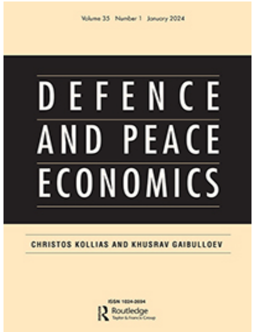

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Collective Action and Intra-group Conflict with Fixed Budgets

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ABSTRACT
We study collective action under adverse incentives: each member of the group has a given budget ('use-it-or-lose-it') that is private information and can be used for contributions to make the group win a prize and for internal fights over this very prize. Even in the face of such rivalry in resource use, the group often succeeds in overcoming the collective action problem in the non-cooperative equilibrium. In one type of equilibrium, all group members jointly contribute; in the other type of equilibrium, volunteers make full standalone contributions. Both types of equilibrium exist for larger and partially overlapping parameter ranges.

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Introduction

In economic analysis, conflicts are often considered and analyzed as an isolated phenomenon even though they are usually embedded in a much more complex web of political interests and disputes. Faced with budget constraints, the actors have to anticipate the course of action of their rivals as well as their allies and decide for themselves how to allocate their resources to different conflict arenas. We study the specific problem in which two players face an internal conflict, while, at the same time, they are in an alliance of convenience in a conflict with a third party.

Alliances of convenience between parties that are actually rivals are not a rare phenomenon. The situation is well illustrated by the conflict between China and Japan. The period of the Sino-Japanese War (1937–1945) was accompanied by an internal conflict between the Kuomintang (KMT) led by Chiang Kai-shek and the Chinese Communist Party (CCP) led by Mao Zedong which evolved before, during, and after that war. Historians such as van de Ven (2018) highlight and study the phenomenon of civil war taking place between the CCP and the KMT simultaneously with the Sino-Japanese war, which allows an understanding of these events only in the face of strong mutual interactions. Mitter (2013) also describes the difficult and fluid relationship in which the CCP and the KMT partly fought with each other against the Japanese invasion during the Sino-Japanese War (1937–1945). The CCP and the KMT were seemingly well aware of the strategic implications of a change in each other's relative strength as a result of military fighting with Japan. Mitter (2013, p. 324) provides indirect evidence of this when he writes:

> Soviet adviser Peter Vladimirov recorded that " the CCP leadership rejoices at the news of the defeat suffered by Chiang Kai-shek's troops in Honan [Henan] and Hunan [. . .]."

Outcomes of fights between Japanese forces and the KMT such as the Japanese success that led to the occupation of large parts of Henan and Hunan are informative about the relative strength of the

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internal rival. In this example, the weakening of the KMT is to the advantage of the CCP. Another related example is the alliance of convenience between Russia and the United States during the Second World War. The alliance members were very aware of the ongoing conflict between them. As the former Secretary of State James F. Byrnes (1947, p. 44) describes in his memories of the Yalta conference:

> One statement of Stalin's that interested me was: " It is not so difficult to keep unity in time of war since there is a joint aim to defeat the common enemy, which is clear to everyone. The difficult task will come after the war when diverse interests tend to divide the Allies. It is our duty to see that our relations in peacetime are as strong as they have been in war."

Byrnes (1947, p. 45) reminds us that the 'tide of Anglo-Soviet-American friendship had reached a new high' at the Yalta conference, but that the 'tide began to ebb' very quickly.[1] The importance of conflict inside the alliance during and/or after the conflict with an external enemy distinguishes these historical examples from studies of inter-group conflict with peaceful, non-adversarial rules about how to share the spoils of victory within the group.

We focus on intra-group conflict that generates opportunity costs of contributing own resources to the fight against the common enemy. The opportunity cost argument which we study differs from the well-known insights gained in inter-group conflict models. In these frameworks, it is well known that the incentives to contribute effort to the alliance objectives largely depend on how the gains of victory are allocated among group members.[2] Also, unless an alliance follows other institutional provisions (e.g. enforced peaceful prize sharing), the time-consistent behavior of members of a victorious alliance is to fight over the spoils of victory. This reduces the value that alliance members attribute to the victory of their alliance and thus reduces their incentives to contribute to the objective of the alliance.[3] These formal frameworks assume that alliance players can mobilize new resources. They incur additional fighting costs in the internal conflict that follows if they fight internally, and this internal conflict diminishes the players' expected net gains from winning. However, the contributions to group effort do not directly restrict a player's fighting ability or the set from which intra-group fighting effort can be chosen. In contrast, the simultaneous determination of the military resources devoted to the fight against the Japanese invaders and the military resources used in the internal civil war during the Sino-Japanese War led to a strict rivalry of the two types of fighting effort. Military resources deployed by Chiang Kai-shek against Japan reduced his clout in the internal Chinese conflict, and as Mitter's (2013) quote illustrates, the opponents were well aware of this fact.[4]

To explore the case in which contributions to group effort directly infringe upon a player's fighting ability in the intra-group conflict, we build on a structure that has been studied as Colonel Blotto games.[5] In such games, the contestants have a given stock of resources. These resources have no use outside the conflict game (that is, are 'use it or lose it'); put differently, resources not used have a value of zero. Contestants decide how to allocate this stock of resources to different battles. Our analysis focuses on alliance members who do not know the exact size of each other's resource budget and either follow standalone equilibrium strategies or equilibrium strategies in which players with a sufficiently large budget make joint, equal contributions.[6]

The assumption of incomplete information also distinguishes our work from much of the work on inter-alliance conflict, which is predominantly work that assumes complete information.[7] Incomplete information about the group members' budget leads to interesting equilibrium outcomes in which group members frequently provide efficient group effort, despite the public good nature of such contributions and despite the adversarial relation between the players of a group. If players know their own budgets but not that of the other member of their group, each player is able to make plans based on the precise knowledge of their own budgets. But as the player knows only the distribution from which the other player's budget is drawn, the intra-group effort of the other group member becomes uncertain. Even though the internal conflict is sharply resolved in favor of the player who has the higher remaining budget, the uncertainty about the other player's budget makes a player's anticipated equilibrium probability of winning (and expected payoff) a smooth function of her own remaining budget.

Summarizing, we study a resource allocation problem in which the members of an alliance win only if their joint effort contributions to the alliance are high enough, and in which the spoils of victory go to the player who, after making such contributions, is richer than the other player. At the margin, each player can increase the win probability of the alliance at the cost of reducing her own internal win probability if the alliance succeeds. We look at this problem from a game-theory perspective for a group of two players with incomplete information: players know their own budgets but not the other alliance member's overall budget. We identify two types of symmetric threshold equilibria. The equilibria described are inefficient ex ante compared to a coordinated action that maximizes the sum of the payoffs of the group members, but in both types of equilibrium the players achieve the group objective for a wide range of budget combinations. In one of these equilibrium types, the players' joint efforts just match the threshold required to succeed. In another type of equilibrium, the group succeeds because one or potentially both players contribute the full amount of resources needed to acquire the asset. Both types of equilibrium are characterized by an under-provision of group effort for some range of budget combinations.

There are two further lines of literature that are related to our study. The first is the literature on the collective provision of a discrete public good (see, e.g. Bagnoli and Lipman 1989, 1992) and, in particular, the studies that allow for incomplete information about valuations (Menezes, Monteiro and Temini 2000; Barbieri and Malueg 2008) or contribution costs (Bliss and Nalebuff 1984; Fudenberg and Tirole 1986) of potential contributors. In the Blotto game we consider, the group objective is also described by a given threshold. The objective is reached if the group members' contributions reach this threshold. Unlike in the literature mentioned, however, provision occurs with resources that are part of a player's given overall budget: the opportunity cost of higher effort is not a function of the effort itself but emerges indirectly as the resources have an important alternative use if and only if the amounts contributed are sufficient to win the later confrontation with the other group member.

Second, the paper is related to the guns-and-butter models of conflict (see, e.g. Haavelmo 1954; Hirshleifer 1985; Skaperdas 1992 for seminal contributions). These models consider players who have endowments that can be used for different purposes: 'butter' and 'guns' in the simplest case. Butter represents the returns of productive investments which the players fight for using the part of their budget that did not go into the production of 'butter' but has instead been invested in the production of 'guns.'[8] Studies that consider opponent groups and intra-group conflict in the guns-and-butter game include Münster (2007) and Bakshi and Dasgupta (2020a, 2020b). Our analysis shares with the guns-and-butter literature that a player who devotes more resources to the production of consumable output automatically produces fewer guns, i.e. is weaker in the distributional conflict with the other group members. Perhaps most closely related to our analysis is the guns-and-butter model by Hodler and Yektaş (2012) who consider incomplete information about the resource endowment. There, the distributional conflict is an all-pay-auction without noise, as in our analysis, but overall productive output ('butter') is a continuous public good: output is a continuous function of efforts. In our context, the group reaches its objective if the players jointly expend an amount of group effort that at least matches a given threshold. This causes a twofold discontinuity: whether the collective task is achieved, and who wins the internal conflict.

In what follows, we first outline the general framework, then study equilibria in which the group objective is achieved by the contributions of a strong standalone player, and then turn to the equilibria in which both players contribute positive amounts to achieve the collective objective.

The Framework

Consider a group with $n = 2$ members. Each of them has an initial budget, and these budgets are denoted by m_i with $i \in \{1, 2\}$. The budget size characterizes the player's type. These budgets are independent random draws from the same cumulative distribution function F with full support on $[0,\bar{m}]$. We assume F to be continuous, differentiable, and concave on $(0,\bar{m})$.[9] This characterizes the type space and distribution of types. Each player knows her own budget and believes that the other player's budget is a random draw from the distribution F.

The players simultaneously choose their contributions $x_1 \in [0, m_1]$ and $x_2 \in [0, m_2]$ that sum up to joint contributions $x_1 + x_2$ and also determine player i's remaining resources $m_i - x_i$. Given the players' beliefs about their co-player's budget, their strategies are functions that map their own budget into contributions $x_1(m_1)$ and $x_2(m_2)$. Player 1's payoff is determined as

$$\pi_1 = \begin{cases} 1 & \text{if} x_1 + x_2 \geq b \text{and} m_1 - x_1 > m_2 - x_2 \\ \frac{1}{2} & \text{if} x_1 + x_2 \geq b \text{and} m_1 - x_1 = m_2 - x_2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and player 2's payoff is described analogously, replacing 1 by 2 and vice versa. The payoff function (1) shows that the budget is 'use-it-or-lose-it.' Players are in a special type of Blotto game. They are members in a fight of their team for a common threshold goal, but they also fight inside the own team over the allocation of what the team wins by reaching the goal. The parameter $b > 0$ is a given positive constant and is observed by the players prior to their effort choices x_1 and x_2. It is the threshold which needs to be matched or topped by the joint efforts of players 1 and 2. This threshold b captures a common adversary's investment. To defeat this adversary this amount b needs to be matched by the alliance's joint effort. The common adversary's choice behavior is not explicitly modeled in order to focus on the adverse incentives inside the alliance generated by the necessity to allocate resources between the external and the internal conflict.

