MODULE 6: COMPUTER GRAPHICS AND SUBSPACES

TOPIC 1: Computer Graphics

Homogeneous Coordinates

Motivating Questions:

How can we represent translations, and rotations about arbitrary points, using linear transforms?
* transformations of the form $T(\vec{x}) = A\vec{x}$ were explored earlier in this course
* we introduced rotations about the origin, but not about arbitrary points
* we also did not explore the transform $(x,y) \Rightarrow (x+h, y+k)$

Translations of points in $\mathbb{R}^n$ do not correspond directly to a linear transform.
Homogeneous coordinates are used to model translations using matrix multiplication.

Definition ⟶ Homogeneous coordinates in $\mathbb{R}^2$

↳ Each point (x,y) in $\mathbb{R}^2$ can be identified with the point (x,y,H), $H \neq 0$, on the plane in $\mathbb{R}^3$ that lies H units above the xy-plane.

Note: we often set $H=1$.

Example: A translation of the form $(x,y) \to (x+h, y+k)$ can be represented as a matrix multiplication with homogeneous coordinates:

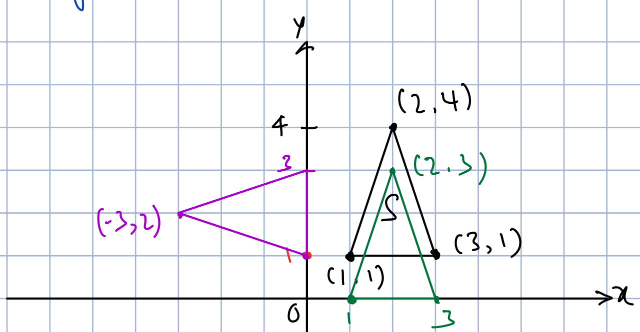
$$\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \\ 1 \end{pmatrix}$$

A Composite Transform with Homogeneous Coordinates

Triangle S is determined by three data points, $(1,1)$, $(2,4)$, $(3,1)$.

Triangle T rotates points by $\frac{\pi}{2}$ radians counterclockwise about the point $(0,1)$.
(a) Represent the data with a matrix, D. Use homogeneous coordinates.
(b) Use matrix multiplication to determine the image of S under T.
(c) Sketch S and its image under T.

(a) $D = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

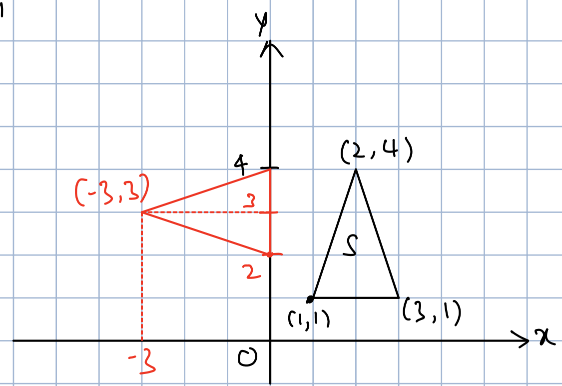


Recall:
Standard matrix for counterclockwise rotation:
$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix} = D'$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = D''$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} D'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -3 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 0 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$



Exercise: Consider the points in $\mathbb{R}^2$, P and Q, whose coordinates are P(3,4) and Q(5,1).

Using homogeneous coordinates, the data matrix, D, for these two points is the following:

$$D = \begin{pmatrix} a & b \\ 4 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 4 & 1 \\ 1 & 1 \end{pmatrix}$$

What does a need to be equal to? Answer: 3

Now suppose that we want to use homogeneous coordinates to reflect points P and Q across the line $x_2 = 2$. We can use a transform of the form $T(D) = AD$, where A is the matrix below.

$$A = \begin{pmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 0 & a_3 \end{pmatrix}\begin{pmatrix} b_1 & b_2 & 0 \\ b_3 & b_4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & c_3 \end{pmatrix}$$

What should c_2 be equal to? Answer: -2. We need $c_2 = -2$ to shift the line of reflection down, so that it passes through the origin.

What should b_1 be equal to? Answer: 1.

