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THE UNIVERSITY *of* EDINBURGH

Fundamentals of Optimization

Homework 1

Instructions

1. You should attempt all questions.
2. The total marks for this assignment are 10.
3. The assignment consists of STACK questions (5/10 marks) and *open-ended* questions (5/10 marks).
4. All STACK questions are duly marked and are available in the STACK quiz. You **must solve those by completing the STACK quiz**.
5. For the open-ended questions, please write down your solutions in a concise and reproducible way and remember to justify every step using appropriate references when necessary. Failing to do so may result in deductions.
6. The strict deadline for completing the quiz and handing-in your solutions for the open-ended questions is **noon (12:00) on Friday, 14 October 2022**.
7. For the open-ended questions, please upload a **single PDF**. For some useful suggestions, please see Course Information → Tips for Creating a PDF File for Submission on the Learn page.

STACK Problems

1 Basic Concepts (3 marks)

STACK question

Decide, for each of the following three optimization problems, whether

(i) the feasible region is *empty*; or *nonempty and bounded*; or *nonempty and unbounded*;

(ii) the feasible region is a *convex set*; or a *nonconvex set*;

(iii) the objective function is *a convex function*; *a concave function*; *both convex and concave*; or *neither convex nor concave*;

(iv) the optimization problem is a *convex optimization problem*; or a *nonconvex optimization problem*;

(v) the optimization problem *is infeasible*, *is unbounded*, or *has a finite optimal value*;

(vi) write down the optimal value using the convention in the lectures (use `+inf` for $+\infty$ and `-inf` for $-\infty$);

(vii) the set of optimal solutions is *empty*; or *nonempty*;

(viii) the set of optimal solutions is a *convex set*; or a *nonconvex set*.

(1.1) $\min\{x^3 - 2x^2 + x - 2 : x^2 - 2x - 8 \geq 0, \quad x \in \mathbb{R}\}$.

(1.2) $\min\{2x^2 - 12x - 6 : x^2 - 6x \geq -5, \quad x \in \mathbb{R}\}$.

[3 marks]

2 Level Sets, Sublevel Sets, Superlevel Sets, and Epigraphs (2 marks)

STACK question

Decide, for each of the two functions,

(i) whether $\mathrm{epi}(f)$ is a *convex set* or *nonconvex set*;

(ii) whether the sublevel set $\mathcal{L}^-_\alpha(f)$, where $\alpha = 0$, is a *convex set* or *nonconvex set*;

(iii) whether the level set $\mathcal{L}_\alpha(f)$, where $\alpha = 1$, is a *convex set* or *nonconvex set*;

(iv) whether the superlevel set $\mathcal{L}^+_\alpha(f)$, where $\alpha = 1$, is a *convex set* or *nonconvex set.*

(2.1) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) \;=\; \min\{|x_1|, |x_2|\}$.

(2.2) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) \;=\; x_1^2 + x_2^2$.

[2 marks]

Open Ended Problems

3 Level Sets and Sublevel Sets (2.5 marks)

Consider the following optimization problem:

$$\text{(P)} \quad \min_x \{f(x) : x \in \mathcal{S}\},$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$, and $\mathcal{S} \subseteq \mathbb{R}^n$. Suppose that the optimal value of (P) is denoted by $z^* \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

(3.1) Prove the following proposition:

(P) is an unbounded optimization problem if and only if

$$\mathcal{S} \cap \mathcal{L}^-_\alpha(f) \neq \emptyset, \quad \text{for all } \alpha \in \mathbb{R},$$

where $\mathcal{L}^-_\alpha(f)$ denotes the sublevel set of f at level $\alpha \in \mathbb{R}$,

[1.5 marks]

(3.2) Suppose that $z^* \in \mathbb{R}$ (i.e., the optimal value is finite). Let $\mathcal{S}^* \subseteq \mathbb{R}^n$ denote the set of optimal solutions of (P). Prove the following identity:

$$\mathcal{S}^* = \mathcal{L}_{z^*}(f) \cap \mathcal{S},$$

where $\mathcal{L}_{z^*}(f)$ denotes the level set of f for $\alpha = z^*$. (Hint: One way of showing that the two sets are equal is to show that each set is a subset of the other one as done in Problem 4.1 in Exercise Set 0.) **[1 marks]**

4 Vertices of Convex Sets (2.5 marks)

Let $\mathcal{C}_1 \subseteq \mathbb{R}^n$ and $\mathcal{C}_2 \subseteq \mathbb{R}^n$ be two nonempty convex sets and let $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$. Suppose that $\mathcal{C} \neq \emptyset$.

(4.1) Prove the following result:

If $\hat{x} \in \mathcal{C}$ and $\hat{x}$ is a vertex of at least one of $\mathcal{C}_1$ and $\mathcal{C}_2$, then $\hat{x}$ is a vertex of $\mathcal{C}$.

[1.5 marks]

(4.2) Consider the following proposition, which is the converse of the proposition in (4.1):

If $\hat{x} \in \mathcal{C}$ is a vertex of $\mathcal{C}$, then $\hat{x}$ is a vertex of at least one of $\mathcal{C}_1$ and $\mathcal{C}_2$.

Either prove this proposition or find a counterexample.

[1 mark]