1. (a) [**4 marks**] Let a_n be the number of compositions of integer n with an even number of parts, where each part has size at least three. Show that the generating function for the sequence $\{a_n\}$ is

$$\sum_{n\geq 0} a_n x^n = \frac{(1-x)^2}{1-2x+x^2-x^6}.$$

Solution

Let $\mathbb{N}_{\geq 3} = \{3,4,5,\ldots\}$. Then the set of compositions is

$$S = \mathbb{N}_{\geq 3}^0 \cup \mathbb{N}_{\geq 3}^2 \cup \mathbb{N}_{\geq 3}^4 \cup \cdots.$$

Hence,

$$\Phi_{\mathbb{N}_{\geq 3}}(x) = x^3 + x^4 + x^5 + \cdots = \frac{x^3}{1-x},$$

and

$$\begin{aligned}
\Phi_S(x) &= \left(\frac{x^3}{1-x}\right)^0 + \left(\frac{x^3}{1-x}\right)^2 + \left(\frac{x^3}{1-x}\right)^4 + \cdots \\
&= \frac{1}{1-\left(\frac{x^3}{1-x}\right)^2} \\
&= \frac{(1-x)^2}{1-2x+x^2-x^6}.
\end{aligned}$$

(b) [**4 marks**] Find a linear recurrence satisfied by the a_n's in (a), and find enough initial conditions to determine the sequence $\{a_n\}$ completely.

Solution

By Theorem 1.26 on page 36 of the course notes, the recurrence relation for the a_n's is

$$a_n - 2a_{n-1} + a_{n-2} - a_{n-6} = 0 \quad \forall n \geq 6.$$

The initial conditions are

$$\begin{aligned}
a_0 &= 1, \\
a_1 - 2a_0 &= -2 \text{ which implies that } \quad a_1 = 0, \\
a_2 - 2a_1 + a_0 &= 1 \text{ which implies that } \quad a_2 = 0, \\
a_3 - 2a_2 + a_1 &= 0 \text{ which implies that } \quad a_3 = 0, \\
a_4 - 2a_3 + a_2 &= 0 \text{ which implies that } \quad a_4 = 0, \\
a_5 - 2a_4 + a_3 &= 0 \text{ which implies that } \quad a_5 = 0,
\end{aligned}$$

(c) [**2 marks**] Using your recurrence, find a_{10}.

Solution

We apply the recurrence relation of part (b) above to evaluate the terms in the following table:

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	0	0	0	0	0	1	2	3	4	5

Therefore, $a_{10} = 5$.

2. (a) **[5 marks]** Give a decomposition that uniquely generates the binary strings in which an odd block of 1's is never followed by an even block of 0's.

Solution

We modify the block decomposition for all binary strings $\{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^*$. The required decomposition is

$$\{0\}^*(\{1\}\{11\}^*\{0\}\{00\}^* \cup \{11\}\{11\}^*\{0\}\{0\}^*)^*\{1\}^*$$

Recall that the empty string is not a block; in particular the string of 0's having length 0 is not a block. For this reason, strings in this set are permitted to end with a a block of 1's of odd length.

(b) **[5 marks]** In assignment 3, we showed that

$$\{1\}^* \left(\{0\}\{0\}^*(\{1\}\{1\}^* \setminus \{111\})\right)^* \{0\}^*\{\varepsilon, 0111\}$$

is a decomposition for the set of strings S that do not contain the substring 01110. Determine the generating function for S as a rational function (i.e., a polynomial divided by a polynomial).

Solution.

