**Fundamentals of Operational Research**
**Tutorial 2**
**School of Mathematics**
**The University of Edinburgh**
**Year 2022/2023**

1. The maintenance of a pump requires the jobs listed in the table below, which also gives the jobs that must be completed before each job can start and the duration in hours of each job.

| Job No. | Job title | Duration | Predecessor jobs |
|---|---|---|---|
| 1 | Remove cover | 2 | – |
| 2 | Disconnect pipes | 2 | – |
| 3 | Assemble test rig | 3 | 2 |
| 4 | Remove seals | 1 | 1 |
| 5 | Remove damaged blades | 2 | 1 |
| 6 | Replace damaged blades | 3 | 5 |
| 7 | Analyse damaged blades | 5 | 5 |
| 8 | Lubricate | 2 | 4 |
| 9 | Test electrics | 4 | 4 |
| 10 | Test running | 3 | 3, 6, 8 |
| 11 | Replace seals and cover | 2 | 9, 10 |
| 12 | Reconnect pipes | 2 | 9, 10 |

(a) Draw a project network showing the above jobs as nodes.

(b) Find the earliest start time, the latest finish time and the slack for each job.

**Solution:**

(a) By following the indications seen in class, we draw the project network that can be seen in Figure 1.

(b) In order to find out the information that we have been asked, we need to compute the critical path using both backward and forward dynamic programming.

Let us start with backward dynamic programming. We know that $f^*_{\text{Finish}} = 0$ and that

$$f^*_i = d_i + \max\{f^*_j\}_{(i,j)\in A},$$

where $d_i$ is the duration of activity $i$ and $A$ is the set of arcs of our AON project network. The information obtained, including latest start and finish times, is on Table 1.

It takes 12 hours to complete the project. The critical path is

$$\text{Start} \to 1 \to 5 \to 6 \to 10 \to 11, 12 \to \text{Finish}.$$

Now, we use forward dynamic programming. In this case we define $f^*_{\text{Start}} = 0$ and

$$f^*_j = d_j + \max\{f^*_i\}_{(i,j)\in A}.$$

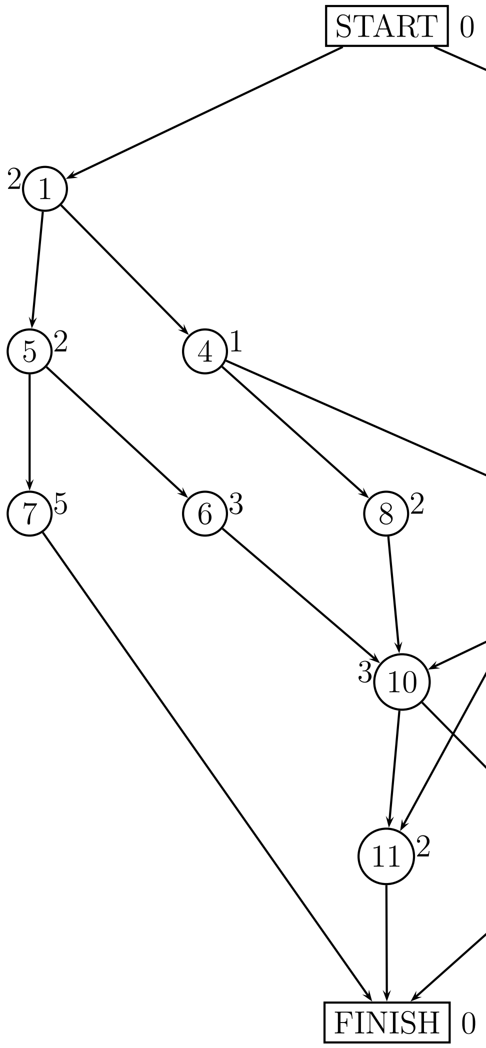


Figure 1: AON project network.

The details are on Table 2.

Obviously, we obtain the same critical path and length as before.

