MODULE 7: DETERMINANTS

Linear Algebra 3: Determinants and Eigenvalues

TOPIC 1: Introduction to Determinants

A Definition of the Determinant

Suppose A is $n \times n$ and has elements a_{ij}.

means 1×1 matrix

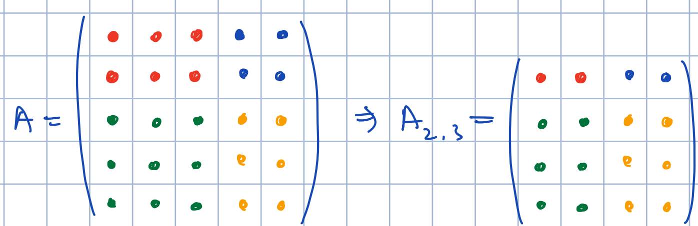
1. If $n = 1$, $A = [a_{11}]$, and has determinant $\det A = a_{11}$.

2. Inductive case: for $n > 1$,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \dots + (-1)^{1+n} a_{1n} \det A_{1n}$$

where A_{ij} is the matrix obtained by eliminating row i and column j of A.

Example:



Example: The Determinant of a 2x2 Matrix

Compute $\det A$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\det A = a A_{11} - b A_{12}$$
$$\underline{\det A = ad - bc}$$

Example: The Determinant of a 3x3 Matrix

Compute $\det \begin{pmatrix} 3 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{vmatrix} 3 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{vmatrix}$.

$$\det \begin{pmatrix} 3 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{pmatrix} = 3 \begin{vmatrix} 4 & -1 \\ 2 & 0 \end{vmatrix} - (-5) \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix}$$
$$= 3[4(0) - (-1)(2)] + 5[2(0) - (-1)(0)] + 0$$
$$= 3(2) + 5(0)$$
$$= 6$$

Exercise: Suppose A is the matrix below.

$$A = \begin{pmatrix} 0 & 4 & 3 \\ 8 & 0 & k \\ 1 & 1 & 0 \end{pmatrix}$$

For what value of k is $\det A = 0$?

$$\det A = 0. \Rightarrow 0(0-k) - 4(0-k) + 3(8-0) = 0.$$
$$4k + 24 = 0 \Rightarrow \underline{k = -6}.$$

Cofactors

Cofactors give us a more convenient notation for determinants.

DEFINITION — Cofactors

↳ The (i,j) cofactor of an $n \times n$ matrix A is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Another way to compute determinants:

Theorem:
The determinant of a matrix A can be computed down any row or column of the matrix. For instance, down the jth column, the determinant is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}.$$

Example: Cofactor Expansion

Compute the determinant of $A = \begin{pmatrix} 5 & 4 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{pmatrix}$.

Solution

$$\det(A) = 5C_{11} - 0C_{21} + 0C_{31} - 0C_{41}$$

$$= 5(-1)^{1+1}\begin{vmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= 5(0C_{13} - 0C_{23} + 3C_{33}) = 15(-1)^{3+3}\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= 15(1+2) = 45 \#$$

Determinant of Triangular Matrices

Theorem: If A is a triangular matrix then $\det A = a_{11}a_{22}a_{33}\cdots a_{nn}$.

Why? → The determinant of the 2x2 triangular matrix $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is the product of the entries on the main diagonal:

$$\begin{vmatrix} a & b \\ 0 & c \end{vmatrix} = ac - 0b = ac$$

Likewise, a 3X3 triangular matrix has the form

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

Using a cofactor expansion down the first column:

$$\det(A) = aC_{11} + 0C_{21} + 0C_{31} = aC_{11}$$
$$= a(-1)^{1+1} \left| \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \right|$$
$$= adf$$

Likewise with larger triangular matrices, the determinant will be the product of the entries on the main diagonal.

Example: compute the determinant of the 5X5 matrix. Empty entries are zero.

$$A = \begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & 1 & \\ & & & 2 & 1 \\ & & & & 2 \end{pmatrix}$$

<u>Solution</u>

The matrix is triangular, so $\det A = |A| = 2^5$.

<u>Computational Efficiency</u>

In general, the computation of a cofactor expansion for an NXN matrix requires roughly N! multiplications.

