MODULE 1: LINEAR SYSTEMS & SPANS

TOPIC 1: Systems of Linear Equations

A linear equation has the form $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ where

- → $a_1, a_2, \dots, a_n$ and b are the coefficients
- → $x_1, x_2, \dots, x_n$ are the variables or the unknowns
- → n is the dimension, or # variables

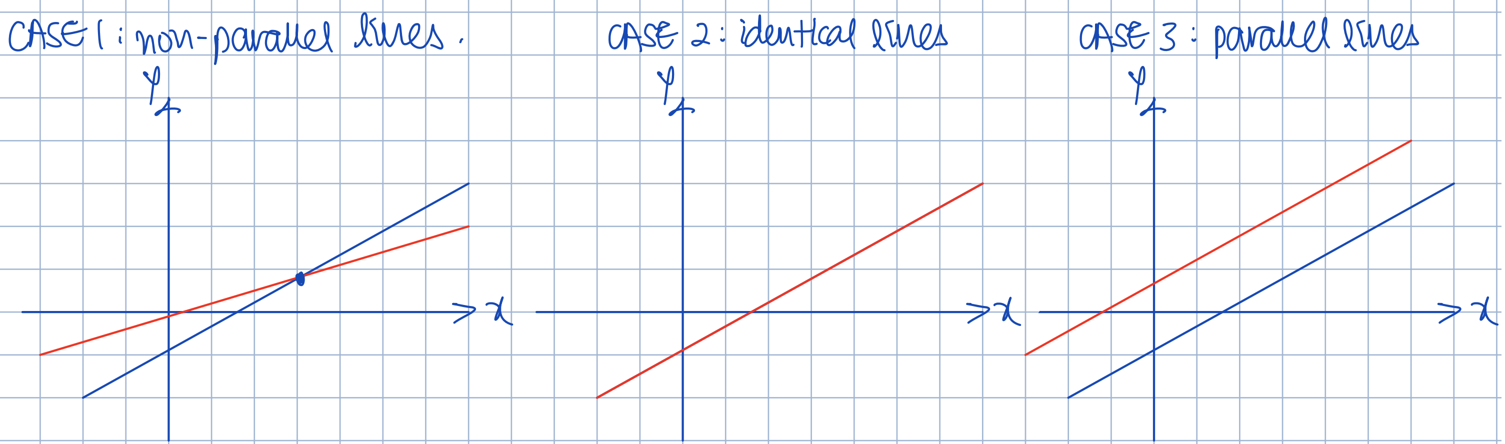
Example: $2x_1 + 4x_2 = 4$ is a line in 2 dimensions
$3x_1 + 2x_2 + x_3 = 6$ is a line in 3 dimensions

When we have ≥ 1 linear equation, we have a linear system of equations. For example, a linear system with 2 equations is

$$\begin{aligned} x_1 + 1.5x_2 + \pi x_3 &= 4 \\ 5x_1 \qquad\qquad + 7x_3 &= 5. \end{aligned}$$

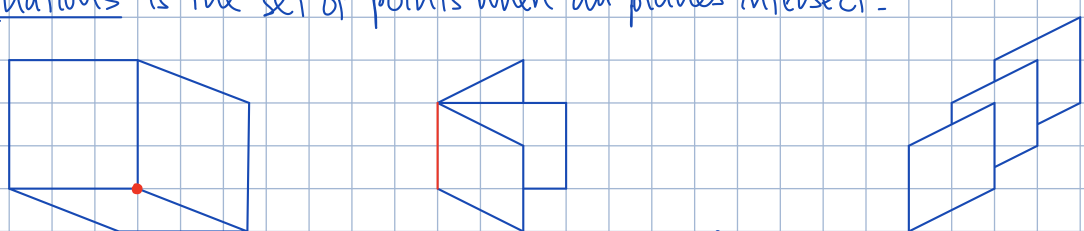
DEFINITION — A Solution of a Linear System
↳ The set of all possible values $x_1, x_2, \dots, x_n$ that satisfy all equations is the solution set of the system. One point in the solution set is a solution.

