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Fundamentals of Optimization

Exercise 3 – Solutions

Remarks

- All questions that are available in the STACK quiz are duly marked. Please solve those using STACK.
- We have added marks for each question. Please note that those are purely for illustrative purposes. The exercise set will not be marked.
- We can derive the inverse of a nonsingular matrix $A \in \mathbb{R}^{2\times 2}$ in closed form:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

STACK Problems

1 Basic Solutions of Polyhedra in Standard Form (2 marks)

(1.1) *STACK question*

By using the enumeration algorithm presented in Section 13.2.5 of the lecture notes, determine the set of all basic solutions and basic feasible solutions of the following polyhedron: $\mathcal{P} = \{x \in \mathbb{R}^4 : x_1 + 2x_2 - x_3 + 4x_4 = 10;\ 2x_1 + 3x_2 - 2x_3 - 2x_4 = 16;\ x \geq \mathbf{0}\}$. For each basic solution and basic feasible solution, determine whether it is degenerate or nondegenerate. You can assume that the coefficient matrix A has full row rank.

[1 mark]

Solution

We have $n = 4$ and $m = 2$. Since A has full row rank, we have $|B| = m = 2$. Moreover,

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 3 & -2 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 16 \end{bmatrix}.$$

To solve this question, we consider all possible subsets $B \subset \{1, 2, 3, 4\}$ such that $|B| = 2$, and check whether the conditions for a basic (feasible) solution hold.

- $B = \{1, 2\}$: The columns A^1 and A^2 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1} b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

  Hence, the basic variables are $\hat{x}_1 = 2$ and $\hat{x}_2 = 4$, and the nonbasic variables are $\hat{x}_3 = 0$ and $\hat{x}_4 = 0$. Therefore, $\hat{x} = [2, 4, 0, 0]^T$ is a basic feasible solution as $\hat{x} \geq \mathbf{0}$. Note that $\hat{x}$ is nondegenerate since both basic variables are different from zero (i.e., $\hat{B} = B$).
- $B = \{1, 3\}$: The columns A^1 and A^3 are linearly dependent since $A^1 + A^3 = \mathbf{0} \in \mathbb{R}^2$. Therefore, this choice does not give rise to a basic solution.

- $B = \{1,4\}$: The columns A^1 and A^4 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & -2 \end{bmatrix}^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 42/5 \\ 2/5 \end{bmatrix}.$$

  Hence, the basic variables are $\hat{x}_1 = 42/5$ and $\hat{x}_4 = 2/5$, and the nonbasic variables are $\hat{x}_2 = 0$ and $\hat{x}_3 = 0$. Therefore, $\hat{x} = [42/5, 0, 0, 2/5]^T$ is a basic feasible solution as $\hat{x} \geq \mathbf{0}$. Note that $\hat{x}$ is nondegenerate since both basic variables are different from zero (i.e., $\hat{B} = B$).
- $B = \{2,3\}$: The columns A^2 and A^3 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

  Hence, the basic variables are $\hat{x}_2 = 4$ and $\hat{x}_3 = -2$, and the nonbasic variables are $\hat{x}_1 = 0$ and $\hat{x}_4 = 0$. Therefore, $\hat{x} = [0, 4, -2, 0]^T$ is a basic solution but not feasible since $\hat{x} \ngeq \mathbf{0}$. Note that $\hat{x}$ is nondegenerate since both basic variables are different from zero (i.e., $\hat{B} = B$).
- $B = \{2,4\}$: The columns A^2 and A^4 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 2 & 4 \\ 3 & -2 \end{bmatrix}^{-1} = \frac{1}{16}\begin{bmatrix} 2 & 4 \\ 3 & -2 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 21/4 \\ -1/8 \end{bmatrix}.$$

