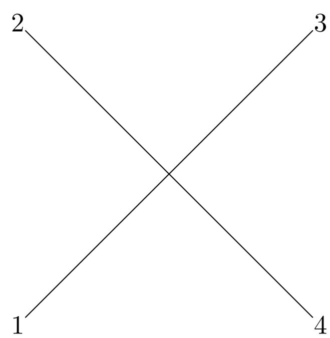
MATH239 Tutorial 6

October 22, 2013

1. For a graph G, we define the complement graph of G, denoted $\bar{G}$, with $V(\bar{G}) = V(G)$ and $\{u, v\} \in E(\bar{G})$ if and only if $\{u, v\} \notin E(G)$.

   (a) Let $G = \{\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}\}$ (i.e. G is the cycle 12341). Draw the complement $\bar{G}$ of G.
   Solution.

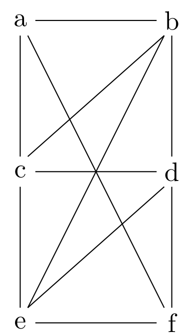
   (b) Find a graph G with 4 vertices such that G is isomorphic to $\bar{G}$.
   Solution. We can choose G to be a path of length 3.

   (c) A subset of vertices S of G is an *independent set* if no two vertices in S are adjacent in G. A subset of vertices T of G is a *clique* if the vertices of T are pairwise adjacent in G. Prove that the size of a largest independent set in G is equal to the size n of a largest clique in $\bar{G}$.
   Solution. Let S be a largest independent set in G, and suppose $|S| = k$. By the definition of the complement, S is a clique of $\bar{G}$. So $n \geq k$.

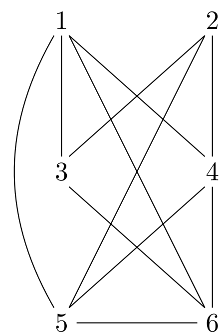
   Suppose $n > k$. Let T be a size n clique of $\bar{G}$. By the definition of the complement, T is an independent set in G. But then $|T| > |S|$, contradicting our choice of S. So $n = k$.

2. Are the following graphs isomorphic?

(a) G



H

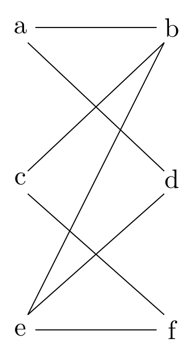


If an isomorphism h from G to H exists, then h must map the degree 3 vertices of G to the degree 3 vertices of H. So we will guess $h(a) = 2$ and $h(f) = 3$. If h is an isomorphism, then h must also map neighbours of a to neighbours of 2, and neighbours of f to neighbours of 3. So we will guess $h(b) = 4, h(c) = 5, h(d) = 1, h(e) = 6$. (If we notice that G is very "symmetrical", we can convince ourselves that we don't have to be very careful about whether we choose $h(e) = 1$ and $h(d) = 6$, or $h(e) = 6$ and $h(d) = 1$. Similar logic applies for assigning $h(a)$ and $h(f)$, as well as $h(c)$ and $h(b)$. So we can be fairly confident that we actually are finding an isomorphism, not just guessing randomly.)

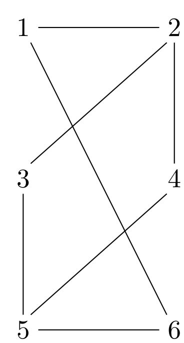
If we check that $\{i, j\}$ is an edge of G if and only if $\{h(i), h(j)\}$ is an edge of H, we can verify that h is an isomorphism:

$$
\begin{array}{ll}
\{h(a), h(b)\} = \{2, 4\} & \{h(a), h(c)\} = \{2, 5\} \\
\{h(a), h(f)\} = \{2, 3\} & \{h(b), h(c)\} = \{4, 5\} \\
\{h(b), h(d)\} = \{4, 1\} & \{h(b), h(e)\} = \{4, 6\} \\
\{h(c), h(d)\} = \{5, 1\} & \{h(c), h(e)\} = \{5, 6\} \\
\{h(d), h(e)\} = \{1, 6\} & \{h(d), h(f)\} = \{1, 3\} \\
\{h(e), h(f)\} = \{6, 3\} &
\end{array}
$$

(b) G



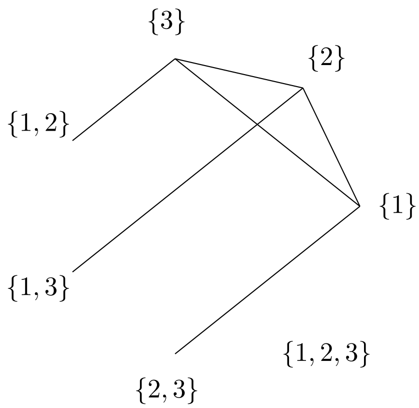
H



The graphs are not isomorphic - the first has two adjacent vertices of degree 3, the second does not.

3. Let G_n be the graph whose vertices are the non-empty subsets of $\{1, 2, ..., n\}$ where two vertices U, W are adjacent if $U \cap W = \emptyset$.

   (a) Draw G_3.



   (b) Let U be a vertex of G_n. Find $\deg(U)$ as a function of $|U|$.
   Solution. Suppose $|U| = k \leq n$. If there exists a vertex W of G_n adjacent to U, we must have $W \subseteq \{1, 2, ..., n\} \setminus U$. Since the only constraint on the size of W is that it be non-empty, there are $\sum_{i=1}^{n-k} \binom{n-k}{i}$ choices for W, and U has degree $\sum_{i=1}^{n-k} \binom{n-k}{i}$.
   Note: Since $\deg(U)$ depends on $|U|$, not all vertices of G have the same degree. So G is not regular.

4. Let G be a graph on $n \geq 10$ vertices, where n is even. Suppose half the vertices of G have degree 2, and half have degree 4. Find $|E(G)|$ in terms of n.
Solution.

$$\begin{aligned} 2|E(G)| &= \sum_{v \in V(G)} \deg(v) \\ &= \frac{n}{2}(2) + \frac{n}{2}(4) \\ &= n + 2n \\ &= 3n. \end{aligned}$$

5. Let G be a graph on n vertices. Prove that some pair of vertices of G must have the same degree.
Solution. Let v be a vertex of G. There are n possibilities for $\deg(v)$: 0, 1, 2, ..., $n-1$. So if no two vertices of G have the same degree, then each of these possibilities is the degree of exactly one vertex of G. In particular, G has a vertex w of degree 0 and a vertex x of degree $n-1$. But then x must be adjacent to every other vertex of G, including w. This is a contradiction. So some two vertices of G must have the same degree.