MATH 239 Assignment 3

DUE: 10am Friday Feb. 1 in the drop boxes opposite the Math Tutorial Centre MC 4067.

1. Let k be a fixed positive integer. Let a_n denote the number of compositions of n with exactly k parts, in which each part is an odd number greater than or equal to 5.

   (a) Find a set S and a weight function w defined on S such that a_n is equal to the number of elements σ of S with $w(\sigma) = n$.

   (b) Find the generating series $\Phi_S(x)$ with respect to the weight function w. Remember to indicate where theorems from class are applied, e.g. Sum and Product Lemmas.

   (c) Find a_n explicitly in terms of n and k.

   SOLUTION.

   (a) Let $U = N^{odd}_{\geq 5} = \{5, 7, 9, \ldots\}$. Then we choose S to be the Cartesian product of k copies of U:

   $$S = U \times U \times \ldots \times U = U^k.$$

   We define w on S by $w(t_1, \ldots, t_k) = t_1 + \ldots + t_k$. Then $(t_1, \ldots, t_k)$ is a composition of n if and only if $w(t_1, \ldots, t_k) = n$.

   (b) The generating series for the 1-part composition U with respect to the usual weight function $w_0(\sigma) = \sigma$ is

   $$\begin{aligned} \Phi_U &= x^5 + x^7 + x^9 + \ldots \\ &= x^5(1 - x^2)^{-1}. \end{aligned}$$

   By our choice of weight function w for S, the conditions of the Product Lemma are satisfied. Thus by the Product Lemma , the generating series for the k-part composition $S = U^k$ is

   $$\begin{aligned} \Phi_S(x) &= \left(x^5(1 - x^2)^{-1}\right)^k \\ &= x^{5k}(1 - x^2)^{-k}. \end{aligned}$$

   (c) We know that a_n is the number of elements of S of weight n, which is

   $$\begin{aligned} [x^n]\Phi_S(x) &= [x^n]x^{5k}(1 - x^2)^{-k} \\ &= [x^{n-5k}](1 - x^2)^{-k} \\ &= [x^{n-5k}] \sum_{i \geq 0} \binom{k + i - 1}{k - 1} x^{2i}. \end{aligned}$$

   The coefficient is zero if $n - 5k$ is odd, or if $n < 5k$. If $n - 5k$ is even then the required coefficient occurs when $i = \frac{n-5k}{2}$ and it is equal to

   $$\binom{k + \frac{n-5k}{2} - 1}{k - 1} = \binom{\frac{n-3k-2}{2}}{k - 1}.$$

   Therefore a_n is $\binom{\frac{n-3k-2}{2}}{k-1}$ when $n - 5k$ is even, and is 0 when $n - 5k$ is odd (or when $n < 5k$).

2. Let a_n be the number of compositions of n with an even number of parts, each of which is at least 6. (Note that the number of parts is not fixed.)

(a) Find a set S and a weight function w defined on S such that a_n is equal to the number of elements σ of S with $w(\sigma) = n$.

(b) Prove that for $n \geq 0$

$$a_n = [x^n]\frac{1-2x+x^2}{1-2x+x^2-x^{12}}.$$

Remember to indicate where theorems from class are applied, e.g. Sum and Product Lemmas.

SOLUTION.

(a) Let $U = N_{\geq 6} = \{6, 7, 8, \ldots\}$. Then we choose S to be the union over all even numbers $2k$ of the Cartesian product of $2k$ copies of U:

$$S = \bigcup_{k\geq 0} U^{2k}.$$

We define w on S by $w(t_1, \ldots, t_{2k}) = t_1 + \ldots + t_{2k}$. Then $(t_1, \ldots, t_{2k})$ is a composition of n with the required properties if and only if $w(t_1, \ldots, t_{2k}) = n$.

(b) The generating series for the 1-part composition U with respect to the standard weight function is

$$\begin{aligned}\Phi_U(x) &= x^6 + x^7 + x^8 + \ldots \\ &= x^6(1-x)^{-1}.\end{aligned}$$

By our choice of weight function w for S, the conditions of the Product Lemma are satisfied for each U^{2k}. Then using the Product Lemma , the generating series for a $2k$-part composition U^{2k} is

$$\begin{aligned}\Phi_{U^{2k}}(x) &= \left(x^6(1-x)^{-1}\right)^{2k} \\ &= x^{12k}(1-x)^{-2k}.\end{aligned}$$

By the Sum Lemma , the generating series for S is

$$\begin{aligned}\Phi_S(x) &= \sum_{k\geq 0} \Phi_U^{2k}(x) \\ &= \sum_{k\geq 0} x^{12k}(1-x)^{-2k} \\ &= \sum_{k\geq 0} \left(x^{12}(1-x)^{-2}\right)^k \\ &= \frac{1}{1-x^{12}(1-x)^{-2}}, \quad \text{by Geometric Series} \\ &= \frac{(1-x)^2}{(1-x)^2 - x^{12}}, \quad \text{by multiplying top and bottom by } (1-x)^2 \\ &= \frac{1-2x+x^2}{1-2x+x^2-x^{12}}.\end{aligned}$$

Therefore, the number of compositions a_n of n into even number of parts, each of which is at least 6 equals

$$[x^n]\frac{1-2x+x^2}{1-2x+x^2-x^{12}}.$$

3. Let $\{a_n : n \geq 0\}$ be the sequence defined in the previous question.

   (a) Prove that $a_0 = 1$ and $a_n = 0$ for $1 \leq n \leq 11$.

