SOS Spring 2012 MATH 239 Midterm Review Package

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This package includes a list of theorems and definitions that you need to know for your midterm, as well as several examples.

ALSO AVAILABLE ON GOOGLE DOCS:
https://docs.google.com/document/d/1A0XuuHAoif8S6X-ruWhsMpwhvFVvGtBf94oqDBEnlZs/edit

Practice, practice, practice, and best of luck! 8D

Topics Covered

Enumeration:

Basic set definitions
Binomial coefficients
Combinatorial proofs
Bijections
Formal power series
Manipulating generating series using sum and product lemmas
Compositions
Recurrence relations
Binary strings
Solving recurrence relations
Asymptotics

Graph Theory:

Graphs (edges, degrees)
Isomorphisms
Bipartite graphs
Paths and cycles
Connectivity (including bridges)

Basic set definitions

For a set S, S denotes the number of elements in S.

Sums and products of sets:

Def: If A and B are sets, then the union AB is defined by
AB={x:xA or xB}
If A and B are disjoint sets, ie. AB=, then AB=A+B.
Def: The Cartesian product AB of sets A and B is the set of all ordered pairs whose first element is an element of A and second element is an element of B, that is
AB = {(a,b):aA, bB}
Then AB=AB.
Def: The Cartesian power Ak is defined inductively by set A1=A and Ak+1=AAk. This is the set of k-tuples of elements from A. Then Ak=Ak.
Note: [n] denotes the set {1, 2, …, n}.

Binomial coefficients

Binomial coefficients are used to prove identities algebraically (as opposed to combinatorial proofs discussed in the next section) and to determine coefficients in generating series, which will be discussed in the section on manipulating generating series.

Theorem 1.3.1: For non-negative integers n and k, the number of k-element subsets of an n-element set is

n choose k=n(n-1)(n-2)...(n-k+1)k!

(Proof on page 3 of course notes)

Fact: When $0 \leq k \leq n$, then

(n choose k) = n!(n-k)!k!

which implies (n choose k)=(n choose n-k)

Theorem 1.3.2 (Binomial Theorem): For any non-negative integer n,

(1+x)n=k=0n(n choose k)xk

(Proof on page 5 of course notes)

Theorem (Problem 1.3.3): Let n and k be non-negative integers. Then

(n+k choose n) =i=0k(n+i-1 choose n-1)

(Proof on page 6 of course notes)

Theorem 1.6.5 (Negative Binomial Theorem): For any positive integer k,

(1-x)-k=n0(n+k-1 choose k-1)xn

(Proof on page 19 of course notes)

Fact: (n choose k) = (n-1 choose k) + (n-1 choose k-1)

Example from Problem set 1.3 question 4 (Page 7 of course notes)

Let n be an integer so that n1. Prove that

n(2)n-1=k=0n-1(k+1)(n choose k+1)

Solution:

Algebraic proof: The Binomial Theorem states that:

(1+x)n=k=0n(n choose k)xk

Differentiating both sides, we get

n(1+x)n-1=k=1nk(n choose k)xk-1

Now let x=1, we get

n(2)n-1=k=1nk(n choose k)

But k=1nk(n choose k)=k=0n-1(k+1)(n choose k+1)

So n(2)n-1=k=0n-1(k+1)(n choose k+1)

Combinatorial proofs

When you write a combinatorial proof:

1. You are given an identity to prove (Left side = Right side)
2. You need to describe the left side as a way of counting all the elements in a certain set A, and the right side as another way of counting all the elements in the same set
3. Since we are counting elements of the same set, then no matter how we count it, we must get the same result. **BOOM Left side = Right side! Q.E.D.**

Note: Finding the right ways to count sets takes creativity and practice. The more you practice, the more situations you'll see, the better you'll be at it.

Example from Problem set 1.3 question 4 (Page 7 of course notes)

Let n be an integer so that n1. Prove that

n(2)n-1=k=0n-1(k+1)(n choose k+1)

Solution:

Combinatorial proof: For each subset A of {1,2,...,n}, and each element a of A, consider the ordered pair (a,A\{a}).

