MATH 239 - Fall 2013

Assignment 3

Due date : Friday, October 4, 2013, at noon (sharp)

Submission Guidelines:

- Total number of marks in this assignment is 30.
- Use a cover page to submit your solutions (available on the course webpage).
- Keep a copy of your manuscript before your submission.
- Assignments submissions are exclusively accepted in the following dropboxes

[Section 001]	Dropbox next to the St Jerome's library, 2nd floor of STJ			
[Section 002]	Math DropBox #18;	Slot #1 A-J,	Slot #2 K-S,	Slot #3 T-Z
[Section 003]	Math DropBox #18;	Slot #4 A-J,	Slot #5 K-S,	Slot #6 T-Z
[Section 004]	Math DropBox #18;	Slot #7 A-J,	Slot #8 K-S,	Slot #9 T-Z

- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- You are not allowed to post this manuscript (or parts of it) online, nor share it (or parts of it) with anyone not enrolled in this course.

Assignment policies: While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. For example, copying or paraphrasing a solution from some fellow student or old solutions from previous offerings of related courses qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. All students found to be cheating will automatically be given a mark of 0 on the assignment. In addition, there will be a 10/100 penalty to their final mark, as well as all academic offenses will be reported to the Associate Dean for Undergraduate Studies and recorded in the student's file (this may lead to further, more severe consequences).

If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that if you see any marking error, then you should return your assignment paper to the TA of your section within one week and with written notes on all the marking errors; please write the notes on a new sheet and attach it to your assignment paper.

NOTE: FOR EACH QUESTION make sure you (1) define a suitable set and weight function, and (2) state clearly which theorems from class you are using (e.g. Sum and Product Lemmas, Binomial Theorem etc.)

Question 1 [Marks 9]
Find the number of positive integers m less than 1,000,000 such that the sum of the digits of m is equal to 14.

Solution. *Note that the set of non-negative integers less than 1,000,000 is in one-to-one correspondence with the Cartesian product $S = N^6$, where $N = \{0, 1, \ldots, 9\}$. We therefore want the number of elements of S with weight 14, where the weight function on S is defined by $w(t_1, \ldots, t_6) = t_1 + \ldots + t_6$.*

The generating series for N with respect to the natural weight function $w_0(\sigma) = \sigma$ is $\Phi_N(x) = 1 + x + x^2 + \cdots + x^9 = \frac{1-x^{10}}{1-x}$ (by Finite Geometric Series). Therefore

$$\begin{aligned} \Phi_S(x) &= (\Phi_N(x))^6, \quad \textit{by Product Lemma} \\ &= \left(\frac{1-x^{10}}{1-x}\right)^6. \end{aligned}$$

We need to find the coefficient of x^{14}:

$$\begin{aligned} [x^{14}]\Phi_S(x) &= [x^{14}](1-x^{10})^6(1-x)^{-6} \\ &= [x^{14}]\left[\sum_{i=0}^{6}(-1)^i\binom{6}{i}x^{10i}\right] \times \left[\sum_{j\geq 0}\binom{6+j-1}{6-1}x^j\right], \quad \textit{By Binomial Theorem/Thm 1.6.5} \\ &= \binom{6+14-1}{5} - \binom{6}{1}\binom{6+4-1}{5} \\ &= \binom{19}{5} - 6\binom{9}{5} \\ &= 10872. \end{aligned}$$

Question 2 [Marks 7]
Let k be a fixed positive integer. Let a_n denote the number of compositions of n with k parts, where each part is even and greater than 5. Find a_n. Express your answer in terms of n and k. (Hint: pay attention to whether n is even or odd.)

Solution. *Let $A = \{6, 8, 10, \ldots\}$ be the set of all even integers which are greater than 5. Let S be the Cartesian product of k copies of A, which we can write as*

$$S = A^k.$$

Then a_n is the number of elements of S of weight n, where the weight function is defined by $w(t_1, \ldots, t_k) = t_1 + \ldots, +t_k$.

