MATH11111: Fundamentals of Optimization **Fall 2022**

Lecture 13 — Basic Solutions of Polyhedra in Standard Form

Lecturer: E. Alper Yıldırım *Week: 4*

13.1 Outline

- Characterisation of Basic Feasible Solutions of Polyhedra in Standard Form
- Enumeration of Vertices
- Review Problems

13.2 Basic Solutions and Basic Feasible Solutions of Polyhedra in Standard Form

Question 1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form. Can we identify simple conditions for basic solutions and basic feasible solutions of $\mathcal{P}$?*

13.2.1 Review of General Polyhedra

Let us recall the definitions of basic solutions and basic feasible solutions for general polyhedra. Consider a general polyhedron $\mathcal{P} \subset \mathbb{R}^n$ given by

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T x \geq b_i, & i \in M_1, \\ (a^i)^T x \leq b_i, & i \in M_2, \\ (a^i)^T x = b_i, & i \in M_3 \end{array} \right\}.$$

For $\hat{x} \in \mathbb{R}^n$, let $I(\hat{x}) = \left\{ i \in M_1 \cup M_2 \cup M_3 : (a^i)^T \hat{x} = b_i \right\}$.

- $\hat{x}$ is a *basic solution* if all of the equality constraints are active at $\hat{x}$ (i.e., $M_3 \subseteq I(\hat{x})$) and the set $\{a^i : i \in I(\hat{x})\}$ contains n linearly independent vectors (i.e., the set $\{a^i : i \in I(\hat{x})\}$ spans $\mathbb{R}^n$).
- $\hat{x}$ is a *basic feasible solution* if $\hat{x}$ is a *basic solution* and $\hat{x}$ is feasible (i.e., $\hat{x} \in \mathcal{P}$).

In this lecture, we will specialise this definition to a polyhedron in standard form.

13.2.2 Polyhedra in Standard Form

Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form.

- If $\hat{x} \in \mathbb{R}^n$ is a basic solution, then $A\hat{x} = b$, i.e., $(a^i)^T \hat{x} = b_i, \; i = 1, \ldots, m$.

- The inequality constraints in $\mathcal{P}$ are given by $x_j \geq 0$, i.e., $(e^j)^T x \geq 0,\ j = 1, \ldots, n$, where $e^j \in \mathbb{R}^n$ is the vector of all zeroes except for 1 in the jth coordinate, $j = 1, \ldots, n$.
- Label the constraints $(a^i)^T x = b_i$ by $i = 1, \ldots, m$ and the constraints $(e^j)^T x \geq 0,\ j = 1, \ldots, n$ using $m + 1, \ldots, m + n$.
- Then, $I(\hat{x}) = \{1, \ldots, m\} \cup \{m + j : (e^j)^T \hat{x} = 0\}$.
- Note that $m + j \in I(\hat{x})$ if and only if $(e^j)^T \hat{x} = \hat{x}_j = 0$, where $j = 1, \ldots, n$.
- Let us define the following index sets:

$$\begin{aligned} \hat{B} &= \{j \in \{1, \ldots, n\} : \hat{x}_j \neq 0\}, \\ \hat{N} &= \{j \in \{1, \ldots, n\} : \hat{x}_j = 0\}. \end{aligned}$$

- We have $\hat{B} \cup \hat{N} = \{1, \ldots, n\}$ and $\hat{B} \cap \hat{N} = \emptyset$.
- Note that $j \in \hat{N}$ if and only if $m + j \in I(\hat{x})$ (i.e., the constraint $x_j \geq 0$ is active at $\hat{x}$).
- Let $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ denote the submatrix of A consisting only of columns A^j, where $j \in \hat{B}$, and define $A_{\hat{N}} \in \mathbb{R}^{m \times |\hat{N}|}$ similarly.

Proposition 13.1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form and let $A\hat{x} = b$. Then, $\hat{x} \in \mathbb{R}^n$ is a basic solution of $\mathcal{P}$ if and only if the submatrix $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ has full column rank (i.e., the columns of $A_{\hat{B}}$ are linearly independent). Furthermore, $\hat{x} \in \mathbb{R}^n$ is a basic feasible solution (i.e., a vertex) of $\mathcal{P}$ if $\hat{x}$ is a basic solution and $\hat{x} \geq \mathbf{0}$.*