A pair of strategies $x_1^*(m_1)$ and $x_2^*(m_2)$ is a Bayesian Nash equilibrium if, given the players' beliefs about the co-players' budgets, these strategies are mutually optimal replies, i.e. if

$$x_1^*(m_1) = \arg \max_{x_1 \in [0, m_1]} \left\{ \int_0^{\bar{m}} \pi_1\left(x_1; x_2^*(m_2)\right) dF(m_2) \right\}$$

and analogously for player 2.

The conflict game studied has two discontinuities for each player. The first discontinuity is on the sum of contributions $x_1 + x_2$ that need to match or exceed b in order to generate a positive payoff inside the group. The second discontinuity is on the determination of the winner inside the group, conditional on $x_1 + x_2$ exceeding b. As players want a group victory but also want to outbid their internal rival exactly in this case of victory, this begs the question of the existence of a (pure-strategy) equilibrium. For the complete-information case, Klumpp and Konrad (2019) show that such equilibria exist with a specific knife-edge property, basically making the discontinuity in whether the group wins interact with the discontinuity for how they split the gain from winning.[10] Incomplete information makes this type of knife-edge equilibrium infeasible, but at the same time smoothens the problem and allows for different types of equilibrium.

We focus on symmetric equilibria in monotonic threshold strategies, i.e. equilibria in which the mutually optimal replies have the following properties: $x_i^*(m_i) = 0$ for all $m_i \in [0, m^*)$ and $x_i^*(m_i) = t$ for all $m_i \in [m^*, \bar{m}]$, for some $m^* \in (0, \bar{m})$ and $t \in (0, m^*]$. The equilibrium concept is Bayesian Nash equilibrium.

Standalone Equilibria

Equilibrium Characterization

First, we consider equilibria in which single players might make contributions that are sufficient to 'stand alone,' i.e. sufficient to match or exceed b. Hence, in the equilibrium with threshold m^*, the players contribute an amount $x_i^*(m_i) = b$ if and only if $m_i \geq m^*$.

Proposition 1 *Let $b < \bar{m}$ and suppose that F is (weakly) concave on the support $[0, \bar{m}]$. There is a unique symmetric standalone equilibrium with threshold m^* defined by*

$$F(m^*) = F(m^* + b) - F(m^* - b), \tag{2}$$

where m^* satisfies $m^* \in (b, 2b]$ and $m^* < \bar{m}$.

When considering a standalone contribution, a player faces a trade-off: the standalone contribution makes sure that the alliance mobilizes sufficient effort to succeed but, at the same time, considerably reduces the player's chances of winning the internal conflict. Equilibrium existence basically requires two conditions. First, types above the threshold m^* (who contribute an amount b) should have no incentive to deviate to a zero contribution in an attempt to free-ride. As we prove in the appendix, such a deviation is most attractive for types m_i just above the threshold m^*, that is, the 'poorest' among the contributing players. Quite intuitively, if they contribute they can only win in the rare cases in which j has a very low budget. This yields a first necessary condition for equilibrium existence and requires the threshold m^* to be rather large since the types just above the threshold would otherwise prefer to free-ride.

Second, types just below the threshold m^* (who free-ride in the equilibrium) should have no incentive to deviate to a standalone contribution.[11] Those types $m_i < m^*$ have a good chance of winning the internal fight since they save their entire resource budget for this fight. They must, however, rely on their alliance partner to provide the standalone contribution to make the alliance successful. Again, as we prove in the appendix, a deviation to a standalone contribution is most attractive for types m_i just below the threshold m^*, that is, the 'richest' among the non-contributing players. Ruling out this deviation yields a second necessary condition for equilibrium existence and requires the threshold m^* to be rather low so that players with budgets just below m^* indeed choose to free-ride. These two restrictions on the incentives of intermediate types just above and just below the threshold m^* characterize a unique symmetric standalone equilibrium with a threshold given by (2).[12]

In order to understand the players' incentives to contribute, it is insightful to consider possible values of m_i and i's payoff depending on the realization of m_j. This also shows that player i's equilibrium expected payoff is non-monotone in m_j for intermediate budgets m_i. Figure 1 illustrates all possible combinations (m_i, m_j) that can occur. In the area to the lower left where $m_i < m^*$ and $m_j < m^*$, both players get zero payoff because no contribution is made. If $m_i \geq m^*$ and/or $m_j \geq m^*$, exactly one of the players i and j wins.

There are four sets of player types m_i. First, there are players i who never win: those with $m_i < m^* - b$ whose resources are not even sufficient to beat a player j who contributes. Note that $m^* \leq 2b$ (compare Proposition 1) implies that $m_i < b$ for those types of players: those players do not have sufficient resources to deviate to a standalone contribution.[13] Second, there are players who free-ride and get a positive expected payoff: those with $m_i \in (m^* - b, m^*)$. Those players would be able to contribute (at least if m_i is sufficiently close to m^*) but they prefer to free-ride, hoping that m_j is in some intermediate interval so that the alliance is successful and they have a chance of beating j in the internal conflict. Moving up vertically in Figure 1 for a given $m_i \in (m^* - b, m^*)$ illustrates the outcome for the possible realizations of m_j, showing that i wins if and only if $m_j \in [m^*, m_i + b)$. Since i's expected payoff is equal to the probability that m_j is in the specified interval, the upper arrow just to the left of m^* indicates the expected equilibrium payoff of types just below the threshold (that is, the 'richest' among the non-contributing players).

Third, there are players with $m_i \in [m^*, m^* + b)$ who contribute: those types can win against free-riders only if m_j is very low (below $m_i - b$) but are beaten by free-riding players j with a budget m_j close to m^*. In addition, they win against contributing players j with a budget below their own but are beaten by richer players. Figure 1 shows the two cases in which those players i win by moving up vertically for a given $m_i \in [m^*, m^* + b)$. The lower arrow just to the right of m^* hence indicates the expected equilibrium payoff of types just above the threshold who never win against other

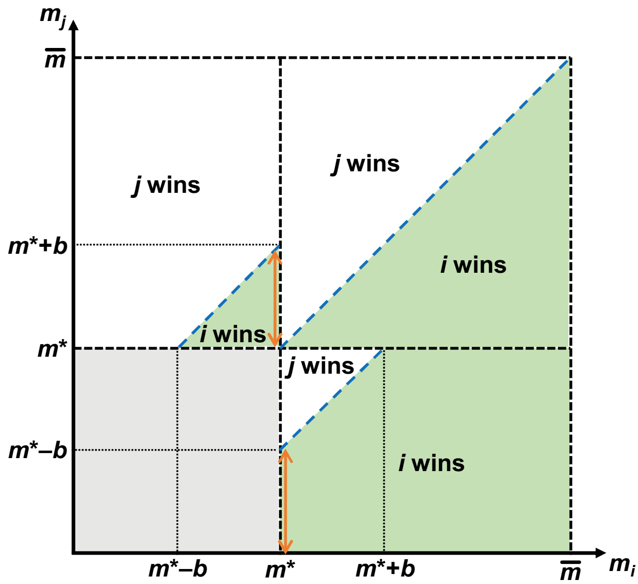

Figure 1. The figure illustrates the different combinations (m_i, m_j) for which either i or j wins in the standalone equilibrium. Focusing on player i, there are four different regions (player types). From left to right, there are (i) players i with $m_i < m^* - b$ who never win since $m_j - b > m_i$ for all j who contribute; (ii) players i with $m_i \in (m^* - b, m^*)$ who win if and only if $m_j \in (m^*, m_i + b)$; (iii) players i with $m_i \in (m^*, m^* + b)$ who win if $m_j < m_i - b$ or if $m_j \in (m^*, m_i)$; (iv) players i with $m_i > m^* + b$ who win if and only if $m_j < m_i$. The expected payoff of a type $\tilde{m}_i$ is obtained by drawing a vertical line at $m_i = \tilde{m}_i$ and calculating the probability that m_j is in the interval where $\tilde{m}_i$ wins. Hence, the arrow just to the left of m^* indicates the expected payoff of (free-riding) types m_i just below m^*: they win if $m_j \in (m^*, m_i + b)$ where $m_i \to m^*$ from below. The arrow just to the right of m^* indicates the expected payoff of (contributing) types m_i just above m^*: they win if $m_j < m_i - b$ where $m_i \to m^*$ from above. In equilibrium, these two payoffs must be identical for deviations of types just below and just above m^* to be non-profitable.

contributing types (that is, of the 'poorest' among the contributing players). Finally, there is a group of players with very high resources. Provision of the standalone effort to the team is best for them, and they never lose against free-riding players j. But still, if such a player fights with another, even stronger team member then this other member also contributes the standalone effort, and still beats player i.

Example

Corollary 1 summarizes the equilibrium strategies for the special case of a uniform distribution for which a closed form solution for $m^*(b)$ exists.

Corollary 1 *Suppose F is a uniform distribution on* $[0, 1]$.
(i) *If* $b < 1/3$, *there is a unique symmetric standalone equilibrium where* $x_i^* = b$ *if* $m_i \geq m^* = 2b$ *and* $x_i^* = 0$ *otherwise.*
(ii) *If* $b \in [1/3, 1)$, *there is a unique symmetric standalone equilibrium where* $x_i^* = b$ *if* $m_i \geq m^* = (1 + b)/2$ *and* $x_i^* = 0$ *otherwise.*

The proof of the more general proposition for weakly concave or concave distribution functions F in the appendix also proves the corollary, but the uniform distribution case offers some illustration, as the probability of m_j being in some interval from $[0, 1]$ is equal to the length of the interval. Let the required contributions be small (Corollary 1(i)). Players contribute if and only if $m_i \geq 2b$. This means that a player gets zero expected payoff if and only if $m_i < m^* - b = b$ (compare Figure 1). Here, only players with resources smaller than b get zero expected payoff. The players who free-ride but still realize a strictly positive expected payoff would all be powerful enough to make a standalone contribution. But doing so reduces their remaining resources so much that they would be defeated by all players who have resources above m^* and contribute. Thus, they prefer to hold back their resources, hoping that the other player has just somewhat more resources, provides the standalone effort and can then be beaten.

To illustrate the reason for the uniqueness of the threshold $m^* = 2b$, consider the case where $b < 1/3$ and suppose that player 2 chooses a threshold different from $2b$. For a smaller equilibrium candidate threshold $m_2^* \in [b, 2b)$ consider player 1 with $m_1 = m_2^*$. This player's probability of winning for $x_1 = b$ is $m_2^* - b$. For $x_1 = 0$, this player's probability of winning is $m_2^* + b - m_2^* = b$. But $b > m_2^* - b$ for all $m_2^* \in [b, 2b)$. Hence, player 1 with $m_1 = m_2^*$ is strictly better off contributing 0 than contributing b. This contradicts a symmetric equilibrium with a threshold $m^* < 2b$. Suppose next that the candidate threshold is $m_2^* \in (2b, 1 - b]$. Consider player 1 with $m_1 = m_2^* - \varepsilon$, $\varepsilon \in (0, b)$ and close to zero. This player's payoff from $x_1 = 0$ is $m_1 + b - m_2^* = b - \varepsilon$. For $x_1 = b$ this player's probability of winning is $m_1 - b = m_2^* - \varepsilon - b$. But $m_2^* - \varepsilon - b > b - \varepsilon$ if $m_2^* > 2b$, so player 1 with $m_1 = m_2^* - \varepsilon$ would prefer $x_1 = b$. This rules out a symmetric equilibrium with a threshold $m^* > 2b$. Similar contradictions can be constructed for the remaining parameter values in the corollary.

