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Fundamentals of Optimization

Homework 4

Instructions

1. You should attempt all questions.
2. The total marks for this assignment are 10.
3. The assignment consists of STACK questions (5/10 marks) and *open-ended* questions (5/10 marks).
4. All STACK questions are duly marked and are available in the STACK quiz. You **must solve those by completing the STACK quiz**.
5. For the open-ended questions, please write down your solutions in a concise and reproducible way and remember to justify every step using appropriate references when necessary. Failing to do so may result in deductions.
6. The strict deadline for completing the quiz and handing-in your solutions for the open-ended questions is **noon (12:00) on Friday, 25 November 2022**.
7. For the open-ended questions, please upload a **single PDF**. For some useful suggestions, please see Course Information → Tips for Creating a PDF File for Submission on the Learn page.

STACK Problems

1 Simplex Method for Nondegenerate LPs (2.5 marks)

STACK question

You are given three linear programming problems below such that the matrix A has full row rank. For each problem, a starting dictionary is given. For each given dictionary, write down the current index sets B and N, the values of the decision variables $\hat{x}_j$, the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether the given dictionary is optimal or not. If applicable, perform only **one** iteration of the simplex method starting from the given dictionary. For the next dictionary (if applicable), write down the current index sets B and N, the values of the decision variables $\hat{x}_j$, the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether this dictionary is optimal or not.

(1.1) LP

$$
\begin{array}{lrcrcrcrcrcl}
\min & -x_1 & - & x_2 & & & & & & & & \\
\text{s.t.} & -x_1 & + & x_2 & + & x_3 & & & & & = & 3 \\
& x_1 & - & 2x_2 & & & + & x_4 & & & = & 2 \\
& x_1 & + & x_2 & & & & & - & x_5 & = & 1 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}
$$

and the starting dictionary is given by

$$\begin{array}{rcrcrcr} z & = & -1 & & & - & x_5 \\ x_2 & = & 1 & - & x_1 & + & x_5 \\ x_3 & = & 2 & + & 2x_1 & - & x_5 \\ x_4 & = & 4 & - & 3x_1 & + & 2x_5 \end{array}$$

(1.2) LP

$$\begin{array}{lrcrcrcrcr} \min & 2x_1 & - & 3x_2 & + & x_3 & & & & \\ \text{s.t.} & 2x_1 & + & x_2 & + & 4x_3 & + & x_4 & = & 7 \\ & x_1 & + & 2x_2 & + & 4x_3 & - & x_4 & = & 5 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 \end{array}$$

and the starting dictionary is given by

$$\begin{array}{rcrcrcr} z & = & \frac{3}{2} & + & \frac{13}{8}x_1 & - & \frac{27}{8}x_2 \\ x_4 & = & 1 & - & \frac{1}{2}x_1 & + & \frac{1}{2}x_2 \\ x_3 & = & \frac{3}{2} & - & \frac{3}{8}x_1 & - & \frac{3}{8}x_2 \end{array}$$

(1.3) LP

$$\begin{array}{lrcrcrcrcrcr} \min & -2x_1 & - & 3x_2 & & & & & & & & \\ \text{s.t.} & x_1 & & & + & x_3 & & & & & = & 4 \\ & x_1 & + & 3x_2 & & & + & x_4 & & & = & 15 \\ & 2x_1 & + & x_2 & & & & & + & x_5 & = & 10 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

and the starting dictionary is given by

$$\begin{array}{rcrcrcr} z & = & 0 & - & 2x_1 & - & 3x_2 \\ x_3 & = & 4 & - & x_1 & & \\ x_4 & = & 15 & - & x_1 & - & 3x_2 \\ x_5 & = & 10 & - & 2x_1 & - & x_2 \end{array}$$

2 The Simplex Method for Degenerate LPs (2.5 marks)

STACK question

You are given the following linear programming problem that has full row rank:

$$\begin{array}{lrcrcrcrcrcrcr} \min & 2x_1 & - & 3x_2 & + & x_3 & & & & & & & & \\ \text{s.t.} & -3x_1 & + & x_2 & - & 4x_3 & + & x_4 & & & & & = & 3 \\ & 2x_1 & + & x_2 & & & & & + & x_5 & & & = & 5 \\ & -x_1 & + & 2x_2 & + & 3x_3 & & & & & + & x_6 & = & 6 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0 \end{array}$$

You are given three different starting dictionaries. For each given dictionary, write down the current index sets B and N, the values of the decision variables $\hat{x}_j$, the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether the given dictionary is optimal or not. If applicable, perform only **one** iteration of the simplex method starting from the given dictionary. If the current dictionary is nondegenerate, use the most negative reduced cost to determine the entering variable. Break ties in favour of the variable with the smallest index. If it is degenerate, use Bland's rule to determine the entering and leaving variables (whenever applicable). For the next dictionary (if applicable), write down the current index sets B and N, the values of the decision variables $\hat{x}_j$, the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether this dictionary is optimal or not.

