University of Edinburgh
School of Mathematics
E. Alper Yıldırım

THE UNIVERSITY *of* EDINBURGH

Fundamentals of Optimization

Homework 4 – Solutions

Instructions

1. You should attempt all questions.
2. The total marks for this assignment are 10.
3. The assignment consists of STACK questions (5/10 marks) and *open-ended* questions (5/10 marks).
4. All STACK questions are duly marked and are available in the STACK quiz. You **must solve those by completing the STACK quiz**.
5. For the open-ended questions, please write down your solutions in a concise and reproducible way and remember to justify every step using appropriate references when necessary. Failing to do so may result in deductions.
6. The strict deadline for completing the quiz and handing-in your solutions for the open-ended questions is **noon (12:00) on Friday, 25 November 2022**.
7. For the open-ended questions, please upload a **single PDF**. For some useful suggestions, please see Course Information → Tips for Creating a PDF File for Submission on the Learn page.

STACK Problems

1 Simplex Method for Nondegenerate LPs (2.5 marks)

STACK question

You are given three linear programming problems below such that the matrix A has full row rank. For each problem, a starting dictionary is given. For each given dictionary, write down the current index sets B and N, the values of the decision variables $\hat{x}_j$, the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether the given dictionary is optimal or not. If applicable, perform only **one** iteration of the simplex method starting from the given dictionary. For the next dictionary (if applicable), write down the current index sets B and N, the values of the decision variables $\hat{x}_j$, the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether this dictionary is optimal or not.

(1.1) LP

$$
\begin{array}{lrcrcrcrcrcl}
\min & -x_1 & - & x_2 & & & & & & & & \\
\text{s.t.} & -x_1 & + & x_2 & + & x_3 & & & & & = & 3 \\
& x_1 & - & 2x_2 & & & + & x_4 & & & = & 2 \\
& x_1 & + & x_2 & & & & & - & x_5 & = & 1 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}
$$

and the starting dictionary is given by

$$
\begin{array}{rcrcrcr}
z & = & -1 & & & - & x_5 \\
x_2 & = & 1 & - & x_1 & + & x_5 \\
x_3 & = & 2 & + & 2x_1 & - & x_5 \\
x_4 & = & 4 & - & 3x_1 & + & 2x_5
\end{array}
$$

Solution

Dictionary 1

We have $B = \{2,3,4\}$, $N = \{1,5\}$, $\hat{x} = [0,1,2,4,0]^T$, $\bar{c} = [0,0,0,0,-1]^T$, and $\hat{z} = -1$.

As $\bar{c} \not\geq \mathbf{0}$, the current vertex is not optimal by Proposition 17.1 (recall that the LP is nondegenerate).

As there is one variable with a negative reduced cost, the current vertex is not optimal and we choose $j^* = 5$. Hence, x_5 will enter the basis.

The min-ratio test yields

$$\lambda^* \;=\; \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} \;=\; \min\left\{\frac{-2}{-1}\right\} \;=\; 2\,.$$

Hence, $k^* = 3$ and x_3 will leave the basis.

Thus, we move x_5 to the left-hand side and x_3 to the right-hand side of Row 2. Afterwards, we substitute x_5 in the right-hand sides of Rows 0, 1, and 3. We performed the first iteration and the new dictionary is shown below.

Dictionary 2

$$
\begin{array}{rcrcrcr}
z & = & -3 & - & 2x_1 & + & x_3 \\
x_2 & = & 3 & + & x_1 & - & x_3 \\
x_5 & = & 2 & + & 2x_1 & - & x_3 \\
x_4 & = & 8 & + & x_1 & - & 2x_3
\end{array}
$$

We have $B = \{2,5,4\}$, $N = \{1,3\}$, $\hat{x} = [0,3,0,8,2]^T$, $\bar{c} = [-2,0,1,0,0]^T$, and $\hat{z} = -3$.

As there is one variable with a negative reduced cost, the current vertex is not optimal by Proposition 17.1. If we were to continue with the simplex method, we would choose x_1 as the entering variable, i.e., $j^* = 1$. Hence, x_1 will enter the basis. When we try to do the min-ratio test, we realize that there are no negative entries in the column underneath x_1 in Rows 1, 2, and 3. Therefore, none of the three rows is eligible for the minimum ratio test. Hence, we can increase x_1 indefinitely without leaving the feasible region. Thus, the problem is unbounded. We can obtain a direction $d \in \mathbb{R}^5$ of unboundedness by defining $d_1 = 1$, $d_3 = 0$, and

$$d_B \;=\; -(A_B)^{-1} A_N d_N \;=\; -(A_B)^{-1} A^1 \;=\; \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

which can also be read from the column of x_1 in Rows 1, 2, and 3. We obtain $d = [1,1,0,1,2]^T$. You can verify that d is a feasible direction at $\hat{x}$, $\hat{x} + \lambda d \in \mathcal{P}$ for any real number $\lambda \geq 0$, and $c^T(\hat{x} + \lambda d) \to -\infty$ as $\lambda \to \infty$.

