MATH 239 - Fall 2013

Assignment 5

Due date : Friday, October 18, 2013, at noon (sharp)

Submission Guidelines:

- Total number of marks in this assignment is 40.
- Use a cover page to submit your solutions (available on the course webpage).
- Keep a copy of your manuscript before your submission.
- Assignments submissions are exclusively accepted in the following dropboxes

[Section 001] Dropbox next to the St Jerome's library, 2nd floor of STJ			
[Section 002] Math DropBox #18;	Slot #1 A-J,	Slot #2 K-S,	Slot #3 T-Z
[Section 003] Math DropBox #18;	Slot #4 A-J,	Slot #5 K-S,	Slot #6 T-Z
[Section 004] Math DropBox #18;	Slot #7 A-J,	Slot #8 K-S,	Slot #9 T-Z

- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- You are not allowed to post this manuscript (or parts of it) online, nor share it (or parts of it) with anyone not enrolled in this course.

Assignment policies: While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. For example, copying or paraphrasing a solution from some fellow student or old solutions from previous offerings of related courses qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. All students found to be cheating will automatically be given a mark of 0 on the assignment. In addition, there will be a 10/100 penalty to their final mark, as well as all academic offenses will be reported to the Associate Dean for Undergraduate Studies and recorded in the student's file (this may lead to further, more severe consequences).

If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that if you see any marking error, then you should return your assignment paper to the TA of your section within one week and with written notes on all the marking errors; please write the notes on a new sheet and attach it to your assignment paper.

Question 1 [Marks 14 = 7+7]
Consider a configuration set L with generating series $\Phi_L(x) = (1 - 2x + x^3)^{-1}$.

(a) Let α_n denote the number of configurations in L of weight n. Derive a linear homogeneous recurrence relation for α_n together with its initial conditions. Then, without finding a closed formula for α_n, determine α_5.

Solution. *We know that* $\Phi_L(x) = \frac{1}{1-2x+x^3}$, *or in other words that*

$$\begin{aligned} 1 &= (1 - 2x + x^3) \sum_{l \geq 0} \alpha_l x^l \\ &= \alpha_0 + (\alpha_1 - 2\alpha_0)x + (\alpha_2 - 2\alpha_1)x^2 + \sum_{l \geq 3} (\alpha_l - 2\alpha_{l-1} + \alpha_{l-3})\, x^l. \end{aligned}$$

We conclude that

$$\begin{aligned} 1 &= \alpha_0 \\ 0 &= \alpha_1 - 2\alpha_0 \\ 0 &= \alpha_2 - 2\alpha_1 \end{aligned}$$

from which we derive

$$\alpha_0 = 1,\ \alpha_1 = 2,\ \alpha_2 = 4.$$

These are the initial conditions of our recurrence

$$\alpha_n = 2\alpha_{n-1} - \alpha_{n-3},\ \forall n \geq 3.$$

Now we compute

$$\begin{aligned} \alpha_3 &= 2\alpha_2 - \alpha_0 = 8 - 1 = 7 \\ \alpha_4 &= 2\alpha_3 - \alpha_1 = 14 - 2 = 12 \\ \alpha_5 &= 2\alpha_4 - \alpha_2 = 24 - 4 = 20 \end{aligned}$$

(b) Without utilizing the recurrence of part (a), find all coefficients α_n of the generating series of L. Then verify that the value you found for α_5 in part (a) agrees with the new expression. (The expression for α_n can be in the form of a finite sum).