On the one hand, there are n possibilities for the element a and then 2n-1 possibilities for the subset A\{a} (this is any subset of {1,2,...,n}\{a}). Thus, there are n2n-1 such pairs.
On the other hand, given A, there are A choices for a. So for each k=A, there are (n choose k) ways to choose A and k ways to choose a, which is k(n choose k). Summing over all the possible values of k (from 1 to n, since we need to have a least 1 element in A to select a from A) we get n(2)n-1=k=1nk(n choose k)
But k=1nk(n choose k)=k=0n-1(k+1)(n choose k+1)
So we get n(2)n-1=k=0n-1(k+1)(n choose k+1).

Bijections

We use bijections to prove that two sets, S and T have the same cardinality. Ie. S=T
<u>Def</u>: Let S and T be sets. Let f:ST be a function (or mapping).
f is <u>1-1</u> or <u>injective</u> if for any x1,x2S, f(x1)=f(x2) implies x1=x2.
f is <u>onto</u> or <u>surjective</u> if for all yT, there exists xS such that f(x)=y.
f is a <u>bijection</u> if it is both 1-1 and onto.
<u>Theorem (from Supplementary 1)</u>: If a function f:ST has an inverse, then f is a bijection. (Proof on Supplementary 1 on your course website)

Proving a bijection:

- Show that f is 1-1 and onto through the definition.
- Show that f has an inverse (by providing the inverse g:TS) and proving that the inverse works by showing that for some xS, g(f(x))=x. (Write out the steps explicitly to ensure that you earn full marks!)

Example from Tutorial 1 Additional exercises question 2

Consider the k-tuples (T1,...,Tk)where each Ti[n]. In other words, if P is the set of all subsets of [n], then such a k-tuple is in the Cartesian product Pk. We define the following two subsets of Pk:
a) S is all such k-tuples where T1T2...Tk
b) T is all such k-tuples that are mutually disjoint, ie TiTj= for any ij.
Find a bijection between S and T. What is the cardinality?

Solution:

Let s=(S1,...,Sk)S. We define f:ST as follows:
f((S1,...,Sk)) = (S1,S2\S1,S3\S2,...,Sk\Sk-1) =(U1,...,Uk)
Note that since SiSi+1 for all i, then for all ji, SjSi. Furthermore,SjSiSi+1. Then Si+1\Si does not contain the elements in Sjeither. ie. Sj(Si+1\Si)= for all ji.
Since for all i, UiSi, then Uj(Si+1\Si)=UjUi+1= for all ji. So (U1,...,Uk)T.
Let v=(V1,...,Vk)T define g:TS as follows:
g((V1,...,Vk))=(V1,V2V1,V3V2V1,...,Vk...V1)=(R1,...,Rk)
It is easy to see that R1R2...Rk, so (R1,...,Rk)S.
Then for s=(S1,...,Sk)S, we have
g(f((S1,...,Sk)))=g((S1,S2\S1,S3\S2,...,Sk\Sk-1))
=g(S1,(S2\S1)S1,(S3\S2)(S2\S1)S1,...,(Sk\Sk-1)...(S2\S1)S1)
=(S1,...,Sk)
Note that (S2\S1)S1=S2 , (S3\S2)(S2\S1)S1= (S3\S2)S2=S3, and so on.
So the inverse of f is g. Then f is a bijection.
Consider the cardinality of set T. Consider a k-tuple in T, v=(V1,...,Vk)T. Since v is a k-tuple of mutually disjoint subsets, then for each element in [n], it is in one of V1,...,Vk or it is not in any of them at all. So there are k+1 ways to place each element in [n]. Hence there are (k+1)n ways to form a k-tuple in T, ie. the cardinality of T is (k+1)n.

Formal power series

<u>Def</u>: Let (a0, a1, a2, ...)be a sequence of rational numbers; then A(x)=a0+a1x+a2x2+...

is called a <u>formal power series</u>. We say that an is the <u>coefficient</u> of xn in A(x) and write an=[xn]A(x).

