MATH 239 MIDTERM
SUGGESTED SOLUTIONS

MIDTERM QUESTIONS:

1 (a) Prove that [3]

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

(b) Give a combinatorial proof that [4]

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

(c) Use (a) and (b) to find an expression for [3]

$$\sum_{i=0}^{n} \binom{\binom{n}{i}}{2}.$$

2(a) Let $c_{n,k}$ be the number of k-subsets of $\{1, \ldots, n\}$ in which consecutive elements do not differ by 2 (we assume that the subsets are listed in increasing order). For each $k \geq 1$, prove that [3]

$$\sum_{n \geq 1} c_{n,k} x^n = \frac{x}{(1-x)^2} \left(\frac{x}{1-x} - x^2 \right)^{k-1}.$$

(b) Prove that [4]

$$\sum_{k \geq 1} \sum_{n \geq 1} c_{n,k} x^n y^k = \frac{xy}{(1-x)(1-x-xy(1-x+x^2))}.$$

(c) Let $d_{n,k}$ be the number of k-subsets of $\{1, \ldots, n\}$ in which no pair of elements differs by 2. Prove that [3]

$$\sum_{k \geq 1} \sum_{n \geq 1} d_{n,k} x^n y^k = \frac{xy(1+xy)}{(1-x)(1-x-x^3y(1+xy))}.$$

(HINT: Consider the list of differences as strings on $\{1, 2, 3, \ldots\}$ with no 2's, and no substring "11".)

3. Let $\mathcal{A}$ be the set of $\{0,1\}$-strings in which every block of 0s has even length, and every block of 1s has length at least two.

(a) Give an expression that uniquely creates the strings in $\mathcal{A}$. [3]

(b) For each $n \geq 0$, let a_n be the number of $\{0,1\}$-strings in $\mathcal{A}$ of length n. Prove that [4]

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1 - x + x^2}{1 - x - x^2 + x^3 - x^4}.$$

(c) Give initial conditions and a linear recurrence relation that determine the value of a_n for all $n \geq 0$. Calculate a_8. [3]

4. Let $a_n = (3n+1)2^n$, $n \geq 0$.

(a) Find a homogeneous, linear recurrence equation for a_n, together with initial conditions that determine the value of a_n for all $n \geq 0$. [4]

(b) Let $b_n = a_n^2$, $n \geq 0$. Find a homogeneous, linear recurrence equation for b_n, together with initial conditions that determine the value of b_n for all $n \geq 0$. [6]

5(a) Prove that every 3-regular graph has an even number of vertices. [2]

(b) Prove that every graph on 2 or more vertices has a pair of vertices of the same degree. [4]

(c) Prove that every graph on 6 vertices either contains a 3-cycle or a set of three vertices that are pairwise non-adjacent. [4]

6. Let G_n be the graph whose vertices are the $\{0,1\}$-strings of length n. Two strings are adjacent if they have the same number of 1's.

(a) Draw G_n for $n = 1$, 2, 3, and 4. [2]

(b) How many components does G_n have? [3]

(c) How many edges does G_4 have? [2]

(d) How many edges does G_n have? [3]

MIDTERM SOLUTIONS:

1(a) (FIRST SOLUTION) The number of all subsets of $S = \{1, ..., n\}$ is 2^n. The number of i–element subsets of S is $\binom{n}{i}$. Every subset of S has i elements for exactly one value of i in the range $0 \leq i \leq n$. Therefore

$$2^n = \sum_{i=0}^{n} \binom{n}{i}.$$

(SECOND SOLUTION) By the Binomial Theorem,

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

Upon substitution of $x = 1$ into this equation, we obtain

$$2^n = \sum_{i=0}^{n} \binom{n}{i}.$$

(b) Let A and B be n–element sets with $A \cap B = \emptyset$, and let $S = A \cup B$. Thus S is a $(2n)$–element set, and the number of n–element subsets of S is $\binom{2n}{n}$.

