Week 5

Facility location problems

Methodology, Modelling, and Consulting Skills
The University of Edinburgh

Discrete location problems

Discrete location problems consist of choosing a subset of locations, among a finite set of candidates, in which to establish facilities which are used to satisfy the demand of a finite set of customers. The choice of the locations must be made to minimise some objective function.
Two types of decisions must be made. Location decisions determine where to establish the facilities and allocation decisions determine how to satisfy the users demand from the established facilities.

> *Laporte G, Nickel S and da Gama FS, 2020, Location Science, Springer Nature, Cham.*

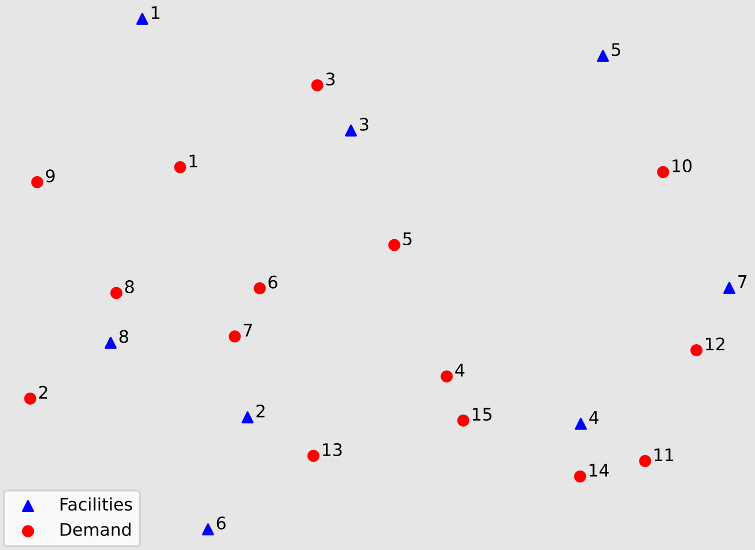
Discrete location problems – Lecture outline

1. The p-median problem
2. The p-centre problem
3. The fixed charge problem
4. The set-covering problem
5. Multi-period facility location problems

The p-median facility location problem

- If we are given a set $\mathcal{I} = \{1, \ldots, m\}$ of potential facilities,
- and a set $\mathcal{J} = \{1, \ldots, n\}$ of customers each with a demand d_j to be supplied,
- find p facilities to open that will satisfy all customer demand at the minimum possible cost.
- The unit costs of supplying customers from candidate facilities are arranged in a matrix $\boldsymbol{C} = c_{i,j}$, where
- $c_{i,j}$ is the cost of serving a unit of demand for customer $j \in \mathcal{J}$ from facility $i \in \mathcal{I}$.

The p-median facility location problem



The p-median facility location problem formulation

Decision variables

$$x_{i,j} = \begin{cases} 1 & \text{if customer } j \in \mathcal{J} \text{ is supplied from facility } i \in \mathcal{I}, \\ 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if facility } i \in \mathcal{I} \text{ is opened,} \\ 0 & \text{otherwise} \end{cases}$$

Objective function

$$\min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} d_j c_{i,j} x_{i,j}$$

Constraints

$$\sum_{i \in \mathcal{I}} x_{i,j} = 1 \quad \forall j \in \mathcal{J}$$

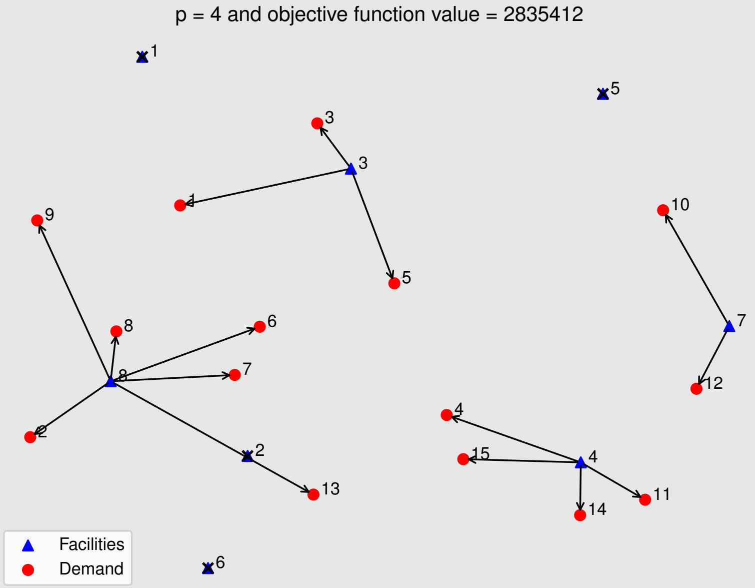
$$x_{i,j} \leq y_i \quad \forall i \in \mathcal{I}, j \in \mathcal{J}$$

