MODULE 3: LINEAR TRANSFORMS

TOPIC 1: An Introduction to Linear Transforms

Let A be a $m \times n$ matrix. We define the function

$$T: \mathbb{R}^n \to \mathbb{R}^m, \quad T(\vec{x}) = A\vec{x}$$

This is called a matrix transformation.

* domain of T is $\mathbb{R}^n$
* codomain of T is $\mathbb{R}^m$.
* vector $T(\vec{x})$ is the image of $\vec{x}$ under T.
* the set of all possible images $T(\vec{x})$ is the range

Example: Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $T(\vec{x}) = A\vec{x}$.

(a) What is the domain and codomain of T?

(b) Compute the image of $\vec{u}$ under T.

(c) What is the range of T?

(a) For $A\vec{x}$ to be defined, $\vec{x} \in \mathbb{R}^2$. $\therefore$ Domain $= \mathbb{R}^2$.

Also, $A\vec{x} \in \mathbb{R}^3$. $\therefore$ Codomain $= \mathbb{R}^3$.

(b) $T(\vec{u}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3+4 \\ 0+4 \\ 3+4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$.

(c) $T = A\vec{x} = x_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. $\Rightarrow$ Range is span $\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$

The function $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\vec{x}) = A\vec{x}$ gives another interpretation of $A\vec{x} = \vec{b}$.

5 ways of representing $A\vec{x} = \vec{b}$:
1. set of linear equations
2. augmented matrix
3. matrix equation
4. vector equation
5. linear transformation equation.

Example: Consider again the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, and associated transform $T(\vec{x}) = A\vec{x}$.

(a) Calculate $\vec{v} \in \mathbb{R}^2$ so that $T(\vec{v}) = \vec{b} = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$.

(b) Give a $\vec{c} \in \mathbb{R}^3$ so that there is no $\vec{v}$ with $T(\vec{v}) = \vec{c}$.
or: Give a $\vec{c}$ that is not in the range of T.
or: Give a $\vec{c}$ that is not in the span of the columns of A.

(a) If this is possible, then $T = A\vec{x} = \vec{b}$.

$$\left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array}\right) \xrightarrow[OR_3 \to R_3]{R_1 - R_2 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right). \quad \vec{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

(b) Let $\vec{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. $T(\vec{v}) = \vec{c} \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{array}\right)$

$R_1: v_1 + v_2 = 1$
$R_3: v_1 + v_3 = 1$ } inconsistent $\Rightarrow \vec{c}$ not in range of T.

Exercise: Consider the linear transform, $T(\vec{x})$.

$$T(\vec{x}) = A\vec{x}. \quad A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) What is the domain of $T(\vec{x})$? $\mathbb{R}^3$

(b) What is the codomain of $T(\vec{x})$? $\mathbb{R}^2$

(c) What is the range of $T(\vec{x})$? $T(\vec{x}) = x_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\Rightarrow$ Range: x_1-axis.

Geometric Interpretations of Linear Transforms

A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u> if

Fact: Every matrix transformation T_A is linear.

* $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$
* $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

} follows from properties of $A\vec{v}$.

So, if T is linear, then

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \dots + c_k T(\vec{v}_k)$$

This is called the <u>principle of superposition</u>.

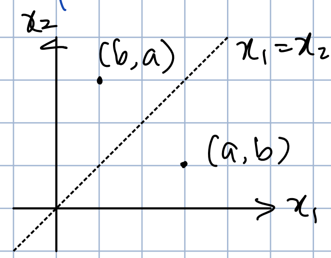
Geometric Interpretations of Transforms in $\mathbb{R}^2$

Example:
Suppose T is the linear transformation $T(\vec{x}) = A\vec{x}$. Give a short geometric interpretation of what $T(\vec{x})$ does to vectors in $\mathbb{R}^2$.

1. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 2. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ 3. $A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ for $k \in \mathbb{R}$.

1. Let $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix}$. $T(\vec{x}) = A\vec{x}$

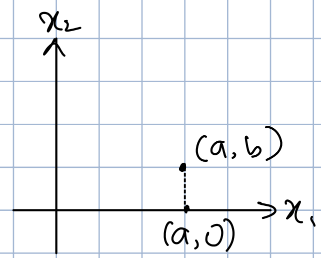
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}.$$



$T(\vec{x})$ is a reflection through the line $x_1 = x_2$.

