MATH 239 Assignment 1

- This assignment is due on Friday, January 18th, 2013, at 10am in the drop boxes outside MC 4067. Late assignments will not be graded.
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.

Exercise 1 (10 pts)**.**

(a) In how may ways is it possible to rearrange the letters in "MISSISSAUGA"?

HINT: Consider in how many ways the letter "M" can be placed, and then in how many ways the letters "I" can be placed, etc

(b) Consider a string using only letters $A_1, \ldots, A_n$. Let $\mathcal{S}$ be the family of strings with exactly i_1 letters A_1; i_2 letters A_2; $\ldots$; i_n letters A_n (where $i_1, \ldots, i_n$ are some non-negative integers). Find a formula for the cardinality of $\mathcal{S}$.

HINT: This generalizes the problem in part (a), the proof is similar.

Solution:

Give 5 points for each parts. Note, for both (a) and (b) two possible proofs are possible (see alternate proof in part (b)). It suffices for students to give the solution in terms of either products of binomials or factorials. It is fine for students to prove (b) first and apply it to get (a).

(a) Consider how many choices we have for the positions of each of the letters,

(1) there are $\binom{11}{1}$ choices for the letter M,

(2) there are $\binom{10}{2}$ choices for the letter I,

(3) there are $\binom{8}{4}$ choices for the letter S,

(4) there are $\binom{4}{2}$ choices for the letter A,

(5) there are $\binom{2}{1}$ choices for the letter U,

(6) there are $\binom{1}{1}$ choices for the letter G,

where in (1), 11 is the total number of letters in "MISSISSAUGA", 1 is the number of letters M; in (2), 10 is the number of remaining letters to be selected after choosing the letter M, and 2 is the number of letters I; in (3), 8 is the number of remaining letters to be selected after choosing the letters M, I, and 4 is the number of letters S; etc. Thus the total number of choices is the product of (1)-(6), i.e.,

$$\binom{11}{1}\binom{10}{2}\binom{8}{4}\binom{4}{2}\binom{2}{1}\binom{1}{1} = 415800.$$

(b) Consider how many choices we have for the positions of each of the letters,

- there are $\binom{i_1+i_2+\ldots+i_n}{i_1}$ choices for the letter A_1,
- there are $\binom{i_2+\ldots+i_n}{i_2}$ choices for the letter A_2,

$\ldots$

- there are $\binom{i_{n-1}+i_n}{i_{n-1}}$ choices for the letter A_{n-1},

- there are $\binom{i_n}{i_n}$ choices for the letter A_n,

where $i_1+i_2+\ldots+i_n$ is the total number of letters in the string and i_1 is the number of letters A_1; $i_2+\ldots+i_n$ is the number of remaining letters to be selected after choosing the letter A_1, and i_2 is the number of letters A_2, etc. Thus the total number of choices is given by,

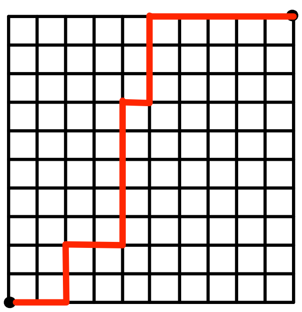
$$\binom{i_1+i_2+\ldots+i_n}{i_1}\binom{i_2+\ldots+i_n}{i_2}\cdots\binom{i_{n-1}+i_n}{i_{n-1}}\binom{i_n}{i_n}=\frac{(i_1+i_2+\ldots+i_n)!}{i_1!i_2!\ldots i_n!}.$$

Alternate solution. Add distinguishing marks to make all the letters distinct. Then we have $n!$ different arrangements of the letters (the number of permutation). We over-counted however, as all possible permutations of the A_1 letters give the same string. We thus need to divide by $i_1!$. Similarly we also need to divide by $i_2!$, …, $i_n!$, which yields,

$$\frac{(i_1+i_2+\ldots+i_n)!}{i_1!i_2!\ldots i_n!}.$$

Exercise 2 (15 pts)**.**

Consider a grid where the lower left corner corresponds by the point $(0,0)$ and the upper right corner to the point (a,b). Thus to reach (a,b) from $(0,0)$ we can travel up b steps and to the right a steps. Let $\mathcal{P}$ be the set of paths starting at location $(0,0)$, ending at (a,b) where we only travel either up or to the right. The following figure gives an example where $(a,b)=(10,10)$ and a member $P\in\mathcal{P}$ is indicated by the thick red path.



