MATH 239 — Fall 2011 Assignment 5

DUE: NOON Friday 28 October 2011 in the drop boxes opposite the Math Tutorial Centre MC 4067 or next to the St. Jerome's library for the St. Jerome's section.

1. Given a graph G, the line graph $L(G)$ is defined in the following way:

$$V(L(G)) = E(G),$$
$$E(L(G)) = \{\{e_1, e_2\} \mid |e_1 \cap e_2| = 1\}.$$

Prove that the line graph $L(K_{m,n})$ to the complete bipartite graph $K_{m,n}$ is regular and find the common degree to all its vertices.

SOLUTION. Let (A, B) be a bipartition of $V(K_{m,n})$, with $|A| = m$ and $|B| = n$. Let $e \in V(L(K_{m,n})) = E(K_{m,n})$. We have $e = \{a, b\}$ for some $a \in A$ and some $b \in B$. Since $\{a, b'\} \in E(K_{m,n})$ and $\{a', b\} \in E(K_{m,n})$ for all $b' \in B$ and $a' \in A$, and since those are all the edges containing a and b, we have

$$\{e' \in E(K_{m,n}) \mid |e' \cap e| = 1\} = \Big\{\{a, b'\} \mid b' \in B \setminus \{b\}\Big\} \cup \Big\{\{a', b\} \mid a' \in A \setminus \{a\}\Big\}.$$

Hence $\deg(e) = |B| - 1 + |A| - 1 = m + n - 2$. Since that result is independent of e, $L(K_{m,n})$ is $(m + n - 2)$-regular.

2. A sequence of decreasing integers is called *graphic* if it corresponds to the degrees of the vertices a graph. Which of these sequences are graphic? Justify your answer.

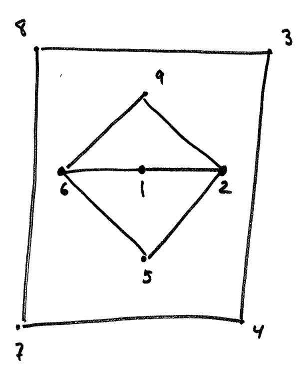
   (a) $3, 2, 2, 2, 1, 1$ **SOLUTION.** Not a graphic sequence since the number of vertices of odd degree in a graph must be even.

   (b) $3, 2, 1, 1, 1$ **SOLUTION.** A graphic sequence. Let $V(G) = \{A, B, c, d, e\}$ and $E(G) = \big\{\{A, c\}, \{A, d\}, \{A, B\}, \{B, e\}\big\}$.

   (c) $7, 5, 3, 2, 1, 1, 1$ **SOLUTION.** Cannot be graphic. In a graph with 7 vertices, the maximal degree of a vertex is 6.

3. Let G_n be the graph with vertex set $\{1, \ldots, n\}$ and such that u and v are adjacent if and only if $u + v \equiv 3 \mod 4$.

   (a) Draw a drawing of G_9.
   SOLUTION. Here is one example:



(b) Show that G_n is bipartite.

SOLUTION. Let's see what are all the possibilities, mod 4, to get 3 as a sum:

$$\begin{aligned}0+0\equiv 0, 0+1\equiv 1, 0+2\equiv 2, \boxed{0+3\equiv 3},\\ 1+1\equiv 2, \boxed{1+2\equiv 3}, 1+3\equiv 0,\\ 2+2\equiv 0, 2+3\equiv 1,\\ 3+3\equiv 2.\end{aligned}$$

So $u+v\equiv 3$ if and only if

- $u\equiv 0$ and $v\equiv 3$ (or vice-versa), or
- $u\equiv 2$ and $v\equiv 1$ (or vice-versa).

We can therefore use the bipartition

$$\begin{aligned}A&=\{u\in V(G_n)\mid u\text{ is even}\},\\ B&=\{v\in V(G_n)\mid v\text{ is odd}\}.\end{aligned}$$

(c) We have a natural weight function deg on $V(G_n)$. Find a formula for $\Phi_{V(G_n)}(x)$. Be careful, your answer depends on the class of n mod 4.