Solution. *We have*

$$\begin{aligned}
\Phi_L(x) &= (1-2x+x^3)^{-1} \\
&= \sum_{l\geq 0} \left(2x - x^3\right)^l \\
&= \sum_{l\geq 0} 2^l x^l \left(1 - \frac{1}{2}x^2\right)^l \\
&= \sum_{l\geq 0} 2^l x^l \sum_{k=0}^{l} \binom{l}{k} (-1)^k \frac{1}{2^k} x^{2k} \\
&= \sum_{l\geq 0} \sum_{k=0}^{l} (-1)^k 2^{l-k} \binom{l}{k} x^{l+2k}.
\end{aligned}$$

Now we want to evaluate $[x^n]\Phi_L(x)$. Observe that $k \leq \lfloor n/2 \rfloor$, which together with $l = n-2k$ make $2k+l=n$. Hence

$$[x^n]\Phi_L(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-3k} \binom{n-2k}{k}.$$

(actually the upper bound in the summation could be written as $\lfloor n/3 \rfloor$). Hence,

$$\begin{aligned}
[x^5]\Phi_L(x) &= \sum_{k=0}^{\lfloor 5/2 \rfloor} (-1)^k 2^{5-3k} \binom{5-2k}{k} \\
&= 2^5 \binom{5}{0} - 2^{5-3} \binom{5-2}{1} \\
&= 2^5 - 2^2 \cdot 3 \\
&= 20,
\end{aligned}$$

in agreement with part (b).

Solution (Alternative solution)**.**

We observe that the polynomial $1-2x+x^3$ has roots $1, \rho_1 = (-1+\sqrt{5})/2$ and $\rho_2 = (-1-\sqrt{5})/2$, and as such we can find constants A, B, C such that

$$\begin{aligned}
\frac{1}{1-2x+x^3} &= \frac{A}{x-1} + \frac{B}{x-\rho_1} + \frac{C}{x-\rho_2} \\
&= \frac{A(x-\rho_1)(x-\rho_2) + B(x-1)(x-\rho_2) + C(x-1)(x-\rho_1)}{1-2x+x^3} \\
&= \frac{(A+B+C)x^2 + (A+\rho_1 B + \rho_2 C)x + A\rho_1\rho_2 + B\rho_2 + C\rho_1}{1-2x+x^3}
\end{aligned}$$

We conclude that

$$\begin{aligned} A + B + C &= 0 \\ A + \rho_1 B + \rho_2 C &= 0 \\ A\rho_1\rho_2 + B\rho_2 + C\rho_1 &= 1 \end{aligned}$$

from which we calculate that $A = 1, B = (-5 - 3\sqrt{5})/10, C = (-5 + 3\sqrt{5})/10$. *With these values, we can easily find that*

$$\begin{aligned} \frac{1}{1 - 2x + x^3} &= A(x-1)^{-1} + B(x-\rho_1)^{-1} + C(x-\rho_2)^{-1} \\ &= -A(1-x)^{-1} - B(\rho_1 - x)^{-1} - C(\rho_2 - x)^{-1} \\ &= -A\sum_{n\geq 0} x^n - \frac{B}{\rho_1}\sum_{n\geq 0}\frac{1}{\rho_1^n}x^n - \frac{C}{\rho_2}\sum_{n\geq 0}\frac{2}{\rho_1^n}x^n. \end{aligned}$$

We conclude that

$$[x^n]\frac{1}{1-2x+x^3} = -A - \frac{B}{\rho_1^{n+1}} - \frac{C}{\rho_1^{n+1}}.$$

By substituting all known values, and for $n = 5$, *we do find* $[x^5]\frac{1}{1-2x+x^3} = 20$ *as expected.*

Question 2 [Marks 15=7+8]

In this question you will find the coefficients of a formal power series with two different methods. You are not allowed to use any subquestion in order to solve the other. However at the end you are expected to compare your answers and verify that they agree.

(a) Use Lemma 3.1.2 and the expansion of known formal power series in order to find the coefficients of the formal power series

$$\frac{3 - 8x + 12x^2 - 8x^3}{(1-x)(1-2x)^3}.$$

Solution. *We observe that the degree of the numerator is 3, and that of the denominator is 4. Since* $(1-x)$ *and* $(1-2x)^3$ *do not have the same roots, Lemma 3.1.2 says that there exist reals* A, B, C, D *such that*

$$\begin{aligned} \frac{3 - 8x + 12x^2 - 8x^3}{(1-x)(1-2x)^3} &= \frac{A}{1-x} + \frac{B + Cx + Dx^2}{(1-2x)^3} \\ &= \frac{A + B + (-6A - B + C)x + (12A - C + D)x^2 + (-8A - D)x^3}{(1-x)(1-2x)^3} \end{aligned}$$