<u>Def</u>: For formal power series A(x) and B(x), define A(x)+B(x)=i0(ai+bi)xi

<u>Def</u>: For formal power series A(x) and B(x), defineA(x)B(x)=i0(j=0iajbi-j)xi

<u>Def</u>: A(x)=B(x) means that ai=bi for all i0

<u>Theorem 1.5.2</u>: Let A(x)=a0+a1x+a2x2+... , P(x)=p0+p1x+p2x2+... , Q(x)=1-q1x-q2x2-... be formal power series. Then Q(x)A(x)=P(x) iff for each n0, an=pn+q1an-1+q2an-2+...+qn-1a0

(Proof on page 13 of course notes)

<u>Corollary 1.5.3</u>: Let P(x) and Q(x) be formal power series. If the constant term of Q(x) is non-zero, then there is a formal power series A(x) satisfying Q(x)A(x)=P(x)… Moreover, A(x) is unique.

(Proof on page 14 of course notes)

<u>Def</u>: We say that B(x) is the <u>inverse</u> of A(x) if A(x)B(x)=1. We denote this by B(x)=A(x)-1 or by B(x)=1A(x).

<u>Useful fact</u>: The inverse of 1+x+x2+...is 1-x, ie 1+x+x2+...=11-x.

<u>Finite Geometric Series Theorem</u>: Let k be a non-negative integer. Then

1+x+x2+...+xk=1-xk+11-x

<u>Theorem 1.5.7</u>: A formal power series has an inverse if and only if it has a non-zero constant term. Moreover, if an inverse exists, then the inverse is unique.

(Proof on page 15 of course notes)

<u>Def</u>: The <u>composition</u> of formal power series A(x)=a0+a1x+a2x2+... and B(x) is defined by A(B(x))=a0+a1B(x)+a2(B(x))2+...

<u>Theorem 1.5.9</u>: If A(x) and B(x) are formal power series with the constant term of B(x) equal to zero, then A(B(x)) is a formal power series.

(Proof on page 16 of course notes)

Example

What is the inverse of A(x)=2+6x-14x2?

Solution:

We have A(x)=2(1-(-3x+7x2))

Then A(x)-1=12(1+(-3x+7x2)+(-3x+7x2)2+...)

Example

Find [xn](1+x2)6(1-2x)-3.

Solution:

[xn](1+x2)6(1-2x)-3=[xn]i0(6 choose i)x2ij0(2+j choose 2)2jxj

=pairs (i,j) 2i+j=n(6 choose i)(2+j choose 2)2j

=i=0floor(n/3)(6 choose i)(2+n-2i choose 2)2n-2i

Manipulating generating series using sum and product lemmas

The typical enumeration problem goes like this: “How many Xs are there (in some set)?”

1. Define a set S.
2. Define a weight function w on set S that assigns to each ∊ S a non-negative integer w() called the <u>weight</u> of .

The question translates to: How many elements of S have weight k?

<u>Def</u>: Let S be a set with weight function w. The <u>generating series</u> for S with respect to w

S(x)=Sxw()

or S(x)=k0akxk
where ak is the number of elements of S with weight k.
Theorem 1.4.3: Let S(x) be the generating series for a finite set S with respect to a weight function w. Then,
1) S(1)=S
2) the sum of the weights of the elements in S is 'S(1)
3) the average weight of the elements in S is 'S(1)/S(1).
(Proof on page 9 of course notes)
Theorem 1.6.1 The Sum Lemma: Let (A, B) be a partition of a set S where AB = and AB = S
Then,S(x) = A(x) + B(x).
More generally,AB (x) = A(x) +B(x) - AB(x).
(Proof on page 17 of course notes)
Theorem 1.6.2 The Product Lemma:
Let A,B be sets of configurations with weight functions and respectively.
If () = (a) + (b) for each = (a,b) A B
Then AB(x) = A(x) B(x)
(Proof on page 18 of course notes)

Example from Tutorial 2 Additional exercises question 1

How many ways can you make up n cents using an unlimited supply of pennies, nickels, dimes and quarters? For example, 7 cents can be made up in two ways: 7 pennies, or 2 pennies and 1 nickel. How would this change if you are allowed to use up to 42 nickels? Express your answers as coefficients of generating series.

