Week 4

Quadratic Programming

Portfolio Optimization

Methodology, Modelling, and Consulting Skills
The University of Edinburgh

Where are we?

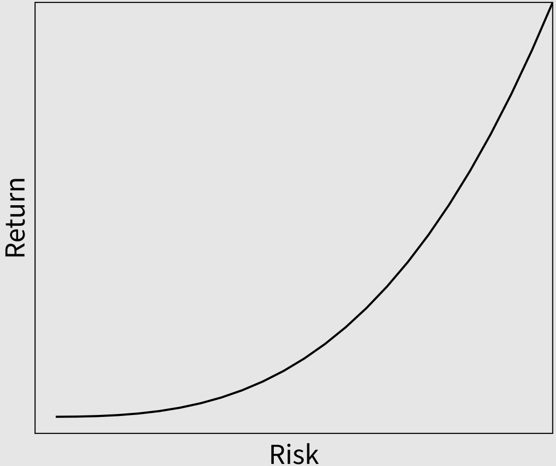
1. Linear programming
2. More linear programming
3. Integer programming
4. Quadratic programming
   - Assignment 1 due 10 October
   - Assignment 2 released
5. Facility location problems
   - I will release the project
6. How to innovate
   - Guest lecture Christina Starko – Edinburgh Innovations
   - Assignment 2 due 24 October
   - Release assignment 3 – due 7 November

7.–10. Project
   - Report due 28 November
   - Presentations on 29 November

Investments

Risk vs Return

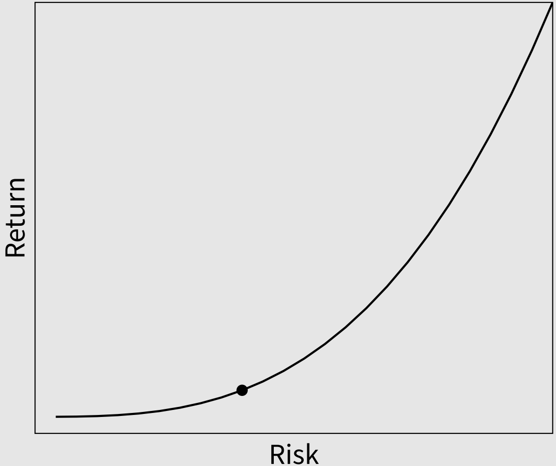
- We would like to earn as much as possible.
- But profitable investments are risky.



Investments

Risk vs Return

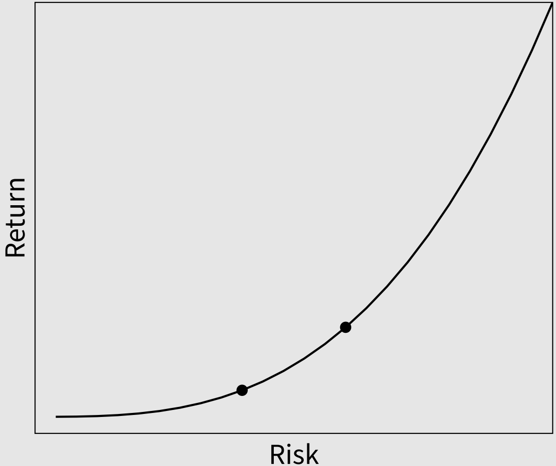
- We would like to earn as much as possible.
- But profitable investments are risky.



Investments

Risk vs Return

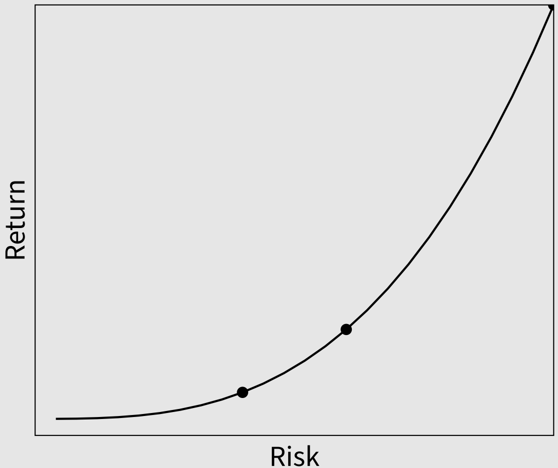
- We would like to earn as much as possible.
- But profitable investments are risky.