These standalone equilibria can be compared with several generic games of non-cooperative provision of a discrete public good with standalone contributions: the various versions of the volunteer's dilemma. A static version of this problem was studied by Diekmann (1985). Its dynamic version is the waiting game, as in Bliss and Nalebuff (1984) and Fudenberg and Tirole (1986). In these games, the provision of the public good has a direct cost for the standalone contributor, and the contributors' individual benefit from the provision of the public good exceeds this cost of standalone provision. Players are willing to incur the cost of standalone provision if the alternative is that the public good is not provided. But if several players are willing to volunteer, players prefer to free-ride if they anticipate that others will make the provision. Players randomize in Diekmann's (1985) static volunteer's game. Zero, one or multiple players might then expend the cost of standalone provision in the equilibrium. In the dynamic versions, the problem turns into a waiting game, and incomplete information about other players' contribution costs and the choice of timing for own action resolves the coordination problem between the players.

The standalone equilibria here start with a very different framework and focus on a different trade-off. Also here, a major goal is the provision of a discrete public good, but contributions to it are made from a given individual budget and do not cause a genuine cost for the contributing player. They are costly only insofar as they reduce the resource endowment of the contributing player in the internal conflict that might follow the public good provision. This opportunity cost matters if the players' contributions are sufficiently high to be successful. As players do not know the resource endowment of the other players in their group, they face strategic uncertainty. This can make none, one or both players independently decide to make a stand-alone contribution. Overprovision occurs if both players are very resource-rich. For uniformly distributed budgets with a maximum budget $\bar{m} > 3b$, this happens if both have budgets that exceed $2b$. Their independent decisions can also lead to what can be seen as underprovision from the perspective of the group. Think of the case with a uniform distribution F and $\bar{m} > 3b$. If players have budgets in the range between b and $2b$, each of them has sufficient resources to make the group win, but none of them contributes the standalone effort and the group does not win. This is precisely the range in which the player prefers to free-ride and hopes that the other group member is sufficiently rich to make the contribution but not too rich so that she could still beat this other player.

Before considering joint contribution equilibria as a second type of equilibrium, we briefly discuss how incentives for standalone contributions change in our framework if (i) the cost of contributing is reduced in the sense that not all resources contributed to alliance success are lost for the internal conflict, and (ii) the number of alliance members goes up.

Standalone Contributions with Partial Carryover

Our framework assumes that resources contributed to the common goal of fighting an external threat are fully lost for the internal conflict within the alliance. This is a plausible assumption under many circumstances. If, for instance, the internal and external conflicts take place simultaneously, a player with given resources must decide which part of her resources to send to which of the two conflicts. The resources may or may not be destroyed in the conflicts but even if all resources largely survive the conflicts, the Blotto trade-off in their use results from the simultaneity of their use.

If external and internal conflicts take place one after the other, a relevant question is whether the resources used to avert the external threat are lost in the process or are still partly available for the internal conflict later on. If, for example, a military threat is averted by paid mercenaries, then the resources expended for this purpose are used up and are no longer available for the internal conflict. Recruits of a national army, on the other hand, who survive the external conflict can later fight in the internal conflict as well.

To incorporate the latter possibility into our analysis, we briefly discuss the implications if the resource contributions to the group effort are not ultimately lost and a player can use a share of these resources in the internal conflict. Such a variation of our framework can most easily be captured by a parameter $\alpha \in [0, 1]$ that reflects the share of contributions x_i that is lost for the internal conflict. Hence, our main framework maps the case of $\alpha = 1$; a weaker trade-off between contributing to the common goal and being prepared for the internal conflict is reflected by a lower value of α. Contributions x_i to alliance success are still subject to the constraint $x_i \leq m_i$ since players cannot contribute more than their resource budget.

Intuitively, a lower α makes contributions to the common goal less costly and should therefore lower the equilibrium threshold m^* for a standalone contribution. With uniformly distributed budgets on $[0, 1]$, for instance, the equilibrium threshold m^* is equal to $m^* = \max\{2\alpha b, b\}$ if b is small (precisely, if $b < \min\{1/(3\alpha), 1\}$). For larger values of b (precisely, if $b \in [\min\{1/(3\alpha), 1\}, 1)$), the equilibrium threshold is equal to $m^* = \max\{(1 + \alpha b)/2, b\}$. This means that m^* becomes lower if α decreases, that is, a lower opportunity cost of contributing leads to less free-riding. If α is very low, then $m^* = b$ so that the equilibrium is efficient in that players make a standalone contribution whenever they have sufficient resources to do so. This is most obvious for $\alpha \to 0$ where there is no opportunity cost of contributing.

Standalone Contributions with More than Two Players

The main analysis of the case with two players strikes us as particularly relevant for many real-world examples in which there are two main rivals facing an internal conflict when making their contributions to a common goal. This maps, for instance, the internal conflict between the communist movement and the Kuomintang in China who jointly faced the Japanese invaders during the Second World War. If instead the alliance consists of $n \geq 3$ players (with budgets drawn independently from the same distribution function F), this affects the players' incentives to make a standalone contribution in two ways. First, as in classic models of the private provision of a discrete public good, the probability of mobilizing sufficient resources b would be increasing in the number of players if the threshold m^* for a standalone contribution was kept unchanged. But if it becomes more likely that someone else will provide the effort (which happens with probability $1 - (F(m^*))^{n-1}$ from the point of view of player i, a probability that increases in n), the incentive to free-ride is strengthened. Second, and in contrast to classic voluntary provision models, the opportunity cost of contributing is a function of the number of players, too. With many alliance members, it becomes more difficult to win the internal fight since more than one other player has to be beaten: keeping the same winner-take-all structure, i now wins the internal fight only if $m_i - x_i > \max_{j \neq i}\{m_j - x_j\}$. This further reduces the incentive to contribute.

In an analogy to condition (2) in case of $n = 2$ players, candidate thresholds m^* of a symmetric standalone equilibrium must satisfy the following condition in the case of a general n:

$$(F(m^* - b))^{n-1} = (F(m^* + b))^{n-1} - (F(m^*))^{n-1} \tag{3}$$

Intuitively, this condition is obtained from the requirement of continuity of the expected equilibrium payoffs when comparing types just below or above the threshold m^*.[14] The left-hand side of equation (3) is equal to the expected equilibrium payoff of contributing types at the threshold: those types are the poorest among the contributing types, so they never win if there is at least one other contributing type. This means they only win if all other (free-riding) players have budgets below $m_i - b = m^* - b$, which is what type $m_i = m^*$ has left after contributing. The right-hand side of equation (3) is equal to the limit of i's expected equilibrium payoff when m_i approaches m^* from below. Those types m_i are the richest among the free-riding players, so they win against all free-riding players. Moreover, they win against contributing players j with a budget $m_j \in (m_i, m_i + b)$. That is, if m_i approaches m^* from below, they will win if all other players have budgets below $m^* + b$, unless everyone else has a budget below m^* and free-rides, too.

In addition to m^* being a solution to (3), equilibrium existence requires $m^* \leq 2b$, just as in the case of $n = 2$ above.[15] For $n = 2$, Proposition 1 shows that a unique solution $m^* \in (b, 2b]$ to (3) exists and that (3) is sufficient for the characterization of equilibrium. For $n > 2$, a solution $m^* \in [b, 2b]$ to (3) need not exist.

For uniformly distributed budgets on $[0, 1]$, for instance, (3) simplifies to

$$(\min\{m^* + b, 1\})^{n-1} - \left((m^*)^{n-1} + (m^* - b)^{n-1}\right) = 0. \tag{4}$$

Let $n > 2$. A candidate solution to (4) must fulfill $m^* > b$. Since $(m^*)^{n-1} + (m^* - b)^{n-1} < (m^* + m^* - b)^{n-1} \leq (m^* + 2b - b)^{n-1}$ for all $m^* \in (b, 2b]$, the left-hand side of (4) is strictly positive for all m^* under consideration if $m^* + b < 1$ holds. This means any solution $m^* \in [b, 2b]$ must satisfy $m^* + b \geq 1$, which directly rules out that such a solution and hence a symmetric standalone threshold equilibrium exist if $b < 1/3$, in the case of $n > 2$ and uniformly distributed budgets.

It may be unexpected that non-existence of a standalone equilibrium can be obtained for small values of b, that is, when standalone contributions are rather 'cheap.' An intuition for this finding relies again on the deviation incentives of types above and below the threshold m^*: with $n > 2$, the threshold m^* is required to be larger in order to prevent deviations to a zero contribution of types with budgets just above m^*. But if m^* is rather high and the resource requirement b is small, this strengthens the deviation incentives of types below the threshold who, in the candidate equilibrium, need to hope that another player will make the alliance win even though only a small contribution b would be required to ensure alliance victory.

Joint Contribution Equilibria

Equilibrium Characterization

Now let us come back to the main framework and turn to a different type of symmetric threshold equilibrium: an equilibrium in which players might contribute zero or half of the necessary joint amount b. Hence, in the equilibrium with contribution threshold $\hat{m}$, the players contribute an amount $\hat{x}_i(m_i) = b/2$ if and only if $m_i \geq \hat{m}$.

Proposition 2 *Let $b < 2\bar{m}$ and suppose that F is (weakly) concave on the support $[0, \bar{m}]$.*

(i) *Suppose that* $b < 2\bar{m}/3$. *If*

$$F(\bar{m}) - F(\bar{m} - b/2) - F(b/2) \geq 0, \tag{5}$$

there is a unique symmetric joint contribution equilibrium characterized by $\hat{m} = b/2$. *If (5) is violated, no symmetric joint contribution equilibrium exists.*
(ii) *Suppose that* $b \in [2\bar{m}/3, \bar{m})$ *and define* $\tilde{b}$ *as the (unique) solution to*

$$F(\bar{m}) - F\left(\bar{m} - \tilde{b}\right) - F\left(\bar{m} - \frac{\tilde{b}}{2}\right) = 0, \tag{6}$$

where $\tilde{b}$ *satisfies* $\tilde{b} \in [2\bar{m}/3, \bar{m})$. *If* $b < \tilde{b}$, *no symmetric joint contribution equilibrium exists. If* $b \geq \tilde{b}$, *the set of symmetric joint contribution equilibria is characterized by* $\hat{m} \in [b/2, z]$ *where* $z \in (b/2, b]$ *is the solution to*

$$F(\bar{m}) - F(\bar{m} - b) - F(z) = 0. \tag{7}$$

(iii) *If* $b \in [\bar{m}, 2\bar{m})$, *the set of symmetric joint contribution equilibria is characterized by* $\hat{m} \in [b/2, \bar{m})$.

