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Fundamentals of Optimization

Homework 2 – Solutions

Instructions

1. You should attempt all questions.
2. The total marks for this assignment are 10.
3. The assignment consists of STACK questions (5/10 marks) and *open-ended* questions (5/10 marks).
4. All STACK questions are duly marked and are available in the STACK quiz. You **must solve those by completing the STACK quiz**.
5. For the open-ended questions, please write down your solutions in a concise and reproducible way and remember to justify every step using appropriate references when necessary. Failing to do so may result in deductions.
6. The strict deadline for completing the quiz and handing-in your solutions for the open-ended questions is **noon (12:00) on Friday, 28 October 2022**.
7. For the open-ended questions, please upload a **single PDF**. For some useful suggestions, please see Course Information → Tips for Creating a PDF File for Submission on the Learn page.

1 Basic Solutions and Basic Feasible Solutions (3 marks)

STACK question

Consider the following polyhedron

$$\mathcal{P} = \{x \in \mathbb{R}^3 : 2x_1 - x_2 + x_3 \leq -1,\ -x_1 + 2x_2 \leq 2,\ x_1 + x_3 \geq -1,\ -2x_1 + x_2 + x_3 = 0,\ x_2 + x_3 \geq -1\}.$$

Decide, for each of the points $\hat{x}$ given below, whether $\hat{x}$ is *infeasible and not a basic solution*, *feasible but not a basic feasible solution*, *a basic solution but infeasible*, or *a basic feasible solution.*

(1.1) $\hat{x} = [0, 1, -1]^T$.

(1.2) $\hat{x} = [-1/3, 0, -2/3]^T$.

(1.3) $\hat{x} = [-1/2, 3/4, -7/4]^T$.

(1.4) $\hat{x} = [-1, -1, 0]^T$.

(1.5) $\hat{x} = [-1/2, -1/2, -1/2]^T$.

[3 marks]

Solution

By labelling the constraints $1, \ldots, 5$, respectively, we have $M_1 = \{3, 5\}$, $M_2 = \{1, 2\}$, and $M_3 = \{4\}$.

(1.1) We have $I(\hat{x}) = \{2, 3, 4\}$. Note that $M_3 \subseteq I(\hat{x})$. As $|I(\hat{x})| = 3 = n$, the set $\{a^i : i \in I(\hat{x})\}$ may or may not span $\mathbb{R}^3$. We need to check whether the three vectors are linearly independent.

$$\det \begin{pmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -5 \neq 0\,.$$

Therefore, $\hat{x}$ is a basic solution. You can easily check that $\hat{x}$ satisfies the remaining two constraints. Therefore, $\hat{x}$ is a basic feasible solution.

(1.2) We have $I(\hat{x}) = \{3, 4\}$. Note that $M_3 \subseteq I(\hat{x})$. The set $\{a^i : i \in I(\hat{x})\}$ contains only two elements. Therefore, the set $\{a^i : i \in I(\hat{x})\}$ cannot span $\mathbb{R}^3$. As $\hat{x}$ satisfies each of the remaining three constraints, $\hat{x}$ is feasible but not a basic solution.

(1.3) We have $I(\hat{x}) = \{2, 4, 5\}$. Note that $M_3 \subseteq I(\hat{x})$. As $|I(\hat{x})| = 3 = n$, the set $\{a^i : i \in I(\hat{x})\}$ may or may not span $\mathbb{R}^3$. We need to check whether the three vectors are linearly independent.

$$\det \begin{pmatrix} -1 & -2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 4 \neq 0\,.$$

Therefore, $\hat{x}$ is a basic solution. You can easily check that $\hat{x}$ violates the third constraint. Therefore, $\hat{x}$ is a basic solution but infeasible.

(1.4) We have $I(\hat{x}) = \{1, 3, 5\}$. Note that $M_3 = \{4\}$ and $M_3 \not\subseteq I(\hat{x})$. It follows that $\hat{x}$ is not a basic solution. Furthermore, $\hat{x}$ clearly violates the fourth constraint. Therefore, $\hat{x}$ is infeasible and not a basic solution. (Note that we do not need to check the linear independence condition in this case.)