2. Let $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix}$. $T(\vec{x}) = A\vec{x}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$



$T(\vec{x})$ is a projection onto the x_1-axis.

3. Let $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix}$. $T(\vec{x}) = A\vec{x}$

$$= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix} = k\begin{pmatrix} a \\ b \end{pmatrix}$$

$T(\vec{x})$ is an enlargement by scale factor of k.

Geometric Interpretations of Transforms in $\mathbb{R}^3$

Example: What does T_A do to vectors in $\mathbb{R}^3$?

(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Let $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. $T(\vec{x}) = A\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$

$T(\vec{x})$ is a projection onto the x_1x_2-plane.

(b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Let $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. $T(\vec{x}) = A\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ -b \\ c \end{pmatrix}$

$T(\vec{x})$ is a reflection through the x_1x_2-plane.

Constructing the Matrix of the Transformation

Example: A linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ satisfies

$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}5\\-7\\2\end{pmatrix}, \quad T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}-3\\8\\0\end{pmatrix}$$

What is the matrix A, so that $T = Ax$?

Matrix A is a 3×2 matrix. Let $A = (\vec{a}_1 \ \vec{a}_2)$.

$$\left.\begin{aligned} \text{Then } A\begin{pmatrix}1\\0\end{pmatrix} &= 1\vec{a}_1 + 0\vec{a}_2 = \vec{a}_1 = \begin{pmatrix}5\\-7\\2\end{pmatrix} \\ A\begin{pmatrix}0\\1\end{pmatrix} &= 0\vec{a}_1 + 1\vec{a}_2 = \vec{a}_2 = \begin{pmatrix}3\\8\\0\end{pmatrix}. \end{aligned}\right\} \therefore A = \begin{pmatrix}5 & 3\\-7 & 8\\2 & 0\end{pmatrix}.$$

Exercise: The linear transform $T(\vec{x}) = A\vec{x}$ reflects vectors across the x_1-axis. What is matrix A?

Let there be 2 vectors $\vec{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$, such that

$$T(\vec{e}_1) = A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\text{and } T(\vec{e}_2) = A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}.$$

Matrix A is a 2×2 matrix. Let $A = (\vec{a}_1 \ \vec{a}_2)$.

$$\left.\begin{aligned} T(\vec{e}_1) &= A\begin{pmatrix}1\\0\end{pmatrix} = \vec{a}_1 = \begin{pmatrix}1\\0\end{pmatrix}. \\ T(\vec{e}_2) &= A\begin{pmatrix}0\\1\end{pmatrix} = \vec{a}_2 = \begin{pmatrix}0\\-1\end{pmatrix}. \end{aligned}\right\} \therefore A = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}.$$

TOPIC 2: Linear Transforms

Standard Vectors

DEFINITION —— The Standard Vectors

↳ The standard vectors in $\mathbb{R}^n$ are the vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$, where:

$$\vec{e}_1 = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}, \quad \dots, \quad \vec{e}_n = \begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix}.$$

For example, in $\mathbb{R}^3$, $\quad \vec{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \ \vec{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \ \vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$

A Property of the Standard Vectors:

Note: If A is a $m \times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then

$$A\vec{e}_i = \vec{v}_i, \text{ for } i = 1, 2, \ldots, n.$$

So, multiplying a matrix by $\vec{e}_i$ gives column i of A.

Example: $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \vec{e}_2 = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$= 0\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 0\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

Theorem: The Standard Matrix

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that

$$T(\vec{x}) = A\vec{x}, \quad \vec{x} \in \mathbb{R}^n.$$

In fact, A is a $m \times n$ matrix, and its jth column is the vector $T(\vec{e}_j)$.

$$A = \big(T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3) \quad \cdots \quad T(\vec{e}_n) \big)$$

The matrix A is the <u>standard matrix</u> for a linear transformation.