Now that we have shifted everything down by 2 units, we are reflecting points across the line $x_2 = 0$. So, b_1, b_2, b_3 and b_4 are chosen so that we can reflect across that line, which gives us the sub-matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We place this 2X2 matrix into the 3X3 matrix so that we are using homogeneous coordinates to obtain $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

What should b_4 be equal to? Answer: -1. As above, we have $b_4 = -1$.

What should a_2 be equal to? Answer: 2. We need $c_2 = +2$ to shift the line of reflection up, so that it corresponds to the line $x_2 = 2$.

3D Homogeneous Coordinates

Homogeneous coordinates in 3D are analogous to our 2D coordinates.

DEFINITION ⟶ Homogeneous Coordinates in $\mathbb{R}^3$

↳ $(x, y, z, 1)$ are homogeneous coordinates for (x, y, z) in $\mathbb{R}^3$.

Homogeneous Coordinates Example:

A translation of the form $(x,y,z) \to (x+h, y+k, z+l)$ can be represented as a matrix multiplication with homogeneous coordinates:

$$\begin{pmatrix} 1 & 0 & 0 & h \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \\ z+l \\ 1 \end{pmatrix}$$

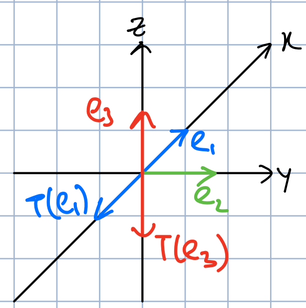
3D Transformation Matrices

Construct matrices for the following transformations:

(a) A translation in $\mathbb{R}^3$ specified by the vector $\vec{p} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$.

Answer: $\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) A rotation in $\mathbb{R}^3$ about the x-axis by π radians.



$T = A\vec{x}$, $A = (a_1\ a_2\ a_3)$, $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{aligned} T(e_1) &= Ae_1 \\ &= 1a_1 + 0a_2 + 0a_3 \\ &= a_1 \\ &= \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

$T(e_2) = Ae_2 = a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. $T(e_3) = Ae_3 = a_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$$\Rightarrow A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(c) A projection onto the plane $x_3=4$.

project onto x_3-plane

Use homogeneous coordinates: $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&4\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-4\\0&0&0&0\end{pmatrix}$

Exercise:

Consider the points in $\mathbb{R}^3$, P and Q, whose coordinates are $P(1,4,2)$ and $Q(7,1,4)$.
Using homogeneous coordinates, the data matrix, D, for these two points is the following:

$$D=\begin{pmatrix}1&b\\4&1\\a&4\\1&1\end{pmatrix}$$

What does a need to be equal to? Answer: 2.

Now suppose that we want to use homogeneous coordinates to project points P and Q onto the plane $x_3=2$.

To create this projection, we can use a transform of the form $T(D)=AD$, where A is the matrix below:

$$A=\begin{pmatrix}1&0&0&a_1\\0&1&0&a_2\\0&0&1&a_3\\0&0&0&1\end{pmatrix}\begin{pmatrix}b_1&b_2&b_3&0\\b_4&b_5&b_6&0\\b_7&b_8&b_9&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&c_1\\0&1&0&c_2\\0&0&1&c_3\\0&0&0&0\end{pmatrix}$$

What should c_2 be equal to? Answer: 0. We need $c_2=0$ because we are not shifting in the x_2 direction.

What should c_3 be equal to? Answer: -2. We need $c_3=-2$ to shift the plane we are projecting onto, so that it passes through the origin.

What should b_1 be equal to? Answer: 1.
Now that we have shifted everything over by 2 units, we are projecting points onto the plane $x_3=0$. So b_1 to b_9 are chosen so that we can perform that projection. This gives us the sub-matrix $\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$. We place this 3X3 matrix into the 4X4 matrix so that we are using homogeneous coordinates: $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$

What should b_5 be equal to? Answer: 1. As above, we have $b_5=1$.

What should b_9 be equal to? Answer: 0. As above, we have $b_9=0$.

What should a_3 be equal to? Answer: 2. We need $a_3 = 2$ to shift the projection plane back to $x_3 = 2$.

TOPIC 2: Subspaces of $\mathbb{R}^n$

DEFINITION — Subsets of $\mathbb{R}^n$

↳ A subset of $\mathbb{R}^n$ is any collection of vectors that are in $\mathbb{R}^n$.

