MATH11111: Fundamentals of Optimization	**Fall 2022**
Lecture 20	Simplex Method and Degeneracy
Lecturer: E. Alper Yıldırım	*Week: 7*

20.1 Outline

- Effects of Degeneracy on the Simplex Method
- Bland's Rule
- Review Problems

20.2 Overview

- Recall that a nondegenerate vertex is an optimal solution if and only if all reduced costs of nonbasic variables are nonnegative.
- At a degenerate vertex, if all reduced costs of nonbasic variables are nonnegative, then it is optimal.
- However, a degenerate vertex may be an optimal solution and yet reduced costs of some nonbasic variables may still be negative.
- In this lecture, we will discuss if and how degeneracy affects the simplex method.

20.3 An Example

Consider the following linear programming problem:

$$
\begin{array}{llllllllll}
\min & -10x_1 & + & 57x_2 & + & 9x_3 & + & 24x_4 & & \\
\text{s.t.} & & & & & & & & & \\
& 0.5x_1 & - & 5.5x_2 & - & 2.5x_3 & + & 9x_4 & \leq & 0 \\
& 0.5x_1 & - & 1.5x_2 & - & 0.5x_3 & + & x_4 & \leq & 0 \\
& x_1 & & & & & & & \leq & 1 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Since the problem is not in standard form, we introduce nonnegative slack variables x_5, x_6, and x_7 for the first, second, and the third inequality constraints, respectively.

We obtain the following equivalent linear programming problem in standard form:

$$
\begin{array}{llllllllllllllll}
\min & -10x_1 & + & 57x_2 & + & 9x_3 & + & 24x_4 & & & & & & & & \\
\text{s.t.} & & & & & & & & & & & & & & & \\
& 0.5x_1 & - & 5.5x_2 & - & 2.5x_3 & + & 9x_4 & + & x_5 & & & & & = & 0 \\
& 0.5x_1 & - & 1.5x_2 & - & 0.5x_3 & + & x_4 & & & + & x_6 & & & = & 0 \\
& x_1 & & & & & & & & & & & + & x_7 & = & 1 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & , & x_7 & \geq & 0
\end{array}
$$

We therefore have

$$A = \begin{bmatrix} 0.5 & -5.5 & -2.5 & 9 & 1 & 0 & 0 \\ 0.5 & -1.5 & -0.5 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} -10 \\ 57 \\ 9 \\ 24 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}.$$

Note that $m = 3$ and $n = 7$. We use $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$ as the starting vertex, with $B = \{5, 6, 7\}$ and $N = \{1, 2, 3, 4\}$.

Dictionary 1

$$\begin{array}{rcrcrcrcrcr} z & = & 0 & - & 10x_1 & + & 57x_2 & + & 9x_3 & + & 24x_4 \\ x_5 & = & 0 & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\ x_6 & = & 0 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\ x_7 & = & 1 & - & x_1 & & & & & & \end{array}$$

- $B = \{5, 6, 7\}$ and $N = \{1, 2, 3, 4\}$.
- The values of basic variables are given by $\hat{x}_5 = 0$, $\hat{x}_6 = 0$, and $\hat{x}_7 = 1$.
- The values of nonbasic variables are given by $\hat{x}_1 = 0$, $\hat{x}_2 = 0$, $\hat{x}_3 = 0$, and $\hat{x}_4 = 0$.
- The objective function value is $\hat{z} = 0$.
- Note that $\hat{x}$ is degenerate since there is at least one basic variable with a value of zero.
- By Row 0, x_1 is the entering variable. By Rows 1 and 2, x_5 and x_6 both achieve the minimum ratio.
- We will break ties in favour of the basic variable with the smallest index, i.e., x_5 is the leaving variable.
- By Row 1, $x_1 = 0 + 11x_2 + 5x_3 - 18x_4 - 2x_5$. Substitute this expression for x_1 in Rows 0, 2, and 3.

Dictionary 2

$$\begin{array}{rcrcrcrcrcr} z & = & 0 & - & 53x_2 & - & 41x_3 & + & 204x_4 & + & 20x_5 \\ x_1 & = & 0 & + & 11x_2 & + & 5x_3 & - & 18x_4 & - & 2x_5 \\ x_6 & = & 0 & - & 4x_2 & - & 2x_3 & + & 8x_4 & + & x_5 \\ x_7 & = & 1 & - & 11x_2 & - & 5x_3 & + & 18x_4 & + & 2x_5 \end{array}$$

- $B = \{1, 6, 7\}$ and $N = \{2, 3, 4, 5\}$.
- The values of basic variables are given by $\hat{x}_1 = 0$, $\hat{x}_6 = 0$, and $\hat{x}_7 = 1$.