The proof of Proposition 2 is in the appendix. To understand the result intuitively, it is crucial to note that only incentives to deviate to standalone contributions need to be considered. First of all, players who contribute in equilibrium reduce the probability of winning to zero should they decide to free-ride and contribute zero: free-riding incentives are absent by construction in an equilibrium where both players are required to contribute in order to reach the common goal. Moreover, deviations to a contribution of $b/2$ are not profitable for players with $m_i < \hat{m}$ who free-ride in the candidate equilibrium: if they contribute $b/2$, player j is required to contribute as well in order to reach the common goal. Players with $m_i < \hat{m}$ would, however, always lose the internal conflict if they contributed $b/2$ and faced another contributing player as their internal rival who must consequently be rather rich. Therefore, the only relevant alternative in a joint contribution equilibrium is a deviation to a standalone contribution. In fact, since types below the threshold prevent joint success with their free-riding strategy and get zero expected equilibrium payoff, the threshold $\hat{m}$ cannot be larger than b since otherwise types $m_i \in (b, \hat{m})$ would deviate to a standalone contribution with which they can win at least if m_j is very low. Due to the requirement $\hat{m} \leq b$, a deviation to a standalone contribution is feasible only for types above the threshold $\hat{m}$. The proof in the appendix shows that due to these considerations, it is sufficient to focus on the 'richest' types: for types $m_i \to \bar{m}$, incentives to deviate to a standalone contribution are strongest.[16]

For large amounts of resource investments b required by the players (Proposition 2(ii) and (iii)), there is a continuum of joint contribution equilibria characterized by contribution thresholds $\hat{m} \in [b/2, z]$ with $z \in (b/2, b]$. This becomes most obvious for the case where $b \geq \bar{m}$ (Proposition 2(iii)). Here, standalone contributions are not feasible for types m_i in the support of F. Since no other contribution can constitute a profitable deviation, any threshold $\hat{m} \in [b/2, \bar{m})$ can be supported as part of a symmetric joint contribution equilibrium. Hence, the set of equilibria includes the joint contribution equilibrium with efficient participation (the one with $\hat{m} = b/2$ where players contribute their share whenever they are able to do so) as well as joint contribution equilibria with inefficiently low participation (where $\hat{m} > b/2$).

For investment thresholds $b \in [2\bar{m}/3, \bar{m})$ as in Proposition 2(ii), a continuum of joint contribution equilibria can exist because the incentives to deviate from a candidate equilibrium are, in some range, independent of the threshold $\hat{m}$. To see why, consider the player with the maximum endowment (with $m_i \to \bar{m}$) and suppose that $\hat{m} \in (\bar{m} - b, \bar{m} - b/2)$. The candidate equilibrium payoff of this player i is $1 - F(\hat{m})$: she wins if and only if j contributes (i wins the internal conflict against all those players j). If i deviates and contributes $x_i = b$, she also wins against some non-contributing players (those with $m_j < m_i - b$) but now wins against the contributing players j only if m_j is small (that is, if $m_j \in (\hat{m}, m_i - b/2)$). Comparing candidate equilibrium payoff and deviation

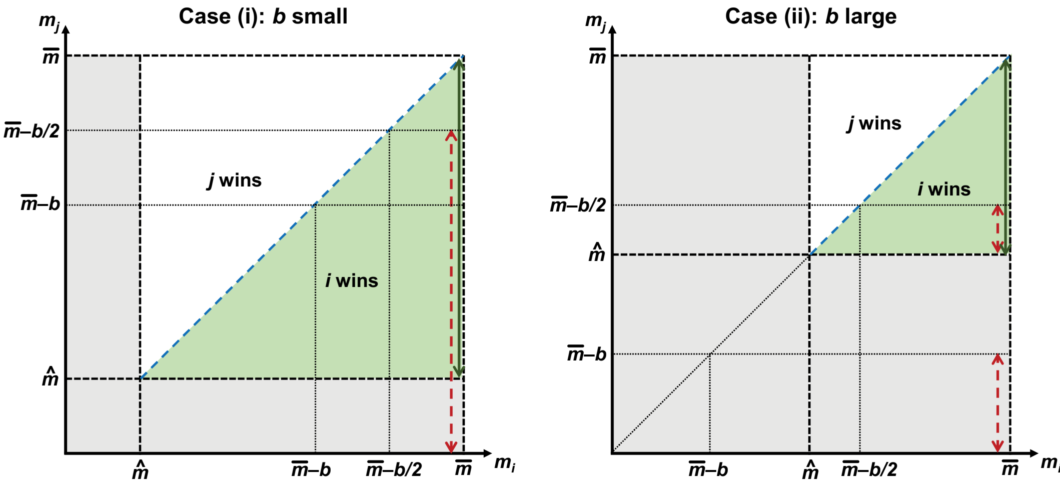

Figure 2. The figure illustrates the different combinations (m_i, m_j) for which either i or j wins in the joint contribution equilibrium, for small vs. large amounts b of resources required. The expected payoff of a type $\tilde{m}_i$ is obtained by drawing a vertical line at $m_i = \tilde{m}_i$ and calculating the probability that m_j is in the interval where $\tilde{m}_i$ wins. Thus, the solid arrow at the far right of each of the panels indicates the expected equilibrium payoff of types $m_i \to \bar{m}$, which is equal to the probability that j contributes. The dashed arrow indicates the expected deviation payoff of types $m_i \to \bar{m}$ when choosing $x_i = b$. Upon deviating, the highest types $m_i \to \bar{m}$ win if $m_j < \bar{m} - b$ (j does not contribute but has fewer resources than i) or if $m_j \in (\hat{m}, \bar{m} - b/2)$ (j contributes $b/2$ and has fewer resources left than i). In case (i) where b is small, high types m_i win against all non-contributing types of j when deviating to a contribution $x_i = b$. In case (ii) where b is large, high types m_i lose against some non-contributing types of j when deviating to a contribution $x_i = b$.

payoff shows that the incentive to deviate is independent of $\hat{m}$ in this range.[17] If $\hat{m}$ becomes too large, however, then the equilibrium participation is low and so are the equilibrium payoffs. Thus, condition (7) defines an upper bound on the contribution thresholds $\hat{m}$ that can be supported in equilibrium.

For small amounts b of resource investments required, the existence of a joint contribution equilibrium is not guaranteed since deviations to standalone contributions are particularly attractive if the required resources are small. For $b < 2\bar{m}/3$, Proposition 2(i) shows that a joint contribution equilibrium may only exist if it involves efficient participation, that is, if $\hat{m} = b/2$ so that all players with $m_i \geq b/2$ contribute in equilibrium. As the two corollaries below show formally, such an equilibrium exists for uniformly distributed budgets but does not exist for strictly concave probability distributions where low budgets are more likely and thus deviations to standalone contributions are more attractive for resource-rich players.

Figure 2 illustrates equilibrium payoffs. Types $m_i < \hat{m}$ get zero equilibrium expected payoff (joint effort is always below b). Types $m_i > \hat{m}$ win in equilibrium whenever $m_j \in (\hat{m}, m_i)$ so that j contributes but has fewer remaining resources than i. Figure 2 also illustrates the incentive to deviate for the highest possible type. The deviation payoff of types $m_i \to \bar{m}$ depends on whether the threshold $\hat{m}$ is smaller or larger than $\bar{m} - b$ and $\bar{m} - b/2$, respectively. (Recall that, in equilibrium, $\hat{m}$ cannot be larger than b.) Case (i) in the left panel considers the case where b is small. Here, $\hat{m} < \bar{m} - b$ ensures that, when deviating to a standalone contribution, the 'richest' types $m_i \to \bar{m}$ still win against all non-contributing types j. But a deviation to a standalone contribution means that i no longer wins against contributing types $m_j \in (m_i - b/2, m_i)$: those who are just a bit 'poorer' than i. The solid and the dashed arrow in the left panel of Figure 2 show the candidate equilibrium payoff and the deviation payoff, respectively, of the highest type $m_i = \bar{m}$; the arrows indicate the range of m_j in which i wins, under the candidate choice and the deviation. In case (i), if low values of m_j are particularly likely (as for strictly concave distribution functions F), a symmetric joint contribution equilibrium does not exist.[18]

The right panel of Figure 2 considers a case where b is large and, hence, $\hat{m} > \bar{m} - b$.[19] For larger values of b, deviations to standalone contributions are less attractive since, upon deviating to $x_i = b$, i not only loses against contributing players j who are just a bit poorer than i (with budgets $m_j \in (m_i - b/2, m_i)$) but also loses against the richest of the non-contributing players j (with budgets $m_j \in (m_i - b, \hat{m})$). Whether types $m_i \to \bar{m}$ with $x_i = b$ can still win against contributing players j depends on whether $\hat{m} < \bar{m} - b/2$. As seen in case (ii) of Figure 2 from the solid arrow (candidate equilibrium payoff of types $m_i \to \bar{m}$) and the dashed arrow (deviation payoff of types $m_i \to \bar{m}$ when choosing $x_i = b$), the incentive to deviate is reduced if b is increased (keeping $\hat{m}$ fixed). In this case, a continuum of thresholds $\hat{m}$ can be supported as part of a joint contribution equilibrium.[20]

Example

The characterization of joint contribution equilibria is simplified when considering the case of uniformly distributed budgets. Corollary 2 illustrates the result of Proposition 2 for this case.

Corollary 2 *Suppose* $F(m) = m$ *on* $[0, 1]$.
(i) *If* $b < 2/3$, *there is a unique symmetric joint contribution equilibrium characterized by* $\hat{m} = b/2$.
(ii) *If* $b \in [2/3, 1)$, *the set of symmetric joint contribution equilibria is characterized by* $\hat{m} \in [b/2, b]$.
(iii) *If* $b \in [1, 2)$, *the set of symmetric joint contribution equilibria is characterized by* $\hat{m} \in [b/2, 1)$.

