Fundamentals of Operational Research
Assignment 1
School of Mathematics
The University of Edinburgh
Year 2022/2023

INSTRUCTIONS:

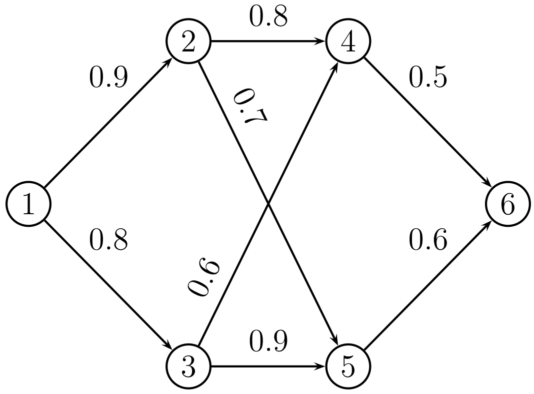
1. This assignment is individual and it is worth 10% of the mark of the course.

2. You must consider this as an exam that you do at home. The same policy than in an exam will be applied for assignments with signs of plagiarism. This is a serious academic offence.

3. The submission deadline is Wednesday 19 October at 14:00 on Gradescope.

4. Provide clear answers and justify every argument that you use. This does not mean to explain in twenty lines what can be said in two. Neither to explain in two words what requires two or three lines. There is no minimum or maximum number of pages that you can submit, but take into account these guidelines.

1) *(10 marks)* We have a data file that needs to be sent between to terminals numbered as 1 and 6 through a network of intermediate computers. The graph below shows for each arc the probability that the file is correctly sent between two computers. If there is no arc between two computers is because they are not directly connected and thus no direct transmission between them is possible (although they might be connected through intermediate computers).



If the file is sent from node i to node j, the probability of being sent correctly is equal to the product of the probabilities of successful transmission for the arcs in that path. For example, the probability of successful transmission from node 3 to node 6 passing through node 4 is $0.6 \times 0.5 = 0.3$.
Use backward dynamic programming to find the route that maximizes the probability of

successful transmission from node 1 to node 6. More specifically, answer the following questions:

(a) *(3 marks)* Describe the stages, states and objective function for this problem.

(b) *(2 marks)* Write a recursive expression that you will use later (in part c) to solve this problem.

(c) *(4 marks)* Use that recursive function to find the maximum probability of successful transmission and a route associated to that probability.

(d) *(1 marks)* Without doing any calculation: Does the maximum probability or the route change if link $(2,4)$ is hacked and the probability of successful transmission lowers from 0.8 to 0.2? Why?

Solution:

(a) *(1 mark)*The objective function is the product of the probabilities of the arc on the chosen path. If p_{ij} is the probability of successful transmission for arc (i,j), then our objective function is

$$\prod_{(ij)\in P(1,6)} p_{ij},$$

where $P(1,6)$ is a path from node 1 to node 6. We need to maximize this function over all the possible paths $P(1,6)$.

*(1 mark)*The states are the different nodes, 1 to 6.

*(1 mark)*The stages are the nodes again.

Option 2: It is also correct if the problem is solved by defining 4 stages, where stage 1 consists of node 1, stage 2 consists of nodes 2 and 3, stage 3 consists of nodes 4 and 5, and stage 4 consists of node 4. This is based on the fact that any path from 1 to¬6 has exactly 3 arcs. The way of solving the problem would differ slightly at some points from what is written here. In that case, it would be as on the hiker example that the Dynamic Programming chapter on the lecture notes.

(b) *(1 mark)*If we are on node i, we need to choose an arc (i,j). Then from j we will choose the path to 6 with the maximum probability of transmission.

*(1 mark)*If we denot by $F(i)$ the maximum probability of transmission from node i to node 6, we have that

$$F(i) = \max\left\{p_{ij} \times F(j)\right\}_{(i,j)\in A},$$

where A is the set of arcs of the graph.

(c) *(1 mark)*We start by defining $F_6 = 1$ because for transmitting from node 6 to node 6 there is no actual transmission. The data file is already there.

(2 marks) Now we do the calculations on the following table.

From node	Maximum transmission probability to node 6	Next node is
6	1	-
5	$0.6 \times F(6) = 0.6 \times 1 = 0.6$	6
4	$0.5 \times F(6) = 0.5 \times 1 = 0.5$	6
3	$\max\{0.6 \times F(4), 0.9 \times F(5)\} = \max\{0.6 \times 0.5, 0.9 \times 0.6\} = \max\{0.3, \underline{0.54}\} = 0.54$	5
2	$\max\{0.8 \times F(4), 0.7 \times F(5)\} = \max\{0.8 \times 0.5, 0.7 \times 0.6\} = \max\{0.4, \underline{0.42}\} = 0.42$	5
1	$\max\{0.9 \times F(2), 0.8 \times F(3)\} = \max\{0.9 \times 0.42, 0.8 \times 0.54\} = \max\{0.378, \underline{0.432}\} = 0.432$	3

*(1 mark)*The maximum probability is 0.432. The only optimal path is $1 \to 3 \to 5 \to 6$.

(d) *(1 mark)*As the transmission probability for arc $(2, 4)$ is now a smaller value, all the paths that use that arc will see its total transmission probability reduced (we do not do the calculations). However, the optimal path that we found does not use this arc. Therefore, the optimal solution is still the same and the optimal value does not change either.