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Fundamentals of Optimization

Exercise 1 – Solutions

Remarks

- All questions that are available in the STACK quiz are duly marked. Please solve those using STACK.
- We have added marks for each question. Please note that those are purely for illustrative purposes. The exercise set will not be marked.

STACK Problems

1 Basic Concepts (3 marks)

STACK question

Decide, for each of the following three optimization problems, whether

(i) the feasible region is *empty*; or *nonempty and bounded*; or *nonempty and unbounded*;

(ii) the feasible region is a *convex set*; or a *nonconvex set*;

(iii) the objective function is *a convex function*; *a concave function*; *both convex and concave*; or *neither convex nor concave*;

(iv) the optimization problem is a *convex optimization problem*; or a *nonconvex optimization problem*;

(v) the optimization problem *is infeasible*, *is unbounded*, or *has a finite optimal value*;

(vi) write down the optimal value using the convention in the lectures (use `+inf` for $+\infty$ and `-inf` for $-\infty$);

(vii) the set of optimal solutions is *empty*; or *nonempty*;

(viii) the set of optimal solutions is a *convex set*; or a *nonconvex set.*

(1.1) $\max\{1/(x^2+1) : |x-1| \geq 2, \quad x \in \mathbb{R}\}$.

(1.2) $\min\{x^2 - 4x + 3 : -x^2 - 10x \geq 24, \quad x \in \mathbb{R}\}$.

[3 marks]

Solution

(1.1) Note that $|x-1| \geq 2$ if and only if $x \in (-\infty, -1] \cup [3, \infty)$. The feasible region is therefore given by $\mathcal{S} = (-\infty, -1] \cup [3, \infty)$. The feasible region is nonempty and unbounded since there does not exist any finite number $K \in \mathbb{R}$ such that $\mathcal{S} \subseteq [-K, K]$. $\mathcal{S}$ is a nonconvex set since $-1 \in \mathcal{S}$, $3 \in \mathcal{S}$ but $(1/2)(-1) + (1/2)(3) = 1 \notin \mathcal{S}$. The objective function is given by $f(x) = 1/(x^2+1)$, whose graph is symmetric around the y-axis. Let $x = 1$, $y = -1$, and $\lambda = 1/2$. Then,

$$f(\lambda x + (1-\lambda)y) = f(0) = 1 > \lambda f(x) + (1-\lambda) f(y) = (1/2)(1/2) + (1/2)(1/2) = 1/2,$$

which implies that f is not a convex function. Similarly, if $x = 0$, $y = 2$, and $\lambda = 1/2$, then,

$$f(\lambda x + (1-\lambda)y) = f(1) = 1/2 < \lambda f(x) + (1-\lambda)f(y) = (1/2)(1) + (1/2)(1/5) = 3/5,$$

which implies that f is not a concave function. Therefore, f is neither convex nor concave. Recall that maximizing $f(x)$ is equivalent to minimizing $-f(x)$. Since $-f$ is not a convex function (i.e., f is not a concave function), the optimization problem is a nonconvex optimization problem. By computing the first derivative of the objective function given by

$$f'(x) = -\frac{2x}{x^2+1},$$

you can easily see that f is strictly decreasing on $[3, \infty)$ and strictly increasing on $(-\infty, -1]$. Therefore, the maximum objective function value is given by $\max\{f(-1), f(3)\} = \max\{1/2, 1/10\} = 1/2$, which implies that the optimal value is given by $z^* = 1/2$. Therefore, the optimization problem has a finite optimal value. In this example, only one feasible solution attains the optimal value, i.e., $\mathcal{S}^* = \{-1\}$. Therefore, the set of optimal solutions is nonempty. Finally, $\mathcal{S}^*$ is a singleton (i.e., it contains only one element). Therefore, it is a convex set by Remark 2 in Section 3.2 in the lecture notes.