For the uniform distribution, there is a unique joint contribution equilibrium with efficient participation in case b is small. Once b is large, there is a continuum of equilibria with threshold $\hat{m} \in [b/2, \min\{b, 1\}]$. This set includes the equilibrium where players' participation is efficient as well as equilibria where the threshold is not reached even though the players jointly have sufficient resources. It can also include equilibria where the threshold is not reached even though one player alone has sufficient resources. The latter again arises because the risk of losing the internal conflict

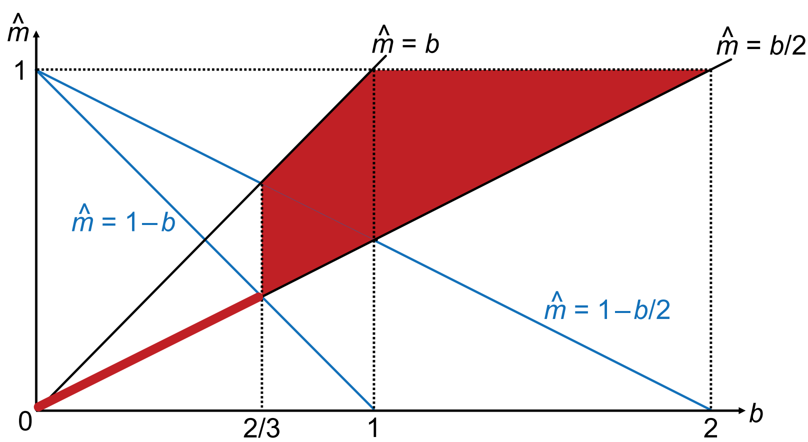

Figure 3. The figure illustrates the set of equilibria characterized in Corollary 2, i.e. for a uniform distribution $F(m_i) = m_i$ on the unit interval $[0, 1]$. The equilibrium combinations of $(b, \hat{m}(b))$ are represented by the highlighted (thick) line for small values of b and the highlighted area for larger values of b. Any equilibrium combination must necessarily be inside the cone generated by the feasibility constraint $\hat{m} \geq b/2$ and the condition $\hat{m} \leq b$ which is a necessary equilibrium condition (see step 3 in the proof of Proposition 2). For $b \in [2/3, 1)$ all $\hat{m}(b) \in [b/2, b]$ are thresholds for which $x_i(m_i) = 0$ if $m_i < \hat{m}$ and $x_i(m_i) = b/2$ if $m_i \geq \hat{m}$ characterize the equilibrium choices. Since $\hat{m}$ cannot be larger than $\bar{m} = 1$ by definition, for $b \in [1, 2)$ the interval for equilibrium thresholds becomes $\hat{m}(b) \in [b/2, 1)$. For $b < 2/3$, however, there is one single threshold $\hat{m}(b) = b/2$ for which a symmetric joint contribution equilibrium exists. For small b, if a higher threshold than $\hat{m} = b/2$ is chosen, it is too attractive for resource-rich players to deviate to a standalone contribution which has low cost and still ensures a sufficiently high probability of winning.

makes the players refrain from increasing their contribution up to b even when they are very resource-rich.

To illustrate this equilibrium further, consider the function Δ that describes the difference between the equilibrium payoff from $x_i = b/2$ and the deviation payoff from $x_i = b$ for $F(m_i) = m_i$ and $m_i = \bar{m} = 1$, which corresponds to equation (11) in the appendix. As explained above in the context of Figure 2, for the deviation payoff of types $m_i \rightarrow \bar{m}$ we need to distinguish whether $\hat{m} > \bar{m} - b$ and $\hat{m} > \bar{m} - b/2$.[21] This yields

$$\Delta(m_i = 1) = \begin{cases} b - \hat{m} & \text{if} 1 > \hat{m} > 1 - \frac{b}{2} \\ \frac{3}{2}b - 1 & \text{if} \hat{m} \in \left[1 - b, 1 - \frac{b}{2}\right] \\ \frac{b}{2} - \hat{m} & \text{if} \hat{m} < 1 - b \end{cases}$$

so that the requirement $\Delta(m_i = 1) \geq 0$ together with the necessary condition $\hat{m} \in [b/2, b]$ can be mapped into Figure 3. The dark (red) line and the dark (red) area are combinations of b and $\hat{m}(b)$ for which $\Delta(m_i = 1) \geq 0$ holds, i.e. combinations of b and $\hat{m}$ for which a deviation to $x_i = b$ does not pay, not even for player types $m_i = 1$ for which this deviation is most attractive among all player types. Figure 3 illustrates that for $b \in [0, 2/3)$ there is precisely one corresponding value of $\hat{m}$ that is feasible and does not invite a profitable deviation. This identifies the unique equilibrium for a given b, for all $b < 2/3$, along the red line.[22] For any larger $b \in (2/3, 2)$, there is a whole set of thresholds $\hat{m}(b) \in [b/2, \min\{b, 1\})$ for which no profitable deviations exist.

The uniform distribution is a special case in that joint contribution equilibria also exist for low values of b. This is no longer true for strictly concave probability distributions. Intuitively, low budgets become more likely if F is concave; the resulting low probability that the required resources b are met and the strengthened incentives to deviate to standalone contributions cause the non-existence of a symmetric joint contribution equilibrium.

Corollary 3 *Suppose F is strictly concave on $(0, 1)$. If $b < \tilde{b}$ where $\tilde{b}$ is defined by (6) and satisfies $\tilde{b} > 2/3$, no symmetric joint contribution equilibrium exists.*

The proof of this corollary is in the appendix. For strictly concave distribution functions F of the budgets, there is either no joint contribution equilibrium (if b is small) or a continuum of equilibria (if b is large), the latter including the one with $\hat{m} = b/2$ as well as joint contribution equilibria with inefficiently low participation.

The joint-contribution results are reminiscent of Bagnoli and Lipman (1989, 1992) who study how several players might efficiently fund a threshold public good. In their framework, this is a possible equilibrium outcome if their joint benefits from provision exceed the total cost. A natural non-cooperative equilibrium in their setup is the one in which the players share the necessary contribution costs evenly.

In the context here, the provision of joint effort that is larger or equal to b can be seen as the provision of a threshold public good. If players' resources are sufficient, there can be a symmetric equilibrium in which they share the burden. An even split is a symmetric equilibrium that is part of the set of symmetric equilibria, unless the players' budgets are likely to be large compared to the amount of team effort needed. Of course, the incentives and conditions are quite different from the standard analysis of the non-cooperative provision of a threshold public good. In the provision game, the quantity of resources can be freely chosen, but each unit contributed has a deterministic provision cost that enters into the contributor's budget. In the Blotto alliance, the provision occurs with resources that are useless if the provision does not occur: players allocate a 'use-it-or-lose-it' budget. The opportunity cost emerges indirectly because the resources have an important alternative use if and only if the amounts contributed are sufficient to reach the group goal. If the total provisions sum up to b or more, then, and only then, do the resources become very valuable for a confrontation that emerges with the other member of the group. A player cares about the relative

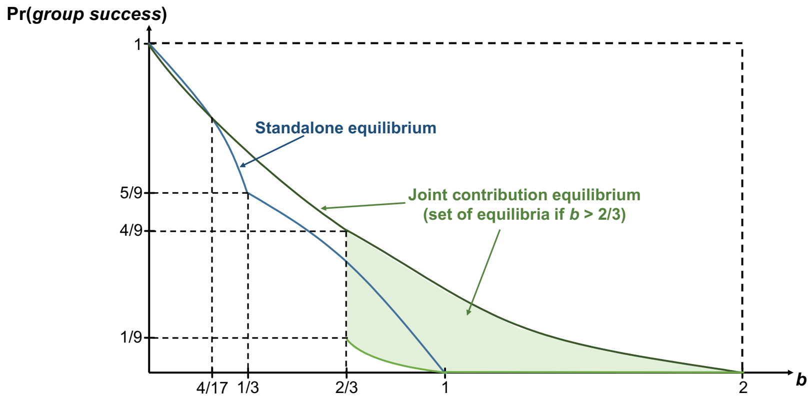

Figure 4. The figure shows the probability that the group is successful ($\Pr(x_1 + x_2 \geq b)$) in the standalone and the joint contribution equilibria, for the case of uniformly distributed budgets on $[0, 1]$. A (unique) standalone equilibrium exists if $b < 1$. A joint contribution equilibrium exists if $b < 2$. If $b > 2/3$, there is a set of joint contribution equilibria; the shaded area in the figure maps the probability of success for all possible thresholds $\hat{m}$.

resource endowment in this confrontation. She might increase the effort contribution to group success, and this might be crucial for reaching a given group benefit. But it is exactly this increased contribution of a player that would make the player weaker than the other group members in the intra-group fight over the benefit. This trade-off differs and is stronger than the free-riding problem in standard voluntary contribution games. There, a player might weigh the additional cost of own contributions and the higher likelihood of enjoyment of the public good. The player has private costs of contributing, but they generate a public benefit if they succeed in providing the threshold public good. Here, the player who withholds resources brings not only the provision of the public good into question. The player also harms the other group members directly because she becomes a stronger rival in the internal conflict once they jointly reach the group goal.

Comparison of Types of Equilibria

The previous analysis identified two types of equilibria: standalone equilibria where only one player is needed for group success but the likelihood of making a contribution may be rather low, and joint contribution equilibria where both players are required for group success but participation rates may be higher. We conclude this analysis with a brief 'welfare comparison' in the range where both types of equilibria exist. Figure 4 shows the probability that the alliance is successful, $\Pr(\sum x_i \geq b)$, for the case of uniformly distributed budgets $F(m) = m$ on $[0, 1]$.

For low resource requirements b, Figure 4 shows that the standalone equilibrium yields a higher success probability than the joint contribution equilibria. Intuitively, if only low effort is required for group success, the threshold m^* is low in the standalone equilibrium, which yields a high success probability (and overcontribution of alliance members with a high probability). For a low b, the threshold $\hat{m}$ in the joint contribution equilibrium is low, too, but here both players are required for alliance success. This causes the standalone equilibrium to be welfare-superior (from the point of view of total alliance payoffs) in case of small resource requirements. This ranking changes for higher resource requirements b. Here, standalone contributions become unlikely, whereas joint contributions can still be supported with higher probability. (The upper boundary of the shaded area represents the efficient joint contribution equilibrium with $\hat{m} = b/2$ whereas the lower boundary represents the least efficient joint contribution equilibrium in the set of equilibria that exist.) This makes the joint contribution equilibrium preferable for the alliance if b is larger and the alliance members' contributions are based on a low threshold $\hat{m}$, that is, the players implicitly coordinate on a low contribution threshold.

Conclusions

Acting as a volunteer in the interest of the group often has major disadvantages: this action dissipates resources that could be used in the power struggle within the group. In a fully non-cooperative world, a player who contributes much to the group effort might generate benefits to the group but could be left with too few resources to succeed in the internal fight that decides the allocation of these benefits. So, while a volunteer acquires some desirable goods for the group, protects or rescues the group or accomplishes other tasks that benefit the group, this very act might be unattractive as it weakens this player in an internal power struggle. We show that even in the face of such high direct opportunity costs of contributions to the objective of the alliance, group members can often manage to overcome the free-riding problem and provide the effort necessary to achieve their common goal. A player might expend the standalone effort to ensure the group wins if the player anticipates that there is a sufficiently large probability that her remaining resources are sufficient to beat the other alliance member. A core result of this work is a precise description of the determinants of such non-cooperative but joint action in equilibrium for the case where the players know their own resources but not that of their group members.

The analysis focuses on two kinds of equilibria: those in which the group members achieve the common goal by joining forces and those in which the group achieves the goal because there are members who are willing to ensure group success by their own efforts alone. Both types of equilibrium exist for larger and partially overlapping parameter ranges. The joint-contribution equilibrium is particularly relevant if each single player's resources are insufficient to make a standalone contribution; the volunteer's (standalone) equilibrium is particularly relevant if the range of possible resource endowments includes large endowments.

