# MODULE 2: SOLUTION SETS & LINEAR INDEPENDENCE

Linear Algebra 1: Linear Equations

"Mathematics is the art of giving the same name to different things." – H. Poincaré

## TOPIC 1: The Matrix Equation

DEFINITION → Matrix-Vector Product as a Linear Combination

| symbol | meaning |
|---|---|
| $\in$ | belongs to |
| $\mathbb{R}^n$ | the set of vectors with n-valued elements |
| $\mathbb{R}^{m\times n}$ | the set of real-valued matrices with m rows and n columns |

→ If $A \in \mathbb{R}^{m\times n}$ has columns $\vec{a}_1, \dots, \vec{a}_n$ and $\vec{x} \in \mathbb{R}^n$, then the matrix vector product $A\vec{x}$ is a linear combination of the columns of A.

$$A\vec{x} = \begin{pmatrix} | & | & \cdots & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & \cdots & | \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n$$

Note: $A\vec{x}$ is in the span of columns of A.

Example: Suppose $A = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

The following product can be written as a linear combination of vectors:

$$A\vec{x} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

Is $\vec{b} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$ in the span of the columns of A?

If $\vec{b}$ is in the span of columns of A, then $b = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

By inspection, $c_1 = 2$ and $c_2 = -3$. $\therefore \vec{b} \in \text{span}\{\text{columns of } A\}$.

Example: Suppose $A = \begin{pmatrix} 2 & 4 \\ 2 & 5 \\ 4 & 9 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 1 \\ t \\ 1 \end{pmatrix}$

$$(A|\vec{y}) = \left(\begin{array}{cc|c} 2 & 4 & 1 \\ 2 & 5 & t \\ 4 & 9 & 1 \end{array}\right)$$

$$\begin{array}{l} R_2 - R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array} \sim \left(\begin{array}{cc|c} 2 & 4 & 1 \\ 0 & 1 & t-1 \\ 0 & 1 & -1 \end{array}\right). \qquad t-1 = -1 \Rightarrow \underline{t=0}.$$

$\therefore$ For vector $\vec{y}$ to be a linear combination of the columns of A, $t = 0$. ✱

### Equivalence Solution Sets

Note that if A is a $m \times n$ matrix with columns $\vec{a}_1, \dots, \vec{a}_n$, and $\vec{x} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$, then the solutions to

$$A\vec{x} = \vec{b}$$

has the same set of solutions as the vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b}$

which has the same set of solutions as the set of linear equations with the augmented matrix

$$[\vec{a}_1 \;\; \vec{a}_2 \;\; \cdots \;\; \vec{a}_n \mid \vec{b}]$$

THEOREM: Linear Combinations and the Existence of Solutions

The equation $A\vec{x} = \vec{b}$ has a solution of $A\vec{x}$ being a linear combination of the columns of A.

* Note: Follows directly from earlier definition of $A\vec{x}$ being a linear combination of the columns of A.

Example: For what vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the equation have a solution?

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 8 & 4 \\ 0 & 1 & -2 \end{pmatrix} \vec{x} = \vec{b}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 2 & 8 & 4 & b_2 \\ 0 & 1 & -2 & b_3 \end{array}\right) \xrightarrow{R_2 - 2R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 2 & -4 & b_2 - 2b_1 \\ 0 & 1 & -2 & b_3 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 1 & -2 & b_3 \\ 0 & 2 & -4 & b_2 - 2b_1 \end{array}\right) \xrightarrow{R_3 - 2R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 1 & -2 & b_3 \\ 0 & 0 & 0 & b_2 - 2b_1 - 2b_3 \end{array}\right)$$

For there to be a solution \ for the solution to be consistent,

$b_2 - 2b_1 - 2b_3 = 0$. Let $b_1$ be the subject: $b_1 = \frac{1}{2}b_2 - b_3$.

Therefore, our $\vec{b}$ must be of the form $\vec{b} = \begin{pmatrix} \frac{1}{2}b_2 - b_3 \\ b_2 \\ b_3 \end{pmatrix}$.

Multiple Representations of Linear Systems

We now have 4 equivalent ways of representing a linear system.

1. A list of equations: $2x_1 + 3x_2 = 7$
   $x_1 - x_2 = 5$.

3. A vector equation: $x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

2. An augmented matrix: $\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$

4. A matrix equation: $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

Each representation gives us a different way to think about linear systems.

TOPIC 2: Solution Sets of Linear Systems

DEFINITION — Homogenous Systems

↳ Linear systems of the form $A\vec{x} = \vec{0}$ are homogenous.

↳ Linear systems of the form $A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$, are inhomogenous.

Because homogenous systems always have the trivial solution, $\vec{x} = \vec{0}$, the interesting question is whether they have non-trivial solutions.

Observation: $A\vec{x}=\vec{0}$ has a nontrivial solution $\iff$ there is a free variable
$\iff$ A has a column with no pivot.

