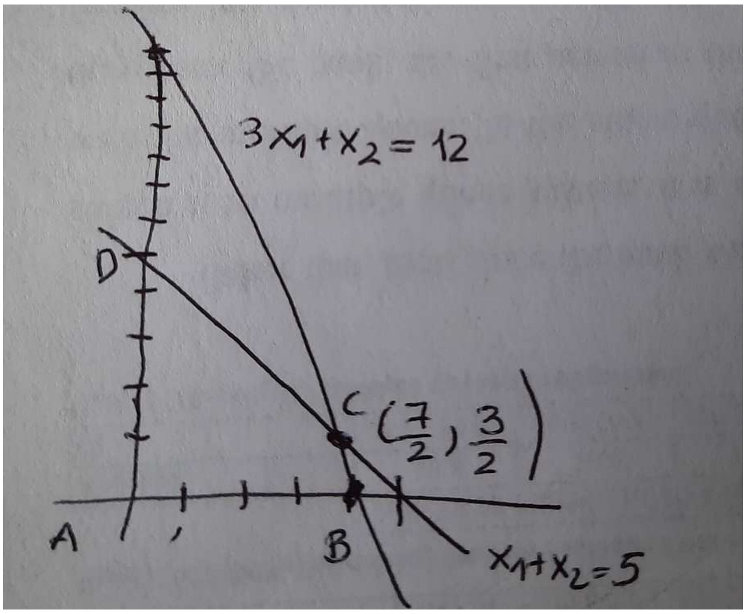
Fundamentals of Operational Research
Tutorial 5
School of Mathematics
The University of Edinburgh
Year 2022/2023

1. Use branch-and-bound to solve the following problem:

$$\begin{array}{ll} \text{Max.} & 5x_1 + 2x_2 \\ \text{s.t.} & 3x_1 + x_2 \le 12, \\ & x_1 + x_2 \le 5, \\ & x_1, x_2 \in \mathbb{Z}^+. \end{array}$$

Solution:
First we draw the feasible region of the linear relaxation:



The feasible region is given by the polyhedron ABCD, where $A = (0,0)$, $B = (4,0)$, $C = (7/2, 3/2)$, and $D = (0,5)$.

When we draw different levels of the objective function $5x_1 + 2x_2$, we see that the optimal solution is $\tilde{x} = (7/2, 3/2)$ and the optimal value is $\tilde{z} = 41/2$.

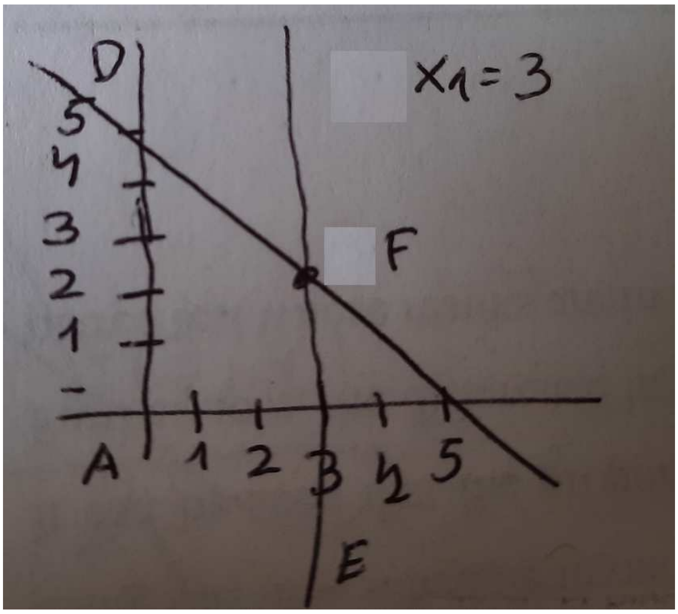
As x_1 and x_2 are fractional but must be integer, we have to branch. Arbitrarily, we choose to branch on variable x_1. If we name the original linear relaxation as Subproblem 1, then we create these two subproblems:

- Subproblem 2: Subproblem 1 + constraint $x_1 \le 3$.
- Subproblem 3: Subproblem 1 + constraint $x_1 \ge 4$.

We choose to solve first Subproblem 3 because we see that the feasible region reduces to a single point: $(4, 0)$. Thus, $\tilde{x} = (4, 0)$ and $\tilde{z} = 20$.

This is an integer solution. So, we prune this node. Moreover, as it is our first integer solution, we record it as our incumbent solution: $\bar{x} = (4, 0)$ and $\bar{z} = 20$.

Now, we need to solve Subproblem 2, which is the only unsolved problem. We draw the feasible region:



The feasible region is polyhedron AEFD, where $E = (3, 0)$ and $F = (3, 2)$. Constraint $3x_1 + x_2 \leq 12$ has no influence after having added $x_1 \leq 3$.

The optimal solution is $\tilde{x} = (3, 2)$ with optimal value $\tilde{z} = 19$. This is an integer solution. So, we prune the current node. Moreover, it does not improve the current incumbent.

As there are no more nodes to explore, the incumbent solution is the optimal solution to the problem: $x^* = (4, 0)$ and $z^* = 20$.