- The values of nonbasic variables are given by $\hat{x}_2 = 0$, $\hat{x}_3 = 0$, $\hat{x}_4 = 0$, and $\hat{x}_5 = 0$.
- The objective function value is $\hat{z} = 0$.
- Note that the objective function value has not improved!
- In fact, we are still at the same vertex given by $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$.
- The only change is the new index sets B and N.
- By Row 0, we can either increase x_2 or x_3.
- Since x_2 has the most negative reduced cost, we will choose x_2 as the entering variable.
- By Row 2, x_6 is the leaving variable.
- Using Row 2, we obtain $x_2 = 0 - 0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6$.
- We substitute this expression for x_2 in Rows 0, 1, and 3.

Dictionary 3

$$
\begin{array}{rcrcrcrcrcr}
z & = & 0 & - & 14.5x_3 & + & 98x_4 & + & 6.75x_5 & + & 13.25x_6 \\
x_1 & = & 0 & - & 0.5x_3 & + & 4x_4 & + & 0.75x_5 & - & 2.75x_6 \\
x_2 & = & 0 & - & 0.5x_3 & + & 2x_4 & + & 0.25x_5 & - & 0.25x_6 \\
x_7 & = & 1 & + & 0.5x_3 & - & 4x_4 & - & 0.75x_5 & + & 2.75x_6
\end{array}
$$

- $B = \{1, 2, 7\}$ and $N = \{3, 4, 5, 6\}$.
- The values of basic variables are given by $\hat{x}_1 = 0$, $\hat{x}_2 = 0$, and $\hat{x}_7 = 1$.
- The values of nonbasic variables are given by $\hat{x}_3 = 0$, $\hat{x}_4 = 0$, $\hat{x}_5 = 0$, and $\hat{x}_6 = 0$.
- The objective function value is $\hat{z} = 0$.
- We are still at the same vertex $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$ with new index sets B and N.
- By Row 0, x_3 is the entering variable.
- By Rows 1 and 2, each of x_1 and x_2 achieves the minimum ratio. We will break ties in favour of the basic variable with the smallest index, i.e., x_1 is the leaving variable.
- By Row 1, $x_3 = 0 - 2x_1 + 8x_4 + 1.5x_5 - 5.5x_6$. Substitute this in Rows 0, 2, and 3.

Dictionary 4

$$
\begin{array}{rcrcrcrcrcr}
z & = & 0 & + & 29x_1 & - & 18x_4 & - & 15x_5 & + & 93x_6 \\
x_3 & = & 0 & - & 2x_1 & + & 8x_4 & + & 1.5x_5 & - & 5.5x_6 \\
x_2 & = & 0 & + & x_1 & - & 2x_4 & - & 0.5x_5 & + & 2.5x_6 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}
$$

- $B = \{3, 2, 7\}$ and $N = \{1, 4, 5, 6\}$.
- The values of basic variables are given by $\hat{x}_3 = 0$, $\hat{x}_2 = 0$, and $\hat{x}_7 = 1$.
- The values of nonbasic variables are given by $\hat{x}_1 = 0$, $\hat{x}_4 = 0$, $\hat{x}_5 = 0$, and $\hat{x}_6 = 0$.
- The objective function value is $\hat{z} = 0$.
- We are still at the same vertex $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$ with new index sets B and N.
- By Row 0, x_4 is the entering variable due to the most negative reduced cost.
- By Row 2, x_2 is the leaving variable. We obtain $x_4 = 0 + 0.5x_1 - 0.5x_2 - 0.25x_5 + 1.25x_6$. Substitute this in Rows 0, 1, and 3.