(1.2) LP

$$
\begin{array}{lrcrcrcrcr}
\min & 2x_1 & - & 3x_2 & + & x_3 & & & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & 4x_3 & + & x_4 & = & 7 \\
 & x_1 & + & 2x_2 & + & 4x_3 & - & x_4 & = & 5 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

and the starting dictionary is given by

$$\begin{array}{rcrcrcr} z & = & \frac{3}{2} & + & \frac{13}{8}x_1 & - & \frac{27}{8}x_2 \\ x_4 & = & 1 & - & \frac{1}{2}x_1 & + & \frac{1}{2}x_2 \\ x_3 & = & \frac{3}{2} & - & \frac{3}{8}x_1 & - & \frac{3}{8}x_2 \end{array}$$

Solution

Dictionary 1

We have $B = \{4, 3\}$, $N = \{1, 2\}$, $x = [0, 0, \frac{3}{2}, 1]^T$, $\bar{c} = [\frac{13}{8}, -\frac{27}{8}, 0, 0]^T$, and $z = 1.5$.

As there is one variable with a negative reduced cost, the current vertex is not optimal by Proposition 17.1, and we choose $j^* = 2$. Hence, x_2 will enter the basis.

The min-ratio test yields

$$\lambda^* = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} = \min \left\{ \frac{-3/2}{-3/8} \right\} = 4 \, .$$

Hence, $k^* = 3$ and x_3 will leave the basis.

Thus, we move x_2 to the left-hand side and x_3 to the right-hand side of Row 2. Afterwards, we substitute x_2 in the right-hand sides of Rows 0 and 1. We performed the first iteration and the new dictionary is shown below.

Dictionary 2

$$\begin{array}{rcrcrcr} z & = & -12 & + & 5x_1 & + & 9x_3 \\ x_4 & = & 3 & - & x_1 & - & \frac{4}{3}x_3 \\ x_2 & = & 4 & - & x_1 & - & \frac{8}{3}x_3 \end{array}$$

We have $B = \{4, 2\}$, $N = \{1, 3\}$, $\hat{x} = [0, 4, 0, 3]^T$, $\bar{c} = [5, 0, 9, 0]^T$, and $\hat{z} = -12$.

As all reduced costs are non-negative, $\bar{c} \geq \mathbf{0}$, the current vertex is optimal by Proposition 17.1. Therefore, we found an optimal solution after performing one iteration of the simplex method.

(1.3) LP

$$\begin{array}{lrcrcrcrcrcl} \min & -2x_1 & - & 3x_2 & & & & & & & & \\ \text{s.t.} & x_1 & & & + & x_3 & & & & & = & 4 \\ & x_1 & + & 3x_2 & & & + & x_4 & & & = & 15 \\ & 2x_1 & + & x_2 & & & & & + & x_5 & = & 10 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

and the starting dictionary is given by

$$\begin{array}{rcrcrcr} z & = & 0 & - & 2x_1 & - & 3x_2 \\ x_3 & = & 4 & - & x_1 & & \\ x_4 & = & 15 & - & x_1 & - & 3x_2 \\ x_5 & = & 10 & - & 2x_1 & - & x_2 \end{array}$$

Solution

Dictionary 1

We have $B = \{3, 4, 5\}$, $N = \{1, 2\}$, $\hat{x} = [0, 0, 4, 15, 10]^T$, $\bar{c} = [-2, -3, 0, 0, 0]^T$, and $\hat{z} = 0$.

As $\bar{c} \ngeq \mathbf{0}$, the current vertex is not optimal by Proposition 17.1.

As there are two variables with negative reduced costs, we pick the one with the most negative reduced cost, i.e. $j^* = 2$. Hence, x_2 will enter the basis.

The min-ratio test yields

$$\lambda^* = \min_{j \in B: d_j < 0} \frac{-\hat{x}_j}{d_j} = \min \left\{ \frac{-15}{-3}, \frac{-10}{-1} \right\} = \frac{15}{3}.$$

Hence, $k^* = 4$ and x_4 will leave the basis.