Proof. $\Rightarrow$: Suppose that $\hat{x} \in \mathbb{R}^n$ is a basic solution of $\mathcal{P}$ and let $I(\hat{x}) = \{1, \ldots, m\} \cup \{m + j : (e^j)^T \hat{x} = 0\}$. Then, $\text{span}\left(\{a^i : i = 1, \ldots, m\} \cup \{e^j : \hat{x}_j = 0\}\right) = \mathbb{R}^n$. Suppose, for a contradiction, that $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ does not have full column rank. Therefore, there exists a vector $d_{\hat{B}} \in \mathbb{R}^{|\hat{B}|} \backslash \{\mathbf{0}\}$ such that $A_{\hat{B}} d_{\hat{B}} = \mathbf{0}$. Then, if we define $d_{\hat{N}} = \mathbf{0} \in \mathbb{R}^{|\hat{N}|}$, we obtain $A_{\hat{B}} d_{\hat{B}} + A_{\hat{N}} d_{\hat{N}} = \mathbf{0}$. Since $d_{\hat{B}} \neq \mathbf{0}$, we obtain a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $Ad = \mathbf{0}$. Hence, $(a^i)^T d = 0,\ i = 1, \ldots, m$. Furthermore, for each $j \in \hat{N}$, since $d_j = 0$, we obtain $(e^j)^T d = d_j = 0$. Since $d \neq \mathbf{0} \in \mathbb{R}^n$, it follows that $\text{span}\left(\{a^i : i = 1, \ldots, m\} \cup \{e^j : \hat{x}_j = 0\}\right) \neq \mathbb{R}^n$. We obtain a contradiction.

$\Leftarrow$: Suppose that $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ has full column rank. Suppose, for a contradiction, that $\hat{x}$ is not a basic solution. Then, $\text{span}\left(\{a^i : i = 1, \ldots, m\} \cup \{e^j : \hat{x}_j = 0\}\right) \neq \mathbb{R}^n$, i.e., there exists a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $(a^i)^T d = 0,\ i = 1, \ldots, m$, and $(e^j)^T d = d_j = 0$ for each $j \in \{1, \ldots, n\}$ such that $\hat{x}_j = 0$. It follows that $Ad = \mathbf{0}$, or equivalently, $A_{\hat{B}} d_{\hat{B}} + A_{\hat{N}} d_{\hat{N}} = \mathbf{0}$, where $d_{\hat{B}} \in \mathbb{R}^{|\hat{B}|}$ and $d_{\hat{N}} \in \mathbb{R}^{|\hat{N}|}$ are the subvectors of d that contain only the coordinates in $\hat{B}$ and $\hat{N}$, respectively. Since $(e^j)^T d = d_j = 0$ for each $j \in \hat{N}$, we obtain $d_{\hat{N}} = \mathbf{0} \in \mathbb{R}^{|\hat{N}|}$. Therefore, we have $A_{\hat{B}} d_{\hat{B}} + A_{\hat{N}} d_{\hat{N}} = A_{\hat{B}} d_{\hat{B}} = \mathbf{0}$. Since $d \neq \mathbf{0}$ and $d_{\hat{N}} = \mathbf{0} \in \mathbb{R}^{|\hat{N}|}$, it follows that $d_{\hat{B}} \in \mathbb{R}^{|\hat{B}|} \backslash \{\mathbf{0}\}$. This contradicts our hypothesis that $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ has full column rank. $\square$

13.2.3 A Simpler Characterisation Under the Full Row Rank Assumption

Recall that if $\mathcal{L} = \{x \in \mathbb{R}^n : Ax = b\} \neq \emptyset$, we can assume that A has full row rank.

Question 2. *Under the full row rank assumption on A, can we obtain a simpler characterisation of basic solutions and basic feasible solutions?*

Proposition 13.2. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form and suppose that A has full row rank. Then, $\hat{x} \in \mathbb{R}^n$ is a basic solution of $\mathcal{P}$ if and only if there exist two index sets $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ such that all of the following conditions hold:*

(i) $B \cup N = \{1, \ldots, n\}$, $|B| = m$ *and* $|N| = n - m$.

(ii) *The matrix* $A_B \in \mathbb{R}^{m \times m}$ *is invertible (i.e., nonsingular).*

(iii) $\hat{x}_B = (A_B)^{-1} b$ *and* $\hat{x}_N = \mathbf{0}$.

Furthermore, $\hat{x} \in \mathbb{R}^n$ is a basic feasible solution (i.e., a vertex) of $\mathcal{P}$ if $\hat{x}$ is a basic solution and $\hat{x} \geq \mathbf{0}$.