Dictionary 5

$$\begin{array}{rcrcrcrcrcr}
z & = & 0 & + & 20x_1 & + & 9x_2 & - & 10.5x_5 & + & 70.5x_6 \\
x_3 & = & 0 & + & 2x_1 & - & 4x_2 & - & 0.5x_5 & + & 4.5x_6 \\
x_4 & = & 0 & + & 0.5x_1 & - & 0.5x_2 & - & 0.25x_5 & + & 1.25x_6 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}$$

- $B = \{3, 4, 7\}$ and $N = \{1, 2, 5, 6\}$.
- The values of basic variables are given by $\hat{x}_3 = 0$, $\hat{x}_4 = 0$, and $\hat{x}_7 = 1$.
- The values of nonbasic variables are given by $\hat{x}_1 = 0$, $\hat{x}_2 = 0$, $\hat{x}_5 = 0$, and $\hat{x}_6 = 0$.
- The objective function value is $\hat{z} = 0$.
- We are still at the same vertex $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$ with new index sets B and N.
- By Row 0, x_5 is the entering variable.
- By Rows 1 and 2, x_3 and x_4 both achieve the minimum ratio. We choose x_3 as the leaving variable.
- By Row 1, we obtain $x_5 = 0 + 4x_1 - 8x_2 - 2x_3 + 9x_6$. Substitute this in Rows 0, 2, and 3.

Dictionary 6

$$\begin{array}{rcrcrcrcrcr}
z & = & 0 & - & 22x_1 & + & 93x_2 & + & 21x_3 & - & 24x_6 \\
x_5 & = & 0 & + & 4x_1 & - & 8x_2 & - & 2x_3 & + & 9x_6 \\
x_4 & = & 0 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_6 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}$$

- $B = \{5, 4, 7\}$ and $N = \{1, 2, 3, 6\}$.
- The values of basic variables are given by $\hat{x}_5 = 0$, $\hat{x}_4 = 0$, and $\hat{x}_7 = 1$.
- The values of nonbasic variables are given by $\hat{x}_1 = 0$, $\hat{x}_2 = 0$, $\hat{x}_3 = 0$, and $\hat{x}_6 = 0$.

- The objective function value is $\hat{z} = 0$.
- We are still at the same vertex $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$ with new index sets B and N.
- By Row 0, x_6 is the entering variable due to the most negative reduced cost.
- By Row 2, x_4 is the leaving variable. We obtain $x_6 = 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4$. Substitute this in Rows 0, 1, and 3.

Dictionary 7

$$
\begin{array}{rcrcrcrcrcr}
z & = & 0 & - & 10x_1 & + & 57x_2 & + & 9x_3 & + & 24x_4 \\
x_5 & = & 0 & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\
x_6 & = & 0 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}
$$

- $B = \{5, 6, 7\}$ and $N = \{1, 2, 3, 4\}$.
- The values of basic variables are given by $\hat{x}_5 = 0$, $\hat{x}_6 = 0$, and $\hat{x}_7 = 1$.
- The values of nonbasic variables are given by $\hat{x}_1 = 0$, $\hat{x}_2 = 0$, $\hat{x}_3 = 0$, and $\hat{x}_4 = 0$.
- The objective function value is $\hat{z} = 0$.
- We are still at the same vertex $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$ with new index sets B and N.
- Note that this is the same dictionary as in Dictionary 1!

20.3.1 Discussion

- In the presence of degeneracy, the simplex method may go through a number of dictionaries and come back to an earlier dictionary.
- This phenomenon is called *cycling*.
- In fact, the vertex remains the same and only the index sets B and N change at each iteration.
- Therefore, the simplex method may cycle indefinitely among a set of dictionaries without making any progress and may not terminate under the existence of degenerate vertices.
- Note that cycling cannot occur under nondegeneracy.
- Fortunately, there is a simple rule that prevents cycling.

20.4 Bland's Rule

Proposition 20.1. *At a degenerate vertex, consider the following rule, known as Bland's rule: If there is more than one nonbasic variable with a negative reduced cost, choose the variable with the smallest subscript as the entering variable. If there is more than one basic variable that achieves the minimum ratio, choose the variable with the smallest subscript as the leaving variable. Then, the simplex method never visits an earlier dictionary and therefore terminates after a finite number of iterations.*

Proof. The proof is based on a contradiction argument. Conceptually, it is not too difficult but the details are somewhat tedious and we therefore omit the proof. □

20.5 Example Revisited

We will now reconsider the same example as in Section 20.3. This time, we will choose the entering and leaving variables using Bland's rule at every degenerate vertex.

Dictionary 1

$$\begin{array}{rcrcrcrcrcr}
z & = & 0 & - & 10x_1 & + & 57x_2 & + & 9x_3 & + & 24x_4 \\
x_5 & = & 0 & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\
x_6 & = & 0 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}$$

- $B = \{5, 6, 7\}$ and $N = \{1, 2, 3, 4\}$.
- By Row 0, x_1 is the entering variable.
- By Rows 1 and 2, x_5 and x_6 are both candidates for the leaving variable. Choose x_5 as the leaving variable by Bland's rule.