Solution:

Let S be the set of ways to make up any number of cents out of the given coins. Let P, N, D, Q, be the respective sets of numbers of pennies, nickles, dimes, and quarters. Let the weight function be the value of the coins.
Then S=PNDQ
We have P(x)=1+x+x2+x3+...=(1-x)-1
N(x)=1+x5+x10+x15+...=(1-x5)-1
D(x)=1+x10+x20+x30+...=(1-x10)-1
Q(x)=1+x25+x50+x75+...=(1-x25)-1
Then by the Product Lemma,
S(x)=P(x)N(x)D(x)Q(x)=(1-x)-1(1-x5)-1(1-x10)-1(1-x25)-1
Then the number of ways to make up n cents is
[xn]S(x)=[xn](1-x)-1(1-x5)-1(1-x10)-1(1-x25)-1
If we had only up to 42 nickels, then by the Finite Geometric Series Theorem,
N(x)=1+x5+x10+x15+...+x425=1-x4351-x5
Then[xn]S(x)=[xn](1-x215)(1-x)-1(1-x5)-1(1-x10)-1(1-x25)-1

Compositions

Def: For non-negative integer n and k, a composition of n with k parts is an ordered list (c1,...,ck) of positive integers c1,...,ck such that c1+...+ck=n..
The integers c1,...,ck are the parts of the composition.
Def: There is an empty composition of 0 with 0 parts.
Useful fact: There are ((n-1) choose (k-1)) compositions of n with k parts for nk1.
(Proof on page 25 of course notes)
Useful fact: There are ((n+k-22) choose (n-k2)) k-part compositions of n in which each part is odd.
(Proof on page 26 of course notes)

Example

Let n and k be positive integers and let an,k denote the number of compositions of n with precisely k parts, in which each part is an element of the set U={1, 5, 9, 13,..., 89}. Express an,kas the coefficient of a generating function.

Solution:

Let SUk be the set of compositions with k parts, where each part is in U.

Then SUk={U}k. (k factors in the Cartesian product)

We have U(x)=x+x5+x9+...+x89

=x(1+x4+x8+...+x88)

=x(1+x4+x24+...+x224)

=x1-x2341-x4

=x1-x921-x4

Then by Product Lemma,

SUk(x)=xk1-x921-x4k

So an,k=[xn]SUk(x)

=[xn]xk1-x921-x4k

=[xn-k]1-x921-x4k

Recurrence relations

Suppose we have

S(x)=n0anxn=P(x)Q(x).

Then Q(x)n0anxn=P(x).

We expand this and compare the coefficients to find a <u>recurrence relation</u> (ie. an in terms of an-1, an-2, and so on) and its initial conditions (the number of initial conditions depends on P(x) and Q(x)).

Example

Let S be a set with weight function w whose generating function is

S(x)=x2+x31-3x3+x4

Derive a recurrence relation with initial conditions for the sequence an=[xn]S(x) for all n0 and determine a5.

Solution:

Let

n0anxn=x2+x31-3x3+x4

Then n0anxn-3n0anxn+3+n0anxn+4=x2+x3

n0anxn-3n3an-3xn+n4an-4xn=x2+x3

Comparing coefficients:

For x0: LHS=a0, RHS=0a0=0

x1: LHS=a1, RHS=0a1=0

x2: LHS=a2, RHS=1a2=1

x3: LHS=a3-3a0, RHS=1a3=1

For n4, LHS=an-3an-3+an-4, RHS=0an=3an-3-an-4

This is the recurrence relation with initial conditions a0=0, a1=0, a2=1, a3=1.

Then a5=3a2-a1=3(1)-0=3.