(1.2) Note that $-x^2 - 10x \geq 24$ if and only if $-x^2 - 10x - 25 \geq -1$ if and only if $-(x+5)^2 \geq -1$ if and only if $|x+5| \leq 1$. The feasible region is therefore given by $\mathcal{S} = [-6, -4]$. The feasible region is nonempty and bounded since $\mathcal{S} \subseteq [-K, K]$ for $K = 6$. $\mathcal{S}$ is a convex set since $\mathcal{S} = \{x \in \mathbb{R} : x \geq -6, \quad x \leq -4\}$, i.e., $\mathcal{S}$ is given by the intersection of the two half spaces in $\mathbb{R}$, each of which is a convex set by Corollary 4.7, and convexity is preserved under taking intersections by Remark 3 in Section 3.2 in the lecture notes. The objective function is given by $f(x) = x^2 - 4x + 3$. Let $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $\lambda \in [0, 1]$. Then,

$$\begin{aligned}
f(\lambda x + (1-\lambda)y) &= (\lambda x + (1-\lambda)y)^2 - 4(\lambda x + (1-\lambda)y) + 3 \\
&= \lambda^2 x^2 + 2\lambda(1-\lambda)xy + (1-\lambda)^2 y^2 - 4\lambda x - 4(1-\lambda)y + 3 \\
&= \lambda(x^2 - 4x + 3) + (1-\lambda)(y^2 - 4y + 3) \\
&\quad + (\lambda^2 - \lambda)x^2 + 2\lambda(1-\lambda)xy + ((1-\lambda)^2 - (1-\lambda))y^2 \\
&= \lambda f(x) + (1-\lambda)f(y) + \lambda(1-\lambda)[-x^2 + 2xy - y^2] \\
&= \lambda f(x) + (1-\lambda)f(y) - \lambda(1-\lambda)(x-y)^2 \\
&\leq \lambda f(x) + (1-\lambda)f(y),
\end{aligned}$$

where we used $\lambda \in [0, 1]$ and $(x-y)^2 \geq 0$ to derive the inequality in the last line. It follows that f is a convex function. By using a similar argument, one can show that $-x^2 - 10x$ is a concave function. Therefore, this is a convex optimization problem since the objective function is convex and the feasible region is given by the superlevel set of a concave function. By computing the first derivative of the objective function given by

$$f'(x) = 2x - 4,$$

you can easily see that f is strictly decreasing on $[-6, -4]$. Therefore, the minimum objective function value is given by $f(-4) = 35$, which implies that the optimal value is given by $z^* = 35$. Therefore, the optimization problem has a finite optimal value. In this example, only one feasible solution attains the optimal value, i.e., $\mathcal{S}^* = \{-4\}$. Therefore, the set of optimal solutions is nonempty. Finally, $\mathcal{S}^*$ is a singleton (i.e., it contains only one element). Therefore, it is a convex set by Remark 2 in Section 3.2 in the lecture notes.

2 Level Sets, Sublevel Sets, Superlevel Sets, and Epigraphs (2 marks)

STACK question

Decide, for each of the two functions,

(i) whether $\text{epi}(f)$ is a *convex set* or *nonconvex set*;

(ii) whether the sublevel set $\mathcal{L}^-_\alpha(f)$, where $\alpha = 0$, is a *convex set* or *nonconvex set*;

(iii) whether the level set $\mathcal{L}_\alpha(f)$, where $\alpha = 1$, is a *convex set* or *nonconvex set*;

(iv) whether the superlevel set $\mathcal{L}^+_\alpha(f)$, where $\alpha = 1$, is a *convex set* or *nonconvex set*.

(2.1) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = \max\{|x_1|, |x_2|\}$.

(2.2) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x_1^2 - x_2^2$.

[2 marks]

Solution

(2.1) Considering the function in (a), we have

$$\text{epi}(f) = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : z \geq \max\{|x_1|, |x_2|\}\}.$$

Let $(x, z_1) \in \text{epi}(f)$, $(y, z_2) \in \text{epi}(f)$, and let $\lambda \in [0, 1]$. We need to show that $\lambda(x, z_1) + (1 - \lambda)(y, z_2) = (\lambda x + (1 - \lambda)y, \lambda z_1 + (1 - \lambda)z_2) \in \text{epi}(f)$, i.e.,

$$\lambda z_1 + (1 - \lambda)z_2 \geq \max\{|\lambda x_1 + (1 - \lambda)y_1|, |\lambda x_2 + (1 - \lambda)y_2|\}. \tag{1}$$

