THE GRAVITATIONAL WAVE MEMORY EFFECT

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Declaration

I affirm that I have identified all my sources. No part of my dissertation paper uses unacknwoledged materials.

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Contents

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Abstract

In this dissertation we intend to study the background related to the memory effect that leads to "gravitational-wave memory effect" and two types of memory effect:(1) We intend to study a whole outline of what is memory effect.(2) We intend to solve the linear memory for N Gravitationally Unbound Particles where we will study different kinds of spherical harmonics,mass quadrapole leading to linear memory effect.(3)Then we try to examine the memory effect for the individual radiated neutrinos[9] (4) Then we will discuss briefly about the introduction of non linear memory effect.

CHAPTER 1

Introduction

1.1 History of Gravitational Waves:

Einstein first postulated the existence of gravitational waves in 1916 as a consequence of his theory of General Relativity, but no direct detection of such waves has been made yet. The best evidence thus far for their existence is due to the work of 1993 Nobel laureates Joseph Taylor and Russell Hulse. They observed, in 1974, two neutron stars orbiting faster and faster around each other, exactly what would be expected if the binary neutron star was losing energy in the form of emitted gravitational waves. The predicted rate of orbital acceleration caused by gravitational radiation emission according to general relativity was verified observationally, with high precision. Cosmic gravitational waves, upon arriving on earth, are much weaker than the corresponding electromagnetic waves. The reason is that strong gravitational waves are emitted by very massive compact sources undergoing very violent dynamics. These kinds of sources are not very common and so the corresponding gravitational waves come from large astronomical distances. On the other hand, the waves thus produced propagate essentially unscathed through space, without being scattered or absorbed from intervening matter [7]

1.2 Theory of Gravitational Wave:

General relativity corrected Newton's theory and is recognized as one of the most ingenious creations of the human mind. The laws of general relativity,though, in the case of slowly moving bodies and weak gravitational fields reduce to the standard laws of Newtonian theory. Nevertheless, general relativity is conceptually different from Newton's theory as it introduces the notion of spacetime and its geometry. One of the basic differences of the two theories concerns the speed of propagation of any change in a gravitational field. As the apple falls from the tree, we have a rearrangement of the distribution of mass of the earth, the gravitational field changes, and a distant observer with a high-precision instrument will detect this change.

Gravitational waves are 'ripples' in space-time caused by some of the most violent and energetic processes in the Universe. The fundamental geometrical framework of relativistic metric theories of gravity is spacetime, which mathematically can be described as a four-dimensional manifold whose points are called events.Every event is labeled by four coordinates x_μ where μ=(0,1,2,3); the three coordinates x^i i=(1,2,3) give the spatial position of the event, while x^0 is related to the coordinate time t ($x^0 = 0$) where c is the speed of light, which unless otherwise stated will be set equal to 1).The motion of a test particle is described by a curve in spacetime. The distance ds between two neighboring events, one with coordinates x_μ and the other with coordinates $x_\mu + dx_\mu$ can be expressed as a function of the coordinates via a symmetric tensor

$$g_{\mu\nu}(x^\lambda) = g_{\nu\mu}(x^\lambda)$$

(1.2.1)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad (1.2.2)$$

This is a generalization of the standard measure of distance between two points in Euclidian space. For the Minkowski spacetime (the spacetime of special relativity),$g_{\mu\nu} = \eta_{\mu\nu} = diag(-1, 1, 1, 1)$.The symmetric tensor is called the metric tensor or simply the metric of the spacetime. In general relativity the gravitational field is described by the metric tensor alone, but in many other theories one or more supplementary fields may be needed as well. According to general relativity, any distribution of mass bends the spacetime fabric and the Riemann tensor $R_{\kappa\lambda\mu\nu}$. is a measure of the spacetime curvature. The Riemann tensor has 20 independent components. When it vanishes the corresponding spacetime is flat.For a perfect fluid the stress-energy tensor is given by the following expression:

$$T^{\mu\nu}(x^{\lambda}) = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu} \tag{1.2.3}$$

where$p(x^{\lambda})$is the local pressure, $\rho(x^{\lambda})$ is the local energy density and $u^{\mu}(x)$ is the four velocity of the infinitesimal fluid element characterized by the event x Einstein's gravitational field equations connect the curvature tensor and the stress-energy tensor through the fundamental relation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \tag{1.2.4}$$

This means that the gravitational field, which is directly connected to the geometry of spacetime, is related to the distribution of matter and radiation in the universe.By solving the field equations, both the gravitational field (the $g_{\mu}\nu$ and the motion of matter is determined. $R_{\mu\nu}$ is the so- called Ricci tensor and comes from a contraction of the Riemann tensor., R is the scalar curvature.$G_{\mu\nu}$ is the so-called Einstein tensor, $k = \frac{8\pi G}{c^4}$ is the coupling constant of the theory and is the gravitational constant, which, unless otherwise stated will be considered equal to 1.As a consequence, in the empty space far from any matter distribution, the Ricci tensor will vanish while the Riemann tensor can be nonzero; this means that the effects of a propagating gravitational wave in an empty spacetime will be described via the Riemann tensor.

1.2.1 Linearized Theory:

Now let's assume that an observer is far away from a given static matter distribution, and the spacetime in which he or she lives is described by a metric $g_{\mu\nu}$.Any change in the matter distribution, i.e., in$T_{\mu\nu}$, will induce a change in the gravitational field, which will be recorded as a change in metric. The new metric will be:

$$\overline{g_{\mu\nu}} = g_{\mu\nu} + h_{\mu\nu} \tag{1.2.5}$$

where $h_{\mu\nu}$ is a tensor describing the variations induced in the spacetime metric. As we will describe analytically later, this new tensor describes the propagation of ripples in spacetime curvature, i.e., the gravitational waves.In order to calculate the new tensor we have to solve Einsteins equations for the varying matter distribution.However, there is a convenient, yet powerful, way to proceed, namely to assume that $h_{\mu\nu}$ is small ($|h_{\mu\nu}| \ll 1$), so that we need only in our calculations. In making this approximation we are effectively assuming that the disturbances produced in spacetime are not huge keep terms linear in. This linearization approach has proved extremely useful for calculations, and for weak fields at least, gives accurate results for the generation of the waves and for their propagation. The first attempt to prove that in general relativity gravitational perturbations propagate as waves with the speed of light is due to Einstein himself. Shortly after the formulation of his theory - the year after - he proved that by assuming linearized perturbations around a flat metric.e $g_{\mu\nu} = \eta_{\mu\nu}$ then the tensor:

$$\overline{h_{\mu\nu}} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\alpha}{}_{\alpha} \tag{1.2.6}$$

is governed by a wave equation, which admits plane wave solutions similar to the ones of electromagnetism; here $h_{\mu\nu}$ is the metric perturbation and $\overline{h_{\mu\nu}}$ is the gravitational field (or the trace reverse of $h_{\mu\nu}$). Then the linear field equations in vacuum have the form

Albert Einstein predicted the existence of gravitational waves in 1916 in his general theory of relativity. We know that the linearised field equations of general relativity could be written in the form of a wave equation:

$\Box^2 h_{\mu\nu} = -2\kappa T_{\mu\nu}(1.2.7)$

provided that the $h_{\mu\nu}$satisfy the Lorenz gauge condition

$$\partial\overline{h_{\mu\nu}} = 0 \tag{1.2.8}$$

This suggests the existence of gravitational waves in an analogous manner to that in which Maxwell's equations predict electromagnetic waves.

1.3 Polarisation states of Gravitational Waves:

There are two types of states of linear polarisation of GW they are: CROSS(X) AND PLUS(+) POLARISATION: Geodsic deviation in the transverse direction provides a means of studying ad characterizing the polarizations of plane waves. Considering a plane monochromatic wave propogating in the z directon .In the TT gauge the constraints $h_{0\mu}^{TT} = 0$,$h_{ij,}^{TT} = ikh_{ij}^{TT} = 0$, $h_{kk}^{TT} = 0$ reveal that the only non vanishing components of $h_{\mu\nu}^{TT}$ are:

$$h_{xx}^{TT} = -h_{yy}^{TT} = \mathbf{R}(A_{+}e^{-iw(t-z)})(1.3.1)$$

$$h_{xy}^{TT} = h_{yx}^{TT} = \mathbf{R}(A_{x}e^{-iw(t-z)})(1.3.2)$$

The amplitudes A_+ and A_x represent two indepedent modes of polarization. If you rotate gravitational wave (GW) of one polarization by 90 degrees along its axis of propagation, you get the GW of same polarization, only shifted in phase. Therefore, GW of another polarization, to be linearly independent, must be rotated between 0 and 90 degrees and 45 degrees are exactly between those two.