$$\sum_{i \in \mathcal{I}} y_i = p$$

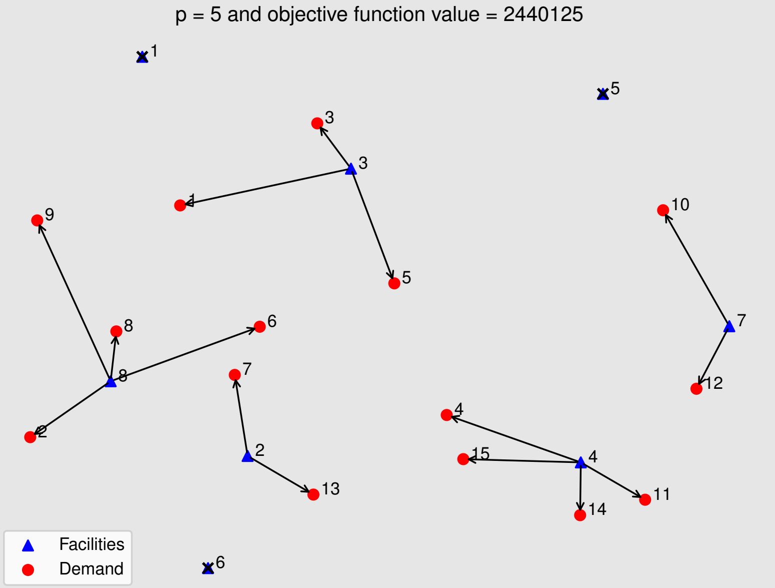
$$x_{i,j} \in \{0, 1\} \quad \forall i \in \mathcal{I}, j \in \mathcal{J}$$

$$y_i \in \{0, 1\} \quad \forall i \in \mathcal{I}$$

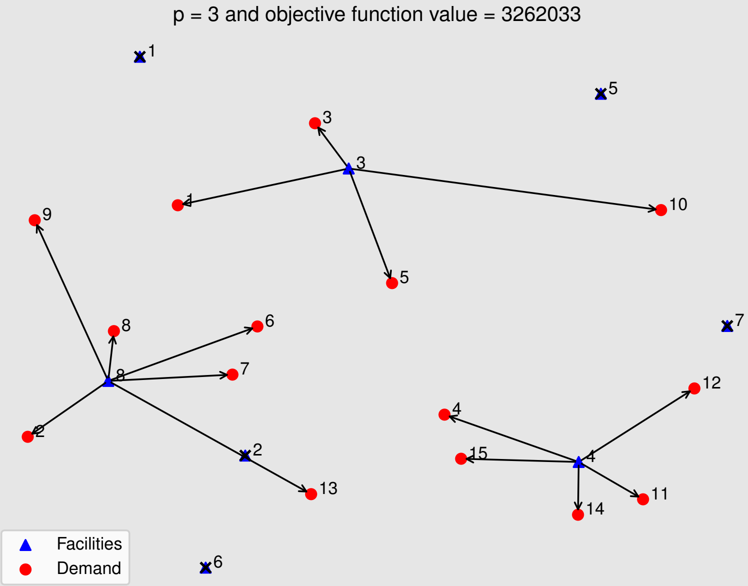
The *p*-median facility location problem – solution



The p-Median problem – solution



The p-Median problem – solution



The p-centre facility location problem

- Minimising the total cost of supplying each customer may not always be the correct objective function.
- Minimising the total cost tends to favour customers who are clustered in population centres to the detriment of clients who are spatially dispersed.
- In many applications, specifically for emergency services we want to minimise the cost of serving the furthest customer.

The p-centre facility location problem formulation

Additional variable

Let $w \in \mathbb{R}$ be the largest distance from any facility to any customer

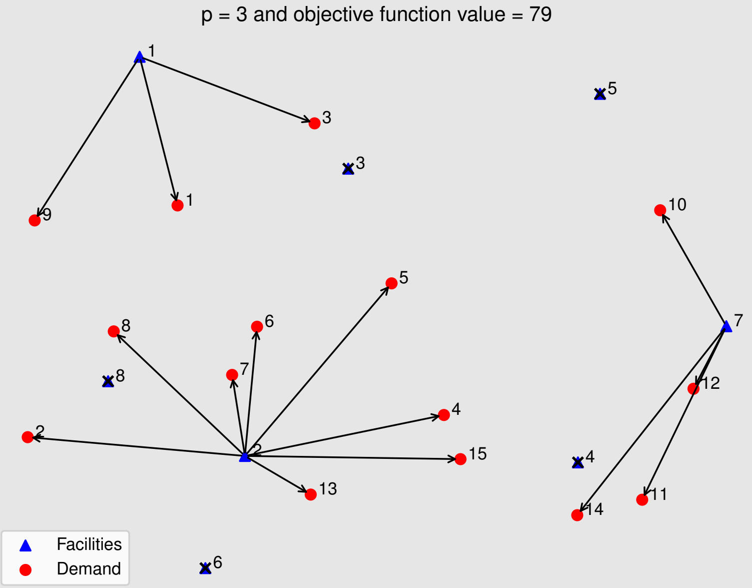
Objective function

$$\min w$$

Constraints

$$
\begin{aligned}
w &\geq \sum_{i \in \mathcal{I}} c_{i,j} x_{i,j} && \forall j \in \mathcal{J} \\
\sum_{i \in \mathcal{I}} x_{i,j} &= 1 && \forall j \in \mathcal{J} \\
x_{i,j} &\leq y_i && \forall i \in \mathcal{I}, j \in \mathcal{J} \\
\sum_{i \in \mathcal{I}} y_i &= p && \\
x_{i,j} &\in \{0, 1\} && \forall i \in \mathcal{I}, j \in \mathcal{J} \\
y_i &\in \{0, 1\} && \forall i \in \mathcal{I} \\
w &\in \mathbb{R} &&
\end{aligned}
$$