→ A 10 x 10 matrix requires roughly $10! = 3.6$ million multiplications.
→ A 20 X20 matrix requires roughly $20! \simeq 2.4 \times 10^{18}$ multiplications

Cofactor expansions may not be practical, but determinants are still useful.
→ We will explore other methods for computing determinants that are more efficient.
→ Determinants are very useful in multivariable calculus for solving certain integration problems.

Exercise: Suppose A is the triangular matrix below.

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

What is the determinant of A equal to?

$\det(A) = 2 \times 3 \times 4 \times 5 = 120$

Exercise:
Suppose X is a 2x2 matrix whose determinant is equal to 100. Suppose also that $\vec{0}$ is the zero vector with 2 entries, and A is the block matrix below.

$$A = \begin{pmatrix} \vec{0}^T & 2 \\ X & \vec{0} \end{pmatrix}$$

Calculate the value of det A. (Hint: use a cofactor expansion)

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $A = \begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ c & d & 0 \end{pmatrix}$.

$$\det(A) = 2(-1)^{1+3}\begin{vmatrix} a & b \\ c & d \end{vmatrix} + 0(-1)^{2+3}\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} + 0(-1)^{3+3}\begin{vmatrix} 0 & 0 \\ a & b \end{vmatrix}$$
$$= 2\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Given $\det(X) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 100$,
$$\Rightarrow \det(A) = 2 \times 100$$
$$= 200$$

TOPIC 2: Properties of the Determinant

"A problem isn't just finished just because you've found the right answer." — Yōko Ogawa

Understanding Relationships Between Ideas

We have a method for computing determinants, but without some of the strategies we explore in this section, the algorithm can be very inefficient.

How can we compute determinants more efficiently?
↳ we saw how determinants are difficult or impossible to compute with a cofactor expansion for large N.
↳ row operations give us a more efficient way to compute determinants

Row Operations

Theorem: Row Operations and the Determinant

Let A be a square matrix.
1. If a multiple of a row of A is added to another row to produce B, then det B = det A.
2. If two rows are interchanged to produce B, then det B = - det A.
3. If one row of A is multiplied by a scalar k to produce B, then det B = k det A.

Examples for theorem:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, |A| = ad - bc.$

① $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_2 + kR_1 \to R_2} \begin{pmatrix} a & b \\ c+ka & d+kb \end{pmatrix} = B.$

$|B| = a(d+kb) - b(c+ka)$
$= ad + \cancel{kab} - bc - \cancel{kab}$
$|B| = ad - bc \Rightarrow \det(B) = \det(A)$ *

② $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} c & d \\ a & b \end{pmatrix} = C$

$|C| = cb - ad = -(ad - bc). \Rightarrow \det(C) = -\det(A)$ *

③ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{kR_1 \to R_1} \begin{pmatrix} ka & kb \\ c & d \end{pmatrix} = D$

$|D| = kad - kbc = k(ad - bc) \Rightarrow \det(D) = k\det(A)$ *

Using Row Operations to Compute a 3X3 Determinant

Compute $\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{pmatrix} \xrightarrow[R_3 + R_1 \to R_3]{R_2 + 2R_1 \to R_2} \begin{pmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{pmatrix} = B$$

$\det A = -\det B$
$= -(1 \times 3 \times (-5)) = 15$ *

Invertibility

Important practical implication: If A is reduced to echelon form, by r interchanges of rows and columns, then

$$|A| = \begin{cases} (-1)^r \times (\text{product of pivots}), & \text{where } A \text{ is invertible} \\ 0, & \text{where } A \text{ is singular} \end{cases}$$

Properties of the Determinant

For any square matrices A and B, we can show the following:

① $\det A = \det A^T$
② A is invertible iff $\det A \neq 0$
③ $\det(AB) = \det A \times \det B$
④ if A is invertible, then $\det(A^{-1}) = \frac{1}{\det A}$.

$A^{-1}A = I$
$\det(A^{-1}A) = \det(I) = 1$
$\det A^{-1} \det A = 1 \Rightarrow \det A^{-1} = \frac{1}{\det A}$

Example: Determinants and Invertibility

Use a determinant to find all values of λ such that matrix C is not invertible.

$$C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \lambda I_3$$

$$C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$$

$$|C| = (5-\lambda)(\lambda^2-1) - 0 + 0$$
$$= (5-\lambda)(\lambda^2-1)$$

C is not invertible if $|C| = 0$. $(5-\lambda)(\lambda^2-1) = 0$

$(5-\lambda)(\lambda+1)(\lambda-1) = 0$. $\lambda = -1, 1$ or 5

$\therefore$ C is not invertible if $\lambda = 5$ or ± 1.