Proof. $\Rightarrow$: Let $\hat{x} \in \mathbb{R}^n$ be a basic solution of $\mathcal{P}$. Then, $A\hat{x} = b$. Recall the sets

$$\begin{aligned} \hat{B} &= \{j \in \{1, \ldots, n\} : \hat{x}_j \neq 0\}, \\ \hat{N} &= \{j \in \{1, \ldots, n\} : \hat{x}_j = 0\}. \end{aligned}$$

By Proposition 13.1, the submatrix $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ has full column rank, which implies that $|\hat{B}| \leq m$.

Case 1: If $|\hat{B}| = m$, then $A_{\hat{B}} \in \mathbb{R}^{m \times m}$ and $\text{rank}(A_{\hat{B}}) = m$, which implies that $A_{\hat{B}}$ is an invertible matrix. Define $B = \hat{B}$ and $N = \hat{N}$. Since $A_{\hat{B}} \hat{x}_{\hat{B}} = A_B \hat{x}_B = b$, $\hat{x}_N = \hat{x}_{\hat{N}} = \mathbf{0}$ and A_B is invertible, the claim follows.

Case 2: If $|\hat{B}| < m$, then since the maximum number of linearly independent columns of A is equal to m and the submatrix $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ has full column rank, we can find $m - |\hat{B}|$ columns in $A_{\hat{N}}$ so that the set of all columns of $A_{\hat{B}}$ and the $m - |\hat{B}|$ columns in $A_{\hat{N}}$ are linearly independent. Let $J \subseteq \hat{N}$ denote the indices of those $m - |\hat{B}|$ columns in $A_{\hat{N}}$. Define $B = \hat{B} \cup J$ and $N = \hat{N} \backslash J$. Note that $|B| = m$ and $A_B \in \mathbb{R}^{m \times m}$ is an invertible matrix. Since $A_{\hat{B}} \hat{x}_{\hat{B}} = A_B \hat{x}_B = b$, $\hat{x}_N = \mathbf{0}$ and A_B is invertible, the claim follows.

$\Leftarrow$: Let $\hat{x} \in \mathbb{R}^n$ be such that there exist two index sets $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ that satisfy conditions (i), (ii), and (iii). By condition (iii), we have $A\hat{x} = A_B \hat{x}_B + A_N \hat{x}_N = A_B \hat{x}_B = A_B (A_B)^{-1} b = b$. Therefore, $A\hat{x} = b$. Recall the sets

$$\begin{aligned} \hat{B} &= \{j \in \{1, \ldots, n\} : \hat{x}_j \neq 0\}, \\ \hat{N} &= \{j \in \{1, \ldots, n\} : \hat{x}_j = 0\}. \end{aligned}$$

By condition (iii), we readily obtain $\hat{B} \subseteq B$ and $N \subseteq \hat{N}$. By condition (ii), $A_{\hat{B}}$ has full column rank since $\hat{B} \subseteq B$ and the columns of A_B are linearly independent. By Proposition 13.1, $\hat{x}$ is a basic solution of $\mathcal{P}$. The claim follows. □

13.2.4 Terminology

Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form. Let $\hat{x} \in \mathbb{R}^n$ be a basic solution of $\mathcal{P}$. Let

$$\begin{aligned} \hat{B} &= \{j \in \{1, \ldots, n\} : \hat{x}_j \neq 0\}, \\ \hat{N} &= \{j \in \{1, \ldots, n\} : \hat{x}_j = 0\}. \end{aligned}$$

and let B and N be two index sets such that $\hat{B} \subseteq B \subseteq \{1, \ldots, n\}$ and $N \subseteq \hat{N} \subseteq \{1, \ldots, n\}$, $B \cup N = \{1, \ldots, n\}$, $|B| = m$, and $|N| = n - m$.

Definition 13.1. 1. *For each $j \in B$, the variable x_j is called a* basic variable. *Note that $\hat{x}_j \geq 0$ for each $j \in B$ if $\hat{x}$ is a basic feasible solution.*

2. *For each $j \in N$, the variable x_j is called a* nonbasic variable. *Note that $\hat{x}_j = 0$ for each $j \in N$ if $\hat{x}$ is a basic solution or a basic feasible solution.*

3. *The invertible matrix $A_B \in \mathbb{R}^{m \times m}$ is called the* basis matrix.

13.2.5 Enumeration of Vertices of Polyhedra in Standard Form

Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form and suppose that A has full row rank. Below is an algorithm for enumerating (i.e., computing) all vertices of $\mathcal{P}$.