(2.1) The starting dictionary is given by

$$\begin{array}{rcrcrcrcr} z &=& 0 &+& 2x_1 &-& 3x_2 &+& x_3 \\ x_4 &=& 3 &+& 3x_1 &-& x_2 &+& 4x_3 \\ x_5 &=& 5 &-& 2x_1 &-& x_2 && \\ x_6 &=& 6 &+& x_1 &-& 2x_2 &-& 3x_3 \end{array}$$

(2.2) The starting dictionary is given by

$$\begin{array}{rcrcrcrcr} z &=& -9 &-& 2x_1 &+& x_4 &+& x_6 \\ x_2 &=& 3 &+& \frac{13}{11}x_1 &-& \frac{3}{11}x_4 &-& \frac{4}{11}x_6 \\ x_3 &=& 0 &-& \frac{5}{11}x_1 &+& \frac{2}{11}x_4 &-& \frac{1}{11}x_6 \\ x_5 &=& 2 &-& \frac{35}{11}x_1 &+& \frac{3}{11}x_4 &+& \frac{4}{11}x_6 \end{array}$$

(2.3) The starting dictionary is given by

$$\begin{array}{rcrcrcrcr} z &=& -9 &-& 7x_1 &-& 11x_3 &+& 3x_4 \\ x_2 &=& 3 &+& 3x_1 &+& 4x_3 &-& x_4 \\ x_5 &=& 2 &-& 5x_1 &-& 4x_3 &+& x_4 \\ x_6 &=& 0 &-& 5x_1 &-& 11x_3 &+& 2x_4 \end{array}$$

Open Ended Problems

3 Puzzle (1 mark)

Consider the following linear program

$$\begin{array}{lrcrcrcrcrcr} \min & -2x_1 &+& c_2\,x_2 &+& c_3\,x_3 &-& x_4 &+& c_5\,x_5 && \\ \text{s.t} & a_{11}\,x_1 &+& a_{12}\,x_2 &+& a_{13}\,x_3 &+& x_4 &&& = & 12 \\ & -x_1 &+& a_{22}\,x_2 &+& a_{23}\,x_3 &&&+& x_5 & = & b_2 \\ & x_1 &,& x_2 &,& x_3 &,& x_4 &,& x_5 & \geq & 0. \end{array}$$

and an intermediate dictionary given by

$$\begin{array}{rcrcrcrcr} z &=& 52 &-& 4.25x_1 &+& \bar{c}_4\,x_4 &-& 0.75x_5 \\ x_3 &=& \hat{x}_3 &-& 0.75x_1 &-& 0.5x_4 &-& 0.25x_5 \\ x_2 &=& 12 &-& \bar{a}_{21}\,x_1 &-& 0.5x_4 &-& 0.75x_5 \end{array}$$

Determine the unknown values for c_2, c_3, c_5, a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, b_2, $\bar{c}_4$, $\hat{x}_3$, and $\bar{a}_{21}$. Justify your solution.

[1 mark]

4 Duality (4 marks)

(4.1) Consider the following polyhedron in standard form:

$$\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are given and $x \in \mathbb{R}^n$. Suppose that $b \geq \mathbf{0}$.

(i) Consider the Phase 1 problem for solving (P):

$$\text{(P1)} \quad \min\{e^T a : Ax + a = b, \quad x \geq \mathbf{0}, \quad a \geq \mathbf{0}\},$$

where $a \in \mathbb{R}^m$ and $e \in \mathbb{R}^m$ denotes the vector of all ones, i.e., $e = [1, 1, \ldots, 1]^T$. Write down the dual of (P1) and justify your solution.

[1 mark]

(ii) Prove that the dual of (P1) has a finite optimal value.

[1 mark]

(4.2) Determine the set of all optimal solutions for the following linear program using the graphical method:

$$
\begin{array}{llrcrcrcrcl}
(P) & \min & x_1 & + & x_2 & + & x_3 & - & x_4 & & \\
 & \text{s.t.} & x_1 & - & x_2 & + & x_3 & - & x_4 & = & 2 \\
 & & x_1 & + & 2x_2 & - & 2x_3 & - & 2x_4 & = & 2 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0.
\end{array}
$$

[2 marks]