  Hence, the basic variables are $\hat{x}_2 = 21/4$ and $\hat{x}_4 = -1/8$, and the nonbasic variables are $\hat{x}_1 = 0$ and $\hat{x}_3 = 0$. Therefore, $\hat{x} = [0, 21/4, 0, -1/8]^T$ is a basic solution but not feasible since $\hat{x} \ngeq \mathbf{0}$. Note that $\hat{x}$ is nondegenerate since both basic variables are different from zero (i.e., $\hat{B} = B$).
- $B = \{3,4\}$: The columns A^3 and A^4 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} -1 & 4 \\ -2 & -2 \end{bmatrix}^{-1} = \frac{1}{10}\begin{bmatrix} -2 & -4 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} -42/5 \\ 2/5 \end{bmatrix}.$$

  Hence, the basic variables are $\hat{x}_3 = -42/5$ and $\hat{x}_4 = 2/5$, and the nonbasic variables are $\hat{x}_1 = 0$ and $\hat{x}_2 = 0$. Therefore, $\hat{x} = [0, 0, -42/5, 2/5]^T$ is a basic solution but not feasible since $\hat{x} \ngeq \mathbf{0}$. Note that $\hat{x}$ is nondegenerate since both basic variables are different from zero (i.e., $\hat{B} = B$).

Therefore, $\mathcal{P}$ has two basic feasible solutions (i.e., vertices) and three basic solutions that are not feasible, all of which are nondegenerate.

(1.2) *STACK question*

By using the enumeration algorithm presented in Section 13.2.5 of the lecture notes, determine the set of all basic solutions and basic feasible solutions of the following polyhedron: $\mathcal{P} = \{x \in \mathbb{R}^3 : 3x_1 + 2x_2 + 4x_3 = 4;\ -x_1 + x_2 - 2x_3 = 2;\ x \geq \mathbf{0}\}$. For each basic solution and basic feasible solution, determine whether it is degenerate or nondegenerate. You can assume that the coefficient matrix A has full row rank.

[1 mark]

Solution

We have $n = 3$ and $m = 2$. Since A has full row rank, we have $|B| = m = 2$. Moreover,

$$A = \begin{bmatrix} 3 & 2 & 4 \\ -1 & 1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

To solve this question, we consider all possible subsets $B \subset \{1,2,3\}$ such that $|B| = 2$ and check whether the conditions for a basic (feasible) solution hold.

- $B = \{1,2\}$: The columns A^1 and A^2 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

  Hence, the basic variables are $\hat{x}_1 = 0$ and $\hat{x}_2 = 2$, and the only nonbasic variable is $\hat{x}_3 = 0$. Therefore, $\hat{x} = [0,2,0]^T$ is a basic feasible solution as $\hat{x} \geq \mathbf{0}$. Note that $\hat{x}$ is degenerate since there is at least one basic variable that is equal to zero (i.e., $\hat{B} \subset B$ or $1 = |\hat{B}| < |B| = 2$).
- $B = \{1,3\}$: The columns A^1 and A^3 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 8 \\ -5 \end{bmatrix}.$$

  Hence, the basic variables are $\hat{x}_1 = 8$ and $\hat{x}_3 = -5$, and the only nonbasic variable is $\hat{x}_2 = 0$. Therefore, $\hat{x} = [8,0,-5]^T$ is a basic solution but not feasible since $\hat{x} \not\geq \mathbf{0}$. Note that $\hat{x}$ is nondegenerate since both basic variables are different from zero (i.e., $\hat{B} = B$).
- $B = \{2,3\}$: The columns A^2 and A^3 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix}^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

  Hence, the basic variables are $\hat{x}_2 = 2$ and $\hat{x}_3 = 0$, and the only nonbasic variable is $\hat{x}_1 = 0$. Therefore, $\hat{x} = [0,2,0]^T$ is a basic feasible solution as $\hat{x} \geq \mathbf{0}$. Note that $\hat{x}$ is degenerate since there is at least one basic variable that is equal to zero (i.e., $\hat{B} \subset B$ or $1 = |\hat{B}| < |B| = 2$). Note that this is the same basic feasible solution as the one given by $B = \{1,2\}$.

