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Fundamentals of Optimization

Exercise 4

Remarks

- All questions that are available in the STACK quiz are duly marked. Please solve those using STACK.
- We have added marks for each question. Please note that those are purely for illustrative purposes. The exercise set will not be marked.
- We can derive the inverse of a nonsingular matrix $A \in \mathbb{R}^{2\times 2}$ in closed form:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

STACK Problems

1 Simplex Method for Nondegenerate LPs (2.5 marks)

STACK question

You are given three linear programming problems below such that the matrix A has full row rank. For each problem, a starting dictionary is given. For each given dictionary, write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether the given dictionary is optimal or not. If applicable, perform only **one** iteration of the simplex method starting from the given dictionary. For the next dictionary (if applicable), write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether this dictionary is optimal or not.

(1.1) LP

$$\begin{array}{lrcrcrcrcrcl}
\min & -4x_1 & - & 3x_2 & & & & & & & & \\
\text{s.t.} & & & x_2 & + & x_3 & & & & & = & 6 \\
& x_1 & + & x_2 & & & + & x_4 & & & = & 7 \\
& 3x_1 & + & 2x_2 & & & & & + & x_5 & = & 18 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}$$

and the starting dictionary is given by

$$\begin{array}{rcrcrcr}
z & = & -22 & - & x_3 & + & 4x_4 \\
x_1 & = & 1 & + & x_3 & - & x_4 \\
x_2 & = & 6 & - & x_3 & & \\
x_5 & = & 3 & - & x_3 & + & 3x_4
\end{array}$$

(1.2) LP

$$\begin{array}{lrcrcrcrcrcl}
\min & 5x_1 & + & 2x_2 & - & 6x_3 & & & & & & \\
\text{s.t.} & -3x_1 & + & x_2 & + & 3x_3 & + & x_4 & & & = & 31 \\
& 4x_1 & - & 2x_2 & + & 2x_3 & & & + & x_5 & = & 18 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}$$

and the starting dictionary is given by

$$\begin{array}{rcrcrcrcr} z & = & 0 & + & 5x_1 & + & 2x_2 & - & 6x_3 \\ x_4 & = & 31 & + & 3x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 18 & - & 4x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

(1.3) LP

$$\begin{array}{lrcrcrcrcrcl} \min & -x_1 & - & 2x_2 & & & & & & & & \\ \text{s.t.} & -6x_1 & + & 3x_2 & + & 3x_3 & & & & & = & 15 \\ & x_1 & - & 3x_2 & & & + & x_4 & & & = & 1 \\ & -2x_1 & & & & & & & + & x_5 & = & 8 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

and the starting dictionary is given by

$$\begin{array}{rcrcrcr} z & = & -10 & - & 5x_1 & + & 2x_3 \\ x_2 & = & 5 & + & 2x_1 & - & x_3 \\ x_4 & = & 16 & + & 5x_1 & - & 3x_3 \\ x_5 & = & 8 & + & 2x_1 & & \end{array}$$

2 The Simplex Method for Degenerate LPs (2.5 marks)

STACK question

You are given the following linear programming problem that has full row rank:

$$\begin{array}{lrcrcrcrcrcl} \min & -x_1 & - & 2x_2 & + & x_3 & & & & & & \\ \text{s.t.} & x_1 & & & + & 2x_3 & + & x_4 & & & = & 2 \\ & & & x_2 & + & 2x_3 & & & + & x_5 & = & 2 \\ & & & & & & & \multicolumn{3}{r}{x_1,\, x_2,\, x_3,\, x_4,\, x_5} & \geq & 0 \end{array}$$

You are given three different starting dictionaries. For each given dictionary, write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether the given dictionary is optimal or not. If applicable, perform only **one** iteration of the simplex method starting from the given dictionary. If the current dictionary is nondegenerate, use the most negative reduced cost to determine the entering variable. If it is degenerate, use Bland's rule to determine the entering and leaving variables (whenever applicable). For the next dictionary (if applicable), write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether this dictionary is optimal or not.