(a) Find a formula for the cardinality of $\mathcal{P}$.

HINT: Very little work involved here.

(b) For $i\in\{0,\ldots,a\}$, let $\mathcal{P}_i$ be the set of paths in $\mathcal{P}$ that first reach the top of the grid in position $x=i$. For instance the solid red path $P\in\mathcal{P}$ in the figure is in $\mathcal{P}_5$. Find a formula for the cardinality of $\mathcal{P}_i$.

HINT: The last part of $P\in\mathcal{P}_i$ is going up.

(c) Use the fact that $\mathcal{P}_0,\mathcal{P}_1,\ldots,\mathcal{P}_n$ form a partition of $\mathcal{P}$ to give a combinatorial proof that for any non-negative integers a,b the following relation holds,

$$\binom{a+b}{a}=\sum_{i=0}^{a}\binom{i+b-1}{i}.$$

Solution:

Give 5 points for each parts. Make sure that part (a) and (b) are justified. (Give 2/5 for (a) and (b) if no justification).

(a) Since we are starting at position $(0,0)$ and ending at position (a,b) the path has length $a+b$. In order to end at position $a+b$ we must go to the right exactly a times (resp. up b times), moreover, if we choose any path (never going down or to the left) going to the right a times (resp. up b times) the path will end in (a,b). Thus the number of choices is $\binom{a+b}{a}$ (resp. $\binom{a+b}{b} = \binom{a+b}{a}$). **(b)** Every path P in $\mathcal{P}_i$ consists of a path Q from $(0,0)$ to the position $(i, b-1)$ followed by a unique path going up once. Thus it suffices to count the paths of type Q. The answer is obtained by part (a) and is $\binom{i+b-1}{i}$. **(c)** We have

$$\binom{a+b}{a} = |\mathcal{P}_0| = \sum_{i=0}^{a} |\mathcal{P}_i| = \sum_{i=0}^{a} \binom{i+b-1}{i},$$

where the first equality follows from (a), the second as $\mathcal{P}_0, \ldots, \mathcal{P}_a$ is a partition of $\mathcal{P}$, and the third by (b).

Exercise 3 (10 pts)**.** Consider the relation

$$k^n = \sum_{i=0}^{n} \binom{n}{i} (k-1)^{n-i}, \tag{$\star$}$$

where $k \geq 2$ is an integer.

(a) Use the binomial theorem to prove ($\star$).

HINT: Very little work involved here.

(b) Find a combinatorial proof of ($\star$).

HINT: Consider strings of length n with k distinct letters.

Solution:

Give 3 points for part (a), 7 points for part (b)

(a) We have,

$$k^n = (1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i = \sum_{i=0}^{n} \binom{n}{i} (k-1)^i = \sum_{i=0}^{n} \binom{n}{n-i} (k-1)^{n-i} = \sum_{i=0}^{n} \binom{n}{i} (k-1)^{n-i},$$

where the first equality is obtained by setting $x = k-1$, the second by the binomial theorem, the third by relabelling the index i by $n-i$ and the last equality by the identity $\binom{n}{n-i} = \binom{n}{i}$. **(b)** Let $\mathcal{S}$ be the set of all strings of length n with k different letters, say $A_1, \ldots, A_k$. Since for each position we have k independent choices $|\mathcal{S}| = k^n$. Let $\mathcal{S}_i$ be the set of strings in $\mathcal{S}$ that have exactly i letters A_1.

Claim. $|\mathcal{S}_i| = \binom{n}{i}(k-1)^{n-i}$.

Proof. There are $\binom{n}{i}$ possible choices for placing the letter A_1 in strings of $\mathcal{S}_i$. Then for each of the remaining $n-i$ position we can place any of the remaining letters $A_2, \ldots, A_k$. For each of the $n-i$ remaining position we have thus $k-1$ independent choices. □

Thus we have,

$$k^n = |\mathcal{S}| = \sum_{i=0}^{n} |\mathcal{S}_i| = \sum_{i=0}^{n} \binom{n}{i} (k-1)^{n-i},$$

where the first equality follows by (a), the second as $\mathcal{S}_0, \ldots, \mathcal{S}_n$ forms a partition of $\mathcal{S}$, and the third by the Claim.

