MODULE 10: ORTHOGONAL PROJECTIONS

TOPIC 1: Dot Products and Length

The Dot Product

The dot product between two vectors, $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, is defined as

$$\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v} = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

Example: For what values of k is $\vec{u}\cdot\vec{v} = 0$?

$$\vec{u} = \begin{pmatrix} -1 \\ k \\ 2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$0 = -1(4) + k + 2(3)$$
$$= -4 + k + 6$$
$$\underline{k = -2}$$

Theorem: Dot Product Identities
Let $\vec{u}, \vec{v}, \vec{w}$ be three vectors in $\mathbb{R}^n$, and $c \in \mathbb{R}$.

Linearity: $(\vec{v} + \vec{w})\cdot\vec{u} = \vec{v}\cdot\vec{u} + \vec{w}\cdot\vec{u}$
Scalars: $(c\vec{u})\cdot\vec{w} = c(\vec{u}\cdot\vec{w})$
Symmetry: $\vec{u}\cdot\vec{w} = \vec{w}\cdot\vec{u}$
Positivity: $\vec{u}\cdot\vec{u} \geq 0$
$\vec{u}\cdot\vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$

$*$ $\vec{u}\cdot\vec{u}$ is a sum of squares

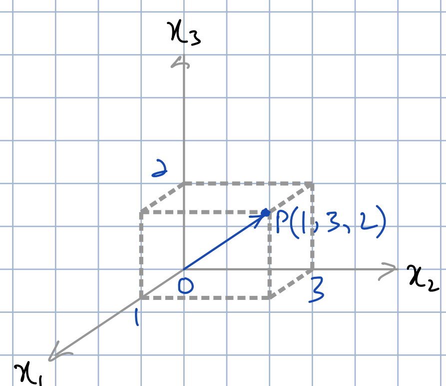
The Length of a Vector

DEFINITION
↳ The length of a vector $\vec{u} \in \mathbb{R}^n$ is

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

Example:
If P is the point $(1, 3, 2)$, the length of vector $\overrightarrow{OP}$ is

$$\sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$



Example:
Let $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^n$ with $\|\vec{u}\| = 5$, $\|\vec{v}\| = \sqrt{3}$, and $\vec{u} \cdot \vec{v} = -1$. Compute the value of $\|\vec{u} + \vec{v}\|$.

$$\begin{aligned}
\|\vec{u} + \vec{v}\|^2 &= \left(\sqrt{(\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v})}\right)^2 \\
&= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\
&= \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \\
&= 5^2 + 2(-1) + (\sqrt{3})^2 \\
&= 25 - 2 + 3 = 26. \\
\therefore \|\vec{u} + \vec{v}\| &= \sqrt{26}.
\end{aligned}$$

Length of Vectors and Unit Vectors

Note: for any vector $\vec{v}$ and scalar c, the length of $c\vec{v}$ is

$$\|c\vec{v}\| = |c| \, \|\vec{v}\|$$

DEFINITION
↳ If $\vec{v} \in \mathbb{R}^n$ has length one, we say that it is a unit vector.

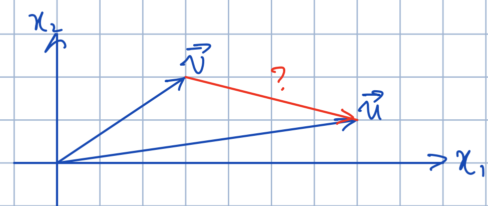
For example, each of the following vectors are unit vectors.

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{y} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{v} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Distance in $\mathbb{R}^n$

DEFINITION
↳ For $\vec{u}, \vec{v} \in \mathbb{R}^n$, the distance between $\vec{u}$ and $\vec{v}$ is given by the formula $\|\vec{u} - \vec{v}\|$.

Example:
Compute the distance from $\vec{u} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

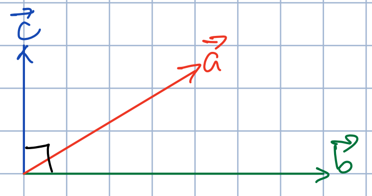


$$\begin{aligned}
\|\vec{u} - \vec{v}\| &= \left\|\begin{pmatrix} 4 \\ -1 \end{pmatrix}\right\| \\
&= \sqrt{4^2 + (-1)^2} = \sqrt{17}
\end{aligned}$$

Angles

Theorem: $\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$. Thus, if $\vec{a} \cdot \vec{b} = 0$, then:
* $\vec{a}$ and\or $\vec{b}$ are zero vectors, or
* $\vec{a}$ and $\vec{b}$ are perpendicular to each other,
or both.
↳ or orthogonal

For example, consider the vectors below:



Exercise:
Suppose $\vec{u}$ and $\vec{v}$ are the vectors below.

$$\vec{u}=\begin{pmatrix}12\\5\end{pmatrix},\quad \vec{v}=\begin{pmatrix}18\\-3\end{pmatrix}$$

What is the distance between these two vectors?

$$\|\vec{u}-\vec{v}\|$$
$$=\left\|\begin{pmatrix}-6\\8\end{pmatrix}\right\| = \sqrt{(-6)^2+8^2}$$
$$=\sqrt{100} = 10 \text{ units}$$

Exercise:
Suppose $\|\vec{u}\|=3$, $\|\vec{v}\|=4$ and the angle between the two vectors is $\frac{\pi}{2}$ radians.

What is $\|\vec{u}+\vec{v}\|$ equal to?