Two Variable Case



Three Variable Case

An equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ defines a plane in $\mathbb{R}^3$. The solution to a system of 3 equations is the set of points where all planes intersect.



planes intersect at a point
→ unique solution

planes intersect on a line
→ infinite # solutions

parallel lines
→ no solution

How can we find the solution set to a set of linear equations?

→ manipulate equations in a linear system using row operations

① Replacement\Addition : Add a multiple of one row to another.

② Interchange : Interchange 2 rows

③ Scaling : Multiply a row by a nonzero scalar.

Example:

$R_1: \; x_1 - 2x_2 + x_3 = 0$
$R_2: \; 2x_2 - 8x_3 = 8$
$R_3: \; 5x_1 - 5x_3 = 10$

$R_1 + R_2 \Rightarrow R_1: \quad x_1 + 0x_2 - 7x_3 = 8$ —— ①

$\frac{1}{2}R_2 \Rightarrow R_2: \quad x_2 - 4x_3 = 4$ —— ②

$R_3 - 5R_1 \Rightarrow R_3: \quad 10x_2 - 10x_3 = 10$

$\frac{1}{10}R_3 \Rightarrow R_3: \quad x_2 - x_3 = 1$

$R_3 - R_2 \Rightarrow R_3: \quad 3x_3 = -3 \Rightarrow \underline{x_3 = -1}.$

Substitute $x_3 = -1$ into ①: $\quad x_1 + 0x_2 - 7(-1) = 8$
$x_1 + 7 = 8 \Rightarrow \underline{x_1 = 1}.$

Substitute $x_3 = -1$ into ②: $\quad x_2 - 4(-1) = 4$
$x_2 + 4 = 4 \Rightarrow \underline{x_2 = 0}.$

∴ Solution is point $(1, 0, -1)$.

It is redundant to write x_1, x_2, x_3 again and again, so we rewrite systems using matrices. For example,

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 7 \end{aligned} \xrightarrow[\text{augmented matrix}]{\text{can be written as the}} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 7 \end{array}\right)$$

The vertical line reminds that the first three columns are the coefficients to our variables x_1, x_2 and x_3.

DEFINITION — Consistent

↳ A linear system is consistent if it has at least one solution.

DEFINITION — Row Equivalence

↳ Two matrices are row equivalent if a sequence of row operations transforms one matrix into the other.

NOTE: If the augmented matrices of 2 linear systems are row equivalent, then they have the same solution set.

Example for Consistent Systems & Row Equivalence:

Suppose

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

① Are A and B row equivalent? Are A & C row equivalent?

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad \underset{\sim}{R_1 - R_2 \to R_1:} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B.$$

∴ A and B are row equivalent.

A and C are not row equivalent.

② Are the augmented matrices $(A|\vec{b})$ and $(C|\vec{b})$ consistent?

$$(A|\vec{b}) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right) \underset{\sim}{R_1 - R_2 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 0, \\ x_2 = 1. \end{array}$$

∴ $A\vec{x} = \vec{b}$ is consistent.

$$(C|\vec{b}) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

2nd row is $0x_1 + 0x_2 = 1$

$\Rightarrow C\vec{x} = \vec{b}$ is not consistent.

TOPIC 2: Row Reduction and Echelon Forms

DEFINITION — Echelon Form

↳ A rectangular matrix is in echelon form if

① All zero rows (if any are present) are at the bottom.

② The first nonzero entry (or leading entry) of a row is to the right of any leading entries in the row above it (if any).

③ All entries below a leading entry (if any) are zero.

Examples: Matrix A is in echelon form. Matrix B is not in echelon form.

$$A = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

A matrix in echelon form is in <u>row reduced echelon form</u> (RREF) if

① All leading entries, if any, are equal to 1.
② Leading entries are the only nonzero entry in their respective column.