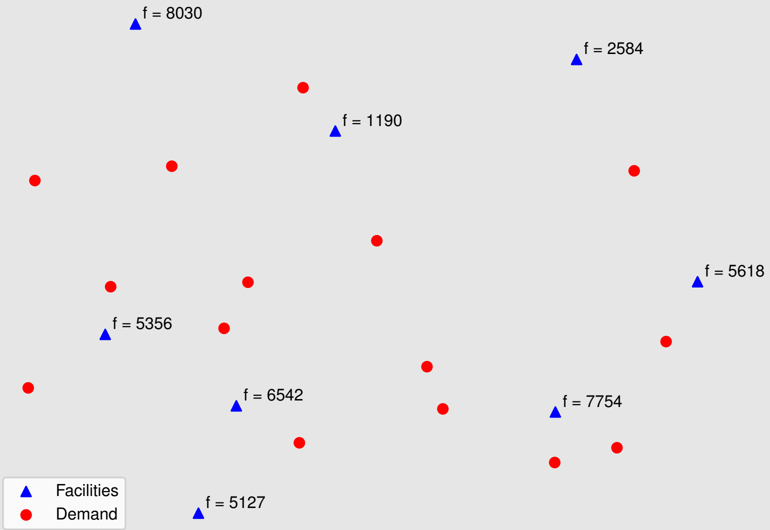
The p-centre facility location problem – solution



The fixed-charge facility location problem

- If we are given a set $\mathcal{I} = \{1, \ldots, m\}$ of potential facilities,
- and a set $\mathcal{J} = \{1, \ldots, n\}$ of customers each with a demand d_j to be supplied,
- determine facilities to open that will satisfy all customer demand at the minimum possible cost if each facility has a fixed opening cost of f_i.
- The unit costs of supplying users from candidate facilities are arranged in a matrix $\boldsymbol{C} = c_{i,j}$, where
- $c_{i,j}$ is the cost of serving a unit of demand for customer $j \in \mathcal{J}$ from facility $i \in \mathcal{I}$.

The fixed-charge facility location problem



The fixed-charge facility location problem formulation

Decision variables

$$x_{i,j} = \begin{cases} 1 & \text{if customer } j \in \mathcal{J} \text{ is supplied from facility } i \in \mathcal{I}, \\ 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if facility } i \in \mathcal{I} \text{ is opened,} \\ 0 & \text{otherwise} \end{cases}$$

Objective function

$$\min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} y_i f_i + d_j c_{i,j} x_{i,j}$$

Constraints

$$\sum_{i \in \mathcal{I}} x_{i,j} = 1 \quad \forall j \in \mathcal{J}$$

$$x_{i,j} \leq y_i \quad \forall i \in \mathcal{I}, j \in \mathcal{J}$$

$$\sum_{i \in \mathcal{I}} y_i = p$$

$$x_{i,j} \in \{0, 1\} \quad \forall i \in \mathcal{I}, j \in \mathcal{J}$$

$$y_i \in \{0, 1\} \quad \forall i \in \mathcal{I}$$

The fixed-charge facility location problem formulation

Decision variables

$$x_{i,j} = \begin{cases} 1 & \text{if customer } j \in \mathcal{J} \text{ is supplied from facility } i \in \mathcal{I}, \\ 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if facility } i \in \mathcal{I} \text{ is opened,} \\ 0 & \text{otherwise} \end{cases}$$

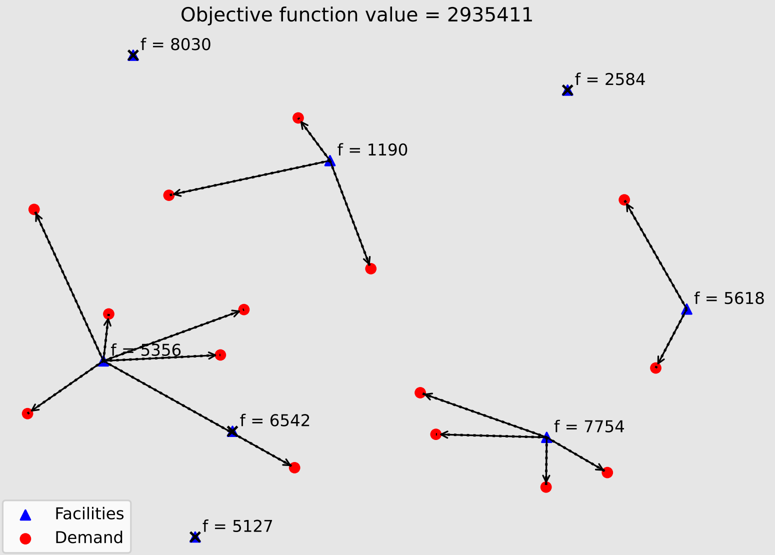
Objective function

$$\min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} y_i f_i + d_j c_{i,j} x_{i,j}$$

Constraints

$$\begin{aligned} \sum_{i \in \mathcal{I}} x_{i,j} &= 1 && \forall j \in \mathcal{J} \\ x_{i,j} &\leq y_i && \forall i \in \mathcal{I}, j \in \mathcal{J} \\ x_{i,j} &\in \{0, 1\} && \forall i \in \mathcal{I}, j \in \mathcal{J} \\ y_i &\in \{0, 1\} && \forall i \in \mathcal{I} \end{aligned}$$