Thus, we move x_2 to the left-hand side and x_4 to the right-hand side of Row 2. Afterwards, we substitute x_2 in the right-hand sides of Rows 0 and 3. We performed the first iteration and the new dictionary is shown below.

Dictionary 2

$$\begin{array}{rcrcrcr} z & = & -15 & - & x_1 & + & x_4 \\ x_3 & = & 4 & - & x_1 & & \\ x_2 & = & 5 & - & \frac{1}{3}x_1 & - & \frac{1}{3}x_4 \\ x_5 & = & 5 & - & \frac{5}{3}x_1 & + & \frac{1}{3}x_4 \end{array}$$

We have $B = \{3, 2, 5\}$, $N = \{1, 4\}$, $\hat{x} = [0, 5, 4, 0, 5]^T$, $\bar{c} = [-1, 0, 0, 1, 0]^T$, and $\hat{z} = -15$.

As there is one variable with a negative reduced cost, the current vertex is not optimal by Proposition 17.1. If we were to continue with the simplex method, we would choose x_1 as the entering variable and perform the min-ratio test to determine the leaving variable.

2 The Simplex Method for Degenerate LPs (2.5 marks)

STACK question

You are given the following linear programming problem that has full row rank:

$$\begin{array}{lrcrcrcrcrcrcl} \min & 2x_1 & - & 3x_2 & + & x_3 & & & & & & & & \\ \text{s.t.} & -3x_1 & + & x_2 & - & 4x_3 & + & x_4 & & & & & = & 3 \\ & 2x_1 & + & x_2 & & & & & + & x_5 & & & = & 5 \\ & -x_1 & + & 2x_2 & + & 3x_3 & & & & & + & x_6 & = & 6 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0 \end{array}$$

You are given three different starting dictionaries. For each given dictionary, write down the current index sets B and N, the values of the decision variables $\hat{x}_j$, the reduced costs $\bar{c}_j$, $j = 1, \dots, n$, and the objective function value $\hat{z}$. Determine whether the given dictionary is optimal or not. If applicable, perform only **one** iteration of the simplex method starting from the given dictionary. If the current dictionary is nondegenerate, use the most negative reduced cost to determine the entering variable. Break ties in favour of the variable with the smallest index. If it is degenerate, use Bland's rule to determine the entering and leaving variables (whenever applicable). For the next dictionary (if applicable), write down the current index sets B and N, the values of the decision variables $\hat{x}_j$, the reduced costs $\bar{c}_j$, $j = 1, \dots, n$, and the objective function value $\hat{z}$. Determine whether this dictionary is optimal or not.

(2.1) The starting dictionary is given by

$$\begin{array}{rcrcrcrcr} z & = & 0 & + & 2x_1 & - & 3x_2 & + & x_3 \\ x_4 & = & 3 & + & 3x_1 & - & x_2 & + & 4x_3 \\ x_5 & = & 5 & - & 2x_1 & - & x_2 & & \\ x_6 & = & 6 & + & x_1 & - & 2x_2 & - & 3x_3 \end{array}$$

Solution

Dictionary 1 We have $B = \{4,5,6\}$, $N = \{1,2,3\}$, $\hat{x} = [0,0,0,3,5,6]^T$, $\bar{c} = [2,-3,1,0,0,0]^T$, and $\hat{z} = 0$.

As there is one variable with a negative reduced cost and the vertex is nondegenerate, the current vertex is not optimal by Proposition 17.1, and we choose $j^* = 2$. Hence, x_2 will enter the basis.

The min-ratio test yields

$$\lambda^* = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} = \min\left\{\frac{-3}{-1}, \frac{-5}{-1}, \frac{-6}{-2}\right\} = 3\,.$$

As we have two variables that achieve the minimum ratio, we break the tie in favour of the basic variable with the smallest index, i.e. x_4. Hence, $k^* = 4$ and x_4 will leave the basis.

Thus, we move x_2 to the left-hand side and x_4 to the right-hand side of Row 1. Afterwards, we substitute x_2 in the right-hand sides of Rows 0, 2, and 3. We performed the first iteration and the new dictionary is shown below.