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= \left(\sqrt{(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})}\right)^2\\ &= \vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v} = \|\vec{u}\|^2+2(\vec{u}\cdot\vec{v})+\|\vec{v}\|^2\\ &= 3^2+2(0)+4^2\\ &= 5^2\\ \|\vec{u}+\vec{v}\| &= 5\end{aligned}$$

$\because$ If $\vec{u}\perp\vec{v}$, $\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\frac{\pi}{2}$
$=\|\vec{u}\|\,\|\vec{v}\|\times 0 = 0$

Orthogonality

DEFINITION — Orthogonal Vectors
↳ Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u}\cdot\vec{v}=0$. This is equivalent to:

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2+\|\vec{v}\|^2$$

This is because:

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})\\ &= \vec{u}\cdot\vec{u}+\vec{u}+\vec{v}+\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v}\\ &= \|\vec{u}\|^2+\|\vec{v}\|^2+2\vec{u}\cdot\vec{v}\end{aligned}$$

But if $\vec{u}$ and $\vec{v}$ are orthogonal, $\vec{u}\cdot\vec{v}=0$.

Orthogonality and the Pythagorean Theorem

If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, the expression

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2+\|\vec{v}\|^2$$

is an n-dimensional version of the Pythagorean Theorem.

Example: If $\vec{u}=\begin{pmatrix}1\\1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}$, then $\vec{u}\cdot\vec{v}=0$, and

$$\|\vec{u}+\vec{v}\|^2 = \left\|\begin{pmatrix}2\\0\end{pmatrix}\right\|^2 = \left(\sqrt{2^2+0^2}\right)^2 = 4$$
$$\|\vec{u}\|^2+\|\vec{v}\|^2 = 2+2 = 4$$

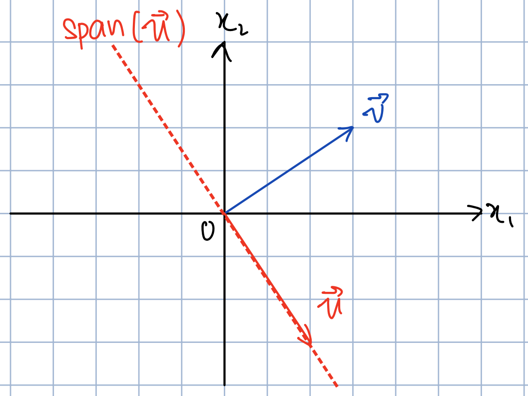
The 2-dimensional Pythagorean Theorem is satisfied.

Orthogonality and the zero vector

* The zero vector in $\mathbb{R}^n$ is orthogonal to every vector $\mathbb{R}^n$.
* We usually mean nonzero vectors when discussing orthogonality.

Example:

Sketch the set of all vectors that are orthogonal to $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Is our set also a subspace?



$\vec{v} \cdot \vec{u} = 0$

$\vec{u} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

Exercise:

Suppose that $\vec{u}$ and $\vec{v}$ are orthogonal vectors, and $\|\vec{u}\| = 5$, $\|\vec{v}\| = 12$.
What is $\|\vec{u} + \vec{v}\|$ equal to?

$$\|\vec{u} + \vec{v}\| = \sqrt{5^2 + 12^2} = 13$$

Exercise:

Suppose $\vec{u}$ is the vector below.

$$\vec{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

A basis for the set of all vectors that are orthogonal to $\vec{u}$ is given by $\vec{w}$, where:

$$\vec{w} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

If $x_1 = 1$, what does x_2 need to be equal to?

$\vec{u} \cdot \vec{w} = 0.$

$3x_1 + x_2 = 0$

$\therefore$ Given $x_1 = 1$, $x_2 = -3$

Orthogonal Compliments

DEFINITIONS

↳ Let W be a subspace of $\mathbb{R}^n$. Vector $\vec{z} \in \mathbb{R}^n$ is orthogonal to W if $\vec{z}$ is orthogonal to every vector in W.

↳ The set of all vectors orthogonal to W is a subspace, the orthogonal compliment of W, or $W^{\perp}$.

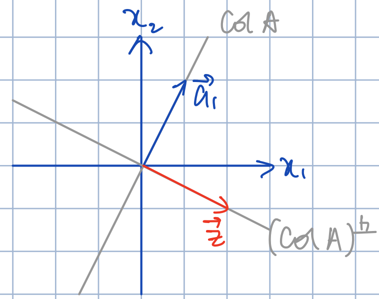
$$W^{\perp} = \{\vec{z} \in \mathbb{R}^n : \vec{z} \cdot \vec{w} = 0 \ \forall \ \vec{w} \in W\}$$

Example: $(\text{Col } A)^\perp$

suppose $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$.

Col A is the span of $\vec{a}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$(\text{Col } A)^\perp$ is the span of $\vec{z} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.



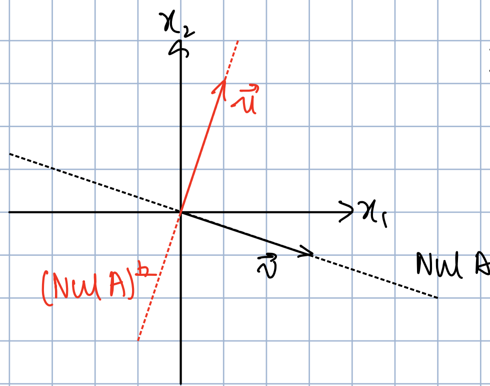
Example: $(\text{Nul } A)^\perp$

For $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$, sketch Nul A and $(\text{Nul } A)^\perp$ on the grid below.

$\vec{v} \in \text{Nul } A$, $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

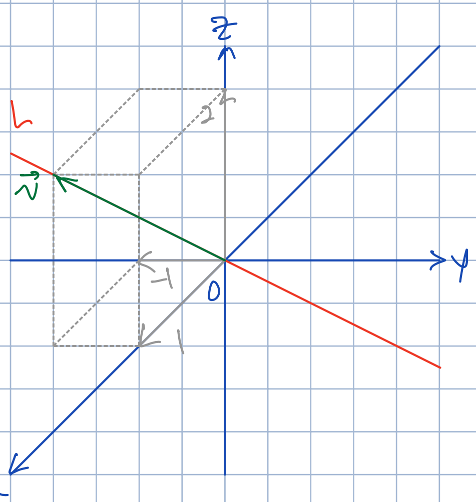


If $\vec{u} \cdot \vec{v} = 0$, then $\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Example: Equation of a Plane

Line L is a subspace of $\mathbb{R}^3$ spanned by $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Then the space $L^\perp$ is a plane.