It follows that $\mathcal{P}$ has one degenerate basic feasible solution (i.e., vertex) and one nondegenerate basic solution that is not feasible.

2 Optimality Conditions and Degeneracy (3 marks)

(2.1) *STACK question*

Consider the following linear program in standard form

$$\min\{-x_1 - 4x_2 - x_3 + 2x_4 : x_1 + 4x_2 + x_3 = 8;\ x_1 + 2x_2 + x_4 = 4;\ x \geq \mathbf{0}\}$$

and the vertices

(a) $\hat{x} = [4,0,4,0]^T$.
(b) $\hat{x} = [0,2,0,0]^T$,
(c) $\hat{x} = [0,0,8,4]^T$.

You can assume that the coefficient matrix A has full row rank. For each vertex, decide whether the vertex is optimal or not, and whether it is degenerate or not.

[3 marks]

Write down a valid choice for the index set B, the reduced costs $\bar{c}_j$, $j \in \{1,\ldots,n\}$, for that basis, and a "candidate" improving direction $d \in \mathbb{R}^n$ if one exists. For the latter, if $\bar{c} \not\geq \mathbf{0}$, use $d_{j^*} = 1$ and $d_j = 0$, $j \in N \setminus \{j^*\}$ to derive the direction, where $j^* \in N$ is the index with the smallest reduced cost $\bar{c}_j$. Verify whether the candidate improving direction d is indeed an improving feasible direction at that vertex. If $\bar{c} \geq \mathbf{0}$, then enter $d = \mathbf{0}$.

If the vertex is degenerate, write down all possible choices of the indices for the index set B, together with the corresponding reduced costs $\bar{c}$ and candidate improving directions.

Solution

First observe that $n = 4$ and $m = 2$. Since A has full row rank, $|B| = 2$. Moreover,

$$A = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 4 \end{bmatrix}, \quad c = [-1, -4, -1, 2]^T.$$

(a) We have $\hat{x}_1 > 0$ and $\hat{x}_3 > 0$ and, hence, $\hat{B} = \{1, 3\}$. As $|\hat{B}| = m = 2$, $\hat{x}$ is nondegenerate and we obtain $B = \hat{B}$ and $N = \{2, 4\}$. Furthermore, we obtain

$$(A_B)^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

and

$$\begin{aligned} \bar{c}_2 &= c_2 - c_B^T (A_B)^{-1} A^2 = -4 - [-1, -1] \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 0 \\ \bar{c}_4 &= c_4 - c_B^T (A_B)^{-1} A^4 = 2 - [-1, -1] \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \end{aligned}$$

Since $\bar{c} \geq \mathbf{0}$, we conclude that $\hat{x}$ is an optimal nondegenerate vertex by Corollary 15.4 and $\hat{d} = \mathbf{0}$.

(b) We only have $\hat{x}_2 > 0$ and, hence, $\hat{B} = \{2\}$. As $|\hat{B}| = 1 < m = 2$, $\hat{x}$ is a degenerate vertex. There are now three possibilities to extend $\hat{B}$ to a basis: $B = \{1, 2\}$, $B = \{2, 3\}$, and $B = \{2, 4\}$.

- $B = \{1, 2\}$ and $N = \{3, 4\}$: We obtain

  $$(A_B)^{-1} = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} -2 & 4 \\ 1 & -1 \end{bmatrix}$$

  and

  $$\begin{aligned} \bar{c}_3 &= c_3 - c_B^T (A_B)^{-1} A^3 = -1 - [-1, -4] \begin{bmatrix} -1 & 2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \\ \bar{c}_4 &= c_4 - c_B^T (A_B)^{-1} A^4 = 2 - [-1, -4] \begin{bmatrix} -1 & 2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \end{aligned}$$

  Since $\bar{c} \geq \mathbf{0}$, we conclude that $\hat{x}$ is an optimal vertex by Corollary 15.4 and $d = \mathbf{0}$.