Dictionary 2

$$\begin{array}{rcrcrcrcr}
z & = & -9 & - & 7x_1 & - & 11x_3 & + & 3x_4 \\
x_2 & = & 3 & + & 3x_1 & + & 4x_3 & - & x_4 \\
x_5 & = & 2 & - & 5x_1 & - & 4x_3 & + & x_4 \\
x_6 & = & 0 & - & 5x_1 & - & 11x_3 & + & 2x_4
\end{array}$$

We have $B = \{2,5,6\}$, $N = \{1,3,4\}$, $\hat{x} = [0,3,0,0,2,0]^T$, $\bar{c} = [-7,0,-11,3,0,0]^T$, and $\hat{z} = -9$.

Note that x_1 has a negative reduced cost. This dictionary is therefore not optimal. As $x_6 = 0$, the current vertex is degenerate. Therefore, we cannot conclude anything about the optimality of this vertex yet. If we were to continue with the simplex method, we would switch to Bland's rule, i.e., we would choose x_1 (as opposed to x_3) as the entering variable, and perform the min-ratio test to determine the leaving variable.

(2.2) The starting dictionary is given by

$$\begin{array}{rcrcrcrcr}
z & = & -9 & - & 2x_1 & + & x_4 & + & x_6 \\
x_2 & = & 3 & + & \frac{13}{11}x_1 & - & \frac{3}{11}x_4 & - & \frac{4}{11}x_6 \\
x_3 & = & 0 & - & \frac{5}{11}x_1 & + & \frac{2}{11}x_4 & - & \frac{1}{11}x_6 \\
x_5 & = & 2 & - & \frac{35}{11}x_1 & + & \frac{3}{11}x_4 & + & \frac{4}{11}x_6
\end{array}$$

Solution

We have $B = \{2,3,5\}$, $N = \{1,4,6\}$, $\hat{x} = [0,3,0,0,2,0]^T$, $\bar{c} = [-2,0,0,1,0,1]^T$, and $\hat{z} = -9$.

As there is one variable (x_1) with a negative reduced cost, the dictionary is not optimal. Since the vertex is degenerate, we cannot conclude anything about the optimality of this vertex yet. Since there is only one nonbasic variable with a negative reduced cost, x_1 is the entering variable, i.e., we choose $j^* = 1$. Hence, x_1 will enter the basis.

The min-ratio test yields

$$\lambda^* = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} = \min\left\{\frac{0}{-5/11}, \frac{-2}{-35/11},\right\} = 0\,.$$

Therefore, x_3 is the variable that achieves the minimum ratio. Hence, $k^* = 3$ and x_3 will leave the basis.

Thus, we move x_1 to the left-hand side and x_3 to the right-hand side of Row 2. Afterwards, we substitute x_1 in the right-hand sides of Rows 0, 1, and 3. We performed the first iteration and the new dictionary is shown below.

Dictionary 2

$$\begin{array}{rcrcrcrcr} z & = & -9 & + & \frac{22}{5}x_3 & + & \frac{1}{5}x_4 & + & \frac{7}{5}x_6 \\ x_2 & = & 3 & - & \frac{13}{5}x_3 & + & \frac{1}{5}x_4 & - & \frac{3}{5}x_6 \\ x_1 & = & 0 & - & \frac{11}{5}x_3 & + & \frac{2}{5}x_4 & - & \frac{1}{5}x_6 \\ x_5 & = & 2 & + & 7x_3 & - & x_4 & + & x_6 \end{array}$$

We have $B = \{2, 1, 5\}$, $N = \{3, 4, 6\}$, $\hat{x} = [0, 3, 0, 0, 2, 0]^T$, $\bar{c} = [0, 0, 22/5, 1/5, 0, 7/5]^T$, and $\hat{z} = -9$. Note that this is the same vertex as in the previous dictionary. As $\bar{c} \geq \mathbf{0}$, the current dictionary is optimal by Corollary 15.4. Therefore, $\hat{x}$ is a degenerate optimal vertex and the simplex method will be terminated at this dictionary with an optimal solution.

(2.3) The starting dictionary is given by

$$\begin{array}{rcrcrcrcr} z & = & -9 & - & 7x_1 & - & 11x_3 & + & 3x_4 \\ x_2 & = & 3 & + & 3x_1 & + & 4x_3 & - & x_4 \\ x_5 & = & 2 & - & 5x_1 & - & 4x_3 & + & x_4 \\ x_6 & = & 0 & - & 5x_1 & - & 11x_3 & + & 2x_4 \end{array}$$

Solution

We have $B = \{2, 5, 6\}$, $N = \{1, 3, 4\}$, $\hat{x} = [0, 3, 0, 0, 2, 0]^T$, $\bar{c} = [-7, 0, -11, 3, 0, 0]^T$, and $\hat{z} = -9$.

