MATH 239 Assignment 5

- This assignment is due on Friday, October 19, 2012, at 10 am in the drop boxes in St. Jerome's (section 1) or outside MC 4067 (the other two sections).
- You may collaborate with other students in the class, provided that you list your collaborators. However, you MUST write up your solutions individually. Copying from another student (or any other source) constitutes cheating and is strictly forbidden.
- The first problem is optional and may be solved for bonus marks.

1. **(Bonus problem)** Find the generating series (with respect to length) for the set of binary strings that do not contain the substring 110011.

Solution:

Let L denote the set of strings that do not contain the substring 110011. Let M denote the set of strings that contain the substring 110011 exactly once, as a suffix. We claim that

$$\begin{aligned}\{\epsilon\} \cup L\{0,1\} &= L \cup M \\ L\{110011\} &= M \cup M\{0011\} \cup M\{10011\}.\end{aligned}$$

The first equation holds for the same reasons outlined in the solution of Problem 2.8.2.

For the second equation, if $a \in L$, then $b = a110011$ clearly contains 110011 as a substring, but it might contain multiple copies. If a ends in 1100 then b contains two copies of 110011, and dropping the final 0011 from b gives us a string in M. If a ends in 11001 then b again contains two copies of 110011, and dropping the final 10011 from b gives us a string in M. Otherwise, a contains only one copy of 110011, i.e., $a \in M$. Therefore $L\{110011\} \subseteq M \cup M\{0011\} \cup M\{10011\}$. On the other hand, it is clear that $M \subseteq L\{110011\}$, $M\{0011\} \subseteq L\{110011\}$, and $M\{10011\} \subseteq L\{110011\}$, since strings in M, $M\{0011\}$, and $M\{10011\}$ can all be formed by appending 110011 to a string with no occurrence of 110011. Therefore $M \cup M\{0011\} \cup M\{10011\} \subseteq L\{110011\}$, and the second equation holds.

Now by the sum and composition rules, we have

$$\begin{aligned}1 + 2x\Phi_L(x) &= \Phi_L(x) + \Phi_M(x) \\ x^6\Phi_L(x) &= (1 + x^4 + x^5)\Phi_M(x).\end{aligned}$$

From the second equation, $\Phi_M(x) = \frac{x^6}{1+x^4+x^5}\Phi_L(x)$. Using this in the first equation gives

$$1 = \left[(1-2x) + \frac{x^6}{1+x^4+x^5}\right]\Phi_L(x),$$

so

$$\Phi_L(x) = \left[(1-2x) + \frac{x^6}{1+x^4+x^5}\right]^{-1}.$$

If desired, this can be simplified to rational form, giving

$$\frac{1+x^4+x^5}{1-2x+x^4-x^5-x^6}.$$

2. Prove Lemma 3.1.1: If $f(x)$ is a polynomial of degree less than r, then there is a polynomial $P(x)$ with degree less than r such that

$$[x^n]\frac{f(x)}{(1-\theta x)^r} = P(n)\theta^n.$$

Solution: Let $f(x) = \sum_{i=0}^{r-1} a_i x^i$. By Theorem 1.6.5, we have

$$\frac{1}{(1-\theta x)^r} = \sum_{m\geq 0} \binom{m+r-1}{r-1} (\theta x)^m,$$

so

$$\frac{f(x)}{(1-\theta x)^r} = \sum_{i=0}^{r-1}\sum_{m\geq 0} \binom{m+r-1}{r-1} a_i \theta^m x^{m+i}.$$

The coefficient of x^n comes from the terms in this sum where $m+i=n$, so

$$\begin{aligned}[x^n]\frac{f(x)}{(1-\theta x)^r} &= \sum_{i=0}^{r-1} \binom{n-i+r-1}{r-1} a_i \theta^{n-i} \\ &= P(n)\theta^n\end{aligned}$$

where

$$P(n) = \sum_{i=0}^{r-1} \binom{n-i+r-1}{r-1} a_i \theta^{-i}.$$

This is a polynomial in n of degree less than r because for each i,

$$\binom{n-i+r-1}{r-1} = \frac{(n-i+r-1)(n-i+r-2)\cdots(n-i+1)}{(r-1)!}$$

is a polynomial in n of degree $r-1$, and $P(n)$ is a linear combination of such polynomials, so it is a polynomial in n of degree at most $r-1$.

3. (a) Find values of a and b so that

$$\frac{x+8}{(x-3)(2x+5)} = \frac{a}{x-3} + \frac{b}{2x+5}.$$

(b) Find a closed-form expression for

$$[x^n]\frac{x+8}{(x-3)(2x+5)}.$$

Solution:

(a) Expanding the right-hand side gives

$$\frac{a}{x-3} + \frac{b}{2x+5} = \frac{a(2x+5) + b(x-3)}{(x-3)(2x+5)} = \frac{x(2a+b) + (5a-3b)}{(x-3)(2x+5)},$$

so we have

$$2a + b = 1 \quad \text{and} \quad 5a - 3b = 8,$$

which has the solution $a = 1$, $b = -1$.

