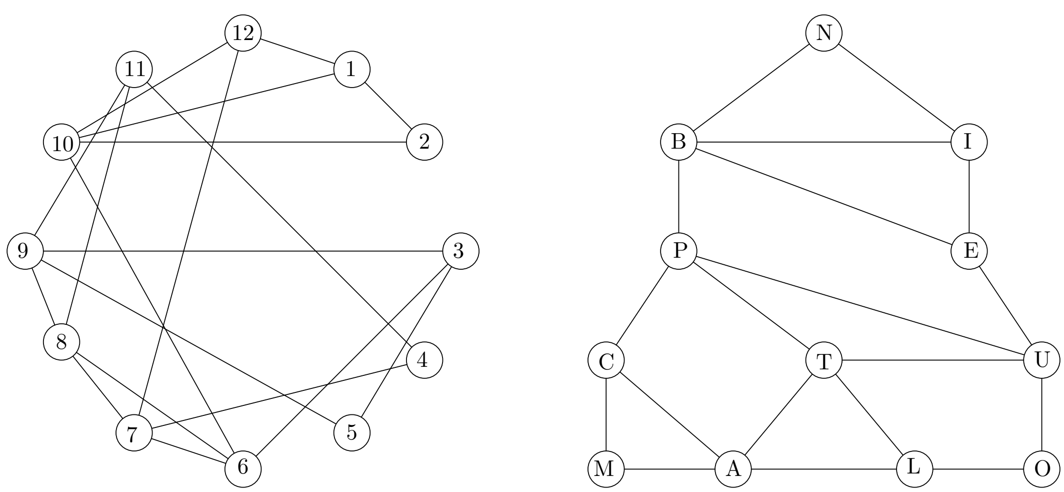
So $\alpha = 1/4$. So a specific solution is then $c_n = n2^{n-2}$. For b_n, an explicit formula has the form

$$b_n = n2^{n-2} + A \cdot 2^n$$

Using $b_1 = 1$, we get $A = 1/4$. So an explicit formula is

$$b_n = (n+1)2^{n-2}.$$

2. {5 marks} The following two graphs G and H are isomorphic. Find an isomorphism.

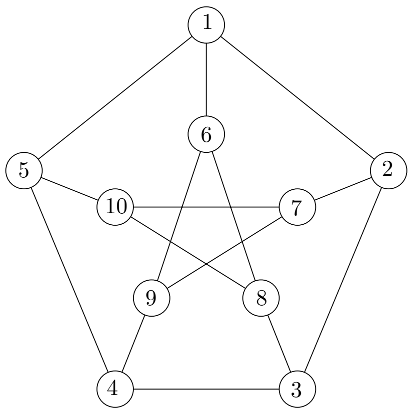


Graph G Graph H

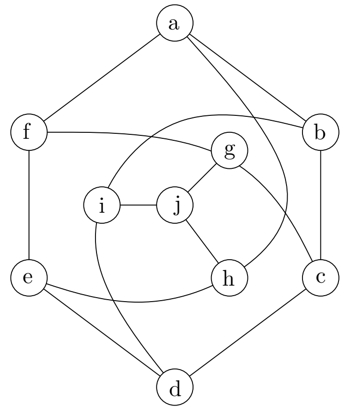
Solution. There is only one possible isomorphism $f : V(G) \to V(H)$, where

v	1	2	3	4	5	6	7	8	9	10	11	12
$f(v)$	I	N	C	O	M	P	U	T	A	B	L	E

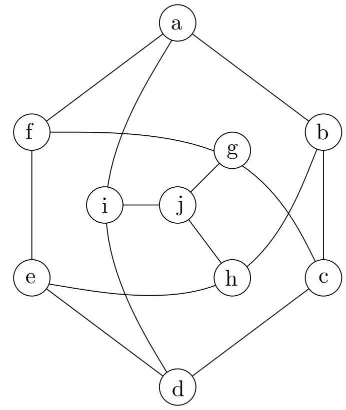
3. {8 marks} Graph G is the Petersen graph. Among the graphs H_1 and H_2, one of them is isomorphic to the Petersen graph, and the other is not. For the one that is isomorphic, give an isomorphism. For the one that is not isomorphic, explain why it is not isomorphic to G.