To choose an n–element subset of S we can fix a value of i in the range $0 \leq i \leq n$, then choose an i–element subset of A and an $(n-i)$–element subset of B (and then take the union of these two sets). Every n–element subset of S is constructed exactly once in this way. For a fixed value of i, there are $\binom{n}{i}$ choices for the subset of A, and $\binom{n}{n-i}$ choices for the subset of B. Since these choices are independent, there are $\binom{n}{i}\binom{n}{n-i}$ choices for this pair of subsets. Since $\binom{n}{n-i} = \binom{n}{i}$, the number above equals $\binom{n}{i}^2$. Since i has any one value in the range $0 \leq i \leq n$, the total number of n–element subsets of S is

$$\sum_{i=0}^{n} \binom{n}{i}^2.$$

This proves that

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2.$$

(c) First notice that

$$\binom{\binom{n}{i}}{2} = \frac{1}{2}\left[\binom{n}{i}^2 - \binom{n}{i}\right].$$

Therefore, using parts (a) and (b),

$$\begin{aligned}
\sum_{i=0}^{n}\binom{\binom{n}{i}}{2} &= \sum_{i=0}^{n}\frac{1}{2}\left[\binom{n}{i}^2 - \binom{n}{i}\right] \\
&= \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}^2 - \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i} \\
&= \frac{1}{2}\binom{2n}{n} - 2^{n-1}.
\end{aligned}$$

Comment: The solution to question 2 contains more detail than we expected you to give in the examination. However, the details are included so that you can read through them carefully and find most of your questions about the problem answered. At the very least, you should have *stated*: a) which decomposition you were using, b) its elementwise action, c) that it is a bijection, and d) that it is additively weight-preserving, so the Product Lemma applies. Each of these points is addressed in the •-paragraphs.

2(a) We use the decomposition Ω that encodes sets in terms of i) the first element, ii) the list of successive differences, and iii) the different between the largest element and n. This is the *Set-differences Decomposition*.

• **Definition of Ω and its bijectivity:** Let $\mathcal{U}_k$ be the set of all k-subsets of $\{1, 2, 3, \ldots\}$. Let $\alpha = \{a_1, \ldots, a_k\} \in \mathcal{U}_k$ be a subset of $\{1, \ldots, n\}$ and, to make this explicit, we denote α by $\{a_1, \ldots, a_k\}_n$. Let $\beta = (b_1, \ldots, b_{k+1})$ be such that $b_1 = a_1 \in \mathcal{N}_{\geq 0}$, $b_i = a_i - a_{i-1} \in \mathcal{N}_{\geq 1}$ for $i = 1, \ldots, k$, and $b_{k+1} = n - a_k \in \mathcal{N}_{\geq 0}$. These equations uniquely define β in terms of α, so

$$\Omega\colon \mathcal{U}_k \longrightarrow \mathcal{N}_{\geq 1} \times \mathcal{N}_{\geq 1}^{k-1} \times \mathcal{N}_{\geq 0} \colon \alpha \longmapsto \beta$$

is injective (one-to-one). Moreover, let $\beta = (b_1, \ldots, b_{k+1}) \in \mathcal{N}_{\geq 1} \times \mathcal{N}_{\geq 1}^{k-1} \times \mathcal{N}_{\geq 0}$. Then $a_i = b_1 + \cdots + b_i$ for $i = 1, \ldots, k$, and $n = b_1 + \cdots + b_{k+1}$. Then $1 \leq a_1 < \ldots < a_k \leq n$ and $a_i - a_{i-1} \neq 0$ for $i = 2, \ldots, k$, so $\{a_1, \ldots, a_k\} \in \mathcal{U}_k$ and $\Omega\alpha = \beta$. Thus Ω is surjective (onto). It follows that Ω is **bijective.**

• **Ω is additively ω-preserving:** Let ω be a weight function on $\mathcal{U}_k$ such that $\omega(\{a_1, \ldots, a_k\}_n) = n$. Let θ be a weight function on $\mathcal{N}_{\geq 1} \times \mathcal{N}_{\geq 1}^{k-1} \times \mathcal{N}_{\geq 0}$

such that $\theta((b_1,\dots,b_{k+1})) = b_1+\cdots+b_{k+1}$. Then

$$\omega(\{a_1,\dots,a_k\}_n) = n = b_1+\cdots+b_{k+1} = \theta((b_1,\dots,b_{k+1})) = \theta\Omega(\{a_1,\dots,a_k\}_n),$$

so $\omega = \theta\Omega$. Then Ω is an ω-preserving bijection. Let ρ be a weight function on $\mathcal{N}_{\geq 0}$ such that $\rho(i) = i$ for all $i \geq 0$. Then $\theta(\beta) = b_1+\cdots+b_{k+1} = \rho(b_1)+\cdots+\rho(b_{k+1})$ so ω is also additively preserved. We conclude that Ω is an **additively ω-preserving bijection**, and therefore, by the Product Lemma, that

$$\Phi^{\omega}_{\mathcal{U}_k} = \Phi^{\rho}_{\mathcal{N}_{\geq 1}}(x)\left(\Phi^{\rho}_{\{1,2,3,4,\dots\}}(x)\right)\Phi^{\rho}_{\mathcal{N}_{\geq 0}}(x).$$

We may now specialise this expression.