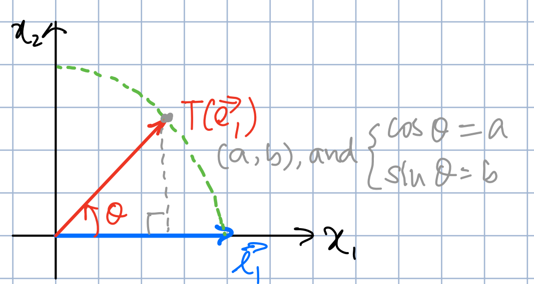
Standard Matrix for a Counterclockwise Rotation

What is the linear transform $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$T(\vec{x}) = \vec{x} \text{ rotated counterclockwise by angle } \theta \text{ about } (0,0).$$

$T(\vec{x}) = A\vec{x}$. Find A.

Matrix A is 2×2. $A = (\vec{a}_1 \;\; \vec{a}_2)$. Find $\vec{a}_1$ and $\vec{a}_2$.



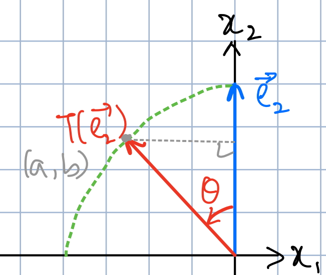
$A\vec{e}_1 = 1\vec{a}_1 + 0\vec{a}_2 = \vec{a}_1$

$T(\vec{e}_1) = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \vec{a}_1$

$A\vec{e}_2 = 0\vec{a}_1 + 1\vec{a}_2 = \vec{a}_2$

$T(\vec{e}_2) = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \vec{a}_2$

$\cos\theta = b$
$\sin\theta = -a$
$\Rightarrow a = -\sin\theta$.



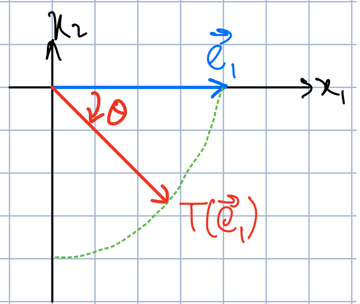
$$\therefore A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

https://xkcd.com/184

Standard Matrix for a Clockwise Transformation

Linear transform $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} \text{ rotated clockwise by angle } \theta \text{ about } (0,0).$$

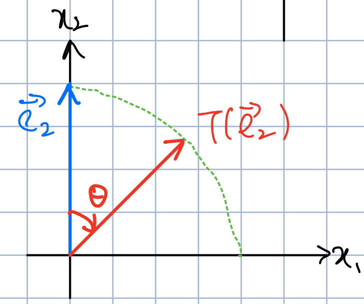


$T(\vec{x}) = A\vec{x}$. A is 2×2, $A = (\vec{a}_1 \ \ \vec{a}_2)$.

$A\vec{e}_1 = 1\vec{a}_1 + 0\vec{a}_2 = \vec{a}_1$. $T(\vec{e}_1) = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$

$A\vec{e}_2 = 0\vec{a}_1 + 1\vec{a}_2 = \vec{a}_2$. $T(\vec{e}_2) = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$

$$\therefore A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$



Example: constructing a standard matrix.

Define a linear transformation by

$$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$$

Is T one-to-one? Is T onto?

$T = A\vec{x}$, and $T(\vec{e}_1) = $ 1st column of A.

$$T(\vec{e}_1) = T(1,0) = (3(1)+0,\ 5(1)+7(0),\ 1+3(0)) = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$$

$$T(\vec{e}_2) = T(0,1) = (3(0)+1,\ 5(0)+7(1),\ 0+3(1)) = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}.$$

$$\Rightarrow T(\vec{x}) = A\vec{x}. \quad A = \begin{pmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{pmatrix}$$

Exercise: Suppose $T(\vec{x}) = A\vec{x}$ is the linear transform that satisfies the following:

$$T(\vec{e}_1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ T(\vec{e}_2) = \begin{pmatrix} 3 \\ 4 \end{pmatrix},\ \vec{e}_1 \in \mathbb{R}^2,\ \vec{e}_2 \in \mathbb{R}^2.$$

Construct the standard matrix for this transform, A.

$$\left.\begin{aligned} T(\vec{e}_1) &= A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \\ T(\vec{e}_2) &= A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}. \end{aligned}\right\} \quad A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}. \quad A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

Exercise: Suppose T is the linear transform below:

$$T(x_1, x_2, x_3) = (x_1 - x_2,\ x_2 - x_3,\ x_3 - x_1)$$

Construct the standard matrix of the transform A.

$$T(\vec{e}_1) = T(1,0,0) = (1-0,\ 0-0,\ 0-1) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

$$T(\vec{e}_2) = T(0,1,0) = (0-1,\ 1-0,\ 0-0) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

$$T(\vec{e}_3) = T(0,0,1) = (0-0,\ 0-1,\ 1-0) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

$$A\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

$$\therefore A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

How many pivot columns does A have?