(2.1) The starting dictionary is given by

$$\begin{array}{rcrcrcrcr} z & = & 1 & - & x_1 & - & \frac{5}{2}x_2 & - & \frac{1}{2}x_5 \\ x_3 & = & 1 & & & - & \frac{1}{2}x_2 & - & \frac{1}{2}x_5 \\ x_4 & = & 0 & - & x_1 & + & x_2 & + & x_5 \end{array}$$

(2.2) The starting dictionary is given by

$$\begin{array}{rcrcrcrcr} z & = & -2 & - & 2x_2 & + & 3x_3 & + & x_4 \\ x_1 & = & 2 & & & - & 2x_3 & - & x_4 \\ x_5 & = & 2 & - & x_2 & - & 2x_3 & & \end{array}$$

(2.3) The starting dictionary is given by

$$\begin{array}{rcrcrcrcr} z & = & 1 & - & \frac{3}{2}x_1 & - & 2x_2 & - & \frac{1}{2}x_4 \\ x_3 & = & 1 & - & \frac{1}{2}x_1 & & & - & \frac{1}{2}x_4 \\ x_5 & = & 0 & + & x_1 & - & x_2 & + & x_4 \end{array}$$

Open Ended Problems

3 Puzzle (1 mark)

Consider the following linear program

$$
\begin{array}{lrcrcrcrcrcr}
\min & c_1\,x_1 & + & c_2\,x_2 & - & 4x_3 & & & & & & \\
\text{s.t} & 3x_1 & + & a_{12}\,x_2 & + & 3x_3 & + & x_4 & & & = & 30 \\
 & 2x_1 & + & a_{22}\,x_2 & + & a_{23}\,x_3 & & & + & x_5 & = & 18 \\
 & & & & & & & & & x_1,\, x_2,\, x_3,\, x_4,\, x_5 & \geq & 0
\end{array}
$$

and the final dictionary with the optimal solution

$$
\begin{array}{rcrcrcrcr}
z & = & -27 & + & 5x_1 & + & \frac{1}{2}x_3 & + & \bar{c}_5\,x_5 \\
x_2 & = & 9 & - & x_1 & - & \frac{3}{2}x_3 & - & \frac{1}{2}x_5 \\
x_4 & = & 21 & - & 2x_1 & - & \frac{3}{2}x_3 & + & \frac{1}{2}x_5
\end{array}
$$

Determine the unknown values for a_{12}, a_{22}, a_{23}, c_1, c_2, and $\bar{c}_5$. Justify your solution.

[1 mark]

4 Duality (4 marks)

(4.1) Consider the following polyhedron in standard form:

$$\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given and $x \in \mathbb{R}^n$.

(i) Suppose that $\mathcal{P}$ is nonempty. Prove that there does not exist a vector $\hat{y} \in \mathbb{R}^m$ such that $A^T\hat{y} \leq \mathbf{0}$ and $b^T\hat{y} > 0$.

[1 mark]

(ii) Conversely, suppose that there does not exist a vector $\hat{y} \in \mathbb{R}^m$ such that $A^T\hat{y} \leq \mathbf{0}$ and $b^T\hat{y} > 0$. Prove that $\mathcal{P}$ is nonempty.

Hint: Try to set up an appropriate pair of primal-dual problems and use the appropriate duality relations.

[1 mark]

(4.2) Determine the set of all optimal solutions for the following linear program without using the simplex method:

$$
\begin{array}{llrcrcrcrcr}
(P) & \min & -2x_1 & + & 6x_2 & + & 6x_3 & - & x_4 & & \\
 & \text{s.t.} & 2x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 1 \\
 & & -3x_1 & + & 2x_2 & + & x_3 & - & x_4 & = & 1 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0.
\end{array}
$$

[2 marks]