# Bayesian Theory Workshop 4

(20 marks total)

1. (5 marks.) A simple random sample of size $n$=8 was taken and yielded the following data, $\mathbf{y}$:

$$0.2,\ -0.1,\ -0.5,\ 1.3,\ -1.0,\ 0.6,\ 1.2,\ 0.4.$$

The data are known to be independent and have a $N(\mu, \sigma^2)$ distribution, where $\sigma^2 = 1$. We specify the prior: $\mu \sim N(0, 1)$.

   (a) Calculate the posterior distribution for $\mu$, $p(\mu|\mathbf{y})$.

   (b) Calculate 90% and 95% HPDIs for $\mu$.

   (c) Calculate the *posterior* predictive distribution for $Y^{new}$, $p(Y^{new}|\mathbf{y}^{old})$.

Hint: Lecture Notes 3 might be useful.

2. (2 marks.) Consider the general hypothesis testing problem, where we have,

$$H_0 : \theta \in \Theta_0 \quad \text{vs} \quad H_1 : \theta \in \Theta_1,$$

such that the union of $\Theta_0 \cup \Theta_1 = \Theta$, the complete parameter space. Letting $p_0$ and $p_1$ denote the prior probabilities for the null hypothesis and alternative hypothesis, respectively, show that the posterior probability of $H_0$ is given by,

$$\Pr(H_0|\mathbf{y}) = \Pr(\theta \in \Theta_0|\mathbf{y}) = \frac{p_0}{p_0 + p_1/BF_{01}},$$

where $BF_{01}$ denotes the Bayes factor of $H_0$ to $H_1$.

3. (5 marks.) We observe data $\mathbf{y} = \{y_1, \ldots, y_n\}$, such that, $Y_i \overset{iid}{\sim} N(\mu, \sigma^2)$, where $\sigma^2$ is known. We wish to test the two simple hypotheses,

$$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_1 : \mu = \mu_1.$$

   (a) Show that the Bayes factor is given by,

$$BF_{01} = \exp\left(-\frac{n(\mu_0 - \mu_1)(\mu_0 + \mu_1 - 2\bar{y})}{2\sigma^2}\right).$$

   (b) Calculate $BF_{01}$ for $H_0$ against $H_1$, when $\mu_0 = 0$, $\mu_1 = 1$, $\sigma^2 = 1$, $n = 9$ and $\bar{y} = 0.645$.

   (c) Interpret the result.

   (d) What happens as $n$ increases but all other numerical terms retain the same values?

Hint: Lecture Notes 5.2 might be helpful.

4. (8 marks.) We have random variables $Y_1, \ldots, Y_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ where both $\mu$ and $\sigma^2$ are unknown. We specify the priors: $\mu \sim N(0, \sigma_o^2)$ and $\sigma \sim U[0, T]$, where $T$ is "large".

   (a) Using a transformation of variables, calculate the corresponding prior on $\sigma^2$. (Section 4.2 of Lecture Notes might be helpful.)

   (b) Calculate the posterior conditional distributions of $\mu$ and $\sigma^2$ (i.e. the posterior distribution for $\mu$ treating $\sigma^2$ as fixed; and the posterior distribution of $\sigma^2$, assuming $\mu$ is fixed).

   Hint: To find the conditional for $\mu$, write down the joint posterior distribution for $\mu$ and $\sigma^2$, then remove all terms in the joint posterior distribution not involving $\mu$, and you will see a recognizable kernel for the distribution of $\mu$. Likewise for $\sigma^2$.