MODULE 4: MATRIX ADDITION AND SCALAR SUBTRACTION

TOPIC 1: Matrix Operations

DEFINITION — Zero Matrix

↳ A zero matrix is any matrix whose every entry is zero.

$$0_{2\times3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 0_{2\times1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

DEFINITION — Identity Matrix

↳ The $n \times n$ identity matrix has ones on the main diagonal, otherwise all zeros.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note: any matrix with dimensions $n \times n$ is square. Zero matrices need not be square, identity matrices must be square.

Matrix Addition and Scalar Multiples

Suppose A and B are $m \times n$ matrices. $a_{i,j}$ is the entry of A in row i and column j, and $b_{i,j}$ is the entry of B in row i and column j.

→ The entries of $A+B$ are $a_{i,j} + b_{i,j}$.
→ If $c \in \mathbb{R}$, then the entries of cA are $ca_{i,j}$.

For example, if

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + c\begin{pmatrix} 7 & 4 & 7 \\ 0 & 0 & k \end{pmatrix} = \begin{pmatrix} 15 & 10 & 17 \\ 4 & 5 & 16 \end{pmatrix}$$

What are the values of c and k?

$$1+7c = 15 \Rightarrow 7c = 14 \Rightarrow c = 2$$
$$6+ck = 16 \Rightarrow 6 + 2k = 16 \Rightarrow 2k = 10 \Rightarrow k = 5.$$

Properties of Sums and Scalar Multiples

Scalar multiples and matrix addition have the expected properties:

If $r, s \in \mathbb{R}$ are scalars, and A, B and C are $m \times n$ matrices, then

1. $A + 0_{m\times n} = A$
2. $(A+B)+C = A+(B+C)$
3. $r(A+B) = rA + rB$
4. $(r+s)A = rA + sA$
5. $r(sA) = (rs)A$

DEFINITION — Matrix Multiplication

↳ Let A be a $m \times n$ matrix, and B be a $n \times p$ matrix. The product AB is a $m \times p$ matrix, equal to

$$AB = A(\vec{b}_1 \ \cdots \ \vec{b}_p) = (A\vec{b}_1 \ \cdots \ A\vec{b}_p)$$

Example: Compute the following product:

$$\begin{aligned} C = AB &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \end{pmatrix} \\ &= \left(\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \ \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} \ \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 4 & 0 & 0 \\ 5 & 4 & 0 \end{pmatrix} \end{aligned}$$

Row Column Rule for Matrix Multiplication

The Row Column Rule is a convenient way to calculate the product AB that many students have encountered in pre-requisite courses.

DEFINITION — Row Column Method

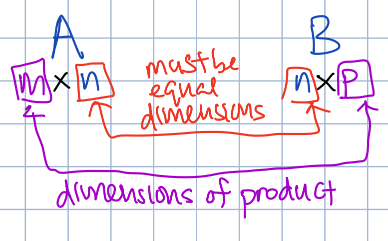
↳ If $A \in \mathbb{R}^{m \times n}$ has rows $\vec{a}_i$, and $B \in \mathbb{R}^{n \times p}$ has columns $\vec{b}_j$, each element of the product $C = AB$ is the dot product $c_{ij} = \vec{a}_i \cdot \vec{b}_j$.

Example: Compute the following using the row-column method.

$$\begin{aligned} C = AB &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2(2)+0(3) & 2(0)+0(4) & 2(0)+0(0) \\ 1(2)+1(3) & 1(0)+1(4) & 1(0)+0(0) \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 & 0 \\ 5 & 4 & 0 \end{pmatrix} \end{aligned}$$

Matrix Dimensions and Matrix Multiplication

Note: the dimensions of A and B determine whether AB is defined, and what its dimensions will be.



Properties of Matrix Multiplication

Let A, B, C be matrices of the sizes needed for the matrix multiplication to be defined, and A is a $m \times n$ matrix.

1. (Associative) $(AB)C = A(BC)$
2. (Left Distributive) $A(B+C) = AB + AC$
3. (Right Distributive) $(A+B)C = AC + BC$
4. (Identity for Matrix Multiplication) $I_m A = A I_n$.