Investments

Risk vs Return

- We would like to earn as much as possible.
- But profitable investments are risky.



Markowitz Portfolio Model

Mean Variance Optimization

Vocabulary

Fund	money that can be invested.
Asset class	Stock, Bond etc that can be bought.
Portfolio	Combination of assets bought.
Return	Value gained of the portfolio.
Risk	?

Markowitz’s idea

Use *variance* of the expected return as measure for risk

The goals

- We would like to:
  - **Maximize return.**
  - **Minimize risk.**

The goals

- We would like to:
  - **Maximize return.**
  - **Minimize risk.**
- We cannot (usually) do both at the same time!

The goals

- We would like to:
  - **Maximize return.**
  - **Minimize risk.**
- We cannot (usually) do both at the same time!
- What we do is:
  - **Fix the level of risk and maximize return**, or
  - **Fix the level of return and minimize risk.**

The problem

- Given a set of assets, select the **portfolio with minimum risk**.
- The risk is the variance of the portfolio.
- The expected returns and the covariance matrix are known:
  - $\boldsymbol{r}$ is the array of expected returns.
  - $\boldsymbol{Q}$ is the covariance matrix.
- Model introduced by Markowitz in 1950s.

Formulation

- Variables:

$$x_i = \text{"fraction of capital invested in asset } i\text{"}.$$

Formulation

- Variables:

$$x_i = \text{“fraction of capital invested in asset } i\text{”.}$$

- We define the risk of the portfolio as

$$\boldsymbol{x}^T Q \boldsymbol{x}.$$

Formulation

- Variables:

$$x_i = \text{“fraction of capital invested in asset } i\text{”.}$$

- We define the risk of the portfolio as

$$\boldsymbol{x}^T Q \boldsymbol{x}.$$

- We impose that there is a minimum expected return m:

$$\boldsymbol{r}^T \boldsymbol{x} \geq m.$$

Formulation

- Variables:
$$x_i = \text{“fraction of capital invested in asset } i\text{”}.$$
- We define the risk of the portfolio as
$$\boldsymbol{x}^T Q \boldsymbol{x}.$$
- We impose that there is a minimum expected return m:
$$\boldsymbol{r}^T \boldsymbol{x} \geq m.$$
- We cannot exceed the available capital:
$$\boldsymbol{e}^T \boldsymbol{x} \leq 1,$$
where $\boldsymbol{e}^T = (1, 1, \ldots, 1)$.

Formulation

- Variables:

$$x_i = \text{“fraction of capital invested in asset } i\text{”}.$$

- We define the risk of the portfolio as

$$\boldsymbol{x}^T Q \boldsymbol{x}.$$

- We impose that there is a minimum expected return m:

$$\boldsymbol{r}^T \boldsymbol{x} \geq m.$$

- We cannot exceed the available capital:

$$\boldsymbol{e}^T \boldsymbol{x} \leq 1,$$

  where $\boldsymbol{e}^T = (1, 1, \ldots, 1)$.
- We can add some additional requirements:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}.$$

Formulation

- The full model is

$$
\begin{array}{ll}
\text{Min.} & \boldsymbol{x}^T Q \boldsymbol{x} \\
\text{s.t.} & \boldsymbol{r}^T \boldsymbol{x} \geq m, \\
 & \boldsymbol{e}^T \boldsymbol{x} \leq 1, \\
 & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \\
 & \boldsymbol{x} \geq \boldsymbol{0}.
\end{array}
$$