(1.5) We have $I(\hat{x}) = \{1, 3, 4, 5\}$. Note that $M_3 \subseteq I(\hat{x})$. The set $\{a^i : i \in I(\hat{x})\}$ contains 4 vectors, and may or may not span $\mathbb{R}^3$. We just need to check if this set contains three linearly independent vectors. For instance, by choosing a^1, a^3, and a^4, we obtain:

$$\det \begin{pmatrix} 2 & 1 & -2 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 2 \neq 0\,.$$

Therefore, $\hat{x}$ is a basic solution. You can easily check that $\hat{x}$ satisfies the remaining (i.e., second) constraint. Therefore, $\hat{x}$ is a (degenerate) basic feasible solution.

2 Graphical Method (2 marks)

STACK question

Consider the following polyhedron:

$$\mathcal{P} \;=\; \{[x_1, x_2]^T \in \mathbb{R}^2 : x_1 \geq 0,\, x_1 + x_2 \geq 1,\, -x_1 + x_2 \leq 3,\, x_2 \geq 0\}\,.$$

Using the graphical method, determine, for each of the following objective functions, the optimal value denoted by z^* (use `+inf` for $+\infty$ and `-inf` for $-\infty$), and whether the set of optimal solutions, denoted by $\mathcal{P}^*$, is either *empty*, a *single vertex*, a *line segment*, a *half line*, or $\mathcal{P}^* = \mathcal{P}$.

[2 marks]

(2.1) $\min\{2x_1 + 2x_2 : x \in \mathcal{P}\}$.

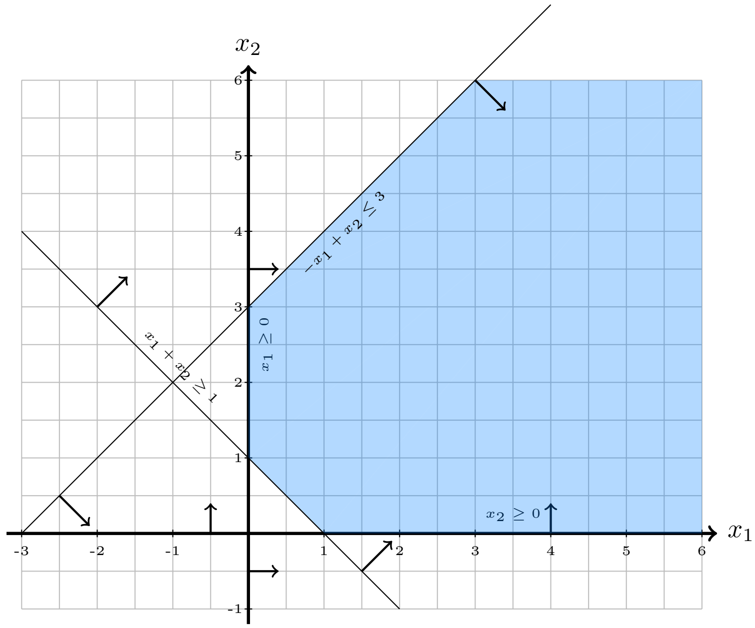

Figure 1: Feasible region for Question 3.

Solution

Since this is a minimization problem, the improving direction is given by $-c = [-2, -2]^T$, which points towards the southwest direction and is perpendicular to the boundary of the constraint given by $x_1 + x_2 \geq 1$. Therefore, all the points on the line segment between $[0, 1]^T$ and $[1, 0]^T$ are optimal, i.e., $\mathcal{P}^* = \{\lambda[1, 0]^T + (1 - \lambda)[0, 1]^T : \lambda \in [0, 1]\}$. Therefore, it is a line segment. Substituting any point on this line segment into the objective function, say $x^* = [1, 0]^T$, we obtain $z^* = 2 \cdot (1) + 2 \cdot 0 = 2$.

(2.2) $\max\{-4x_1 + 2x_2 : x \in \mathcal{P}\}$.