For comparison, if you rotate EM wave of one polarization by 180 degrees, you get EM wave of same polarization, only shifted in phase.

That's why polarizations of EM waves are at 90 degrees to each other, while polarizations of GWs are at 45 degrees to each other.

1.4 Why it is called Memory Effect?

The memory effect has been known since the 1970's in its linear form. The linear memory generally arises in systems with unbound components: a binary on a hyperbolic orbit (two-body scattering) , matter or neutrinos ejected from a supernova , or gamma-ray burst jets . In the 1990's a nonlinear form of memory was discovered independently by Blanchet Damour and Christodoulou . The nonlinear memory arises from the contribution of the emitted GWs to the changing quadrupole and higher mass moments. Thorne has discussed, the nonlinear memory can be described in terms of a linear memory in which the unbound masses are the individual radiated gravitons. This implies that nearly all GW sources are sources with memory (even if the component objects remain bound).

1.5 What is Memory Effect?

General Relativity predicts that gravitational waves will have an oscillatory component as well as a memory component. The memory and oscillatory components are polarised in the plus and cross direction. These polarizations are related by a 45 degree rotation comparison to the 90 degree rotation for electromagnetic radiation. The polarizations have an oscillatory and non-oscillatory component. For example, in the case of GW'S produced during inspiral of binary stars, there is a non-oscillatory component to the "+" polarization which makes the amplitude of the gravitational wave end with a non-zero value. The "non-zero" amplitude represents the "gravitational wave memory", a weak stretching that permanently alters the space time fabric.All gravitational-wave sources possess some form of gravitational-wave memory. The GWsignal from a 'source with memory' has the property that the late-time and early-time values of at least one of the GW polarizations differ from zero:

$$\Delta h^{\mathrm{mem}}_{+,\times} = \lim_{t\to+\infty} h_{+,\times}(t) - \lim_{t\to-\infty} h_{+,\times}(t)$$

(1.5.1)

where h_+ and $h_\times$ represent the plus and cross polarization, and t is the time at the observer.

Gravitational-wave memory refers to the permanent displacement of an "ideal" GW detector after the GW has passed . An ideal detector is one which is only sensitive to gravitational forces—e.g., a ring of freely-falling test-masses—and is isolated from local tidal interactions. After the passage of a GW without memory, the detector returns to its initial state of internal displacement (its state long before the passage of the wave). After the passage of a GW with memory, the initial and final displacement states differ. There are two types of GW memory: linear and nonlinear In this dissertation, we will try to study mostly the linear memory effect, and the formalism associated with it, and try to examine the memory effect for hyperbolic binaries, individual radiated neutrinos, etc. Alongside this, we will also try to briefly familiarised with the non-linear memory effect as an extension of linear memory effect through exploring the ideas of gravitons [12].

1.6 Detection of Memory Effect:

In general, memory can arise both in the linearized Einstein field equations and in their full non-linear form. Early research focused on the production of linear memory from unbound systems such as supernovae or triple black hole interactions . Non-linear contributions to memory were originally thought to be negligibly small . However, further investigations showed that bound systems such as binary black holes produce significant non-linear memory . A detector like LISA is capable of detecting the memory because of its good sensitivity in the low-frequency band where typical memory sources are stronger. The interferometers like LIGO which are bars and ground basedare ot capable of 'storing' a memory signal as they are not truly free: internal elastic forces push a bar back to its equilibrium shape, and magnets on the LIGO test masses (as well as its pendulum wires) push them back to their equilibrium positions. Due to it's proof-masses are really freely-floating, a detector like LISA is able to maintain a permanent displacement. However, this latetime displacement caused by the GW memory cannot be directly observable without information on the prior state of the detector: the spacetime metric near a detector long after a GW with memory has passed is equivalent to flat spacetime in nonstandard coordinates . It it is the build-up of the memory (the difference in the metric between late and early times) that is observable [3]

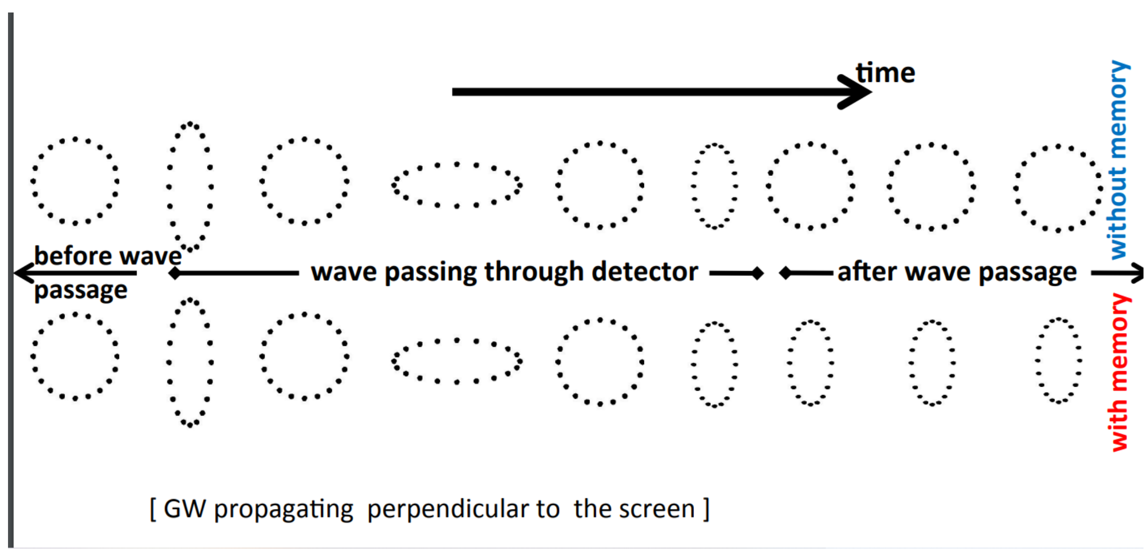

Figure 1.1: GW propagating perpendicular to the screen

CHAPTER 2

FORMALISM:

2.1 POST NEWTONIAN WAVE GENERATION FORMALSM:

The asymptotic waveform h_{ij}^{TT} can be decomposed into two sets of symmetric trace-free (STF) radiative multipole moments as

$$h_{ij}^{TT} = \frac{4G}{c^2 R}\Pi_{ijmn}\sum_{\ell=2}^{\infty}\left\{\frac{1}{c^\ell \ell!}\mathcal{U}_{mnL-2}\left(T_R\right)N_{L-2}\right.$$
$$\left. + \frac{2\ell}{c^{\ell+1}(\ell+1)!}\epsilon_{pq(m}\mathcal{V}_{n)pL-2}\left(T_R\right)N_{qL-2}\right\}$$

(2.1.1)

Here $\mathcal{U}_L\left(T_R\right)$ are the mass-type moments and $\nu_L\left(T_R\right)$ are the current-type moments. Here we are trying to derive Post-Newtonian wave generation formalism. The transverse-traceless (TT) projection operator Π_{ijmn} is given by

$$\Pi_{ijmn} = P_{im}P_{jn} - \tfrac{1}{2}P_{ij}P_{mn}$$

(2.1.2)

where $P_{ij} = \delta_{ij} - N_i N_j$ Given an orthonormal triad (N,P,Q) the polarization waveforms can be given by

$$h_+ = \frac{1}{2}\left(P_mP_n - Q_mQ_n\right)h^{TT}_{mn}$$
$$h_\times = \frac{1}{2}\left(P_mQ_n + P_nQ_m\right)h^{TT}_{mn} \quad (2.1.3)$$

A natural (but by no means unique) choice for the triad is $\vec{P} = \vec{e}_\Theta$ and$\vec{Q} = \vec{e}_\Phi$

It is then straightforward to show that:

h$_+ - ih_\times = m^*_m m^*_n h^{TT}_{mn}$(2.1.4)

where denotes complex conjugation. It will now be shown how

$$h_+ - ih_\times$$

can be decomposed into modes using spin-weighted spherical harmonics of weight -2

$$h_+ - ih_\times = \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} h^{\ell m}{}_{-2}Y^{\ell m}(\Theta, \Phi)$$

