MATH 239

TUTORIAL 7

Question 4:
Let G be a connected graph with 5 vertices of degree 10, and the rest of the vertices of G have degree 1. Find the minimum and maximum number of vertices possible.

Answer 4:
Let the set of degree 10 vertices be X and the set of other vertices be Y. Let s be the number of edges joining 2 vertices of X together. Since each vertex of Y must join to a vertex of X, we have of $|Y| = 5 \cdot 10 - 2s$ by counting neighbours of X.

Now, in the minimum case, we want to maximize s, so the vertices of X should join to each other to get the complete grah K_5. As K_5 has 10 edges, we have $|Y| = 5 \cdot 10 - 2 \cdot 10 = 30$. Hence, $|V(G)| = 35$.

On the other hand, a path from a vertex of X to another must only use vertices in X, so the subgraph of those 5 vertices are connected. Since adding an edge can degree the number of component by at most 1, there must be at least 4 edges joining those 5 vertices. For example, they can form a path of length 4. This means $|Y| = 5 \cdot 10 - 2 \cdot 4 = 42$. Hence, $|V(G)| = 47$.

Question 5:
Prove that, if G is connected, any two longest paths have a vertex in common.

Answer 5:
Suppose for contradiction $P = v_0 v_1 \cdots v_n$ and $P' = u_0 u_1 \cdots u_n$ are both longest paths of G. Since G is connected, there exists a path from v_i to u_j for all $0 \leq i, j \leq n$. Let $Q = v_i q_1 q_2 \cdots q_{m-1} u_j$ be a shortest path that satisfies this property. Without loss of generality, assume $i \geq j$. Now, none of the internal vertices q_s can be in P or P', as otherwise this contradicts with Q being a shortest path. Then, $v_0 v_1 \cdots v_i q_1 q_2 \cdots q_{m-1} u_j u_{j+1} \cdots u_n$ is path since v_i and u_j are the only common vertices of P, P', and Q . Furthermore, this path contains at least $(i+1)+(n-j+1) = n+(i-j)+2 \geq n+2$ vertices, so it is a path of length at least $n+1$

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