- $B = \{2, 3\}$ and $N = \{1, 4\}$: We obtain

  $$(A_B)^{-1} = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1/2 \\ 1 & -2 \end{bmatrix}$$

  and

  $$\begin{aligned} \bar{c}_1 &= c_1 - c_B^T (A_B)^{-1} A^1 = -1 - [-4, -1] \begin{bmatrix} 0 & 1/2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \\ \bar{c}_4 &= c_4 - c_B^T (A_B)^{-1} A^4 = 2 - [-4, -1] \begin{bmatrix} 0 & 1/2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \end{aligned}$$

  Since $\bar{c} \geq \mathbf{0}$, we conclude that $\hat{x}$ is an optimal vertex by Corollary 15.4 and $d = \mathbf{0}$.
- $B = \{2, 4\}$ and $N = \{1, 3\}$: We obtain

  $$(A_B)^{-1} = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/4 & 0 \\ -1/2 & 1 \end{bmatrix}$$

and

$$\begin{aligned}
\bar{c}_1 &= c_1 - c_B^T (A_B)^{-1} A^1 = -1 - [-4, 2] \begin{bmatrix} 1/4 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 \\
\bar{c}_3 &= c_3 - c_B^T (A_B)^{-1} A^3 = -1 - [-4, 2] \begin{bmatrix} 1/4 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1
\end{aligned}$$

Since $\bar{c} \ngeq \mathbf{0}$ and $\hat{x}$ is a degenerate vertex, we cannot infer any information about the optimality of $\hat{x}$. The candidate improving direction is obtained by setting $d_1 = 1$ and $d_3 = 0$, and

$$d_B = -(A_B)^{-1} A_N d_N = -\begin{bmatrix} 1/4 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/4 \\ -1/2 \end{bmatrix}$$

Therefore,

$$d = [1, -1/4, 0, -1/2]^T .$$

Note that d is an improving direction since $c^T d = [-1, -4, -1, 2]^T [1, -1/4, 0, -1/2] = -1 = \bar{c}_1 < 0$. Let us check if d is indeed a feasible direction at $\hat{x}$. Note that $A(\hat{x} + \lambda d) = b$ for any $\lambda \in \mathbb{R}$ since $Ad = \mathbf{0}$. Consider $\hat{x} + \lambda d = [0, 2, 0, 0]^T + \lambda[1, -1/4, 0, -1/2]^T = [\lambda, 2 - (1/4)\lambda, 0, (-1/2)\lambda]^T$. Note that there does not exist a real number $\lambda^* > 0$ such that $\hat{x} + \lambda d \geq \mathbf{0}$ if $\lambda \in [0, \lambda^*]$ since the fourth component becomes negative whenever $\lambda > 0$. Therefore, d is not a feasible direction at $\hat{x}$. Note that we have $d \in \mathcal{D} \backslash \hat{\mathcal{D}}$.

(c) We have $\hat{x}_3 > 0$ and $\hat{x}_4 > 0$ and, hence, $\hat{B} = \{3, 4\}$. As $|\hat{B}| = m = 2$, $\hat{x}$ is nondegenerate and we obtain $B = \hat{B}$ and $N = \{1, 2\}$. Furthermore, we obtain

$$(A_B)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned}
\bar{c}_1 &= c_1 - c_B^T (A_B)^{-1} A^1 = -1 - [-1, 2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2 \\
\bar{c}_2 &= c_2 - c_B^T (A_B)^{-1} A^2 = -4 - [-1, 2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = -4
\end{aligned}$$

Since $\bar{c} \ngeq \mathbf{0}$ and $\hat{x}$ is a nondegenerate vertex, we can conclude that $\hat{x}$ is not optimal by Proposition 17.1. The candidate improving direction is obtained by setting $d_1 = 0$ and $d_2 = 1$, and

$$d_B = -(A_B)^{-1} A_N d_N = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