An alternative expression for the waveform is given by:

$\mathrm{h}_{ij}^{TT} = \frac{G}{c^2R}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\frac{1}{c^\ell}U^{\ell m}\left(T_R\right)T_{ij}^{E2,\ell m}\right.$
$\left.+\frac{1}{c^{\ell+1}}V^{\ell m}\left(T_R\right)T_{ij}^{B2,\ell m}\right\}$ (2.1.5)

where $T_{ij}^{E2,\ell m}$ and $T_{ij}^{B2,\ell m}$ are pure-spin tensor harmonics, and where the mass multipole moments $U^{\ell m}\left(T_R\right)$ and current multipole moments $V^{\ell m}\left(T_R\right)$ are related to their STF counterparts by

$$U^{\ell m} = \frac{16\pi}{(2\ell+1)!!}\sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}}\mathcal{U}_L\mathcal{Y}_L^{\ell m*}$$
$$V^{\ell m} = \frac{-32\pi\ell}{(2\ell+1)!!}\sqrt{\frac{(\ell+2)}{2\ell(\ell+1)(\ell-1)}}\mathcal{V}_L\mathcal{Y}_L^{\ell m*}$$

where $\mathcal{Y}_L^{\ell m}$ are the STF spherical harmonics which are related to the scalar spherical harmonics by

$$Y^{\ell m}(\Theta,\Phi) = \mathcal{Y}_L^{\ell m}N_L$$

The pure-spin tensor harmonics are related to the spinweighted spherical harmonics by

$$T_{ij}^{E2,\ell m} = \frac{1}{\sqrt{2}}\left({}_{-2}Y^{\ell m}m_i m_j + {}_{2}Y^{\ell m}m_i^* m_j^*\right) \quad (2.1.6)$$

$$\mathrm{T}_{ij}^{B2,\ell m} = \frac{-i}{\sqrt{2}}\left({}_{-2}Y^{\ell m}m_i m_j - {}_{2}Y^{\ell m}m_i^* m_j^*\right) \quad (2.1.6)$$

Combining Eqs. (2.1.4), (2.1.5), and (2.1.6) yields:

$$\mathrm{h}_+ - ih_\times = \frac{G}{c^2R}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\frac{1}{c^\ell\sqrt{2}}U^{\ell m}\left(T_R\right)\right.$$
$$\left. -\frac{i}{c^{\ell+1}\sqrt{2}}V^{\ell m}\left(T_R\right)\right\}_{-2}Y^{\ell m}(\Theta,\Phi) \tag{2.1.7}$$

We know the spin-weighted spherical harmonic components of the waveform are given by $h^{\ell m} = \frac{G}{\sqrt{2}Rc^{\ell+2}}\left(U^{\ell m}\left(T_R\right) - \frac{i}{c}V^{\ell m}\left(T_R\right)\right)$

The post-Newtonian approximation is a slow-motion, weak-field approximation to general relativity. In order to produce a post Newtonian waveform, we have to solve both the post-Newtonian equations of motion describing the binary, and the post-Newtonian equations describing the generation of gravitational wave. But for the most important (or most studied) GW source — the quasi-circular compact binaries — the nonlinear memory has quite a large contribution to the time-domain waveform amplitude: in a post-Newtonian (PN) expansion of the waveform polarizations, the memory effect enters at leading-(Newtonian)-order. That the memory enters at such low PN order is related to the fact that it is a hereditary effect — the memory amplitude at any retarded time depends on the entire past motion of the source (and not just on the source's instantaneous retarded-time configuration). The nonlinear memory is a unique nonlinear effect because it's non-oscillatory nature makes it distinctly visible in the waveform. For these reasons We can say that the memory should be studied further and its prospects for detection reassessed.

2.1.1 Spherical Harmonics:

There are three kinds of Spherical Harmonics those are:

- Scalar Spherical Harmonics
- Vector Spherical Harmonics
- Tensor Spherical Harmonics

Here for calculation of memory effect we only require Scalar Spherical Harmonics and Spin-weighted spherical harmonics.

Scalar Spherical Harmonics:

The usual representation of scalar spherical harmonics is in terms of complex functions of θ and ϕ Explicit form for the spherical harmonics Y_{2m} are: Following the usual convention as shown in [5]:

$$Y_{2-2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}(sin^2\theta)e^{-2i\phi} \tag{2.1.8}$$

$$Y_{2-2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}(sin\theta cos\theta)e^{-2i\phi} \tag{2.1.9}$$

$$Y_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}}(cos^2\theta - 1)$$

(2.1.10)

The usual representation of scalar spherical harmonics is in terms of complex functions of θ and ϕ [11].

$$Y^{lm} = C^{lm} e^{im\phi} P^{lm}(cos\theta) m \geq 0 \tag{2.1.11}$$

$$= C^{lm}(e^{i\phi} sin\theta)^2 \sum_{j=0}^{[(l-m)/2]} a^{lmj}(cos\theta)^{l-m-2j} m \geq 0 \tag{2.1.12}$$

$$= (-1)^m Y^{l|m|*} m < 0 \tag{2.1.13}$$

$$C^{lm} \equiv (-1)^m (\frac{2l = 1}{4\pi} \frac{(l-m)!}{(l+m)!})^{1/2} \tag{2.1.14}$$

$$a^{lmj} \equiv \frac{(-1)^j}{2^l j|(l-j)|} \frac{(2l-2j)!}{(l-m-2j)!} \tag{2.1.15}$$

These scalar harmonics are orthonormal

$$\int Y^{lm} Y^{l'm'*} d\Omega = \delta_{ll'} \delta_{mm'}$$

and have parity $\pi = (-1)^l$ ('electric-type' parity). Under complex conjugation, they transform as

$$Y^{lm*} = (-1)^m Y^{l-m} \tag{2.1.16}$$

The set of all symmetric trace-free tensors of rank l ("STF-l tensors") generates an irreducible representation of the rotation group, of weight l.Hence, there exists a one-to-one mapping between them and the spherical harmonics of order l. To exhibit that mapping one expresses the Cartesian components of the unit radial vector n in the form: $n_x + in_y = e^{i\phi} sin\theta$, $n_z = cos\theta$

Inserting these expressions in (2.1.16) we obtain:

$$Y^{lm}(\theta, \phi) = \mathcal{Y}^{lm}_{K_l} N_{k_l} \tag{2.1.17}$$

where $N_{k_l}$is the tensor product of l radial vectors by

$$N_{k_l} = n_{k_1} n_{k_2} ... n_{k_l}$$

. where $\mathbf{n} = \mathbf{e_r} = \frac{\partial}{\partial r}$, in spherical polar co-ordinates. Here is the following (location-independent) STF-l tensor $Y^{lm}_{K_l}$

The tensors $Y^{lm}_{K_l}$ with -l<=m<=l serve two roles: first, they generate the spherical harmonics of order l second, they form a basis for the (2l +1)-dimensional vector space of STF-l tensors,i.e, any STF-l tensor F can be expanded as

$$\mathcal{F}_{kl} = \sum_{m=-1}^{l]} F^{lm} \mathcal{Y}^{lm}_{K_l} \tag{2.1.18}$$

One can then expand any scalar function $f(\theta, \phi)$ in spherical harmonics with complex number coefficients as

$$f(\theta, \phi) = \sum_{l=0}^{\infty]} \sum_{m=-1}^{l]} F^{lm} Y lm(\theta, \phi) \tag{2.1.19}$$

Alternatively,

$$f(\theta, \phi) = \sum_{l=0}^{\infty]} \mathcal{F}_{K_l}^{lm} N_{kl} \tag{2.1.20}$$

2.1.2 Explicit form for Spin-Weighted Spherical Harmonics:

Several different conventions for vector spherical harmonics exist in literature. We will be using the spin-weighted spherical harmonics Y_s^{lm} devised by Newman-Penrose(1966). These will be used to calculate the multi-pole expansion for hyperbolic binaries (from [11]).