The fixed-charge facility location problem – solution

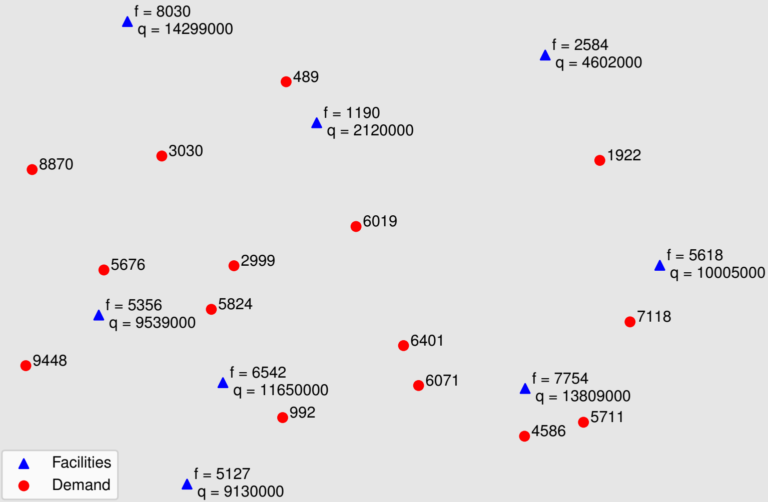


The fixed-charge facility location problem with capacities

- Suppose that each facility has a capacity q_i,
- this would usually depend on the storage size of the facility or the processing capacity.
- For this example lets let

$$q_i = \frac{\sum_{j \in \mathcal{J}} d_f f_i}{\sum_{i \in \mathcal{I}} f_i} \alpha$$

The fixed-charge facility location problem with capacities



The fixed-charge facility location with capacities problem formulation

Decision variables

$$x_{i,j} = \begin{cases} 1 & \text{if customer } j \in \mathcal{J} \text{ is supplied from facility } i \in \mathcal{I}, \\ 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if facility } i \in \mathcal{I} \text{ is opened,} \\ 0 & \text{otherwise} \end{cases}$$

Objective function

$$\min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} y_i f_i + d_j c_{i,j} x_{i,j}$$

Constraints

$$\sum_{i \in \mathcal{I}} x_{i,j} = 1 \quad \forall j \in \mathcal{J}$$

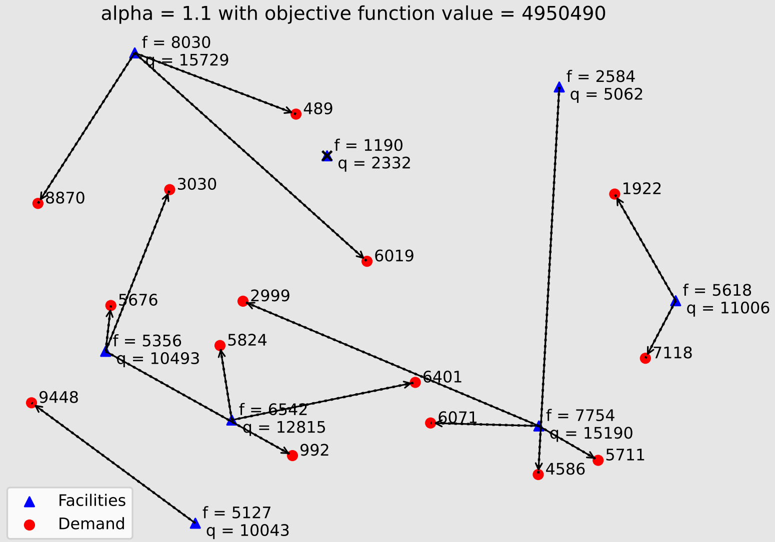
$$x_{i,j} \leq y_i \quad \forall i \in \mathcal{I}, j \in \mathcal{J}$$

$$\sum_{j \in \mathcal{J}} d_j x_{i,j} \leq q_i \quad \forall i \in \mathcal{I}$$

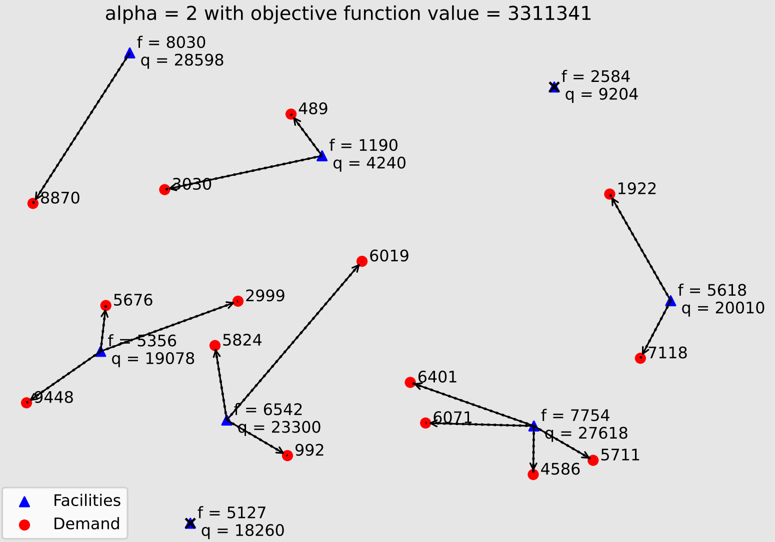
$$x_{i,j} \in \{0, 1\} \quad \forall i \in \mathcal{I}, j \in \mathcal{J}$$