Therefore,

$$d = [0, 1, -4, -2]^T .$$

Note that d is an improving direction since $c^T d = [-1, -4, -1, 2]^T [0, 1, -4, -2] = -4 = \bar{c}_2 < 0$. Let us check if d is indeed a feasible direction at $\hat{x}$. Note that $A(\hat{x} + \lambda d) = b$ for any $\lambda \in \mathbb{R}$ since $Ad = \mathbf{0}$. Consider $\hat{x} + \lambda d = [0, 0, 8, 4]^T + \lambda[0, 1, -4, -2]^T = [0, \lambda, 8 - 4\lambda, 4 - 2\lambda]^T$. Note that letting $\lambda^* = 2 > 0$, we obtain $\hat{x} + \lambda d \geq \mathbf{0}$ if $\lambda \in [0, \lambda^*]$. Therefore, d is a feasible direction at $\hat{x}$. Note that we have $d \in \hat{\mathcal{D}}$ since $\mathcal{D} = \hat{\mathcal{D}}$ for a nondegenerate vertex by Section 17.4.1 in the lecture notes.

As illustrated by this example, a linear programming problem may have multiple vertices that are optimal (i.e., (a) and (b)) such that one of them is nondegenerate and another one is degenerate.

Open Ended Problems

3 Feasible Directions and Optimality Conditions (2.5 marks)

Consider the following linear programming problem (P) in standard form:

$$(\mathrm{P}) \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Let $\bar{x} \in \mathbb{R}^n$ be an optimal solution of (P) such that $\bar{x}$ is not a vertex.

(3.1) Prove that there exists a feasible direction $\bar{d} \in \mathbb{R}^n$ at $\bar{x}$ such that $\bar{d} \neq \mathbf{0}$ and $c^T \bar{d} = 0$.

[1.5 marks]

Solution

Note that (P) is a linear programming problem (P) in standard form. By the hypothesis, the optimal value is finite since $\bar{x} \in \mathbb{R}^n$ is an optimal solution (i.e., $z^* = c^T\bar{x}$). Then, by the Fundamental Theorem of Linear Programming, there exists a basic feasible solution (i.e., a vertex), say x^*, which is also an optimal solution and we know that $x^* - \bar{x} \neq 0$ since $\bar{x}$ is not a vertex, and that $z^* = c^T\bar{x} = c^T x^*$. Let $\bar{d} = x^* - \bar{x}$. Then, since $\bar{x}$ and $x^* = \bar{x} + \bar{d}$ are feasible solutions, $\bar{d}$ is clearly a feasible direction at $\bar{x}$ (using $\lambda^* = 1$). Now, observe that $\bar{d} = x^* - \bar{x} \neq 0$ and $c^T\bar{d} = c^T(x^* - \bar{x}) = c^T x^* - c^T\bar{x} = 0$, which proves the claim.

(3.2) Prove that (P) has an infinite number of optimal solutions.

[1 mark]

Solution

From (3.1), we know that there exists a vertex x^* which is also an optimal solution and $x^* \neq \bar{x}$ since $\bar{x}$ is not a vertex. Since, the feasible region of a linear programming problem is always a polyhedron, and every polyhedron is a convex set by Remark 6.2, $(\lambda x^* + (1-\lambda)\bar{x})$ is also a feasible solution of (P) for all $\lambda \in [0, 1]$. Now we want to show that $(\lambda x^* + (1-\lambda)\bar{x})$ is an optimal solutions for each $\lambda \in [0, 1]$. By the following equality,

$$c^T(\lambda x^* + (1-\lambda)\bar{x}) = \lambda c^T x^* + (1-\lambda)c^T\bar{x} = \lambda z^* + (1-\lambda)z^* = z^*.$$

Clearly $\lambda x^* + (1-\lambda)\bar{x}$ is also an optimal solution for any $\lambda \in [0, 1]$. Since we have infinitely many real numbers in any interval $[a, b] \subseteq \mathbb{R}$ such that $b > a$, we conclude that we can find infinitely many λ values such that $\lambda \in [0, 1]$. Therefore, we have infinitely many optimal solutions.