The explicit form for the s = 2 spin-weighted l = 2 spherical harmonics are given by :.They can be computed using equations (4) and (5) of Kidder (2008).Inserting an extra factor of $(-1)^m$ into their formula, to ensure the usual normalisation convention for spin-weight zero spherical harmonics is used .These are forms are given in the paper [5]

${}_{-2}Y^{2-2} = \frac{1}{8}\frac{\sqrt{5}}{\sqrt{\pi}}(1 - cos\theta)^2)e^{-2i\phi}$

(2.1.21)

$$_{-2}Y^{2-1} = \frac{1}{4}\frac{\sqrt{5}}{\sqrt{\pi}} sin\theta(1 - cos\theta)e^{-i\phi}$$

(2.1.22)

$$_{-2}Y^{20} = \frac{1}{8}\sqrt{\frac{30}{\pi}}(sin^2\theta)$$

(2.1.23)

$$_{-2}Y^{21} = \frac{1}{4}\sqrt{\frac{5}{\pi}} sin\theta(1 + \cos\theta)e^{i\phi} \qquad (2.1.24)$$

$$_{-2}Y^{22} = \frac{1}{8}\sqrt{\frac{5}{\pi}}(1 + \cos\theta)^2 e^{2i\phi} \qquad (2.1.25)$$

2.1.3 Tensor Spherical Harmonics:

We would require to use tensor spherical harmonics of the pure-spin set, and those who are Transverse and traceless. Hence, we would require to use: $T_{ij}^{E2,\ell m}$ and $T_{ij}^{B2,\ell m}$ which are connected to spin-weighted harmonics as:

$$T_{ij}^{E2,\ell m} = \frac{1}{\sqrt{2}} \left({}_{-2}Y^{\ell m} m_i m_j + {}_{2}Y^{\ell m} m_i^* m_j^* \right) \tag{2.1.26}$$

and

$$T_{ij}^{B2,\ell m} = \frac{-i}{\sqrt{2}} \left({}_{-2}Y^{\ell m} m_i m_j - {}_{2}Y^{\ell m} m_i^* m_j^* \right) \tag{2.1.27}$$

where $\mathbf{m} \equiv 2^{-1/2}(\mathbf{e}_\theta + i\mathbf{e}_\phi)$ and $\mathbf{m}^* \equiv 2^{-1/2}(\mathbf{e}_\theta - i\mathbf{e}_\phi)$; $\mathbf{e}_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}$ and $\mathbf{e}_\phi = \frac{1}{rsin\theta}\frac{\partial}{\partial\phi}$

CHAPTER 3

The Gravitational-Wave Memory Effect

In experimental terms, Gravitational-Wave (GW) memory effect refers to the permanent change of displacement of an "ideal" GW detector after a GW has passed. GW's are often described using a dimensionless, symmetric, second rank, purely special, transverse, and traceless tensor field ${h_{jk}}^{TT}(t - z)$ defined in the rest frame of the detector propagating along z-direction in this case with the speed of light. When a burst of gravitational waves passes through an idealized detector, having two free masses having a vectorial separation l_k, for a finite duration it can cause a permanent change in the test-mass separation, i.e., it can have ${h_{jk}}^{TT} = 0$ before the burst arrives, and ${h_{jk}}^{TT} = \Delta {h_{jk}}^{TT} 0$ after the burst has passed. This permanent change in

${h_{jk}}^{TT}$ is called the “memory” of the burst.

Theoretically, GW memory effect is a result of a non-oscillatory component to one of the two polarisation states ‘$+$’ and ‘$\times$’ of a GW. Typically, GWs are thought to oscillate with an amplitude about a zero value that rises slowly building up to a maximum, then decaying to zero later. In actuality, the GW signal displays a “memory” that causes the signal amplitude to decay to a nonzero value. The GW signal from a ‘source with memory’ has the property that the late-time and early-time values of at least one of the GW polarizations differ from zero, corresponding to an observed difference in the metric perturbation:

$$\delta h_{+,\times}^{mem} = \lim_{t\to+\infty} h_{+,\times}(t) - \lim_{t\to-\infty} h_{+,\times}(t)$$

t being the time for observer, $h_{+,\times}$ are the usual plus and cross polarization [3].

GW memory effect is either linear or non-linear memory.

3.1 BACKGROUND 0F LINEAR MEMORY:

Linear memory was discovered in the 1970s, arises from near-zero-frequency changes in the time derivatives of the source’s multipole moments. Multipole moments are a combination of the mass moment, the extent to which an object resists rotational acceleration about a particular axis, and the mass-current moment which corresponds to the star's spin angular momentum (the star's moment of inertia about its magnetic poles multiplied by its spin angular frequency Ω) .Linear memory also appears in systems that experience kicks such as a rogue black hole or systems that eject particles such as neutrinos from supernovae.Since linear and nonlinear memory depend on the form of General Theory field equations, a set of ten coupled non-linear differential equations that describe gravity as a result of spacetime being curved by mass and energy, it is then possible that different forms of memory could be uncovered if general relativity were to be modified . Since memory is difficult for LIGO to detect, it has mostly been disregarded by scientists studying gravitational waves. However, the memory scales linearly with the black hole's mass which means that there will likely be a detectable contribution to the calculated waveform amplitude of the resulting gravitational waves . This memory effect is computed to be non-negligible as the order it enters the waveform at approximately the same order as the quadrupole. Due to this we can conclude that the memory effect should not be impossible to detect with the proper equipment and analysis techniques. Now that LIGO has published seven CBC (compact binary coalescence) events, there is more data to explore and a greater potential to detect memory.[13]

3.2 Examples and Applications of Linear Memory

3.2.1 Hyperbolic Binary:

As an example of linear memory we need to study the waveform from a hyperbolic binary The leading order multipolar contribution to the polarization is as shown in [3]:

$$h_+ - ih_x = \sum_{m=-2}^{m=2} \frac{I_{2m}^{(2)}}{R\sqrt{2}} {}_{-2}Y^{\ell m}(\Theta, \Phi)$$

(3.2.1)

In the xy plane for a Keplerian binary orbit ,with relative orbital separation r(t) ,total mass M = m1 + m2, reduced mass ratio $\eta = \frac{m1m2}{M^2}$ and orbital phase angle $\phi(t)$

The mass quadrupole is

$$I_{2m} = \frac{16\pi}{5\sqrt{3}} \eta M r(t)^2 Y_{2m}^*(\pi/2, \phi(t)) \tag{3.2.2}$$

For Keplerian orbits with semi-latus rectum p, eccentricity e0, and true anomaly $v = \phi - w_p$ ($w_p = 0$ sets the periastron direction on the x-axis), the orbital motion is described by

$$r = \frac{p}{1 + e_0 cos v} \tag{3.2.3}$$

$$\dot{v} = \dot{\phi}(t) = \frac{\sqrt{pM}}{r^2} \tag{3.2.4}$$

The waveforms are written as;

$$\boxed{I_{20}^{(2)} = -8\sqrt{\frac{\pi}{15}}\frac{\eta M^2}{p} e_0 \left(e_0 + \cos v\right)} \tag{3.2.5}$$

The detailed calculation of $I_{20}^{(2)}$ can be found in appendix I Now another waveform obtained from ;

$$\boxed{I_{2\pm2}{}^{(2)} = (-4\sqrt{\frac{2\pi}{5}}\eta\frac{M^2}{p}e^{\pm 2i\phi(t)})(((-e_0{}^2 + 1 + (1 + e_0 cos v)(1 + 2e_0 e^{\mp iv})))} \tag{3.2.6}$$

The detailed calculation of $I_{2-2}^{(2)}$ can be found in appendix II For0 <e$_0$ $< 1 these wave forms are clearly oscillatory. But for a hyperbolic orbit$(e$_0$ >1, with $(w_p = 0)$the phase angle approaches$\phi_- = v_- = arccos(e_0^{-1})$at early times $(t \to -\infty)$, while at late times $(t \to +\infty)$ it approaches $\phi_+ = v_+ = arccos(e_0^{-1})$.