Examples: * the span of the columns of a 3x4 matrix is a subset of $\mathbb{R}^3$

* the set of all vectors of the form $\begin{pmatrix} 1 \\ k \end{pmatrix}$ is a subset of $\mathbb{R}^2$

DEFINITION — Subspaces in $\mathbb{R}^n$

↳ A subset H of $\mathbb{R}^n$ is a subspace if it is closed under scalar multiplication and vector addition. That is: for any $c \in \mathbb{R}$ and for $\vec{u}, \vec{v} \in H$,

① $c\vec{u} \in H$ → NOTE: condition 1 implies that the zero vector must be in H.

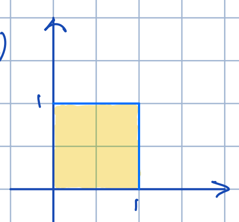
② $\vec{u} + \vec{v} \in H$

The span of any vector in $\mathbb{R}^n$ is a subspace. – True.
The span of a single vector is a subspace: the set of vectors in the span is closed under vector addition and scalar multiplication.

Example: Subspaces

Which of the following subsets could be a subspace of $\mathbb{R}^2$?

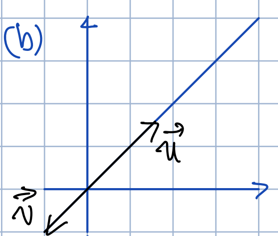
(a)



the unit square

If $\vec{u} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, $c\vec{u}$ is not in subset for all c.

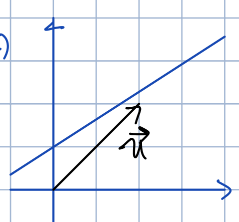
(b)



a line passing through the origin

If $\vec{u}$ and $\vec{v}$ are in subset, $c\vec{u}$ and $\vec{u}+\vec{v}$ are also in subset.
⇒ we have a subspace.

(c)



a line that does not pass through the origin

$c\vec{u}$ not necessarily in subset ⇒ not a subspace

Example: Set Builder Notation

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; ab = 0 \right\}$.

① Give an example of at least two vectors that are in V. $\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

② Give an example of at least two vectors that are not in V. $\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 10 \\ 4 \end{pmatrix}, \dots$

③ Is the zero vector in V? Yes

④ Is V a subspace? $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, but $\vec{u} + \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
$\therefore$ Not a subspace.

Exercise: Assume that k can be any real number. Select all sets that are subspaces.

✗ The set S_1, which is the set of all vectors of the form $\begin{pmatrix} 1 \\ k \end{pmatrix}$.

✗ The set S_2, which is the set of all vectors of the form $\begin{pmatrix} k \\ k^2 \end{pmatrix}$.

✓ The set S_3, which is the set of all vectors of the form $\begin{pmatrix} 0 \\ k \end{pmatrix}$.

The first set, S_1, is not a subspace because it does not include the zero vector. The second set, S_2, does include the zero vector, but is not closed under vector addition. The vectors $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ are in S_2, but $\vec{a} + \vec{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is not in S_2. The third set is a subspace.

Exercise: Assume that k can be any real number. Select all sets that are subspaces.

✗ The set of all solutions to the linear system $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

✓ The set of all solutions to the linear system $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The solution to the first system is a set of points that does not pass through the origin, so it cannot be a subspace.
The solution set to the second system is a line that passes through the origin, so it is a subspace. Every line that passes through the origin is a subspace.

Exercise: Consider the set of vectors, $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$.

Which of the following vectors are in S?

$\vec{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ ✓ $\quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{c} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ ✓ $\quad \vec{d} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ✓

Is S a subspace? Yes, because it is closed under vector addition and scalar multiplication.

The Column Space and the Null Space of a Matrix

Recall: For $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$, that span $\{\vec{v}_1, \dots, \vec{v}_p\}$ is: the set of all possible linear combinations of the vectors $\vec{v}_j$.

This is a subspace, spanned by $\vec{v}_1, \dots, \vec{v}_p$.

DEFINITION — Column Space and Null space of a Matrix.