A possible illustration in which the equilibrium considerations might shed light on the conflict parties' incentives and actions is the historical episode in China when a period of internal civil war between the Chinese Communist Party (CCP) and the Kuomintang (KMT) occurred in parallel with the Sino-Japanese war. Van Ven (2018, p. 146) quotes from a report according to which Mao Zedong allegedly had called the Sino-Japanese war a great opportunity and described CCP policy as focusing '70% on expansion, 20% on dealing with the KMT, and 10% on resisting Japan.' We take the genuine resource rivalry in this conflict between the deployment of resources to repel the Japanese invasion on the one hand and the power struggle between the CCP and KMT on the other as a motivation for a systematic formal analysis of this problem. One should state clearly that the full picture of the conflict theater was much richer than our model. The two conflicts have a pre-war history, the United States and the Soviet Union played changing roles in the conflict, the civil war heated up after the end of the Second World War, and the Chinese parties might not have known the precise amount of resources needed to stop the Japanese invasion, among other unmodeled aspects. However, our analysis alludes to the possibility that such conflict situations might have at least two types of equilibrium outcomes. In one of these types, the parties fighting in the civil war make contributions of military resources to jointly defend against the Japanese invasion in an alliance of convenience, provided that their endowments are above some threshold. Another type of equilibrium suggests that very resource-rich players mobilize sufficiently high resources to stop the invasion with a standalone effort, whereas players with fewer resources concentrate their effort on the internal struggle.

From the point of view of alliance success as a measure of the total welfare of the alliance, the latter type of standalone equilibrium has the advantage that only one alliance member is needed for the common goal to be achieved. From an ex ante point of view, this can be beneficial for the alliance as a whole in particular when only comparably little effort is required to fight the external threat. The disadvantage of such a standalone equilibrium is, however, that the contribution threshold goes up and a higher individual resource budget is required for making such a standalone contribution, compared to a joint defense contribution. Weighting these two aspects, our analysis shows that a joint defense strategy yields a higher overall alliance welfare if the common threat is substantial, i.e. the resource requirement for successful defense is high. In this case, even very resource-rich players would need to invest a large share of their budget as a standalone contribution to fighting the invasion so that even the richest players are reluctant to make such a contribution, as this would make them highly vulnerable in the internal conflict.

Notes

1. More generally, alliance members think about the continuation game in which they might have to solve their internal quarrels over how to split the rents from reaching their group objective. Historians discuss evidence in the context of the Napoleonic wars (O'Connor 1969), the Great War (Bunselmeyer 1975, p. 15) and the great alliance in the Second World War (Weinberg 1994, p. 736).
2. Olson and Zeckhauser (1966), Katz, Nitzan, and Rosenberg (1990), Ursprung (1990) and Esteban and Ray (2001) focus on free-riding incentives. Nitzan (1991) discusses alliance members' rewards as a function of their and other alliance members' contributions. Konrad (2014) surveys this literature and the literature that explains why alliances might be formed despite the collective-good problem. Bakshi and Dasgupta (2018) embed this problem into a dynamic framework.
3. For theoretical analyses of inter-group alongside intra-group conflict see, e.g. Katz and Tokatlidu (1996), Wärneryd (1998), Konrad (2004), and Münster (2007). Ke, Konrad and Morath (2013) and Ke, Konrad and Morath (2015) present experimental findings in this context that are much in line with theory predictions.
4. One cannot rule out institutional contexts in which an alliance member who fights for the common good receives recognition or political support in the internal conflict. However, in the example we highlighted, the quote by Mitter (2013) suggests that this was not the case.
5. The study of how a player and her adversary would allocate given amounts of military resources across several battlefields has been studied in many variants. The Colonel Blotto name for this game is attributed to Gross and Wagner (1950). More recent major contributions are Roberson (2006) and Roberson and Kvasov (2012). For a survey see Kovenock and Roberson (2010). The multiple battles typically have a geographical interpretation and efforts in the various battles are chosen simultaneously. An exception that accounts for sequentiality in multi-battle contests is by Klumpp, Konrad and Solomon (2019).
6. In a conference paper, Klumpp and Konrad (2019) modified the Blotto framework to study an alliance problem that is structurally related to the current paper. They consider players that each have a given and commonly known overall budget. They can use these resources to support their alliance's military objectives. What they do not contribute to this purpose can be used for internal fighting. Their analysis considers a fight between two alliances and identifies a very large set of equilibria. One of these equilibria attributes certain victory to the alliance that has the larger sum of resources and attributes victory inside the victorious group with equal probability among a set of players with the largest budgets.
7. Within the general class of Colonel Blotto games, some elements of incomplete information have been studied. Incomplete information about competing players' valuations is analyzed by Kovenock and Roberson (2011); incomplete information about the players' resource endowments is studied by Adamo and Matros (2009). However, these frameworks are constant-sum games in which players are rivals throughout, whereas in our framework the game is not a constant-sum game: players have a common goal ('winning the asset for the group') and adversarial goals ('winning the intra-group contest').
8. The literature is growing rapidly. See Garfinkel and Skaperdas (2007) for a comprehensive survey by two main contributors to the field.
9. In addition to $F(m)$ being a smooth and atomless cumulative distribution function, the assumption of (weak) concavity deserves to be highlighted. The uniform distribution and many right-skewed distributions comply with this assumption. Prominent examples are the exponential distribution and the Pareto distribution that is empirically particularly relevant as a characterization of incomes (or endowments).
10. For instance, for $m_1 + m_2 > b$ and $\max\{m_1, m_2\} - \min\{m_1, m_2\} < b$ an equilibrium exists for which $m_1 - x_1 = m_2 - x_2$ and $x_1 + x_2 = b$.
11. Starting from the candidate equilibrium where contributions are either $x_i = 0$ or $x_i = b$, the only relevant deviations are deviations to b or 0. Any contribution $x_i \in (0, b)$ is dominated by $x_i = 0$ since $x_i \in (0, b)$ does not change the probability of alliance success but reduces the resources available for the internal fight. Similarly, any contribution $x_i > b$ is dominated by $x_i = b$.
12. Concavity of F ensures that considering incentives to deviate reduces to considering the types around the threshold m^*. For details see the proof in the appendix.

13. This explains why $m^* \leq 2b$ must hold in equilibrium. If those types $m_i < m^* - b$ with zero expected equilibrium payoff could deviate to a standalone contribution, they would do so and would win if the alliance partner has a very low budget and can still be beaten.
14. As m_i of a non-contributor approaches m^* from below, or of a contributor approaches m^* from above, i's expected equilibrium payoff must converge to the expected equilibrium payoff of a type $m_i = m^*$ at the threshold. To be precise, (3) is a necessary condition in an equilibrium with a threshold $m^* > b$. Requiring continuity of expected equilibrium payoffs at m^* is equivalent to requiring non-deviation of types just above and below the threshold, provided that a deviation of types just below the threshold to a standalone contribution is feasible, that is, $m^* > b$. But $m^* > b$ must hold in any equilibrium: if instead $m^* = b$, type $m_i = m^*$ would get zero expected payoff but could ensure a strictly positive payoff when free-riding (precisely, a payoff which is equal to the right-hand side of (3) evaluated at $m^* = b$).
15. To repeat the argument, suppose to the contrary that $m^* > 2b$. Then, types m_i in the (non-empty) interval $(b, m^* - b)$ get zero expected payoff: they do not contribute themselves and lose against all contributing types m_j due to $m_i < m^* - b \leq m_j - b$. But for $m_i \in (b, m^* - b)$, i could deviate to a standalone contribution, in which case she would win with probability $(F(m_i - b))^{n-1} > 0$. This contradicts $m^* > 2b$.
16. In the context of contributions to (continuous) public goods, it may seem surprising that the relevant deviations are deviations to standalone contributions rather than free-riding. This is caused by the nature of the 'public good' being a threshold public good whose consumption value is zero for the group if the threshold b is not met.
17. For details see the proof in the appendix. Formally, the left-hand side of equation (6) is the difference between candidate payoff and deviation payoff for the highest possible budget $m_i = \bar{m}$, in case the required resources are $\tilde{b}$. Since the incentive to deviate is stronger, the lower b (the left-hand side of (6) is increasing in b), the threshold $\tilde{b}$ defined by (6) is a lower bound for the existence of a joint contribution equilibrium in the range of Proposition 2(ii).
18. In case (i) of Figure 2, a no-deviation condition is $F(\bar{m}) - F(\bar{m} - b/2) \geq F(\hat{m})$ which, as $\hat{m} \geq b/2$, is violated if F is strictly concave.
19. Again, necessary condition in equilibrium is $\hat{m} \in [b/2, b]$.
20. In the right panel of Figure 2, if $\hat{m} > \bar{m} - b/2$ then the equilibrium payoff for types $m_i \to \bar{m}$ remains $F(\bar{m}) - F(\hat{m})$ whereas the payoff from deviating to $x_i = b$ is $F(\bar{m} - b)$ and, hence, independent of $\hat{m}$. This explains the upper bound z in Proposition 2(ii) for equilibrium thresholds $\hat{m}$.
21. Referring back to the two cases illustrated in Figure 2, $\Delta(m_i = 1)$ for $\hat{m} < 1 - b$ corresponds to the payoff difference illustrated by the solid and the dashed arrow in case (i) of Figure 2 and $\Delta(m_i = 1)$ for $\hat{m} \in [1 - b, 1 - b/2]r2)\text{if}m_i \in [\min\{\hat{m} + b, \bar{m}\}, \bar{m}]$ corresponds to the payoff difference illustrated by the solid and the dashed arrow in case (ii) of Figure 2.
22. With $\hat{m} \in [b/2, b]$, $\Delta(m_i = 1) \geq 0$ requires $\hat{m} \leq b/2$ if $\hat{m} < 1 - b$. If $\hat{m} \geq 1 - b$, $\Delta(m_i = 1) \geq 0$ is violated for $b < 2/3$ (compare the respective conditions illustrated in Figure 3).
23. A similar argument shows that there is no symmetric joint contribution equilibrium if F is strictly concave on some non-empty interval $(m', m'') \subseteq [0, \bar{m}]$ and weakly concave otherwise.

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A Appendix

A.1 Proof of Proposition 1

Before we show equilibrium existence, we show that there is a unique solution $m^* \in (b, 2b]$ to (2). For all $m^* \leq b$, the left-hand side (LHS) is strictly smaller than the right-hand side (RHS) of equation (2). Moreover, the LHS strictly increases in m^* whereas the RHS weakly decreases in m^* (given that $F'' \leq 0$). If m^* approaches $\min\{2b, \bar{m}\}$, the LHS is weakly larger than the RHS. To show the latter, suppose first that $2b < \bar{m}$. We need to show that

$$F(2b) \geq F(2b+b) - F(2b-b)$$

which is equivalent to

$$\int_0^{2b} F'(x)dx - \int_0^{2b} F'(x+b)dx \geq 0.$$

This inequality is true since F is weakly concave; it holds with strict inequality if F is strictly concave on some non-empty interval. If $2b \geq \bar{m}$ and $m^* \to \bar{m}$, the LHS of (2) approaches one, while the RHS of (2) approaches $1 - F(\bar{m} - b) < 1$. Thus, there is a unique solution $m^* > b$ to (2) which is weakly smaller than $2b$ if $2b < \bar{m}$ and strictly smaller than $\bar{m}$ (and $2b$) if $2b \geq \bar{m}$.