   (b) Prove that for each $n \geq 12$ the number a_n satisfies

   $$a_n = 2a_{n-1} - a_{n-2} + a_{n-12}.$$

   (c) Find the exact value of a_{15}.

SOLUTION.

(a) There is exactly one composition of zero, namely the empty composition with 0 parts. Since 0 is even, this shows that $a_0 = 1$. Apart from the empty composition, any composition satisfying the conditions has at least 2 parts, each of which is at least 6. Hence we get a composition of n only when $n \geq 12$. Therefore $a_n = 0$ for $1 \leq n \leq 11$.

Alternatively, we could find the values of $a_0, \ldots, a_{11}$ by comparing coefficients as in the solution to the next part below.

(b)

$$\begin{aligned}
\sum_{n\geq 0} a_n x^n &= \frac{1-2x+x^2}{1-2x+x^2-x^{12}} \\
(1-2x+x^2-x^{12})\sum_{n\geq 0} a_n x^n &= 1-2x+x^2 \\
\sum_{n\geq 0} a_n x^n - 2\sum_{n\geq 0} a_n x^{n+1} + \sum_{n\geq 0} a_n x^{n+2} - \sum_{n\geq 0} a_n x^{n+12} &= 1-2x+x^2 \\
\sum_{n\geq 0} a_n x^n - 2\sum_{n\geq 1} a_{n-1} x^n + \sum_{n\geq 2} a_{n-2} x^n - \sum_{n\geq 12} a_{n-12} x^n &= 1-2x+x^2.
\end{aligned}$$

Now we compare the coefficient of x^n on both sides for $n \geq 12$ to get:

$$a_k - 2a_{k-1} + a_{k-2} - a_{k-12} = 0 \quad \Longrightarrow \quad a_k = 2a_{k-1} - a_{k-2} + a_{k-12}.$$

(c) Using the result of the previous part we find

$$\begin{aligned}
a_{12} &= 2a_{11} - a_{10} + a_0 = 1, \\
a_{13} &= 2a_{12} - a_{11} + a_0 = 2, \\
a_{14} &= 2a_{13} - a_{12} + a_1 = 3, \\
a_{15} &= 2a_{14} - a_{13} + a_2 = 4.
\end{aligned}$$

Alternatively, we could observe that each valid composition of 15 must have exactly 2 parts, so the solutions are $(6,9)$, $(7,8)$, $(8,7)$ and $(9,6)$.

4. Let b_n be the number of compositions of n with an even number of parts, such that at least one part is less than or equal to 5. Prove that for $n \geq 0$

$$b_n = [x^n](1 + \frac{x^2}{1-2x} - \frac{1-2x+x^2}{1-2x+x^2-x^{12}}).$$

SOLUTION. We use exactly the same approach as in Question 2 to find c_n, the TOTAL number of compositions of n with an even number of parts. Then $b_n = c_n - a_n$.

Let $U = N_{\geq 1} = \{1, 2, 3, \ldots\}$. Then we choose S to be the union over all even numbers $2k$ of the Cartesian product of $2k$ copies of U:

$$S = \bigcup_{k \geq 0} U^{2k}.$$

We define w on S by $w(t_1, \ldots, t_{2k}) = t_1 + \ldots + t_{2k}$. Then $(t_1, \ldots, t_{2k})$ is a composition of n with the required properties if and only if $w(t_1, \ldots, t_{2k}) = n$.

The generating function for the 1-part composition U with respect to the standard weight function is

$$\begin{aligned} \Phi_U(x) &= x + x^2 + x^3 + \ldots \\ &= x(1-x)^{-1}. \end{aligned}$$

By our choice of weight function w for S, the conditions of the Product Lemma are satisfied for each U^{2k}. Then using the Product Lemma , the generating series for a $2k$-part composition U^{2k} is

$$\begin{aligned} \Phi_{U^{2k}}(x) &= \left(x(1-x)^{-1}\right)^{2k} \\ &= x^{2k}(1-x)^{-2k}. \end{aligned}$$

By the Sum Lemma , the generating series for S is

$$\begin{aligned} \Phi_S(x) &= \sum_{k \geq 0} \Phi_U^{2k}(x) \\ &= \sum_{k \geq 0} x^{2k}(1-x)^{-2k} \\ &= \sum_{k \geq 0} \left(x^2(1-x)^{-2}\right)^k \\ &= \frac{1}{1 - x^2(1-x)^{-2}}, \quad \text{by Geometric Series} \\ &= \frac{(1-x)^2}{(1-x)^2 - x^2}, \quad \text{by multiplying top and bottom by } (1-x)^2 \\ &= \frac{1-2x+x^2}{1-2x} = 1 + \frac{x^2}{1-2x}. \end{aligned}$$

Therefore $c_n = [x^n](1 + \frac{x^2}{1-2x})$. Thus

$$\begin{aligned} b_n &= c_n - a_n = [x^n](1 + \frac{x^2}{1-2x}) - [x^n]\frac{1-2x+x^2}{1-2x+x^2-x^{12}} \\ &= [x^n](1 + \frac{x^2}{1-2x} - \frac{1-2x+x^2}{1-2x+x^2-x^{12}}). \end{aligned}$$