- The column space of A, Col A, is the subspace of $\mathbb{R}^m$ spanned by $\vec{a}_1, \dots, \vec{a}_n$
- The null space of A, Null A, is the subspace of $\mathbb{R}^n$ spanned by the set of all vectors $\vec{x}$ that solve $A\vec{x} = \vec{0}$.

Example: Column Space

Is $\vec{b}$ in the column space of A?

$$A = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$$

If $\vec{b} \in$ Col A, then $\vec{b} \in$ Span$\{$columns of $A\}$.

check if system is consistent.

To check, $c_1 \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 6 \\ 7 \end{pmatrix} + c_3 \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix} = \vec{b} \Rightarrow \left(\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array}\right)$

$$\left(\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array}\right) \xrightarrow[R_3 + 3R_1 \to R_3]{R_2 + 4R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{array}\right)$$

$$\xrightarrow{R_2 - 3R_3 \to R_2} \left(\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -6 & 5 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -2 & -6 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{System is consistent.}$$

$\therefore b \in$ Col A.

Example: Null space

Using the same matrix as in the previous example, is $\vec{v}$ in the null space of A?

$$A = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -5\lambda \\ -3\lambda \\ \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

If $\vec{v} \in \text{Null } A$, then $A\vec{v} = \vec{0}$.

$$A\vec{v} = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix}\begin{pmatrix} -5\lambda \\ -3\lambda \\ \lambda \end{pmatrix}$$

$$= \begin{pmatrix} -5\lambda + 9\lambda - 4\lambda \\ 20\lambda - 18\lambda - 2\lambda \\ 15\lambda - 21\lambda + 6\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \vec{v} \in \text{Null } A$.

Example: Column and Null Space

Give an example of a vector in the column space of A, and a vector in the null space of A.

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is in Col A.

If $\vec{x} \in$ null space, then $A\vec{x} = \vec{0}$. $\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix}, \ldots$$

Example construction:

Give an example of a matrix whose column space is spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and whose null space is spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}, \quad A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$2 + a = 0 \Rightarrow a = -2.$

$2 + b = 0 \Rightarrow b = -2$

$$\therefore A = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}.$$

Exercise: Suppose A is the matrix below.

$$A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}.$$

If $\vec{x} = \begin{pmatrix} 1 \\ k \end{pmatrix}$ is in the null space of A, what does k need to be?

$$\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad 6 + 2k = 0 \Rightarrow \underline{\underline{k = -3}}$$

If $\vec{y} = \begin{pmatrix} k \\ 4 \end{pmatrix}$ is in the column space of A, what does k need to be?

$$\alpha \begin{pmatrix} k \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}. \qquad \begin{aligned} 4\alpha &= 6 \Rightarrow \alpha = \tfrac{3}{2}. \\ \alpha k &= \tfrac{3}{2}k = 3 \Rightarrow \underline{\underline{k = 2}} \end{aligned}$$

Basis of a Subspace

DEFINITION — Basis

↳ A <u>basis</u> for a subspace H is a set of linearly independent vectors in H that span H.

Example:

The set $H \in \{\vec{x} \in \mathbb{R}^4 \mid x_1 - 3x_2 - 5x_3 + 7x_4 = 0\}$ is a subspace.

(a) H is the null space for what matrix A?
(b) Construct a basis for H.

(a) $x_1 - 3x_2 - 5x_3 + 7x_4 = 0$. $$\overbrace{\begin{pmatrix} 1 & -3 & -5 & 7 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}^{\vec{x}} = A\vec{x} = 0.$$

(b) To construct a basis for H, use parametric vector form:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3x_2 + 5x_3 - 7x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{v}_1} + x_3 \underbrace{\begin{pmatrix} 5 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_2} + x_4 \underbrace{\begin{pmatrix} -7 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}_3}$$

By inspection, v_1, v_2 and v_3 are linearly independent.