Since $(x, z_1) \in \text{epi}(f)$ and $(y, z_2) \in \text{epi}(f)$, we have

$$z_1 \geq \max\{|x_1|, |x_2|\}, \quad z_2 \geq \max\{|y_1|, |y_2|\}.$$

This implies that $z_1 \geq |x_1|, z_1 \geq |x_2|, z_2 \geq |y_1|, z_2 \geq |y_2|$. Since $\lambda \in [0, 1]$, we have $\lambda z_1 \geq \lambda|x_1|, \lambda z_1 \geq \lambda|x_2|, (1-\lambda)z_2 \geq (1-\lambda)|y_1|, (1-\lambda)z_2 \geq (1-\lambda)|y_2|$. Therefore, $\lambda z_1+(1-\lambda)z_2 \geq \lambda|x_1| + (1 - \lambda)|y_1|$ and $\lambda z_1 + (1 - \lambda)z_2 \geq \lambda|x_2| + (1 - \lambda)|y_2|$, i.e.,

$$\lambda z_1 + (1 - \lambda)z_2 \geq \max\{\lambda|x_1| + (1 - \lambda)|y_1|, \lambda|x_2| + (1 - \lambda)|y_2|\}.$$

Since, for any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, we have

$$|\lambda\alpha + (1 - \lambda)\beta| \leq |\lambda\alpha| + |(1 - \lambda)\beta| = \lambda|\alpha| + (1 - \lambda)|\beta|$$

by the triangle inequality, we obtain

$$\begin{aligned} \lambda z_1 + (1 - \lambda)z_2 &\geq \max\{\lambda|x_1| + (1 - \lambda)|y_1|, \lambda|x_2| + (1 - \lambda)|y_2|\} \\ &\geq \max\{|\lambda x_1 + (1 - \lambda)y_1|, |\lambda x_2 + (1 - \lambda)y_2|\}, \end{aligned}$$

which establishes (1). It follows that $\text{epi}(f)$ is a convex set. Note that this implies that f is a convex function by Proposition 3.1.

Note that $f(x) \geq 0$ for each $x \in \mathbb{R}^2$, which implies that $\mathcal{L}^-_\alpha(f) = \emptyset$ for each $\alpha < 0$, which is a convex set by Remark 1 in Section 3.2 in the lecture notes. For each $\alpha \geq 0$, $f(x) \leq \alpha$ if and only if $\max\{|x_1|, |x_2|\} \leq \alpha$ if and only if $|x_1| \leq \alpha$ and $|x_2| \leq \alpha$. Therefore, we obtain all the points in the interior and the boundary of the square of side length 2α centred at the origin whose four corner points are given by $[\pm\alpha, \pm\alpha]^T$. Therefore, for $\alpha = 0$, we get $\mathcal{L}^-_\alpha(f) = \{[0, 0]^T\}$, which is a convex set since it is a singleton. You can easily show that $\mathcal{L}^-_\alpha(f)$ is a convex set for each $\alpha \geq 0$ since it is given by the intersection of four half spaces in $\mathbb{R}^2$.

Similarly, we obtain $\mathcal{L}_\alpha(f) = \emptyset$ for each $\alpha < 0$, which is a convex set. For $\alpha \geq 0$, note that $f(x) = \alpha$ if and only if (i) $|x_1| = \alpha$ and $|x_2| \leq \alpha$; or (ii) $|x_2| = \alpha$ and $|x_1| \leq \alpha$. Therefore, we obtain (boundaries of) squares of side length 2α centred at the origin whose four corner points are given by $[\pm\alpha, \pm\alpha]^T$. Therefore, for $\alpha = 1$, we get

$$\mathcal{L}_\alpha(f) = \{x \in \mathbb{R}^2 : x_1 \in \{-1, 1\}, \quad x_2 \in [-1, 1]\} \cup \{x \in \mathbb{R}^2 : x_1 \in [-1, 1], \quad x_2 \in \{-1, 1\}\},$$

which is a nonconvex set since $x = [1,1]^T \in \mathcal{L}_\alpha(f)$, $y = [-1,-1]^T \in \mathcal{L}_\alpha(f)$, but the midpoint $[0,0]^T \notin \mathcal{L}_\alpha(f)$. Similarly, you can show that $\mathcal{L}_\alpha(f)$ is a nonconvex set for each $\alpha > 0$.