As there are two nonbasic variables with negative reduced costs, the current dictionary is not optimal. As $x_6 = 0$, the current vertex is degenerate! Therefore, we cannot conclude anything about the optimality of this vertex yet. Hence, we switch to Bland's rule.

There are two variables with a negative reduced cost: x_1 and x_3.

Following Bland's rule, we choose the variable with the smaller index, i.e. $j^* = 1$. Hence, x_1 will enter the basis.

The min-ratio test yields

$$\lambda^* = \min_{j \in B: d_j < 0} \frac{-\hat{x}_j}{d_j} = \min \left\{ \frac{-2}{-5}, \frac{0}{-5} \right\} = 0 .$$

Hence, $k^* = 6$ and x_6 will leave the basis.

Thus, we move x_1 to the left-hand side and x_6 to the right-hand side of Row 3. Afterwards, we substitute x_1 in the right-hand sides of Rows 0, 1, and 2.

Dictionary 2

$$\begin{array}{rcrcrcrcr} z & = & -9 & + & \frac{22}{5}x_3 & + & \frac{1}{5}x_4 & + & \frac{7}{5}x_6 \\ x_2 & = & 3 & - & \frac{13}{5}x_3 & + & \frac{1}{5}x_4 & - & \frac{3}{5}x_6 \\ x_5 & = & 2 & + & 7x_3 & - & x_4 & + & x_6 \\ x_1 & = & 0 & - & \frac{11}{5}x_3 & + & \frac{2}{5}x_4 & - & \frac{1}{5}x_6 \end{array}$$

We have $B = \{2, 5, 1\}$, $N = \{3, 4, 6\}$, $\hat{x} = [0, 3, 0, 0, 2, 0]^T$, $\bar{c} = [0, 0, \frac{22}{5}, \frac{1}{5}, 0, \frac{7}{5}]^T$, and $\hat{z} = -9$. Note that this is the same vertex as in the previous dictionary. As $\bar{c} \geq \mathbf{0}$, the current dictionary is optimal by Corollary 15.4. Therefore, $\hat{x}$ is a degenerate optimal vertex and the simplex method will be terminated at this dictionary with an optimal solution.

Open Ended Problems

3 Puzzle (1 mark)

Consider the following linear program

$$\begin{array}{lrcrcrcrcrcl}
\min & -2x_1 & + & c_2\,x_2 & + & c_3\,x_3 & - & x_4 & + & c_5\,x_5 & & \\
\text{s.t} & a_{11}\,x_1 & + & a_{12}\,x_2 & + & a_{13}\,x_3 & + & x_4 & & & = & 12 \\
 & -x_1 & + & a_{22}\,x_2 & + & a_{23}\,x_3 & & & + & x_5 & = & b_2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0.
\end{array}$$

and an intermediate dictionary given by

$$\begin{array}{rclcrcrcr}
z & = & 52 & - & 4.25x_1 & + & \bar{c}_4\,x_4 & - & 0.75x_5 \\
x_3 & = & \hat{x}_3 & - & 0.75x_1 & - & 0.5x_4 & - & 0.25x_5 \\
x_2 & = & 12 & - & \bar{a}_{21}\,x_1 & - & 0.5x_4 & - & 0.75x_5
\end{array}$$

Determine the unknown values for c_2, c_3, c_5, a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, b_2, $\bar{c}_4$, $\hat{x}_3$, and $\bar{a}_{21}$. Justify your solution.

[1 mark]

Solution

From the dictionary, we can observe that x_3 and x_2 are basic variables, so we have $B = \{3, 2\}$ and $N = \{1, 4, 5\}$.
Note that

$$A_B = \begin{bmatrix} a_{13} & a_{12} \\ a_{23} & a_{22} \end{bmatrix},$$

and all of its entries are unknown. By looking at the entries in Rows 1 and 2 underneath the nonbasic variables x_4 and x_5, we obtain

$$-A_B^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \quad -A_B^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.25 \\ -0.75 \end{bmatrix},$$

respectively, which implies that

$$A_B^{-1} = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}.$$

Therefore,

$$A_B = \left(A_B^{-1}\right)^{-1} = \frac{1}{0.25}\begin{bmatrix} 0.75 & -0.25 \\ -0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix},$$

i.e.,

$$a_{12} = -1, \quad a_{13} = 3, \quad a_{22} = 2, \quad a_{23} = -2.$$

In order to obtain b_2 and $\hat{x}_3$, we use $\hat{x}_B = (A_B)^{-1}b$, i.e.,

$$\begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}\begin{bmatrix} 12 \\ b_2 \end{bmatrix} = \begin{bmatrix} \hat{x}_3 \\ 12 \end{bmatrix}.$$

The second equation yields $b_2 = 8$ and substituting this value into the first equation, we obtain $\hat{x}_3 = 8$.