(b) We have

$$\frac{1}{x-3} = -\frac{1}{3} \cdot \frac{1}{1 - x/3} = -\frac{1}{3} \sum_{n \geq 0} (x/3)^n$$

and

$$\frac{1}{2x+5} = \frac{1}{5} \cdot \frac{1}{1 + 2x/5} = \frac{1}{5} \sum_{n \geq 0} (-2x/5)^n,$$

so

$$\begin{aligned} [x^n] \frac{x+8}{(x-3)(2x+5)} &= [x^n] \left(\frac{1}{x-3} - \frac{1}{2x+5} \right) \\ &= -\frac{1}{3} \left(\frac{1}{3} \right)^n - \frac{1}{5} \left(-\frac{2}{5} \right)^n \\ &= -\left(\frac{1}{3^{n+1}} + \frac{(-2)^n}{5^{n+1}} \right). \end{aligned}$$

4. Suppose $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_3 = 4$, and

$$a_n = 8a_{n-2} - 16a_{n-4}$$

for all integers $n \geq 4$. Determine a_n explicitly for all non-negative integers n.

Solution: The characteristic polynomial is

$$\begin{aligned} x^4 + 0x^3 - 8x^2 + 0x + 16 &= (x^2)^2 - 8(x^2) + 16 \\ &= (x^2 - 4)^2 \\ &= (x-2)^2(x+2)^2, \end{aligned}$$

which has roots $x = \pm 2$, each with multiplicity 2. Therefore

$$a_n = (A + Bn)2^n + (C + Dn)(-2)^n$$

for some constants A, B, C, D. From the initial conditions, we have

$$\begin{aligned} 1 &= A + C \\ 2 &= 2A + 2B - 2C - 2D \\ 3 &= 4A + 8B + 4C + 8D \\ 4 &= 8A + 24B - 8C - 24D. \end{aligned}$$

This could be solved by Gaussian elimination, but another approach is to define $E_\pm := A \pm C$ and $F_\pm := B \pm D$, giving

$$\begin{aligned}
1 &= E_+ \\
2 &= 2E_- + 2F_- \\
3 &= 4E_+ + 8F_+ \\
4 &= 8E_- + 24F_-.
\end{aligned}$$

Then it is easy to see that $E_+ = 1$, $F_+ = -1/8$, $E_- = 5/4$, $F_- = -1/4$, so $A = (E_+ + E_-)/2 = 9/8$, $B = (F_+ + F_-)/2 = -3/16$, $C = (E_+ - E_-)/2 = -1/8$, $D = (F_+ - F_-)/2 = 1/16$. Therefore

$$\begin{aligned}
a_n &= (\tfrac{9}{8} - \tfrac{3}{16}n)2^n + (-\tfrac{1}{8} + \tfrac{1}{16}n)(-2)^n \\
&= \frac{2^n}{16}\left[(18 - 3n) + (-2 + n)(-1)^n\right].
\end{aligned}$$

5. Let n be a fixed positive integer. Suppose $b_i = (-1)^i i$ for $i = 0, 1, \ldots, n-1$ and

$$b_i = -\sum_{k=1}^{n} \binom{n}{k} b_{i-k}.$$

for all integers $i \geq n$. Determine b_i explicitly for all non-negative integers i.

Solution: The characteristic polynomial is

$$\begin{aligned}
x^n + \sum_{k=1}^{n} \binom{n}{k} x^{n-k} &= \sum_{k=0}^{n} \binom{n}{k} x^k \\
&= (1 + x)^n
\end{aligned}$$

(by the Binomial Theorem), which has the single root $x = -1$ with multiplicity n. Therefore $b_i = (-1)^i p(i)$ where $p(i)$ is a polynomial of degree $n - 1$ that satisfies $(-1)^i p(i) = b_i$ for $i = 0, 1, \ldots, n - 1$. Since $p(i) = i$ satisfies the initial conditions, we have $b_i = (-1)^i i$.

6. Suppose $c_0 = 0$, $c_1 = -1$, and

$$c_i = -2c_{i-1} - c_{i-2} + 4i - 4$$

for all integers $i \geq 2$. Determine c_i explicitly for all non-negative integers i.

Solution: Consider a solution of the form $c_i = \alpha i + \beta$. Then to have

$$\begin{aligned}
\alpha i + \beta = c_i = &= -2c_{i-1} - c_{i-2} + 4i - 4 \\
&= -2(\alpha(i-1) + \beta) - (\alpha(i-2) + \beta) + 4i - 4 \\
&= (-3\alpha + 4)i + (4\alpha - 3\beta - 4)
\end{aligned}$$

for all i, we must have $\alpha = -3\alpha + 4$ and $\beta = 4\alpha - 3\beta - 4$, which has the solution $\alpha = 1$ and $\beta = 0$. From problem 5, the corresponding homogeneous equation has a solution of the form $(-1)^i(A + Bi)$. Thus the general solution has the form

$$c_i = i + (-1)^i(A + Bi)$$

for some constants A and B. We have

$$\begin{aligned} c_0 &= A = 0 \\ c_1 &= 1 - (A + B) = -1, \end{aligned}$$

so $A = 0$ and $B = 2$. Thus the solution is

$$c_i = i(1 + 2(-1)^i).$$