Construct an equation of the plane $L^\perp$.



If $\vec{u} \in L^\perp$, and $\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$,

then $\vec{u} \cdot \vec{v} = x - y + 2z = 0$.

$z = -\frac{1}{2}x + \frac{1}{2}y$

Exercise:

Suppose W is spanned by vector $\vec{u}$ where:

$\vec{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

If $W^\perp$ is spanned by the vector $\vec{v}$, where

$\vec{v} = \begin{pmatrix} 1 \\ k \end{pmatrix}$

What does k need to be?

$W \cdot W^\perp = 0$.

$\begin{pmatrix} 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ k \end{pmatrix} = 0$.

$4 + k = 0 \Rightarrow k = -4$

Subspace W is the span of vector u, where:

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

What dimension of W equal to?

The dimension of a subspace is the number of vectors needed to form a basis for that space. In this case, one vector spans W, so there is only one vector in the basis.
$\therefore \dim(W) = 1$.

What dimension of $W^{\perp}$ equal to?

If $\vec{x} \in W^{\perp}$, then $\vec{x} \cdot \vec{u} = 0$.

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = x_1 + 2x_2 = 0. \Rightarrow x_1 = -2x_2$.
x_3 is free.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

There are 2 independent vectors in our parametric vector form, they form a basis for $W^{\perp}$, and so $\dim(W^{\perp}) = 2$.

The Four Fundamental Subspaces

DEFINITION
↳ Row A is the space spanned by the rows of matrix A.

We can show that:
* a basis for Row A is given by the pivot rows of A.
* $\dim(\text{Row } A) = \dim(\text{Col } A)$
* $\text{Row } A = \text{Col } A^T$
* in general, Row A and Col A are not related to each other.

Recall: if N is the number of columns in a matrix, that

$$N = \dim(\text{Col } A) + \dim(\text{Nul } A)$$

but if $\dim(\text{Row } A) = \dim(\text{Col } A)$, then we could express this as

$$N = \dim(\text{Row } A) + \dim(\text{Nul } A)$$

In fact, there are many other equivalent ways of expressing the above theorem.

① Relationships between Row A and Nul A:

Suppose vector $\vec{v}$ is in Nul A. Then $A\vec{v} = \vec{0}$.

We compute $A\vec{v}$ by taking dot products between each row of A and $\vec{v}$.
All of these dot products are zero, so $\vec{v}$ is orthogonal to the rows of A.

$\therefore$ Row A is orthogonal to Nul A. In other words, $(\text{Row } A)^{\perp} = \text{Nul } A$, or
$\text{Row } A = (\text{Nul } A)^{\perp}$

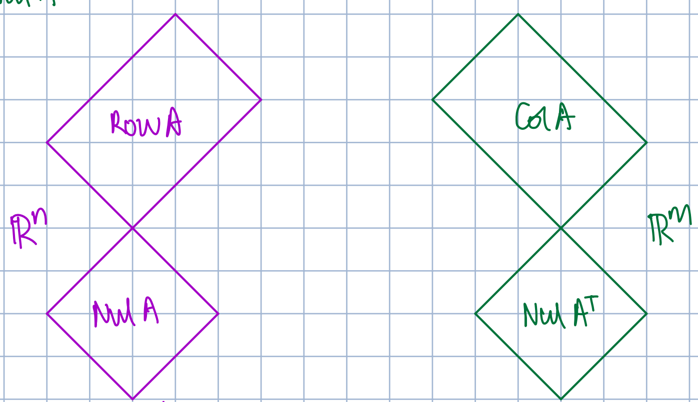
② Relationships between Col A and Nul A^T:

There is a similar relationship between Col A and Nul A^T.
Suppose vector $\vec{x}$ is in Nul A^T. Then $A^T\vec{x} = \vec{0}$.

This implies that → $\vec{x}$ is orthogonal to the rows of A^T
↳ $\vec{x}$ is orthogonal to the columns of A.

$\therefore$ Col A is orthogonal to Nul A^T. In other words, $(\text{Col } A)^{\perp} = \text{Nul } A^T$

Theorem (The Four Subspaces):
For any $A \in \mathbb{R}^{m \times n}$, the orthogonal complement of Row A is Nul A, and the orthogonal complement of Col A is Nul A^T.



Vectors in Col A and Nul A^T are in $\mathbb{R}^m$.

Vectors in Row A and Nul A are in $\mathbb{R}^n$.

Example: Suppose $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Construct a basis for the following:

(a) Row A $\quad \{(1 \ \ 3 \ \ 0), (0 \ \ 0 \ \ 1)\}$

(c) Col A $\quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

(b) $(\text{Row } A)^{\perp} = \text{Nul } A$, $\left\{ \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \right\}$

(d) $(\text{Col } A)^{\perp}$ $\quad \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Exercise: Suppose A is 30×10 and $\dim((\text{Row } A)^{\perp}) = 2$. What is the rank of A equal to?

$$\begin{aligned}\#\text{columns of } A &= \dim(\text{Col } A) + \dim(\text{Nul } A)\\ &= \text{rank } A + \dim((\text{Row } A)^{\perp})\\ 10 &= \text{rank } A + 2\\ \therefore \text{rank } A &= 8\end{aligned}$$

$\because (\text{Row } A)^{\perp} = \text{Nul } A$, $\text{rank } A = \dim(\text{Col } A)$

Exercise: If A is 20×25 and $\dim((\text{Col } A)^{\perp}) = 8$, what is the rank of A equal to?