Dictionary 2

$$\begin{array}{rcrcrcrcrcr}
z & = & 0 & - & 53x_2 & - & 41x_3 & + & 204x_4 & + & 20x_5 \\
x_1 & = & 0 & + & 11x_2 & + & 5x_3 & - & 18x_4 & - & 2x_5 \\
x_6 & = & 0 & - & 4x_2 & - & 2x_3 & + & 8x_4 & + & x_5 \\
x_7 & = & 1 & - & 11x_2 & - & 5x_3 & + & 18x_4 & + & 2x_5
\end{array}$$

- $B = \{1, 6, 7\}$ and $N = \{2, 3, 4, 5\}$.
- By Row 0, x_2 and x_3 are the candidates for entering variable. By Bland's rule, we choose x_2 as the entering variable.
- By Row 2, x_6 is the leaving variable.

Dictionary 3

$$\begin{array}{rcrcrcrcrcr}
z & = & 0 & - & 14.5x_3 & + & 98x_4 & + & 6.75x_5 & + & 13.25x_6 \\
x_1 & = & 0 & - & 0.5x_3 & + & 4x_4 & + & 0.75x_5 & - & 2.75x_6 \\
x_2 & = & 0 & - & 0.5x_3 & + & 2x_4 & + & 0.25x_5 & - & 0.25x_6 \\
x_7 & = & 1 & + & 0.5x_3 & - & 4x_4 & - & 0.75x_5 & + & 2.75x_6
\end{array}$$

- $B = \{1, 2, 7\}$ and $N = \{3, 4, 5, 6\}$.
- By Row 0, x_3 is the entering variable.
- By Rows 1 and 2, x_1 and x_2 are both candidates for the leaving variable. Choose x_1 as the leaving variable by Bland's rule.

Dictionary 4

$$\begin{array}{rcrcrcrcrcr}
z & = & 0 & + & 29x_1 & - & 18x_4 & - & 15x_5 & + & 93x_6 \\
x_3 & = & 0 & - & 2x_1 & + & 8x_4 & + & 1.5x_5 & - & 5.5x_6 \\
x_2 & = & 0 & + & x_1 & - & 2x_4 & - & 0.5x_5 & + & 2.5x_6 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}$$

- $B = \{3, 2, 7\}$ and $N = \{1, 4, 5, 6\}$.
- By Row 0, x_4 and x_5 are candidates for the entering variable. We choose x_4 as the entering variable by Bland's rule.
- By Row 2, x_2 is the leaving variable.

Dictionary 5

$$\begin{array}{rcrcrcrcrcr}
z & = & 0 & + & 20x_1 & + & 9x_2 & - & 10.5x_5 & + & 70.5x_6 \\
x_3 & = & 0 & + & 2x_1 & - & 4x_2 & - & 0.5x_5 & + & 4.5x_6 \\
x_4 & = & 0 & + & 0.5x_1 & - & 0.5x_2 & - & 0.25x_5 & + & 1.25x_6 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}$$

- $B = \{3, 4, 7\}$ and $N = \{1, 2, 5, 6\}$.
- By Row 0, x_5 is the entering variable.
- By Rows 1 and 2, x_3 and x_4 are both candidates for the leaving variable. Choose x_3 as the leaving variable by Bland's rule.

Dictionary 6

$$\begin{array}{rcrcrcrcrcr}
z & = & 0 & - & 22x_1 & + & 93x_2 & + & 21x_3 & - & 24x_6 \\
x_5 & = & 0 & + & 4x_1 & - & 8x_2 & - & 2x_3 & + & 9x_6 \\
x_4 & = & 0 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_6 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}$$

- $B = \{5, 4, 7\}$ and $N = \{1, 2, 3, 6\}$.