Solution

Since this is a maximization problem, the improving direction is given by $c = [-4, 2]^T$, which points in the northwest direction. You can see from Figure 1 that the unique optimal solution is given by the intersections of the boundary of the constraints $-x_1 + x_2 \leq 3$ and $x_1 \geq 0$. Replacing each inequality by equality and solving for the resulting system, we obtain $x^* = [0, 3]^T$, i.e., $\mathcal{P}^* = \{x^*\}$. Therefore, it is given by a unique vertex. Substituting it into the objective function, we obtain $z^* = (-4) \cdot 0 + 2 \cdot 3 = 6$.

(2.3) $\min\{-4x_1 + 2x_2 : x \in \mathcal{P}\}$.

Solution

Since this is a minimization problem, the improving direction is given by $-c = [4, -2]^T$, which points towards the southeast direction. You can verify that you can move the level sets of the objective function along this direction indefinitely without ever leaving the feasible region. It follows that the problem is unbounded, i.e., $z^* = -\infty$ since this is a minimization problem. Therefore, there is no optimal solution, i.e., $\mathcal{P}^* = \emptyset$.

(2.4) $\max\{-4x_1 + 4x_2 : x \in \mathcal{P}\}$.

Solution

Since this is a maximization problem, the improving direction is given by $c = [-4, 4]^T$, which points in the northwest direction. Note that the improving direction is perpendicular to the boundary of the constraint $-x_1 + x_2 \leq 3$. Therefore, any feasible solution on the half line starting at $[0, 3]^T$ towards the direction $[1, 1]^T$ is an optimal solution, i.e., $\mathcal{P}^* = \{[0, 3]^T + \lambda[1, 1]^T : \lambda \geq 0\}$. Therefore, it is a half line. Substituting any point on this half line into the objective function, say $x^* = [0, 3]^T$, we obtain $z^* = (-4) \cdot (0) + 4 \cdot 3 = 12$.

Open Ended Problems

3 Polyhedra in Standard Form (1 mark)

(3.1) Convert the following general linear programming problem into standard form:

$$
\begin{array}{lllllllllll}
\max & -2x_1 & - & x_2 & + & 3x_3 & - & 2x_4 & & \\
\text{s.t.} & & & & & & & & & \\
 & 2x_1 & - & x_2 & + & x_3 & - & x_4 & = & 1 \\
 & -x_1 & + & 2x_2 & - & 3x_3 & + & x_4 & \leq & 2 \\
 & x_1 & + & 2x_2 & - & x_3 & + & 2x_4 & \geq & 4 \\
 & x_1 \geq 0 & , & x_2 \leq 0 & , & x_3 \leq 0 & . & & &
\end{array}
$$

Remark It is irrelevant whether the problem is actually feasible or not.

[1 mark]

Solution

- Since it is a maximization problem, we need to negate the objective function to convert it into a minimization problem (i.e., $\min 2x_1 + x_2 - 3x_3 + 2x_4$).
- For the second and third constraints, we need to define two new nonnegative variables s_1 and s_2 that we add and subtract, respectively.
- Concerning the nonpositive variables x_2 and x_3, we can replace x_2 by the new nonnegative variable $x_2^- = -x_2$, and x_3 by the new nonnegative variable $x_3^- = -x_3$.
- Concerning the free variable x_4, we can replace x_4 by the difference of two new nonnegative variables x_4^+ and x_4^- through $x_4 = x_4^+ - x_4^-$.

This results in the following equivalent linear programming problem in standard form:

$$
\begin{array}{llllllllllllllll}
\min & 2x_1 & - & x_2^- & + & 3x_3^- & + & 2x_4^+ & - & 2x_4^- & & & & & & \\
\text{s.t.} & & & & & & & & & & & & & & & \\
 & 2x_1 & + & x_2^- & - & x_3^- & - & x_4^+ & + & x_4^- & & & & & = & 1 \\
 & -x_1 & - & 2x_2^- & + & 3x_3^- & + & x_4^+ & - & x_4^- & + & s_1 & & & = & 2 \\
 & x_1 & - & 2x_2^- & + & x_3^- & + & 2x_4^+ & - & 2x_4^- & & & - & s_2 & = & 4 \\
 & x_1 \geq 0 & , & x_2^- \geq 0 & , & x_3^- \geq 0 & , & x_4^+ \geq 0 & , & x_4^- \geq 0 & , & s_1 \geq 0 & , & s_2 \geq 0. & &
\end{array}
$$

Note that this linear programming problem, which is in standard form, is equivalent to the original one in the sense that there is a one-to-one correspondence between their feasible solutions. Therefore, there is also a one-to-one correspondence between their optimal solutions (if any).