The generating function for $\{1\}\{1\}^* \setminus \{111\}$ is

$$\frac{x}{1-x} - x^3 = \frac{x - x^3 + x^4}{1-x},$$

and the generating function for $\{0\}\{0\}^*$ is $\frac{x}{1-x}$. Therefore, the generating function for this set of strings is

$$\begin{aligned} f(x) &= \frac{1}{1-x}\left(\frac{1}{1-\left(\frac{x}{1-x}\right)\left(\frac{x-x^3+x^4}{1-x}\right)}\right)\frac{1}{1-x}(1+x^4) \\ &= \frac{1+x^4}{1-2x+x^4-x^5}. \end{aligned}$$

3. Let L denote the set of binary strings that do not contain the string 101. Let M denote the set of binary strings that contain exactly one copy of 101, with that copy occupying the last three positions.

(a) **[5 marks]** Show that $L\{101\} = M \cup M\{01\}$.

Solution.

To establish this result, we must show that

$$L\{101\} \subseteq M \cup M\{01\} \text{ and } L\{101\} \supseteq M \cup M\{01\}.$$

Let ℓ be a string of L. If ℓ ends in 10, $\ell = k10$ where k does not contain the substring 101 because ℓ does not contain the substring 101. Therefore $k101 \in M$, and

$$\ell 101 = k10101 \in M\{01\}.$$

If ℓ does not end with 10, then it must end with 11, 01, or 00. In each case, it is easy to confirm that $\ell 1 \notin M$. Therefore,

$$\ell 101 \in M.$$

Hence,

$$L\{101\} \subseteq M \cup M\{01\}.$$

Let m be any string in the set M. By definition, $m = n101$ and n has no occurrence of the substring 101. Hence $n \in L$, and $m = n101 \in L\{101\}$. This establishes that $M \subseteq L\{101\}$. Furthermore $m01 = n10101$. Since n has no occurrence of the substring 101, neither does the string $n10$. Hence, $n10 \in L$ and we can conclude that $m01 = (n10)101 \in L\{101\}$. This establishes that

$$M \cup M\{01\} \subseteq L\{101\}.$$

Combining these two facts, we get that

$$L\{101\} = M \cup M\{01\}.$$

(b) **[5 marks]** Using the previous relation and the identity (which you do not need to prove)

$$\{\varepsilon\} \cup L\{0,1\} = L \cup M,$$

derive the generating function for L. (As usual, the weight of a string is just its length.)

Solution.

Let $\ell(x)$ be the generating function for the set of strings L and let $m(x)$ be the generating function for the set of strings M. From part (a), we get the functional equation

$$x^3\ell(x) = m(x) + x^2 m(x) = m(x)(1 + x^2).$$

Simplifying this expression, we get

$$m(x) = \frac{x^3\ell(x)}{1 + x^2}.$$

From the equation in part (b), we get the functional equation

$$1 + 2x\ell(x) = \ell(x) + m(x).$$

Substituting for $m(x)$, we get

$$1 + 2x\ell(x) = \ell(x) + \frac{x^3\ell(x)}{1 + x^2}.$$

Therefore,

$$\ell(x) = \frac{1 + x^2}{1 - 2x + x^2 - x^3}.$$

4. **[10 marks]** Find the solution to the recurrence

$$a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3}, \quad n \geq 3$$

with $a_0 = 1$, $a_1 = 1$ and $a_2 = 25$.

Solution.
The characteristic polynomial for this recurrence relation is

$$x^3 - x^2 - 8x + 12 = (x-2)^2(x+3).$$

Therefore, the general solution for this recurrence relation is of the form

$$a_n = (A + Bn)2^n + C(-3)^n \quad \forall n \geq 0.$$

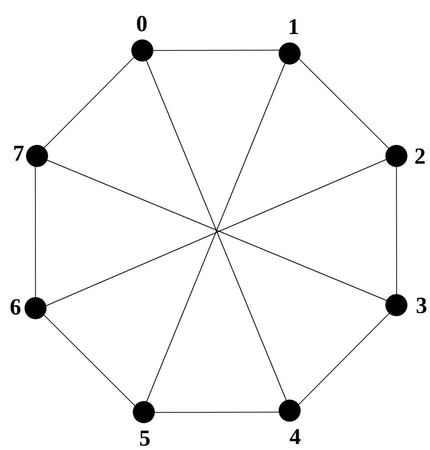
Applying the initial conditions, we get the following 3 equations in 3 unknowns:

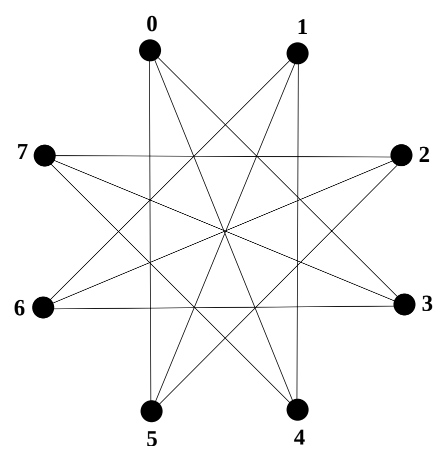
$$\begin{aligned} n = 0: \quad A + C &= 1 \\ n = 1: \quad 2A + 2B - 3C &= 1 \\ n = 2: \quad 4A + 8B + 9C &= 25. \end{aligned}$$

The solution to these equations is $(A, B, C) = (0, 2, 1)$. Therefore, $a_n = 2n2^n + (-3)^n \quad \forall n \geq 0$.

5. **[4 marks]** We construct graphs G and H with vertex set $V = \{0, 1, \ldots, 7\}$ as follows. Two vertices i and j of V are adjacent in G if and only if the difference $i - j$ modulo 8 lies in $\{1, 4, 7\}$. Two vertices i and j of V are adjacent in H if and only if the difference $i - j$ modulo 8 lies in $\{3, 4, 5\}$.

(a) Draw the graphs G and H.





(b) **[6 marks]** Show that the map f given by

$$f(i) = 3i, \quad \text{modulo } 8$$

is an isomorphism from G to H.

Solution.
Let us re-label every vertex i of graph G by $3i$ reduced modulo 8.

$i \pmod 8$	0	1	2	3	4	5	6	7
$3i \pmod 8$	0	3	6	1	4	7	2	5

By inspection, this is a bijection from the vertices of G to the vertices of H. The edge $\{i,j\}$ in G having $i - j \equiv 1 \pmod 8$ is mapped to the edge $\{3i, 3j\}$ having $3i - 3j \equiv 3 \pmod 8$ in H. Similarly, the edge $\{i,j\}$ in G having $i - j \equiv 7 \pmod 8$ is mapped to the edge $\{3i, 3j\}$ having $3i - 3j \equiv 5 \pmod 8$ in H, and the edge $\{i,j\}$ in G having $i - j \equiv 4 \pmod 8$ is mapped to the edge $\{3i, 3j\}$ having $3i - 3j \equiv 4 \pmod 8$ in H.

Therefore, the bijection f maps every edge of G to an edge of H. Since G and H both have 12 edges, every pair of non-adjacent vertices of G is mapped to a non-adjacent pair of vertices in H. Therefore, graphs G and H are isomorphic.

6. (a) **[5 marks]** Let G be a graph where every vertex has degree at least k, $k \geq 2$. Show that G contains a path of length at least k.

Solution.
See the solution to problem 4(a) in assignment 5.

(b) **[5 marks]** Let H be a connected graph where every vertex has even degree. Does H have a bridge? Justify your answer.

Solution.
We will prove that such a graph H does not have a bridge.

To obtain a contradiction, we assume there is a graph H in which every vertex has even degree having a bridge $e = \{x, y\}$. By Lemma 2.23, graph $H - e$ has 2 components, component H_x containing vertex x and component H_y containing y. Now in graph H_x, vertex x is of odd degree, and every other vertex has even degree. But this is impossible because every graph has an even number of vertices of odd degree by Theorem 2.7. Since our assumption that H has a bridge leads to a contradiction, it follows that H does not have a bridge. This establishes the result.