MODULE 5: THE INVERTIBLE MATRIX THEOREM AND APPLICATIONS

TOPIC 1: The Invertible Matrix Theorem

"A synonym is a word you use when you can't spell the other one." — Baltasar Gracián

Theorem: The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. These statements are all equivalent.

(a) A is invertible.
(b) A is row equivalent to I_n.
(c) A has n pivotal columns (all columns are pivotal)
(d) $A\vec{x} = \vec{0}$ has only the trivial solution.
(e) The columns of A are linearly independent.
(f) The equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$.
(g) The columns of A span $\mathbb{R}^n$.
(h) There is a $n \times n$ matrix C so that $CA = I_n$ (A has a left inverse)
(i) There is a $n \times n$ matrix D so that $AD = I_n$ (A has a right inverse)
(j) A^T is invertible.

Invertibility and Composition

The diagram below gives another perspective on the role of A^{-1}:

$$\begin{tikzcd} \boxed{\vec{x}} \arrow[r, bend right, "\text{multiplication by } A"'] & \boxed{A\vec{x}} \arrow[l, bend right, "\text{multiplication by } A^{-1}"'] \end{tikzcd}$$

The matrix inverse A^{-1} transforms $A\vec{x}$ back to $\vec{x}$. This is because

$$A^{-1}(A\vec{x}) = A(A^{-1}\vec{x}) = I\vec{x} = \vec{x}$$

Items (h) and (i) of the Invertible Matrix Theorem (IMT) lead us directly to the following theorem.

Theorem: If A and B are $n \times n$ matrices and $AB = I$, then A and B are invertible, and $B = A^{-1}$ and $A = B^{-1}$.

Example 1: Identifying whether a Matrix is Invertible

Is this matrix invertible? $\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ 0 & -1 & -1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{R_2 - 3R_1 \to R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{R_3 + R_2 \to R_3} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

$\Rightarrow$ Every column is pivotal.
$\therefore$ A is invertible.

Example 2: Constructing an Expression for the Inverse

Suppose A is an invertible square matrix and

$$A^2 + 4A = I$$

Give an expression for A^{-1}.

$$A^2 + 4A = I$$
$$A(A + 4I) = I$$
$$\Rightarrow A^{-1} = A + 4I$$

Example 3: Matrix Completion Problem (MCP)

Netflix Problem is another example of MCP.

singular = not invertible

If possible, fill in the missing elements of the matrices below with numbers so that each of the matrices are $\boxed{\text{singular}}$. If this is not possible to do so, state why.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

not possible, pivot in every column.

Netflix Problem:
Given a ratings matrix in which each entry (i, j) represents the rating of movie j by customer i if customer i has watched movie j, and is otherwise missing, predict the remaining matrix entries in order to make recommendations to customers on what to watch next.

TOPIC 2: Application – Partitioned Matrices

"Mathematics is not about numbers, equations, computations, or algorithms. Mathematics is about understanding."
– William Paul Thurston

Partitioned Matrices and Matrix Multiplication

This matrix:

$$A = \begin{pmatrix} 3 & 1 & 4 & 1 & 0 \\ 1 & 6 & 1 & 0 & 1 \\ 0 & 0 & 0 & 4 & 2 \end{pmatrix}$$

can also be written as:

$$A = \begin{pmatrix} \begin{pmatrix} 3 & 1 & 4 \\ 1 & 6 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 4 & 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

We partitioned our matrix into four <u>blocks</u>, each of which has different dimensions.

Why are Partitioned Matrices useful? $\rightarrow$ Partitioned matrices give a succinct representation of a matrix.

$$\begin{pmatrix} 1 & 0 & 0 & * & \cdots & * \\ 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 1 & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} I_3 & F \\ 0 & 0 \end{pmatrix}$$

This can be useful when studying something called the <u>null space</u> of A.

Recall the Row Column Method for Matrix Multiplication:

Theorem: Let A be $m \times n$ and B be $n \times p$ matrix. Then, the (i,j) entry of AB is

$$\text{row}_i A \cdot \text{col}_j B.$$

This is the <u>Row Column Method</u> for matrix multiplication.