Exercise 4 (10 pts)**.** Consider the expression

$$(1+x+y)^{27}.$$

What is the coefficient of x^7y^9?

HINT: The argument is similar to that of the proof of the binomial theorem.

Solution:

10 points. As an alternate proof we can show that there is a bijection between the possible monomials and the strings of length 27 with letters $X, Y, 1$ where the strings have exactly 7 letter X and 9 letter Y. Then use the result in exercise 1.

We have,

$$(1+x+y)^{27} = \underbrace{(1+x+y)}_{(1)}\underbrace{(1+x+y)}_{(2)}\ldots\underbrace{(1+x+y)}_{(27)}.$$

To obtain a monomial x^7y^9 from $(1+x+y)^{27}$ we must select x from 7 of the terms (1)-(27). There are $\binom{27}{7}$ ways of doing this. We then must select y from 9 of the remaining $27-7$ terms. There are $\binom{20}{9}$ ways of doing this. Thus the answer is,

$$\binom{27}{7}\binom{20}{9} = 149153518800.$$

Exercise 5 (10 pts)**.** The n children of the Von Trapp family all have different ages. Whenever they sing, they stand, shoulder to shoulder, on a line. Indicate in how many ways they can line up under the following conditions,

(1) The youngest child is never on the left-most position.

HINT: Look at the complementary case.

(2) The oldest child is to the right of the youngest child.

NOTE: it does not have to be directly to the right, for example, the oldest child could be 3 to the right of the youngest child.

Solution:

Give 5 points for each part. Note, there is an alternate proof for (b) obtained first selecting the position of the youngest child, then the oldest one, then all the remaining ones.

(a) Let $\mathcal{P}$ be the set of all permutations of the n children of the Von Trapp family and let $\mathcal{P}_0$ be the set of permutations that keep the youngest child in position 1. Then $|\mathcal{P}_0| = (n-1)!$ as we have all possible permutations of the remaining children. We are looking for $|\mathcal{P}| - |\mathcal{P}_0| = n! - (n-1)!$.

(b) Let $\mathcal{P}$ be the set of all permutations of the n children of the Von Trapp family. Let i, j denote respectively the youngest and oldest child. Let us partition $\mathcal{P}$ into $\mathcal{P}^-$ and $\mathcal{P}^+$ where $\mathcal{P}^-$ is the set of permutations where i occurs before j and $\mathcal{P}^+$ is the set of permutations where i occurs after j.

Claim. $|\mathcal{P}^+| = |\mathcal{P}^-|$.

Proof. Consider the following function f that takes as input an arbitrary permutation of $\mathcal{P}$ and interchanges the order of i and j. The result follows from the fact that f defines a bijection from $|\mathcal{P}^-|$ to $|\mathcal{P}^+|$ (f is is its own inverse). $\square$

We have $|\mathcal{P}| = n!$. As $\mathcal{P}^-, \mathcal{P}^+$ is a partition of $\mathcal{P}$, $|\mathcal{P}| = |\mathcal{P}^-| + |\mathcal{P}^+|$. By the Claim, $|\mathcal{P}^-| = \frac{n!}{2}$.

Exercise 6 (10 pts)**.** Let $\mathcal{S}$ be the set of all finite subsets of the positive integers. For each $A \in S$, define the weight $w(A)$ to be the largest element in A (we define $w(\emptyset) = 0$).

(a) Determine the generating series for $\mathcal{S}$ with weights w.

(b) Suppose we change the word "largest" to "smallest" in part (a). Can we still define a generating series for $\mathcal{S}$ with weights w that is a formal power series?

Solution:

Give 5 points for each parts.

(a) For the generating series of $\mathcal{S}$ with weights w, the coefficient of x^n represents the number of sets whose largest element is n. For $n \geq 1$, there are 2^{n-1} such sets: any subset of $[n-1]$ union with $\{n\}$. For $n = 0$, there is one such set, namely the empty set. Thus the generating series is,

$$\phi_S(x) = 1 + \sum_{n \geq 1} (2^{n-1}) x^n.$$

(b) The generating function for $\mathcal{S}$ is of the form $\sum_{n \geq 0} a_n x^n$ where a_n is the number of sets of $\mathcal{S}$ of weight n. As there are an infinite number of sets containing $n = 1$ for instance, the coefficient of 1 should be infinite, in particular, the generating function is not a formal power series.