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THE UNIVERSITY *of* EDINBURGH

Fundamentals of Optimization

Homework 2

Instructions

1. You should attempt all questions.
2. The total marks for this assignment are 10.
3. The assignment consists of STACK questions (5/10 marks) and *open-ended* questions (5/10 marks).
4. All STACK questions are duly marked and are available in the STACK quiz. You **must solve those by completing the STACK quiz**.
5. For the open-ended questions, please write down your solutions in a concise and reproducible way and remember to justify every step using appropriate references when necessary. Failing to do so may result in deductions.
6. The strict deadline for completing the quiz and handing-in your solutions for the open-ended questions is **noon (12:00) on Friday, 28 October 2022**.
7. For the open-ended questions, please upload a **single PDF**. For some useful suggestions, please see Course Information → Tips for Creating a PDF File for Submission on the Learn page.

1 Basic Solutions and Basic Feasible Solutions (3 marks)

STACK question

Consider the following polyhedron

$$\mathcal{P} = \{x \in \mathbb{R}^3 : 2x_1 - x_2 + x_3 \leq -1,\ -x_1 + 2x_2 \leq 2,\ x_1 + x_3 \geq -1,\ -2x_1 + x_2 + x_3 = 0,\ x_2 + x_3 \geq -1\}.$$

Decide, for each of the points $\hat{x}$ given below, whether $\hat{x}$ is *infeasible and not a basic solution*, *feasible but not a basic feasible solution*, *a basic solution but infeasible*, or *a basic feasible solution*.

(1.1) $\hat{x} = [0, 1, -1]^T$.

(1.2) $\hat{x} = [-1/3, 0, -2/3]^T$.

(1.3) $\hat{x} = [-1/2, 3/4, -7/4]^T$.

(1.4) $\hat{x} = [-1, -1, 0]^T$.

(1.5) $\hat{x} = [-1/2, -1/2, -1/2]^T$.

[3 marks]

2 Graphical Method (2 marks)

STACK question

Consider the following polyhedron:

$$\mathcal{P} \;=\; \{[x_1, x_2]^T \in \mathbb{R}^2 : x_1 \geq 0,\, x_1 + x_2 \geq 1,\, -x_1 + x_2 \leq 3,\, x_2 \geq 0\}\,.$$

Using the graphical method, determine, for each of the following objective functions, the optimal value denoted by z^* (use `+inf` for $+\infty$ and `-inf` for $-\infty$), and whether the set of optimal solutions, denoted by $\mathcal{P}^*$, is either *empty*, a *single vertex*, a *line segment*, a *half line*, or $\mathcal{P}^* = \mathcal{P}$.

[2 marks]

(2.1) $\min\{2x_1 + 2x_2 : x \in \mathcal{P}\}$.

(2.2) $\max\{-4x_1 + 2x_2 : x \in \mathcal{P}\}$.

(2.3) $\min\{-4x_1 + 2x_2 : x \in \mathcal{P}\}$.

(2.4) $\max\{-4x_1 + 4x_2 : x \in \mathcal{P}\}$.

Open Ended Problems

3 Polyhedra in Standard Form (1 mark)

(3.1) Convert the following general linear programming problem into standard form:

$$\begin{array}{llllllllll}
\max & -2x_1 & - & x_2 & + & 3x_3 & - & 2x_4 & & \\
\text{s.t.} & & & & & & & & & \\
& 2x_1 & - & x_2 & + & x_3 & - & x_4 & = & 1 \\
& -x_1 & + & 2x_2 & - & 3x_3 & + & x_4 & \leq & 2 \\
& x_1 & + & 2x_2 & - & x_3 & + & 2x_4 & \geq & 4 \\
& x_1 \geq 0 & , & x_2 \leq 0 & , & x_3 \leq 0 & . & & &
\end{array}$$

Remark It is irrelevant whether the problem is actually feasible or not.

[1 mark]

4 Polytopes vs Polyhedra (2 marks)

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron given by

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T x \geq b_i, & i \in M_1, \\ (a^i)^T x \leq b_i, & i \in M_2, \\ (a^i)^T x = b_i, & i \in M_3 \end{array} \right\},$$

where M_1, M_2, and M_3 are finite index sets, $a^i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for each $i \in M_1 \cup M_2 \cup M_3$. Let us define the following set:

$$\mathcal{R} = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T d \geq 0, & i \in M_1, \\ (a^i)^T d \leq 0, & i \in M_2, \\ (a^i)^T d = 0, & i \in M_3 \end{array} \right\}.$$

(4.1) Prove the following result:

> For each $\hat{x} \in \mathcal{P}$ and each $\hat{d} \in \mathcal{R}$, we have $\hat{x} + \lambda\hat{d} \in \mathcal{P}$ for each real number $\lambda \geq 0$.

[1 mark]

(4.2) Prove the following result:

> If $\mathcal{P} \subseteq \mathbb{R}^n$ is a polytope, then $\mathcal{R} = \{\mathbf{0}\}$ (i.e., $\mathcal{R}$ consists only of the vector of all zeroes $\mathbf{0} \in \mathbb{R}^n$).

[1 mark]

5 Existence of Vertices in Polyhedra (2 marks)

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron given by

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T x \geq b_i, & i \in M_1, \\ (a^i)^T x \leq b_i, & i \in M_2, \\ (a^i)^T x = b_i, & i \in M_3 \end{array} \right\},$$

where M_1, M_2, and M_3 are finite index sets, $a^i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for each $i \in M_1 \cup M_2 \cup M_3$. Let us define the following set:

$$\mathcal{R} = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T d \geq 0, & i \in M_1, \\ (a^i)^T d \leq 0, & i \in M_2, \\ (a^i)^T d = 0, & i \in M_3 \end{array} \right\}.$$

(5.1) Prove the following result:

> $\mathcal{P} \subseteq \mathbb{R}^n$ has no vertices if and only if there exists $\hat{d} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $\hat{d} \in \mathcal{R}$ and $-\hat{d} \in \mathcal{R}$.

[2 marks]