Example:
$$AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

<u>The Row and Column Method for Block Matrices</u>

Partitioned matrices can be multiplied using this method, as if each block were a scalar (provided each block has appropriate dimensions so that products are defined)

Example: compute the matrix product using the given partitioning.

$$AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_2 & x \end{pmatrix}\begin{pmatrix} u \\ y \end{pmatrix}$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $u = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 1 \end{pmatrix}$

$$\begin{aligned} AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} I_2 & x \end{pmatrix}\begin{pmatrix} u \\ y \end{pmatrix} \\ &= I_2 u + x y \\ &= \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} * \end{aligned}$$

How might we use this approach to determine an expression for the inverse of a matrix?

Recall, using our formula for a 2x2 matrix, $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \frac{1}{ac}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$.

Example: Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{n\times n}$ are invertible matrices. Construct the inverse of $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$.

Note: If A is a $n\times n$ matrix, $n>1$, then $\frac{1}{A}$ is undefined.

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = I = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$$

$$0W + CY = 0 \Rightarrow CY = 0 \Rightarrow C^{-1}CY = C^{-1}0 \Rightarrow Y = 0.$$
$$0X + CZ = I \Rightarrow CZ = I \Rightarrow C^{-1}CZ = C^{-1}I \Rightarrow Z = C^{-1}$$
$$AW + BY = I \Rightarrow AW = I \Rightarrow A^{-1}AW = A^{-1}I \Rightarrow W = A^{-1}$$
$$AX + BZ = 0 \Rightarrow AX = -BZ = -BC^{-1}$$
$$A^{-1}AX = -A^{-1}BC^{-1} \Rightarrow X = -A^{-1}BC^{-1}$$

$$\therefore \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

Exercise: Suppose A, B, C, X, Y and Z are all invertible $n\times n$ matrices and $XY = I_n$. Suppose also that P is equal to the matrix product below:

$$P = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}.$$

The inverse of P has the form below:

$$P^{-1} = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$$

Note, again, that 0 is the $n\times n$ zero matrix.
What is Y equal to in terms of A, B and C?

$B0 + CZ = 1$
$CZ = 1 \Rightarrow Z = C^{-1}$.

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BX + CY = 0 \Rightarrow CY = -BX \Rightarrow Y = -C^{-1}BX$$

$$AX + 0Y = 1 \Rightarrow AX = 1 \Rightarrow A^{-1}AX = A^{-1} \Rightarrow X = A^{-1}.$$
$$\therefore Y = -C^{-1}BA^{-1} \quad = ZBA^{-1}$$

TOPIC 3: Application – Solving Linear Systems with the LU Factorization

"Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity." — Alan Turing

The use of the LU Decomposition to solve linear systems was one of the areas of mathematics that Turing helped to develop. The decomposition is widely used to solve linear systems of equations.

Recall that we could solve $A\vec{x}=\vec{b}$ by using $\vec{x}=A^{-1}\vec{b}$.
However, this requires computation of the inverse of an $n\times n$ matrix, which is especially difficult for large n. Instead we could solve $A\vec{x}=\vec{b}$ with Gaussian Elimination, but this is not efficient for large n. There are more efficient and accurate methods for solving linear systems that rely on matrix factorizations.

Matrix Factorizations

Matrix factorization\decomposition: factorization of a matrix into a product of matrices.

Factorizations can be useful for solving $A\vec{x}=\vec{b}$, or understanding the properties of a matrix.

In this section, we factor a matrix into lower and into upper triangular matrices.

Triangular Matrices

Rectangular matrix A is upper triangular if $a_{i,j}=0$ for $i>j$. Examples:

$$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Elements below the main diagonal are 0.

Rectangular matrix A is lower triangular if $a_{i,j}=0$ for $i<j$. Examples:

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 4 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$$

Elements above the main diagonal are 0.

Matrix that is both upper-triangular and lower-triangular? I_n, $O_{m,n}$

Theorem: The LU Factorization

If A is an $m\times n$ matrix that can be row reduced to echelon form without row exchanges, then $A=LU$. L is a lower triangular $m\times m$ matrix with 1's on the diagonal, U is an echelon form of A.

