MATH 239 - Assignment 7

DUE: 10am Friday March. 15th in the drop boxes opposite the Math Tutorial Centre MC 4067.

Exercise 1 (20pts).

Let $G = (V, E)$ be a graph and suppose that every vertex has degree at least $k \geq 2$.

(a) Show that G has a path with at least k edges.

(b) Show that G has a cycle with at least k edges.

Solution:

(a) Among all paths in G choose a path P that contains as many edges as possible (such a path exists as there are only a finite number of vertices). Suppose that the path P consists of edges,

$$v_0v_1, v_1v_2, \ldots, v_{r-1}v_r. \qquad (\star)$$

We may assume that $r < k$ for otherwise the path P contains at least k edges as required. Since $deg(v_r) \geq k$, there are k vertices adjacent to v_r. As $r < k$ at least one of these vertices, say w, is distinct from $v_0, \ldots, v_{r-1}$. But then the path obtained from P by adding the edge v_rw has more edges than P, contradicting our choice of P. **(b)** Let P be a path with as many edges as possible and assume that P has edges as in ($\star$). As proved in part (a), $r \geq k$. Since P is a longest path all vertices adjacent to v_r are in $\{v_0, \ldots, v_{r-1}\}$. Since v_r has degree at least k, there exists v_i adjacent to v_r where $i \in \{0, r-k\}$. But then the path contained in P between v_i and v_r together with the edge v_iv_r forms a cycle with at least $k+1$ (hence k) edges.

Exercise 2 (20pts).

A walk is *closed* if the first vertex and the last vertex of the walk are the same.

(a) Show that in a bipartite graph, every closed walk has an even number of edges.
(Note, edges are counted as many time as they appear in the walk.)

(b) Show that every closed walk that does not repeat an edge contains a cycle.

Solution:

(a) Consider a walk $W = v_0, e_1, v_1, \ldots, v_{i-1}, e_i, v_i, \ldots, v_{n-1}, e_n, v_n$ where $e_i = v_{i-1}v_i$. Suppose that W is a closed walk, i.e. that $v_0 = v_n$, and suppose that G is a bipartite graph with partition X, Y. We may assume that $v_0 \in X$. Because of e_1, $v_1 \in Y$. Similarly, because of e_2, $v_2 \in X$. More generally, for all $i \in \{1, \ldots, n\}$, $v_i \in X$ if and only if i is even. Since $v_0 = v_n$ it follows that n is even and the edges traversed by the walk are exactly the edges $e_1, \ldots, e_n$. **(b)** Consider a walk $W = v_0, e_1, v_1, \ldots, v_{i-1}, e_i, v_i, \ldots, v_{n-1}, e_n, v_n$ where $e_i = v_{i-1}v_i$. Suppose that W is a closed walk, i.e. that $v_0 = v_n$. It follows that $W' = v_1, e_2, v_2, \ldots, v_{i-1}, e_i, v_i, \ldots,$ v_{n-1}, e_n, v_n is a walk from v_1 to v_n. Hence, by Theorem 4.6.2 there exists a path P from v_1 to v_n only using edges of W'. But then P together with edge e_1 form a cycle C.

Exercise 3 (20pts).

Prove that the following statements are equivalent for a graph $G = (V, E)$,

(i) G is connected and G has exactly one cycle,

(ii) G is connected and $|E| = |V|$,

(iii) G has exactly one cycle and $|E| = |V|$.

Solution:

Suppose (i) holds. Let C denote the unique cycle in G and let e be any edge of C. Let $T = (V, E')$ be obtained from G by deleting edge e, i.e. $E' = E \setminus \{e\}$. Since e is in C, e is not a bridge of G (see Lemma 4.9.2). It follows that T is connected. Moreover, since C was the unique cycle of G, T is has no cycle. It follows that T is a tree. Hence, by Theorem 5.1.5 $|E'| = |V| - 1$, thus $|E| = |V|$. In particular, (ii) and (iii) holds.

Suppose (ii) holds. As G is connected, by Theorem 5.2.1 it contains a spanning tree $T = (V, E')$. Since T is a tree, $|E'| = |V| - 1$ (Theorem 5.1.5). It follows that G is obtained from T by adding a single edge $e = uv$. Since T is connected, there is a path P between u and v. Then P together uv is a cycle. Suppose G has two distinct cycle C_1, C_2 then $C_1 \setminus \{e\}, C_2 \setminus \{e\}$ are the edges of two distinct paths of T between u and v, a contradiction with Lemma 5.1.2. Thus (i) holds.

