MATH 239 Assignment 7

- This assignment is due on Friday, November 2, 2012, at 10am in the drop boxes in St. Jerome's (section 1) or outside MC 4067 (the other two sections).
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.

1. Let G be a bipartite graph with vertex classes A and B. Suppose that every vertex in A has degree k and every vertex in B has degree ℓ. Prove that $|B| = \frac{k}{\ell}|A|$.

2. Let n be a positive integer. We define a graph G_n as follows. The vertex set of G_n is the set of all permutations of $\{1, 2, \ldots, n\}$. (Recall that a *permutation* of $\{1, 2, \ldots, n\}$ is just an ordering of the elements of $\{1, 2, \ldots, n\}$. Thus in particular $|V(G_n)| = n!$.) Two permutations σ and σ' are joined by an edge of G_n if and only if σ' can be obtained from σ by interchanging two positions. (For example, 3241 and 1243 are adjacent in G_4.)

   (a) Draw G_3 and label the vertices.

   (b) Prove that G_n is bipartite for every n. (Hint: consider partitioning the vertex set according to the function T, where for $\sigma = a_1 a_2 \ldots a_n$, the value of $T(\sigma)$ is the number of pairs $s < t$ such that $a_s > a_t$.)

3. Let G be a graph that has 20 vertices of degree 25 and 300 vertices of degree 5, and no other vertices. Prove that for every vertex x of degree 25 there exists a path in G from x to a vertex of degree 5.

4. Let $W = v_0 e_1 v_1 \ldots e_n v_n$ be a walk in a graph G, such that $v_0 = v_n$ and all the edges $e_1, \ldots, e_n$ are distinct. Prove that there exists a set $\{C_1, \ldots, C_m\}$ of cycles in G such that $\{e_1, \ldots, e_n\} = E(C_1) \cup \ldots \cup E(C_m)$, and $E(C_s) \cap E(C_r) = \emptyset$ for all $s \neq r$. (Hint: use induction on n.)

5. Let G be a graph with p vertices. Suppose every vertex in G has degree at least $\frac{p-1}{2}$. Prove that G is connected.