2. Consider the problem

$$\begin{array}{ll} \text{Max.} & 3x_1 + 2x_2 \\ \text{s.t.} & x_1 + 2x_2 \leq 7, \\ & 5x_1 + 2x_2 \leq 10, \\ & x_1, x_2 \in \mathbb{Z}^+. \end{array}$$

The optimal solution to the linear relaxation is $\tilde{x} = \left(\frac{3}{4}, \frac{25}{8}\right)$. Find Gomory cuts that cut off this point.

Solution:

The point is the intersection of the two constraints given. Thus, we add slack constraints s_3 and s_4:

$$x_1 + 2x_2 + s_3 = 7,$$
$$5x_1 + 2x_2 + s_4 = 10.$$

Solving for x_1 and x_2 we have that:

$$x_1 = \frac{3}{4} + \frac{1}{4}s_3 - \frac{1}{4}s_4,$$
$$x_2 = \frac{25}{8} - \frac{5}{8}s_3 + \frac{1}{8}s_4.$$

Now, we apply splitting of coefficients for each of the equalities and each of the two possible inequalities in order to obtain Gomory cuts as seen in class.

(a) First inequality, $\leq$.
$x_1 - \frac{1}{4}s_3 + \frac{1}{4}s_4 \leq \frac{3}{4}$;

$x_1 + \left(-1\frac{3}{4}\right) s_3 + \frac{1}{4}s_4 \leq \frac{3}{4}$;
$x_1 - s_3 \leq \frac{3}{4} - \frac{3}{4}s_3 - \frac{1}{4}s_4 \leq \frac{3}{4}$.
Using that s_3 is integer:
$x_1 - s_3 \leq \left\lfloor \frac{3}{4} \right\rfloor = 0$;
$x_1 - s_3 \leq 0$.
Substituting $s_3 = 7 - x_1 - 2x_2$ we have that
$x_1 - (7 - x_1 - 2x_2) \leq 0$;
$2x_1 + 2x_2 \leq 7$.

(b) First inequality, $\geq$.
$x_1 - \frac{1}{4}s_3 + \frac{1}{4}s_4 \geq \frac{3}{4}$;
$x_1 - \frac{1}{4}s_3 + \left(1 - \frac{3}{4}\right) s_4 \geq \frac{3}{4}$;
$x_1 + s_4 \geq \frac{3}{4} + \frac{1}{4}s_3 + \frac{3}{4}s_3 \geq \frac{3}{4}$.
Thus, $x_1 + s_4 \geq \left\lceil \frac{3}{4} \right\rceil_= 1$.
Replacing $s_4 = 10 - 5x_1 - 2x_2$ we have that
$x_1 + 10 - 5x_1 - 2x_2 \geq 1$;
$-4x_1 - 2x_2 \geq -9$;
$4x_1 + 2x_2 \leq 9$.

(c) Second inequality, $\leq$.
$x_2 + \frac{5}{8}s_3 - \frac{1}{8}s_4 \leq \frac{25}{8}$;
$x_2 + \frac{5}{8}s_3 + \left(-1 + \frac{7}{8}\right) s_4 \leq \frac{25}{8}$;
$x_2 - s_4 \leq \frac{25}{8} - \frac{5}{8}s_3 - \frac{7}{8}s_4 \leq \frac{25}{8}$.
Thus, $x_2 - s_4 \leq \left\lfloor \frac{25}{8} \right\rfloor = 3$.
Now, we substitute s_3:
$x_2 - (10 - 5x_1 - 2x_2) \leq 3$;
$5x_1 + 3x_2 \leq 13$.

(d) Second inequality, $\geq$.
$x_2 + \frac{5}{8}s_3 - \frac{1}{8}s_4 \geq \frac{25}{8}$;
$x_2 + \left(1 - \frac{3}{8}\right) s_3 - \frac{1}{8}s_4 \geq \frac{25}{8}$.
Thus, $x_2 + s_3 \geq \left\lceil \frac{25}{8} \right\rceil = 4$.
Then:
$x_2 + 7 - x_1 - 2x_2 \geq 4$;
$-x_1 - x_2 \geq -3$;
$x_1 + x_2 \leq 3$.

3. For the game having the following payoff table, determine the optimal strategy for each player.

Strategy		Player 2		
		1	2	3
	1	-3	1	2
Player 1	2	1	2	1
	3	1	0	-2

Solution:
Let us reduce the table by eliminating the dominated strategies.

- Strategy 3 of player 1 is dominated by strategy 2.
- Strategy 3 of player 2 is dominated by strategy 1.
- Strategy 1 of player 1 is dominated by strategy 2.
- Strategy 2 of player 2 is dominated by strategy 1.

Therefore, the optimal strategy is for player 1 to play strategy 2 and for player 2 to play strategy 1. The value of the game is 1.

4. Find a saddle point for the following game using the minimax criterion to find the best strategy for each player.

Strategy		Player 2		
		1	2	3
	1	1	-1	1
Player 1	2	-2	0	3
	3	3	1	2

Solution:
For player 1, we maximize the minimum payoff:

- Strategy 1: $\min\{1, -1, 1\} = -1$.
- Strategy 2: $\min\{-2, 0, 3\} = -2$.
- Strategy 3: $\min\{3, 1, 2\} = 1$.