4 Polytopes vs Polyhedra (2 marks)

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron given by

$$
\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T x \geq b_i, & i \in M_1, \\ (a^i)^T x \leq b_i, & i \in M_2, \\ (a^i)^T x = b_i, & i \in M_3 \end{array} \right\},
$$

where M_1, M_2, and M_3 are finite index sets, $a^i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for each $i \in M_1 \cup M_2 \cup M_3$. Let us define the following set:

$$\mathcal{R} = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T d \geq 0, & i \in M_1, \\ (a^i)^T d \leq 0, & i \in M_2, \\ (a^i)^T d = 0, & i \in M_3 \end{array} \right\}.$$

(4.1) Prove the following result:

For each $\hat{x} \in \mathcal{P}$ and each $\hat{d} \in \mathcal{R}$, we have $\hat{x} + \lambda \hat{d} \in \mathcal{P}$ for each real number $\lambda \geq 0$.

[1 mark]

Solution

Let $\hat{x} \in \mathcal{P}$ and $\hat{d} \in \mathcal{R}$. Then, for any $\lambda \geq 0$, we have

$$\begin{array}{rcll} (a^i)^T(\hat{x} + \lambda \hat{d}) & = & \underbrace{(a^i)^T \hat{x}}_{\geq b_i} + \underbrace{\lambda}_{\geq 0} \underbrace{(a^i)^T \hat{d}}_{\geq 0} \geq b_i, & i \in M_1, \\ (a^i)^T(\hat{x} + \lambda \hat{d}) & = & \underbrace{(a^i)^T \hat{x}}_{\leq b_i} + \underbrace{\lambda}_{\geq 0} \underbrace{(a^i)^T \hat{d}}_{\leq 0} \leq b_i, & i \in M_2, \\ (a^i)^T(\hat{x} + \lambda \hat{d}) & = & \underbrace{(a^i)^T \hat{x}}_{= b_i} + \underbrace{\lambda}_{\geq 0} \underbrace{(a^i)^T \hat{d}}_{= 0} = b_i, & i \in M_3, \end{array}$$

which implies that $\hat{x} + \lambda \hat{d} \in \mathcal{P}$ for each $\lambda \geq 0$.

(4.2) Prove the following result:

If $\mathcal{P} \subseteq \mathbb{R}^n$ is a polytope, then $\mathcal{R} = \{\mathbf{0}\}$ (i.e., $\mathcal{R}$ consists only of the vector of all zeroes $\mathbf{0} \in \mathbb{R}^n$).

[1 mark]

Solution

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polytope (i.e., a bounded polyhedron). Clearly, $d = \mathbf{0} \in \mathcal{R}$ since it satisfies each of the constraints defining $\mathcal{R}$. Therefore, $\{\mathbf{0}\} \subseteq \mathcal{R}$. Suppose, for a contradiction, that $\mathcal{R} \neq \{\mathbf{0}\}$. Then, there exists $\hat{d} \in \mathcal{R} \backslash \{\mathbf{0}\}$. By Problem (4.1), $\hat{x} + \lambda \hat{d} \in \mathcal{P}$ for each $\lambda \geq 0$. It follows that $\mathcal{P}$ is not bounded since it contains a half line and a half line cannot be bounded. To see this, note that there exists $i \in \{1, \ldots, n\}$ such that $\hat{d}_i \neq 0$ since $\hat{d} \in \mathcal{R} \backslash \{\mathbf{0}\}$. Therefore, $\hat{x}_i + \lambda \hat{d}_i \to \pm\infty$ as $\lambda \to \infty$, depending on whether $\hat{d}_i < 0$ or $\hat{d}_i > 0$, which implies that there does not exist $K \in \mathbb{R}$ such that $|x_j| \leq K$ for all $j = 1, \ldots, n$ and for all $x \in \mathcal{P}$. This is a contradiction, which implies that we have $\mathcal{R} = \{\mathbf{0}\}$.