$\dim((\text{Col } A)^{\perp}) = \dim(\text{Nul } A^T) = \#$ non-pivot columns of A^T. $= 8$

A is $20\times 25 \Rightarrow A^T$ is 25×20.

$\Rightarrow \#$ pivot columns in $A^T = 20 - 8 = 12$

$\Rightarrow A$ has 12 pivot rows.

$\therefore \text{rank } A = \#$ pivot columns in $A = \#$ pivot rows in $A = 12$ ※

TOPIC 2: Orthogonal Bases

Orthogonal Vector Sets

DEFINITION

↳ A set of vectors $\{\vec{u}_1, \dots, \vec{u}_p\}$ are an orthogonal set of vectors if for each $j \neq k$, $\vec{u}_j \perp \vec{u}_k$.

Example:

Fill in the missing entries to make $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a set of nonzero orthogonal vectors.

$$\vec{u}_1 = \begin{pmatrix}4\\0\\1\end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix}-2\\0\\a\end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix}0\\b\\c\end{pmatrix}$$

We require $\vec{u}_1 \cdot \vec{u}_2 = 0$:

$$4(-2) + 0(0) + 1(a) = 0. \Rightarrow a = 8.$$

$$\Rightarrow \text{we need } \vec{u}_2 = \begin{pmatrix}-2\\0\\8\end{pmatrix}.$$

We also need $\vec{u}_3 \cdot \vec{u}_1 = \vec{u}_3 \cdot \vec{u}_2 = 0$.

$0(4) + b(0) + c(1) = 0 \Rightarrow c = 0$.

$0(-2) + b(0) + 0(8) = 0 \Rightarrow 0b = 0$. b is free.

$$\Rightarrow \text{We can set } \vec{u}_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Orthogonality and Linear Independence

Theorem: Let $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal set of vectors. If $\vec{u}_i$ are nonzero, then the S is a set of linearly independent vectors.

Proof: Suppose $\{\vec{u}_1, \ldots, \vec{u}_p\}$ are a set of nonzero orthogonal vectors, and

$$\sum_{i=1}^{p} c_i \vec{u}_i = c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_p\vec{u}_p = \vec{0}$$

for scalars $c_1, c_2, \ldots, c_p$. Then

$$\begin{aligned} 0 &= \vec{u}_1 \cdot \vec{0} \\ &= \vec{u}_1 \cdot \sum_{i=1}^{p} c_i \vec{u}_i \\ &= \sum_{i=1}^{p} c_i \vec{u}_i \cdot \vec{u}_1 \\ &= c_1 \vec{u}_1 \cdot \vec{u}_1 \end{aligned}$$

But $\vec{u}_1$ is not the zero vector, so $c_1 = 0$.
Likewise, $c_2, c_3, \ldots, c_p$ must also be zero, which means that S is a set of linearly independent vectors.

Orthogonal Bases

Theorem: Expansion in Orthogonal Basis

Let $\{\vec{u}_1, \ldots, \vec{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then, for any vector $\vec{w} \in W$,

$$\vec{w} = c_1\vec{u}_1 + \ldots + c_p\vec{u}_p$$

Above, the scalars are $c_q = \dfrac{\vec{w} \cdot \vec{u}_q}{\vec{u}_q \cdot \vec{u}_q}$

Obtaining the coefficients, c_q, with the above formula is generally more efficient than row reduction. However, we can only apply this theorem when we have an orthogonal basis for W.

Example: Orthogonal Bases

Suppose W is the subspace of $\mathbb{R}^3$ that is orthogonal to $\vec{x}$.

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{s} = \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$$

1. Confirm that an orthogonal basis for W is given by $\vec{u}$ and $\vec{v}$.

For $\vec{u}$ and $\vec{v}$ to form an orthogonal basis for W, $\vec{u}$ and $\vec{v}$ must be in W and orthogonal.

$\vec{x} \cdot \vec{u} = 1(1) + 1(-2) + 1(1) = 0.$
$\vec{x} \cdot \vec{v} = 1(-1) + 1(0) + 1(1) = 0.$
$\Rightarrow \vec{u}$ and $\vec{v}$ are in W.

$\vec{u} \cdot \vec{v} = 1(-1) - 2(0) + 1(1) = 0.$
$\Rightarrow \vec{u}$ and $\vec{v}$ are orthogonal.

2. Assume $\vec{s} \in W$. Compute the expansion of $\vec{s}$ in the basis for W.

Given that $\vec{s}$ is in W, and we have an orthogonal basis for W, we can use our theorem to write:

$$\vec{s} = \frac{\vec{s}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} + \frac{\vec{s}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$$

$$= \frac{3(1)-4(-2)+1(1)}{1(1)-2(-2)+1(1)}\vec{u} + \frac{3(-1)-4(0)+1(1)}{-1(-1)+0(0)+1(1)}\vec{v} = \frac{12}{6}\vec{u} + \frac{-2}{2}\vec{v}$$

$$\therefore \vec{s} = 2\vec{u} - \vec{v}$$

DEFINITION — Orthonormal Basis

↳ An <u>orthonormal basis</u> for a subspace W is an orthogonal basis $\{\vec{u}_1, \ldots, \vec{u}_p\}$ in which every vector $\vec{u}_k$ has unit length. In this case, for each $\vec{w} \in W$,

$$\vec{w} = (\vec{w}\cdot\vec{u}_1)\vec{u}_1 + \cdots + (\vec{w}\cdot\vec{u}_p)\vec{u}_p$$

$$\|\vec{w}\| = \sqrt{(\vec{w}\cdot\vec{u}_1)^2 + \cdots + (\vec{w}\cdot\vec{u}_p)^2}$$

Example: Orthonormal Basis

W is a subspace of $\mathbb{R}^3$ that is perpendicular to x. Calculate the missing coefficients in the orthonormal basis for W, which is $\{u, v\}$.

$$\vec{x} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \vec{u} = \frac{1}{\sqrt{}}\begin{pmatrix}1\\0\\\end{pmatrix}, \quad \vec{v} = \frac{1}{\sqrt{}}\begin{pmatrix}\\\\\end{pmatrix}$$

For $\vec{u}$ to be in W, we need $\vec{x}\cdot\vec{u} = 0$.

