Math 239 - Tutorial 7

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1. Prove that, if G is connected, then any two longest paths have a vertex in common.

   Solution. Assume the contrary, i.e. assume that there are two longest paths P_1 and P_2 that don't have a vertex in common, and let their length be k. Take the shortest path P that has one endpoint in $V(P_1)$ and the other in $V(P_2)$ and name its endpoints $v \in V(P_1)$ and $w \in V(P_2)$ and its length $l \geq 1$. As G is connected, there is such a path, and as the length of P was chosen to be minimal, v is its only vertex in $V(P_1)$ and w the only vertex in $V(P_2)$: Assume there is another vertex $x \in V(P_1)$ along P, then the path segment from x to w would be a shorter path with endpoints in $V(P_1)$ and $V(P_2)$, contradicting the minimal length. Similarly there is no other vertex $y \in V(P_2)$ apart from w. Note that v divides P_1 up into two parts, let the longer one of the parts be Q_1, and note that it has length at least $\lceil \frac{k}{2} \rceil$. Similarly w divides P_2 up into two parts, name the longer one Q_2 and its length is at least $\lceil \frac{k}{2} \rceil$. Then the new path Q_1PQ_2 (the path obtained by attaching these three paths together) has length at least $\lceil \frac{k}{2} \rceil + l + \lceil \frac{k}{2} \rceil \geq 2\lceil \frac{k}{2} \rceil + 1 \geq k + 1$. But this contradicts the maximality of P_1 and P_2, so the assumption that they had no vertex in common must be wrong.

2. Let G be a graph with p vertices and every vertex of G has degree at least $(p-1)/2$. Prove that G is connected.

   Solution. Let $x \in V(G)$ and $N(x)$ be the set of neighbours of x. Let $S = V(G) - (N(x) \cup \{x\})$. As we have that $|N(x)| \geq (p-1)/2$, then $|S| \leq p - (p-1)/2 - 1 = (p-1)/2$, therefore each $z \in S$ is adjacent to at least one neighbour of G. Hence any vertex is connected to x and we obtained that G is connected.

3. Prove that if every vertex of a graph G has degree at least 3, then G contains a cycle of even length.

   Solution. Let P be a longest path in G. Let v be an end vertex of P. All neighbours of v are in P – otherwise, we could add a neighbour of v to P to make P longer. Suppose a neighbour w of v is at an odd distance from v in the path. Then taking the path in P from v to w (which has an odd number of edges) and adding the edge from v to w gives an even cycle.
   Now suppose no neighbour of v is at an odd distance from v in the path. Then there are two neighbours of v, say u and w, that are both at an even distance from v. Then the path in P from u to w (which has an even number of edges) and the edges uv and vw gives an even cycle.

4. A *forest* is a graph with no cycles . Let G be a graph, a *maximal spanning forest* T is a spanning subgraph of G that is a forest, and that is maximal with respect to the number of edges. Show that if k is the number of components of G, then $|E(T)| = |V(G)| - k$.

   Solution. Let T be a maximal spanning forest of G. Let $G_1, \ldots, G_k$ be the components of G. For any $i = 1, \ldots, k$ let T_i be the induced subgraph of T by $V(G_i)$. Observe that T_i is acyclic and is a spanning subgraph of the connected graph G_i. As T was edge maximal, T_i must be a spanning tree of G_i. Thus we know that $|E(T_i)| = |V(G_i)| - 1$. Observe that for $i \neq j$, there are no T_iT_j-edges. Therefore $|E(T)| = \sum_{i=1}^{k} |E(T_i)| = \sum_{i=1}^{k}(|V(G_i)| - 1) = |V(G)| - k$.

5. Prove that every tree is bipartite.

   Solution. By induction. If tree T has only one vertex, then the result is clearly true (Take $(\{v\}, \emptyset)$ as the bipartition). Suppose any tree with $k \geq 1$ vertices is bipartite. Let T be a tree with $k+1$ vertices. Since $k+1 \geq 2$, T has a leaf v. By induction, $T - v$ is bipartite, with bipartition (A, B). Since

v is a leaf, it has exactly one neighbour, w. We may assume that $w \in A$. Then setting $B := B \cup \{v\}$ gives a partition (A, B) of $V(T)$, where no pair of vertices in the same part of the partition is adjacent. Thus T is bipartite.

6. Prove that the only k-regular trees for any k are K_1 (with $k = 0$) and K_2 (with $k = 1$).

   Solution. Assume we have such a k-regular tree T on n vertices for some k and n. Recall that by Theorem 5.1.5 a tree T on n vertices has exactly $n - 1$ edges. Further recall that by Theorem 4.3.1,

   $$2|E(T)| = \sum_{v \in V(T)} \deg(v)$$

   Expanding this equation using our knowledge about the number of edges and k-regularity, we get:

   $$2n - 2 = 2|E(T)| = \sum_{v \in V(T)} \deg(v) = \sum_{v \in V(T)} k = kn$$

   By transforming the equation formed by the term at the very left and right, we get that $(2 - k)n = 2$. As k and n are non-negative integers, this means that either $2 - k = 2$ and $n = 1$ or $2 - k = 1$ and $n = 2$. In the first case we get $k = 0$ and $n = 1$ and the corresponding graph is K_1, in the second case we get $k = 1$ and $n = 2$ and the corresponding graph is K_2.

   Note: A shorter proof to this is using the knowledge from class that any tree on at least 2 vertices has a leaf. So for a tree on at least 2 vertices to be k-regular, k must be 1. But then the graph can only be a number of disjoint copies of K_2, meaning it is not connected and thus not a tree if it has more than 2 vertices.