MATH 239 – Tutorial 2

1.

Compute $[x^n]\dfrac{1}{(1+x)(1-2x^3)}$.

Solution

We have $\dfrac{1}{1+x} = \displaystyle\sum_{k\geq 0}(-x)^k$ and $\dfrac{1}{1-2x^3} = \displaystyle\sum_{\ell=0} 2^\ell x^{3\ell}$.

Therefore, $\dfrac{1}{(1+x)(1-2x^3)} = \displaystyle\sum_{k\geq 0}(-x)^k \sum_{\ell\geq 0} 2^\ell x^{3\ell} = \sum_{k,\ell\geq 0}(-1)^k 2^\ell x^{k+3\ell}$.

Let $n = k + 3\ell$ and sum over n and ℓ. This gives

$$\sum_{n\geq 0}\sum_{\ell=0}^{\lfloor \frac{n}{3}\rfloor}(-1)^{n-3\ell}2^\ell x^n,$$

which gives

$$[x^n]\frac{1}{(1+x)(1-2x^3)} = \sum_{\ell=0}^{\lfloor \frac{n}{3}\rfloor}(-1)^{n-3\ell}2^\ell = (-1)^n\sum_{\ell=0}^{\lfloor \frac{n}{3}\rfloor}(-2)^\ell.$$

This is a geometric sum, therefore

$$[x^n]\frac{1}{(1+x)(1-2x^3)} = (-1)^n\frac{1-(-2)^{\lfloor \frac{n}{3}\rfloor+1}}{3}.$$

2. (Problem Set 1.5)

Let $A(x)$ and $B(x)$ be formal power series.
(a) Show that if $A(x)B(x) = 0$, then $A(x) = 0$ or $B(x) = 0$.
(b) Show that if $A(x)^2 = B(x)^2$, then $A(x) = \pm B(x)$

Solution

(a) Let $A(x) \neq 0$. We can then write $A(x) = x^k A'(x)$ where $A'(x)$ has a non-zero constant term.
Therefore, $A(x)B(x) = x^k A'(x)B(x)$. By expanding this and combining like terms, we get

$$A(x)B(x) = x^k[a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \ldots] = 0.$$

Since $A'(x)$ has a non-zero constant term, $a_0 b_0 = 0$ implies that $b_0 = 0$. Inserting this in $a_0 b_1 + b_0 a_1 = 0$, we get that $b_1 = 0$. Using the same logic, it can be seen that $B(x) = 0$.

(b) Let $A(x) = \sum_{i \geq 0} a_i x^i$ and $B(x) = \sum_{j \geq 0} b_j x^j$. Expand both $A(x)^2$ and $B(x)^2$ to get

$$A(x)^2 = a_0^2 + 2a_0a_1x + (2a_0a_2 + a_1^2)x^2 + (2a_0a_3 + 2a_1a_2)x^3 + \dots$$

and

$$B(x)^2 = b_0^2 + 2b_0b_1x + (2b_0b_2 + b_1^2)x^2 + (2b_0b_3 + 2b_1b_2)x^3 + \dots$$

By comparing coefficients, observe that $a_0^2 = b_0^2$, which implies that $a_0 = \pm b_0$.
If $a_0 = b_0$, $2a_0a_1 (= 2b_0a_1) = 2b_0b_1$, which implies that $a_1 = b_1$; $2a_0a_2 + a_1^2 (= 2b_0a_2 + b_1^2) = 2b_0b_2 + b_1^2$, which implies that $a_2 = b_2$; etc.. It can be seen that $A(x) = B(x)$ when $a_0 = b_0$.
If $a_0 = -b_0$, $2a_0a_1 (= -2b_0a_1) = 2b_0b_1$, which implies that $a_1 = -b_1$; $2a_0a_2 + a_1^2 (= -2b_0a_2 + (-b_1)^2) = 2b_0b_2 + b_1^2$, which implies that $a_2 = -b_2$; etc.. It can be seen that $A(x) = -B(x)$ when $a_0 = -b_0$.
$\therefore A(x) = \pm B(x)$.

3.

For a binary string s, define its weight $w(s)$ to be the number of 1's in the stirng plus the length of the string itself.

(a) Let S_n be the set of all binary strings of length n. Use the product lemma to determine $\Phi_{S_n}(x)$.

(b) Let T be the set of all binary strings (regardless of lenght). Determine $\Phi_T(x)$.

Solution

(a) Think of each bit in the string as contributing 1 to the length of the string, and another 1 if that bit is 1. For each bit $\{0, 1\}$, use the weight function

$$\alpha(a) = \begin{cases} 1 & a = 0 \\ 2 & a = 1 \end{cases}$$

So $\Phi_{\{0,1\}} = x + x^2$. Using the weight of a string $w(a_1 \dots a_n) = \sum_{i=0}^{n} \alpha(a_i)$, by the product lemma,

$$\Phi_{S_n}(x) = \Phi_{\{0,1\}^n}(x) = (x + x^2)^n.$$

(b) We see that $T = S_0 \cup S_1 \cup S_2 \cup \dots$. Using the sum lemma,

$$\Phi_T(x) = \sum_{n \geq 0} \Phi_{S_n}(x) = \sum_{n \geq 0} (x + x^2)^n = \frac{1}{1 - x - x^2}.$$

4.

Determine the number of compositions of n into k parts, where every part is a multiple of 2.

Solution
Let $T = \{2, 4, 6, 8, \dots\}$ be the set of positive multiples of 2. It's generating series is given by

$$\Phi_T(x) = x^2 + x^4 + x^6 + \dots = \frac{x^2}{1 - x^2}.$$

By the product lemma, the generating function is

$$\Phi_S(x) = \Phi_{T^k}(x) = \left(\frac{x^2}{1 - x^2}\right)^k.$$

Therefore,

$$\begin{aligned}
[x^n]\left(\frac{x^2}{1 - x^2}\right)^k &= [x^n]x^{2k}(1 - x^2)^k, \\
&= [x^{n-2k}]\sum_{i \geq 0}\binom{i + k - 1}{k - 1}(x^2)^i, \\
&= [x^{n-2k}]\sum_{i \geq 0}\binom{i + k - 1}{k - 1}x^{2i}.
\end{aligned}$$

We are looking for the coefficient of x^{n-2k}, so $i = \frac{n-2k}{2}$. This implies that

$$\begin{aligned}
[x^n]\left(\frac{x^2}{1 - x^2}\right)^k &= \tbinom{\frac{n-2k}{2} + k - 1}{k-1}, \\
&= \tbinom{\frac{n-2}{2}}{k-1}.
\end{aligned}$$