Week 3

Integer Programming

Methodology, Modelling, and Consulting Skills
The University of Edinburgh

Easy problem?

$$
\begin{aligned}
\min \quad & x_2 \\
\text{s.t.} \quad & 31013x_1 = 41014x_2 + 51015x_3, \\
& x_1 \geq 1, \\
& x_1, x_2, x_3 \in \mathbb{Z}.
\end{aligned}
$$

Goals for today

There is more than Linear Programming …

- (Mixed) Integer Programming

Two examples

- Knapsack problem
- Lot Sizing

Knapsack problem – an example

If I am going on a hike and I am considering taking with four items. Each item has an associated benefit and weight denoted by b_i and w_i, respectively for $i = 1, 2, 3, 4$. Suppose that my knapsack has a maximum capacity denoted by W, which items should I pack into my knapsack to maximise my benefit?

Knapsack problem – an example

Gather data

benefits

Item	1	2	3	4
b_i	16	22	12	8
w_i	5	7	4	3

Additional data

W My knapsack has a capacity of 14

Model

Decision variable

$x_i = 1$ if I pack item i and 0 otherwise,

Model

$$\begin{array}{ll} \max & \sum_{i=1}^{4} b_i x_i \\ \text{s.t.} & \sum_{i=1}^{4} x_i w_i \leq W \end{array}$$

New modelling tool

- Binary variables: $x \in \{0, 1\}$

Lot Sizing – an example

From: Pochet, Wolsey; Production planning by mixed integer programming

BikeCo produces a variety of different bikes. To plan the production of racing bikes, we have a demand prediction for the next year.
We can produce at most one batch of bikes per month. To produce a batch in a month we need to set up the factory for a cost of £5000. The marginal cost of each bike is £100. We have warehouse to store bikes which costs us £5 per bike per month.
As racing bikes are just a tiny part of BikeCo's production, we can assume that we have no upper bound on the production and place in storage. Peak demand period is January to August, so we need to plan for that time.

Lot Sizing – an example

Gather data

Sales forecasts

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug
400	400	800	800	1200	1200	1200	1200

Additional data

Starting inventory 200 bikes in stock

Problem – Setup costs

Observations

- Main difference to last week: Setup costs
- Setup costs cannot be modelled as an LP!

Model without setup costs

$$
\begin{array}{lll}
\min & \sum_{t=1}^{8} 100x_t + \sum_{t=1}^{8} 5i_t & \\
\text{s.t.} & i_0 = 200, & \\
& i_t = i_{t-1} + x_t - y_t, & t \in \{1, \ldots, 8\} \\
& y_t \geq d_t, & t \in \{1, \ldots, 8\} \\
& x_t, y_t, i_t \geq 0, & t \in \{1, \ldots, 8\}.
\end{array}
$$

Variables

x_t	Production level in month t,
y_t	Amount of available for supply sold in month t,
i_t	Inventory level at the end of month t.

Parameter

d_t	Demand in month t.

How to integrate setup costs?

New variables

z_t $\quad$ $z_t = 1$ if there is production in period t and 0 otherwise

New objective function

$$\sum_{t=1}^{8} 100x_t + \sum_{t=1}^{8} 5i_t + \sum_{t=1}^{8} 5000z_t$$

New constraints

$$x_t \leq M z_t, \quad t \in \{1, \ldots, 8\}$$

where M is a *very* big number.

How to integrate setup costs?

Observations

This implies

$$z_t = 1 \implies x_t \leq M,$$
$$z_t = 0 \implies x_t \leq 0.$$

This constraint type is known as *variable upper bound constraint* or *big-M constraint*.

New model

$$
\begin{array}{lll}
\min & \sum_{t=1}^{8} 100x_t + \sum_{t=1}^{8} 5i_t + \sum_{t=1}^{8} 5000z_t & \\
\text{s.t.} & i_0 = 200, & \\
& i_t = i_{t-1} + x_t - y_t, & t \in \{1, \ldots, 8\} \\
& y_t \geq d_t, & t \in \{1, \ldots, 8\} \\
& x_t \leq M z_t, & t \in \{1, \ldots, 8\} \\
& x_t, y_t, i_t \geq 0, & t \in \{1, \ldots, 8\}, \\
& z_t \in \{0, 1\}, & t \in \{1, \ldots, 8\},
\end{array}
$$

How to choose M?

- M needs to be larger than the largest possible production
- $M \geq \sum_{t=1}^{8} d_t$ sufficient.

Optimal solution

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Total
Demand	400	400	800	800	1,200	1,200	1,200	1,200	7,200
Production	800	0	1,600	0	1,200	1,200	1,200	1,200	7,000
unit cost	60,000	0	160,000	0	120,000	120,000	120,000	120,000	700,000
set-up cost	5,000	0	5,000	0	5,000	5,000	5,000	5,000	30,000
End Inventory	400	0	800	0	0	0	0	0	
Inv. cost	2,000	0	4,000	0	0	0	0	0	6,000

Discussion

What if …

- … we replace $z_t \in \{0, 1\}$ with $z_t \in [0, 1]$?

Then …

- … setup costs get underestimated and
- the solver typically returns a zero inventory solution (depends on the choice of M).

Refinement

$$
\begin{array}{lll}
\min & \sum_{t=1}^{8} 100x_t + \sum_{t=1}^{8} 5i_t + \sum_{t=1}^{8} 5000z_t & \\
\text{s.t.} & i_0 = 200, & \\
& i_t = i_{t-1} + x_t - y_t, & t \in \{1,\ldots,8\} \\
& y_t \geq d_t, & \\
& x_t \leq M_t z_t, & t \in \{1,\ldots,8\} \\
& x_t, y_t, i_t \geq 0, & t \in \{1,\ldots,8\}, \\
& z_t \in \{0,1\}, & t \in \{1,\ldots,8\},
\end{array}
$$

How to choose M_t?

