MATH 239 – Tutorial 11

1. (Problem Set 8.2, Q5, 6)

(a) Show that the 64 squares of a chessboard can be covered with 32 dominoes, each of which covers two adjacent squares.
(b) Show that if two opposite corner squares of a chessboard are removed, then the resulting board cannot be covered with 31 dominoes.

Solution

(a) A chessboard consists of a checkered pattern of alternating light and dark squares. A domino will cover both a white square and a light square. We can therefore think of a chessboard as a bipartite graph G with bipartition A, B, where A consists of all the light squares and B consists of all the dark squares. There is an edge in G if two squares are adjacent on the chessboard. The idea of covering the board with dominoes can be thought of as finding a perfect matching on G.
The simplest way to cover the chessboard is to put 4 dominoes, end to end, on each of the 8 rows. This covers the board perfectly using 32 dominoes.

(b) Removing two opposite corner squares will remove two squares of the same color, i.e. two vertices from the same bipartition. This reduces to the problem of finding a perfect matching over G with, w.l.o.g., $|A| = 30$ and $|B| = 32$, which cannot be done. The size of the maximum matching will be of 30, where all the vertices of A are matched.

2. (Problem Set 8.2, Q7)

Let G be a graph with even number of vertices. Prove that if G has a Hamilton cycle, then G has a perfect matching.

Solution

Let the graph G have a Hamilton cycle H. Since G has an even number of vertices, H will have an even number of edges. From this, follow around the cycle alternately picking edges to be in and out of a matching M. Since H has even length, this will be possible. Therefore, M is a perfect matching.

3. (Problem Set 8.3, Q5)

Find a maximum matching and a minimum cover in the graph of Figure 1, by applying the algorithm, beginning with the maching indicated.