Set $\vec{u} = \frac{1}{\sqrt{a}}\begin{pmatrix}1\\0\\b\end{pmatrix} \Rightarrow \vec{x}\cdot\vec{u} = \frac{1}{\sqrt{a}}(1+0+b) = 0 \Rightarrow b = -1$

$\therefore$ For $\vec{u}$ to have unit length, we need $a = 2$

For $\vec{u}$ and $\vec{v}$ to form an orthonormal basis, we need $\vec{x}\cdot\vec{u} = 0$.

Set $\vec{v} = \frac{1}{\sqrt{k}}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} \Rightarrow \vec{u}\cdot\vec{v} = \frac{1}{\sqrt{2k}}(c_1 - c_3) = 0 \Rightarrow c_1 = c_3$

If $\vec{v}$ is in W we need $\vec{x}\cdot\vec{v} = 0$.

$$\vec{x}\cdot\vec{v} = \frac{1}{\sqrt{k}}(c_1 + c_2 + c_3) = 0.$$

$$2c_1 + c_2 = 0 \Rightarrow c_2 = -2c_1.$$

Choosing $c_1 = 1$, then $c_2 = -2$ and $c_3 = 1$.

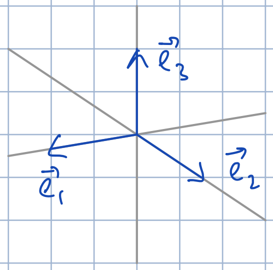
Our vectors are:

$$\vec{x} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \vec{u} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad \vec{v} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

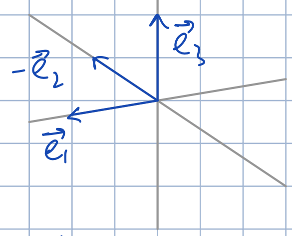
As required, $\vec{u}$ and $\vec{v}$ are orthonormal, and they form a basis for W, which is the set of vectors orthogonal to $\vec{x}$.

Note: Bases are not unique.

For example, any vector $\vec{w} \in \mathbb{R}^3$ can be written as a linear combination of $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, or any other orthogonal basis for $\mathbb{R}^3$.



an orthogonal basis for $\mathbb{R}^3$



another orthogonal basis for $\mathbb{R}^3$

Exercise:

Suppose $\vec{u}$ and $\vec{v}$ below, give an orthogonal basis for subspace W.

$$\vec{u} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}.$$

Suppose also that $\vec{x} \in W$ is the vector below.

$$\vec{x} = \begin{pmatrix} 8 \\ 11 \\ 11 \end{pmatrix}$$

Because we know that $x \in W$, we should be able to determine c_1 and c_2 so that

$$c_1\vec{u} + c_2\vec{v} = \vec{x}$$

We could set up an augmented matrix and row reduce to determine c_1 and c_2, but we also know that $\vec{u}$ and $\vec{v}$ are orthogonal, so there is a more efficient approach.

Compute the ratio of dot products $\frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$. What is the ratio equal to?

$$\frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} = \frac{8(4) + 11(1) + 11(1)}{4^2 + 1^2 + 1^2} = \frac{54}{18} = 3$$

Compute the ratio of dot products $\frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$. What is the ratio equal to?

$$\frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = \frac{8(-1) + 11(2) + 11(2)}{(-1)^2 + 2^2 + 2^2} = \frac{36}{9} = 4$$

We now know the values of c_1 and c_2 for us to write $c_1\vec{u} + c_2\vec{v} = \vec{x}$. Find c_1 and c_2.

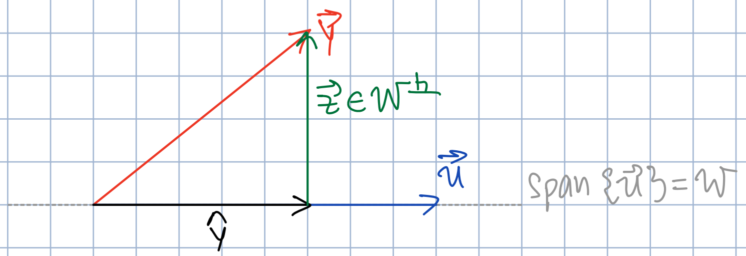
$$\vec{u} \cdot (c_1\vec{u} + c_2\vec{v}) = \vec{u} \cdot \vec{x}.$$
$$c_1 \vec{u} \cdot \vec{u} + c_2 \vec{v} \cdot \vec{u} = 54$$
$$18c_1 + 0c_2 = 54 \Rightarrow c_1 = 3.$$

$$\vec{v} \cdot (c_1\vec{u} + c_2\vec{v}) = \vec{v} \cdot \vec{x}$$
$$c_1 \vec{u} \cdot \vec{v} + c_2 \vec{v} \cdot \vec{v} = \vec{v} \cdot \vec{x}$$
$$3(0) + 9c_2 = 36 \Rightarrow c_2 = 4$$

Projection — Motivating Questions

Suppose $\vec{y}$ and $\vec{u}$ are nonzero vectors in $\mathbb{R}^n$, and W to be the span of $\vec{u}$.
* Let the closest vector to W to $\vec{y}$ be $\hat{y}$. How can we identify $\hat{y}$?
* We would like to write $\vec{y} = \hat{y} + \vec{z}$, where $\vec{z} \in W^{\perp}$. How do we identify vectors $\hat{y}$ and $\vec{z}$?