$$y_i \in \{0, 1\} \quad \forall i \in \mathcal{I}$$

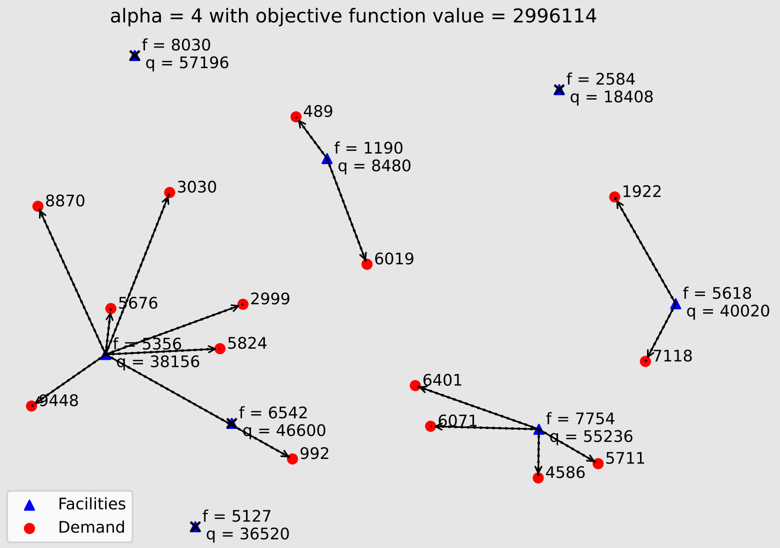
The fixed-charge facility location problem with capacities – solution



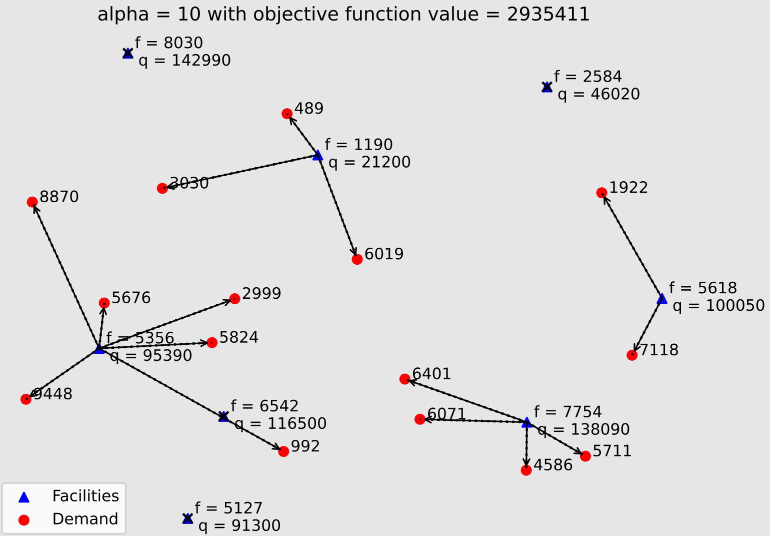
The fixed-charge facility location problem with capacities – solution



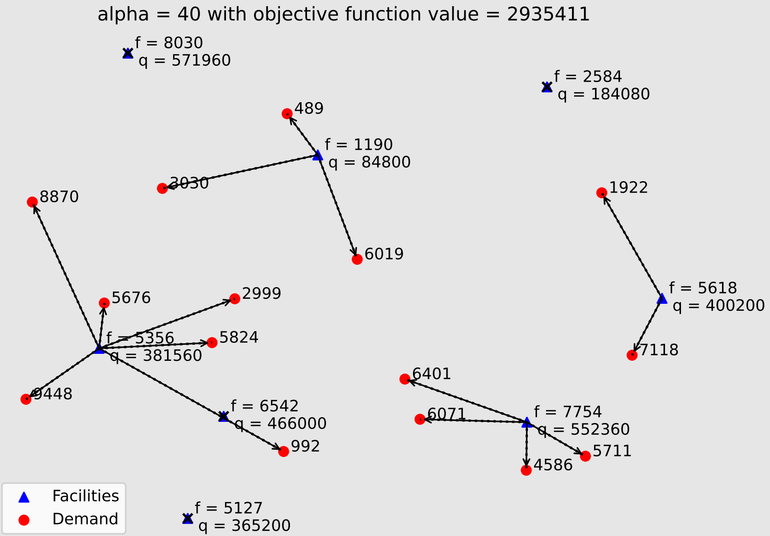
The fixed-charge facility location problem with capacities – solution