Suppose (iii) holds. Let $G_1, \ldots, G_k$ be the connected components of G. Suppose that for $i = 1, \ldots, k$, $G_i = (V_i, E_i)$. We may assume that G_1 has exactly one cycle, and that $G_2, \ldots, G_k$ each have no cycles, i.e. are trees. It follows from the fact that (i) implies (ii) that $|V_1| = |E_1|$. For all $i = 2, \ldots, k$, $|V_i| = |E_i| + 1$ by Theorem 5.1.5. Thus

$$|V| = |V_1| + \sum_{i=2}^{k} |V_i| = |E_1| + \sum_{i=2}^{k} (|E_i| + 1) = |E| + (k - 1).$$

It follows $k = 1$, i.e. that G is connected.

Exercise 4 (20pts)**.**

Let $G = (V, E)$ be a graph with distinct vertices s and t. We say that a set of st-paths are *internally disjoint* if no two of these paths share a common vertex aside from s and t. A set of vertices X is a *vertex st-cut* if $X \subseteq V \setminus \{s, t\}$ and the graph obtained from G by removing all vertices in X has no path from s to t. Show that statement (i) implies statement (ii).

(i) There exists k internally disjoint paths from s to t.

(ii) Every vertex st-cut contains at least k vertices.

Note, these statements are in fact equivalent but you are not asked to prove this.

Solution:

Let $P_1, \ldots, P_k$ be a set of internally disjoint paths between s and t and let X be any vertex st-cut. Let H be the graph obtained from G by removing all vertices X (as well as all edges incident to X). Since by definition of st-cut, there exists no path between s and t in H, we must have a component $H' = (V', E')$ of H where $s \in V'$ but $t \notin V'$. For any $i \in \{1, \ldots, k\}$, the path P_i must be of the form $P_i = v_0, e_1, v_1, \ldots, v_{i-1}, e_i, v_i, \ldots, v_{n-1}, e_n, v_n$ where $e_i = v_{i-1}v_i$ and $v_0 = s$, $v_n = t$. Since $v_0 \in V'$ but $v_n \notin V'$, there is a vertex, say w_i that is the first vertex of P_i that is not in V'. But then $w_i \in X$. Since the paths $P_1, \ldots, P_k$ are internally disjoint, $w_1, \ldots, w_k$ are all distinct. It follows that $X \geq \{w_1, \ldots, w_k\}$ has cardinality at least k as required.

Exercise 5 (20pts)**.**

Let $G = (V, E)$ be a graph that is k-regular. Denote by $\delta(S)$ the set of edges with exactly one endpoint in S and by $\gamma(S)$ the set of edges with two endpoints in S.

(a) Show that for every $S \subseteq V$ we have

$$\sum_{v \in S} deg(v) = |\delta(S)| + 2|\gamma(S)|.$$

(b) Using (a) show that if a connected graph is k-regular where k is even then G has no bridge.

Solution:
(a) Consider the sum

$$\sum_{v \in S} deg(v) \tag{$\star$}$$

If $ab \in E$ has both endpoints in S, then both a and b contribute 1 to $(\star)$. If $ab \in E$ has exactly one endpoint in S, say a, then a contributes 1 to $(\star)$ but b does not. If $ab \in E$ has neither endpoint in S, then neither a or b contribute to $(\star)$. Hence, (a) holds. **(b)** Suppose for a contradiction that G is connected but has a bridge ab. Then Lemma 4.9.2 implies that $G \setminus \{ab\}$ has two components $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $a \in V_1$ and $b \in V_2$. In particular $\delta(V_1) = \{ab\}$. Since G is k-regular and by (a) we have,

$$k|V_1| = \sum_{v \in V_1} deg(v) = |\delta(V_1)| + 2|\gamma(V_1)|.$$

Thus $|\delta(V_1)| = k|V_1| - 2|\gamma(V_1)|$. As k is even $|\delta(V_1)|$ is even, a contradiction as $\delta(V_1) = \{ab\}$.

Exercise 6 (20pts)**.**

(a) Show that if a tree has a vertex of degree r then it has at least r vertices of degree 1.

(b) Show that if a tree with at least two vertices has k vertices of degree r then it has at least $k(r-2)+2$ vertices of degree 1.

Solution:
Observe that (a) is a special case of (b) (set $k = 1$), thus it suffices to show (b). Let $G = (V, E)$ be a tree. Recall that k denotes the number of vertices of degree r, and let k' the number of vertices of degree 1. Then

$$|V| - 1 = |E| = \frac{1}{2}\sum_{v \in V} deg(v) \geq \frac{1}{2}\big[kr + k' + 2(|V| - k - k')\big], \tag{1}$$

where the first equality follows from the fact that in a tree the number of edges equal to the number of vertices minus one (Theorem 5.1.5), the second equality follows from the fact that the sum of the degree of a graph is equal to twice the number of edges (Theorem 4.3.1); and the inequality follows from the fact that, (since G has at least two vertices) there are $|V| - k - k'$ vertices that have degree greater than one and smaller than r. Multiplying (1) by 2 on both sides we deduce,

$$2|V| - 2 \geq kr + k' + 2|V| - 2k - 2k'.$$

After simplifying the terms we obtain $k' \geq (r-2)k + 2$.