Binary Strings

Def: A binary string, or {0,1}-string, is a string of 0's and 1's, its length is the number of occurences of 0 and 1 in the string. Usually the weight function on binary strings is the length.
Def: There is a unique binary string of length 0 denoted , the empty string.
Def: If a=a1a2...an and b=b1b2...bm where each ai{0,1} and each bi{0,1}, then the concatenation ab of a and b is a1a2...anb1b2...bm.
Def: Let A and B be sets of binary strings. Then AB={ab,aA,bB}.
Note that AB={ab:(a,b)AB}
Def: Let A be a set of binary strings. Define A*={}AAAAAA...={}AA2A3...
Fact: The set of all binary strings is then given by {0,1}*.
Def: A block in a binary string is a maximal nonempty substring consisting entirely of 0's or entirely of 1's.
Def: For binary strings a and b, b is a substring of a if a=cbd for some binary strings c and d.
Def: AB is ambiguous if there exist distinct pairs (a1,b1) and a2,b2 in AB with a1b1=a2b2. Otherwise, AB is unambiguous. ie. AB is unambiguous if every string in AB uniquely decomposes into a string in A concatenated with a string in b.
Def: A* is unambiguous if the sets {}, A, A2, … are disjoint and for each iN0, Ai is unambiguous. Alternatively, for every sA*, there is exactly one k such that sAk and there exists exactly one k-tuple with a1,a2,...,ak with a1,a2,...,akA and s=a1a2...ak.
Sum Lemma for Binary Strings: If A and B are sets of binary strings and AB= , then AB(x)=A(x)+B(x)
Theorem 2.6.1 (Product Lemma and *-Lemma for Binary Strings): Let A, B be sets of binary strings.
1) If AB is unambiguous, then AB(x)=A(x)B(x)
2) If A* is unambiguous, then A*(x)=(1-A(x))-1
(Proof on page 37 of course notes)
0-decomposition: {0,1}* = {1}*({0}{1}*)* and this is unambiguous.
1-decomposition: {0,1}* = {0}*({1}{0}*)* and this is unambiguous.
Block decomposition: {0,1}*={1}*({0}{0}*{1}{1}*)*{0}* or {0}*({1}{1}*{0}{0}*)*{1}* and this is unambiguous.
Def: A recursive definition of a set of binary strings, S, defines S in terms of itself. When writing a recursive definition, consider what every element of S begins with or ends with.

Example

Let S be be the set of binary strings in which every 0 that has a 1 somewhere to its right is in a block of at most two 0's. Find an unambiguous expression for S and the generating function for S with respect to the weight function, which is the length of the string.

Solution:

The block decomposition (which is unambiguous) is {1}*({0}{0}*{1}{1}*)*{0}*.
Then by making a restriction on the block decomposition, we have
S={1}*({0,00}{1}{1}*)*{0}* (Note that a block of 0's without any 1's to the right of it can be of any length.)
Then S is unambiguous.
We have {1}*(x)={0}*(x)=(1-x)-1
{0,00}(x)=x+x2
{1}{1}*(x)=x(1-x)-1 by the Product Lemma for binary strings

{0,00}{1}{1}*(x)=(x2+x3)(1-x)-1 by the Product Lemma for binary strings
Then by the *-Lemma and the Product Lemma, we have
S(x)=(1-x)-1(1-(x2+x3)(1-x)-1)-1(1-x)-1
=(1-2x+x4)-1 (Algebraic details omitted)

Solving recurrence relations

Homogeneous relations

Def: For a general homogeneous recurrence relation cn+q1cn-1+q2cn-2+...+qkcn-k=0 for all nk, with given initial conditions c0,c1,...,ck-1, the characteristic polynomial is
C(y)=yk+q1yk-1+q2yk-2+...+qk-1y+qk=0

Theorem 3.2.2: Suppose (cn)n0 satisfies the a homogeneous recurrence relation. If the characteristic polynomial of this recurrence has root i with multiplicity mi, for i=1,...,j, then the solution to the recurrence is
cn=P1(n)1n+...+Pj(n)jn
where each Pi(n) is a polynomial in n with degree < mi, and these polynomials are determined by the initial conditions c0,...,ck-1.