Examples: Matrix A is in RREF. B is not in RREF.

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 6 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix in Echelon form:

$$\begin{pmatrix} 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ in echelon, not RREF

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in RREF

$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ not echelon

$\begin{pmatrix} 0 & 6 & 3 & 0 \end{pmatrix}$ in echelon, not RREF

$\begin{pmatrix} 1 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in RREF

DEFINITION — Pivot Position, Pivot column

↳ A <u>pivot position</u> in a matrix A is a location in A that corresponds to a leading 1 in the row reduced echelon form of A

↳ A <u>pivot column</u> is a column of A that contains a pivot position.

Example: Express the matrix in RREF and identify the pivot columns.

$$\begin{pmatrix} 0 & -3 & -6 & 9 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -3 & -6 & 9 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \end{pmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{pmatrix} -1 & -2 & -1 & 3 \\ 0 & -3 & -6 & 9 \\ -2 & -3 & 0 & 3 \end{pmatrix}$$

$$\underset{\substack{-R_1 \to R_1 \\ R_3 - 2R_2 \to R_3}}{\sim} \begin{pmatrix} 1 & 2 & 1 & -3 \\ 0 & -3 & -6 & 9 \\ 0 & 1 & 2 & -3 \end{pmatrix}$$

$$\underset{R_2 + 3R_3 \to R_2}{\sim} \begin{pmatrix} 1 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -3 \end{pmatrix}$$

$$\underset{R_2 \leftrightarrow R_3}{\sim} \begin{pmatrix} 1 & 2 & 1 & -3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underset{R_1 - 2R_2 \to R_1}{\sim} \begin{pmatrix} 1 & 0 & -3 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

First & second rows are pivotal

Row Reduction Algorithm:

① Swap the first row with a lower one so the leftmost nonzero entry is in the first row.

② Scale the first row so that its leading entry is equal to 1.

③ Use row replacement so all entries above and below this leading entry (if any) are equal to zero.

Then, repeat these steps for row 2, then row 3, and so on, for the remaining rows of the matrix.

Notes on Row Reduction Algorithm:

*There are many algorithms for reducing a matrix to echelon form, or to RREF.
*If we only need to count pivots, we do not need RREF. Echelon form is sufficient.

Basic and Free Variables:

consider the augmented matrix $(A|\vec{b}) = \left(\begin{array}{ccccc|c} 1 & 3 & 0 & 7 & 0 & 4 \\ 0 & 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right)$

The leading one's are in the first, third and fifth columns.
→ The pivot columns of A are the first, third, and fifth columns.
→ The corresponding variables of the system $A\vec{x} = \vec{b}$ are x_1, x_3 and x_5.
Variables that correspond to a pivot are basic variables.
→ Variables that are not basic are free variables. They can take any value.
→ The free variables are x_2 and x_4. Any choice of the free variables leads to a solution of the system.

Note: A matrix, on its own, does not have basic variables or free variables. Systems have variables.

→ If A has n columns, then the linear system $(A|\vec{b})$ must have n variables. One variable for each column of the matrix.

→ There are 2 types of variables: basic and free.
A variable cannot be both free and basic at the same time.

$$\begin{aligned} n &= \text{number of columns of } A \\ &= (\text{number of basic variables}) + (\text{number of free variables}) \end{aligned}$$

THEOREM: Existence & Uniqueness

→ A linear system is consistent iff (exactly when) the last column of the augmented matrix does not have a pivot. This is the same as saying that the RREF of the augmented matrix does not have a row of the form

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 \cdots & 0 & 1 \end{array}\right)$$

Moreover, if a linear system is consistent, then it has

1. a unique solution iff there are no free variables, and
2. infinitely many solutions that are parameterized by free variables.

Example: If possible, determine the coefficients of the polynomial $y(t) = a_0 t + a_1 t^2$ that passes through the points that are given in the form (t, y).

