MATH 239

TUTORIAL 3

Question 1:
Let S be a set of configurations, and w be a weight function on S. Let S_e be the subset of S such that its elements have even weight, and S_o be the subset of S such that its elements have odd weight. Show that
a) $\Phi_{S_e}(x) = \frac{\Phi_S(x)+\Phi_S(-x)}{2}$
b) $\Phi_{S_o}(x) = \frac{\Phi_S(x)-\Phi_S(-x)}{2}$

Answer 1:
a) Let $\Phi_S(x) = \sum_{i\geq 0} s_i x^i$. Since S_e is the set of all even weighed elements of S, we have

$$[x^n]\,\Phi_{S_e}(x) = \begin{cases} s_n & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Now, notice that

$$\begin{aligned} [x^n]\,\frac{\Phi_S(x)+\Phi_S(-x)}{2} &= \frac{1}{2}\left([x^n]\,\Phi_S(x) + [x^n]\,\Phi_S(-x)\right) \\ &= \frac{1}{2}\left(s_n + (-1)^n s_n\right) \\ &= \begin{cases} s_n & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

This gives us that $[x^n]\,\Phi_{S_e}(x) = [x^n]\,\frac{\Phi_S(x)+\Phi_S(-x)}{2}$ for all n, so we have $\Phi_{S_e}(x) = \frac{\Phi_S(x)+\Phi_S(-x)}{2}$ as desired.
b) Since $S = S_e \cup S_o$, by the sum lemma we have

$$\begin{aligned} \Phi_S(x) &= \Phi_{S_e}(x) + \Phi_{S_o}(x) \\ \Phi_{S_o}(x) &= \Phi_S(x) - \Phi_{S_e}(x) \\ &= \Phi_S(x) - \frac{\Phi_S(x)+\Phi_S(-x)}{2} \\ &= \frac{\Phi_S(x)-\Phi_S(-x)}{2} \end{aligned}$$

as desired.

Question 2:
Show that for $n \geq 1$, the number of composition of n with all parts odd is the same as the number of compositions of $n+1$ such that each part is at least 2.

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Answer 2:
Let $O = \{1, 3, 5, 7, \dots\}$. Then,

$$\Phi_O(x) = x + x^3 + x^5 + x^7 + \cdots = \frac{x}{1 - x^2}$$

Next, let O_k be the set of k odd tuples. Then by the product lemma,

$$\begin{aligned} \Phi_{O_k}(x) &= (\Phi_O(x))^k \\ &= \left(\frac{x}{1 - x^2}\right)^k \end{aligned}$$

Finally, let S_O be the set odd tuples of arbitrary length. Then $S_O = \bigcup_{k \ge 0} O_k$. By the sum lemma, its generating function is given by

$$\begin{aligned} \Phi_{S_O}(x) &= \sum_{k \ge 0} \Phi_{O_k}(x) \\ &= \sum_{k \ge 0} \left(\frac{x}{1 - x^2}\right)^k \\ &= \frac{1}{1 - \frac{x}{1 - x^2}} \\ &= \frac{1 - x^2}{1 - x - x^2} \\ &= 1 + \frac{x}{1 - x - x^2} \end{aligned}$$

We do the same thing to get the generating function for compositions with each part at least 2. Let $N_{\ge 2} = \{2, 3, 4, 5, \dots\}$. Then,

$$\Phi_{N_{\ge 2}}(x) = x^2 + x^3 + x^4 + x^5 + \cdots = \frac{x^2}{1 - x}$$

Next, let N_k be the set of k tuples with each part at least 2. Then by the product lemma,

$$\begin{aligned} \Phi_{N_k}(x) &= \left(\Phi_{N_{\ge 2}}(x)\right)^k \\ &= \left(\frac{x^2}{1 - x}\right)^k \end{aligned}$$