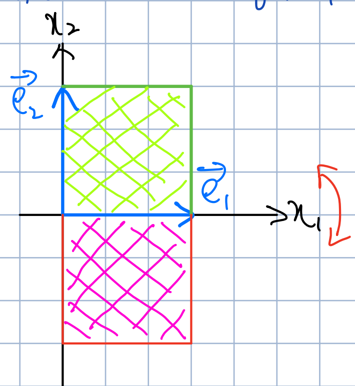
$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 + R_1 \to R_3} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_3 + R_2 \to R_3} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

A has 2 pivot columns.

Standard Matrices of Linear Transforms

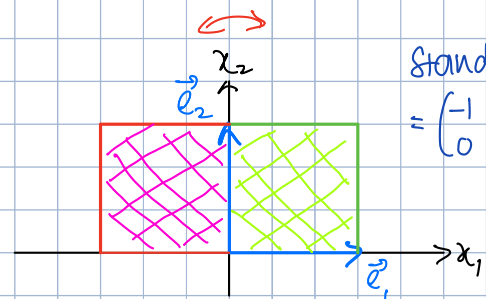
Standard matrices in $\mathbb{R}^2$:
- reflections
- contractions & expansions
- projections

① Reflection through x_1-axis



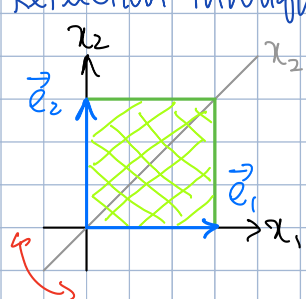
Standard matrix $= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

② Reflection through x_2-axis.



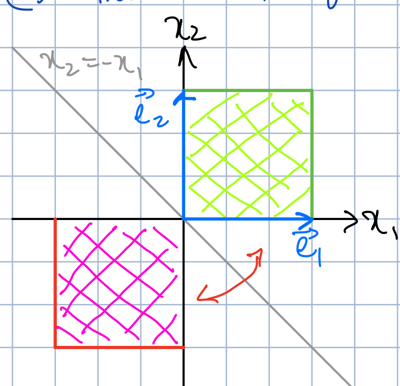
Standard matrix $= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

③ Reflection through $x_2 = x_1$



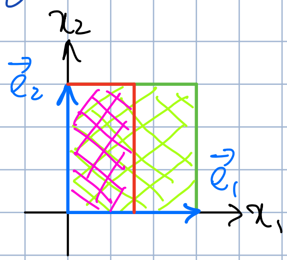
Standard matrix $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

④ Reflection through $x_2 = -x_1$



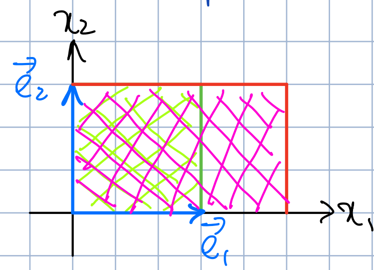
Standard matrix $= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

⑤ Horizontal Contraction



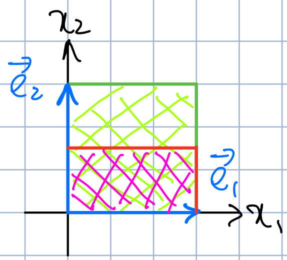
Standard matrix $= \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $|k|<1$

⑥ Horizontal Expansion



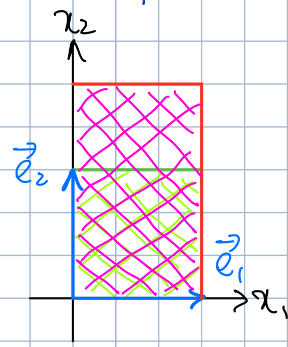
Standard matrix $= \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $k>1$