We first show existence. Consider the candidate standalone equilibrium with threshold $m^* > b$ as given in (2). Suppose first that $m_i \geq m^*$ where i is supposed to choose $x_i^* = b$ and realize an expected payoff of

$$F(m_i) - F(m^*) + F(\min\{m^*, m_i - b\}).$$

This expected payoff consists of two probabilities: (i) the probability $F(m_i) - F(m^*)$ that j contributes (i.e. $m_j \geq m^*$) but has a budget lower than m_i (which implies $m_j - x_j < m_i - x_i$ so that i wins the internal fight); and (ii) the probability $F(\min\{m^*, m_i - b\})$ that j does not contribute (i.e. $m_j < m^*$) and, in addition, has a budget lower than $m_i - b$ (which implies $m_j - x_j = m_j < m_i - x_i$). Any $x_i > b$ is strictly dominated by $x_i = b$ and any $x_i \in (0, b)$ leads to a strictly lower payoff than $x_i = 0$ if j follows the candidate strategy. If i deviates to $x_i = 0$, her deviation payoff is

$$F(m_i + b) - F(m^*)$$

since she gets a positive payoff whenever j contributes (i.e. $m_j \geq m^*$) but has a budget lower than $m_i + b$ (in which case $m_j - b < m_i - 0$). The candidate strategy $x_i^* = b$ is a best reply if and only if

$$F(m_i) - F(m^*) + F(\min\{m^*, m_i - b\}) \geq F(m_i + b) - F(m^*).$$

If $m_i \in [m^*, m^* + b)$, this no-deviation condition becomes

$$F(m_i) + F(m_i - b) - F(m_i + b) \geq 0.$$

Since the left-hand side of this inequality is (weakly) increasing in m_i if F is (weakly) concave, a necessary condition for the existence of the equilibrium is

$$F(m^*) \geq F(m^* + b) - F(m^* - b), \tag{8}$$

and this inequality holds if m^* is given by (2). Note that (8) requires $m^* > b$: at $m^* = b$, (8) is equivalent to $F(b) \geq F(2b) - F(0)$ which is violated due to $b < \bar{m}$.

If instead $m_i \in [m^* + b, \bar{m})$ (and this interval is non-empty, which, due to $m^* > b$, requires $\bar{m} > 2b$), the no-deviation condition is equivalent to

$$F(m^*) + F(m_i) - F(m_i + b) \geq 0.$$

If F is (weakly) concave, the left-hand side of this inequality is (weakly) increasing in m_i and is, thus, larger than

$$F(m^*) + F(m^* + b) - F(m^* + b + b) \geq F(m^*) + F(m^*) - F(m^* + b)$$
$$> F(m^*) + F(m^* - b) - F(m^* + b) = 0$$

where the first (weak) inequality uses (weak) concavity of F and the equality uses (2). Altogether, the candidate strategy $x_i^* = b$ is a best reply if and only if $m^* > b$ and (8) holds.

Now suppose that $m_i < m^*$ where i is supposed to choose $x_i^* = 0$ and realize an expected payoff of

$$\max\{F(m_i + b) - F(m^*), 0\}.$$

This expected payoff equals the probability j contributes (i.e. $m_j \geq m^*$) but has a budget lower than $m_i + b$ (in which case $m_j - x_j < m_i - x_i = m_i$), provided this interval is non-empty. Consider possible deviations. Any $x_i > b$ is strictly dominated by $x_i = b$; any $x_i \in (0, b)$ does not change the probability that joint contributions are at least b but lowers the probability that i wins against j, as compared to $x_i = 0$. Thus, if $m_i \in [0, b)$, no profitable deviation exists.

From the case of $m_i \geq m^*$ above, it follows that equilibrium existence requires $m^* > b$. If $m_i \in [b, m^*)$ and i deviates to $x_i = b$, i gets a positive payoff if and only if j does not contribute and, in addition, has a budget below $m_i - b$ (in which case $m_j - x_j = m_j < m_i - x_i$). (If j contributes, too, m_j must be larger than m^* so that $m_j - x_j > m_i - x_i$ in the case of $m_i < m^*$.) Thus, i's deviation payoff is

$$F(\min\{m^*, m_i - b\}) = F(m_i - b),$$

which is strictly positive if m_i is in the (non-empty) interval (b, m^*). The candidate strategy $x_i^* = 0$ is a best reply if and only if

$$\max\{F(m_i + b) - F(m^*), 0\} \geq F(m_i - b)$$

for all $m_i \in (b, m^*)$. First of all, since $F(m_i - b) > 0$ if $m_i > b$, this requires $m_i + b > m^*$ for all $m_i \in (b, m^*)$, that is, requires

$$m^* \leq 2b,$$

which holds for m^* as given by (2), where $m^* \in (b, 2b]$ if $2b < \bar{m}$ and $m^* \in (b, \bar{m})$ if $2b \geq \bar{m}$ (see above). In this case, the no-deviation condition is equivalent to $F(m^*) \leq F(m_i + b) - F(m_i - b)$ for all $m_i \in (b, m^*)$. Since the right-hand side of this inequality is (weakly) decreasing in m_i if F is (weakly) concave, a necessary condition for the existence of the equilibrium is

$$F(m^*) \leq F(m^* + b) - F(m^* - b), \tag{9}$$

and this inequality holds if for a threshold m^* as given by (2).

Uniqueness follows directly from the arguments above. Since (8) and (9) are necessary conditions for equilibrium existence, m^* must be given by (2) in any equilibrium. Since there is a unique solution m^* to (2), the equilibrium must be unique in the class of symmetric standalone equilibria.

A.2 Proof of Corollary 1

From Proposition 1 it follows that there is a unique solution $m^* \in (b, \bar{m})$ to (2). For a uniform distribution, condition (2) is equivalent to

$$\frac{m^*}{\bar{m}} = \min\left\{\frac{m^* + b}{\bar{m}}, 1\right\} - \frac{m^* - b}{\bar{m}} \Leftrightarrow 2m^* = \min\{m^* + 2b, \bar{m} + b\}. \tag{10}$$

Suppose $m^* + 2b < \bar{m} + b$. Then, (10) is solved for $m^* = 2b$. In order for $m^* + 2b < \bar{m} + b$ to hold at $m^* = 2b$, we must have $b < \bar{m}/3$. This shows part (i).

Now suppose $m^* + 2b \geq \bar{m} + b$. Then, (10) is solved for $m^* = (\bar{m} + b)/2$. In order for $m^* + 2b \geq \bar{m} + b$ to hold at $m^* = (\bar{m} + b)/2$, we must have $(\bar{m} + b)/2 \geq \bar{m} - b$ or, equivalently, $b \geq \bar{m}/3$. This shows part (ii). Note that $(\bar{m} + b)/2 \leq 2b$ if $b \geq \bar{m}/3$. Uniqueness follows from Proposition 1.

A.3 Proof of Proposition 2

Before we derive the equilibrium set for different values of b, we derive some preliminary results.

Step 1: We show that investigating equilibrium existence reduces to considering deviations to $x_i = b$ (standalone contributions). To see why, consider first types $m_i < \hat{m}$ so that $\hat{x}_i = 0$ in the candidate equilibrium. If $m_i < b/2$, $\hat{x}_i = 0$ is strictly preferred to any $x_i > 0$ if j follows the candidate strategy. If $m_i \geq b/2$, $\hat{x}_i = 0$ is strictly preferred to any $x_i \in (0, b/2)$, and $x_i = b/2$ is strictly preferred to any $x_i \in (b/2, b)$. A deviation to $\breve{x}_i = b/2$, however, yields zero expected payoff: the threshold b would only be met if $m_j \geq \hat{m}$, in which case it must hold that $m_j - b/2 \geq \hat{m} - b/2 > m_i - b/2$. Hence, the only choice that may constitute a profitable deviation is $x_i = b$. (All $x_i > b$ are strictly dominated by $x_i = b$.)

Similarly, for types $m_i \geq \hat{m}$, deviations to $x_i < b/2$ cannot be profitable since the threshold b would never be met in this case. Any $x_i \in (b/2, b)$ is strictly worse than the candidate strategy $\hat{x}_i = b/2$ and any $x_i > b$ is strictly dominated by $x_i = b$. Again, the only choice that may constitute a profitable deviation is $x_i = b$.

Step 2: Let $b \geq \bar{m}$. From Step 1 it follows that there are no profitable deviations (since the set of types that can deviate to $x_i = b$ has mass zero). By definition of the joint contribution equilibrium, $\bar{m} > \hat{m} \geq b/2$ and, hence, $b < 2\bar{m}$, which shows part (iii) of Proposition 2.

Step 3: Let $b < \bar{m}$. Suppose that $\hat{m} > b$ and consider types $m_i \in (b, \hat{m})$. Those types' candidate equilibrium payoff is zero (the threshold is never reached since they do not contribute). Deviations to $x_i = b$ yield an expected payoff of at least $\Pr(m_j < m_i - b) = F(m_i - b) > 0$ so that $x_i = 0$ cannot be a best reply for types $m_i \in (b, \hat{m})$. Since $\hat{m} \geq b/2$ by assumption, it follows that, for $b < \bar{m}$, the contribution threshold $\hat{m}$ must satisfy $\hat{m} \in [b/2, b]$ in any symmetric joint contribution equilibrium. With Step 1, $\hat{m} \in [b/2, b]$ implies that the candidate strategy $\hat{x}_i = 0$ is a best reply for types $m_i < \hat{m}$.

Step 4: Let $b < \bar{m}$. For types $m_i \geq \max\{\hat{m}, b\}$, the difference between the candidate equilibrium payoff (under $\hat{x}_i$) and expected payoff when deviating to $x_i = b$, defined as $\Delta(m_i) := \pi_i(m_i; \hat{x}_i) - \pi_i(m_i; x_i = b)$, is (weakly) decreasing in m_i. To show this, note first that the candidate equilibrium payoff is $F(m_i) - F(\hat{m})$ since i wins if and only if $m_j \in [\hat{m}, m_i)$. The deviation payoff under $x_i = b$ depends on whether (i) $m_i - b/2 > \hat{m}$ (so that i can still win against contributors j) and (ii) $m_i - b > \hat{m}$ (so that i wins against all non-contributors j). Formally, if $m_i \in [b, \hat{m} + b/2)$ and this interval is non-empty, i wins with $x_i = b$ if and only if $m_j < m_i - b$. If $m_i \in [\hat{m} + b/2, \hat{m} + b)$ and this interval is non-empty, i wins with $x_i = b$ if $m_j < m_i - b$ or if $m_j \in [\hat{m}, m_i - b/2)$. If $m_i \in [\hat{m} + b, \bar{m}]$ and this interval is non-empty, i wins with $x_i = b$ if $m_j < \hat{m}$ or if $m_j \in [\hat{m}, m_i - b/2)$, that is, if and only if $m_j < m_i - b/2$. Thus, for $m_i \geq b$, the difference $\Delta(m_i)$ between candidate equilibrium payoff and expected payoff when deviating to $x_i = b$ is:

$$\Delta(m_i) = \begin{cases} F(m_i) - F(\hat{m}) - F(m_i - b) & \text{if} m_i \in \left[b, \min\left\{\hat{m} + \frac{b}{2}, \bar{m}\right\}\right) \\ F(m_i) - F(m_i - b) - F\left(m_i - \frac{b}{2}\right) & \text{if} m_i \in \begin{matrix}\left[\min\left\{\hat{m} + \frac{b}{2}, \bar{m}\right\},\right. \\ \left.\min\{\hat{m} + b, \bar{m}\}\right)\end{matrix} \\ F(m_i) - F(\hat{m}) - F\left(m_i - \frac{b}{2}\right) & \text{if} m_i \in [\min\{\hat{m} + b, \bar{m}\}, \bar{m}] \end{cases} \tag{11}$$

$\Delta(m_i)$ (weakly) decreases in m_i if F is (weakly) concave. Thus, a necessary and sufficient condition for equilibrium existence is obtained when investigating $\Delta(\bar{m})$ (the incentive to deviate to $x_i = b$ for the highest possible budget).