Example: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1=0, \\ x_2=0 \end{matrix} \Rightarrow \vec{x}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

Example: A Homogenous System

Identify the free variables, and the solution set, of the system.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ 2x_1 - x_2 - 5x_3 &= 0 \\ x_1 \quad\quad - 2x_3 &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & -1 & -5 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \xrightarrow[R_3-R_1\to R_3]{R_2-2R_1\to R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right)$$

$$\xrightarrow[-\frac{1}{7}R_2\to R_2]{0R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1-3R_2\to R_1} \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right) \\ \begin{array}{ccc} p & p & f \end{array} \end{array}$$

$x_1 + 0x_2 - 2x_3 = 0 \Rightarrow x_1 = 2x_3$
$0x_1 + x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

$$\text{solution set} = \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -x_3 \\ 1 \end{pmatrix} = x_3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Exercise: Matrix A is 5x4 and has 3 pivotal columns. The only solution to the homogenous linear system $A\vec{x}=\vec{0}$ is the trivial solution.

$\Rightarrow$ IMPOSSIBLE. If A is 5x4 and has 3 pivot columns then one column is not pivotal. There must be a free variable, so there must be an infinite number of solutions.

Exercise: Matrix A is 3x2. The homogenous linear system $A\vec{x}=\vec{0}$ has an infinite number of solutions and every column of A is pivotal.

$\Rightarrow$ IMPOSSIBLE. If A is 3x2 and $A\vec{x}=\vec{0}$ has an infinite number of solutions, there must be a free variable.

Homogenous and inhomogenous systems are related to each other in a way that is easier to see with parametric vector form.

# Parametric Vector form of the solution of a Non-homogenous System.

Example: Write the solution as a sum of vectors. Give a geometric interpretation of the solution.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 9 \\ 2x_1 - x_2 - 5x_3 &= 11 \\ x_1 \quad\quad - 2x_3 &= 6 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 2 & -1 & -5 & 11 \\ 1 & 0 & -2 & 6 \end{array}\right) \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -7 & -7 & -7 \\ 0 & -3 & -3 & -3 \end{array}\right)$$

$$\xrightarrow[0R_3 \to R_3]{-\frac{1}{7}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 - 3R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 - 2x_3 = 6 \Rightarrow x_1 = 6 + 2x_3$$
$$x_2 + x_3 = 1 \Rightarrow x_2 = 1 - x_3$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 + 2x_3 \\ 1 - x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

In general, suppose the free variables for $A\vec{x} = \vec{0}$ are $x_k, \dots, x_n$. Then all solutions to $A\vec{x} = \vec{0}$ can be written as

$$\vec{x} = x_k\vec{v}_k + x_{k+1}\vec{v}_{k+1} + \dots + x_n\vec{v}_n$$

for some $\vec{v}_k, \dots, \vec{v}_n$. This is the parametric form of the solution.

Exercise: Consider the matrix $A$ below.

$$A = \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

If the solution to $A\vec{x} = \vec{0}$ has the parametric vector form below,

$$\vec{x} = x_3\vec{v}_1 + x_4\vec{v}_2 = x_3 \begin{pmatrix} h \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} p \\ k \\ 0 \\ 1 \end{pmatrix}$$

Find the values of $h$, $p$, and $k$.

$$A = \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \quad A\vec{x} = \vec{0}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right) \qquad \begin{array}{l} x_1 + 4x_3 + x_4 = 0 \Rightarrow x_1 = -4x_3 - x_4 \\ x_2 + x_4 = 0 \Rightarrow x_2 = -x_4. \end{array}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4x_3 - x_4 \\ -x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}. = x_3 \begin{pmatrix} h \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} p \\ k \\ 0 \\ 1 \end{pmatrix}$$

$\therefore h = -4, p = k = -1.$ *

## TOPIC 3: Linear Independence

### Linear Independence

What is the smallest number of vectors needed in a parametric solution to a linear system?

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ are linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$$

has only the trivial solution. It is linearly dependent otherwise.

In other words, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ are linearly dependent if there are real numbers $c_1, c_2, \ldots, c_k$ not all zero so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$$

Consider the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.

To determine whether the vectors are linearly independent, we can set the linear combination to the zero vector:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = [\vec{v}_1 \quad \vec{v}_2 \quad \ldots \quad \vec{v}_k] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = V\vec{c} \overset{??}{=} \vec{0}$$

Linear independence: there is NO non-zero solution $\vec{c}$

Linear dependence: there is a nonzero solution

Example: For what values of h, if any, is the set of vectors linearly independent?

$$\begin{bmatrix}1\\1\\h\end{bmatrix}, \begin{bmatrix}1\\h\\1\end{bmatrix}, \begin{bmatrix}h\\1\\1\end{bmatrix}$$

If independent, then $c_1\begin{pmatrix}1\\1\\h\end{pmatrix} + c_2\begin{pmatrix}1\\h\\1\end{pmatrix} + c_3\begin{pmatrix}h\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

for only $c_1 = c_2 = c_3 = 0$, then independent.

$$\left(\begin{array}{ccc|c}1 & 1 & h & 0\\1 & h & 1 & 0\\h & 1 & 1 & 0\end{array}\right) \xrightarrow[R_3 - hR_1 \to R_3]{R_2 - R_1 \to R_2} \left(\begin{array}{ccc|c}1 & 1 & h & 0\\0 & h-1 & 1-h & 0\\0 & 1-h & 1-h^2 & 0\end{array}\right)$$

$$\xrightarrow{R_3 + R_2 \to R_3} \left(\begin{array}{ccc|c}1 & 1 & h & 0\\0 & h-1 & 1-h & 0\\0 & 0 & 2-h-h^2 & 0\end{array}\right)$$

If $2-h-h^2=0$, vectors are dependent because $c_3$ is free.