5 Existence of Vertices in Polyhedra (2 marks)

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron given by

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T x \geq b_i, & i \in M_1, \\ (a^i)^T x \leq b_i, & i \in M_2, \\ (a^i)^T x = b_i, & i \in M_3 \end{array} \right\},$$

where M_1, M_2, and M_3 are finite index sets, $a^i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for each $i \in M_1 \cup M_2 \cup M_3$. Let us define the following set:

$$\mathcal{R} = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T d \geq 0, & i \in M_1, \\ (a^i)^T d \leq 0, & i \in M_2, \\ (a^i)^T d = 0, & i \in M_3 \end{array} \right\}.$$

(5.1) Prove the following result:

> $\mathcal{P} \subseteq \mathbb{R}^n$ has no vertices if and only if there exists $\hat{d} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $\hat{d} \in \mathcal{R}$ and $-\hat{d} \in \mathcal{R}$.

[2 marks]

Solution

Since this is an if and only if statement, we need to prove both implications:

$\Rightarrow$: Suppose that $\mathcal{P} \subseteq \mathbb{R}^n$ is a nonempty polyhedron with no vertices. By Proposition 8.1, $\mathcal{P}$ contains a line, i.e., there exists $\hat{x} \in \mathcal{P}$ and $\hat{d} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $\hat{x} + \lambda\hat{d} \in \mathcal{P}$ for each $\lambda \in \mathbb{R}$. Then, we claim that $(a^i)^T\hat{d} = 0$ for each $i \in M_1 \cup M_2 \cup M_3$. For any $i \in M_1$, we have

$$(a^i)^T(\hat{x} + \lambda\hat{d}) = \underbrace{(a^i)^T\hat{x}}_{\geq b_i} + \underbrace{\lambda}_{\in\mathbb{R}}(a^i)^T\hat{d} \geq b_i, \quad i \in M_1, \quad \lambda \in \mathbb{R}.$$

If $(a^i)^T\hat{d} < 0$, then the inequality above would be violated for sufficiently large and positive values of λ. Similarly, if $(a^i)^T\hat{d} > 0$, then the inequality above would be violated for sufficiently large (in absolute value) and negative values of λ. We therefore obtain that $(a^i)^T\hat{d} = 0$ for each $i \in M_1$. A similar argument shows that $(a^i)^T\hat{d} = 0$ for each $i \in M_2 \cup M_3$. Therefore, we obtain $(a^i)^T\hat{d} = 0$ for each $i \in M_1 \cup M_2 \cup M_3$. It follows that $\hat{d} \in \mathcal{R}$. Clearly, we also have $(a^i)^T(-\hat{d}) = 0$ for each $i \in M_1 \cup M_2 \cup M_3$, which implies that $-\hat{d} \in \mathcal{R}$. Since $\hat{d} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, the claim follows.

$\Leftarrow$: Suppose that there exists $\hat{d} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $\hat{d} \in \mathcal{R}$ and $-\hat{d} \in \mathcal{R}$. Then, by Problem (4.1), we have $\hat{x} + \lambda\hat{d} \in \mathcal{P}$ and $\hat{x} + \lambda(-\hat{d}) = \hat{x} - \lambda\hat{d} \in \mathcal{P}$ for each $\lambda \geq 0$ and each $\hat{x} \in \mathcal{P}$. Note that

$$\begin{aligned}\{\hat{x} + \lambda\hat{d} : \lambda \in \mathbb{R}\} &= \{\hat{x} + \lambda\hat{d} : \lambda \geq 0\} \cup \{\hat{x} + \lambda\hat{d} : \lambda < 0\} \\ &= \{\hat{x} + \lambda\hat{d} : \lambda \geq 0\} \cup \{\hat{x} - \lambda\hat{d} : \lambda \geq 0\}.\end{aligned}$$

Since $\mathcal{P}$ is nonempty, it follows that there exists $\hat{x} \in \mathcal{P}$ and $\hat{d} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $\hat{x} + \lambda\hat{d} \in \mathcal{P}$ for each $\lambda \in \mathbb{R}$. Therefore, $\mathcal{P}$ contains a line, which implies that $\mathcal{P}$ contains no vertices by Proposition 8.1.