The set of restricted subsets: Now let $\mathcal{V}_k$ be the set of all subsets of $\{1,2,3,\dots\}$ that contain no *consecutive* elements that differ by exactly 2. Then $\mathcal{V}_k \subset \mathcal{U}_k$. In this notation, we seek $\Phi^{\omega}_{\mathcal{V}_k}(x)$, the generating series of $\mathcal{V}_k$ with respect to the weight ω. We therefore specialise the above decomposition to $\mathcal{V}_k$ and recompute the codomain to preserve bijectivity. We get a new bijection Θ that is related to Ω by our having dropped 2 from the set of differences. This information is summarised in the following diagram:

$$\begin{array}{ccc}
\Omega\colon \mathcal{U}_k & \longrightarrow & \mathcal{N}_{\geq 1}\times\{1,2,3,4,\dots\}^{k-1}\times\mathcal{N}_{\geq 0} \\
\uparrow & & \uparrow \\
\Theta\colon \mathcal{V}_k & \longrightarrow & \mathcal{N}_{\geq 1}\times\{1,3,4,5,\dots\}^{k-1}\times\mathcal{N}_{\geq 0}.
\end{array}$$

The decomposition Θ is therefore an additively ω-preserving bijection, since this property is inherited from Ω. Thus

$$\Phi^{\omega}_{\mathcal{V}_k} = \Phi^{\rho}_{\mathcal{N}_{\geq 1}}(x)\left(\Phi^{\rho}_{\{1,3,4,\dots\}}(x)\right)\Phi^{\rho}_{\mathcal{N}_{\geq 0}}(x) = \frac{x}{1-x}\left(\frac{x}{1-x}-x^2\right)^{k-1}\frac{1}{1-x},$$

which is the result.

b) From part (a)

$$\begin{aligned}
\sum_{k,n\geq 1} c_{n,k}x^n y^k &= \frac{x}{(1-x)^2}\sum_{k\geq 1}\left(\frac{x}{1-x}-x^2\right)^{k-1} = \frac{xy}{(1-x)^2}\sum_{k\geq 0}\left(\frac{x}{1-x}-x^2\right)^{k} y^k \\
&= \frac{xy}{(1-x)^2}\frac{1}{1-y\left(\frac{x}{1-x}-x^2\right)}
\end{aligned}$$

and the result follows.

c) Let $\mathcal{W}_k$ be the set of all subsets of $\{1, 2, 3, \ldots\}$ that contain no elements that differ by exactly 2. Note that these elements are not necessarily consecutive (so $\{1, 4, 5, 6, 9\}$ is disallowed since 4 and 6 differ by 2). This additional case is recognised by the occurrence of two consecutive 1s in the list of differences. It is therefore convenient to regard the $(k-1)$-tuple of differences as a *string*. We shall need to remove strings that have 2 or 11 as substrings.

Another specialisation of Ω: We therefore restrict the decomposition Ω now to $\mathcal{W}_k \subset \mathcal{U}_k$ and recompute the codomain to preserve bijectivity. We get a new bijection Ψ that is related to Ω by our having eliminated 2 *and* the substring 11 from strings formed by the differences between consecutive elements in subsets. This is summarised in the following diagram, where $\mathcal{E}_k$ denotes the set of all such strings of length k :

$$\begin{array}{ccc} \Omega\colon \mathcal{U}_k & \longrightarrow & \mathcal{N}_{\geq 1} \times \{1, 2, 3, 4, \ldots\}^{k-1} \times \mathcal{N}_{\geq 0} \\ \uparrow & & \uparrow \\ \Psi\colon \mathcal{W}_k & \longrightarrow & \mathcal{N}_{\geq 1} \times \mathcal{E}_k \times \mathcal{N}_{\geq 0}. \end{array}$$

The second line gives Ψ, which is an additively ω-preserving bijection since this property is inherited from Ω. Thus, by the Product Lemma,

$$\Phi^{\omega}_{\mathcal{W}_k}(x) = \frac{x}{1-x}\ \Phi^{\omega}_{\mathcal{E}_k}(x)\ \frac{1}{1-x},$$

and $d_{n,k}$ is identified as $d_{n,k} = [x^n]\Phi^{\omega}_{\mathcal{W}_k}(x)$.

