NAME:____________________

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Math 239 Quiz

Wednesday October 12, 4:30 pm

No calculators or other aids may be used. Show all your work.

Question 1. Let n be an integer so that $n \geq 1$. Prove that

$$n2^{n-1} = \sum_{k=0}^{n-1}(k+1)\binom{n}{k+1}.$$

(You may give either a combinatorial proof or a proof using the binomial theorem.)

Solution: **Combinatorial proof.** For each subset A of $\{1, 2, \ldots, n\}$, and each element a of A, consider the ordered pair $(a, A \setminus \{a\})$.

On the one hand, there are n possibilities for the element a and then 2^{n-1} possibilities for the subset $A \setminus \{a\}$ (this is any subset of $\{1, 2, \ldots, n\} \setminus \{a\}$). Thus, there are $n2^{n-1}$ such pairs.

On the other hand, given A, there are $|A|$ choices for a. So for each k, there are $k\binom{n}{k}$ possibilities for the pair $(a, A \setminus \{a\})$, with $a \in A$ and $|A| = k$.

Summing over all possible values of k, namely 0 to n, yields (realizing $0\binom{n}{0} = 0$ and setting $j = k - 1$ for the second equality)

$$n2^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} = \sum_{j=0}^{n-1}(j+1)\binom{n}{k+1},$$

as required.

Binomial Theorem proof.

The Binomial Theorem states that

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k.$$

Differentiating yields

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}x^{k-1}.$$

Substituting $x = 1$ yields the desired result.

Question 2. Find the generating function for the compositions of n in which each part is an odd positive integer at most 20. (Note the number of parts is not fixed. Your solution should specify the set to be counted, and you should indicate each place where you apply a theorem from class.)

Solution: As usual, the set S_k of such compositions with k parts is $S_1 \times S_1 \times \cdots \times S_1$ (k times). Since the weight function is additive over the parts, the Product Lemma implies $\Phi_{S_k}(x) = (\Phi_{S_1})^k$. In this solution, we allow the empty composition. (It is not necessary to do so.)

Let S denote the set of all compositions in which each part is odd and at most 20. Since the S_k are disjoint, the Sum Lemma implies

$$\Phi_S(x) = \sum_{k=0}^{\infty} \Phi_{S_k}(x) = \sum_{k=0}^{\infty} (\Phi_{S_1}(x))^k = \frac{1}{1 - \Phi_{S_1}(x)} .$$

It remains to determine $\Phi_{S_1}(x)$. Since the possibilities for the parts are $1, 3, 5, 7, \ldots, 15, 17, 19$,

$$\Phi_{S_1}(x) = x + x^3 + x^5 + \cdots + x^{19} ,$$

so

$$\Phi_S(x) = \frac{1}{1 - x - x^3 - x^5 - \cdots - x^{19}} .$$

Alternate solution:

Since the possibilities for the parts are $1, 3, 5, 7, \ldots, 15, 17, 19$,

$$\Phi_{S_1}(x) = x + x^3 + x^5 + \cdots + x^{19} = x(1 + x^2 + \cdots + x^{18}) = \frac{x(1 - x^{20})}{1 - x^2} ,$$

so

$$\Phi_S(x) = \frac{1}{1 - \frac{x(1-x^{20})}{1-x^2}} = \frac{1 - x^2}{1 - x^2 - x(1 - x^{20})} = \frac{1 - x^2}{1 - x - x^2 + x^{21}} .$$

Question 3. **(a)** Let S be a set with weight function w whose generating function is

$$\Phi_S(x) = \frac{x^2 + x^3}{1 - 3x^3 + x^4}.$$

Derive a recurrence relation with initial conditions for the sequence $a_n = [x^n]\Phi_S(x)$ for all $n \geq 0$.

(b) Determine a_5.

Solution: (a)

Solution 1 (using theorems from class): Applying the general principle, , $a_n - 2a_{n-3} + a_{n-4} = 0$ is the recurrence relation for all $n \geq 4$.

Since the numerator x^2+x^3 has degree 3 and the denominator $1-3x^3+x^4$ has degree 4, we need to evaluate a_0, a_1, a_2, a_3 in order to determine the sequence.

Now $x^2 + x^3 = (1 - 3x^3 + x^4)(\sum_{n\geq 0} a_n x^n)$. Comparing the coefficients of x^0, x^1, x^2, and x^3 on both sides of this equation, we find

$$\begin{aligned}
x^0 &: \quad 0 = a_0 \\
x^1 &: \quad 0 = a_1 \\
x^2 &: \quad 1 = a_2 \\
x^3 &: \quad 1 = a_3 - 3a_0 \,.
\end{aligned}$$

Solving, we get $a_0 = 0 = a_1$, $a_2 = 1$, and $a_3 = 1$.

Solution 2 (from first principles):

Multiplying by the denominator we get

$$\begin{aligned}
(1 - 3x^3 + x^4)(\sum_{n\geq 0} a_n x^n) &= x^2 + x^3 \\
\sum_{n\geq 0} a_n x^n - 3\sum_{n\geq 0} a_n x^{n+3} + \sum_{n\geq 0} a_n x^{n+4} &= x^2 + x^3 \\
\sum_{n\geq 0} a_n x^n - 3\sum_{n\geq 3} a_{n-3} x^n + \sum_{n\geq 4} a_{n-4} x^n &= x^2 + x^3.
\end{aligned}$$

Comparing the coefficients of x^0, x^1, x^2, and x^3 on both sides of this equation, we find

$$\begin{aligned}
x^0 &: \quad a_0 = 0 \\
x^1 &: \quad a_1 = 0 \\
x^2 &: \quad a_2 = 1 \\
x^3 &: \quad a_3 - 3a_0 = 1 \,.
\end{aligned}$$

Solving, we get the initial conditions $a_0 = 0 = a_1$, $a_2 = 1$, and $a_3 = 1$.

Comparing the coefficient of x^n on both sides of this equation for every $n \geq 4$, we find the general form of the recurrence

$$a_n - 2a_{n-3} + a_{n-4} = 0.$$

(b) We compute $a_4 = 2a_1 - a_0 = 0$, and $a_5 = 2a_2 - a_1 = 2$.