- By Row 0, x_1 and x_6 the candidates for entering variable. By Bland's rule, we choose x_1 as the entering variable.
- By Row 2, x_4 is the leaving variable.
- In the previous case, we had used x_6 as the entering variable due to the most negative reduced cost. **Since the current vertex is degenerate, we ignore the most negative reduced cost and apply Bland's rule instead to choose the entering variable.**

Dictionary 7

$$\begin{array}{rcrcrcrcrcr} z & = & 0 & + & 27x_2 & - & x_3 & + & 44x_4 & + & 20x_6 \\ x_5 & = & 0 & + & 4x_2 & + & 2x_3 & - & 8x_4 & + & x_6 \\ x_1 & = & 0 & + & 3x_2 & + & x_3 & - & 2x_4 & - & 2x_6 \\ x_7 & = & 1 & - & 3x_2 & - & x_3 & + & 2x_4 & + & 2x_6 \end{array}$$

- $B = \{5, 1, 7\}$ and $N = \{2, 3, 4, 6\}$.
- By Row 0, x_3 is the entering variable.
- By Row 3, x_7 is the leaving variable.

Dictionary 8

$$\begin{array}{rcrcrcrcrcr} z & = & -1 & + & 30x_2 & + & 42x_4 & + & 18x_6 & + & x_7 \\ x_5 & = & 2 & - & 2x_2 & - & 4x_4 & + & 5x_6 & - & 2x_7 \\ x_1 & = & 1 & & & & & & & - & x_7 \\ x_3 & = & 1 & - & 3x_2 & + & 2x_4 & + & 2x_6 & - & x_7 \end{array}$$

- $B = \{5, 1, 3\}$ and $N = \{2, 4, 6, 7\}$.
- Note that $\hat{z} = -1 < 0$ and we therefore moved to a new vertex!
- By Row 0, the current vertex is optimal since all reduced costs are nonnegative.
- Therefore, an optimal solution is given by $x^* = [1, 0, 1, 0, 2, 0, 0]^T$ and the optimal value is $z^* = -1$.

20.6 Concluding Remarks

- Degeneracy may cause problems in the implementation of the simplex method.
- These issues can easily be circumvented by using Bland's rule.
- **Note that Bland's rule needs to be applied only when the current vertex is degenerate.**
- For a nondegenerate vertex, it is still reasonable to choose the nonbasic variable with the most negative reduced cost.

- Therefore, by using Bland's rule at each degenerate vertex, we conclude that the simplex method always terminates after a finite number of iterations.

- In the next lecture, we will address the issue of detecting infeasibility and computing an initial vertex if one is not easily identifiable.

Exercises

Question 20.1. *Explain what leads to cycling at a degenerate vertex and why the simplex method cannot cycle if every vertex is nondegenerate.*

MATH11111: Fundamentals of Optimization **Fall 2022**

Lecture 21 The Two-Phase Method

Lecturer: E. Alper Yıldırım *Week: 7*

21.1 Outline

- The Two-Phase Method
- Driving Artificial Variables out of the Basis
- Review Problems

21.2 Overview

- The simplex method requires an initial basic feasible solution.
- In this lecture, we will discuss how to construct an initial basic feasible solution if one is not easily identifiable, and how to detect infeasibility.
- We will discuss a method that solves both problems simultaneously.

21.3 Easily Identifiable Basic Feasible Solutions

Suppose that a linear programming problem is in the following form:

$$\min\{c^T x : Ax \leq b, \quad x \geq \mathbf{0}\},$$

where $b \geq \mathbf{0}$.

- Then, we can define a vector of slack variables, one for each constraint and denoted by x^s, and convert the above problem into an equivalent linear programming problem in standard form:

$$\min\{c^T x : Ax + x^s = b, \quad x \geq \mathbf{0}, \quad x^s \geq \mathbf{0}\}.$$

- If we set $\hat{x} = \mathbf{0}$ and $\hat{x}^s = b \geq \mathbf{0}$, then we easily obtain an initial basic feasible solution.
- This is the starting solution we used in all of the previous examples.

21.4 General Case

As before, we will assume that the linear programming problem is in standard form, i.e.,

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\}.$$

- Note that we may assume that $b \geq \mathbf{0}$ by multiplying the ith equality constraint by -1 if $b_i < 0$, where $i = 1, \ldots, m$.
- We also assume that A has full row rank.
- **Question 1:** Does there exist a feasible solution?
- **Question 2:** If the feasible region is nonempty, how can we find a vertex? (Recall that every nonempty polyhedron in standard form has at least one vertex.)