Strings with no 2 or 11 as substrings: Let $\mathcal{D} = \{1, 3, 4, 5, \ldots\}$. Then the decomposition for $\mathcal{D}^\star$ in terms of the blocks of 1s is given by the first line of the following diagram, and the specialisation to the set $\mathcal{E}$ is given in the second line:

$$\begin{array}{ccc} \mathcal{D}^\star & = & (1^\star\,(\mathcal{D} - \{1\}))^\star\, 1^\star \\ \uparrow & & \uparrow \\ \mathcal{E} := \bigcup_{k\geq 0} \mathcal{E}_k & = & (\{\varepsilon, 1\}(\mathcal{D} - \{1\}))^\star\, \{\varepsilon, 1\} \end{array}$$

That is to say, the substring 11 is removed by the decomposition in the second line, so $1^\star$ is replaced by $\{\varepsilon, 1\}$.

The generating series: Then the generating series for $\mathcal{E}_k$ with respect to the sum of the elements in the string, marked by x is, by the Product Lemma,

$$\Phi^{\omega}_{\mathcal{E}_k}(x) = [y^{k-1}]\frac{\Phi_{\{\varepsilon,1\}}}{1 - \Phi_{\{\varepsilon,1\}}\Phi_{\mathcal{D}-\{1\}}},$$

where y marks the length of strings. But $\Phi_{\{\varepsilon,1\}} = 1 + xy$ and $\Phi_{\mathcal{D}-\{1\}} = \Phi_{\{3,4,5,\ldots\}} = yx^3/(1-x)$. Thus

$$\Phi^{\omega}_{\mathcal{E}_k}(x) = [y^{k-1}](1+xy)\left(1-(1+xy)\frac{yx^3}{1-x}\right)^{-1},$$

so, combining these expressions, we have

$$[y^k]\sum_{k,n\geq 1} d_{n,k}x^n y^k = \Phi^{\omega}_{\mathcal{W}_k}(x) = \frac{x}{1-x}[y^k]y(1+xy)\left(1-(1+xy)\frac{yx^3}{1-x}\right)^{-1}\frac{1}{1-x}$$

so

$$\sum_{k,n\geq 1} d_{n,k}x^n y^k = \frac{xy(1+xy)}{(1-x)^2}\left(1-(1+xy)\frac{yx^3}{1-x}\right)^{-1},$$

and the result follows.

3(a) The following are some valid decompositions.

$$\begin{aligned}
&\{00\}^*\left(\{11\}\{1\}^*\{00\}\{00\}^*\right)^*\{\epsilon, 11, 111, 1111, \ldots\}\\
&\{\epsilon, 11, 111, 1111, \ldots\}\left(\{00\}\{00\}^*\{11\}\{1\}^*\right)^*\{00\}^*\\
&\{\epsilon, 11, 111, 1111, \ldots\}\left(\{00\}\{\epsilon, 11, 111, 1111, \ldots\{\right)^*
\end{aligned}$$

Or, you can use the following recursive decomposition. Let S be the set of strings in this question and T be the set of strings that start with a non-empty block of 1's of any length, and the rest of the blocks have the same restriction as the strings in S. Then

$$\begin{aligned}
S &= \{\epsilon\}\cup\{00\}S\cup\{1\}T\cup\{11, 111, 1111, \ldots\}\\
T &= \{1, 11, 111, \ldots\}\{00\}S
\end{aligned}$$

(b) Using the first decomposition in part (a) we get

$$\begin{aligned}
\sum_{i\geq 0} a_n x^n &= \left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-\frac{x^2}{1-x}\frac{x^2}{1-x^2}}\right)\left(\frac{1}{1-x}-x\right)\\
&= \left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-\frac{x^2}{1-x}\frac{x^2}{1-x^2}}\right)\left(\frac{1-x+x^2}{1-x}\right)\\
&= \frac{1-x+x^2}{(1-x)(1-x^2)-x^4}\\
&= \frac{1-x+x^2}{1-x-x^2+x^3-x^4} \qquad (1)
\end{aligned}$$

Of course, the other decompositions give similar calculations and the same result.