Finally, we obtain $\mathcal{L}_\alpha^+(f) = \mathbb{R}^2$ for each $\alpha \le 0$ since $f(x) \ge 0$ for each $x \in \mathbb{R}^2$, which is obviously a convex set. For each $\alpha > 0$, we obtain all the points outside of the square of side length 2α centred at the origin whose four corner points are given by $[\pm\alpha, \pm\alpha]^T$, including the boundary points. Therefore, for $\alpha = 1$, we get

$$\mathcal{L}_\alpha^+(f) = \{x \in \mathbb{R}^2 : x_1 \in (-\infty, -1] \cup [1, \infty)\} \cup \{x \in \mathbb{R}^2 : x_2 \in (-\infty, -1] \cup [1, \infty)\}.$$

This is a nonconvex set since $[1,1]^T \in \mathcal{L}_\alpha^+(f)$ and $[-1,-1]^T \in \mathcal{L}_\alpha^+(f)$ but the midpoint $[0,0]^T \notin \mathcal{L}_\alpha^+(f)$. Similarly, you can show that $\mathcal{L}_\alpha^+(f)$ is a nonconvex set for each $\alpha > 0$.

(2.2) Considering the function in (b), we have

$$\operatorname{epi}(f) = \{(x,z) \in \mathbb{R}^2 \times \mathbb{R} : z \ge x_1^2 - x_2^2\}.$$

Let $x = [0,1]^T$, $z_1 = -1$, $y = [0,-1]^T$, $z_2 = -1$, and $\lambda = 1/2$. Then, $(x, z_1) \in \operatorname{epi}(f)$ and $(y, z_2) \in \operatorname{epi}(f)$. However, $\lambda(x, z_1) + (1-\lambda)(y, z_2) = ([0,0]^T, -1) \notin \operatorname{epi}(f)$, which implies that $\operatorname{epi}(f)$ is a nonconvex set. Note that this implies that f is a nonconvex function by Proposition 3.1.

If we fix $\alpha \in \mathbb{R}$ and $x_2 = \beta$, then we obtain $f(x) \le \alpha$ if and only if $x_1^2 \le \alpha + \beta^2$. If $\alpha + \beta^2 < 0$, then no such $x_1 \in \mathbb{R}$ exists. Otherwise, $|x_1| \le \sqrt{\alpha + \beta^2}$. Therefore,

$$\mathcal{L}_\alpha^-(f) = \bigcup_{\beta \in \mathbb{R}: \alpha + \beta^2 \ge 0} \left\{[x_1, \beta]^T : -\sqrt{\alpha + \beta^2} \le x_1 \le \sqrt{\alpha + \beta^2}\right\}, \quad \alpha \in \mathbb{R}.$$

For $\alpha = 0$, since $\beta^2 + \alpha \ge 0$ for any $\beta \in \mathbb{R}$, we get

$$\mathcal{L}_\alpha^-(f) = \bigcup_{\beta \in \mathbb{R}} \left\{[x_1, \beta]^T : -|\beta| \le x_1 \le |\beta|\right\}.$$

This is a nonconvex set since, for instance, for $\beta = -1$, we have $x = [1,-1]^T \in \mathcal{L}_\alpha^-(f)$ and for $\beta = 1$, we have $y = [1,1]^T \in \mathcal{L}_\alpha^-(f)$. However, for $\lambda = 1/2$, $\lambda x + (1-\lambda)y = [1,0]^T \notin \mathcal{L}_\alpha^-(f)$.

Similarly, if we fix $\alpha \in \mathbb{R}$ and $x_2 = \beta$, then we obtain $f(x) = \alpha$ if and only if $x_1^2 = \alpha + \beta^2$. If $\alpha + \beta^2 < 0$, then no such $x_1 \in \mathbb{R}$ exists. Otherwise, $x_1 = \pm\sqrt{\alpha + \beta^2}$, i.e., there are at most two choices for each choice of x_2, which are symmetric around 0. Therefore,

$$\mathcal{L}_\alpha(f) = \bigcup_{\beta \in \mathbb{R}: \alpha + \beta^2 \ge 0} \left\{[\sqrt{\alpha + \beta^2}, \beta]^T, [-\sqrt{\alpha + \beta^2}, \beta]^T\right\}, \quad \alpha \in \mathbb{R}.$$