⑦ Vertical Contraction



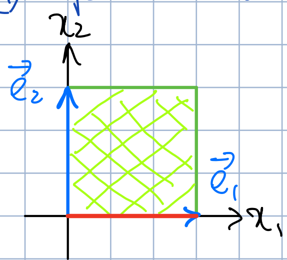
Standard matrix $= \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $|k|<1$

⑧ Vertical Expansion



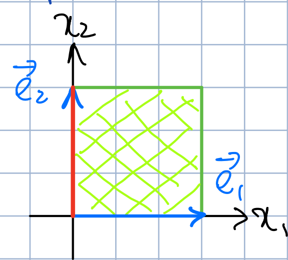
Standard matrix $= \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $k>1$

⑨ Projection onto the x_1-axis



Standard matrix $= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

⑩ Projection onto the x_2-axis



Standard matrix $= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Example: Composite Transform

Construct a matrix $A \in \mathbb{R}^{2\times 2}$, such that $T(\vec{x}) = A\vec{x}$, where T is a linear transformation that rotates vectors in $\mathbb{R}^2$ counterclockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$.

A is 2×2, $A = \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{pmatrix}$

Transformation 1, T_1: counterclockwise rotation by $\pi/2$ rad.
Transformation 2, T_2: reflection through the line $x_1 = x_2$

$T_1(\vec{e}_1) = T_1\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$T_2(T_1(\vec{e}_1)) = T_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$\Big\}\ T(\vec{e}_1) = T_2(T_1(\vec{e}_1)) = \begin{pmatrix}1\\0\end{pmatrix}.$

$T_1(\vec{e}_1) = T_1\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$

$T_2(T_1(\vec{e}_1)) = T_2\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$

$\Big\}\ T(\vec{e}_2) = T_2(T_1(\vec{e}_2)) = \begin{pmatrix}0\\-1\end{pmatrix}$

$\Big\}\ \therefore A = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

Exercise: The linear transform $T(\vec{x}) = A\vec{x}$ maps vectors in $\mathbb{R}^2$ to vectors in $\mathbb{R}^2$. Geometrically, $T(\vec{x})$ first reflects vectors across the line $x_1 = x_2$ and then rotates them clockwise by $\pi/2$ radians.

Find A.

A is 2×2. $A = \left(T(\vec{e}_1) \quad T(\vec{e}_2) \right)$

Transformation 1, T_1: Reflection across line $x_1 = x_2$.
Transformation 2, T_2: Clockwise rotation by $\frac{\pi}{2}$ rad.

$$\left.\begin{array}{l} \left.\begin{array}{l} T_1(\vec{e}_1) = T_1\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \\ T_2(T_1(\vec{e}_1)) = T_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \end{array}\right\} T(\vec{e}_1) = T_2(T_1(\vec{e}_1)) = \begin{pmatrix}1\\0\end{pmatrix} \\ \left.\begin{array}{l} T_1(\vec{e}_2) = T_1\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \\ T_2(T_1(\vec{e}_2)) = T_2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} \end{array}\right\} T(\vec{e}_2) = T_2(T_1(\vec{e}_2)) = \begin{pmatrix}0\\-1\end{pmatrix} \end{array}\right\} \therefore A = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}.$$

Onto and One-to-One

DEFINITION — Onto

In other words, $A\vec{x} = \vec{b}$ is always consistent.

↳ A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto if for all $\vec{b} \in \mathbb{R}^m$ there is a $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = A\vec{x} = b$.

Implications:
* Onto is an existence property: for any $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution.
* T is onto if and only if its standard matrix has a pivot in every row.

example, if $A = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$, then $T(\vec{x}) = A\vec{x}$ is not onto.

ex. $\vec{b} = \begin{pmatrix}0\\1\end{pmatrix}$ is not in range of $T(\vec{x})$

DEFINITION — one-to-one

In other words, $A\vec{x} = \vec{b}$ has at most one solution.

↳ A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if for all $\vec{b} \in \mathbb{R}^m$ there is at most one (possibly no) $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = A\vec{x} = b$.