(c) From (1) we see

$$\sum_{i\geq 0} a_n x^n - \sum_{i\geq 0} a_n x^{n+1} - \sum_{i\geq 0} a_n x^{n+2} + \sum_{i\geq 0} a_n x^{n+3} - \sum_{i\geq 0} a_n x^{n+4} = 1 - x + x^2$$

Equating coefficients we get

$$\begin{aligned} a_0 &= 1 \\ a_1 - a_0 &= -1 \to a_1 = 0 \\ a_2 - a_1 - a_0 &= 1 \to a_2 = 2 \\ a_3 - a_2 - a_1 + a_0 &= 0 \to a_3 = 1 \\ a_n &= a_{n-1} + a_{n-2} - a_{n-3} + a_{n-4}, \text{ for } n \geq 4 \end{aligned}$$

This gives $a_5 = 3, a_6 = a_7 = 8$ and $a_8 = 17$

4(a) Since $3n+1$ is a polynomial in n of degree 1, the characteristic polynomial for this recurrence has 2 as a root, of multiplicity 2, so the characteristic polynomial is

$$(x-2)^2 = x^2 - 4x + 4.$$

Thus a_n satisfies the recurrence $a_n - 4a_{n-1} + 4a_{n-2} = 0$, $n \geq 2$. For the initial conditions, we calculate $a_0 = (3 \cdot 0 + 1)2^0 = 1$, and $a_1 = (3 \cdot 1 + 1)2^1 = 8$.

(b) We have

$$b_n = a_n^2 = (3n+1)^2 2^{2n} = (9n^2 + 6n + 1)4^n, \quad n \geq 0,$$

where $9n^2 + 6n + 1$ is a polynomial in n of degree 2, so the characteristic polynomial for this recurrence has 4 as a root, of multiplicity 3. Therefore the characteristic polynomial is

$$(x-4)^3 - x^3 - 12x^2 + 48x - 64,$$

so b_n satisfies the recurrence equation

$$b_n - 12b_{n-1} + 48b_{n-2} + 64b_{n-3} = 0, \quad n \geq 3.$$

For the initial conditions, we calculate $b_0 = a_0^2 = 1^2 = 1$, $b_1 = a_1^2 = 8^2 = 64$, and $b_2 = a_2^2 = 28^2 = 784$.

5(a) We know that

$$\sum_{v\in V(G)} \deg(v) = 2|E|$$

so, in a 3-regular graph,

$$3|V(G)| = 2|E(G)|.$$

Note that:

$$|V(G)| \equiv 3|V(G)| = 2|E(G)| \equiv 0 \mod 2$$

so $|V(G)|$ is even.

Notes on Solutions Received: Generally, this problem was fairly well-done. Those who did not provide correct solutions often attempted to count the vertices with arguments along the lines of "fix a vertex; it is adjacent to three other vertices; each of these is adjacent to three vertices, etc." However, these constructions often double-counted vertices, or failed to count all vertices in the graph.

5(b) This question was on Assigment 5, so a solution can be found in the Assignment 5 solutions.

Many students wanted to prove this by induction, but did so incorrectly. For those who want to do the problem by induction, this is what a correct inductive proof would look like:

Let n denote the number of vertices, and apply induction on n. For the base case, $n = 2$, the only possible graphs are K_2 (in which there are two vertices of degree 1) or the empty graph (in which there are two vertices of degree 0). So the theorem holds in the base case.

Suppose that $n > 2$ and that the theorem holds for graphs on fewer than n vertices. Let G be a graph with n vertices, and consider two cases. If G has a vertex v of degree 0, then $H := G \setminus \{v\}$ is a graph on $n-1$ vertices, so by the inductive hypothesis, H has two vertices a and b with $\deg_H(a) = \deg_H(b)$. But since $\deg(v) = 0$, then

$$\deg_G(a) = \deg_H(a) = \deg_H(b) = \deg_G(a).$$

In the other case, G has no vertices of degree 0, so there are $n-1$ possible values for the degree of a vertex: $1 \leq \deg(v) \leq n-1$. But G has n vertices, so by the Pigeonhole Principle, there exist two vertices of G of the same degree. Thus, the statement of 5(b) is true by induction.