Example: Determinants and Matrix Powers

Determine the value of $\det A$, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & 7 & 3 \end{pmatrix}^{100}$$

$$\det A = \left| \begin{pmatrix} 0 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & 7 & 3 \end{pmatrix}^{100} \right| = \left| \begin{pmatrix} 0 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & 7 & 3 \end{pmatrix} \right|^{100}$$
$$= [0 - 1(3-2) + 0]^{100}$$
$$= (-1)^{100} = 1.$$

Exercise: Suppose A is the matrix below.

$$A = \begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Suppose also that $\det A = 5$. Determine the value of $\det B$.

$$B = \begin{pmatrix} b & a & c \\ 2 & 1 & 3 \\ 5 & 4 & 6 \end{pmatrix}$$

$$A^T = \begin{pmatrix} a & 1 & 4 \\ b & 2 & 5 \\ c & 3 & 6 \end{pmatrix} \xrightarrow{R_2 \Leftrightarrow R_1} \begin{pmatrix} b & 2 & 5 \\ a & 1 & 4 \\ c & 3 & 6 \end{pmatrix} = A^{T\prime}. \ (A^{T\prime})^T = B.$$

$\Rightarrow \det B = -\det A = -5.$

Exercise: Suppose C is the matrix below.

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

Without any computation, state the value of the determinant C.

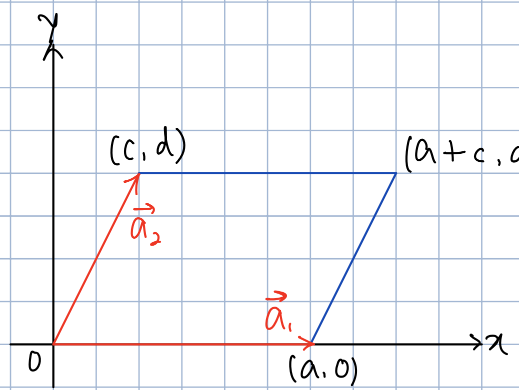
Note: Columns of C are linearly independent.
↳ C is singular and has no inverse.
$\Rightarrow \det(C) = 0$

Exercise: B and C are $n \times n$ matrices with $\det B = -3$ and $\det C = 5$.
What is the value of $\det(B^T C^2)$?

$$\det(B^T C^2) = \det B^T \times \det C \times \det C$$
$$= \det B \times (\det C)^2$$
$$= -3(5)^2 = -75 \text{ ※}$$

TOPIC 3: Determinants and the Area of a Parallelogram

Area and Determinants



Constructing a matrix whose columns are $\vec{a}_1$ and $\vec{a}_2$ yields

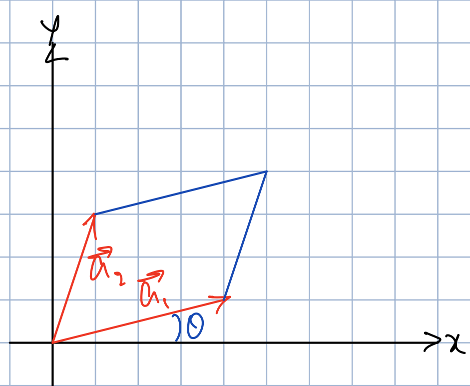
$$A = (\vec{a}_1 \ \ \vec{a}_2) = \begin{pmatrix} a & c \\ 0 & d \end{pmatrix},$$
$$|\det A| = |ad| = \text{area of parallelogram}$$

Also note that changing the value of c (and keeping everything else constant) does not change the area of the parallelogram.

More General Parallelograms

What if one side is not resting on a coordinate axes? Can we still use the determinant to give us the area of our parallelogram?

Suppose we take the parallelogram used in the previous example, but rotate it about the origin by θ radians counterclockwise. The rotation will not change the area of the region.



Will $|\det A| = |\det(\vec{a}_1 \ \ \vec{a}_2)|$ still give us the area of our parallelogram?