The fixed-charge facility location problem with capacities – solution



The fixed-charge facility location problem with capacities – solution

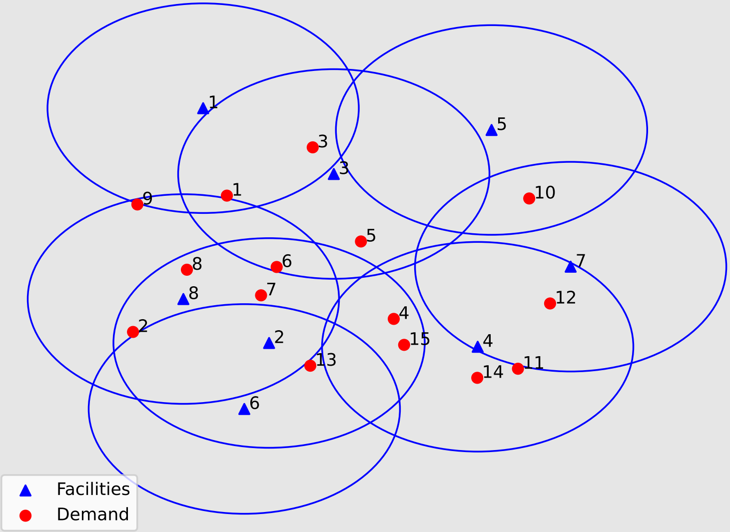


The set-covering problem

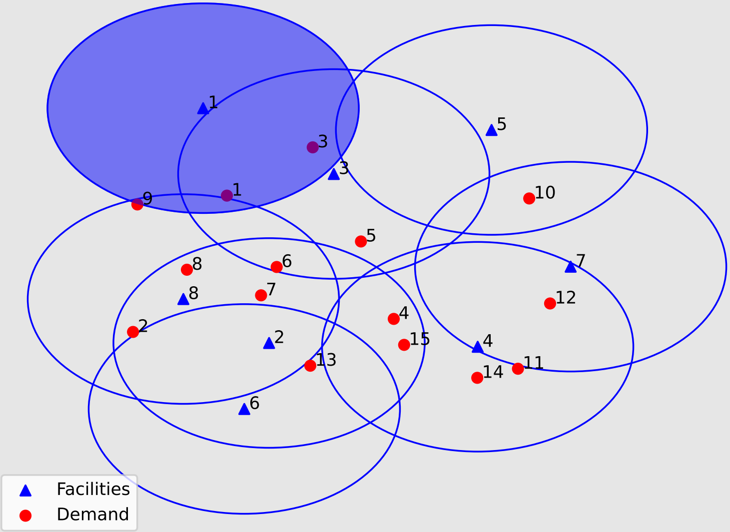
- In some applications a facility can only service a customer if it are within a certain distance of the customer.
- If we are given a set $\mathcal{I} = \{1, \ldots, m\}$ of potential facilities and a set $\mathcal{J} = \{1, \ldots, n\}$ of customers.
- If $c_{i,j}$ denotes the distance from facility i to customer j then we say that facility i can cover customer j if $c_{i,j} \leq D$.

$$a_{i,j} = \begin{cases} 1 & \text{if customer } j \in \mathcal{J} \text{ can be covered by facility } i \in \mathcal{I}, \\ 0 & \text{otherwise} \end{cases}$$

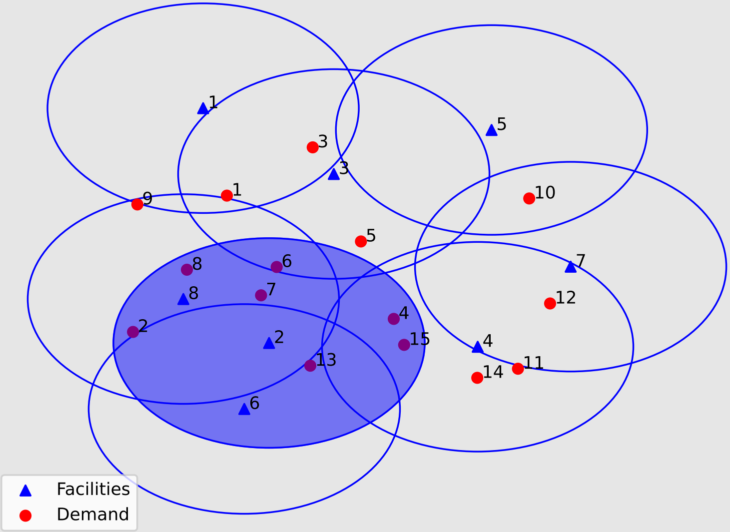
The set-covering problem



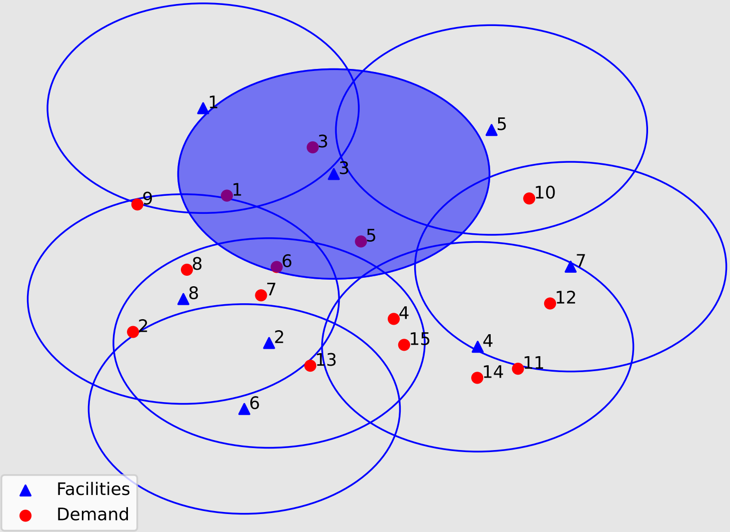
The set-covering problem



The set-covering problem



The set-covering problem



The set-covering problem formulation

Decision variables

$$x_{i,j} = \begin{cases} 1 & \text{if customer } j \in \mathcal{J} \text{ is covered by facility } i \in \mathcal{I}, \\ 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if facility } i \in \mathcal{I} \text{ is opened,} \\ 0 & \text{otherwise} \end{cases}$$