Enumeration of Vertices:

(i) Choose m linearly independent columns from A and let $B \subseteq \{1, \ldots, n\}$ denote the set of indices of these columns.

(ii) Define $N = \{1, \ldots, n\} \backslash B$.

(iii) Solve the system $A_B \, \hat{x}_B = b$, i.e., compute $\hat{x}_B = (A_B)^{-1} b$ and obtain the values of basic variables.

(iv) Define $\hat{x}_N = \mathbf{0} \in \mathbb{R}^{|N|}$, i.e., set all nonbasic variables to zero.

(v) $\hat{x} \in \mathbb{R}^n$ is a basic solution.

(vi) If $\hat{x}_B \geq \mathbf{0}$ (i.e., values of all basic variables are nonnegative), then $\hat{x} \in \mathbb{R}^n$ is a basic feasible solution.

Remarks

1. Recall that every polyhedron has at most a finite number of vertices.

2. Every nonempty polyhedron in standard form has at least one vertex.

3. Under the full row rank assumption on A, the number of basic solutions of a polyhedron in standard form with m equations and n variables is bounded above by $\binom{n}{m}$.

4. Therefore, if A has full row rank, then the number of basic feasible solutions is bounded above by $\binom{n}{m}$ since the set of basic feasible solutions is a subset of basic solutions.

5. However, if $m = 10$ and $n = 20$ (considered a "tiny" linear programming problem), we have $\binom{n}{m} = 184756$.

Exercises

Question 13.1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form and suppose that A has full row rank. Suppose that $\mathcal{P} = \emptyset$.*

(i) Does $\mathcal{P}$ have any basic feasible solutions?

(ii) Does $\mathcal{P}$ have any basic solutions?

MATH11111: Fundamentals of Optimization	**Fall 2022**
Lecture 14	Fundamental Theorem of Linear Programming
Lecturer: E. Alper Yıldırım	*Week: 4*

14.1 Outline

- Existence of an Optimal Solution
- Fundamental Theorem of Linear Programming
- Review Problems

14.2 Overview

In this lecture, we will establish the fundamental theorem of linear programming. This theorem will form the theoretical basis to develop a solution method for a general linear programming problem.

Let

$$\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x \geq b_i,\ i \in M_1; (a^i)^T x \leq b_i,\ i \in M_2; (a^i)^T x = b_i,\ i \in M_3\}$$

be a nonempty polyhedron that contains at least one vertex and let $c \in \mathbb{R}^n$.

Consider the general linear programming problem given by

$$\text{(P)} \quad \min\{c^T x : x \in \mathcal{P}\}.$$

Earlier, we established the following result: If $\mathcal{P}$ contains at least one vertex, and the set of optimal solutions of (P), denoted by $\mathcal{P}^*$, is nonempty, then $\mathcal{P}^*$ contains at least one vertex of $\mathcal{P}$ (i.e., there exists at least one optimal solution which is a vertex of $\mathcal{P}$).

However, note that this is still a partial result since (i) we need to assume that $\mathcal{P}$ contains at least one vertex, and (ii) that $\mathcal{P}^* \neq \emptyset$.

If a linear programming problem is infeasible or unbounded, then it does not have any optimal solutions. In general, an optimization problem may not have any optimal solutions even if it has a finite optimal value:

$$\min\{1/x : x \geq 1\}$$

Note that the optimal value is 0, and even though the problem has a finite optimal value, this value is not attained by any feasible solution.

Question 1. *Does every linear programming problem with a finite optimal value have at least one optimal solution?*

14.3 Existence of an Optimal Solution

The following proposition establishes that every linear programming problem in standard form with a finite optimal value has at least one optimal solution (i.e., the optimal value is attained by at least one feasible solution).

Proposition 14.1. *Let (P) denote a linear programming problem in standard form.*

$$\begin{array}{llrcl} (P) & \min & c^T x & & \\ & \text{s.t.} & & & \\ & & Ax & = & b, \\ & & x & \geq & \mathbf{0}. \end{array}$$

If (P) has a finite optimal value, then the set of optimal solutions, denoted by $\mathcal{P}^*$, is nonempty. Furthermore, $\mathcal{P}^*$ contains at least one vertex of $\mathcal{P}$.

