MATH 239 Spring 2012: Assignment 9

Due: 9:29 AM, Friday, July 13 2012 in the dropboxes outside MC 4066

Last Name:

First Name:

I.D. Number:

Section:

Mark (For the marker only): /50

Acknowledgments:

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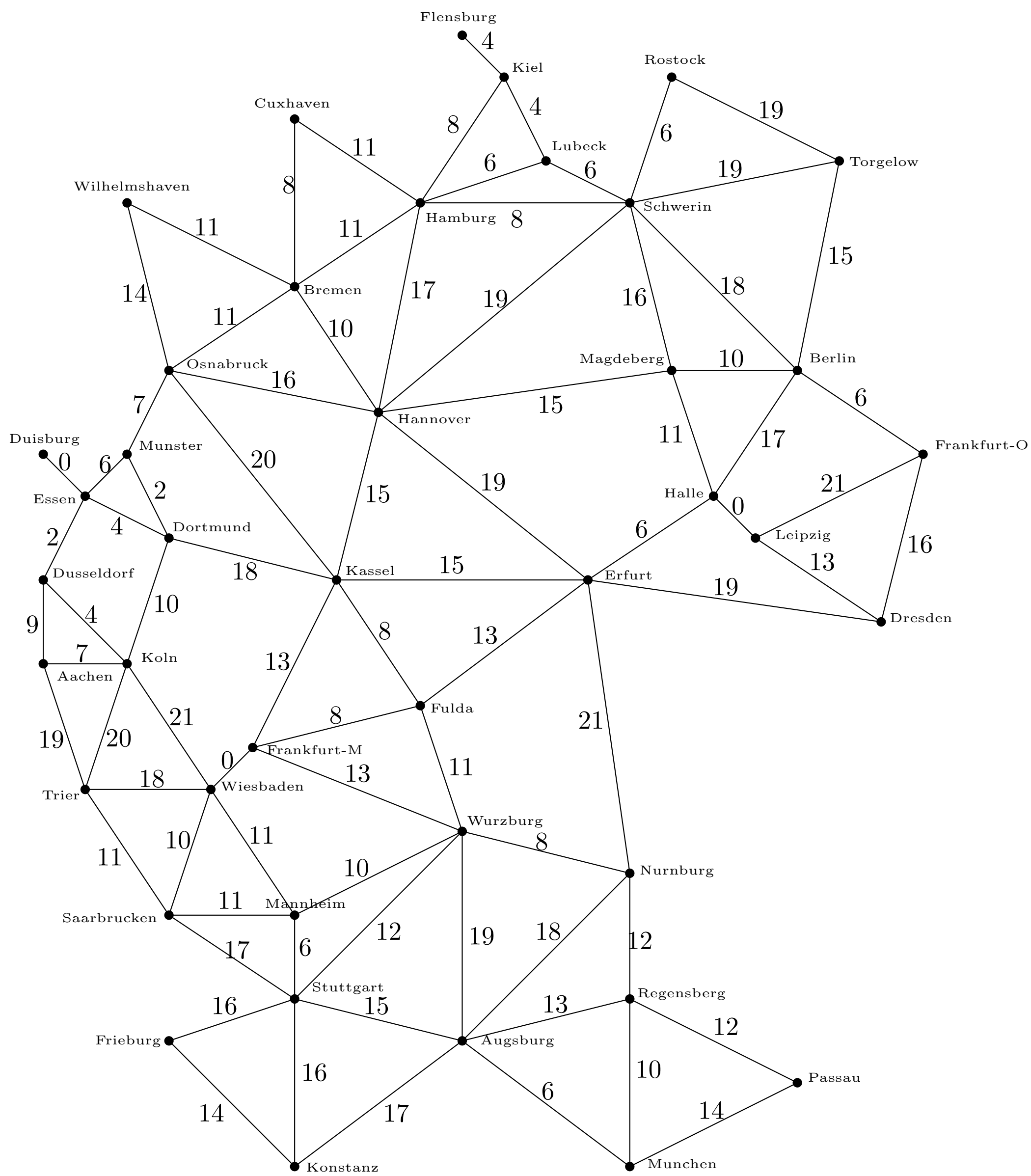
1. {10 marks} Let G be a connected graph, and let T be a spanning tree of G. Let x be a vertex in G. For any vertex v in G, define $d(v)$ to be the length of the unique x, v-path in T. Suppose that all the edges in G that are not in T join two vertices whose d-values have the same parity.