To obtain a_{11} and $\bar{a}_{21}$, we use the fact that the entries in Row 1 and Row 2 underneath the nonbasic variable are given by $-A_B^{-1}A^1$, i.e.,

$$-\begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}\begin{bmatrix} a_{11} \\ -1 \end{bmatrix} = \begin{bmatrix} -0.75 \\ \bar{a}_{21} \end{bmatrix}.$$

The first equation yields $a_{11} = 2$ and substituting this value into the second equation, we obtain $\bar{a}_{21} = -0.25$.

Note that $c_B = [c_3, c_2]^T$, both of which are unknown. We use $\hat{z} = c_B^T(A_B)^{-1}b = c_B^T\hat{x}_B$ and $\bar{c}_1 = c_1 - c_B^T(A_B)^{-1}A^1$, i.e.,

$$12c_2 + 8c_3 = 52,$$

and

$$-4.25 = -2 - [c_3, c_2]^T \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

which gives us two equations in two unknowns:

$$\begin{aligned} 0.25c_2 + 0.75c_3 &= 2.25 \\ 12c_2 + 8c_3 &= 52 \end{aligned}$$

Solving this system simultaneously, we obtain $c_2 = 3$ and $c_3 = 2$.

Finally, to compute $\bar{c}_4$ and c_5, we use $\bar{c}_j = c_j - c_B^T(A_B)^{-1}A^j$, where $j \in N$, i.e.,

$$\begin{aligned} \bar{c}_4 &= -1 - [2, 3]^T \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ -0.75 &= c_5 - [2, 3]^T \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \end{aligned}$$

which yields $\bar{c}_4 = -3.5$ and $c_5 = 2$.

In summary, we get

$c_2 = 3$, $c_3 = 2$, $c_5 = 2$, $a_{11} = 2$, $a_{12} = -1$, $a_{13} = 3$, $a_{22} = 2$, $a_{23} = -2$, $b_2 = 8$, $\bar{c}_4 = -3.5$, $\hat{x}_3 = 8$, and $\bar{a}_{21} = 0.25$.

4 Duality (4 marks)

(4.1) Consider the following polyhedron in standard form:

$$\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given and $x \in \mathbb{R}^n$. Suppose that $b \geq \mathbf{0}$.

(i) Consider the Phase 1 problem for solving (P):

$$\text{(P1)} \quad \min\{e^T a : Ax + a = b, \quad x \geq \mathbf{0}, \quad a \geq \mathbf{0}\},$$

where $a \in \mathbb{R}^m$ and $e \in \mathbb{R}^m$ denotes the vector of all ones, i.e., $e = [1, 1, \ldots, 1]^T$. Write down the dual of (P1) and justify your solution.

[1 mark]

Solution

Note that we can rewrite the Phase 1 problem in the following way:

$$\text{(P1)} \quad \min\{\mathbf{0}^T x + e^T a : Ax + Ia = b, \quad x \geq \mathbf{0}, \quad a \geq \mathbf{0}\},$$

where $I \in \mathbb{R}^{m \times m}$ is the identity matrix. Let us define

$$\tilde{c} = \begin{bmatrix} \mathbf{0} \\ e \end{bmatrix} \in \mathbb{R}^{n+m}, \quad \tilde{b} = b \in \mathbb{R}^m, \quad \tilde{A} = \begin{bmatrix} A & I \end{bmatrix} \in \mathbb{R}^{m \times (n+m)}, \quad \tilde{x} = \begin{bmatrix} x \\ a \end{bmatrix} \in \mathbb{R}^{n+m}.$$

Therefore, the Phase 1 problem is given by

$$\text{(P1)} \quad \min\{\tilde{c}^T \tilde{x} : \tilde{A}\tilde{x} = \tilde{b}, \quad \tilde{x} \geq \mathbf{0}\},$$

which is in standard form. The dual is then given by

$$\text{(D1)} \quad \max\{\tilde{b}^T \tilde{y} : \tilde{A}^T \tilde{y} \leq \tilde{c}\}.$$

Substituting the above expressions for $\tilde{b}$, $\tilde{A}$, and $\tilde{c}$, and defining $y = \tilde{y} \in \mathbb{R}^m$, we obtain

$$\text{(D1)} \quad \max\{b^T y : A^T y \leq \mathbf{0}, \quad y \leq e\}.$$

(ii) Prove that the dual of (P1) has a finite optimal value.

