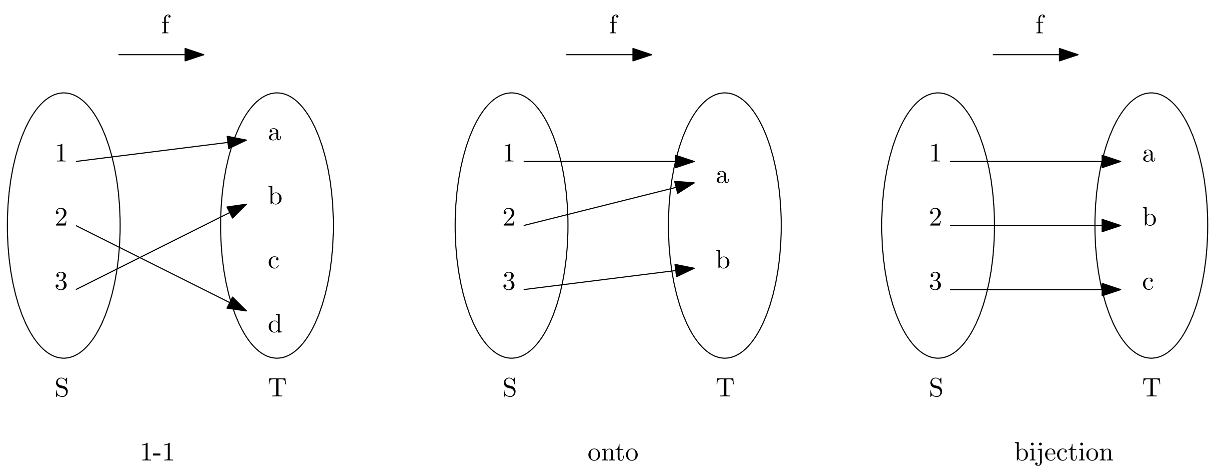
MATH 239 Supplementary 1: Bijections

1 Review (?): Functions

We begin with a few definitions on functions. Let S and T be sets. Let $f : S \to T$ be a function (or mapping).

- f is *1-1* or *injective* if for any $x_1, x_2 \in S$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. In other words, every element in S is being mapped to a unique element in T.
- f is *onto* or *surjective* if for all $y \in T$, there exists $x \in S$ such that $f(x) = y$. In other words, every element in T is being mapped to from some element in S.
- f is a *1-1 correspondence* or *bijection* if it is both 1-1 and onto.

We can visualize these definitions using the following diagram:



The functions that you have seen in high school or calculus are typically defined for $f : \mathbb{R} \to \mathbb{R}$. For example, $f(x) = e^x$ is 1-1, but not onto; $f(x) = x^3 - x$ is onto, but not 1-1; and $f(x) = 2x + 1$ is both 1-1 and onto, hence a bijection. (As an exercise, prove these facts for the three functions.)

2 Functions and cardinalities

In combinatorics, we use mappings to compare the cardinalities of finite sets S and T. If there exists a mapping $f : S \to T$ that is 1-1, then $|S| \leq |T|$. This is because the $|S|$ elements of S must be mapped to distinct elements in T, so there must be at least $|S|$ distinct elements in T. On the other hand, if there exists a mapping $f : S \to T$ that is onto, then $|S| \geq |T|$. This is because for the $|T|$ elements in T, each must be mapped to by a distinct element in S. Therefore, if there exists a bijection $f : S \to T$, then $|S| = |T|$, as f is both 1-1 and onto.

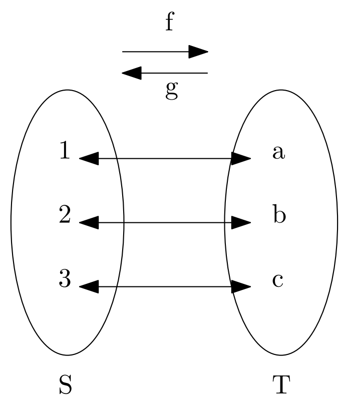
In addition to showing that two sets have equal size, bijections have the nice property that they "pair up" elements of S with elements of T exactly. This gives a *correspondence* between S and T.

3 Inverses

We can prove that a function is a bijection by going through the definition of 1-1 and onto. Alternatively, we could show that the mapping is "reversible", that is, there exists an inverse function. For $f : S \to T$, the inverse (if it exists) of f is a function $g : T \to S$ such that for all $x \in S$, $g(f(x)) = x$, and for all $y \in T$, $f(g(y)) = y$.