These two questions are closely related. A sketch of what we are looking for:



For what $\hat{y}$ is $\vec{z} \in W^{\perp}$?

We wish to write $\vec{y} = \hat{y} + \vec{z}$, where $\vec{u} \in W$, and $\vec{z} \in W^{\perp}$. If $\vec{z}$ is in $W^{\perp}$, then

$$0 = \vec{z} \cdot \vec{w}$$

Give some real number k, $\vec{z} = \vec{y} - k\vec{u}$, so

$$\begin{aligned} 0 &= (\vec{y} - k\vec{u}) \cdot \vec{u} \\ &= \vec{y} \cdot \vec{u} - k\vec{u} \cdot \vec{u} \\ k &= \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}, \quad \vec{u} \neq \vec{0} \end{aligned}$$

Thus, the closest vector in the span of $\vec{u}$ is

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

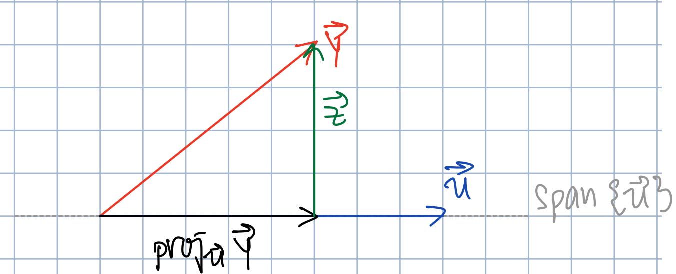
An Orthogonal Projection

Theorem:
Let $\vec{u}$ be a nonzero vector in $\mathbb{R}^n$, and let $\vec{y}$ be any vector in $\mathbb{R}^n$. The orthogonal projection of $\vec{y}$ onto $\vec{u}$ is the vector in the span of $\vec{u}$ that is closest to $\vec{y}$.

$$\text{proj}_{\vec{u}} \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

Moreover, $\vec{y} = \hat{y} + \vec{z}$, where $\vec{z} \in W^{\perp}$.

Schematic:



Geometric Interpretation:

The vector $\vec{z} = \vec{y} - \text{proj}_{\vec{u}} \vec{y}$ is orthogonal to $\vec{u}$, so that

$$\vec{y} = \text{proj}_{\vec{u}} \vec{y} + \vec{z}$$
$$\|\vec{y}\|^2 = \|\text{proj}_{\vec{u}} \vec{y}\|^2 + \|\vec{z}\|^2$$

True or False: if $\vec{u}$ is in one-dimensional subspace S, and $S^\perp$ is also a one-dimensional subspace, then the projection of $\vec{u}$ onto $S^\perp$ is $\vec{0}$.

If $\vec{x} \in S^\perp$, $\vec{x} \neq \vec{0}$, then

$$\text{proj}_{\vec{x}}\vec{u} = \frac{\vec{u}\cdot\vec{x}}{\vec{x}\cdot\vec{x}}\vec{x} = \frac{0}{\vec{x}\cdot\vec{x}}\vec{x} = \vec{0}. \quad \therefore \underline{\text{TRUE}}.$$

Example: Computing Projections and Distances

Suppose L is the line spanned by $\vec{u}$.

$$\vec{u} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \vec{y} = \begin{pmatrix}3\\4\\2\end{pmatrix}$$

1. Calculate the projection of $\vec{y}$ onto line L.

$$\text{proj}_L \vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}} = \frac{9}{3}\vec{u} = 3\vec{u}.$$

2. What is the distance between $\vec{y}$ and line L?

$$\|\vec{y} - \text{proj}_L \vec{y}\| = \|y - 3\vec{u}\| = \left\|\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\|$$
$$= \sqrt{0^2 + 1^2 + (-1)^2}$$
$$= \sqrt{2} \text{ units}$$

Exercise:

Suppose $\vec{u}$ gives a basis for subspace W.

$$\vec{u} = \begin{pmatrix}4\\1\\1\end{pmatrix}.$$

Suppose also that $\vec{y}$ is the vector below.

$$\vec{y} = \begin{pmatrix}16\\-9\\1\end{pmatrix}$$

$\vec{y}$ is not in W, but the closest vector in W to $\vec{y}$ is $\hat{y}$, where $\hat{y} = \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$.

Find $\hat{y}$.

$$\hat{y} = \text{proj}_{\vec{u}}\vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{16(4) - 9(1) + 1(1)}{4^2 + 1^2 + 1^2}\vec{u}$$
$$= \frac{56}{18}\vec{u}$$
$$= 3\vec{u} = 3\begin{pmatrix}4\\1\\1\end{pmatrix} = \begin{pmatrix}12\\3\\3\end{pmatrix}$$

Orthonormal Columns

Theorem: A $m \times n$ matrix U has orthonormal columns iff $U^T U = I_n$.

Can U have orthonormal columns if $n > m$?

Properties of Matrices with Orthonormal Columns

Theorem:
Suppose $m \times n$ matrix U has orthonormal columns and $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^n$.

1. $\|U\vec{x}\| = \|\vec{x}\|$
2. $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$
3. $(U\vec{x}) \cdot (U\vec{y}) = 0$ iff $\vec{x} \cdot \vec{y} = 0$

These properties tell us that the mapping $\vec{x} \to U\vec{x}$ preserves lengths and orthogonality.
Proofs of the above statements are similar. Let's prove the first statement.

1. First consider $\|U\vec{x}\|^2$.

$$\begin{aligned} \|U\vec{x}\|^2 &= (U\vec{x}) \cdot (U\vec{x}) \\ &= \vec{x}^T U^T U \vec{x} \\ &= \vec{x}^T I \vec{x} \qquad \because U^T U = I \\ \|U\vec{x}\|^2 &= \|\vec{x}\|^2 \end{aligned}$$

Taking the square root of both sides yields: $\|U\vec{x}\| = \|\vec{x}\|$.