Finally, we calculate the slack for each activity $i$ as $S(i) = LF(i) - EF(i)$. The results are on Table 3

2. An industrial pump manufacturer has one pump in stock at the start of month 1 and has orders for $d_t$ pumps in months $t = 1, 2, 3, 4$. The company wishes to have one pump in stock at the start of month 5. Orders in a given month may be met from stock or from that month production.

The cost of producing $x$ pumps in any month is $r(x)$ and at most 3 pumps can be produced per month. The cost of having $y$ pumps in stock at the start of any month is $s(y)$. There is no space for more than 2 pumps in stock at the start of any month.

Here is the different data:

- $d_1 = 2,\ d_2 = d_3 = d_4 = 1.$
- $r(0) = 4,\ r(1) = 14,\ r(2) = 20,\ r(3) = 24.$

| $i$ | $f_i^* = LP(i, \text{Finish})$ | Next node is | $LS(i)$ | $LF(i)$ |
|---|---|---|---|---|
| Finish | $0$ | - | $12 - 0 = 12$ | 12 |
| 12 | $2 + 0 = 2$ | Finish | $12 - 2 = 10$ | 12 |
| 11 | $2 + 0 = 2$ | Finish | $12 - 2 = 10$ | 12 |
| 10 | $3 + \max\{2, 2\} = 5$ | 11,12 | $12 - 5 = 7$ | 10 |
| 9 | $4 + \max\{2, 2\} = 6$ | 11,12 | $12 - 6 = 6$ | 10 |
| 8 | $2 + 5 = 7$ | 10 | $12 - 7 = 5$ | 7 |
| 7 | $5 + 0 = 5$ | Finish | $12 - 5 = 7$ | 12 |
| 6 | $3 + 5 = 8$ | 10 | $12 - 8 = 4$ | 7 |
| 5 | $2 + \max\{8, 5\} = 10$ | 6 | $12 - 10 = 2$ | 4 |
| 4 | $1 + \max\{7, 6\} = 8$ | 8 | $12 - 8 = 4$ | 5 |
| 3 | $3 + 5 = 8$ | 10 | $12 - 8 = 4$ | 7 |
| 2 | $2 + 8 = 10$ | 3 | $12 - 10 = 2$ | 4 |
| 1 | $2 + \max\{8, 10\} = 12$ | 5 | $12 - 12 = 0$ | 2 |
| Start | $0 + \max\{12, 10\} = 12$ | 1 | $12 - 12 = 0$ | 0 |

Table 1: Critical path with backward dynamic programming.

| $i$ | $f_i^* = LP(\text{Start}, i) = EF(i)$ | Next node is | $ES(i)$ |
|---|---|---|---|
| Start | $0$ | - | 0 |
| 1 | $2 + 0 = 2$ | Start | 0 |
| 2 | $2 + 0 = 2$ | Start | 0 |
| 3 | $3 + 2 = 5$ | 2 | 2 |
| 4 | $1 + 2 = 3$ | 1 | 2 |
| 5 | $2 + 2 = 4$ | 1 | 2 |
| 6 | $3 + 4 = 7$ | 5 | 4 |
| 7 | $5 + 4 = 9$ | 5 | 4 |
| 8 | $2 + 3 = 5$ | 4 | 3 |
| 9 | $4 + 3 = 7$ | 4 | 3 |
| 10 | $3 + \max\{5, 7, 5\} = 10$ | 6 | 7 |
| 11 | $2 + \max\{7, 10\} = 12$ | 10 | 10 |
| 12 | $2 + \max\{7, 10\} = 12$ | 10 | 10 |
| Finish | $0 + \max\{9, 12, 12\} = 12$ | 11,12 | 12 |

Table 2: Critical path with forward dynamic programming.

- $s(0) = 0,\ s(1) = 4,\ s(2) = 7.$

The order of actions at each month to determine costs is:

1. Pay inventory cost.
2. Produce pumps.
3. Deliver orders.

You are asked the following:

(a) Identify the states of the problem and write down a dynamic programming recurrence.

| $i$ | $EF(i)$ | $LF(i)$ | $S(i)$ |
|---|---|---|---|
| Start | 0 | 0 | 0 |
| 1 | 2 | 2 | 0 |
| 2 | 2 | 4 | 2 |
| 3 | 5 | 7 | 2 |
| 4 | 3 | 5 | 2 |
| 5 | 4 | 4 | 0 |
| 6 | 7 | 7 | 0 |
| 7 | 9 | 12 | 3 |
| 8 | 5 | 7 | 2 |
| 9 | 7 | 10 | 3 |
| 10 | 10 | 10 | 0 |
| 11 | 12 | 12 | 0 |
| 12 | 12 | 12 | 0 |
| Finish | 12 | 12 | 0 |

Table 3: Slacks.