For the two rational functions to be equal, we must have

$$\begin{aligned} A + B &= 3 \\ -6A - B + C &= -8 \\ 12A - C + D &= 12 \\ -8A - D &= -8 \end{aligned}$$

from which we derive that $A = 1, B = 2, C = 0, D = 0$. Therefore

$$\begin{aligned}
[x^n]\frac{3-8x+12x^2-8x^3}{(1-x)(1-2x)^3} &= [x^n]\frac{1}{1-x} + [x^n]\frac{2}{(1-2x)^3} \\
&= [x^n]\sum_{l\geq 0} x^l + [x^n]2\sum_{l\geq 0}\binom{l+2}{2}(2x)^l \\
&= 1 + 2^n(n+2)(n+1).
\end{aligned}$$

(b) Suppose the formal power series $A(x)$ satisfies the identity

$$(8x^4 - 20x^3 + 18x^2 - 7x + 1)A(x) = -8x^3 + 12x^2 - 8x + 3.$$

Determine a formula for $[x^n]A(x)$ for all $n \geq 0$ by establishing a linear homogeneous recurrence relation for the coefficients α_n of $A(x)$, which you have to solve after you also find its initial conditions (see also Example 1.5.1 of your course notes) .
Hint: verify that the roots of the characteristic polynomial are 1, 2 with multiplicities 1,3 respectively.

Solution. *Suppose that $A(x) = \sum_{n\geq 0}\alpha_n x^n$. From the given identity we have that*

$$\begin{aligned}
-8x^3 + 12x^2 - 8x + 3 &= (8x^4 - 20x^3 + 18x^2 - 7x + 1)\sum_{n\geq 0}\alpha_n x^n \\
&= 8\sum_{n\geq 0}\alpha_n x^{n+4} - 20\sum_{n\geq 0}\alpha_n x^{n+3} + 18\sum_{n\geq 0}\alpha_n x^{n+2} - 7\sum_{n\geq 0}\alpha_n x^{n+1} + \sum_{n\geq 0}\alpha_n x^n \\
&= 8\sum_{n\geq 4}\alpha_{n-4} x^n - 20\sum_{n\geq 3}\alpha_{n-3} x^n + 18\sum_{n\geq 2}\alpha_{n-2} x^n - 7\sum_{n\geq 1}\alpha_{n-1} x^n + \sum_{n\geq 0}\alpha_n x^n \\
&= \alpha_0 + (\alpha_1 - 7\alpha_0)x + (\alpha_2 - 7\alpha_1 + 18\alpha_0)x^2 + (\alpha_3 - 7\alpha_2 + 18\alpha_1 - 20\alpha_0)x^3 \\
&\quad + \sum_{n\geq 4}(8\alpha_{n-4} - 20\alpha_{n-3} + 18\alpha_{n-2} - 7\alpha_{n-1} + \alpha_n)x^n
\end{aligned}$$

We conclude that α_n satisfies the linear homogeneous recurrence relation

$$8\alpha_{n-4} - 20\alpha_{n-3} + 18\alpha_{n-2} - 7\alpha_{n-1} + \alpha_n = 0$$

with initial conditions

$$\begin{aligned}
\alpha_0 &= 3 \\
\alpha_1 - 7\alpha_0 &= -8 \\
\alpha_2 - 7\alpha_1 + 18\alpha_0 &= 12 \\
\alpha_3 - 7\alpha_2 + 18\alpha_1 - 20\alpha_0 &= -8
\end{aligned}$$

which can be rewritten as

$$\alpha_0 = 3,\ \alpha_1 = 13,\ \alpha_2 = 49,\ \alpha_3 = 161.$$

In order to solve the recurrence relation, we set up its characteristic polynomial