Step 5: We note that $F(m) \geq m/\bar{m}$ for all $m \in (0, \bar{m})$ if F is weakly concave. With $F(0) = 0$ and $F(\bar{m}) = 1$, concavity forces F to be weakly above the straight line between $(0, 0)$ to $(\bar{m}, 1)$.

To show this formally, note first that weak concavity of F implies $F'(0) \geq 1/\bar{m}$. If, to the contrary, $F'(0) < 1/\bar{m}$, then $\int_0^{\bar{m}} F'(z)dz < \int_0^{\bar{m}} (1/\bar{m})dz = 1$, which contradicts $F(\bar{m}) = 1$. Now suppose that $F(m) < m/\bar{m}$ for some $m \in (0, \bar{m})$. If $F'(0) = 1/\bar{m}$, then, by weak concavity of F, $F(\bar{m}) \leq F(m) + (\bar{m} - m)F'(m) < m/\bar{m} + (\bar{m} - m)/\bar{m} = 1$; contraction. If $F'(0) > 1/\bar{m}$ then $F(m) < m/\bar{m}$ for some $m \in (0, \bar{m})$ implies that F must cross the line $y = m/\bar{m}$ from above at some $\breve{m} \in (0, m)$; hence, $F'\left(\breve{m}\right) < 1/\bar{m}$. With weak concavity of F, it follows that $F(\bar{m}) < 1$; contradiction. Thus, $F(m) \geq m/\bar{m}$ for all $m \in (0, \bar{m})$. A similar argument shows that $F(m) > m/\bar{m}$ for all $m \in (0, \bar{m})$ if F is strictly concave on some non-empty interval (m', m'').

Since part (iii) of the proposition follows from Step 2 above, it remains to prove parts (i) and (ii). Assume $b < \bar{m}$. By Step 3, let $\hat{m} \in [b/2, b]$. By Steps 1 and 3, we only need to consider the behavior of types $m_i \geq \hat{m}$ where, by Step 4, for equilibrium existence it is sufficient to focus on types with budget $m_i \to \bar{m}$.

Part (i): Assume $b < 2\bar{m}/3$ and consider the incentive to deviate from the candidate equilibrium if $m_i \to \bar{m}$. If $b < \bar{m}/2$ then $\bar{m} > 2b \geq \hat{m} + b$ for all $\hat{m}$ under consideration. (In words, this implies that, when deviating to $x_i = b$, types $m_i \to \bar{m}$ win against all non-contributing types of j.) If $b \in [\bar{m}/2, 2\bar{m}/3)$ then $\bar{m} \geq \hat{m} + b$ if $\hat{m} \in [b/2, \bar{m} - b]$ and $\bar{m} \in (\hat{m} + b/2, \hat{m} + b)$ if $\hat{m} \in (\bar{m} - b, b]$. (Here, whether types $m_i \to \bar{m}$ win against non-contributing players j with m_j close to $\hat{m}$ depends on the size of the threshold $\hat{m}$.)

Suppose first that $\hat{m}$ is such that $\bar{m} \geq \hat{m} + b$. Then, with (11), the candidate strategy is a best reply for all $m_i \in [b, \bar{m}]$ if and only if

$$F(\bar{m}) - F(\hat{m}) - F(\bar{m} - b/2) \geq 0. \tag{12}$$

If F is weakly concave, it holds that $F(m) \geq m/\bar{m}$ for all $m \in (0, \bar{m})$ (compare Step 5 above). Thus, the left-hand side of (12) is weakly smaller than

$$1 - \frac{\hat{m}}{\bar{m}} - \frac{\bar{m} - b/2}{\bar{m}} = \frac{1}{\bar{m}}\left(\frac{b}{2} - \hat{m}\right),$$

which is strictly negative if $\hat{m} > b/2$. Thus, (12) is violated for all $\hat{m} > b/2$ and a joint contribution equilibrium can exist only if $\hat{m} = b/2$. Inserting $\hat{m} = b/2$ into (12) yields (5) as necessary and sufficient condition for equilibrium existence. (Existence is ensured under (5) since $\bar{m} \geq \hat{m} + b$ holds at $\hat{m} = b/2$ by assumption of $b < 2\bar{m}/3$ and, hence, (12) is the relevant no-deviation condition.)

Now suppose that $\hat{m}$ is such that $\bar{m} \in (\hat{m} + b/2, \hat{m} + b)$. (This can occur if and only if $b \in [\bar{m}/2, 2\bar{m}/3)$. Here, note that $\hat{m} + b/2 \leq 3b/2 < \bar{m}$ if $b < 2\bar{m}/3$.) With (11), the candidate strategy is a best reply for all $m_i \in [b, \bar{m}]$ if and only if

$$F(\bar{m}) - F(\bar{m} - b) - F\left(\bar{m} - \frac{b}{2}\right) \geq 0. \tag{13}$$

Since weak concavity of F implies $F(m) \geq m/\bar{m}$ for all $m \in (0, \bar{m})$, the left-hand side of (13) is weakly smaller than

$$F(\bar{m}) - \frac{\bar{m} - b}{\bar{m}} - \frac{\bar{m} - \frac{b}{2}}{\bar{m}} = \frac{1}{\bar{m}}\left(\frac{3b}{2} - \bar{m}\right),$$

which is strictly negative due to $b < 2\bar{m}/3$. Thus, (13) is violated and no joint contribution equilibrium can exist with threshold $\hat{m}$ that is such that $\bar{m} \in (\hat{m} + b/2, \hat{m} + b)$. This completes the proof of part (i).

Part (ii): Suppose that $b \in [2\bar{m}/3, \bar{m})$. Here, we have $\hat{m} + b \geq b/2 + b \geq \bar{m}$. If $\hat{m}$ is small such that $\bar{m} \geq \hat{m} + b/2$, the candidate strategy is a best reply for all $m_i \in [b, \bar{m}]$ if and only if (13) holds. The left-hand side of (13) is strictly negative if $b \to 0$, strictly increasing in b, and strictly positive if $b \to \bar{m}$. Thus, there exists a unique solution $\tilde{b} \in (0, \bar{m})$ given by (6) such that the candidate strategy is a best reply if and only if $b \geq \tilde{b}$. Since the left-hand side of (13) is weakly smaller than

$$\frac{1}{\bar{m}}\left(\frac{3b}{2} - \bar{m}\right)$$

(see the previous paragraph), it must hold that $\tilde{b} \in [2\bar{m}/3, \bar{m})$. If $b \in \left[2\bar{m}/3, \tilde{b}\right)$, no symmetric joint contribution equilibrium exists.

If $b \in \left[\tilde{b}, \bar{m}\right)$, a joint contribution equilibrium can be supported for any $\hat{m}$ that is sufficiently small such that $\bar{m} \geq \hat{m} + b/2 \Leftrightarrow \hat{m} \leq \bar{m} - b/2$. (This upper bound, $\bar{m} - b/2$, on $\hat{m}$ approaches b if $b \to 2\bar{m}/3$ and approaches $b/2$ if $b \to \bar{m}$. Hence, taking into account that $\hat{m} \in [b/2, b]$, the interval $\hat{m} \in [b/2, \bar{m} - b/2]$ for which a joint contribution equilibrium exists is non-empty.)

Now consider larger thresholds $\hat{m}$ for which $\bar{m} < \hat{m} + b/2 \Leftrightarrow \hat{m} > \bar{m} - b/2$. With (11), the candidate strategy is a best reply for all $m_i \in [b, \bar{m}]$ if and only if

$$F(\bar{m}) - F(\hat{m}) - F(\bar{m} - b) \geq 0. \tag{14}$$

The left-hand side of (14) strictly decreases in $\hat{m}$. If $\hat{m} \downarrow \bar{m} - b/2$, (14) is equivalent to (13) and holds if and only if $b \geq \tilde{b}$; thus, equilibrium existence again requires $b \geq \tilde{b}$. If $\hat{m} \to b$, the left-hand side of (14) is

$$F(\bar{m}) - F(\bar{m} - b) - F(b) \leq 1 - \frac{\bar{m} - b}{\bar{m}} - \frac{b}{\bar{m}} = 0$$

where the weak inequality holds due to $F(m) \geq m/\bar{m}$ for all $m \in (0, \bar{m})$ (due to weak concavity of F). Thus, there is a unique solution $z \in (\bar{m} - b/2, b]$ to (7) such that any $\hat{m} \in [b/2, z]$ can be supported as part of a joint contribution equilibrium. Since (14) is a necessary condition for equilibrium existence, $\hat{m} \in [b/2, z]$ characterizes the full set of symmetric joint contribution equilibria in the case where $b \in \left[\tilde{b}, \bar{m}\right)$.

A.4 Proof of Corollary 2

Part (i): Suppose $b < 2/3$. From Proposition 2(i), there is a unique symmetric joint contribution equilibrium if and only if (5) holds, which is true (it holds with equality) if F is a uniform distribution.

Part (ii): Suppose $b \in [2/3, 1)$. From Proposition 2(ii), equilibrium existence requires $b \geq \tilde{b}$ as given in (6). For a uniform distribution, $\tilde{b} = 2/3$ so that a joint contribution equilibrium exists for all $b \in [2/3, 1)$. Any $\hat{m}$ with $\hat{m} \leq$

$\bar{m} - b/2$ can be supported as equilibrium since (13) holds for all $b \in [2/3, 1)$. For larger $\hat{m}$ (that is, $\hat{m} > \bar{m} - b/2$), necessary and sufficient condition for equilibrium existence is (14), which, for $F(m) = m$, is equivalent to $\hat{m} \leq b$. This shows part (ii).

Part (iii): Follows from Proposition 2(iii).

A.5 Proof of Corollary 3

Suppose $b < 2/3$. From Proposition 2(i), there is a unique symmetric joint contribution equilibrium if and only if (5) holds. Since strict concavity of F implies $F(m) > m/\bar{m} = m$ for all $m \in (0, 1)$, the left-hand side of (5) is strictly smaller than

$$1 - (1 - b/2) - b/2 = 0$$

so that (5) is violated.[23]

Suppose $b \geq 2/3$. By Proposition 2(ii), equilibrium existence requires $b \geq \tilde{b}$. Using again $F(m) > m$ in condition (6) shows that $\tilde{b} > 2/3$. Altogether, for small thresholds $b < \tilde{b}$, a symmetric joint contribution equilibrium does not exist. (For larger values of b, the equilibrium is as characterized in Proposition 2.)