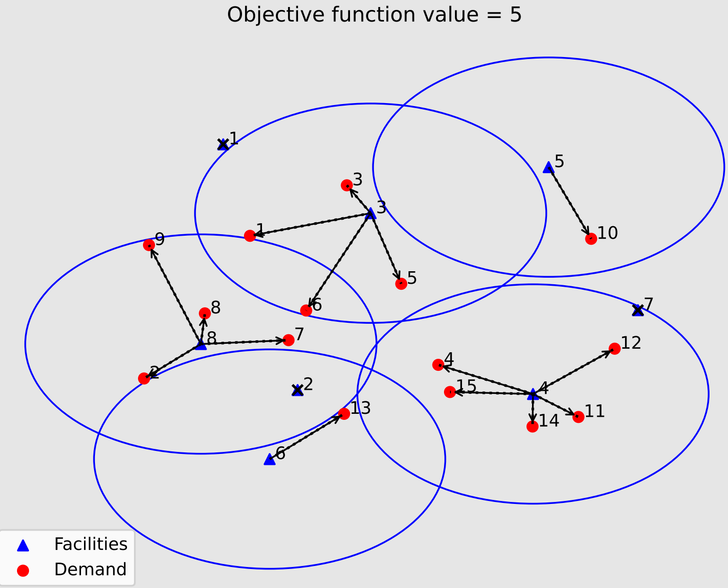
Objective function

$$\min \sum_{i \in \mathcal{I}} \sum_{i \in \mathcal{I}} y_i$$

Constraints

$$\begin{aligned} \sum_{i \in \mathcal{I}} x_{i,j} a_{i,j} = 1 \quad & \forall j \in \mathcal{J} \\ x_{i,j} \leq y_i \quad & \forall i \in \mathcal{I}, j \in \mathcal{J} \\ x_{i,j} \in \{0, 1\} \quad & \forall i \in \mathcal{I}, j \in \mathcal{J} \\ y_i \in \{0, 1\} \quad & \forall i \in \mathcal{I} \end{aligned}$$

The set-covering problem – solution



The set-covering problem formulation – cover limit

Decision variables

$$x_{i,j} = \begin{cases} 1 & \text{if customer } j \in \mathcal{J} \text{ is covered by facility } i \in \mathcal{I}, \\ 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if facility } i \in \mathcal{I} \text{ is opened,} \\ 0 & \text{otherwise} \end{cases}$$

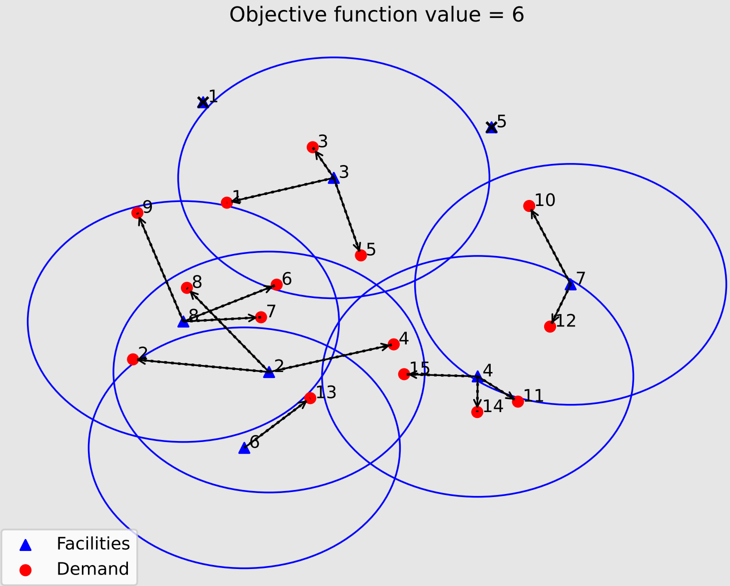
Objective function

$$\min \sum_{i \in \mathcal{I}} y_i$$

Constraints

$$\begin{aligned} \sum_{i \in \mathcal{I}} x_{i,j} a_{i,j} &= 1 && \forall j \in \mathcal{J} \\ x_{i,j} &\leq y_i && \forall i \in \mathcal{I}, j \in \mathcal{J} \\ \sum_{j \in \mathcal{J}} x_{i,j} a_{i,j} &\leq L && \forall i \in \mathcal{I} \\ x_{i,j} &\in \{0, 1\} && \forall i \in \mathcal{I}, j \in \mathcal{J} \\ y_i &\in \{0, 1\} && \forall i \in \mathcal{I} \end{aligned}$$

The set-covering problem – solution



The multi-period p-median facility location problem

- If variations are predictable for such values, it may be desirable to plan for future adjustments in the location of facilities and in other related decisions (e.g., shipment decisions).
- In this case, locating a set of facilities becomes a question not only of “where" but also of “when". A new dimension is introduced in the decision space: the time.

The multi-period p-median facility location problem

- If we are given a time planning horizon $\mathcal{T}$, a set $\mathcal{I} = \{1, \ldots, m\}$ of potential facilities,
- and a set $\mathcal{J} = \{1, \ldots, n\}$ of customers each with a demand $d_{j,t}$ to be supplied at time t,
- find p_t facilities to open that will satisfy all customer demand at the minimum possible cost.
- The unit costs of supplying users from candidate facilities are arranged in a three-dimensional array $\boldsymbol{C} = c_{i,j,t}$, where
- $c_{i,j,t}$ is the cost of serving a unit of demand for customer $j \in \mathcal{J}$ from facility $i \in \mathcal{I}$ in time period $t \in \mathcal{T}$.