This difference in the late and early time values of the orbital phase angle gives us a corresponding difference in the derivatives of the mass multipoles, So we can write that;

$$\triangle {I_{20}}^{(2)} = 0 \tag{3.2.7}$$

$$\triangle {I_{2\pm2}}^{(2)} = \pm i16\sqrt{\frac{2\pi}{5}}\frac{\eta M^2}{p}\frac{({e_0}^2-1)^{3/2}}{{e_0}^2} \tag{3.2.8}$$

Hence we can say that this results in the GW polarization amplitudes shown in 3.1

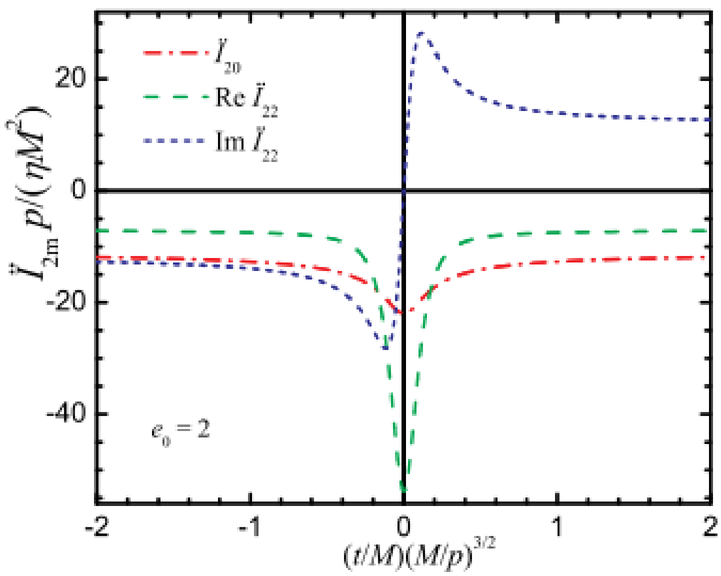

Figure 3.1: Example of gravitational-wave signals with memory

Here in this figure it is shown that the waveform modes $\ddot{I_{2m}}$ for a hyperbolic orbit with eccentricity $e_0 = 2$ as a function of time as seen in (3.2.2).Here we can see linear memory is present in the imaginary part of $\ddot{I_{22}}$

For any parabolic orbit ($e_0 = 1$) there is no memory because the orbital phase angle returns to its early-time value. As we have seen in the above example that due to disturbance in the mass or current source moment derivatives,linear memory arises that is $\triangle I_{lm}^{(l)}$ or $\triangle J_{lm}^{(l)}$ for l$\geq$ 2.

3.2.2 'N' Gravitationally Unbound Particles:

The linear memory for an unbound system is derived by solving the linearized, harmonic gauge Einstein field equations (EFE) for the space-space piece of the metric perturbation hjk:

$$\Box\overline{h_{jk}} = -16\pi T_{jk} \tag{3.2.9}$$

Here T_{jk} is the stress-energy tensor of N gravitationally unbound particles with masses M_A and constant velocities v_A,$\overline{h_{jk}}$ is the trace-reversed metric perturbation, and $\Box$ is the flat-space wave operator. We have solved this equation (via the Lienard-Wiechert solution) Now we have projected to transverse -traceless(TT) gauge and we got the below equation:

$$\boxed{\Delta h_{jk}^{TT} = \triangle\sum_{A=1}^{N}\frac{4M_A}{r\sqrt{1-v_A^2}}[\frac{v_A^j v_A^k}{1-\mathbf{v_A}.\mathbf{N}}]^{TT}} \tag{3.2.10}$$

where **N** points from the source to the observer.

Detailed calculation of derivation of (3.2.10) is shown in APPENDIX-III. In this formula, the masses and velocities could refer to (i)the masses and velocities of the pieces of a disrupted binary, (ii) a gamma-ray-burst jet, or(iii) the individual radiated neutrinos or pieces of ejected material in a supernova explosion. We now try to solve ((3.2.10)) for memory of individual neutrinos

3.2.3 Memory-triggered supernova neutrino detection:

Here we try to formulate the memory of the individual radiated neutrinos. The memory is a non-oscillatory, permanent distortion of the local space time due to the anisotropic emission of matter or energy by a distant source. The memory due to neutrino emission by a supernova at distance r has characteristic strain$hc10231021(10kpc/r)$ and frequencies in the Deci-Hz band, f 0.1 3 Hz . The memory develops 0.1 s from the start of the neutrino emission, thus being an ideal time-trigger. Next generation powerful Deci-Hz GW detectors, like the Deci-hertz Interferometer Gravitational wave Observatory (DECIGO) and the Big Bang Observer (BBO) will provide robust triggers for supernovae at 10 Mpc and beyond . These would result in a nearly pure sample of 10 100 supernova neutrino events from the local universe within a few decades. The starting point is Einstein's field equation,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{3.2.11}$$

Here it suffices to work in the weak-field approximation, where the metric is nearly flat, with small perturbations $h_{\mu\nu}$.In this approximation, the field equations (3.2.11) are still invariant under coordinate transformations that preserve the weak-field condition. where the effective stress-energy tensor (in presence of matter) is $S_{\mu\nu} = T_{\mu\nu} - \frac{\eta_{\mu\nu}T^{\lambda}_{\lambda}}{2}$. One can solve the wave equation by using the retarded Green's function corresponding to the D'Alembert operator in four-space time dimensions,

$$h_{\mu\nu} = 4G\int \frac{S_{\mu\nu}(\vec{x'}, t - |\vec{x} - \vec{x'}|)}{|\vec{x} - \vec{x'}|} d^3\vec{x'} \tag{3.2.12}$$

Now The gauge choice leading us to this solution does not fix completely all the gauge freedom and an additional constraint should be imposed to leave only the physical degrees of freedom. This can be done by projecting the source tensor $S_{\mu\nu}$ into its transverse-traceless (TT) components. We will now use the Source term as:

$$S^{ij}(t, x) = \frac{(n^i n^j)_{TT}}{r^2} \int_{-\infty}^{\infty} \sigma(t') f(\Omega', t') \delta(t - t' - r) dt' \tag{3.2.13}$$

This source term represents the effect of emitted radiation from the source origin at x $= 0$ $\sigma(t')$ denotes the rate of energy loss, is the angular distribution of emission and $f(\Omega', t')$ Now we are substituting the ansatz into the wave form (3.2.12), and integrating, we obtain the following expression for the wave form:

$$h^{ij}_{TT}(t, x) = 4G \int_{-\infty}^{t-r} \int_{4\pi} \frac{(n^i n^j)_{TT} f(\Omega', t') \sigma(t')}{t - t' - r cos\theta} d\Omega' dt' (3.2.14)$$

where where θ is the angle between the observer position and the radiation source point.we assume that the observer is situated at a distance $r = |x| \rightarrow \infty$ from the source. The radiation that reaches the observer at a time t was actually emitted at time, $t' = t - \frac{r}{c}$, physically representing a case where the neutrino pulse itself causes a gravitational wave signal. We can now rewrite (3.2.14)in this approximation as

$$h_{TT}^{ij}(t,x) = \frac{4G}{rc^4}\int_{-\infty}^{t-\frac{r}{c}}\int_{4\pi}\frac{n^in^j{}_{TT}dL_\nu(\Omega',t')}{1-cos\theta d\Omega'}d\Omega' \tag{3.2.15}$$

where $f(\Omega',t')\sigma(t') = \frac{dL_\nu(\Omega',t')}{d\Omega'}$ which is the direction dependent neutrino luminosity. Now,we can write

$(\mathrm{n}^in^j)_{TT} = \frac{(1-cos^2\theta)(2cos^2\phi-1)}{2} = \frac{(1-cos^2\theta)cos2\phi}{2}$

Now substituiting in eqn (3.2.15) we get;

$$h_{TT}^{xx}(t,x) = \frac{2G}{rc^4}\int_{-\infty}^{t-\frac{r}{c}}\int_{4\pi}\frac{(1+cos\theta)cos2\phi dL_\nu(\Omega',t')}{d\Omega'}d\Omega' \tag{3.2.16}$$

as stated in [8]

The '×' polarization $h_{TT}^{xy} = h_{TT}^{+}$ can be found by simply replacing $cos2\phi$ by $sin2\phi$. Where $\alpha(t')$ is the anisotropy parameter. Now combining all these equaions this enables us to write (3.2.16) in the following convenient form,

$$h_{TT}^{xx} = h(r,t) = \frac{2G}{rc^4}\int_{-\infty}^{t-r/c}dt'L_\nu(t')\alpha(t') \tag{3.2.17}$$

where c is the speed of light, t is the time post bounce and G is the universal gravitational constant. L_ν is the all-flavors neutrino luminosity α is the time-varying anisotropy parameter as mentioned in [9]

CHAPTER 4

INTRODUCTION TO NON LINEAR MEMORY EFFECT:

In the 1990's a nonlinear form of memory was discovered independently by Blanchet Damour and Christodoulou .The nonlinear memory appears from the contribution of the emitted GW's to the changing quadrupole and higher mass moments. As discussed in [12] the non linear memory or Christodoulou's effect comes from a linear memory in which the unbound masses are the individual radiated gravitons. This implies that nearly all GW sources are sources with memory. The Christodoulou effect, in fact, is the contribution to (3.2.10) from the gravitationalwave burst's gravitons.