$\therefore$ Basis is the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

Example: Construct a basis for Nul A and a basis for Col A.

$$A = \begin{pmatrix} -3 & 6 & -1 & 0 \\ 1 & -2 & 2 & 0 \\ 2 & -4 & 5 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivotal columns

Null A: $A\vec{x} = \vec{0}$ can be written as

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 = 2x_2 \\ x_3 = 0. \\ x_2, x_4 \text{ are free} \end{cases}$$

Parametric vector form:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \text{ Let } \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\therefore$ Basis for Null A is $\{\vec{v}_1, \vec{v}_2\}$

Col A: Basis for Col A given by pivotal columns of A. $\left\{ \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \right\}$

Exercise: Suppose A is the matrix below:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

How many vectors are needed to form a basis for the null space of A?

$$A\vec{x} = \vec{0} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \quad x_1 + 2x_3 = 0 \Rightarrow x_1 = -2x_3.$$

x_2 is a free variable.

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

If $\vec{v} = \begin{pmatrix} 2 \\ 0 \\ k \end{pmatrix}$ is in the null space of A, what does k need to be equal to?

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad 2 + 2k = 0 \Rightarrow k = -1.$$

Suppose $\vec{u} = \begin{pmatrix} c_1 \\ 1 \\ 0 \end{pmatrix}$. If $\vec{v}$ and $\vec{u}$ form a basis for Null A, what does c_1 need to be equal to?

Answer: $c_1 = 0$.

Exercise: Suppose A is the matrix below.

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which of the following sets form a basis for Col A?

- ✗ the first and second columns.
- ✓ the first, second and fourth columns.
- ✓ the first, third and fourth columns.

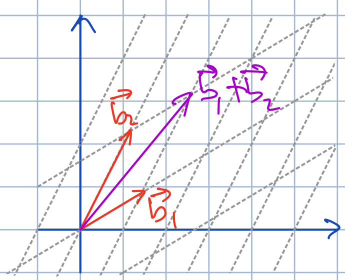
3 vectors are needed in the basis to span the subspace. These three vectors are linearly independent and in the space.

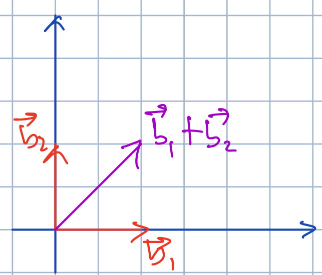
TOPIC 3: Dimension and Rank

Choice of Basis

Key idea: There are many possible choices of basis for a subspace. Our choice can give us dramatically different properties.

Example: sketch $\vec{b}_1 + \vec{b}_2$ for the two different coordinate systems below.





Coordinate Systems

DEFINITION — Coordinate Vector

↳ Let $B = \{\vec{b}_1, \ldots, \vec{b}_p\}$ be a basis for a subspace H. If $\vec{x}$ is in H, then coordinates of $\vec{x}$ relative B are the weights (scalars) $c_1, \ldots, c_p$ so that

$$\vec{x} = c_1\vec{b}_1 + \ldots + c_p\vec{b}_p$$

And

$$[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is the coordinate vector of $\vec{x}$ relative B, or the B-coordinate vector of $\vec{x}$.

Example: Coordinate Vector

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$. Verify that $\vec{x}$ is in the span of $B = \{\vec{v}_1, \vec{v}_2\}$, and calculate $[\vec{x}]B$.

If $\vec{x}$ is in span $\{\vec{v}_1, \vec{v}_2\}$, then there exists c_1 and c_2 such that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x}$, or

$$\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{array}\right) \xrightarrow{R_3 - R_1 \to R_3} \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 - R_2 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \vec{x}$ is in span of $\vec{v}_1, \vec{v}_2$ and $[\vec{x}]B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Exercise:

The vectors $\vec{v}_1$ and $\vec{v}_2$ below form a basis for subspace S.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Vectors $\vec{x}$ is in S, and is equal to $\vec{x} = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}$.

Because $\vec{v}_1$ and $\vec{v}_2$ form a basis for S, we can express $\vec{x}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$. That is, we can write

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x}.$$

Moreover, because $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, the values of c_1 and c_2 will be unique.

What does c_1 need to be equal to?

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & -1 & 8 \\ 1 & 1 & 1 \end{array}\right) \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 - R_2 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right). \ \underline{c_1 = 3}, c_2 = -2$$

By placing c_1 and c_2 into a vector, we obtain the coordinate vector for $\vec{x}$ in the basis given by $\vec{v}_1$ and $\vec{v}_2$. The coordinate vector is:

$$[x]B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

What is the value of the first entry of $[x]B$? Answer: -2.

