1. **[9 marks]** Find the following coefficients:

   (a)
   $$[x^n](1+3x^2)^{-5}$$

   (b)
   $$[x^5](1+3x^3)^3(1-x)^{-2}$$

   (c)
   $$[x^n](1-5x)^{-3}(1+x)^m \quad \text{(where } m \text{ is a positive integer)}$$

   (d)
   $$[x^{73}](1+x^2)^{10}(1-x^4)^2$$

2. **[12 marks]**

   (a) Prove that the generating function for compositions of n in which each part is at least 3, and the number of parts is exactly 100, is
   $$\frac{x^{300}}{(1-x)^{100}}.$$

   (b) Prove that the generating function for compositions of n in which each part is at least 3 is
   $$\frac{1-x}{1-x-x^3}.$$

   (c) Prove that the generating function for compositions of n in which each part is at least 3, and the number of parts is at most 100, is
   $$\frac{(1-x)^{101}-x^{303}}{(1-x)^{100}(1-x-x^3)}.$$

3. **[10 marks]** Give decompositions that uniquely create the following sets of 0/1-strings. In each case, give the corresponding generating function.

   (a) All strings with no odd blocks of length greater than 4.

   (b) All strings with no occurrence of the substring 0011.

4. [**6 marks**]

   (a) Find the generating function for the following set of strings: $\{\epsilon, 00\}\big(\{1\}(\{\epsilon\} \cup \{0\}\{00\}^*)\big)^*$. Write your answer as $\frac{p(x)}{q(x)}$ where p and q are polynomials.

   (b) Prove that the following decomposition does *not* uniquely create the set of all strings where every odd block of 0s is immediately followed by an odd block of 1s. (Show *either* that it does not create them all, *or* that it creates some strings more than once.)

$$\{1\}^* \left(\{0\}\{00\}^*\{1\}\{11\}^* \cup \{00\}\{00\}^*\{11\}\{11\}^*\right)^* \{00\}^*$$

5. [**10 marks**]

   (a) Find the general solution of the recurrence equation

$$b_n = -3b_{n-1} + 4b_{n-2} \ (n \geq 2)$$

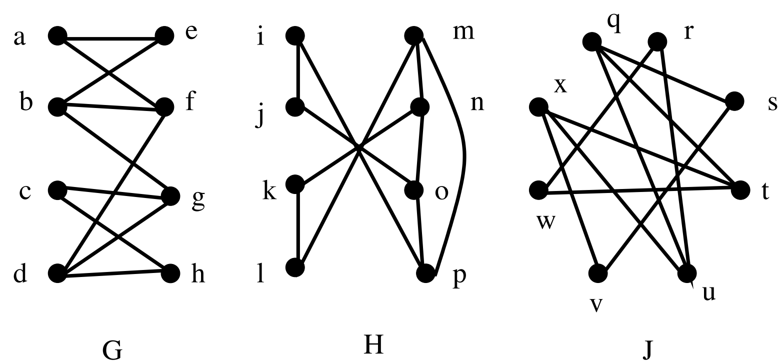
   without initial conditions. That is, find the solution involving some constants which you do not determine but would depend on the initial conditions.

   (b) Let the sequence b_n be defined by $b_0 = 1$, $b_1 = 2$, and

$$b_n + 3b_{n-1} - 4b_{n-2} = 5 \ \text{ for all } n \geq 2.$$

   Solve this recurrence relation to obtain a closed form expression for b_n.

6. [**8 marks**] For each of the three pairs (G, H), (G, J) and (H, J), determine if the two graphs in the pair are isomorphic. Prove your claim in each case.



7. [**8 marks**] Let $n \geq 1$ be an integer. The graph H_n is defined as follows: the vertex set of H_n is the set of all strings of of 0s, 1s and 2s of length n (these are called *ternary* strings), that is

$$V(H_n) = \{t_1 t_2 \cdots t_n : t_i \in \{0, 1, 2\} \text{ for each } i\}.$$

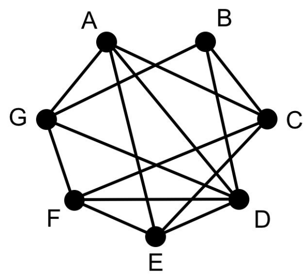
Two vertices of H_n are joined by an edge if they differ in exactly one coordinate.

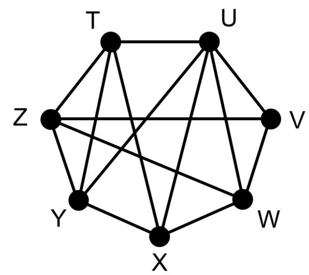
(a) Draw H_1 and H_2.

(b) Find the number of vertices of H_n. Prove your answer is correct.

(c) Find the number of edges of H_n. Prove your answer is correct.

1. Prove that the following two graphs are not isomorphic.





2. Can a graph exist on ten vertices whose vertex degrees are given as follows? In each case, either construct such a graph or explain why it doesn't exist.

   (a) 8, 8, 8, 8, 6, 5, 5, 3, 2, 2.

   (b) 8, 8, 8, 8, 6, 5, 5, 2, 2, 2.

3. A graph G is defined as follows. Its vertices are the binary strings of length n with exactly three 1's, and two vertices are adjacent if they have exactly two 1's in the same position. How many vertices and edges does G have? For which $n \geq 3$ is G bipartite?

4. A graph G is a subgraph of the 237-dimensional cube Q_{237}. Prove that G is bipartite.