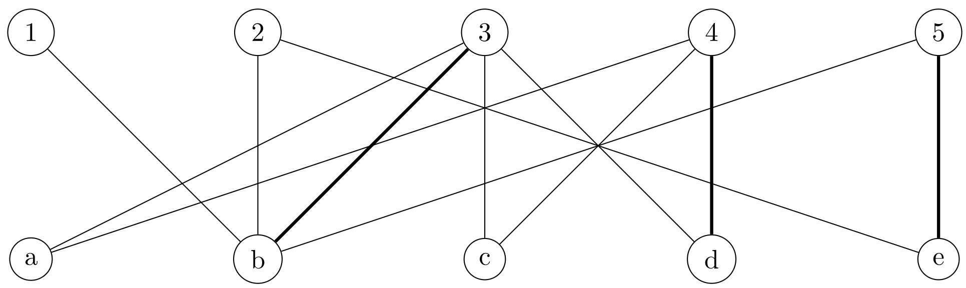


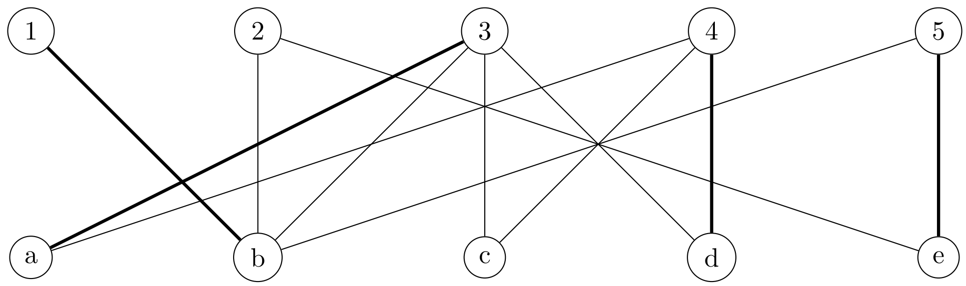
Figure 1: Question 3

Solution

Step 0. Set $M = \{\{3, b\}, \{4, d\}, \{5, e\}\}$.
Step 1. Set $\hat{X} = \{1, 2\}$ and $\hat{Y} = \emptyset$.
Step 2. Add $\{b, e\}$ to $\hat{Y}$, and set $pr(b) = 1, pr(e) = 2$.
Step 3. Step 2 added vertices to $\hat{Y}$.
Step 4. There is no unsaturated vertices in Y.
Step 5. Add $\{3, 5\}$ to $\hat{X}$. $\hat{X} = \{1, 2, 3, 5\}$ and set $pr(3) = b, pr(5) = e$. Go to Step 2
Step 2. Add $\{a, c, d\}$ to $\hat{Y}$. $\hat{Y} = \{a, b, c, d, e\}$ and set $pr(a) = pr(c) = pr(d) = 3$.
Step 3. Step 2 added vertices to $\hat{Y}$.
Step 4. Two vertices in $\hat{Y}$ are unsaturated. Using the pr values, trace the augmenting path $a, 3, b, 1$ and replace M by $M' = \{\{1, b\}, \{3, a\}, \{4, d\}, \{5, e\}\}$. Go to Step 1.

Step 1. Set $\hat{X} = \{2\}$ and $\hat{Y} = \emptyset$.
Step 2. Add $\{b, e\}$ to $\hat{Y}$, and set $pr(b) = pr(e) = 2$.
Step 3. Step 2 added vertices to $\hat{Y}$.
Step 4. No unsaturated vertices were added.
Step 5. Add $\{1, 5\}$ to $\hat{X}$. $\hat{X} = \{1, 2, 5\}$ and set $pr(1) = b, pr(5) = e$. Go to Step 2.
Step 2. No vertices can be added.
Step 3. Output $M = \{\{1, b\}, \{3, a\}, \{4, d\}, \{5, e\}\}$ as a maximum matching, and $C = \hat{Y} \cup (A \backslash \hat{X}) = \{3, 4, b, e\}$ as a minimum cover.

The resulting graph is



4. (Problem Set 8.3, Q8)

Let G be bipartite with bipartition A, B. Suppose that C and C' are both covers of G. Prove that $\hat{C} = (A \cap C \cap C') \cup (B \cap (C \cup C'))$ is also a cover of G.

Solution

Let $e = \{u, v\} \in E(G)$ such that $u \in A$ and $v \in B$. Since C and C' are covers of G, at least one of $u, v \in C$ and at least one of $u, v \in C'$.
If $u \in C$ and $u \in C'$, $u \in A \cap C \cap C'$ and e is covered by $\hat{C}$.
If $v \in C$ or $v \in C'$, $v \in B \cap (C \cup C')$ and e is covered by $\hat{C}$.
If $u \notin C$ (resp. $u \notin C'$), then $v \in C$ (resp. $v \in C'$) and we are in the second situation.
If $v \notin C$ and $v \notin C'$, then $u \in C$ and $u \in C'$ and we are in the first situation.
Therefore, $\hat{C}$ is a cover of G.

5. (Problem Set 8.6, Q3)

Let G be a bipartite graph with bipartition A, B, let M be a matching of G, and let $D \subseteq A$. Prove that $|M| \leq |A| - |D| + |N(D)|$.

Solution

Consider the subset of vertices $C = (A \backslash D) \cup N(D)$.
Let $e = \{u, v\} \in E(G)$, where $u \in A$ and $v \in B$.
If $u \in D$, then $v \in N(D)$ and e is covered by C.
If $u \notin D$, then $u \in A \backslash D$ and e is covered by C.
Therefore, C is a cover of G.
Any matching is at most as large as any cover, therefore $|M| \leq |(A \backslash D) \cup N(D)| = |A| - |D| + |N(D)|$.

6. (Problem Set 8.6 Q5)

Let G be a bipartite graph with bipartition A, B such that $|A| = |B|$, and for every proper nonempty subset D of A, we have $|N(D)| > |D|$. Prove that for every edge $e \in E(G)$ there is a perfect matching containing e.

Solution

Let $e = \{u, v\} \in E(G)$, with $u \in A, v \in B$. Consider the graph H where the vertices $\{u, v\}$ are removed. This removes u from A – Call this new bipartition A'. Since $|N(D)| > |D|$ for all $D \subset A$, $|N(D)| \geq |D|$ for all $D \subseteq A' \subset A$. Therefore Hall's theorem holds and we can find a perfect matching M' in H.
Therefore, a perfect matching in H with the edge $\{u, v\}$ gives a matching containing e.