4 Reduced Costs and Optimality Conditions (2.5 marks)

Consider the following linear programming problem in standard form:

$$(\mathrm{P}) \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Assume that A has full row rank. Let $x^* \in \mathbb{R}^n$ be a vertex with the corresponding index sets B and N.

(4.1) Suppose that $\bar{x} \in \mathbb{R}^n$ is a feasible solution of (P) such that $\bar{x} \neq x^*$. Prove that there exists an index $k \in N$ such that $\bar{x}_k > 0$.

[1 mark]

Solution

Let $x^* \in \mathbb{R}^n$ be a vertex of (P) with the corresponding index sets B and N. We have $x_B^* = (A_B)^{-1}b \geq \mathbf{0}$ and $x_N^* = \mathbf{0} \in \mathbb{R}^{n-m}$ by Proposition 13.2. Let $\bar{x} \in \mathbb{R}^n$ be a feasible solution of (P) such that $\bar{x} \neq x^*$. Suppose, for a contradiction, that the conclusion is false. Then, we have $\bar{x}_j = x_j^* = 0$ for each $j \in N$. Therefore using the same index sets B and N for the vertex x^*, we conclude that $\bar{x}_N = \mathbf{0} \in \mathbb{R}^{n-m}$. Since $A\bar{x} = A_B\bar{x}_B + A_N\bar{x}_N = A_B\bar{x}_B = b$, we obtain $\bar{x}_B = (A_B)^{-1}b$, which implies that $\bar{x}_B = x_B^*$. Since $\bar{x}_B = x_B^*$ and $\bar{x}_N = x_N^*$, we then conclude that $x^* = \bar{x}$, which is a contradiction. Therefore, we conclude that there exists an index $k \in N$ such that $\bar{x}_k > 0$.

(4.2) Consider the vertex x^* again. Suppose that reduced costs of all nonbasic variables are strictly positive, i.e.,

$$\bar{c}_j = c_j - c_B^T(A_B)^{-1}A^j > 0, \quad j \in N.$$

Prove, using (4.1), that x^* is the unique optimal solution.

[1.5 marks]

Solution

Let $x^* \in \mathbb{R}^n$ be an optimal vertex with the corresponding index sets B and N. Suppose that reduced costs of all nonbasic variables are strictly positive.

Suppose, for a contradiction, that x^* is not the unique optimal solution. Then, there exists a feasible solution $\bar{x} \in \mathbb{R}^n$ such that $\bar{x} \neq x^*$ and $\bar{x}$ is also an optimal solution. In the following, we want to derive a contradiction by considering the vector $d = \bar{x} - x^*$.

By (4.1), there exists an index $k \in N$ such that $\bar{x}_k > 0$ whereas $x_k^* = 0$ since $k \in N$, Therefore, $d_k = \bar{x}_k - x_k^* = \bar{x}_k > 0$.

Note that $Ad = A(\bar{x} - x^*) = b - b = \mathbf{0}$, i.e. $A_Bd_B + A_Nd_N = \mathbf{0}$, which implies that $d_B = -(A_B)^{-1}A_Nd_N$. Since $c^T\bar{x} = c^Tx^*$, we obtain

$$0 = c^Td = c_B^Td_B + c_N^Td_N = -c_B^T(A_B)^{-1}A_Nd_N + c_N^Td_N = \sum_{j \in N} \bar{c}_j d_j > 0,$$

where the strict inequality follows from $d_j = \bar{x}_j - x_j^* = \bar{x}_j \geq 0$ and $\bar{c}_j > 0$ for each $j \in N$, and the observation that $d_k = \bar{x}_k > 0$ and $\bar{c}_k > 0$. We obtain a contradiction. Therefore, we conclude that x^* is the unique optimal solution.