Example: If $A\in\mathbb{R}^{3\times 2}$, the LU factorization has the form:

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}\begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \end{pmatrix}$$

Using the LU Decomposition to Solve a Linear System

Goal: given rectangular matrix A and vector $\vec{b}$, we wish to solve $A\vec{x}=\vec{b}$ for $\vec{x}$.

Algorithm: To solve $A\vec{x}=\vec{b}$ for $\vec{x}$,

1. Construct the LU decomposition of A to obtain L and U.
2. Set $U\vec{x}=\vec{y}$. Forward solve for $\vec{y}$ in $L\vec{y}=\vec{b}$.
3. Backwards solve for $\vec{x}$ in $U\vec{x}=\vec{y}$.

Example: Solve the linear system $A\vec{x}=\vec{b}$, given the LU decomposition of A.

$$A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

① Set $U\vec{x}=\vec{y}$. Solve $L\vec{y}=\vec{b}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

$1y_1 + 0y_2 + 2y_3 + 0y_4 = 2. \Rightarrow y_1 = 2.$

$1y_1 + 1y_2 + 0y_3 + 0y_4 = 3 \Rightarrow y_1 + y_2 = 3 \Rightarrow y_2 = 1.$

$2y_2 + 1y_3 = 2 \Rightarrow 2y_2 + y_3 = 2 \Rightarrow y_3 = 0$

$1y_3 + 1y_4 = 0 \Rightarrow y_3 + y_4 = 0 \Rightarrow y_4 = 0.$

$$\Rightarrow \vec{y} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

② Solve $U\vec{x}=\vec{y}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + 0x_2 + 0x_3 = 2 \Rightarrow x_1 = 2.$

$2x_2 + x_3 = 1$

$2x_3 = 0 \Rightarrow x_3 = 0.$

$\Rightarrow 2x_2 = 1 \Rightarrow x_2 = \frac{1}{2}.$

$$\Rightarrow \vec{x} = \begin{pmatrix} 2 \\ 1/2 \\ 0 \end{pmatrix}$$

Exercise: Suppose A as LU factorization below:

$$A = \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{L}\underbrace{\begin{pmatrix} 4 & 5 \\ 0 & 2 \end{pmatrix}}_{U}$$

In this exercise, we wish to solve the system $A\vec{x}=\vec{b}$, for $\vec{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

To solve this linear system using the LU factorization, we first set $U\vec{x}=\vec{y}$. This gives us the system $L\vec{y}=\vec{b}$, where $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. What does y_2 have to be equal to?

$L\vec{y} = \vec{b}$. $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

$y_1 = 1$.
$2y_1 + y_2 = 4 \Rightarrow \underline{\underline{y_2 = 2}}$.

$$\vec{y} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

To solve this linear system using the LU factorization, we next solve $U\vec{x} = \vec{y}$, where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. What does x_1 have to be equal to?

$U\vec{x} = \vec{y}$. $\begin{pmatrix} 4 & 5 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$2x_2 = 2 \Rightarrow x_2 = 1$
$4x_1 + 5x_2 = 1 \Rightarrow 4x_1 + 5 = 1$.
$\underline{\underline{x_1 = -1}}$.

Why We Can Compute the LU Factorization

Suppose A can be row reduced to echelon form U without interchanging rows.

$$\underbrace{E_p \cdots E_1}_{L^{-1}} A = U \Rightarrow L^{-1}A = U \Rightarrow LL^{-1}A = LU.$$

where the E_j are matrices that perform elementary row operations. Because we did not swap rows, each E_j happens to be lower triangular and invertible.

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

Therefore, $A = \underbrace{E_1^{-1} \cdots E_p^{-1}}_{=L} U = LU$.

An algorithm for computing LU

To compute the LU decomposition:

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Example: Compute the LU factorization of A.

$$A = \begin{pmatrix} 4 & -3 & -1 & 5 \\ -16 & 12 & 2 & -17 \\ 8 & -6 & -12 & 22 \end{pmatrix} \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 + 4R_1 \to R_2} \begin{pmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & -10 & 12 \end{pmatrix}$$

$$\xrightarrow{R_3 - 5R_2 \to R_3} \begin{pmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -3 \end{pmatrix} = U.$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix} \quad \therefore A = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix} \begin{pmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Notes on LU computation:

↳ There are other definitions of the LU factorization that you may encounter in future courses or applications.