By the Product Lemma,

$$\begin{aligned}\Phi_S(x) &= (\Phi_A(x))^k = (x^6 + x^8 + x^{10} + ...)^k \\ &= \left(\frac{x^6}{1-x^2}\right)^k.\end{aligned}$$

Then a_n is given by

$$\begin{aligned}[x^n]\Phi_S(x) &= [x^n]x^{6k}(1-x^2)^{-k} = [x^{n-6k}](1-x^2)^{-k} \\ &= [x^{n-6k}]\sum_{i\geq 0}\binom{k+i-1}{k-1}x^{2i} \\ &= \begin{cases} 0 & \textit{if } n \textit{ is odd or } n < 6k, \\ \binom{\frac{n}{2}-2k-1}{k-1} & \textit{if } n \textit{ is even and } n \geq 6k, \end{cases}\end{aligned}$$

where we use the Binomial Theorem (in the form of Theorem 1.6.5).

Question 3 [Marks 7]
Let a_n be the number of compositions of n with an odd number of parts, none of which is equal to 1 or 2. Find the generating series $\sum_{n\geq 0} a_n x^n$. Express your answer as a rational function, i.e. in which both numerator and denominator are products of polynomials in x. (So you do not need to calculate the coefficients a_n explicitly.)

Solution. *The quantity a_n counts the number of elements of $S = \bigcup_{k\ odd} N^k$ of weight n, where $N = \{3, 4, 5, \dots\}$ and the weight function is $w(t_1, \dots, t_r) = t_1 + \dots, +t_r$.*
Then

$$\begin{aligned}\Phi_S(x) &= \sum_{k\ odd}\Phi_{N^k}(x), \quad \textit{By Sum Lemma} \\ &= \sum_{k\ odd}(\Phi_N(x))^k, \quad \textit{By Product Lemma} \\ &= \sum_{k\ odd}(x^3 + x^4 + \cdots)^k \\ &= \sum_{k\ odd}\left(\frac{x^3}{1-x}\right)^k \\ &= \frac{x^3}{1-x} \times \frac{1}{\left(1 - \frac{x^6}{(1-x)^2}\right)} \\ &= \frac{x^3}{1-x} \times \frac{(1-x)^2}{1-2x+x^2-x^6} \\ &= \frac{x^3-x^4}{1-2x+x^2-x^6}.\end{aligned}$$

Question 4 [Marks 7]
Let c_n be the number of compositions of n, $n \geq 0$, with exactly 3 parts, in which the first part is the smallest amongst the three parts. (For example, $(2, 8, 4)$, $(2, 10, 2)$ and $(4, 5, 5)$ are compositions of 14 of this type. Note the smallest part is not necessarily unique.) Find the generating series $\sum_{n\geq 0} c_n x^n$. Express your answer as a rational function (as described in the previous question).

Solution. *For each $i \geq 1$, let $N_i = \{i, i+1, i+2, \dots\}$. Then c_n is the number of elements of $S = \bigcup_{i\geq 1}\{i\} \times N_i \times N_i$ that have weight n, where the weight function is defined by $w(t_1, t_2, t_3) = t_1 + t_2 + t_3$.*

We have

$$\begin{aligned}
\Phi_S(x) &= \sum_{i\geq 1} \Phi_{\{i\}\times N_i\times N_i}(x), \quad \textit{By Sum Lemma} \\
&= \sum_{i\geq 1} x^i \left(\Phi_{N_i}(x)\right)^2, \quad \textit{By Product Lemma} \\
&= \sum_{i\geq 1} x^i \left(x^i + x^{i+1} + x^{i+2} + \cdots\right)^2 \\
&= \sum_{i\geq 1} x^i \left(\frac{x^i}{1-x}\right)^2 \\
&= \sum_{i\geq 1} \frac{x^{3i}}{(1-x)^2} \\
&= \frac{x^3}{(1-x)^2} \sum_{i\geq 0} x^{3i} \\
&= \frac{x^3}{(1-x)^2(1-x^3)}.
\end{aligned}$$