Implications:
* One-to-one is a uniqueness property, it does not assert existence for all $\vec{b}$.
* T is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is the zero vector, $\vec{x} = \vec{0}$.
* T is one-to-one iff every column of A is pivotal.

ex. if $A = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 1\end{pmatrix}$, $T(\vec{x}) = A\vec{x}$ is not one-to-one.

ex. if $\vec{b} = \begin{pmatrix}2\\3\end{pmatrix}$, $\vec{x} = \begin{pmatrix}2\\3\end{pmatrix} - x_3\begin{pmatrix}1\\1\end{pmatrix}$

Example: Matrix Completion, one-to-one and onto.

Complete the matrices by entering numbers into the missing entries so that the properties are satisfied. If it isn't possible to do so, state why.

(a) A is a 2×3 matrix for a one-to-one transform.

$$A = \begin{pmatrix} 1 & 0 & \\ 0 & & 1 \end{pmatrix}$$

not possible.
∵ need every column to be pivotal.

(b) B is a 3×3 standard matrix for a transform that is one-to-one and onto.

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem: Onto Transforms

For a linear transformation, $T: \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A, these are the equivalent statements:

1. T is onto.
2. A has columns that span $\mathbb{R}^m$.
3. Every row of A is pivotal.

ex. $T = A\vec{x}$ where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Theorem: One-to-one Transforms

For a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A, these are equivalent statements:

1. T is one-to-one.
2. The unique solution to $T(\vec{x}) = \vec{0}$ is the trivial one.
3. A has linearly independent columns.
4. Each column of A is pivotal.

eg. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Example: Constructing a standard Matrix, One-to-one and onto

Define a linear transformation by

$$T(x_1, x_2) = (3x_1 + x_2,\ 5x_1 + 7x_2,\ x_1 + 3x_2)$$

Construct the standard matrix for the transformation. Is T one-to-one? Is T onto?

$T = A\vec{x}$, A is 3×2, $A = (T(\vec{e}_1)\ \ T(\vec{e}_2))$

$$T(\vec{e}_1) = T(1,0) = (3,5,1) = \begin{pmatrix}3\\5\\1\end{pmatrix}.$$

$$T(\vec{e}_2) = T(0,1) = (1,7,3) = \begin{pmatrix}1\\7\\3\end{pmatrix}$$

$$A = \begin{pmatrix}3 & 1\\ 5 & 7\\ 1 & 3\end{pmatrix}$$

Since number of rows are more than number of columns, T cannot be pivotal.

For T to be one-to-one, columns need to be linearly independent, and by inspection the columns are not multiples of each other.

$\Rightarrow$ T is one-to-one.

Example: Linear Transform Review

Suppose A is a $m\times n$ standard matrix for transform T, and there are some vectors $\vec{b} \in \mathbb{R}^m$ that are not in the range of $T(\vec{x}) = A\vec{x}$.

(1) $A\vec{x} = \vec{b}$ could be inconsistent.

True. ex. $T = A\vec{x}$, $A = \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$, $\vec{b} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

(3) T could be one-to-one

True. ex. $A = \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$

(2) There cannot be a pivot in every column of A.

False, $A = \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$

Exercise: Indicate whether the following situations are possible or impossible.

(1) T is a one-to-one linear transform that maps vectors in $\mathbb{R}^5$ to $\mathbb{R}^4$.

$\Rightarrow$ Impossible. If T is a linear transform that maps vectors $\mathbb{R}^5$ to $\mathbb{R}^4$ then the standard matrix has five rows and four columns. So there cannot be a pivot in every column. The transform cannot be one-to-one.

(2) T is an onto linear transform that maps vectors in $\mathbb{R}^{26}$ to $\mathbb{R}^{20}$.

$\Rightarrow$ Possible. If T is a linear transform that maps vectors $\mathbb{R}^{26}$ to $\mathbb{R}^{20}$ then the standard matrix has 20 rows and 26 columns. So there could be a pivot in every row. The transform could be onto.

③ T is an onto linear transform that maps vectors in $\mathbb{R}^6$ to $\mathbb{R}^2$. The linear system $A\vec{x}=\vec{b}$ is inconsistent, where $\vec{x} \in \mathbb{R}^6$, $\vec{b} \in \mathbb{R}^2$.

$\Rightarrow$ Impossible. If T is an onto linear transform that maps vectors in $\mathbb{R}^6$ to $\mathbb{R}^2$, then $A\vec{x}=\vec{b}$ is always consistent.