↳ There are several other ways of computing this decomposition.

↳ The only row operation we use to construct L and U: replace a row with a multiple of a row above it.

↳ As for the other two row operations:

* multiplying a row by a nonzero scalar is not needed.
* we cannot swap rows: more advanced linear algebra and numerical analysis courses address this limitation.

Exercise: Suppose A is the matrix below:

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}.$$

The LU factorization of this matrix has the form:

$$A = LU = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & k \end{pmatrix}.$$

What do h and k need to be equal to?

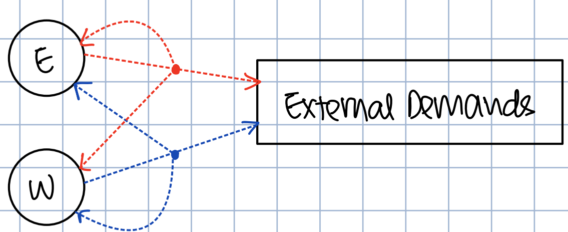
$$A = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix} \xrightarrow{R_2 - 2R_1 \to R_2} \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = U. \quad \therefore k = 3$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2 + 2R_1 \to R_2} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = L. \quad \therefore h = 2$$

TOPIC 4: The Leontif Input-Output Model

"Computers and robots replace humans in the exercise of mental functions in the same way as mechanical power replaced them in the performance of physical tasks." —Wassily Leontif, 1983

Example: An Economy with Two Sectors



* this economy contains two sectors: electricity (E) and water (W)
* external demands do not produce E and W
* how can we represent this economy with a set of linear equations?

The Leontif Model: Output Vector

Suppose economy has N sectors, with outputs measured by $\vec{x} \in \mathbb{R}^N$, where

$$\vec{x} = \text{output vector}$$
$$x_i = \text{entry } i \text{ of vector } \vec{x}$$
$$= \text{number of units produced by sector } i$$

The Leontif Model: Internal Consumption

The consumption matrix, C, describes how units are consumed by sectors to produce output.

Two equivalent ways of defining entries of C:

* sector i sends a proportion of its units to sector j, call it $c_{i,j} x_i$

* sector j requires a proportion of the units created by sector i, call it $c_{i,j} x_i$

Entries of C are $c_{i,j}$ with $c_{i,j} \in [0,1]$ and

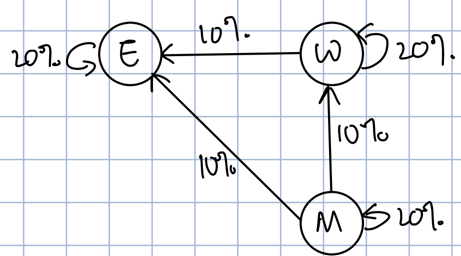
$$C\vec{x} = \text{units consumed}$$
$$\vec{x} - C\vec{x} = \text{units left after internal consumption}$$

Example (with 3 sectors):

An economy contains three sectors, E, W, and M.

For every 100 units of output,
* E requires 20 units from E, 10 units from W, and 10 units from M.
* W requires 0 units from E, 20 units from W, and 10 units from M.
* M requires 0 units from E, 0 units from W, and 20 units from M.

If the output vector is $\vec{x} = \begin{pmatrix} x_E \\ x_W \\ x_M \end{pmatrix}$, construct the consumption matrix for this economy.



$$\text{Units consumed} = C\vec{x}$$
$$= x_E \begin{pmatrix} 0.2 \\ 0.1 \\ 0.1 \end{pmatrix} + x_W \begin{pmatrix} 0 \\ 0.2 \\ 0.1 \end{pmatrix} + x_M \begin{pmatrix} 0 \\ 0 \\ 0.2 \end{pmatrix}$$
$$= \begin{pmatrix} 0.2 & 0 & 0 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0.1 & 0.2 \end{pmatrix} \vec{x}$$

$\therefore$ Consumption matrix is $C = \frac{1}{10} \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

Note:
* total output for each sector is the sum along the outgoing edges for each sector, which generates rows of C.
* elements of C represent percentages with no units, they have values between 0 and 1.
* our output vector has units.