$2-h-h^2=0 \Rightarrow (2+h)(1-h)=0 \Rightarrow h=-2$ or $1$.

$\therefore$ For vectors to be independent, $h \in \mathbb{R}$, $h \neq -2$, $h \neq 1$.

Exercise: Consider the vectors $\vec{a}$ and $\vec{b}$ below:

$$\vec{a} = \begin{pmatrix}2\\4\\12\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}1\\2\\t\end{pmatrix}$$

For what values of $t$ are the two vectors linearly dependent?

For $\vec{a}$ and $\vec{b}$ to be linearly independent, $c_1\vec{a} + c_2\vec{b} = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

$$\left(\begin{array}{cc|c}2 & 1 & 0\\4 & 2 & 0\\12 & t & 0\end{array}\right) \xrightarrow[R_3 - 6R_1 \Rightarrow R_3]{R_2 - 2R_1 \Rightarrow R_2} \left(\begin{array}{cc|c}2 & 1 & 0\\0 & 0 & 0\\0 & t-6 & 0\end{array}\right)$$

If $\vec{a}$ and $\vec{b}$ are to be linearly dependent, $t-6=0 \Rightarrow \underline{\underline{t=6}}$.

NOTE: $\vec{a}$ and $\vec{b}$ are linearly dependent when they are multiples of each other.
If $t=6$, then $\vec{a} = 2\vec{b}$.

## Example: Two Dependent Vectors

Suppose $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$. When is the set $\{\vec{v}_1, \vec{v}_2\}$ linearly independent? Provide a geometric interpretation.

Solution

From our definition of linear dependence, if $\vec{v}_1, \vec{v}_2$ are dependent, then there exists a $c_1$ and a $c_2$, not both zero, so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

Consider 2 cases:

1. If $\vec{v}_1$ and\or $\vec{v}_2$ is the zero vector, then the vectors are dependent. If for example $\vec{v}_1 = \vec{0}$, then $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ is satisfied for $c_2 = 0$ and any $c_1$.

2. If $\vec{v}_1 \neq \vec{0}$ and $\vec{v}_2 \neq \vec{0}$, then $\vec{v}_2 = -\frac{c_1}{c_2}\vec{v}_1$, so $\vec{v}_1$ and $\vec{v}_2$ are multiples of each other. The vectors are parallel (one vector is in the span of the other).

Thus, two vectors in $\mathbb{R}^n$ are dependent when either or both of the following occur:

1) One or both vectors are the zero vector
2) One vector is a multiple of the other

## Linear Independence Theorems

### 1) More Vectors Than Elements

Suppose $\vec{v}_1, \dots, \vec{v}_k$ are vectors in $\mathbb{R}^n$. If $k > n$, then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent.

WHY? Every column of the matrix $A = (\vec{v}_1, \dots, \vec{v}_k)$ would have to be pivotal for the vectors to be independent.

But, A has more columns than rows, so every column cannot be pivotal. The vectors must be linearly dependent.

### 2) Set Contains Zero Vector

If any one or more of $\vec{v}_1, \dots, \vec{v}_k$ is $\vec{0}$, then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent.

WHY? Every column of the matrix $A = (\vec{v}_1, \dots, \vec{v}_k)$ would have to be pivotal for the vectors to be independent.

But, A has a zero column, so every column cannot be pivotal. The vectors must be linearly dependent.

## Application of Linear Independance Theorems

Example: By inspection, which matrices have linearly independent columns?

① $A = \begin{pmatrix} 1 & \boxed{0} \\ 2 & \boxed{0} \end{pmatrix}$

zero column
⇒ dependent

② $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

more columns than rows
⇒ dependent

③ $C = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$

last column is equal to sum of first two
⇒ dependent

④ $D = \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{pmatrix}$

every column is pivotal
⇒ linearly independent.

Exercise: Consider the matrix A below.

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 9 \\ 4 & 3 & k \end{pmatrix}$$

For what value of k are the columns of A linearly dependent?

$$A\vec{x} = \vec{0}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 2 & 3 & 9 & 0 \\ 4 & 3 & k & 0 \end{array}\right) \xrightarrow[R_3 - 4R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & k-12 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 - R_2 \to R_3} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 3 & 0 \\ 0 & \boxed{3} & 3 & 0 \\ 0 & 0 & \boxed{k-15} & 0 \end{array}\right)$$

Echelon form of A is linearly independent if every column is pivotal. If $k-15=0$, echelon form of A will only have 2 pivots. Then the homogenous equation $A\vec{x}=\vec{0}$ will have a free variable, meaning that there is a nontrivial solution to $A\vec{x}=\vec{0}$.

$$\Rightarrow k-15=0 \Rightarrow \therefore k=15$$