(b) Draw a network and find a shortest path that corresponds to the cheapest production schedule.

**Solution:**

(a) The states can be identified as pairs $(t, y)$, where $t$ is the month and $y$ is the number of pumps in stock. For $t = 1$ and $t = 5$ we have $y = 1$. For $t \in \{2, 3, 4\}$ we have that $y \in \{0, 1, 2\}$.

Let$f(t, y)$ be the minimal cost to meet future orders when we start state $(t, y)$. We need to take into account the following things:

- We can produce at most 3 pumps per stage.
- We can store at most 2 pumps.
- The minimum total cost from now until the end of the time horizon is equal to the inventory cost when we start the stage plus production cost of this stage plus the minimum total cost from next stage onwards.

Therefore we can write the following recurrence:

$$f(t, y) = \min \{r(x) + s(y) + f(t + 1, y + x - d_t) \ / \ x \in \{0, 1, 2, 3\}, \ y + x - d_t \in \{0, 1, 2\}\},$$

with $f(5, 1) = 0$. Note that we do not really mind the value of $f(5, 1)$, as this will be common to all the combinations. There is no decision to be taken for $t = 5$.

(b) Now the directed graph has the states $(t, y)$ as nodes. The arcs are the possible transitions. The cost for the arc from $(t, y_1)$ to $(t+1, y_2)$ is $s(y_1)+r(y_2-y_1+d_t)$. The argument in $r(x)$ comes from the fact that $y_2 = y_1 - d_t + x$.

Note also that some transactions are not possible. For example, from $(2, 2)$ to $(3, 0)$. This is because $d_2 = 1$ and we cannot produce $-1$ pumps.

Using the expression stated above we compute the cost labels of the arcs:

- $(1, 1) \to (2, 0): \ s(1) + r(0 - 1 + 2) = s(1) + r(1) = 4 + 14 = 18.$

- $(1,1) \to (2,1):\ s(1) + r(2) = 4 + 20 = 24.$
- $(1,1) \to (2,2):\ s(1) + r(3) = 4 + 24 = 28.$
- $(2,0) \to (3,0):\ s(0) + r(0-0+1) = s(0) + r(1) = 0 + 14 = 14.$
- $(2,0) \to (3,1):\ s(0) + r(2) = 0 + 20 = 20.$
- $(2,0) \to (3,2):\ s(0) + r(3) = 0 + 24 = 24.$
- $(2,1) \to (3,0):\ s(1) + r(0-1+1) = s(1) + r(0) = 4 + 4 = 8.$
- $(2,1) \to (3,1):\ s(1) + r(1) = 4 + 14 = 18.$
- $(2,1) \to (3,2):\ s(1) + r(2) = 4 + 20 = 24.$
- $(2,2) \to (3,1):\ s(2) + r(1-2+1) = s(2) + r(0) = 7 + 4 = 11.$
- $(2,2) \to (3,2):\ s(2) + r(1) = 7 + 14 = 21.$

The directed graph that represents this problem is given in Figure 2.