- M needs to be larger than the largest possible production
- $M_t \geq \sum_{s=t}^{8} d_s$ sufficient.

Refinement

What if include a maximum production in a month?

- Let P_t be the maximum production in month t for all $t \in \{1, \ldots, 8\}$

$$
\begin{array}{lll}
\min & \sum_{t=1}^{8} 100x_t + \sum_{t=1}^{8} 5i_t + \sum_{t=1}^{8} 5000z_t & \\
\text{s.t.} & i_0 = 200, & \\
& i_t = i_{t-1} + x_t - y_t, & t \in \{1, \ldots, 8\} \\
& y_t \geq d_t, & t \in \{1, \ldots, 8\} \\
& x_t \leq M_t z_t, & t \in \{1, \ldots, 8\} \\
& x_t \leq P_t, & t \in \{1, \ldots, 8\} \\
& x_t, y_t, i_t \geq 0, & t \in \{1, \ldots, 8\}, \\
& z_t \in \{0, 1\}, & t \in \{1, \ldots, 8\}
\end{array}
$$

Refinement

What if include a maximum production in a month?

- Let P_t be the maximum production in month t for all $t \in \{1, \ldots, 8\}$

$$
\begin{array}{lll}
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\text{s.t.} & i_0 = 200, & \\
& i_t = i_{t-1} + x_t - y_t, & t \in \{1, \ldots, 8\} \\
& y_t \geq d_t, & t \in \{1, \ldots, 8\} \\
& x_t \leq P_t z_t, & t \in \{1, \ldots, 8\} \\
& x_t \leq P_t, & t \in \{1, \ldots, 8\} \\
& x_t, y_t, i_t \geq 0, & t \in \{1, \ldots, 8\}, \\
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Refinement

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& x_t, y_t, i_t \geq 0, & t \in \{1, \ldots, 8\}, \\
& z_t \in \{0, 1\}, & t \in \{1, \ldots, 8\}
\end{array}
$$

Either-Or constraints

Suppose that we have two constraints

$$f(x_1, x_2, \ldots x_n) \leq 0$$
$$g(x_1, x_2, \ldots x_n) \leq 0$$

and we only require that one of the two constraints must be satisfied.

Either-Or constraints

$$f(x_1, x_2, \ldots x_n) \leq My$$
$$g(x_1, x_2, \ldots x_n) \leq (1 - y)M$$

where $y \in \{0, 1\}$

This implies

$$y = 0 \implies f(x_1, x_2, \ldots x_n) \leq 0 \text{ and } g(x_1, x_2, \ldots x_n) \leq M,$$
$$y = 1 \implies f(x_1, x_2, \ldots x_n) \leq M \text{ and } g(x_1, x_2, \ldots x_n) \leq 0, .$$

Either-Or constraints – example

Dorian Auto is considering manufacturing three types of autos: compact, midsize, and large. The resources required for, and the profits yielded by, each type of car are shown in the table below. Currently, 6 000 tons of steel and 60 000 hours of labour are available. For production of a type of car to be economically feasible, at least 1,000 cars of that type must be produced. Formulate an IP to maximise Dorian's profit.

	Compact	Midsize	Large
Steel required (tons)	1.5	3.0	5.0
Labour required (hours)	30	25	40
Profit yield	2 000	3 000	4 000

Either-Or constraints – example

Decision variable

Let $x_i \in \mathbb{Z}$ be the number of autos produced of type i where 1 denotes compact, 2 denotes midsize and 3 denotes large.
Let $y_i \in \{0, 1\}$ be a binary auxiliary variable

Objective function

$$\max z = \sum_{i=1}^{3} x_i p_i,$$

where p_i denotes the profit yield from product i.

Either-Or constraints – example

Let s_i and l_i denote the steel and labour requirements for product i, respectively.

Constraints

$$\begin{aligned}
\sum_{i=1}^{3} x_i s_i &\leq 6\,000 \\
\sum_{i=1}^{3} x_i l_i &\leq 60\,000 \\
x_i &\leq M_i y_i && \forall i \in \{1, 2, 3\} \\
1\,000 x_i &\leq M_i(1 - y_i) && \forall i \in \{1, 2, 3\}
\end{aligned}$$

where $M_1 = 2\,000$, $M_2 = 2\,000$ and $M_3 = 1\,200$

Integer Programs

Integer Program (IP)

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^n. \end{aligned}$$

- Linear objective
- Linear constraints
- Integer variables

Mixed Integer Programs

Mixed Integer Program (MIP)

$$
\begin{aligned}
\min \quad & c^\top x \\
\text{s.t.} \quad & Ax \leq b \\
& x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}.
\end{aligned}
$$

- Linear objective
- Linear constraints
- Both integer and continuous variables

Integer variables

- Many magnitudes cannot have fractional values: hired workers, chairs made, vehicles needed, etc.
- Integer variables model these decisions.
- Can only have integer values: $x \in \mathbb{Z}$.
- Integer variables allow to model more complex systems.
- Binary variables: only can be 0 or 1.

Difficulty

Problems with integer variables are much harder to solve.

Both in

- theory and
- practice.

Rule-of-thumb

In reasonable time, state-of-the-art solvers can typically solve

- LPs with roughly 1×10^7 variables, but only
- IPs with roughly 1×10^4.

Note: There are nasty exceptions to this rule-of-thumb.