Proof. Since the feasible region, denoted by $\mathcal{P}$, is nonempty, we may assume that A has full row rank. We will show the following result. For every feasible solution $\bar{x} \in \mathcal{P}$, there exists a vertex $\hat{x} \in \mathcal{P}$ such that $c^T\hat{x} \leq c^T\bar{x}$. Let $\bar{x} \in \mathcal{P}$ be an arbitrary feasible solution.

Case 1: If $\bar{x}$ is a vertex, then we set $\hat{x} = \bar{x}$ and this proves our claim.

Case 2: Suppose that $\bar{x}$ is not a vertex. Let $\bar{B} = \{j \in \{1, \ldots, n\} : \bar{x}_j \neq 0\} = \{j \in \{1, \ldots, n\} : \bar{x}_j > 0\}$, and $\bar{N} = \{j \in \{1, \ldots, n\} : \bar{x}_j = 0\}$. Then, $A_{\bar{B}} \in \mathbb{R}^{m \times |\bar{B}|}$ does not have full column rank. Therefore, there exists a nonzero vector $d_{\bar{B}} \in \mathbb{R}^{|\bar{B}|}$ such that $A_{\bar{B}} d_{\bar{B}} = \mathbf{0} \in \mathbb{R}^m$. Define $d_{\bar{N}} = \mathbf{0} \in \mathbb{R}^{|\bar{N}|}$. Let $d \in \mathbb{R}^n$ be a vector that has $d_{\bar{B}}$ and $d_{\bar{N}}$ as its subvectors. Therefore, $Ad = A_{\bar{B}} d_{\bar{B}} + A_{\bar{N}} d_{\bar{N}} = \mathbf{0} \in \mathbb{R}^m$. If $c^T d > 0$, then replace d by $-d$. Note that $Ad = \mathbf{0}$ still holds. We can assume that $c^T d \leq 0$. Consider the line $\bar{x} + \lambda d$, where $\lambda \in \mathbb{R}$. Note that, for any real number $\lambda \in \mathbb{R}$, we have

$$A(\bar{x} + \lambda d) = A\bar{x} + \lambda A d = b,$$

and

$$(\bar{x} + \lambda d)_{\bar{B}} = \bar{x}_{\bar{B}} + \lambda d_{\bar{B}}, \quad (\bar{x} + \lambda d)_{\bar{N}} = \bar{x}_{\bar{N}} + \lambda d_{\bar{N}} = \mathbf{0}.$$

Case 2a: Suppose that $c^T d < 0$. We claim that there exists $j \in \bar{B}$ such that $d_j < 0$ (i.e., the vector $d_{\bar{B}}$ has at least one negative component.

Suppose that the claim is false. Then, $d_{\bar{B}} \geq 0$. Therefore, $\bar{x} + \lambda d \geq \mathbf{0}$ for each $\lambda \geq 0$, which implies that $\bar{x} + \lambda d \in \mathcal{P}$ for each $\lambda \geq 0$.

Since $c^T d < 0$, we obtain

$$c^T(\bar{x} + \lambda d) = c^T\bar{x} + \lambda c^T d \to -\infty$$

as $\lambda \to +\infty$. It follows that (P) is an unbounded problem, which contradicts our hypothesis.

Therefore, our claim holds. Since $\bar{x}_{\bar{B}} > 0$, $d_{\bar{B}}$ has at least one negative component, and $\lambda \geq 0$, there exists a real number $\lambda^* > 0$ such that $\bar{x} + \lambda d \geq \mathbf{0}$ if $\lambda \in [0, \lambda^*]$ and at least one component of $(\bar{x} + \lambda^* d)_{\bar{B}} = \bar{x}_{\bar{B}} + \lambda^* d_{\bar{B}}$ is equal to zero. Letting $\lambda = \lambda^*$, we have $(\bar{x} + \lambda^* d)_{\bar{N}} = \bar{x}_{\bar{N}} + \lambda^* d_{\bar{N}} = \mathbf{0}$. Therefore, we obtain a new feasible solution $\hat{x} = \bar{x} + \lambda^* d$ with a better objective function value since

$$c^T\hat{x} = c^T(\bar{x} + \lambda^* d) = c^T\bar{x} + \lambda^* c^T d < c^T\bar{x},$$

and $\hat{B} \subset \bar{B}$, where $\hat{B} = \{j \in \{1, \ldots, n\} : \hat{x}_j \neq 0\} = \{j \in \{1, \ldots, n\} : \hat{x}_j > 0\}$. If $A_{\hat{B}}$ has full column rank, then $\hat{x}$ is a vertex of $\mathcal{P}$ and we are done. Otherwise, we can repeat the same procedure starting with $\hat{x}$ as our new feasible solution.