$$x^4 - 7x^3 + 18x^2 - 20x + 8 = (x-1)(x-2)^3$$

with roots 1,2 and multiplicities 1 and 3 respectively. We conclude that

$$\alpha_n = A + (B + Cn + Dn^2)2^n$$

for some reals A, B, C, D that we will determine from the initial conditions. Indeed, we have

$$\begin{aligned} 3 =& \alpha_0 = A + B \\ 13 =& \alpha_1 = A + (B + C + D)2 \\ 49 =& \alpha_2 = A + (B + 2C + 4D)4 \\ 161 =& \alpha_3 = A + (B + 3C + 9D)8 \end{aligned}$$

from which we derive

$$A = 1, = 2, C = 3, D = 1.$$

In other words, $\alpha_n = 1 + 2^n(n+2)(n+1)$ exactly as in part (a).

Question 3 [Marks 11=5+6]
Find the general solutions to the following recurrence equations.

(a) $a_0 = 9, a_1 = 36$, and $a_n - 7a_{n-1} + 10a_{n-2} = 3^n$ for $n \geq 2$.

Solution. *The first task is to determine one solution to the non-homogeneous recurrence. From the form of the given equation, we suspect that there must exist some solution that has the form $t_n = C \cdot 3^n$, for some real C. By substituting back to the recurrence, we see that constant C must satisfy (for every $n \geq 3$) that*

$$3^n = C \cdot 3^n - 7C \cdot 3^{n-1} + 10C \cdot 3^{n-2}$$

Solving for C we obtain $C = -9/2$, and hence $t_n = -\frac{9}{2}3^n$.

Next, we study the homogeneous recurrence $a_n - 7a_{n-1} + 10a_{n-2} = 0, n \geq 3$. The characteristic polynomial reads as

$$x^2 - 7x + 10 = (x-2)(x-5).$$

We conclude that a solution r_n to the homogeneous recurrence above has the form

$$r_n = A2^n + B5^n.$$

To determine the constants A, B we utilize the initial conditions of the non-homogeneous recurrence and we have

$$9 + \frac{9}{2} = r_0 - t_0 = A + B$$
$$36 + \frac{9}{2}3 = r_1 - t_1 = 2A + 5B$$

with solution $A = 6, B = 15/2$. Hence

$$a_n = 6 \cdot 2^n + \frac{15}{2} \cdot 5^n - \frac{9}{2}3^n$$

is the general solution to the non-homogeneous recurrence equation.

(b) $b_n - 3\sum_{i=0}^{n-1} b_i = (-1)^n$, for all $n \geq 0$.
Hint: think how two consecutive terms relate.

Solution. *We subtract*

$$b_{n-1} - 3\sum_{i=0}^{n-2} b_i = (-1)^{n-1}$$

from

$$b_n - 3\sum_{i=0}^{n-1} b_i = (-1)^n$$

to obtain

$$b_n - 4b_{n-1} = 2(-1)^n$$

with initial conditions that can be obtained from the original recurrence equation, that is $b_0 = (-1)^0 = 1$.

In order to solve this new recurrence, we need to first find one solution of $b_n - 4b_{n-1} = 2(-1)^n$. We will look for a solution among expressions of the form $t_n = B(-1)^n$. In order to satisfy the recurrence equation, we must have for every $n \geq 1$ that

$$B(-1)^n - 4B(-1)^{n-1} = 2(-1)^n$$

from which we obtain that $B = 2/5$, i.e. $t_n = \frac{2}{5}(-1)^n$.

Next we find find the general solution of the corresponding homogeneous equation, which is $r_n - 4r_{n-1} = 0$. The characteristic polynomial is $x - 4$ with unique root $x_0 = 4$, for the general solution of the homogeneous equation is of the form $r_n = A4^n$, for some real A that can be determined from $r_0 = b_0 - t_0$, i.e. $A = 1 - \frac{2}{5} = \frac{3}{5}$. We conclude that the general solution to the non-homogeneous recurrence is

$$b_n = \frac{3}{5}4^n + \frac{2}{5}(-1)^n.$$