For $\alpha = 1$, since $\beta^2 + 1 \ge 0$ for any $\beta \in \mathbb{R}$, we get

$$\mathcal{L}_\alpha(f) = \bigcup_{\beta \in \mathbb{R}} \left\{\sqrt{1 + \beta^2}, \beta]^T, [-\sqrt{1 + \beta^2}, \beta]^T\right\}.$$

This is a nonconvex set since, for instance, for $\beta = -1$, we have $x = [\sqrt{2}, -1]^T \in \mathcal{L}_\alpha(f)$ and $y = [-\sqrt{2}, -1]^T \in \mathcal{L}_\alpha(f)$. However, for $\lambda = 1/2$, $\lambda x + (1-\lambda)y = [0,-1]^T \notin \mathcal{L}_\alpha(f)$.

Finally,

$$\mathcal{L}_\alpha^+(f) = \bigcup_{\beta \in \mathbb{R}: \alpha + \beta^2 \ge 0} \left\{[x_1, \beta]^T : x_1 \in (-\infty, -\sqrt{\alpha + \beta^2}] \cup [\sqrt{\alpha + \beta^2}, \infty)\right\}, \quad \alpha \in \mathbb{R}.$$

For $\alpha = 1$, since $\beta^2 + \alpha \ge 0$, we get

$$\mathcal{L}_\alpha^+(f) = \bigcup_{\beta \in \mathbb{R}} \left\{[x_1, \beta]^T : x_1 \in (-\infty, -\sqrt{1 + \beta^2}] \cup [\sqrt{1 + \beta^2}, \infty)\right\}, \quad \alpha \in \mathbb{R}.$$

You can easily show that $\mathcal{L}_\alpha^+(f)$ is a nonconvex set by choosing, for $\beta = 0$, $x = [-1,0]^T \in \mathcal{L}_\alpha^+(f)$, $y = [1,0]^T \in \mathcal{L}_\alpha^+(f)$, but for $\lambda = 1/2$, $\lambda x + (1-\lambda)y = [0,0]^T \notin \mathcal{L}_\alpha^+(f)$.

Open Ended Problems

3 Level Sets and Sublevel Sets (2.5 marks)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function.

(3.1) By Proposition 4.1, if f is a linear function, then the level set $\mathcal{L}_\alpha(f)$ is a convex set for each $\alpha \in \mathbb{R}$. Consider the converse proposition given below:

If $\mathcal{L}_\alpha(f)$ is a convex set for each $\alpha \in \mathbb{R}$, then f is a linear function.

Either prove this proposition or give a counterexample (i.e., find an example $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies the hypothesis but does not satisfy the conclusion).

[1.5 marks]

Solution

While it may be tempting to think that the proposition is probably true, we may want to be a bit more careful. In particular, note that there are various examples of convex sets, including the empty set and sets with a single element. Based on this, let us consider the following function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$. We claim that $\mathcal{L}_\alpha(f)$ is a convex set for each $\alpha \in \mathbb{R}$. Indeed, for each $\alpha \in \mathbb{R}$, $\mathcal{L}_\alpha(f) = \{\alpha^{1/3}\}$. Therefore, $\mathcal{L}_\alpha(f)$ is a convex set since it contains a single element (see Remark 2 in Section 3.2 in the lecture notes). However, we claim that f is not a linear function. To see this, let $x = 1$ and $y = 2$. Then, we have $f(x) = 1$ and $f(y) = 8$. However, $f(x + y) = f(3) = 27 \neq 1 + 8 = f(x) + f(y)$. Therefore, f is not a linear function and the proposition is false.

(3.2) By Proposition 4.2, if f is a convex function, then the sublevel set $\mathcal{L}^-_\alpha(f)$ is a convex set for each $\alpha \in \mathbb{R}$. Consider the converse proposition given below:

If $\mathcal{L}^-_\alpha(f)$ is a convex set for each $\alpha \in \mathbb{R}$, then f is a convex function.

Either prove this proposition or give a counterexample (i.e., find an example $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies the hypothesis but does not satisfy the conclusion).