Orthogonal Matrices

An orthogonal matrix is a square matrix whose columns are orthonormal.
A better name might be orthonormal matrix, but orthogonal matrix is a standard term in linear algebra. Also, matrices with columns that are orthogonal, but do not have unit length, are not often used.

Note: If U is an orthogonal matrix, then $U^{-1} = U^T$.

Determinant of an Orthogonal Matrix

If A is a square matrix with orthonormal columns, then $\det A$ is equal to 1 or -1.

Proof: $1 = \det I = \det(A^T A) = \det(A^T)\det(A) = (\det A)^2$.
$\therefore \det A = \pm 1$.

True or False: 1. If U is orthogonal, then its columns are also linearly independent. TRUE
2. If the determinant of a square matrix is 1, then the matrix must be orthogonal.
FALSE. Counterexample: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Exercise: If $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^n$ and the columns of $A \in \mathbb{R}^{n \times n}$ are orthonormal.
What is $(A\vec{u}) \cdot (A\vec{v})$ equal to?

$$(A\vec{u}) \cdot (A\vec{v}) = (A\vec{u})^T \cdot (A\vec{v}) = \vec{u}^T A^T A \vec{v} = \vec{u}^T I \vec{v} = \vec{u}^T \vec{v} = \vec{u} \cdot \vec{v} = 0$$

TOPIC 3: Orthogonal Projections

Best Approximation Theorem

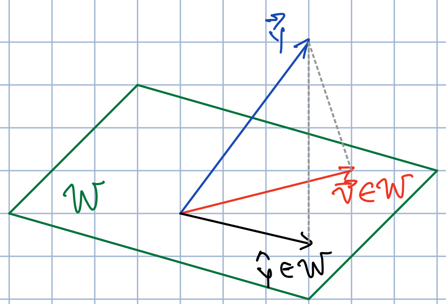
Theorem:
Let W be a subspace of $\mathbb{R}^n$, $\vec{y} \in \mathbb{R}^n$, and $\hat{y}$ is the orthogonal projection of $\vec{y}$ onto W.
Then for any $\vec{v} \neq \hat{y}$, $\vec{v} \in W$, we have

$$\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|$$

That is, $\hat{y}$ is the unique vector in W that is closest to $\vec{y}$.

Explanation for why $\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|$:

Is the orthogonal projection of $\vec{y}$ onto W the closest point in W to $\vec{y}$?



$$\vec{y} - \vec{v} = \vec{y} - \vec{v} + (\hat{y} - \hat{y}) = \underbrace{(\vec{y} - \hat{y})}_{\in W^{\perp}} + \underbrace{(\hat{y} - \vec{v})}_{\in W}$$

Pythagorean Theorem:

$$\|\vec{y} - \vec{v}\|^2 = \|\vec{y} - \hat{y}\|^2 + \|\hat{y} - \vec{v}\|^2 > \|\vec{y} - \hat{y}\|^2$$

True or False: If $\vec{v}$ is a vector in $\mathbb{R}^n$ and W is a subspace, then $\text{proj}_W(\text{proj}_W \vec{v}) = \text{proj}_W \vec{v}$.

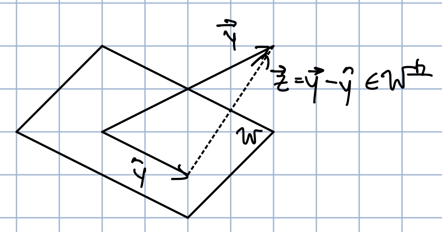
$\text{proj}_W \vec{v} = \hat{v} \in W$. $\quad \text{proj}_W \hat{v} = \hat{v}$. $\quad \Rightarrow$ TRUE.

Example:

$$\vec{y} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

What is the distance between $\vec{y}$ and subspace $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$? Assume $\hat{y}$ is the projection of $\vec{y}$ onto W.

$$\text{Distance between } \vec{y} \text{ and span}\{\vec{u}_1, \vec{u}_2\} = \|\vec{y} - \hat{y}\| = \left\| \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \right\|$$
$$= \sqrt{2^2 + (-2)^2 + 0} = \sqrt{8}.$$



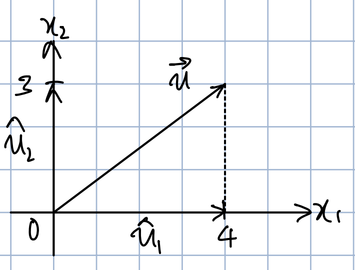
Exercise: Suppose $\vec{u} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

What is the distance between the vector $\vec{u}$ and its projection onto the x_1-axis?

$$\hat{u}_1 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}. \quad \|\vec{u} - \hat{u}_1\| = \sqrt{(4-4)^2 + (3-0)^2} = 3$$

What is the distance between the vector $\vec{u}$ and its projection onto the x_2-axis?

$$\hat{u}_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}. \quad \|\vec{u} - \hat{u}_2\| = \sqrt{(4-0)^2 + (3-3)^2} = 4$$



The Orthogonal Decomposition Theorem

Motivation:
Vector $\vec{y}$ is not in Col A.

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

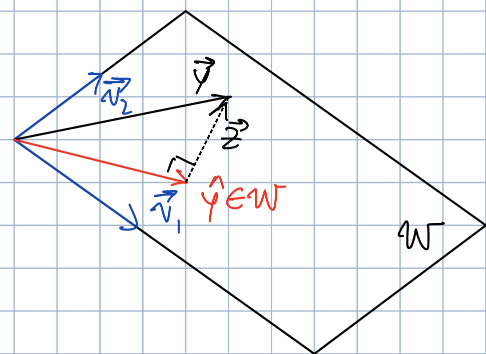
There is no solution to $A\vec{x} = \vec{y}$.