The Dimension of a Subspace

DEFINITION — Dimension of a subspace

↳ The dimension (or cardinality) of a nonzero subspace H, $\dim H$, is the number of vectors in a basis of H. We define $\dim\{0\} = 0$.

Note: zero vector cannot be a basis vector. The dimension of the set that only contains the zero vector is zero.

Example: The dimensions of the column space for each matrix below is 2.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

*basis for subspace is never unique.

↑ can be used instead of 2nd column, it is in the span of the first two.

$\dim \mathbb{R}^n = n$.

ex, if $n=2$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\dim(\text{Col } A) = 0$.

$\dim(\text{Null } A)$ is the number of free variables.
$\dim(\text{Col } A)$ is the number of pivots.

$H = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ has dimension 2.

ex, $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}, \ldots$

$$x_1 + x_2 + x_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A\vec{x} = \vec{0}.$$

$$\vec{x} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Theorem:
Suppose H is a p-dimensional subspace of $\mathbb{R}^n$. Any set of p independent vectors that are in H are automatically a basis for H.

Example:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Two bases for the column space of A are:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Exercise: Indicate whether the following statement is true or false:

The dimension of the null space of a 2×4 matrix will be at least two.

→ True. Any matrix that does not have a pivot in every column will have nonzero vectors in its null space. In this case, there will be at least two linearly independent vectors in the null space because there are at most two pivot columns.

Exercise: The basis for the null space of a 4×5 matrix consists of exactly 2 vectors.

What is the dimension of the column space of that matrix?

Answer: 3. In general, if N is the number of columns of a matrix, then

$$N = \dim(\text{Null } A) + \dim(\text{Col } A)$$

What is the dimension of the null space of that matrix?

Answer: 2. The dimension of a null space is the number of vectors in any basis for that null space. We are given that there are two vectors in the basis, so $\dim(\text{Null } A) = 2$.

Rank and Invertibility

DEFINITION — Rank

↳ The rank of a matrix is the dimension of its column space.

Example: Compute rank(A) and dim(Nul(A)).

$$A = \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & -6 & 5 & -6 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivotal

Rank (A) = 3.
Dim (Nul(A)) = 2

Rank, Basis and Invertibility Theorems

Rank Theorem: If a matrix A has n columns, then Rank A + dim(Nul A) = n.

basic variables + # free variables = n.
pivotal columns + # non-pivotal columns = n.

Example construction:
If possible give an example of a 2x3 matrix A, that is in RREF and has the given properties.

(a) rank(A) = 3 → not possible. 2x3 matrix can have at most two pivots. Rank(A) = 3 requires at least 3 rows and 3 columns.

(b) rank(A) = 2 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(c) dim(Nul(A)) = 2 $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(d) Nul A = $\{\vec{0}\}$ If dimension of the null space + rank is always equal to the number of columns, the dimension of the null space for a 2x3 matrix has to be at least 1. ⇒ Not possible.

Invertibility

Let A be a n×n matrix. These conditions are equivalent.
1) A is invertible.
2) The columns of A are a basis for $\mathbb{R}^n$.
3) Col A = $\mathbb{R}^n$
4) rank A = dim(Col A) = n
5) Null A = $\{\vec{0}\}$.

Example Construction:
If possible, give an example of matrix A, that is in RREF and has the given properties.

(a) A is 2X2, invertible, and rank A = 1. ⇒ not possible. the rank would have to be 2 for the matrix to be invertible.

(b) A is 4x4, invertible, and rank A = dim(Nul A). ⇒ not possible.
For an invertible matrix, we need the rank to be 4.

Exercise: Indicate whether the following situations are possible or impossible.

A is a 4x3 matrix with linearly independent columns and rank 4.
Impossible. The rank of a matrix is the number of pivots that the matrix has, and there are only 3 columns. So the rank of this matrix can be at most 3.

A is a singular n×n matrix with rank n.
Impossible. A singular matrix is a matrix that is not invertible. If the matrix is not invertible, then the rank cannot be equal to the number of columns. Invertible matrices have a rank equal to the number of columns of the matrix.