If one side is not resting on a coordinate axes, will absolute value of the determinant still give us the area of our parallelogram?

※ Rotating the points that define our parallelogram by angle θ about the origin will not change the area of our parallelogram.

※ We can model a rotation of the points that define our parallelogram by multiplying A with a rotation matrix.

$$A' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & c \\ 0 & d \end{pmatrix}$$

Is the determinant of this new matrix also equal to ad?

The determinant of a product of matrices is the product of the determinants.

$$\det A' = \det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \det\begin{pmatrix} a & c \\ 0 & d \end{pmatrix} = 1(ad) = ad$$

Theorem: The absolute value of the determinant of a 2x2 matrix, whose columns determine adjacent edges of a parallelogram, will give the area of the parallelogram.

Example: Area of a Parallelogram

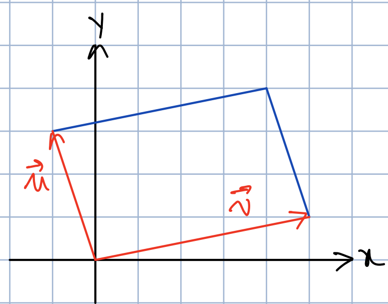
Compute the area of the parallelogram determined by $\vec{0}, \vec{u}, \vec{v}$, and $\vec{u}+\vec{v}$, where

$$\vec{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Solution

The absolute value of the determinant of a matrix whose columns are $\vec{u}$ and $\vec{v}$ yields

$$\begin{aligned} \text{Area} &= |\det A| \\ &= \left|\det\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}\right| \\ &= |-1-6| = 7 \end{aligned}$$



Exercise:

Suppose $\vec{u}$ and $\vec{v}$ are the vectors below.

$$\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 2 \\ k \end{pmatrix}$$

Suppose S is the parallelogram that is determined by the vectors $\vec{0}, \vec{u}, \vec{v}$ and $\vec{u}+\vec{v}$.

(a) If $k=3$, what is the area of S?

$$\begin{aligned} \text{Area} &= \left|\det\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}\right| \\ &= |3-6| = 3 \end{aligned}$$

(b) If $k=4$, what is the area of S?

$$\begin{aligned} \text{Area} &= \left|\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right| \\ &= |4-6| = 2 \end{aligned}$$

(c) If $k=5$, what is the area of S?

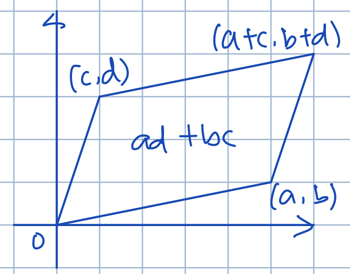
$$\begin{aligned} \text{Area} &= \left|\det\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\right| \\ &= |5-6| = 1 \end{aligned}$$

(c) If $k=6$, what is the area of S?

$$\begin{aligned} \text{Area} &= \left|\det\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\right| \\ &= |6-6| = 0 \end{aligned}$$

Area and Determinants

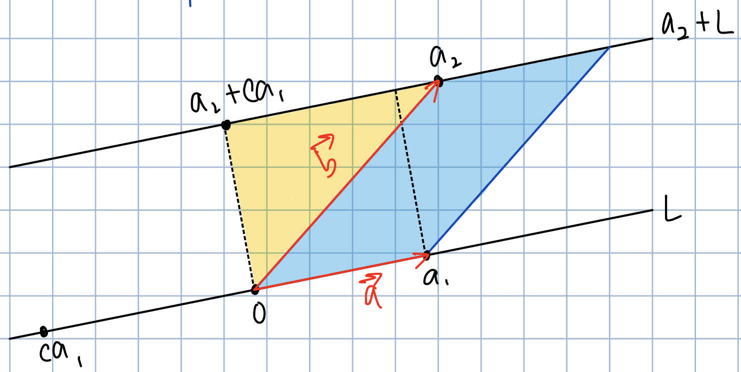
Recall that in $\mathbb{R}^2$, determinants can give us the area of a parallelogram.



Area of parallelogram

$$= \left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = ad - bc$$

Key Geometric Fact (which works in any dimension):
The area of the parallelogram spanned by 2 vectors, $\vec{a}, \vec{b}$ is equal to the area spanned by $\vec{a}$, $c\vec{a} + \vec{b}$, for any scalar c.