21.4.1 Basic Idea

- We define and add a nonnegative "artificial" variable a_i to the left-hand side of each equality constraint i, where $i = 1, \ldots, m$.
- Denote this vector of artificial variables by $a \in \mathbb{R}^m$.
- We obtain the following modified system:

$$Ax + a = b, \quad x \geq \mathbf{0}, \quad a \geq \mathbf{0}.$$

- Note that the set of all vectors $(x, a) \in \mathbb{R}^n \times \mathbb{R}^m$ that satisfy the above system is a polyhedron.
- Let us now define

$$\begin{aligned} \mathcal{P} &= \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\} \\ \tilde{\mathcal{P}} &= \{(x, a) \in \mathbb{R}^n \times \mathbb{R}^m : Ax + a = b, \quad x \geq \mathbf{0}, \quad a \geq \mathbf{0}\} \end{aligned}$$

Proposition 21.1. *Suppose that $b \geq \mathbf{0}$.*

(i) Then, $\tilde{\mathcal{P}}$ is a nonempty polyhedron.

(ii) Furthermore, $\mathcal{P}$ is a nonempty polyhedron if and only if there exists a solution $(\bar{x}, \bar{a}) \in \tilde{\mathcal{P}}$ such that $\bar{a} = \mathbf{0}$.

(iii) Finally, if A has full row rank and $(\hat{x}, \hat{a})$ is a vertex of $\tilde{\mathcal{P}}$ such that $\hat{a} = \mathbf{0}$, then $\hat{x}$ is a vertex of $\mathcal{P}$.

Proof. (i) Since $b \geq \mathbf{0}$, we can set $(\bar{x}, \bar{a}) = (\mathbf{0}, b)$. Then, $A\bar{x} + \bar{a} = b$ and $\bar{x} \geq \mathbf{0}$ and $\bar{a} \geq \mathbf{0}$, i.e., $(\bar{x}, \bar{a}) \in \tilde{\mathcal{P}}$, which implies that $\tilde{\mathcal{P}}$ is a nonempty polyhedron (regardless of whether $\mathcal{P}$ is nonempty or not).

(ii) Suppose that $\mathcal{P}$ is a nonempty polyhedron. Then, there exists $\bar{x} \in \mathbb{R}^n$ such that $A\bar{x} = b$ and $\bar{x} \geq \mathbf{0}$. Then, by defining $(\bar{x}, \bar{a}) = (\bar{x}, \mathbf{0})$, we obtain $(\bar{x}, \bar{a}) \in \tilde{\mathcal{P}}$.

Conversely, if there exists a solution $(\bar{x}, \bar{a}) \in \tilde{\mathcal{P}}$ such that $\bar{a} = \mathbf{0}$, then we clearly have $\bar{x} \in \mathcal{P}$.

(iii) Finally, suppose that A has full row rank and that $(\hat{x}, \hat{a})$ is a vertex of $\tilde{\mathcal{P}}$ such that $\hat{a} = \mathbf{0}$. Since $\hat{a} = \mathbf{0}$, let us define $\hat{B} = \{j \in \{1, \ldots, m\} : \hat{x}_j > 0\}$. Note that $|\hat{B}| \leq m$.

Case 1: If $|\hat{B}| = m$, we set $B = \hat{B}$ and $N = \{1, \ldots, n\} \backslash B$ and we obtain that $\hat{x}$ is a nondegenerate vertex of $\mathcal{P}$.

Case 2: If $|\hat{B}| < m$, since A has full row rank, we can choose $m - |\hat{B}|$ columns of A corresponding to $\hat{x}_j = 0$, add those indices to $\hat{B}$ to obtain an index set with $|B| = m$, and ensure that $A_B \in \mathbb{R}^{m \times m}$ is invertible. Therefore, $\hat{x}$ is a degenerate vertex of $\mathcal{P}$.

□

21.5 The Two-Phase Method

- By Proposition 21.1, we can try to solve the following *auxiliary* linear programming problem, called the Phase 1 Problem:

$$\text{(AUX)} \quad \min \left\{ \sum_{i=1}^{m} a_i : Ax + a = b, \quad x \geq \mathbf{0}, \quad a \geq \mathbf{0} \right\}.$$