Formulation

- The full model is

$$\begin{aligned} \text{Min.} \quad & \boldsymbol{x}^T Q \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{r}^T \boldsymbol{x} \geq m, \\ & \boldsymbol{e}^T \boldsymbol{x} \leq 1, \\ & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

- **The objective function is non-linear:**

$$\sum_i \sigma_{ii} x_i^2 + \sum_{\substack{i,j, \\ i<j}} 2\sigma_{ij} x_i x_j .$$

Demand and supply response

- Decision makers often desire information on the equilibrium price and quantity that would exist in a given market
- Basic economic theory states that when price changes, so does the quantity demanded

Demand and supply response

Consider an industry which produces n products using r different inputs.

Demand and supply response

Consider an industry which produces n products using r different inputs.

Index sets

N	set products
R	set set of resources

Variables

x_i	amount of product i produced ($i \in N$)
y_j	amount of input j required ($j \in R$)

Parameters

b_{ji}	amount of resource j required to produce a unit product i ($i \in N, j \in R$)
p_i	selling price of unit i ($i \in N$)
c_j	cost of producing resource j ($j \in R$)

Demand and supply response

Consider an industry which produces n products using r different inputs.

Index sets

N	set products
R	set set of resources

Variables

x_i	amount of product i produced ($i \in N$)
y_j	amount of input j required ($j \in R$)

Parameters

b_{ji}	amount of resource j required to produce a unit product i ($i \in N, j \in R$)
p_i	selling price of unit i ($i \in N$)
c_j	cost of producing resource j ($j \in R$)

$$
\begin{aligned}
\max \quad & \sum_{i \in N} p_i x_i - \sum_{J \in R} c_j y_j \\
\text{s.t.} \quad & \sum_{i \in N} b_{ji} x_i \leq y_j \quad \forall j \in R \\
& x_i, y_j \geq 0 \quad \forall i \in N, j \in R
\end{aligned}
$$

Demand and supply response

Consider an industry which produces n products using r different inputs.

Index sets

N	set products
R	set set of resources

Variables

x_i	amount of product i produced ($i \in N$)
y_j	amount of input j required ($j \in R$)

Parameters

b_{ji}	amount of resource j required to produce a unit product i ($i \in N, j \in R$)
p_i	selling price of unit i ($i \in N$)
c_j	cost of producing resource j ($j \in R$)

$$
\begin{aligned}
\max \quad & \boldsymbol{p}^T \boldsymbol{x} - \boldsymbol{c}^T \boldsymbol{y} \\
\text{s.t.} \quad & B\boldsymbol{x} \leq \boldsymbol{y} \\
& \boldsymbol{x}, \boldsymbol{y} \geq \boldsymbol{0}
\end{aligned}
$$

Demand and supply response

Suppose that the selling price of the products are given as

$$\boldsymbol{p}^T = \boldsymbol{\alpha} - \boldsymbol{x}^T D$$

and that the production costs of the resources are given as

$$\boldsymbol{c}^T = \boldsymbol{\beta} + \boldsymbol{y} F$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the constant intercept terms, $\boldsymbol{D}$ is the $n \times n$ matrix of demand gradient where the entry $D_{i,j}$ is the effect that the price of product i on product j and $\boldsymbol{F}$ is similarly the $r \times r$ matrix of supply gradient

Demand and supply response

$$
\begin{aligned}
\max \quad & \boldsymbol{\alpha} \boldsymbol{x} - \boldsymbol{\beta} \boldsymbol{y} - \boldsymbol{x}^T \boldsymbol{D} \boldsymbol{x} - \boldsymbol{y}^T \boldsymbol{F} \boldsymbol{y} \\
\text{s.t.} \quad & B \boldsymbol{x} \leq \boldsymbol{y} \\
& \boldsymbol{x}, \boldsymbol{y} \geq \mathbf{0}
\end{aligned}
$$