Finally, let $S_{\ge 2}$ be the set of tuples with each part at least 2. Then $S_{\ge 2} = \bigcup_{k \ge 0} N_k$. By the sum lemma, its generating function is given by

$$\begin{aligned} \Phi_{S_{\ge 2}}(x) &= \sum_{k \ge 0} \Phi_{N_k}(x) \\ &= \sum_{k \ge 0} \left(\frac{x^2}{1 - x}\right)^k \\ &= \frac{1}{1 - \frac{x^2}{1 - x}} \\ &= \frac{1 - x}{1 - x - x^2} \\ &= 1 + \frac{x^2}{1 - x - x^2} \end{aligned}$$

Finally, notice that $\Phi_{S_{\geq 2}}(x) - 1 = x(\Phi_{S_O}(x) - 1)$. Taking coefficients of x^{n+1} on both sides for $n \geq 1$, we get

$$\begin{aligned}
\left[x^{n+1}\right]\Phi_{S_{\geq 2}}(x) - 1 &= \left[x^{n+1}\right]x(\Phi_{S_O}(x) - 1) \\
\left[x^{n+1}\right]\Phi_{S_{\geq 2}}(x) - 1 &= [x^n]\Phi_{S_O}(x) - 1 \\
\left[x^{n+1}\right]\Phi_{S_{\geq 2}}(x) &= [x^n]\Phi_{S_O}(x)
\end{aligned}$$

as $n \geq 1$ means that the constant terms do not matter. Finally, notice that $\left[x^{n+1}\right]\Phi_{S_{\geq 2}}(x)$ gives the number of compositions of $n+1$ with each part at least 2, while $[x^n]\Phi_{S_O}(x)$ gives the number of compositions of n with each part odd. This proves that the two numbers are equal.

Question 3:

In this question, we will consider an alternative decomposition of $\{0,1\}^\star$. Let $A = \{0, 10, 11\}$ and $B = \{\epsilon, 1\}$.

a) Prove that $A^\star B$ is unambiguous.

b) Determine the generating series for $A^\star B$, where the weight of a string s is its length.

c) Conclude that $A^\star B$ generates all binary strings.

Answer 3:

a) We prove this by induction on the length of the string.

Let s be a string of length n generated by $A^\star B$. If $n = 0$, then $s = \epsilon$, and the only way to write the string is to take ϵ from $A^\star$ and ϵ from B. If $n = 1$, then either $s = 0$ or $s = 1$. If $s = 0$, the only way to write the string is to take 0 from $A^\star$ and ϵ from B. If $s = 1$, the only way to write the string is to take ϵ from $A^\star$ and 1 from B.

Finally, suppose $n \geq 2$. As the strings of B have length at most 1, s must start with one or more copies of A. Therefore, either $s = 0s'$, $s = 10s'$, or $s = 11s'$ must hold true, where s' is a (possibly empty) string of shorter length. In any of these three cases, s cannot be written starting with another element of A. Furthermore, by inductive hypothesis, s' is unambiguous. Therefore, s must also be unambiguous.

We will note here that a small modification of this proof can also show that $A^\star B$ generates all binary strings, but we will defer this later to show another way to prove this.

b) First, note that

$$\begin{aligned}
\Phi_A(x) &= x + 2x^2 \\
\Phi_B(x) &= 1 + x
\end{aligned}$$

Using product lemma, we have

$$\begin{aligned}
\Phi_{A^\star B}(x) &= \frac{1}{1 - \Phi_A(x)} \cdot \Phi_B(x) \\
&= \frac{1}{1 - (x + 2x^2)} \cdot (1 + x) \\
&= \frac{1}{1 - 2x}
\end{aligned}$$

c) We can see that

$$\begin{aligned}
[x^n]\Phi_{A^\star B}(x) &= [x^n]\sum_{i \geq 0}(2x)^i \\
&= 2^n
\end{aligned}$$

As there are only 2^n distinct binary strings of length n, and $A^\star B$ generates that many distinct binary strings of length n, it must have in fact generated all of them.