Warnings:
1. (non-commutative) In general, $AB \neq BA$.
2. (non-cancellation) $AB = AC$ does not mean $B = C$.
3. (zero divisors) $AB = 0$ does not mean that either $A = 0$ or $B = 0$.

Example: Suppose $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

1. Give an example of a 2×2 matrix that does not commute with A.

ex. $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = B$.

$BA = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A$.

$AB \neq BA$.

The Associative Property

If $C = \vec{x}$, then the associative property is: $(AB)\vec{x} = A(B\vec{x})$. Schematically,

$$\begin{tikzcd} \boxed{\vec{x}} \arrow[rr, "\text{multiplication by } AB"] \arrow[d, "\text{multiplication by } B"] & & \boxed{AB\vec{x}} \\ \boxed{B\vec{x}} \arrow[urr, "\text{multiplication by } A"'] & & \end{tikzcd}$$

The matrix product $AB\vec{x}$ can be obtained by either:

- multiplying by matrix AB, or
- multiplying by B, then by A.

This means that matrix multiplication corresponds to <u>composition of the linear transformations</u>.

Transpose of A Matrix

A^T is the matrix whose columns are the rows of A.

Example: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \\ 4 & 2 \end{pmatrix}$

Properties of Matrix Transpose: ① $(A^T)^T = A$ ② $(A+B)^T = A^T + B^T$ ③ $(rA)^T = rA^T$ ④ $(AB)^T = B^T A^T$

Matrix Powers

For $n \times n$ matrix and positive integer k, A^k is the product of k copies of A.

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$$

Example: Compute C^2.

$$C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad C^2 = CC = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

Example: Given

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Which of these operations are defined, and what are the dimensions of the result?

① $A + 3C^2 \Rightarrow$ undefined. $\because$ A is 2×2, $3C^2$ is 3×3

② $A(AB)^T$ $\quad AB$ is $2 \times 3 \Rightarrow (AB)^T = 3 \times 2$.
A is 2×2. $\therefore$ undefined

③ $A + ABCB^T$ $\quad AB$ is 2×3.
ABC is 2×3.
$ABCB^T$ is 2×2. A is 2×2. $\Rightarrow A + ABCB^T$ is 2×2. Defined.

TOPIC 2: Inverse of a Matrix

Definition — Matrix Inverse

$A \in \mathbb{R}^{n \times n}$ is invertible (or nonsingular) if there is a $C \in \mathbb{R}^{n \times n}$ so that

$$AC = CA = I_n.$$

If there is, we write $C = A^{-1}$.

A matrix that is not invertible is singular.

Theorem: The Inverse of a 2×2 Matrix

The 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is nonsingular iff $ad - bc \neq 0$, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example: State the inverse of the matrix $\begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}$.

$$A = \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}. \quad A^{-1} = \frac{1}{2(-7)-5(-3)} \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}$$
$$= \frac{1}{1}\begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}$$

Example – Solving a Linear System

Use a matrix inverse to solve the linear system.

$$3x_1 + 4x_2 = 7$$
$$5x_1 + 6x_2 = 7$$

Can write as: $A\vec{x} = \vec{b}$, $A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$
$$= \frac{1}{18-20}\begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 7 \end{pmatrix}$$
$$= -\frac{1}{2}\begin{pmatrix} 42-28 \\ -35+21 \end{pmatrix}$$
$$= -\frac{1}{2}\begin{pmatrix} 14 \\ -14 \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$$

$\Rightarrow$ solution: $\vec{x} = \begin{pmatrix} -7 \\ 7 \end{pmatrix}$. $\underline{x_1 = -7, x_2 = 7}$.

The Inverse of a nxn Matrix

An Algorithm for Computing A^{-1}:

Suppose $A \in \mathbb{R}^{n \times n}$. We can use the following algorithm to compute A^{-1}.

1. Row reduce the augmented matrix $(A \mid I_n)$ to RREF.

2. If reduction has form $(I_n \mid B)$ then A is invertible and $B = A^{-1}$. Otherwise, A is not invertible.

Example: Compute the inverse of $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

To compute A^{-1}, $(A|I) = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 - 2R_3 \to R_2]{R_1 - 3R_3 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) = (I|A^{-1})$$

$$\therefore A^{-1} = \begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Why does this algorithm produce A^{-1}?