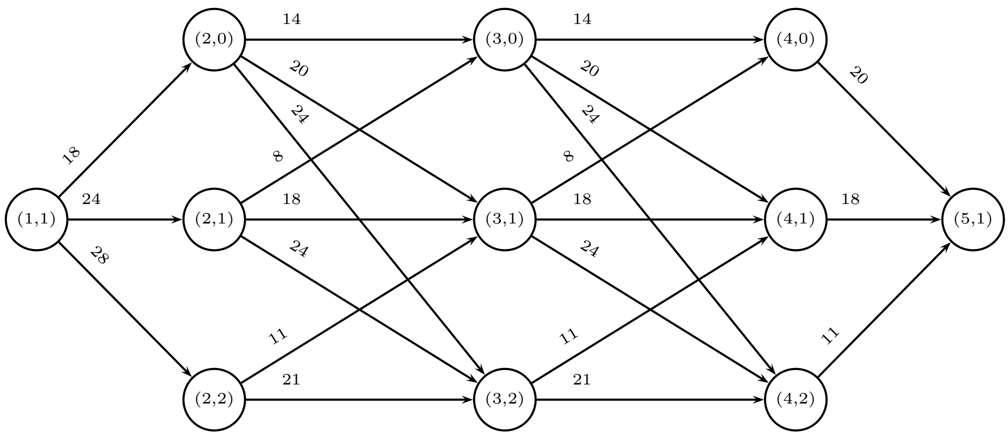


Figure 2: Pump problem network.

We find now the shorts path on the network in the usual way. The results are on Table 4.

| $(t,y)$ | $f(t,y)$ | Next node is |
|---|---|---|
| $(5,1)$ | $0$ | - |
| $(4,0)$ | $20+0=20$ | $(5,1)$ |
| $(4,1)$ | $18+0=18$ | $(5,1)$ |
| $(4,2)$ | $11+0=11$ | $(5,1)$ |
| $(3,0)$ | $\min\{14+20, 20+18, 24+11\} = \min\{\underline{34}, 38, 35\} = 34$ | $(4,0)$ |
| $(3,1)$ | $\min\{8+20, 18+18, 24+11\} = \min\{\underline{28}, 36, 35\} = 28$ | $(4,0)$ |
| $(3,2)$ | $\min\{11+18, 21+11\} = \min\{\underline{29}, 32\} = 29$ | $(4,1)$ |
| $(2,0)$ | $\min\{14+34, 20+28, 24+29\} = \min\{\underline{48}, \underline{48}, 53\} = 48$ | $(3,0)$ or $(3,1)$ |
| $(2,1)$ | $\min\{8+34, 18+28, 24+29\} = \min\{\underline{42}, 46, 53\} = 42$ | $(3,0)$ |
| $(2,2)$ | $\min\{11+28, 21+29\} = \min\{\underline{39}, 50\} = 39$ | $(3,1)$ |
| $(1,1)$ | $\min\{18+48, 24+42, 28+39\} = \min\{\underline{66}, \underline{66}, 67\} = 66$ | $(2,0)$ or $(2,1)$ |

Table 4: Shortest paths.

The cheapest cost is 66. There are 3 optimal paths with this cost. They are shown in Figure 3. For example, if we consider the path on the top, then the optimal production is:

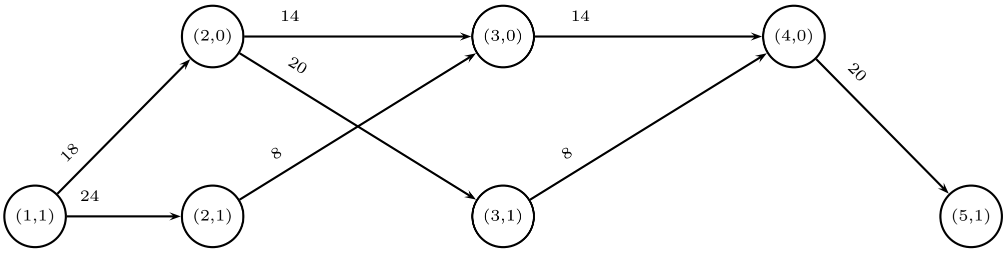


Figure 3: Shortest paths.

- On month 1 start with 1 pump and on month 2 start with 0 pumps. So, produce 1 pump on month 1.
- On month 2 start with 0 pumps and on month 3 start with 0 pumps. So, produce 1 pump on month 2.
- On month 3 start with 0 pumps and on month 4 start with 0 pumps. So, produce 1 pump on month 3.
- On month 4 start with 0 pumps and on month 5 start with 1 pump. So, produce 2 pumps on month 4.