MATH 239 Tutorial 9 Problems

1. Let G be a weighted connected graph, and let T be a minimum spanning tree of G. Prove that if C is a cycle in G, then some edge in C of maximum weight is not in T.

2. Suppose we want to find a minimum spanning tree whose largest weight is as small as possible. Prove that Kruskal's algorithm produces such a tree.

3. Prove the more general version of Euler's formula: If G is a graph with a planar embedding with n vertices, m edges, s faces and c components, then

$$n - m + s = 1 + c.$$

4. Let G be a 3-regular connected planar graph which has an embedding where each face has degree either 4 or 6, and no two faces of degree 4 are adjacent. Determine an example of such a graph with the fewest number of vertices.

5. In class, we have proved that any planar graph with n vertices have at most $3n - 6$ edges. What can you say about those graphs with exactly $3n - 6$ edges? Draw one where $n = 8$.

Additional exercises

1. Let T be a tree with n vertices, and let x be a vertex in T. For any vertex v in T, define $d(v)$ to be the length of the unique v, x-path in T. Prove that

$$\sum_{v \in V(G)} d(v) \leq \binom{n}{2}.$$

   When does eqauality hold?

2. Suppose we want to find a minimum spanning path in a weighted complete graph. We use a greedy algorithm as follows: Start with an edge of minimum weight. At each stage, we consider the edges incident with the two ends of our current path, pick one that has the smallest weight and extend our path by one edge. We repeat until we get a spanning path. Does this algorithm work? Either prove that this algorithm works, or provide an example of a weighted K_n for each n for which this does not work.