Each graviton can be regarded as an individual system and thus must be included in the final $\sum_{A}$", but, of course, not in the initial $\sum_{A}$. In the final sum, the quantity $\frac{M_A}{\sqrt{1-v_A{}^2}}$, for graviton A should be interpreted as the graviton's energy E_A, as measured in the detector's rest frame, the graviton speed v_A,must be set to 1, and correspondingly the gravitons' contribution to the memory (1) can be written as

$$\triangle h_{jk}^{TT} = \frac{4}{r} \int \frac{dE}{d\Omega'} [\frac{\xi^{j'} \xi^{k'}}{1 - cos\theta'}]^{TT} d\Omega'$$

(4.0.1)

Here $d\Omega'$ is the integral over solid angle $\xi j'$ is a unit vector pointing from the source toward $d\Omega'$ θ' is the angle between $\xi^{j'}$ and the direction of the detector .

CHAPTER 5

CONCLUSION:

- We have studied the whole background related to gravitational-wave memory effect.
- We have studied brief introduction to Post Newtonian memory corrections.
- We have studied post-Newtonian memory corrections to the GW polarizations.
- We have studied the linear memory considering the waveform from a hyperbolic binary . We have derived the linear memory of N Gravitationally Unbound particles.

- We tried to examine the memory effect of individual radiated neutrinos
- We have given a brief idea of non-linear memory effect

CHAPTER 6

APPENDIX

6.1 APPENDIX-I:

6.1.1 CALCULATION OF $\mathbf{I}_{20}^{(2)}$:

We know, the mass quadrupole moment I_{2m} is given by:

$$I_{2m} = \frac{16\pi}{5\sqrt{3}}\eta M r(t)^2 Y^*_{2m}(\pi/2, \phi(t)) \tag{6.1.1}$$

Here, total mass is $M = m_1 + m_2$, reduced mass ratio is $\eta = \frac{m_1 m_2}{M^2}$, and orbital phase angle is $\phi(t)$. Using the relation,

$$Y_{2m}^*(\pi/2, \phi(t)) = (-1)^m Y_{2-m}(\pi/2, \phi(t))$$

and putting $m = 0$, we get:

$$I_{20} = \frac{16\pi}{5\sqrt{3}} \eta M r(t)^2 Y_{20}(\pi/2, \phi(t)) \tag{6.1.2}$$

Now, using orbits with semi-latus rectum p, eccentricity e_0, and true anomaly $v = \phi(t) - \omega_p$ ($\omega_p = 0$ sets the periastron direction on the x-axis), the orbital motion is described by,

$$r = \frac{p}{1 + e_0 cosv} \tag{6.1.3}$$

and

$$\dot{v} = \dot{\phi}(t) = \frac{\sqrt{pM}}{r^2} \tag{6.1.4}$$

Now, differentiating (6.2.3) with respect to time, we get:

$$\dot{r} = \frac{\sqrt{pM}}{p} e_0 sinv \tag{6.1.5}$$

Twice differentiating (6.2.3) with respect to time, we get:

$$\ddot{r} = \frac{\sqrt{pM}}{p}\dot{v}e_0 cosv \tag{6.1.6}$$

and using (6.2.4),

$$\ddot{r} = \frac{pM}{pr^2}e_0 cosv \tag{6.1.7}$$

Now, differentiating $\phi(t)$ with respect to time, we get the same result as $\dot{v}$:

$$\dot{\phi} = \frac{\sqrt{pM}}{r^2}$$

Twice differentiating (6.2.4) with respect to time, we get:

$$\ddot{\phi} = \frac{\sqrt{pM}}{r^3}(-2\dot{r}) \tag{6.1.8}$$

We know the explicit form for the spherical harmonic, $Y_{20}(\theta, \phi(t))$, is given by :

$$Y_{20}(\theta, \phi(t)) = \frac{1}{4}\sqrt{\frac{5}{\pi}}(cos^2\theta - 1)$$

Putting $\theta = \pi/2$, we have,

$$Y_{20}(\pi/2) = -\frac{1}{4}\sqrt{\frac{5}{\pi}} \tag{6.1.9}$$

The mass quarupole moment from (6.2.1), I_{2-2}, is re-written as:

$$I_{2-2} = Kr(t)^2 Y_{20} \tag{6.1.10}$$

Where, the constant K is denoted by,

$$K = \frac{16\pi}{5\sqrt{3}}\eta M$$

From (6.1.10), we see that Y_{20} has no dependence on $\phi(t)$. Hence, differentiating I_{20} once with respect to time, we get:

$$I_{20}{}^{(1)} = K 2 r(t) \dot{r} Y_{20} \tag{6.1.11}$$

Twice differentiating I_{2-2} with respect to time, we get:

$$I_{20}{}^{(2)} = K(2\dot{r}^2 Y_{20} + 2r\ddot{r}Y_{20}) \tag{6.1.12}$$

Hence, putting (6.2.5) and (6.2.7) in (6.2.10), we have:

$$I_{20} = K\left(\frac{M}{p}\sin^2 v(-\frac{1}{2}\sqrt{\frac{5}{\pi}}) - \frac{2}{4}\frac{p^2 M\left(-e_0\cos v\right)\left(1+e_0\cos v\right)^2}{p^3\left(1+e_0\cos v\right)}\sqrt{\frac{5}{\pi}}\right)$$

Or,

$$I_{20} = K\left(\frac{-1}{2}\frac{\sqrt{5}}{\sqrt{\pi}}\left(\frac{Me_0^2\sin^2 v}{p} + \frac{e_0 M\cos v\left(1+e_0\cos v\right)}{p}\right)\right)$$

Or,

$$I_{20}^{(2)} = -\frac{8\pi\sqrt{5}}{5\sqrt{3}\sqrt{\pi}}\eta M\left(\frac{Me_0^2\sin^2 v + e_0\cos v + e_0^2 M\cos^2 v}{p}\right)$$

Or,

$$I_{20}^{(2)} = -8\sqrt{\frac{\pi}{15}}\eta M\left(\frac{Me_0^2 + Me_0\cos v}{p}\right)$$

Hence,

$$\boxed{I_{20}^{(2)} = -8\sqrt{\frac{\pi}{15}}\frac{\eta M^2}{p}e_0\left(e_0 + \cos v\right)} \tag{6.1.13}$$

6.2 APPENDIX-II

6.2.1 CALCUATION OF $\mathbf{I}_{2-2}^{(2)}$

We know, the mass quadrupole moment I_{2m} is given by:

$$I_{2m} = \frac{16\pi}{5\sqrt{3}}\eta M r(t)^2 Y_{2m}^{*}(\pi/2, \phi(t)) \tag{6.2.1}$$

Here, total mass is $M = m_1 + m_2$, reduced mass ratio is $\eta = \frac{m_1 m_2}{M^2}$, and orbital phase angle is $\phi(t)$. Using the relation,

$$Y_{2m}^{*}(\pi/2, \phi(t)) = (-1)^m Y_{2-m}(\pi/2, \phi(t))$$

and putting $m = 2$, we get:

$$I_{2-2} = \frac{16\pi}{5\sqrt{3}}\eta M r(t)^2 Y_{2-2}(\pi/2, \phi(t)) \tag{6.2.2}$$

Now, using orbits with semi-latus rectum p, eccentricity e_0, and true anomaly $v = \phi(t) - \omega_p$ ($\omega_p = 0$ sets the periastron direction on the x-axis), the orbital motion is described by,

$$r = \frac{p}{1 + e_0 cosv} \tag{6.2.3}$$

and

$$\dot{v} = \dot{\phi}(t) = \frac{\sqrt{pM}}{r^2} \tag{6.2.4}$$

Now, differentiating (6.2.3) with respect to time, we get:

$$\dot{r} = \frac{\sqrt{pM}}{p} e_0 sinv \tag{6.2.5}$$

Twice differentiating (6.2.3) with respect to time, we get:

$$\ddot{r} = \frac{\sqrt{pM}}{p} \dot{v} e_0 cosv \tag{6.2.6}$$

and using (6.2.4),

$$\ddot{r} = \frac{pM}{pr^2} e_0 cosv \tag{6.2.7}$$

Now, differentiating $\phi(t)$ with respect to time, we get the same result as $\dot{v}$:

$$\dot{\phi} = \frac{\sqrt{pM}}{r^2}$$

Twice differentiating (6.2.4) with respect to time, we get:

$$\ddot{\phi} = \frac{\sqrt{pM}}{r^3}(-2\dot{r}) \tag{6.2.8}$$

We know the explicit form for the spherical harmonic, $Y_{2-2}(\theta, \phi(t))$, is given by :

$$Y_{2-2}(\theta, \phi(t)) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} sin^2\theta e^{-2i\phi}$$