(a) $L(-1,0)$ and $M(1,1)$

$L: a_0(-1) + a_1(-1)^2 = -a_0 + a_1 = 0$

$M: a_0(1) + a_1(1)^2 = a_0 + a_1 = 1$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right) \xrightarrow{R_1+R_2 \Rightarrow R_1} \left(\begin{array}{cc|c} 0 & 2 & 1 \\ 1 & 1 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}R_2 \Rightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \end{array}\right) \xrightarrow{R_1 - R_2 \Rightarrow R_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{array}\right)$$

⇒ unique solution, $y = \frac{1}{2}t + \frac{1}{2}t^2$.

(b) $P(2,0)$, $Q(1,1)$ and $R(0,2)$

$P: a_0(2) + a_1(2)^2 = 2a_0 + 4a_1 = 0$

$Q: a_0(1) + a_1(1)^2 = a_0 + a_1 = 1$

$R: a_0(0) + a_1(0)^2 = 2$

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 1 & 1 & 1 \\ \boxed{0 \quad 0} & \boxed{2} \end{array}\right)$$

System is inconsistent.
⇒ No solution.

TOPIC 3: Vector Equations

Motivation: We want to think about the algebra in linear algebra (system of equations & their solution sets) in terms of geometry (points, lines, planes, etc.)

↳ gives us deeper insight into the properties of systems and their solutions
↳ introduce n-dimensional space $\mathbb{R}^n$, and vectors inside it.

DEFINITION — $\mathbb{R}^n$

↳ let n be a positive whole number, i.e. $n \in \mathbb{Z}^+$

Recall: $\mathbb{R}$ denotes the collection of all real numbers.

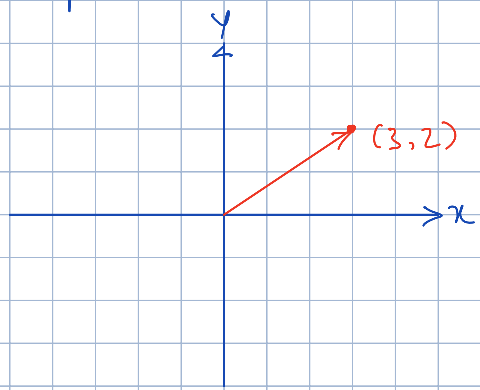
↳ $\mathbb{R}^n$ = all ordered n-tuples of real numbers $(x_1, x_2, \dots, x_n)$

When $n=1$, we get $\mathbb{R}$ back: $\mathbb{R}^1 = \mathbb{R}$. Geometrically, this is the number line.

When $n=2$, we can think of $\mathbb{R}^2$ as a plane.

* every point in this plane can be represented by an ordered pair of real numbers, its x- and y- coordinates

Example: Sketch the point $(3, 2)$ and the vector $\begin{pmatrix}3\\2\end{pmatrix}$.



So far, elements of $\mathbb{R}^n$ have been imagined as points: in the line, plane, space, etc.

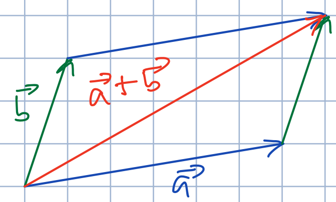
↳ elements of $\mathbb{R}^n$ can also be imagined as vectors: arrows with a given length and direction

Example, vector $\begin{pmatrix}3\\2\end{pmatrix}$ points horizontally in the amount of its x-coordinate, and vertically in the amount of its y-coordinate.