To solve a homogeneous recurrence relation:

1. Find the characteristic polynomial
2. Find the roots of the characteristic polynomial (the i) and their multiplicities (the mi).
3. Write out the general form of the solution, which is cn=P1(n)1n+...+Pj(n)jn
where each Pi(n) is a polynomial in n with degree < mi
4. Use the initial conditions to determine the coefficients of each polynomial by substituting n=0 for c0, n=1 for c1, n=2 for c2, and so on, then solving the system of equations. (Depending on how many initial conditions you have.)

To find a homogeneous recurrence from a solution:

1. Look at which numbers have the power of n (these are the roots of the characteristic polynomial), and look at the polynomial in n that they are multiplied with (the degree of the polynomial is the multiplicity of the root).
2. Expand the characteristic polynomial, and find the general homogeneous recurrence relation based on the definition of characteristic polynomial.
3. Check that the general recurrence is satisfied by the solution you are given.
4. Specify the initial conditions by substituting n=0,1,... into the solution. If your recurrence is an+q1an-1+q2an-2+...+qkan-k=0, then you should have k-1 initial conditions.

Non-homogeneous relations

The general form of the recurrence is bn+q1bn-1+q2bn-2+...+qkbn-k=f(n) (*)
for all nk, where f(n) is a function of n, and b0,...,bk-1 are the initial conditions.

Theorem 3.3.1: Suppose that the sequence an is a solution to (*) for all nk. (ie. The initial conditions don't apply to an.)
Then the general solution to (*) is given by bn=cn+an, n0
where cn is the solution to the homogeneous recurrence relationcn+q1cn-1+q2cn-2+...+qkcn-k=0, where the coefficients in cn are chosen so that:
b0=c0+a0, b1=c1+a1, …, bk-1=ck-1+ak-1
(Proof on page 62 of course notes)

To solve a non-homogeneous recurrence relation:

1. Make an educated guess about what the specific solution an is, and check that an satisfies the recurrence relation.
2. Find cn, the solution to the homogeneous recurrence relation as outlined in "To solve a homogeneous recurrence relation", but instead of step 4, set the coefficients of cn so that b0=c0+a0, b1=c1+a1, …, bk-1=ck-1+ak-1

3. Then the solution is bn=cn+an, n0.

Example

Solve bn-4bn-1+5bn-2-2bn-3=24(-1)n for all n3, where b0=-1, b1=-3, b2=2.

Answer

bn=2(-1)n-2+3n-2n for all n0

Details provided at session.

Asymptotics

Def: The sequence cn is asymptotic to the function g(n) as n, if

ncng(n)=1

We write cn~g(n).

Note: The dominating term in cn is g(n).

Graphs

Def: A graph G is a finite nonempty set, V(G), of objects, called vertices, together with a set, E(G), of unordered pairs of distinct vertices. The elements of E(G) are called edges.

Def: If e = {u, v} then we say that u and v are adjacent vertices, and that edge e is incident with vertices u and v. We can also say that the edge e joins u and v.

Note: In the graphs we study, edges are unordered pairs of vertices, ie {u,v} and {v,u} represent the same edge. Also, multiple edges and loops are forbidden.

Def: Graphs with multiple edges and loops are multigraphs.

Def: Vertices adjacent to a vertex u are called neighbours of u. The set of neighbours of u is denoted N(u).

Def: The number of edges incident with a vertex v is called the degree of v, and is denoted by deg(v).

Def: The maximum degree is the largest degree of any vertex in G.

Theorem 4.3.1: For any graph G, we have

vV(G)deg(v)=2E(G)

(Proof on page 93 of course notes)

Corollary 4.3.2 (The Handshake Theorem):The number of vertices of odd degree in any graph is even.

(Proof on page 93 of course notes)

Corollary 4.3.3: The average degree of a vertex in the graph G is2E(G)V(G).

Def: A graph in which every vertex has degree k, for some fixed k, is called a k-regular graph (or just a regular graph).

Def: A complete graph is one in which all pairs of distinct vertices are adjacent. (Thus each vertex is joined to every other vertex). The complete graph with p vertices is denoted by Kp, p1.