Demand and supply response

$$
\begin{aligned}
\max \quad & \boldsymbol{\alpha} \boldsymbol{x} - \boldsymbol{\beta} \boldsymbol{y} - \gamma \boldsymbol{x}^T \boldsymbol{D} \boldsymbol{x} - \gamma \boldsymbol{y}^T \boldsymbol{F} \boldsymbol{y} \\
\text{s.t.} \quad & B \boldsymbol{x} \leq \boldsymbol{y} \\
& \boldsymbol{x}, \boldsymbol{y} \geq \mathbf{0}
\end{aligned}
$$

- We often introduce the parameter $\gamma \in [0.5, 1]$ to model the effects of market competition
- In the case where $\gamma = 1$ we have the market monopoly
- In the case where $\gamma = 0.5$ we haver the case of perfect competition

Demand and supply response example

A firm in Florida producing oranges has two uses for the oranges it produces. One is to convert them into juice, the other is sell fresh oranges. If y denotes the number of oranges produced then the supply price of oranges is given as

$$c = 3 + 0.0005y.$$

If x_1 denotes the number of fresh oranges sold and x_2 denotes the quantity of orange juice sold then the selling price of fresh oranges and orange juice is given as

$$p_1 = 4 - 0.002x_2 - 0.001x_1$$

and

$$p_2 = 7.5 - 0.005x_2 - 0.0002x_1,$$

respectively. Assuming a purely competitive market where all prices are those received by the producer and that the producer makes 1 unit of juice from 2 oranges. Formulate a QP that maximises the firms profits.

Demand and supply response example

A firm in Florida producing oranges has two uses for the oranges it produces. One is to convert them into juice, the other is sell fresh oranges. If y denotes the number of oranges produced then the supply price of oranges is given as

$$c = 3 + 0.0005y.$$

If x_1 denotes the number of fresh oranges sold and x_2 denotes the quantity of orange juice sold then the selling price of fresh oranges and orange juice is given as

$$p_1 = 4 - 0.0002x_2 - 0.001x_1$$

and

$$p_2 = 7.5 - 0.005x_2 - 0.0002x_1,$$

respectively. Assuming a purely competitive market where all prices are those received by the producer and that the producer makes 1 unit of juice from 2 oranges. Formulate a QP that maximises the firms profits.

Demand and supply response example

$$
\begin{aligned}
\max \quad & \boldsymbol{\alpha}\boldsymbol{x} - \boldsymbol{\beta} y - 0.5\boldsymbol{x}^T \boldsymbol{D}\boldsymbol{x} - 0.5 y^2 F \\
\text{s.t.} \quad & x_1 + 2x_2 \leq y \\
& \boldsymbol{x}, y \geq \mathbf{0}
\end{aligned}
$$

where $\boldsymbol{\alpha} = [4, 7.5]$, $\boldsymbol{\beta} = [3]$,

$$
\boldsymbol{D} = \begin{bmatrix} 0.001 & 0.0002 \\ 0.0002 & 0.005 \end{bmatrix}
$$

and $\boldsymbol{F} = [0.0005]$. The optimal solution is to supply 806 oranges to produce 116 units of orange juice and sell 574 as fresh oranges.

Nonlinear Programs

General form

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \qquad \text{for all } i \in \{1, \cdots, m\}. \end{aligned}$$

where $f, g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$.

Difficulty

- Finding global optima *very* difficult.
- Finding local optima much easier, when f, g_i suitably regular.

Important case: f, g_i *convex*, then local optima are global optima.

Linear vs non-linear

Linear is nicer

- Linear models have good properties: there is an optimal solution on a vertex.
- Non-linear models do not have this property.

But . . .

- sometimes we need nonlinearities.

Quadratic Programming

Quadratic program

$$\begin{aligned} \min \quad & x^\top Q x \\ \text{s.t.} \quad & Ax \leq b. \end{aligned}$$

Notes

- Some authors also allow quadratic constraints.
- If Q is positive semidefinite, then objective is convex $\Rightarrow$ local minimum is global.
- If Q is a covariance matrix, then Q is positive semidefinite.