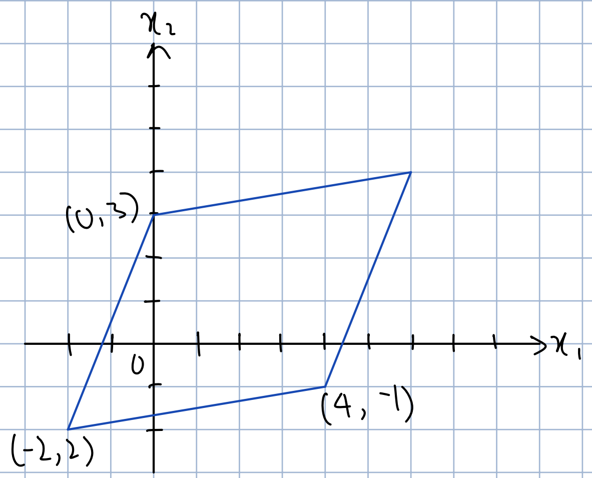


Area of parallelogram = base × height

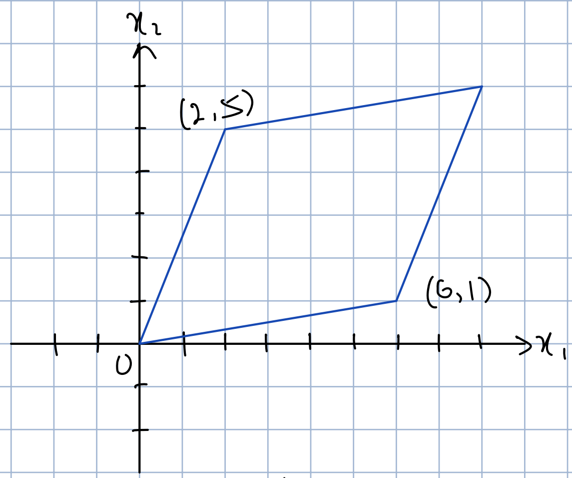
$|A| = |\vec{a} \ \vec{b}|$

Example: Area of a Parallelogram

Calculate the area of the parallelogram determined by the points $(-2,2)$, $(0,3)$, $(4,-1)$, $(6,4)$, shown in A.



(a)



(b)

Note:
Translating a region in $\mathbb{R}^2$ does not change its area.

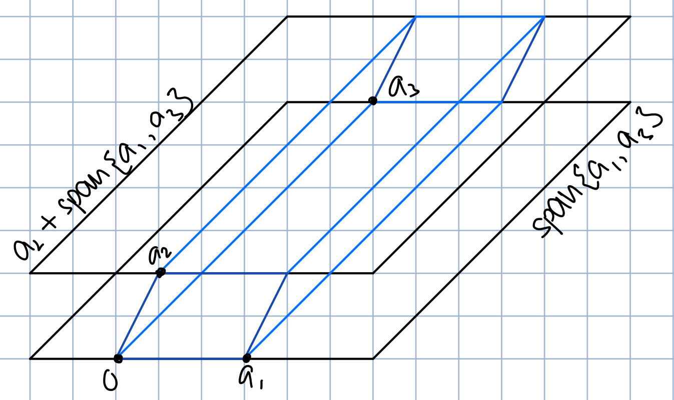
$$\text{Area} = \left| \det \begin{pmatrix} 6 & 2 \\ 1 & 5 \end{pmatrix} \right| = |30 - 2| = 28$$

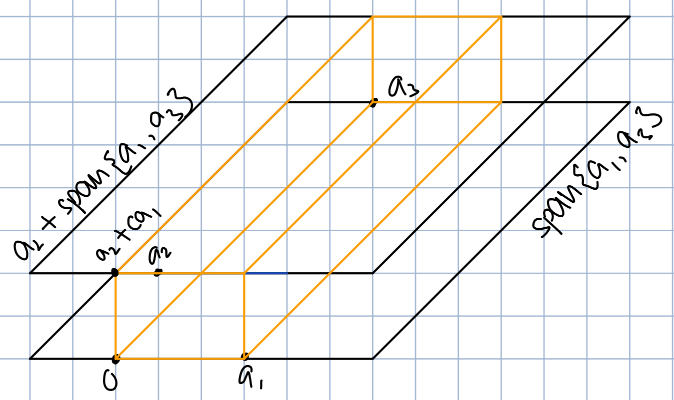
Determinants of $n \times n$ Matrices

Theorem: The volume of the parallelepiped spanned by the columns of an $n \times n$ matrix A is $|\det A|$.