[1 marks]

Solution

Again, it may be tempting to think that the proposition is probably true. However, by following the discussion in (3.1), we may want to be a bit more careful. In particular, let us consider the same function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ as in (6.1). We claim that $\mathcal{L}^-_\alpha(f)$ is a convex set for each $\alpha \in \mathbb{R}$. Indeed, for each $\alpha \in \mathbb{R}$, $\mathcal{L}^-_\alpha(f) = (-\infty, \alpha^{1/3}]$, which is a half line. You can easily show that a half line is a convex set. Therefore, $\mathcal{L}^-_\alpha(f)$ is a convex set for each $\alpha \in \mathbb{R}$. However, we claim that f is not a convex function. To see this, let $x = -1$, $y = 0$, and $\lambda = 1/2$. Then, we have

$$f(\lambda x + (1-\lambda)y) = f(-1/2) = -1/8 \not\leq \lambda f(x) + (1-\lambda)f(y) = (1/2)\cdot(-1) + (1/2)\cdot 0 = -1/2.$$

Therefore, f is not a convex function and the proposition is false.

Note that any increasing or decreasing one-dimensional nonconvex function would actually yield a counterexample to this claim.

4 Vertices of Convex Sets (2.5 marks)

(4.1) Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a nonempty convex set and let $\hat{x} \in \mathcal{C}$. Prove the following result:

If $\hat{x}$ is a vertex of $\mathcal{C}$, then there does not exist a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $\hat{x} - d \in \mathcal{C}$ and $\hat{x} + d \in \mathcal{C}$.

[1.5 marks]

Solution

Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a nonempty convex set and let $\hat{x} \in \mathcal{C}$ be a vertex of $\mathcal{C}$. Then, we know from Definition 6.1 in the lecture that there exists a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : a^T x = \alpha\}$ and a halfspace $\mathcal{H}^+ = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}$, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$, such that $\mathcal{C} \cap \mathcal{H} = \{\hat{x}\}$ and $\mathcal{C} \subseteq \mathcal{H}^+$.

Suppose, for a contradiction, that there exists a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $\hat{x} - d \in \mathcal{C}$ and $\hat{x} + d \in \mathcal{C}$. Then, since $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, $\hat{x} - d \in \mathcal{C}$, $\hat{x} + d \in \mathcal{C}$, $\mathcal{C} \cap \mathcal{H} = \{\hat{x}\}$ and $\mathcal{C} \subseteq \mathcal{H}^+$, we know from the proof of Proposition 6.1 that $a^T(\hat{x} - d) > \alpha$ and $a^T(\hat{x} + d) > \alpha$. Since $a^T \hat{x} = \alpha$, we obtain $a^T d < 0$ and $a^T d > 0$, respectively. Clearly, this is a contradiction. Therefore, there does not exist a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $\hat{x} - d \in \mathcal{C}$ and $\hat{x} + d \in \mathcal{C}$.

(4.2) Consider the following set

$$\mathcal{C} = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 1\}.$$

By relying on (4.1), show that $\mathcal{C}$ does not contain any vertices.

[1 mark]

Solution

Suppose, for a contradiction, that there exists $\hat{x} \in \mathcal{C}$ such that $\hat{x}$ is a vertex. Then, $\hat{x}_1 + \hat{x}_2 \leq 1$. Define $d = [-1, 1]^T \in \mathbb{R}^2$. Then, we claim that (i) $\hat{x} - d \in \mathcal{C}$ and (ii) $\hat{x} + d \in \mathcal{C}$. Indeed, (i) $(\hat{x}_1 - d_1) + (\hat{x}_2 + d_2) = \hat{x}_1 + 1 + \hat{x}_2 - 1 = \hat{x}_1 + \hat{x}_2 \leq 1$, which implies that $\hat{x} - d \in \mathcal{C}$ and (ii) $(\hat{x}_1 + d_1) + (\hat{x}_2 - d_2) = \hat{x}_1 - 1 + \hat{x}_2 + 1 = \hat{x}_1 + \hat{x}_2 \leq 1$, which implies that $\hat{x} + d \in \mathcal{C}$. By using the contrapositive of (4.1), we conclude that $\hat{x}$ cannot be a vertex of $\mathcal{C}$, which is a contradiction. Therefore, we conclude that $\mathcal{C}$ does not contain any vertices.