Graph G



Graph H_1



Graph H_2

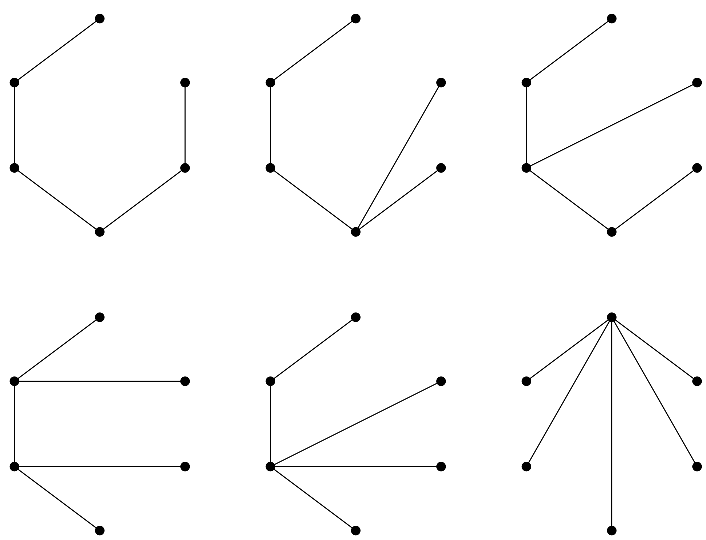
Solution. Graph G is not isomorphic to graph H_1, because there is a cycle of length 4 in H_1 (e.g. $ahef$), but G does not have any cycle of length 4.

Graph G is isomorphic to graph H_2, we give one possible isomorphism $f : V(G) \to V(H_2)$:

v	1	2	3	4	5	6	7	8	9	10
$f(v)$	a	b	c	d	i	f	h	g	e	j

4. {6 marks} Draw all non-isomorphic graphs on 6 vertices and 5 edges which do not contain a cycle.

   Solution.



5. {4 marks} Prove that if k is odd, then any k-regular graph has even number of vertices.

   Solution. Every vertex of this graph has odd degree. Since there is always an even number of odd-degree vertices in a graph, this graph has even number of vertices.

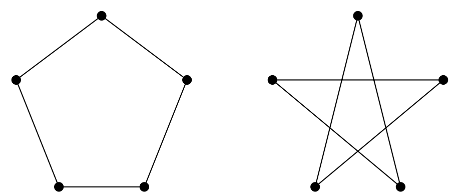
6. {7 marks} At Jochen's party of combinatorialists (of more than 2 guests, of course), a series of handshakes were generated among the guests. We are not a narcissistic bunch, so no shaking onto oneself is allowed. Prove that at least two of the guests shook hands with the same number of people.

Solution. We use a graph with n vertices to simulate this problem. An edge exists between each pair of people that shook hands. Suppose by way of contradiction the degree of every vertex is different. Since there are n vertices and the maximum degree is $n-1$, all n possibilities of degrees $0, 1, \ldots, n-1$ must be used, each by a different vertex. However, now there is a vertex of degree $n-1$, meaning it is adjacent to every other vertex, but one of them have degree 0, which have no neighbours. This is a contradiction. Hence, at least two vertices must have the same degree.

7. {12 marks} Let G be a graph. The complement of G, denoted $\overline{G}$, is the graph where $V(\overline{G}) = V(G)$, and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

   (a) Draw a graph G on 5 vertices such that G is isomorphic to $\overline{G}$.

   Solution.



G $\overline{G}$

   (b) Suppose G has p vertices where $p \equiv 2 \pmod 4$. Prove that G is not isomorphic to $\overline{G}$.

   Solution. Since every pair of vertices uv contributes an edge in either G or $\overline{G}$, the total number of edges $|E(G)| + |E(\overline{G})| = \binom{p}{2} = \frac{1}{2}p(p-1)$. If $p \equiv 2 \pmod 4$, then $p = 2 + 4k$ for some integer k. Then $\frac{1}{2}p(p-1) = \frac{1}{2}(2+4k)(2+4k-1) = (1+2k)(1+4k)$. Since both $1+2k$ and $1+4k$ are odd numbers, $|E(G)| + |E(\overline{G})|$ is odd. However, if G is isomorphic to $\overline{G}$, then $|E(G)| = |E(\overline{G})|$, so their total must be even. This is impossible, hence G cannot be isomorphic to $\overline{G}$.

   (c) Prove that if G is regular, then $\overline{G}$ is also regular.

   Solution. Suppose G has p vertices. For each vertex v in G, the degree of v in G is k, and v is not adjacent to $p-1-k$ vertices. This means that v is adjacent to $p-1-k$ vertices in $\overline{G}$, so v has degree $p-1-k$ in $\overline{G}$. This applies to every vertex in G, hence $\overline{G}$ is $(p-1-k)$-regular.

8. {Extra credit: 4 marks} Prove that given any set of 6 people, there exist either 3 mutual friends or 3 mutual strangers. We adopt the unusual social convention that a pair of people is either friends or strangers. (A proof involving lengthy case analysis will not be accepted.)

Solution. This is the same as saying given any colouring of the edges of K_6 with red or blue, there is a triangle of the same colour. Suppose we have an arbitrary colouring, and let v be a vertex. Now v has degree 5, and among the 5 edges incident with v, 3 of them must have the same colour. Without loss of generality, suppose this colour is red, and let v_1, v_2, v_3 be three vertices that are joined to v with red edges. If there is any red edge among v_1v_2, v_1v_3, v_2v_3, then they form a red triangle with v. Otherwise, all three edges are blue, in which case they form a blue triangle.