The Leontif Model: Demand

There is also an external demand given by $\vec{d} \in \mathbb{R}^n$. We ask if there is an $\vec{x}$ such that

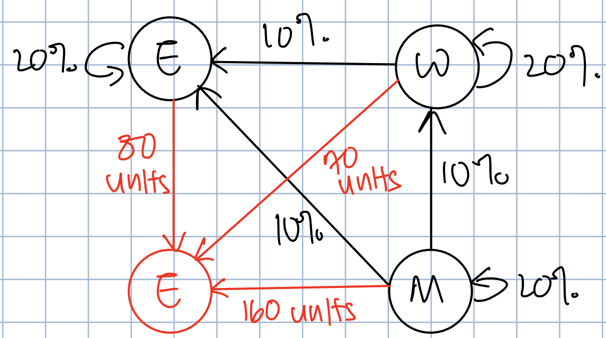
$$\vec{x} - C\vec{x} = \vec{d}.$$

We can rewrite this as $(I - C)\vec{x} = \vec{d}$. $\Rightarrow \vec{x} = (I-C)^{-1}\vec{d}$.

This matrix equation is the <u>Leontif Input-Output Model</u>. Solving for $\vec{x}$ gives the output that meets external demand exactly.

Example Revisited:

Now suppose there is an external demand: what production level is required to satisfy a final demand of 80 units of E, 70 units of W, and 160 units of M?



The production level would be found by solving:

$$(I - C)\vec{x} = \vec{d}$$

$$\frac{1}{10}\begin{pmatrix} 10-2 & 0-0 & 0-0 \\ 0-1 & 10-2 & 0-0 \\ 0-1 & 0-1 & 10-2 \end{pmatrix}\vec{x} = \begin{pmatrix} 80 \\ 70 \\ 160 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 0 & 0 \\ -1 & 8 & 0 \\ -1 & -1 & 8 \end{pmatrix}\vec{x} = \begin{pmatrix} 800 \\ 700 \\ 1600 \end{pmatrix}$$

$8x_E = 800 \Rightarrow x_E = 100$

$-x_E + 8x_W = 700$

$8x_W = 800 \Rightarrow x_W = 100$

$-x_E - x_W + 8x_M = 1600$

$8x_M = 1800 \Rightarrow x_M = 225$

Exercise: Consider the production model $\vec{x} = C\vec{x} + \vec{d}$

for an economy with two sectors, where

$$C = \frac{1}{10}\begin{pmatrix} 2 & 6 \\ 4 & 2 \end{pmatrix}, \quad \vec{d} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}.$$

The augmented matrix which can be used to solve the system for $\vec{x}$ has the form below:

$$\left(\begin{array}{cc|c} .8 & a & 8 \\ b & .8 & 2 \end{array}\right).$$

What does a equal to? Upon solving this linear system, what is x_1 equal to?

$$\vec{x} - C\vec{x} = \vec{d}$$

$$(I - C)\vec{x} = \vec{d}$$

$$\frac{1}{10}\begin{pmatrix} 10-2 & -6 \\ -4 & 10-2 \end{pmatrix}\vec{x} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 8 & -6 \\ -4 & 8 \end{pmatrix}\vec{x} = \begin{pmatrix} 80 \\ 20 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 8 & -6 & 80 \\ -4 & 8 & 20 \end{array}\right) \qquad \therefore a = -0.6$$

$$\xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{cc|c} 4 & -3 & 40 \\ -4 & 8 & 20 \end{array}\right)$$

$$\xrightarrow{R_2 + R_1 \to R_2} \left(\begin{array}{cc|c} 4 & -3 & 40 \\ 0 & 5 & 60 \end{array}\right)$$

$$5x_2 = 60 \Rightarrow x_2 = 12$$

$$4x_1 - 3x_2 = 40$$

$$4x_1 - 3(12) = 40$$

$$4x_1 = 76 \Rightarrow x_1 = 19$$