   (a) Prove that if uv is an edge in T, then $|d(u) - d(v)| = 1$.

   (b) Prove that any cycle of G contains an even number of edges from T.

2. {7 marks} Produce a minimum spanning tree of the following graph. You do not need to show your work. (Source: The map of Germany from the board game Power Grid.)



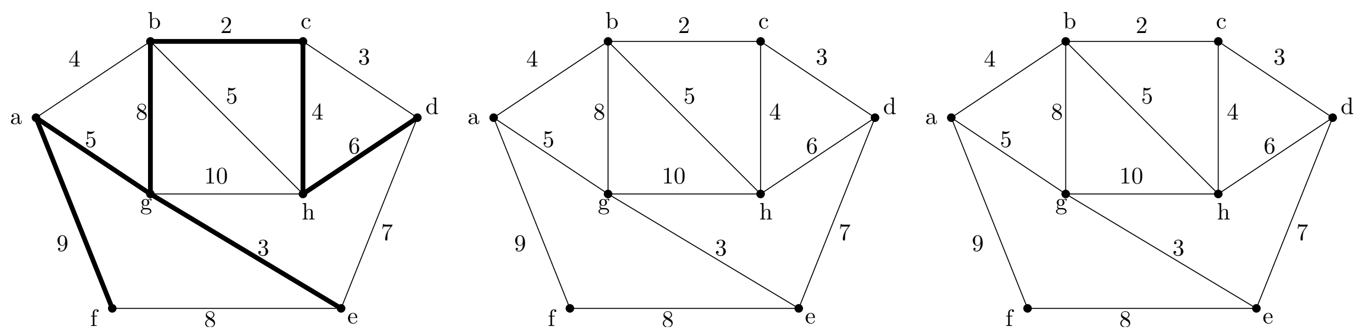
3. {5 marks} Let T be a minimum spanning tree of a weighted graph G. For any two vertices u, v in G, is it true that the unique u,v-path in T is a path of minimum weight among all u,v-paths in G? Give a proof or a counterexample.

4. {14 marks} We propose another algorithm for finding a minimum spanning tree of a connected graph G. Let $w(e)$ be the weight of an edge. Start with any spanning tree T.

   Find a pair of edges (e, e') such that $e \in E(G) \setminus E(T)$, e' is in the unique cycle of $T + e$, and $w(e) < w(e')$. Replace T by $T - e + e'$.

   The algorithm repeats this process, and it terminates when no such pair of edges can be found.

   (a) Perform 2 iterations of this algorithm on the graph below, using the bolded edges as the starting tree. Indicate which pair of edges you are choosing.



   (b) Prove that when the algorithm terminates, it produces a spanning tree.

   (c) Prove that when the algorithm terminates, it produces a minimum spanning tree. (You may start this way if you wish: Let T be the tree produced by the algorithm, and let T^* be a minimum spanning tree that has the most number of edges in common with T.)

5. {8 marks} Let G be a 4-regular connected planar graph with an embedding where every face has degree 3 or 4, and adjacent faces have different face degrees. Determine the number of vertices, edges, faces of degree 3, and faces of degree 4 in G. Draw a planar embedding of G.

6. {6 marks} Is it true that any planar embedding of any simple connected planar graph has either a vertex of degree at most 3 or a face of degree at most 3? Give a proof or a counterexample.