Example: Volume of a Parallelepiped

Any 3×3 matrix A can be transformed into a diagonal matrix using row operations that do not change $|\det(A)|$.





Example: Volume of a Parallelpiped

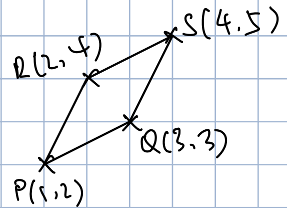
Compute the volume of the parallelepiped that has one vertex at the origin, and adjacent vertices at $(2,0,0)$, $(1,3,0)$, and $(0,1,4)$.

$$\left|\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}\right| = |2 \times 3 \times 4| = 24$$

Exercise:

Consider points $P(1,2)$, $Q(3,3)$, $R(2,4)$ and $S(4,5)$. The four points together can be connected to create parallelogram S.

What is the area of S?



Area of S

$$= \left|\begin{pmatrix} 3-1 & 2-1 \\ 3-2 & 4-2 \end{pmatrix}\right| = \left|\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right|$$

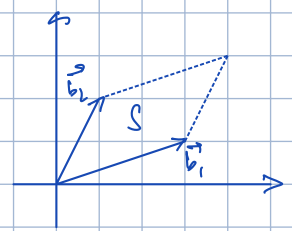
$$= |4-1| = 3$$

Determinants and Linear Transformations

Theorem:
If $T_A : \mathbb{R}^n \to \mathbb{R}^n$, and S is some parallelogram in $\mathbb{R}^n$, then

$$\text{volume}(T_A(S)) = |\det(A)| \times \text{volume}(S)$$

where $T_A(\vec{x}) = A\vec{x}$.



Any point in S can be represented as $c_1\vec{b}_1 + c_2\vec{b}_2$, $0 \le c_1, c_2 \le 1$
We transform S using T_A:

$$T_A(c_1\vec{b}_1 + c_2\vec{b}_2) = A(c_1\vec{b}_1 + c_2\vec{b}_2)$$
$$= c_1 A\vec{b}_1 + c_2 A\vec{b}_2$$

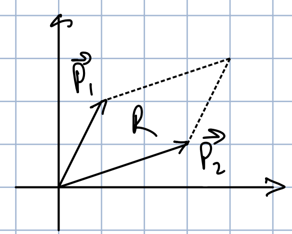
$$(A\vec{b}_1 \quad A\vec{b}_2) = A(\vec{b}_1 \quad \vec{b}_2) = AB,$$
$$|\det AB| = |\det A||\det B|$$

Example: Determinants and Linear Transformations

R is the parallelogram determined by $\vec{p}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. and $\vec{p}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. If $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, what is the area of the image of R under the map $\vec{x} \to A\vec{x}$?

$$A(\vec{p}_1 \quad \vec{p}_2) = (A\vec{p}_1 \quad A\vec{p}_2)$$

$$|\det(A\vec{p}_1 \quad A\vec{p}_2)| = |\det A \times \det(\vec{p}_1 \quad \vec{p}_2)|$$
$$= |\det A||\det(\vec{p}_1 \quad \vec{p}_2)|$$
$$= |1+1||3-8|$$
$$= 2 \times 5 = 10$$



Example:
$T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times2}$, is a linear transformation that first reflects vectors in $\mathbb{R}^2$ through the line $x_1 = x_2$, then projects them onto the line $x_1 = 0$. Calculate the value of the determinant of A.

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det A = \det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \times \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= 0(-1) = 0$$

Exercise:
$T_A(\vec{x}) = A\vec{x}$, where $A \in \mathbb{R}^{2\times2}$. Is a linear transform that rotates vectors in $\mathbb{R}^2$ clockwise by π radians about the origin, then projects them onto the x_1-axis. What is the value of $\det(A)$?

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \det(A) = \det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= 0(1) = 0.$$