Case 2b: Suppose that $c^T d = 0$. Since $\mathcal{P}$ is a nonempty polyhedron in standard form, it cannot contain a line. Therefore, the line $\bar{x} + \lambda d$ should eventually exit $\mathcal{P}$. Let $\lambda^* \in \mathbb{R}$ denote the value of λ corresponding to the last point of contact with $\mathcal{P}$ before it exits $\mathcal{P}$. Let $\hat{x} = \bar{x} + \lambda^* d$. By a similar reasoning, there should exist $j \in \bar{B}$ such that $\hat{x}_j = 0$. Therefore,

$$c^T \hat{x} = c^T(\bar{x} + \lambda^* d) = c^T \bar{x} + \lambda^* c^T d = c^T \bar{x},$$

i.e., the objective function value of $\hat{x}$ is no worse than that of $\bar{x}$ and $\hat{B} \subset \bar{B}$, where $\hat{B} = \{j \in \{1, \ldots, n\} : \hat{x}_j \neq 0\} = \{j \in \{1, \ldots, n\} : \hat{x}_j > 0\}$. If $A_{\hat{B}}$ has full column rank, then $\hat{x}$ is a vertex of $\mathcal{P}$ and we are done. Otherwise, we can repeat the same procedure starting with $\hat{x}$ as our new feasible solution.

Therefore, it follows from both cases that, for any feasible solution $\bar{x} \in \mathcal{P}$, there exists a vertex $\hat{x} \in \mathcal{P}$ such that $c^T \hat{x} \leq c^T \bar{x}$. Since $\mathcal{P}$ contains at most a finite number of vertices denoted by $\{x^1, \ldots, x^k\}$, we obtain that the optimal value, denoted by z^*, satisfies:

$$z^* = \min_{i=1,\ldots,k} c^T x^i.$$

Therefore, $\mathcal{P}^* \neq \emptyset$ and $\mathcal{P}^*$ contains at least one vertex of $\mathcal{P}$.

□

14.4 A Generalisation

In this section, we give a partial extension of Proposition 14.1 to general linear programming problems.

Corollary 14.1. *Every linear programming problem with a finite optimal value has at least one optimal solution.*

Proof. Note that every linear programming problem can be converted into an equivalent linear programming problem in standard form. Therefore, the result follows from Proposition 14.1, and the fact that there is a one-to-one correspondence between the feasible solutions of the two linear programming problems. □

Remark 14.1. *Note that Corollary 14.1 holds even if the feasible region does not contain any vertices. However, we can no longer guarantee that the set optimal solutions contains at least one vertex since the feasible region may not necessarily contain any vertices in the general case.*

14.5 Fundamental Theorem of Linear Programming

By combining all the results thus far, we can state the following fundamental theorem.

Theorem 14.1 (Fundamental Theorem of Linear Programming)**.** *Let (P) denote a linear programming problem in standard form.*

$$\begin{array}{lll} (P) & \min & c^T x \\ & s.t. & \\ & & Ax = b, \\ & & x \geq 0. \end{array}$$

Then, exactly one of the following three statements is true for (P):

1. *(P) is infeasible and* $z^* = +\infty$.

2. *The optimal value is finite* $(-\infty < z^* < +\infty)$ *and there is at least one vertex* $x^* \in \mathcal{P}$ *in the set of optimal solutions (i.e.,* $c^T x^* = z^*$*).*
3. *(P) is unbounded and* $z^* = -\infty$.

Theorem 14.1 has many interesting implications. Recall that every linear programming problem can be converted into an equivalent linear programming problem in standard form. By Theorem 14.1, if the optimal value is finite, it suffices to search only over the finite number of vertices.

However, the following three problems still remain:

1. What if there are too many vertices?
2. How do we detect that (P) is infeasible?
3. How do we detect that (P) is unbounded?

We will attempt to answer these questions in the following lectures.

Exercises

Question 14.1. *Let (P) denote a linear programming problem in standard form.*

$$\begin{array}{llrcl} (P) & \min & c^T x & & \\ & s.t. & & & \\ & & Ax & = & b, \\ & & x & \geq & 0. \end{array}$$

Suppose that the optimal value of (P) is finite, $(-\infty < z^* < +\infty)$. *Show that the set of optimal solutions, denoted by* $\mathcal{P}^*$*, is a nonempty polyhedron that does not contain a line.*