MATH 239 Tutorial 2 Solution Outline

1. (a) Let S be the set of all subsets of $[3]$. Let w be the weight function on S such that for each $A \in S$, $w(A)$ is the sum of all elements of A. Determine the generating series $\Phi_S(x)$ with respect to w.

   (b) Let w' be the weight function on S such that for each $A \in S$, $w'(A)$ is twice the sum of all elements of A. Determine the generating series $\Phi'_S(x)$ with respect to w'.

   (c) What is the relationship between $\Phi_S(x)$ and $\Phi'_S(x)$?

   Solution.

   (a) $\Phi_S(x) = 1 + x + x^2 + 2x^3 + x^4 + x^5 + x^6$.

   (b) $\Phi'_S(x) = 1 + x^2 + x^4 + 2x^6 + x^8 + x^{10} + x^{12}$.

   (c) $\Phi'_S(x) = \Phi_S(x^2)$.

2. Let w be the weight function defined on $\mathbb{N}_0$ as follows: for each $a \in \mathbb{N}_0$,

$$w(a) = \begin{cases} a/2 & a \text{ is even} \\ 2a & a \text{ is odd} \end{cases}$$

   Determine the generating series of $\mathbb{N}_0$ with respect to w.

   Solution. Let $E = \{0, 2, 4, 6, \ldots\}$ and $O = \{1, 3, 5, 7, \ldots\}$. Then $\mathbb{N}_0 = E \cup O$ is a disjoint union. Using the sum lemma, we have

$$\begin{aligned} \Phi_E(x) &= 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \\ \Phi_O(x) &= x^2 + x^6 + x^{10} + x^{14} + \cdots = \frac{x^2}{1-x^4} \\ \Phi_{\mathbb{N}_0}(x) &= \frac{1}{1-x} + \frac{x^2}{1-x^4} = \frac{1 + x^2 - x^3 - x^4}{1 - x - x^4 + x^5}. \end{aligned}$$

3. For a binary string s, define its weight $w(s)$ to be the number of 1's in the string plus the length of the string itself. For example, $w(110100001) = 13$.

   (a) Let S_n be the set of all binary strings of length n. Use the product lemma to determine $\Phi_{S_n}(x)$.

   (b) Let T be the set of all binary strings (regardless of length). Determine $\Phi_T(x)$.

   Solution.

   (a) Think of each bit in the string as contributing 1 to the length of the string, and another 1 if that bit is 1. For each bit $\{0, 1\}$, use the weight function

$$\alpha(a) = \begin{cases} 1 & a = 0 \\ 2 & a = 1 \end{cases}$$

   So $\Phi_{\{0,1\}}(x) = x + x^2$. Using the weight of a string $w(a_1 \ldots a_n) = \sum_{i=0}^{n} \alpha(a_i)$, by the product lemma,

$$\Phi_{S_n}(x) = \Phi_{\{0,1\}^n}(x) = (x + x^2)^n.$$

(b) We see that $T = S_0 \cup S_1 \cup S_2 \cup \cdots$. Using the sum lemma,

$$\Phi_T(x) = \sum_{n\geq 0} \Phi_{S_n}(x) = \sum_{n\geq 0} (x+x^2)^n = \frac{1}{1-(x+x^2)}.$$

4. Let S_n be the set of all subsets of $[n]$, and for each $A \in S_n$, define $w(A)$ to be the sum of the elements in A. Give a combinatorial interpretation of the following:

$$\Phi_{S_n}(x) = (1+x^n)\Phi_{S_{n-1}}(x).$$

Solution. We can partition $S_n = A \cup B$ where A is the set of all subsets of $[n]$ which contains the element n and B is the set of all subsets of $[n]$ which does not contain the element n. This is a disjoint union. Clearly, $B = S_{n-1}$, so $\Phi_B(x) = \Phi_{S_{n-1}}(x)$. Any element of A consists of an element in S_{n-1} union with $\{n\}$. So we can find a bijection between A and S_{n-1} as follows: $f : A \to S_{n-1}$ where for any $T \in A$, $f(T) = T \setminus \{n\}$. For each element $T \in A$, $w(T) = w(f(T)) + n$. So

$$\Phi_A(x) = \sum_{T\in A} x^{w(T)} = \sum_{f(T)\in S_{n-1}} x^{w(f(T))+n} = x^n \Phi_{S_{n-1}}(x).$$

By the sum lemma,

$$\Phi_{S_n}(x) = \Phi_A(x) + \Phi_B(x) = (1+x^n)\Phi_{S_{n-1}}(x).$$

Additional exercises

1. How many ways can you make up n cents using an unlimited supply of pennies, nickels, dimes and quarters? For example, 7 cents can be made up in two ways: 7 pennies, or 2 pennies and 1 nickel. How would this change if you are allowed to use up to 42 nickels? Express your answers as coefficients of generating series.

Solution. We use the set $\mathbb{N}_0^4$ where $w(a,b,c,d) = a + 5b + 10c + 25d$. Then

$$\Phi_{\mathbb{N}_0^4} = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}.$$

If there are only 42 nickels, then we have the set $\mathbb{N}_0 \times \{0, 1, \ldots, 42\} \times \mathbb{N}_0 \times \mathbb{N}_0$. The generating series is

$$\frac{1-x^{215}}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}.$$

2. Let S be the set of all finite subsets of $\mathbb{N}$, and suppose the weight of a subset is the sum of all its elements. Find a "nice" expression for the generating series of S.

Solution. Use Q4 above and induction, $\Phi_S(x) = \prod_{n\geq 0}(1+x^n)$.

3. Using mathematical induction on k, prove that

$$(1-x)^{-k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n.$$

Solution. Clearly true when $k = 1$.

$$\begin{aligned}
(1-x)^{-k} &= (1-x)(1-x)^{-(k-1)} \\
&= (1-x) \sum_{n \geq 0} \binom{n+k-2}{k-2} x^n \\
&= 1 + \sum_{n \geq 1} \left(\binom{n+k-2}{k-2} + \binom{n+k-1}{k-2} \right) x^n \\
&= 1 + \sum_{n \geq 1} \binom{n+k-1}{k-1} x^n \\
&= \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n.
\end{aligned}$$