SOLUTION. The solution will of course depend on class of n mod 4. From the previous solution, we see that

$$\deg(u)=\begin{cases}|\{v\in\{1,\dots,n\}\mid v\equiv 3\mod 4\}|, & \text{if } u\equiv 0\mod 4,\\ |\{v\in\{1,\dots,n\}\mid v\equiv 2\mod 4\}|, & \text{if } u\equiv 1\mod 4,\\ |\{v\in\{1,\dots,n\}\mid v\equiv 1\mod 4\}|, & \text{if } u\equiv 2\mod 4,\\ |\{v\in\{1,\dots,n\}\mid v\equiv 0\mod 4\}|, & \text{if } u\equiv 3\mod 4.\end{cases}$$

Let's check the various cases. Suppose first that $n=4m$. Then all the classes $[0],[1],[2],[3]$ have the same number of elements, m, since

$$\begin{aligned}[0]&=\{4,\dots,4m\}\\ [1]&=\{1,\dots,4(m-1)+1\},\\ [2]&=\{2,\dots,4(m-1)+2\},\\ [3]&=\{3,\dots,4(m-1)+3\}.\end{aligned}$$

Hence all the vertices have degree m, and

$$\Phi_{V(G_{4m})}(x)=4mx^m.$$

If $n=4m+1$, the class $[1]$ has one more element than earlier, $4m+1$, so has $m+1$ elements. The extra vertex will connect the m vertices in $[2]$, so all the vertices in $[2]$ have degree $m+1$, while the vertices in $[0]\cup[1]\cup[3]$ have degree m. So

$$\Phi_{V(G_{4m+1})}(x)=(3m+1)x^m+mx^{m+1}.$$

If $n = 4m + 2$, the classes [1] and [2] have now $m + 1$ elements (we have added $4m + 2$ to [2]) while the classes [0] and [3] still have m elements. The vertices in $[1] \cup [2]$ have degree $m + 1$ while the vertices in $[3] \cup [0]$ have degree m. Hence

$$\Phi_{V(G_{4m+2})}(x) = 2mx^m + (2m + 2)x^{m+1}.$$

Lastly, if $n = 4m+3$, the classes [1], [2] and [3] now all have $m+1$ elements while [0] still has m elements. The vertices in [0], [1] and [2] therefore have degree $m+1$ while the vertices in [3] have degree m. Hence

$$\Phi_{V(G_{4m+3})}(x) = (m + 1)x^m + (3m + 2)x^{m+1}.$$

4. Let $\mathcal{G}_n$ be the set of all finite graphs with vertices $\{1, \ldots, n\}$, and let $\mathcal{G} = \bigcup_{n\geq 0} \mathcal{G}_n$. Let the weight function $w \colon \mathcal{G} \to \mathbb{N}_{\geq 0}$ be given by $w(G) = |V(G)|$. Find $\Phi_{\mathcal{G}}(x)$. (Note that we consider here actual graphs, not isomorphism classes.)

   SOLUTION. By definition of graphs, if $G \in \mathcal{G}_n$, then $E(G) \subseteq C_{2,n} := \{S \subseteq \{1, \ldots, n\} \mid |S| = 2\}$. Note that $|C_{2,n}| = \binom{n}{2}$, hence the number of possible graphs with n vertices is $2^{\binom{n}{2}} = 2^{n(n-1)/2}$. So

$$\Phi_{\mathcal{G}}(x) = \sum_{n\geq 0} 2^{n(n-1)/2} x^n.$$

5. Consider now $w(G) = |E(G)|$ on $\mathcal{G}_{100}$. Find $\Phi_{\mathcal{G}_{100}}(x)$.

   SOLUTION. The number of vertices is fixed, so we know that any $G \in \mathcal{G}_{100}$ have $E(G) \subseteq \{\{a, b\} \mid 1 \leq a < b \leq n\}$. If $P_2(\{1, \ldots, 100\}) = \{S \subset \{1, \ldots, 100\} \mid |S| = 2\}$, we have

$$|\{G \in \mathcal{G}_{100} \mid |E(G)| = n\}| = |\{T \subset P_2(\{1, \ldots, 100\}) \mid |T| = n\}.$$

   Since $|P_2(\{1, \ldots, 100\})| = \binom{100}{2} = \frac{100\cdot 99}{2} = 4950$, we have

$$|\{G \in \mathcal{G}_{100} \mid |E(G)| = n\}| = \binom{4950}{n}.$$

   Hence

$$\Phi_{\mathcal{G}_{100}}(x) = \sum_{n=0}^{4950} \binom{4950}{n} x^n = (1 + x)^{4950}.$$