

DUE: 10am Friday Feb. 1 in the drop boxes opposite the Math Tutorial Centre MC 4067.

1. Let k be a fixed positive integer. Let a_n denote the number of compositions of n with exactly k parts, in which each part is an odd number greater than or equal to 5.
 - (a) Find a set S and a weight function w defined on S such that a_n is equal to the number of elements σ of S with $w(\sigma) = n$.
 - (b) Find the generating series $\Phi_S(x)$ with respect to the weight function w . Remember to indicate where theorems from class are applied, e.g. Sum and Product Lemmas.
 - (c) Find a_n explicitly in terms of n and k .

2. Let a_n be the number of compositions of n with an even number of parts, each of which is at least 6. (Note that the number of parts is not fixed.)
 - (a) Find a set S and a weight function w defined on S such that a_n is equal to the number of elements σ of S with $w(\sigma) = n$.
 - (b) Prove that for $n \geq 0$

$$a_n = [x^n] \frac{1 - 2x + x^2}{1 - 2x + x^2 - x^{12}}.$$

Remember to indicate where theorems from class are applied, e.g. Sum and Product Lemmas.

3. Let $\{a_n : n \geq 0\}$ be the sequence defined in the previous question.

- (a) Prove that $a_0 = 1$ and $a_n = 0$ for $1 \leq n \leq 11$.
 - (b) Prove that for each $n \geq 12$ the number a_n satisfies

$$a_n = 2a_{n-1} - a_{n-2} + a_{n-12}.$$

- (c) Find the exact value of a_{15} .
4. Let b_n be the number of compositions of n with an even number of parts, such that at least one part is less than or equal to 5. Prove that for $n \geq 0$

$$b_n = [x^n] \left(1 + \frac{x^2}{1 - 2x} - \frac{1 - 2x + x^2}{1 - 2x + x^2 - x^{12}} \right).$$