Now, $\max\{-1, -2, 1\}$.

Thus, the optimal strategy for player 1 is strategy 3. The maximin payoff happens for strategy 2 of player 2.

For player 2, we minimize the maximum payoff:

- Strategy 1: $\max\{1, -2, 3\} = 3$.
- Strategy 2: $\max\{-1, 0, 1\} = 1$.
- Strategy 3: $\max\{1, 3, 2\} = 3$.

Now, $\min\{3, 1, 3\} = 1$.

Thus, the optimal strategy for player 2 is strategy 2. The minimax payoff happens for strategy 3 of player 1.

We see that $(3, 2)$ is a saddle point. There is a stable solution: player 1 plays strategy 3 and player 2 plays strategy 2. The value of the game is 1.

5. For the game having the following payoff table, use the graphical solution method to determine the value of the game and the optimal mixed strategies.

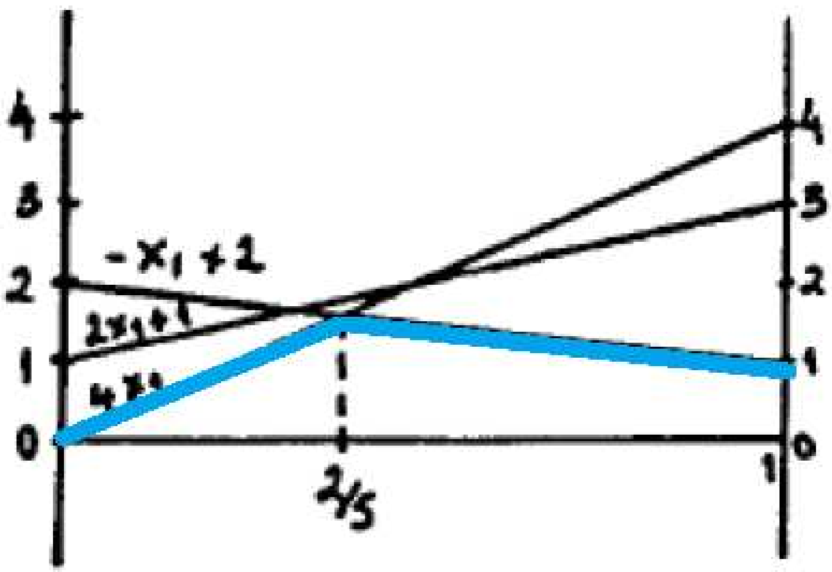
Strategy		Player 2		
		1	2	3
	1	4	3	1
Player 1	2	0	1	2

Solution:
As we have only two nondominated strategies for player 1, we can apply the graphical solution method studied in class.

First we calculate the expected payoff lines for player 1.

(y_1, y_2, y_3)	Expected Payoff for Player 1
$(1, 0, 0)$	$4 \cdot x_1 + 0(1 - x_1) = 4x_1$
$(0, 1, 0)$	$3 \cdot x_1 + 1 \cdot (1 - x_1) = 1 + 2x_1$
$(0, 0, 1)$	$1 \cdot x_1 + 2(1 - x_1) = 2 - x_1$

Now we draw these three lines:



The maximum minimum expected payoff for player 1 is achieved at the point x_1 where lines $4x_1$ and $2 - x_1$ cut. When we solve $4x_1 = 2 - x_1$, we obtain that $x_1^* = \frac{2}{5}$.

Therefore, the optimal mixed strategy for player 1 is $(x_1^*, x_2^*) = \left(\frac{2}{5}, \frac{3}{5}\right)$ and the value of the game is $v = 4 \cdot \frac{2}{5} = \frac{8}{5}$.

Next we need to determine the optimal mixed strategy for player 2.

The expected payoff for player 1 is

$$y_1 \cdot 4x_1 + y_2(1 + 2x_1) + y_3(2 - x_1).$$

We know that for $x_1 = 2/5$ this amount is equal to $8/5$. Thus:

$$\begin{aligned} \frac{8}{5}y_1 + \frac{9}{5}y_2 + \frac{8}{5}y_3 &= \frac{8}{5}; \\ 8y_1 + 9y_2 + 8y_3 &= 8. \end{aligned}$$

As $y_1 + y_2 + y_3 = 1$, we have that $y_2^* = 0$.

Now we have that player 2 has only strategies 1 and 3 to consider. Thus, we repeat the same procedure as before.

(x_1, x_2)	Expected Payoff for Player 1
$(1, 0)$	$4y_1 + 1 \cdot (1 - y_1) = 1 + 3y_1$
$(0, 1)$	$0 \cdot y_1 + 2(1 - y_1) = 2 - 2y_1$

The point where these two lines cut is $y_1^* = \frac{1}{5}$.

Therefore, the optimal mixed strategy for player 2 is $(y_1^*, y_2^*, y_3^*) = \left(\frac{1}{5}, 0, \frac{4}{5}\right)$.