Putting $\theta = \pi/2$, we have,

$$Y_{2-2}(\pi/2, \phi(t)) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} e^{-2i\phi} \tag{6.2.9}$$

Now, differentiating with respect to ϕ, we get:

$$\frac{dY_{2-2}}{d\phi} = -2i\frac{1}{4}\sqrt{\frac{15}{2\pi}} e^{-2i\phi} \tag{6.2.10}$$

Or,

$$\frac{dY_{2-2}}{d\phi} = -2iY_{2-2} \tag{6.2.11}$$

Twice differentiating w.r.t ϕ we get:

$$\frac{d^2Y_{2-2}}{d\phi^2} = -4\frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{-2i\phi} \tag{6.2.12}$$

Or,

$$\frac{d^2Y_{2-2}}{d\phi^2} = -4Y_{2-2} \tag{6.2.13}$$

The mass quarupole moment from (6.2.1), I_{2-2}, is re-written as:

$$I_{2-2} = Kr(t)^2 Y_{2-2} \tag{6.2.14}$$

Where, the constant K is denoted by,

$$K = \frac{16\pi}{5\sqrt{3}}\eta M$$

Now differentiating I_{2-2} once with respect to time, we get:

$$I_{2-2}{}^{(1)} = K(2r(t)\dot{r}Y_{2-2} + r^2\frac{dY_{2-2}}{d\phi}\dot{\phi}) \tag{6.2.15}$$

Twice differentiating I_{2-2} with respect to time, we get:

$$I_{2-2}{}^{(2)} = K(2\dot{r}^2 Y_{2-2} + 2r\ddot{r}Y_{2-2} + 2r\dot{r}\frac{dY_{2-2}}{d\phi} + 2r\dot{r}\frac{dY_{2-2}}{d\phi} + r^2\frac{d^2Y_{2-2}}{\phi^2}\dot{\phi}^2) \tag{6.2.16}$$

Or,

$$I_{2-2}{}^{(2)} = K(Y_{2-2}(2\dot{r}^2 + 2r\ddot{r}) + \frac{dY_{2-2}}{d\phi}(4r\dot{r}\dot{\phi} + r^2\ddot{\phi}) + \frac{d^2Y_{2-2}}{d^2\phi}(r^2\ddot{\phi}) \tag{6.2.17}$$

Now, putting (6.2.13) in (6.2.17), we get:

$$I_{2-2}{}^{(2)} = K(Y_{2-2}(2\dot{r}^2 + 2r\ddot{r} - 4r^2\dot{\phi}^2) + \frac{dY_{2-2}}{d\phi}(4r\dot{r}\dot{\phi} + r^2\ddot{\phi})) \tag{6.2.18}$$

Again, putting (6.2.11) in (6.2.18), we get:

$$I_{2-2}{}^{(2)} = K(Y_{2-2}(2\dot{r}^2 + 2r\ddot{r} - 4r^2\dot{\phi}^2) - 2ir^2\ddot{\phi} - 8ir\dot{r}\dot{\phi}) \qquad (6.2.19)$$

Now,

$$r^2\ddot{\phi} + 4r\dot{r}\dot{\phi} = r^2(\frac{\sqrt{pM}}{r^3}(-2\dot{r}) + 4\dot{r}r\frac{pM}{r^2})$$

Or,

$$r^2\ddot{\phi} + 4r\dot{r}\dot{\phi} = \frac{\sqrt{pM}}{r}(4\dot{r} - 2\dot{r})$$

Or,

$$r^2\ddot{\phi} + 4r\dot{r}\dot{\phi} = \frac{\sqrt{pM}}{r}(2\dot{r})$$

Or,

$$r^2\ddot{\phi} + 4r\dot{r}\dot{\phi} = 2\frac{\sqrt{pM}}{r}e_0 sinv\frac{\sqrt{pM}}{p}$$

Therefore, using (6.2.3),

$$r^2\ddot{\phi} + 4r\dot{r}\dot{\phi} = \frac{2e_0 sinv(1 + e_0 cosv)pM}{p^2} \tag{6.2.20}$$

Also,

$$2\dot{r}^2 + 2r\ddot{r} - 4r^2\dot{\phi}^2 = \frac{2pM}{p^2}{e_0}^2 sin^2 v + 2\frac{pM}{pr}e_0 cosv - 4\frac{pM}{r^2}$$

Or,

$$2\dot{r}^2 + 2r\ddot{r} - 4r^2\dot{\phi}^2 = 2pM(\frac{e_0 sin^2 v}{p^2} + \frac{e_0 cosv}{p^2}(1 + e_0 cosv) - \frac{2}{p^2}((1 + e_0 cosv)^2)$$

Or,

$$2\dot{r}^2 + 2r\ddot{r} - 4r^2\dot{\phi}^2 = (\frac{2pM}{p^2})((e_0)^2 + e_0 cosv - 4e_0 cosv - 2 - 2(e_0)^2 cos^2 v)$$

Or,

$$2\dot{r}^2 + 2r\ddot{r} - 4r^2\dot{\phi}^2 = (\frac{2pM}{p^2})((e_0)^2 - 3e_0 cosv - 2 - 2(e_0)^2 cos^2 v)$$

Therefore,

$$2\dot{r}^2 + 2r\ddot{r} - 4r^2\dot{\phi}^2 = (\frac{2pM}{p^2})((e_0)^2 - 2e_0 cosv(1 + e_0 cosv) - e_0 cosv - 2)) \tag{6.2.21}$$

Now, putting (6.2.20) and (6.2.21) in (6.2.19) we get:

$$I_{2-2}{}^{(2)} = K(\frac{2pM}{p^2})((e_0)^2 - 2e_0 cosv(1 + e_0 cosv) - e_0 cosv - 2)) - (2i)\frac{2pM}{p^2} e_0 sinv(1 + e_0 cosv)$$

Or,

$$I_{2-2}{}^{(2)} = K(\frac{2pM}{p^2})({e_0}^2 - e_0 cosv - 2 - 2e_0(cosv + isinv)(1 + e_0 cosv))$$

Or,

$$I_{2-2}{}^{(2)} = K(\frac{2pM}{p^2})({e_0}^2 - e_0 cosv - 2 - 2e_0(e^{iv})(1 + e_0 cosv))$$

Or,

$$I_{2-2}{}^{(2)} = K((\frac{2pM}{p^2})({e_0}^2 - 1 - 1(1 + e_0 cosv) - 2e_0 e^{iv}(1 + e_0 cosv)))$$

Or,

$$I_{2-2}{}^{(2)} = K((\frac{2pM}{p^2})({e_0}^2 - 1 - (1 + e_0 cosv)(1 + 2e_0 e^{iv})))$$

Or,

$$I_{2-2}{}^{(2)} = K((\frac{-2pM}{p^2})(-{e_0}^2 + 1 + (1 + e_0 cosv)(1 + 2e_0 e^{iv})))$$

Now putting the value of K, we get:

$$I_{2-2}{}^{(2)} = (-4\sqrt{\frac{2\pi}{5}}\eta\frac{M^2}{p}e^{-2i\phi})(((-{e_0}^2 + 1 + (1 + e_0 cosv)(1 + 2e_0 e^{iv})))$$
(6.2.22)

Which is the required expression for the quadrupole mass moment I_{2-2}.