The multi-period *p*-median facility location problem formulation

Decision variables

$$x_{i,j,t} = \begin{cases} 1 & \text{if customer } j \in \mathcal{J} \text{ is supplied from facility } i \in \mathcal{I} \text{ in time period } t \in \mathcal{T}, \\ 0 & \text{otherwise} \end{cases}$$

$$y_{i,t} = \begin{cases} 1 & \text{if facility } i \in \mathcal{I} \text{ is open in time period } t \in \mathcal{T}, \\ 0 & \text{otherwise} \end{cases}$$

Objective function

$$\min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{t \in \mathcal{T}} d_j c_{i,j} x_{i,j}$$

Constraints

$$\sum_{i \in \mathcal{I}} x_{i,j,t} = 1 \quad \forall j \in \mathcal{J},\ t \in \mathcal{T}$$

$$x_{i,j,t} \leq y_{i,t} \quad \forall i \in \mathcal{I},\ j \in \mathcal{J}, t \in \mathcal{T}$$

$$\sum_{i \in \mathcal{I}} y_{i,t} = p_t \quad \forall t \in \mathcal{T}$$

$$x_{i,j} \in \{0, 1\} \quad \forall i \in \mathcal{I},\ j \in \mathcal{J},\ t \in \mathcal{T}$$

$$y_{i,t} \in \{0, 1\} \quad \forall i \in \mathcal{I},\ t \in \mathcal{T}$$

The multi-period p-median facility location problem

- What if we cannot change our decision?
- If we open a facility then it should remain opened

The multi-period p-median facility location problem formulation

Constraints

$$
\begin{aligned}
\sum_{i \in \mathcal{I}} x_{i,j,t} = 1 &\quad \forall j \in \mathcal{J},\ t \in \mathcal{T} \\
x_{i,j,t} \leq y_{i,t} &\quad \forall i \in \mathcal{I},\ j \in \mathcal{J},\ t \in \mathcal{T} \\
\sum_{i \in \mathcal{I}} y_{i,t} = p_t &\quad \forall t \in \mathcal{T} \\
y_{i,t} \geq y_{i,t-1} &\quad \forall i \in \mathcal{I},\ t \in \mathcal{T} \setminus \{1\} \\
x_{i,j} \in \{0, 1\} &\quad \forall i \in \mathcal{I},\ j \in \mathcal{J},\ t \in \mathcal{T} \\
y_i \in \{0, 1\} &\quad \forall i \in \mathcal{I},\ t \in \mathcal{T}
\end{aligned}
$$

Restrictions on p_t?

$$1 \leq p_1 \leq p_2 \leq p_3 \leq p_{|\mathcal{T}|}$$

The multi-period fixed charge facility location problem

- What is a fixed charge?
- Operating a warehouse for a year?
- Opening a warehouse?
- Closing a warehouse?

The multi-period fixed charge facility location problem – formulation

Decision variables

$$x_{i,j,t} = \begin{cases} 1 & \text{if customer } j \in \mathcal{J} \text{ is supplied from facility } i \in \mathcal{I} \text{ in time period } t \in \mathcal{T}, \\ 0 & \text{otherwise} \end{cases}$$

$$y_{i,t} = \begin{cases} 1 & \text{if facility } i \in \mathcal{I} \text{ is open in time period } t \in \mathcal{T}, \\ 0 & \text{otherwise} \end{cases}$$

$$z'_{i,t} = \begin{cases} 1 & \text{if facility } i \in \mathcal{I} \text{ is opened at the start of time period } t \in \mathcal{T}, \\ 0 & \text{otherwise} \end{cases}$$

$$z''_{i,t} = \begin{cases} 1 & \text{if facility } i \in \mathcal{I} \text{ is closed at the start of time period } t \in \mathcal{T}, \\ 0 & \text{otherwise} \end{cases}$$

Parameters

- Let $f_{i,t}$ be the operational cost of facility $i \in \mathcal{I}$ in time period $t \in \mathcal{T}$,
- $g_{i,t}$ be the cost of opening facility $i \in \mathcal{I}$ at the start of time period $t \in \mathcal{T}$
- and $h_{i,t}$ be the cost of closing facility $i \in \mathcal{I}$ at the start of time period $t \in \mathcal{T}$

The multi-period fixed charge facility location problem – formulation

Objective function

$$\min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{t \in \mathcal{T}} d_j c_{i,j} x_{i,j} + f_{i,t} y i, t + g_{i,t} z' i, t + h_{i,t} z'' i, t$$

Constraints

$$\begin{aligned}
\sum_{i \in \mathcal{I}} x_{i,j,t} &= 1 && \forall j \in \mathcal{J},\ t \in \mathcal{T} \\
x_{i,j,t} &\leq y_{i,t} && \forall i \in \mathcal{I},\ j \in \mathcal{J},\ t \in \mathcal{T} \\
y_{i,y} - y_{i,t-1} &= z'_{i,t} - z''_{i,t} && \forall i \in \mathcal{I},\ t \in \mathcal{T} \setminus \{1\} \\
y_{i,y} &= z'_{i,t} && \forall i \in \mathcal{I},\ t \in \{1\} \\
x_{i,j} &\in \{0, 1\} && \forall i \in \mathcal{I},\ j \in \mathcal{J},\ t \in \mathcal{T} \\
y_i &\in \{0, 1\} && \forall i \in \mathcal{I},\ t \in \mathcal{T}
\end{aligned}$$

To summarise

- Most facility locations are NP-hard, they can be difficult to solve for large problems
- When we consider temporal elements the size of the problem can quickly explode
- Chose your battles wisely

  Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show — Bertrand Russell

  If you fail to prepare, prepare to fail — Unknown