Theorem 3.1. *If a function $f : S \to T$ has an inverse, then f is a bijection.*

Proof. Let $g : T \to S$ be the inverse of f. We need to prove that f is 1-1 and onto. Suppose $f(x_1) = f(x_2)$. Then $g(f(x_1)) = g(f(x_2))$. By definition of inverse, $x_1 = x_2$, so f is 1-1. Let $y \in T$. Since g is a function, $g(y) = x$ for some $x \in S$. Then $f(g(y)) = f(x)$, and by the definition of inverse, $y = f(x)$. Therefore, x is mapped to y, and f is onto. □



bijection

4 Examples

Example 1. For some $0 \leq k \leq n$, let S be the set of k-subsets of $[n]$, and let T be the set of $(n-k)$-subsets of $[n]$. We can define the following mapping:

$$f : S \to T, \quad f(A) = [n] \setminus A \quad \forall\, A \in S.$$

First, we need to check that this is a proper mapping. For any $A \in S$, $|A| = k$. So $f(A) = [n] \setminus A$ has cardinality $n - k$, meaning $f(A) \in T$.

To prove that f is a bijection, we provide its inverse mapping:

$$g : T \to S, \quad g(B) = [n] \setminus B \quad \forall B \in T.$$

Then, for each $A \in S$,

$$g(f(A)) = g([n] \setminus A) = [n] \setminus ([n] \setminus A) = A.$$

And for each $B \in T$,

$$f(g(B)) = f([n] \setminus B) = [n] \setminus ([n] \setminus B) = B.$$

So g is an inverse, hence f is a bijection.

Since $|S| = \binom{n}{k}$ and $|T| = \binom{n}{n-k}$, this bijection is a combinatorial proof of the identity $\binom{n}{k} = \binom{n}{n-k}$

We illustrate this bijection by matching the elements of A with elements of B for the case where $n = 5, k = 2$.

S		T
$\{1,2\}$	$\longleftrightarrow$	$\{3,4,5\}$
$\{1,3\}$	$\longleftrightarrow$	$\{2,4,5\}$
$\{1,4\}$	$\longleftrightarrow$	$\{1,3,5\}$
$\{1,5\}$	$\longleftrightarrow$	$\{2,3,4\}$
$\{2,3\}$	$\longleftrightarrow$	$\{1,4,5\}$
$\{2,4\}$	$\longleftrightarrow$	$\{1,3,5\}$
$\{2,5\}$	$\longleftrightarrow$	$\{1,3,4\}$
$\{3,4\}$	$\longleftrightarrow$	$\{1,2,5\}$
$\{3,5\}$	$\longleftrightarrow$	$\{1,2,4\}$
$\{4,5\}$	$\longleftrightarrow$	$\{1,2,3\}$

The function f maps in the $\rightarrow$ direction, the inverse g maps in the $\leftarrow$ direction. For example, $f(\{1,5\}) = \{2,3,4\}$ while $g(\{1,2,5\}) = \{3,4\}$.

Example 2. We can establish a bijection between two completely different looking sets of objects in order to find a correspondence between the two sets. Let S be the set of all subsets of $[n]$, and let T be the set of all $\{0,1\}$-strings of length n. We define $f : S \to T$ in the following way: For a subset A of $[n]$, we can create a string $f(A) = a_1 a_2 \cdots a_n$ length n where

$$a_i = \begin{cases} 0 & i \notin A \\ 1 & i \in A \end{cases}$$

This mapping is reversible: Let $t = b_1 b_2 \cdots b_n \in T$. Define $g : T \to S$ where

$$g(t) = \{i \in [n] \mid b_i = 1\}.$$

You can check that $g(f(A)) = A$ and $f(g(t)) = t$, so f is a bijection.

This bijection tells us that the number of subsets of $[n]$ is equal to the number of binary strings of length n, which is 2^n.

We illustrate this bijection for $n = 3$.

S		T
$\{\emptyset\}$	$\longleftrightarrow$	000
$\{1\}$	$\longleftrightarrow$	100
$\{2\}$	$\longleftrightarrow$	010
$\{3\}$	$\longleftrightarrow$	001
$\{1,2\}$	$\longleftrightarrow$	110
$\{1.3\}$	$\longleftrightarrow$	101
$\{2,3\}$	$\longleftrightarrow$	011
$\{1,2,3\}$	$\longleftrightarrow$	111