Similarly, the expression for I_{2+2} can be arrived at by putting $m = -2$ in (6.2.1). We get the same result, but the signs are interchanged, and hence:

$$I_{2+2}{}^{(2)} = (-4\sqrt{\frac{2\pi}{5}}\eta\frac{M^2}{p}e^{+2i\phi})(((-{e_0}^2 + 1 + (1 + e_0 cosv)(1 + 2e_0 e^{-iv})))$$
(6.2.23)

Therefore, we have:

$$\boxed{I_{2\pm2}{}^{(2)} = (-4\sqrt{\frac{2\pi}{5}}\eta\frac{M^2}{p}e^{\pm2i\phi(t)})(((-{e_0}^2 + 1 + (1 + e_0 cosv)(1 + 2e_0 e^{\mp iv})))}$$
(6.2.24)

Infinitesimal co-ordinate transform:

In Linearized Theory of Gravity, if we choose infinitesimal co-ordinate transforms such that:

$$x^{\mu\prime}(\mathcal{P}) = x^{\mu}(\mathcal{P}) + \xi^{\mu}(\mathcal{P}) \tag{6.2.25}$$

where $\xi^{\mu}(\mathcal{P})$ are four arbitrary functions small enough to leave $|h_{\mu'\nu'}| \ll 1$. For such transformations, the metric perturbations for the new($x^{\mu\prime}$) and the old (x^{μ}) co-ordinate systems are related by

$$h_{\mu\nu}^{new} = h_{\mu\nu}^{old} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \tag{6.2.26}$$

Gauge transformation and gauge invariance:

In linearized theory, one usually regards gauge transformation as (6.2.26), analogous to those in electromagnetic theory given by

$$A_{\mu}^{new} = A_{\nu}^{old} + \psi_{,\mu} \tag{6.2.27}$$

We know, without loss of generality, one can impose the 'the gauge condition':

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0 \tag{6.2.28}$$

6.3 APPENDIX - III:

6.3.1 LINEAR MEMORY FOR N GRAVITATIONALLY UNBOUND PARTICLES:

These gauge conditions are the tensor analogue of the Lorentz gauge $A^{\alpha}{}_{,\alpha} = 0$ of electromagnetic theory. The field equations then become

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 16\pi T_{\mu\nu} \tag{6.3.1}$$

The gauge conditions (6.2.28),the field equations((6.3.1), and the definition of metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} - \frac{\eta_{\mu\nu}\bar{h}}{2} \tag{6.3.2}$$

are the fundamental equations of Linearized Theory of Gravity written in Lorentz Gauge.

Green's Function of the wave equation:

Our starting point is the linearised Einstein equation, re-written from (6.2.28), using the D'Alembertian operator

$$\Box h_{\mu\nu} = -16\pi T_{\mu\nu} \tag{6.3.3}$$

which assumes that both the source, in the guise of the energy momentum tensor $T_{\mu\nu}$ and the perturbed metric $h_{\mu\nu}$ are small. This is simply a bunch of decoupled wave equations.

We will consider a situation in which matter fields are localised to some spatial region Σ. In this region, there is a time-dependent source of energy and momentum $T_{\mu\nu}(x', t)$, such as two orbiting black holes. Outside of this region, the energy-momentum tensor vanishes: $T_{\mu\nu}(x', t) = 0$ for $x' \neq \Sigma$. We want to know what the metric $h_{\mu\nu}$ looks like a long way from the region Σ.

The general solution to (6.3.3), the linearised field equations in Lorentz gauge, outside of Σ can be given using the (retarded) Green's function; it is

$$\bar{h}_{\mu\nu}(t, \vec{x}) = \int_{\Sigma} \frac{4T_{\mu\nu}(t - |x - x'|, x')}{|x - x'|} d^3x' \tag{6.3.4}$$

For nearly Newtonian sources: $T_{00} \gg |T_{oj}|, T_{00} \gg |T_{jk}|$, and velocities low enough that retardation is negligible, then (6.3.4) reduces to

$$\bar{h}_{00} = -4\Phi \tag{6.3.5}$$

$$\bar{h}_{0j} = \bar{h}_{jk} = 0 \tag{6.3.6}$$

where,

$$\Phi(t, x) = -\int \frac{T_{00}(t, x')}{|x - x'|} d^3x' = Newtonian\ Potential \tag{6.3.7}$$

Now, considering the external gravitational field of a static spherical body as described in the Lorentz frame i.e in nearly rectangular co-ordinate system $|h_{\mu\nu}| << 1$ in which the body is located at x=y=z=0 for all t, and adopting Lorentz gauge

$$\bar{h}_{00} = \frac{4M}{(x^2 + y^2 + z^2)^{1/2}} \tag{6.3.8}$$

$$h_{00} = h_{xx} = h_{yy} = h_{zz} = \frac{2M}{(x^2 + y^2 + z^2)^{1/2}} \tag{6.3.9}$$

where M is a constant given by

$$M = \int \rho\, d^3x = \int T_{00}\, d^3x = total\ mass$$

Now, adopting spherical polar coordinates,

$$x = r sin\theta cos\phi,$$

$$y = r sin\theta sin\phi,$$

$$z = rcos\theta$$

and regarding $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ as components of tensors in flat space-time, and by using tensor transformations laws, we get the solutions (6.3.8), (6.3.10) in the form:

$$\bar{h}_{\mu\nu} = \frac{4M}{r} \tag{6.3.10}$$

$$\bar{h}_{0j} = \bar{h}_{jk} = 0 \tag{6.3.11}$$

$$h_{00} = \frac{2M}{r} \tag{6.3.12}$$

$$h_{0j} = 0 \tag{6.3.13}$$

Now, let us consider the metric perturbation $h_{\mu\nu}$, induced at the field point x by a particle with 4-velocity u, at a point x' on its trajectory. Since we are interested in the long-wavelength limit, we assume that the field point is far from the particle and that space-time is nearly flat there.In the particle's rest frame the perturbation is that due to the gravitational field of a static particle of mass M, given by (6.3.10) and (6.3.11).

Also, from (6.3.2), we have

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{\eta_{\mu\nu}h}{2}$$

However, we need the perturbation in the rest frame of the observer. By a

Lorentz transformation, one finds:

$$\bar{h}_{\mu\nu} = \frac{4Mu_\mu u_\nu}{-k.u} \tag{6.3.14}$$

The denominator occurs because in the particle's rest frame

$$-k.u = k^0 = |x^0 - x'^0| = r_{ret}$$

where $k = r_{ret}(1, \vec{n})$ is the null vector joining the point x and x'.

Bursts with Memory(BWM) for two or more unbound masses:

For sources consisting of either of collision of two or more initially free masses, or an explosion of an initial single mass into several freely and independently moving masses, the permanent change in gravitational-wave field(the burst-memory) Δh_{ij}^{T} is equal to transverse, traceless (TT) part of the time independent, Coulomb type 1/r of the final system minus that of the initial system given by (6.3.14). The superscript TT on means "in the rest frame of the source project out the piece that is transverse to the line between source and detector and remove that piece's trace", i.e., "take the transverse, traceless or TT part".

Now, Δh_{ij}^{TT} can be written in the following form:

$$\Delta h_{ij}^{TT} = \Delta(\sum_A \frac{4P_i^A P_j^A}{\mathbf{k.P^A}})^{TT} \tag{6.3.15}$$

where P^A is the 4-momentum of mass A of the system, and P_i^A is the spatial component of that 4-momentum in the rest frame of a distant observer and $\mathbf{k}$ is the past directed null 4-vector from observer to source. We are considering G=c=1.

And if for body A, mass is M_A and energy per unit mass is $\gamma_A = (1 - v_A^2/c^2)^{-1/2}$, then

$$\Delta h_{xx}^{TT} = -\Delta h_{yy}^{TT} = \sum_A \frac{2GM_A}{c^4 r}\Delta[\frac{\gamma_A(v_A^x v_A^x - v_A^y v_A^y}{1 + v_A^z/c}] \tag{6.3.16}$$

$$\Delta h_{xy}^{TT} = \Delta h_{yx}^{TT} = \sum_A \frac{4GM_A}{c^4 r}\delta[\frac{\gamma_A v_A^x v_A^y}{1 + v_A^z/c}] \tag{6.3.17}$$

and all other components vanishes.

Now, if we consider r is the distance from source to detector, v_A^j is the velocity of the center of mass of system A, and θ_A, is the angle between $v_A^j$and the direction from the source to the detector, as measured in the rest frame of the detector, then

$$\triangle h_{jk}^{TT} = \triangle \sum_{A=1}^{N} \frac{4M_A}{r\sqrt{1-v_A^2}} [\frac{v_A^j v_A^k}{1 - v_A cos\theta_A}]^{TT} \tag{6.3.18}$$

Or,

$$\boxed{\Delta h_{jk}^{TT} = \triangle \sum_{A=1}^{N} \frac{4M_A}{r\sqrt{1-v_A^2}} [\frac{v_A^j v_A^k}{1 - \mathbf{v_A}.\mathbf{N}}]^{TT}} \tag{6.3.19}$$

where N points from the source to the observer.

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