[1 mark]

Solution

Proof I:

First, we argue that the dual of (P1), denoted by (D1), has a nonempty feasible region. Let $\hat{y} = \mathbf{0} \in \mathbb{R}^m$. Then, we have $A^T\hat{y} = \mathbf{0} \leq \mathbf{0}$ and $\hat{y} = \mathbf{0} \leq e$. Therefore, $\hat{y}$ is a feasible solution of (D1). This implies that (D1) is not infeasible.

By the Fundamental Theorem of Linear Programming, (D1) either has a finite optimal value or is unbounded. Suppose, for a contradiction, that it is unbounded. Then, by the Weak Duality Theorem, the dual of (D1) is infeasible. By the Primal-Dual symmetry, the dual of (D1) is given by (P1). However, this is a contradiction, since $\hat{x} = \mathbf{0} \in \mathbb{R}^n$ and $\hat{a} = b \geq \mathbf{0}$ is a feasible solution of (P1). We conclude that (D1) has a finite optimal value.

Proof II:

Note that (P1) has a nonempty feasible region since $\hat{x} = \mathbf{0} \in \mathbb{R}^n$ and $\hat{a} = b \geq \mathbf{0}$ is a feasible solution of (P1). Furthermore, its optimal value is finite since $e^T a = \sum_{i=1}^{m} a_i \geq 0$ whenever $a \geq \mathbf{0}$, which implies that zero is a lower bound on the objective function value of any feasible solution of (P1). It follows that (P1) cannot be unbounded or infeasible. By the Fundamental Theorem of Linear Programming, (P1) has a finite optimal value. By Table 24.1, we conclude that (D1) has a finite optimal value. In fact, by the Strong Duality Theorem, the optimal values of (P1) and (D1) agree.

(4.2) Determine the set of all optimal solutions for the following linear program using the graphical method:

$$
\begin{array}{llrcrcrcrcl}
(P) & \min & x_1 & + & x_2 & + & x_3 & - & x_4 & & \\
 & \text{s.t.} & x_1 & - & x_2 & + & x_3 & - & x_4 & = & 2 \\
 & & x_1 & + & 2x_2 & - & 2x_3 & - & 2x_4 & = & 2 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0.
\end{array}
$$

[2 marks]

Solution

We start by formulating the dual problem, which will only have two variables:

$$
\begin{array}{llrcrcr}
(D) & \max & 2y_1 & + & 2y_2 & & \\
 & \text{s.t.} & y_1 & + & y_2 & \leq & 1 \\
 & & -y_1 & + & 2y_2 & \leq & 1 \\
 & & y_1 & - & 2y_2 & \leq & 1 \\
 & & -y_1 & - & 2y_2 & \leq & -1 \\
 & & y_1 & , & y_2 & \in & \mathbb{R}.
\end{array}
$$

The graphical representation of the dual problem, together with its optimal solution, are shown in Figure 1.

The set of optimal solutions of (D) is given by a line segment, i.e.,

$$
\mathcal{D}^* = \left\{ \lambda \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 1 \\ 0 \end{bmatrix} : \lambda \in [0,1] \right\}.
$$

To find a primal optimal solution using complementary slackness conditions, we can use any dual optimal solution. We will pick $y^* = [1/3, 2/3]^T$ with $z_D^* = 2y_1^* + 2y_2^* = 2$. Since two of the four constraints of (D) are active at y^*, we conclude that y^* is a nondegenerate basic feasible solution of (D).

