MATH 239 Spring 2012: Assignment 1

Due: 9:29 AM, Friday, May 11 2012 in the dropboxes outside MC 4066

Note: Please study the supplementary files on bijections and formal power series. In this assignment, all variables are non-negative integers. Also, $[n] = \{1, 2, \dots, n\}$ ($[0] = \emptyset$).

Last Name:

First Name:

I.D. Number:

Section:

Mark (For the marker only): /50

Acknowledgments:

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1. {6 marks} We start with a few counting problems. You do not need to show your work for this question only. Simplify your answer as much as possible.

   (a) How many $\{0, 1\}$-strings are there with exactly n 0's and m 1's?

   (b) For some $n \geq 2$, how many subsets of $[2n]$ contain at most two odd numbers?

   (c) A permutation of $[n]$ is a bijection $\sigma : [n] \to [n]$ (in other words, it is any rearrangement of $[n]$). How many permutations σ of $[n]$ satisfy the property that $\sigma(1) \neq 1$?

2. {4 marks} For some $0 \leq r \leq k \leq n$, how many subsets of $[n]$ have r elements in common with the set $\{1, \ldots, k\}$? Describe two sets S and T such that the answer to our question is the cardinality of the cartesian product $S \times T$, then determine what is this answer.

3. {8 marks} Let $n \geq 1$. Let S be the set of all subsets of $[n]$ that contains the element 1, and let T be the set of all subsets of $[n]$ that contains the element n. Define a bijection between S and T, and prove that it is a bijection.

4. {12 marks} Consider the following identity:

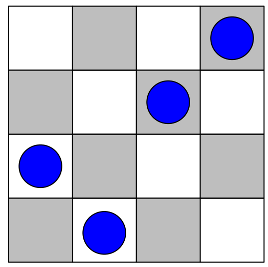
$$3^n = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}.$$

(a) Give a combinatorial proof of this identity.

(b) Give an algebraic proof of this identity.

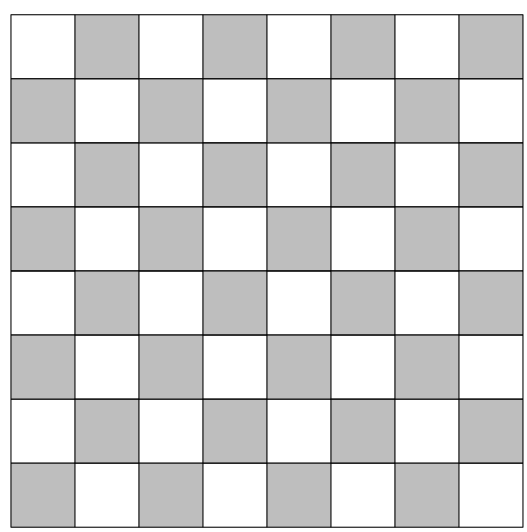
5. {8 marks} We wish to consider the problem of placing n rooks on an $n \times n$ board so that no rook can attack another. This is equivalent to placing them so that no two are on the same column or row. A little bit of thought would hopefully convince you that there are $n!$ ways to do this.

Now let's restrict the problem so that no rook can appear on the main diagonal. Then it is not so obvious how many ways we can do this. Let R_n be the set of all such placements. We can represent a placement by listing all rook positions in a set. For example, $\{(3,1),(4,2),(2,3),(1,4)\}$ is an element of R_4, representing the following placement:



Let $[n] = \{1, 2, \ldots, n\}$. A **derangement** is a bijection $\sigma : [n] \to [n]$ such that $\sigma(x) \neq x$ for each $x \in [n]$. Let S_n be the set of all derangements of $[n]$. We can represent a $\sigma \in S_n$ by $(\sigma(1)\sigma(2)\cdots\sigma(n))$. For example, (3142) is a derangement of [4], but (2431) is not, since $\sigma(3) = 3$. There are 3 derangements of [3], 9 derangements of [4], but the general formula for $[n]$ is a bit more involved (not important for this question).

Find a bijection between R_n and S_n, hence proving that they have the same cardinality. You need to prove that it is indeed a bijection, and illustrate your bijection by drawing the placement corresponding to the derangement (73428561).



6. {12 marks} Read the supplementary file on formal power series. (For a formal treatment of the subject, see Section 1.5 of the course notes.)

   (a) Suppose

   $$f(x) = x^5 + x^8 + x^{11} + x^{14} + \cdots + x^{1337}$$
   $$g(x) = 1 + (x + f(x))^2 + (x + f(x))^4 + (x + f(x))^6 + \cdots$$

   Express $g(x)$ in the form $\frac{p(x)}{q(x)}$ where $p(x), q(x)$ are polynomials.

   (b) Determine $[x^7](1 + x^2)(1 + 2x)^{-36}$.

   (c) For some fixed $k \geq 1$, determine $[x^{3n}](1 - 2x^3)^{-4}(1 - 5x)^{-k}$.

7. {Extra credit: 5 marks} For $0 \leq r \leq n$, prove the following identity (using any correct method):

$$\binom{n}{r} 3^{n-r} = \sum_{k=r}^{n} \binom{n}{k} \binom{k}{r} 2^{n-k}.$$