Note: Kp has (p choose 2) edges, so no graph with p vertices can have more than (p choose 2 edges).

Example from Tutorial 6 Additional exercises question 3

Married couple Mario and Peach invited 3 other couples to the castle on the mountain for a cake party (and it’s no lie). During the party, some handshaking took place with the restriction that a person cannot shake hands with themselves nor with their spouse. After all the shakings were done, Peach went around to ask the 7 others in the party how many people they shook hands with, and she received a different answer from everyone. How many hands did Mario shake? How many hands did Peach shake? What happens if Mario and Peach invited n couples to the party?

Answer:

Mario and Peach both shook 3 hands. If Mario and Peach invite n couples, they each shake n hands. Details are given at the session.

Example

Which of the following sequences are degree sequences of a graph on seven vertices? If it is, give an example of such a graph; if not, explain why not.

a) 3, 3, 3, 3, 3, 3, 3
b) 4, 4, 4, 3, 3, 3, 3
c) 6, 6, 3, 2, 2, 2, 1
d) 6, 5, 5, 5, 5, 5, 5

Solution:

a) The sequence is not the degree sequence of a graph on seven vertices because the sum of the sequence is 21, which is odd and by theorem (4.3.1), the sum of the degrees of a graph must be even.
b) This is the degree sequence of a graph. A drawing is presented at the session.
c) This is not the degree sequence of a graph on seven vertices. Since 6 is the maximum degree and two vertices have the maximum degree, that means all other vertices must be adjacent to each of these two vertices, so the minimum degree in the graph must be 2, whereas this sequence contains 1.
d) This is the degree sequence of a graph. A drawing is presented at the session.

Isomorphism

Def: Two graphs G1,G2 are isomorphic if there exists a bijection f: V(G1) -> V(G2) such that vertices f(u) and f(v) are adjacent in G2 iff u and v are adjacent in G1.

To show 2 graphs are isomorphic: Exhibit an isomorphism and check that adjacency is preserved.

To show 2 graphs are not isomorphic: Find some feature of one graph that is not shared by the other, where the feature does not depend on vertex labels.

Some questions you may consider about the two graphs are:
- Do they have the same number of vertices?
- Do they have the same degree sequence?
- Are they both bipartite/not bipartite? (See below section on bipartite graphs)
- Do they have the same cycle lengths? (See section on cycles and paths)
- Do they have the same subgraphs? (See section on cycles and paths)

Example covered at session

Bipartite graphs

Def: A graph in which the vertices can be partitioned into two sets A and B, so that all edges join a vertex in A to a vertex in B, is called a bipartite graph, with bipartition (A,B).

Def: The complete bipartite graph Km,n has all vertices in A adjacent to all vertices in B, with A=m and B=n.

Def: For n>=0 the n-cube is the graph whose vertices are the {0,1} - strings of length n, and two strings are adjacent iff they differ in exactly one position.

Useful facts: The number of vertices in the n-cube (n 0) is2n

The number of edges in the n-cube (n 0) is n2n.

The *n-cube* is bipartite.

Example from Tutorial 7 Additional exercises question 1

Let k1. Prove that if G is a k-regular bipartite graph with a bipartition (A, B) of the vertices, then A=B

Solution:

Each vertex in A has degree k, and A has A vertices. Similarly, each vertex in B has degree k,

and B has B vertices. Since (A,B) is a bipartition of G, then the number of edges going out from A (sum of degrees of vertices in A) must equal the number of edges going out from B (sum of degrees of vertices in B).
Then we have kA=kB A=B.