2 questions → Can we find a vector $\hat{y}$ that is in Col A, that is closest to $\vec{y}$?
↳ Can we write $\vec{y} = \hat{y} + z$, where
* $\hat{y} \in \text{Col } A$
* $z \in (\text{Col } A)^{\perp}$

Geometric Explanation for Orthogonal Decomposition

Suppose:
* vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^3$ form an orthonormal basis a subspace, W
* $W = \text{Span}\{\vec{v}_1, \vec{v}_2\}$
* vector $\vec{y}$ is not in W

Goals: (1) identify the vector in W that is closest to $\vec{y}$, which we call $\hat{y}$
(2) identify $\vec{z}$ so that $\vec{y} = \hat{y} + \vec{z}$



Example: Orthogonal Decomposition

Suppose $\vec{u}_1, \ldots, \vec{u}_5$ is an orthonormal basis for $\mathbb{R}^5$. Let $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$.

For any vector $\vec{y} \in \mathbb{R}^5$, construct the vectors $\hat{y}$ and z so that $\vec{y} = \hat{y} + \vec{z}$, where $\hat{y} \in W$ and $\vec{z} \in W^{\perp}$.

Note: $\vec{u}_i$ spans $\mathbb{R}^5$, so $\vec{y} = \sum_{i=1}^{5} c_i \vec{u}_i$

$$= \underbrace{(c_1\vec{u}_1 + c_2\vec{u}_2)}_{\in W} + \underbrace{(c_3\vec{u}_3 + c_4\vec{u}_4 + c_5\vec{u}_5)}_{\in W^{\perp}}$$

$$= \hat{y} + \vec{z}$$

proof for this theorem in this example?

Theorem: Orthogonal Decomposition Theorem
Let W be a subspace of $\mathbb{R}^n$. Then, each vector $\vec{y} \in \mathbb{R}^n$ has the unique decomposition

$$\vec{y} = \hat{y} + \vec{z}, \quad \hat{y} \in W, \quad z \in W^{\perp}.$$

And, if $\vec{u}_1, \ldots, \vec{u}_p$ is any orthogonal basis for W, $\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \ldots + \frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p}\vec{u}_p$.

We say that $\hat{y}$ is the orthogonal projection of $\vec{y}$ onto W.

More on the Orthogonal Decomposition Theorem

Why is $z \in W^{\perp}$? Our theorem tells us that

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \dots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p = \sum_{i=1}^{p} \frac{\vec{y}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}\vec{u}_i$$

Then, $\vec{z} = \vec{y} - \hat{y}$ is in $W^{\perp}$ because for any j, we have $\vec{u}_j \cdot \vec{z} = 0$:

$$\vec{u}_j \cdot \vec{z} = \vec{u}_j \cdot (\vec{y} - \hat{y}) = \vec{u}_j \cdot \vec{y} - \vec{u}_j \cdot \hat{y} = \vec{u}_j \cdot \vec{y} - \vec{u}_j \cdot \sum_{i=1}^{p} \frac{\vec{y}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}\vec{u}_i$$

$$= \vec{u}_j \cdot \vec{y} - \vec{u}_j \cdot \frac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}\vec{u}_j$$

$$= \vec{u}_j \cdot \vec{y} - \left(\frac{\vec{y}\cdot\vec{u}_j}{\cancel{\vec{u}_j\cdot\vec{u}_j}}\right)\cancel{\vec{u}_j\cdot\vec{u}_j} = 0.$$

Example: Constructing an Orthogonal Decomposition

$$\vec{y} = \begin{pmatrix}4\\0\\3\end{pmatrix}, \vec{u}_1 = \begin{pmatrix}2\\2\\0\end{pmatrix}, \vec{u}_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Construct the decomposition $\vec{y} = \hat{y} + \vec{z}$, where $\hat{y}$ is the orthogonal projection of $\vec{y}$ onto $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$, and $z \in W^{\perp}$.

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 = \frac{8}{8}\vec{u}_1 + \frac{3}{1}\vec{u}_2 = \begin{pmatrix}2\\2\\3\end{pmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{pmatrix}2\\-2\\0\end{pmatrix}. \quad \therefore \vec{y} = \hat{y} + \vec{z} = \begin{pmatrix}2\\2\\3\end{pmatrix} + \begin{pmatrix}2\\-2\\0\end{pmatrix}$$

Exercise:

Suppose $\vec{u}$ and $\vec{v}$ below gives a basis for subspace W.

$$\vec{u} = \begin{pmatrix}4\\1\\1\end{pmatrix}, \vec{v} = \begin{pmatrix}-1\\2\\2\end{pmatrix}$$

Suppose also that $\vec{y}$ is the vector below:

$$\vec{y} = \begin{pmatrix}1\\11\\3\end{pmatrix}$$

$\vec{y}$ is not in W, but the closest vector in W to $\vec{y}$ is $\hat{y}$, where:

$$\hat{y} = \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$$

Find $\hat{y}$.

$\vec{u}$ and $\vec{v}$ are orthogonal. $\Rightarrow \hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} + \frac{\vec{y}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{4+11+3}{4^2+1^2+1^2}\begin{pmatrix}4\\1\\1\end{pmatrix} + \frac{-1+22+6}{(-1)^2+2^2+2^2}\begin{pmatrix}-1\\2\\2\end{pmatrix}$

$$= \frac{18}{18}\begin{pmatrix}4\\1\\1\end{pmatrix} + \frac{27}{9}\begin{pmatrix}-1\\2\\2\end{pmatrix}$$

$$\therefore \hat{y} = \begin{pmatrix}1\\7\\7\end{pmatrix}.$$

What is the distance between W and $\vec{y}$?

$$z = \|\vec{y} - \hat{y}\| = \left\| \begin{pmatrix} 1 \\ 11 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 7 \\ 7 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} \right\|$$

$$= \sqrt{0^2 + 4^2 + (-4)^2} = \sqrt{32}$$