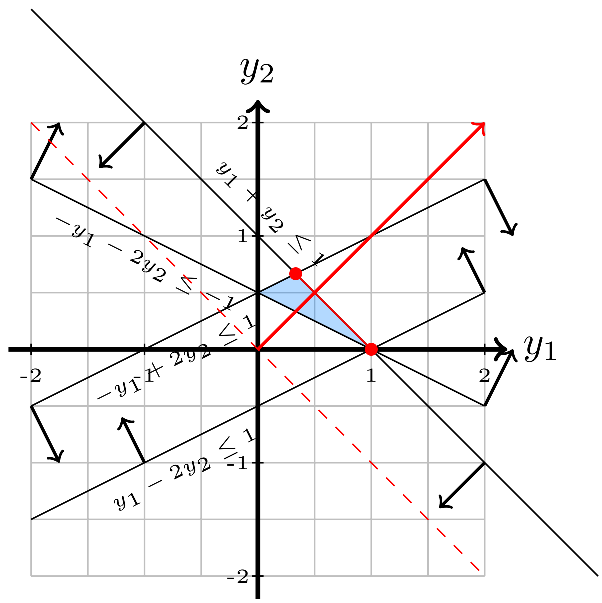

Figure 1: Feasible region of Question 4.1 (blue region), improving direction (solid red arrow), contour lines of the objective function (dashed red lines), and the optimal solution set (red line segment)

Next, we write down and solve the complementary slackness conditions. Since y^* is an optimal solution of (D), every primal optimal solution of (P) should satisfy primal feasibility and complementary slackness conditions together with the unique dual optimal solution.

In general, two feasible solutions $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ of the primal and the dual problem, respectively, must satisfy

$$\begin{aligned}
\bar{x}_1 (1 - \bar{y}_1 - \bar{y}_2) &= 0 \\
\bar{x}_2 (1 + \bar{y}_1 - 2\bar{y}_2) &= 0 \\
\bar{x}_3 (1 - \bar{y}_1 + 2\bar{y}_2) &= 0 \\
\bar{x}_4 (-1 + \bar{y}_1 + 2\bar{y}_2) &= 0
\end{aligned}$$

$$\begin{aligned}
\bar{y}_1 (2 - \bar{x}_1 + \bar{x}_2 - \bar{x}_3 + \bar{x}_4) &= 0 \\
\bar{y}_2 (2 - \bar{x}_1 - 2\bar{x}_2 + 2\bar{x}_3 + 2\bar{x}_4) &= 0
\end{aligned}$$

Substituting the values of the unique dual optimal solution in the complementary slackness conditions, we get that any primal optimal solution $\bar{x}$ must satisfy the following conditions:

$$\begin{aligned}
\bar{x}_1 \cdot 0 &= 0 & & \\
\bar{x}_2 \cdot 0 &= 0 & & \\
\bar{x}_3 \cdot (2) &= 0 & \bar{x}_3 &= 0 \\
\bar{x}_4 \cdot (2/3) &= 0 & \bar{x}_4 &= 0 \\
2 - \bar{x}_1 + \bar{x}_2 - \bar{x}_3 + \bar{x}_4 &= 0 \qquad \Rightarrow & \bar{x}_1 - \bar{x}_2 &= 2 \\
2 - \bar{x}_1 - 2\bar{x}_2 + 2\bar{x}_3 + 2\bar{x}_4 &= 0 & \bar{x}_1 + 2\bar{x}_2 &= 2
\end{aligned}$$

Note that we end up with two equations and two unknowns. Solving the system simultaneously, we obtain

$$\begin{aligned}\bar{x}_1 &= 2\\ \bar{x}_2 &= 0\end{aligned}$$

Clearly, the solution $\bar{x} = [2, 0, 0, 0]^T$ is a feasible solution for (P). By the complementary slackness conditions, we conclude that $\bar{x}$ is an optimal solution of (P). Furthermore, since $\bar{x}$ is the unique solution that satisfies the complementary slackness conditions together with the dual optimal solution y^*, we conclude that it is the unique optimal solution of (P), i.e.,

$$\mathcal{P}^* = \left\{[2, 0, 0, 0]^T\right\}.$$

Note, in particular, that the unique primal optimal solution is a degenerate vertex. Furthermore, you can easily verify that the primal optimal value is given by 2, verifying the Strong Duality Theorem since the optimal value of (D) is also equal to 2.

As an additional remark, if you had used the other vertex given by $y^* = [1, 0]^T$, you would obtain two equations in three unknowns since there are three active constraints at this vertex, i.e., it is degenerate. Then, you can follow a similar argument as in the solution of Problem 4.1 in Exercise Set 4 to find the set of primal optimal solutions.

Finally, if you had used any point on the line segment different from the two vertices, you would find the primal optimal solution directly as there is only one active dual constraint at any such solution, which forces three of the four components of $\bar{x}$ to zero.

In summary, the final solution would not be affected by the choice of the dual optimal solution, but the amount of work may vary slightly. It is, in general, a good idea to choose a dual optimal solution with the fewest number of active constraints (if there is such a choice) since that leads to the fewest number of unknowns in $\bar{x}$.