Vector Algebra

When imagining $\mathbb{R}^n$ as a vector, we write it as a matrix with n rows and one column. For example, suppose

$$\vec{u} = \begin{pmatrix}u_1\\u_2\end{pmatrix}, \quad \vec{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix}$$



Vectors have the following properties:

① Scalar multiples: $c\vec{u} = c\begin{pmatrix}u_1\\u_2\end{pmatrix}$

② Vector Addition: $\vec{u}+\vec{v}$

$$= \begin{pmatrix}u_1\\u_2\end{pmatrix} + \begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}u_1+v_1\\u_2+v_2\end{pmatrix}$$

DEFINITION — Linear combinations

↳ Given vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p \in \mathbb{R}^n$, and scalars $c_1, c_2, \dots, c_p$, the vector $\vec{y}$, where

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p$$

is called a <u>linear combination</u> of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ with weights $c_1, c_2, \dots, c_p$.

Example: Can $\vec{y}$ be represented as a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

$$\vec{y} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Solution: $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{y}$

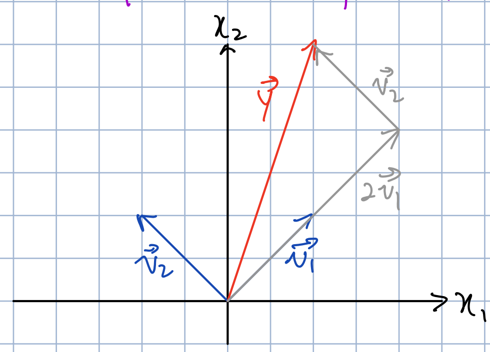
If $\vec{y}$ can be represented as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, we can find c_1 and c_2 so that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{y}$.

$$c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Rewrite this vector equation as a system of equations

$$\begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} -c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} c_1 - c_2 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



$c_1 - c_2 = 1$ — ①

$c_1 + c_2 = 3$ — ②

① + ② → $2c_1 = 4 \Rightarrow c_1 = 2$ *

$\Rightarrow 2 - c_2 = 1 \Rightarrow c_2 = 1$ *

$\therefore$ $\vec{y}$ can be represented as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

Geometric Interpretation of Linear Combinations:

Note: Any 2 vectors in $\mathbb{R}^2$ that are not scalar multiples of each other, span $\mathbb{R}^2$.

↳ This means that any vector in $\mathbb{R}^2$ can be represented as a linear combination of 2 vectors that are not multiples of each other.

Linear Combinations Example in $\mathbb{R}^3$:

Can $\vec{y}$ be represented as a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

$$\vec{y} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Solution: $c_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \begin{aligned} c_1 - c_2 &= 1 \\ c_1 + c_2 &= 3 \\ 0c_1 + 0c_2 &= 1 \end{aligned}$

$\Rightarrow$ The system is inconsistent.

$\Rightarrow$ No solution to this system.

$\therefore$ There are no values of c_1 and c_2 so that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{y}$.

$\therefore$ $\vec{y}$ cannot be expressed as a linear combination of the other 2 vectors.

DEFINITION — Span

↳ Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$, and scalars $c_1, c_2, \ldots, c_p$, the set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is called the <u>span</u> of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

Example: Is $\vec{y}$ in the span of vectors $\vec{v}_1$ and $\vec{v}_2$?

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \quad \text{and } \vec{y} = \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix}$$

Solution:

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{y}$$

$$c_1\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -3 & 6 & 15 \end{array}\right)$$

$$\begin{array}{l} R_2 + 2R_1 \to R_2 \\ R_3 + 3R_1 \to R_3 \end{array} \sim \left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 12 & 36 \end{array}\right)$$

$$\begin{array}{l} \frac{1}{9}R_2 \to R_2 \\ \frac{1}{12}R_3 \to R_3 \end{array} \sim \left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right)$$

$\Rightarrow$ inconsistent. $\therefore$ $\vec{y}$ is not in span $\{v_1, v_2\}$.

The Span of Two Vectors in $\mathbb{R}^3$:

[In general, any 2 nonparallel vectors in $\mathbb{R}^3$ span a plane that passes through the origin.] Any vector in that plane is also in the span of the two vectors.

Plane will pass through origin because origin is always going to be in the span of a set of vectors.

→ The linear combination where all the c's are 0 gives us a point; that point is the origin.