Paths and cycles

Def: A subgraph of a graph G is a graph whose vertex set is a subset U of V(G),and whose edge set is a subset of those edges of G that have both vertices in U.
If V(H) = V(G), that is, H has all vertices of G, we say it is a spanning subgraph of G.
If H is a subgraph of G and H is not equal to G we say it is a proper subgraph of G.
Note: G is a spanning subgraph of itself.
Def: A walk in a graph G from v0to vn ,n0 is an alternating sequence of vertices and edges of G v0e1v1e2...vn-1envn which begins with vertex v0 , ends with vertex vn and ,for 1in,edgeei={vi-1,vi}
The length of a walk is the number of edges in it.
Def: A path is a walk in which all the vertices are distinct.
Theorem 4.6.2: If there's a walk from vertex x to vertex y in G, then there is a path from x to y in G.
(Proof on course notes page 102)
Corollary 4.6.3: Let x,y,z be vertices of G. If there is a path from x to y in G and a path from y to z in G then there is a path from x to z in G.
(Proof on course notes page 103)
Def: A walk is said to be closed if v0=vn.
Def: A cycle is a closed walk with no repeated edges where v0 is the only repeated vertex. The length of a cycle is the number of edges it contains.
Def: A cycle of length n is called an n-cycle.
Note: The shortest cycle we can have is a 3-cycle (a triangle).
Def: A spanning cycle in a graph is known as a Hamilton cycle.
Def: The girth of a graph G is the length of the shortest cycle in G, and is denoted by g(G).

Example from Problem Set 4.6 question 9

Show that if there is a closed walk of odd length in the graph G, then G contains an odd cycle (that is, G has a subgraph which is a cycle on an odd number of vertices).

Solution:

Suppose we have a closed walk of odd length v0...vn where vn=v0. If this walk does not contain any other repeated vertices, then we have an odd cycle. Otherwise, we have vi=vjfor some i<j.
Case 1: If j - i (the number of edges in vivi+1...vj) is odd, then the closed walk vivi+1...vj has odd length and has fewer repeated vertices.
Case 2: If j - i is even, then the closed walk v0...vivj+1...vn has odd length (since we parsed off a section of even length and original closed walk has odd length), and it has fewer repeated vertices.
Either way, we get a closed walk with odd length and fewer repeated vertices. Repeating this process, we end up with a closed walk with odd length and no repeated vertices, ie an odd cycle, in G.

Connectivity (including bridges)

Def: A graph G is connected if for each two vertices x and y, there is a path from x to y.
Theorem 4.8.2: Let G be a graph and let v be a vertex in G.If for each vertex w in G, there is a

path from v to w in G, then G is connected.
(Proof on page 107 of course notes)
Fact: The *n-cube* is connected for each n0.
Def: A component of G is a subgraph C of G such that :
(a) C is connected.
(b) No subgraph of G that properly contains C is connected.
Def: Given a subset X of the vertices of G, the cut induced by X is the set of edges that have exactly one end in X.
Theorem 4.8.5 A graph G is not connected iff there exists a proper nonempty subset X of V(G) such that the cut induced by X is empty.
(Proof on page 109 of course notes)
To show a graph is connected: Use the definition of a connected graph, or find a vertex v joined to all other vertices by paths.
To show a graph is not connected: Find a proper non-empty subset X of V(G) that induces an empty cut.
Def: If e is an edge in G, we denote by G-e (or by G\e) the graph whose vertex set is V(G) and whose edge set is E(G)\{e}. (So G-e is the graph obtained from G by deleting the edge e.)
Def: An edge e of G is a bridge if G\e has more components than G\e*(synonym: cut-edge)*
Lemma 4.9.2: If e= {x,y} is a bridge of a connected graph G then G-e has precisely two components; furthermore, x and y are in different components.
(Proof on page 110 of course notes)
Theorem 4.9.3: An edge e is a bridge for a graph G iff it is not contained in any cycle of G.
(Proof on page 110 of course notes)
Corollary 4.9.4: If there are two distinct paths from vertex u to vertex v in G then G contains a cycle.
(Proof on page 111 of course notes)

Example

Let G be a 6-regular connected graph. Prove that G does not have a bridge.

Solution:

Assume on the contrary that G does have a bridge, e. Then G-e would contain two components, G1and G2where each component contains exactly 1 vertex of degree 5, since we only removed 1 edge from a 6-regular graph. The component G1is a graph itself that contains a vertex of odd degree. This is a contradiction to the Handshake Theorem, so G does not have a bridge.