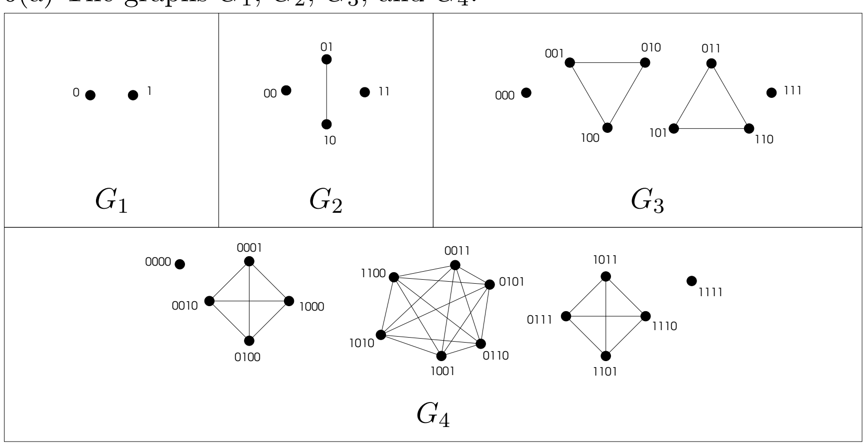
Notes on Solutions received: Common mistakes included assuming without justification that the graph is connected, and neglecting the fact that a vertex can have a degree of zero. In attempts at an inductive proof, a common mistake was to add a vertex and edges to a graph without accounting for the fact that this can change the degrees of the other vertices.

(c) Let G be a graph on 6 vertices. Note that G is not necessarily connected. Consider a vertex v in $V(G)$.

Suppose first, that the degree of v is at least 3. We are done if v has two neighbors u and w that are connected as v, u, w form a 3-cycle in this case. On the other hand, if no two of v's neighbors are adjacent then we have identified a set of 3 vertices which are pairwise non-adjacent.

Now, assume that v has degree at most 2. In this case there are three vertices $u, w, z \in V(G)$ that are not adjacent to v. If there is a pair of vertices in $\{u, w, z\}$ which is not adjacent then, together with v, we have a set of 3 pair-wise non-adjacent vertices. On the other hand, each pair of vertices in $\{u, w, z\}$ is adjacent and therefore u, w, z form a 3-cycle in G.

6(a) The graphs G_1, G_2, G_3, and G_4.



(b) In G_n any two vertices with the same number of 1's are adjacent, and hence in the same component. Since every edge connects two vertices with the same number of 1's, no vertices with different numbers of 1's are in the same component. So there is precisely one component for each possible number of 1's. Since the possible numbers of 1's are the elements of the set $\{0, 1, 2, \ldots, n\}$, the graph G_n has $n+1$ components.

Note: This answer required more than simply extrapolating a pattern from part (a). It was necessary to explain why the pattern continues to hold for values of n greater than 4.

Note: A component need not contain any edges. A single isolated vertex satisfies the definition of being connected, since there is a path between any pair of vertices in the component (there just happen to be zero pairs for which the statement must be true).

(c) The graph G_4 has 4 vertices of degree 3 in the component of vertices with one 1, 6 vertices of degree 5 in the component of vertices with two 1's, and 4 vertices of degree 3 in the component of vertices with three 1's. Combining these, we see the sum of the degrees of the vertices is

$$2q = \sum_{v \in V(G_4)} \deg v = 4 \times 3 + 6 \times 5 + 4 \times 3 = 12 + 30 + 12 = 54.$$

Dividing by 2, we see that G_4 has $q = 27$ edges.

(d) In G_n, there are $\binom{n}{i}$ vertices with i 1's. Every pair of these is adjacent. Since there are $\binom{\binom{n}{i}}{2}$ such pairs, the component with i 1's contributes $\binom{\binom{n}{i}}{2}$ edges to the graph. The total number of edges is thus

$$\sum_{i=0}^{n} \binom{\binom{n}{i}}{2}.$$

This expression can be simplified using question 1(c) to see that G_n has $\frac{1}{2}\binom{2n}{n} - 2^{n-1}$ edges, but such a simplification was not required. As a check,

$$\frac{1}{2}\binom{2(4)}{4} - 2^{4-1} = \frac{1}{2}70 - 2^3 = 35 - 8 = 27,$$

in agreement with part (c).