Suppose A is a 3×3 matrix and $A^{-1} = (\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3)$. The first column of A^{-1} is

$$\vec{x}_1 = A^{-1}\vec{e}_1$$

Recall: $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $A^{-1}\vec{e}_1 = 1\vec{x}_1 + 0\vec{x}_2 + 0\vec{x}_3$

$A\vec{x}_1 = AA^{-1}\vec{e}_1$

$A\vec{x}_1 = \vec{e}_1$

This implies:

$$A\vec{x}_1 = \vec{e}_1, \quad \text{or} \quad (A|\vec{e}_1)$$

Thus,
* if we row reduce to RREF, we obtain the first column of the inverse, $\vec{x}_1$.
* each column of A^{-1} is found by reducing $A\vec{x}_i = \vec{e}_i$.

Think of the algorithm as simultaneously solving n linear systems:

$$\left.\begin{array}{l} A\vec{x}_1 = \vec{e}_1 \\ A\vec{x}_2 = \vec{e}_2 \\ \quad \vdots \\ A\vec{x}_n = \vec{e}_n \end{array}\right\} \begin{array}{l} (A|\vec{e}_1 \ \vec{e}_2 \ \cdots \ \vec{e}_n) \\ = (A|I) \end{array}$$

Each column of A^{-1} is $A^{-1}\vec{e}_i = \vec{x}_i$.

Another perspective on constructing A^{-1} uses elementary matrices.

<u>Properties of the Matrix Inverse</u>

A and B are invertible $n \times n$ matrices.
* $(A^{-1})^{-1} = A$
* $(AB)^{-1} = B^{-1}A^{-1}$ (non-commutative)
* $(A^T)^{-1} = (A^{-1})^T$

$C = (AB)^{-1}$. What is C?

If $C = B^{-1}A^{-1}$, then:

$CAB = B^{-1}A^{-1}AB$
$= B^{-1}IB = B^{-1}B$
$= I$.

$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

True or False: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Let $X = AB$. $(ABC)^{-1} = (XC)^{-1}$
$$= C^{-1}X^{-1} = C^{-1}(AB)^{-1}$$
$$= C^{-1}(B^{-1}A^{-1}) = C^{-1}B^{-1}A^{-1} \text{ (QED)}. \therefore \text{True}.$$

Elementary Matrices

An elementary matrix, E, is one that differs by I_n by one row operation.

Recall our elementary row operations
- → swap rows
- → multiply a row by a nonzero scalar
- → add a multiple of one row to another.

We can represent each operation by a matrix multiplication with an elementary matrix.

Note that: (1) every E is invertible
(2) every E is square

Example: suppose $E\begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

By inspection, what is E? How does it compare to I_3?

E must be 3X3. $\begin{pmatrix} 1 & 1 & 1 \\ \boxed{-2 \quad 1 \quad 0} \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ \boxed{0 \quad 3 \quad 2} \\ 0 & 2 & 1 \end{pmatrix}$

applied $R_2 + 2R_1 \to R_2$.

$E = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, can check by multiplying.

NOTE: differs from I_3 by one row operation $(R_2 + 2R_1 \to R_2)$

Returning to understanding why the algorithm works, we apply a sequence of row operations to A to obtain I_n:

$$\underbrace{(E_k \cdots E_3 E_2 E_1)}_{\text{assuming } A^{-1} \text{ exists}} A = I_n$$

Thus, $\underbrace{E_k \cdots E_3 E_2 E_1}_{A^{-1}}$ is the inverse matrix we seek.

Our algorithm for calculating the inverse of a matrix is the result of the following theorem:

Theorem: Matrix A is invertible iff it is row equivalent to the identity. In this case, the any sequence of elementary row operations that transforms A into I, applied to I, generates A^{-1}.

Using The Inverse to solve a Linear System:

"Just because we can do something a certain way, does not mean that we should."

* We could use A^{-1} to solve a linear system, $A\vec{x} = \vec{b}$.
  We would calculate A^{-1} and then: $\vec{x} = A^{-1}\vec{b}$.

* A^{-1} is seldom used: computing it can take a very long time, and is prone to numerical error.
  ↳ why learn how to compute A^{-1} then? — elementary matrices and properties of A to be used to derived results