- Note that (AUX) has a nonempty feasible region by Proposition 21.1 if $b \geq \mathbf{0}$.
- The optimal value of (AUX) is always nonnegative since $a \geq \mathbf{0}$ and the objective function is given by $\sum_{i=1}^{m} a_i$.
- Furthermore, if we set $\hat{x} = \mathbf{0}$ and $\hat{a} = b \geq \mathbf{0}$, we obtain an easily identifiable vertex, from which we can start the simplex method to solve (AUX).
- There are two possibilities:
  (i) **Case 1:** If the optimal value of (AUX) is positive, then $\mathcal{P} = \emptyset$ by Proposition 21.1 (i.e., the original linear programming problem is infeasible).
  (ii) **Case 2:** If the optimal value of (AUX) is zero, then (AUX) has an optimal vertex $(\hat{x}, \hat{a})$ such that $\hat{a} = \mathbf{0}$. Assuming that A has full row rank, then $\hat{x}$ is a vertex of $\mathcal{P}$ by Proposition 21.1 and can be used to start the simplex method to solve the original linear programming problem, called the Phase 2 Problem.

21.5.1 An Example

Consider the following linear programming problem:

$$\begin{array}{lrcrcrcrcrcl}
\min & 2x_1 & + & 3x_2 & + & 3x_3 & + & x_4 & - & 2x_5 & & \\
\text{s.t.} & & & & & & & & & & & \\
 & x_1 & + & 3x_2 & & & + & 4x_4 & + & x_5 & = & 2 \\
 & x_1 & + & 2x_2 & & & - & 3x_4 & + & x_5 & = & 2 \\
 & -x_1 & - & 4x_2 & + & 3x_3 & & & & & = & 1 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}$$

- Since a vertex is not easily identifiable, we will use the two-phase method.
- We define and add nonnegative artificial variables a_1, a_2, and a_3 corresponding to the three equality constraints and try to minimize $a_1 + a_2 + a_3$ in Phase 1.

Phase 1 Problem

$$
\begin{array}{lrcrcrcrcrcrcrcrcrcl}
\min & & & & & & & & & & & a_1 & + & a_2 & + & a_3 & & \\
\text{s.t.} & & & & & & & & & & & & & & & & & \\
& x_1 & + & 3x_2 & & & + & 4x_4 & + & x_5 & + & a_1 & & & & & = & 2 \\
& x_1 & + & 2x_2 & & & - & 3x_4 & + & x_5 & & & + & a_2 & & & = & 2 \\
& -x_1 & - & 4x_2 & + & 3x_3 & & & & & & & & & + & a_3 & = & 1 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & a_1 & , & a_2 & , & a_3 & \geq & 0
\end{array}
$$

- We will use $\hat{x}_1 = \hat{x}_2 = \hat{x}_3 = \hat{x}_4 = \hat{x}_5 = 0$ and $\hat{a}_1 = 2$, $\hat{a}_2 = 2$, and $\hat{a}_3 = 1$ as the initial vertex for the Phase 1 problem.
- Then, Rows 1, 2, and 3 of the starting dictionary are given by the following:

$$
\begin{array}{rcrcrcrcrcrcr}
a_1 & = & 2 & - & x_1 & - & 3x_2 & & & - & 4x_4 & - & x_5 \\
a_2 & = & 2 & - & x_1 & - & 2x_2 & & & + & 3x_4 & - & x_5 \\
a_3 & = & 1 & + & x_1 & + & 4x_2 & - & 3x_3 & & & &
\end{array}
$$

- The basic variables are a_1, a_2, and a_3.
- We have not yet added Row 0, given by $z = a_1 + a_2 + a_3$.
- **However, recall that we need to express the right-hand side of Row 0 in terms of nonbasic variables only** (i.e., x_1, x_2, x_3, x_4, and x_5).
- Therefore, we will substitute the expressions on the right-hand sides of Rows 1, 2, and 3 for a_1, a_2, and a_3, respectively, in Row 0.
- **We can apply the simplex method only after we satisfy this requirement (i.e., we need to ensure that our initial dictionary is in proper form).**

Phase 1 Dictionary 1

$$
\begin{array}{rcrcrcrcrcrcr}
z & = & 5 & - & x_1 & - & x_2 & - & 3x_3 & - & x_4 & - & 2x_5 \\
a_1 & = & 2 & - & x_1 & - & 3x_2 & & & - & 4x_4 & - & x_5 \\
a_2 & = & 2 & - & x_1 & - & 2x_2 & & & + & 3x_4 & - & x_5 \\
a_3 & = & 1 & + & x_1 & + & 4x_2 & - & 3x_3 & & & &
\end{array}
$$

- By Row 0, x_3 is the entering variable due to the most negative reduced cost.
- By Row 3, a_3 is the leaving variable.

Phase 1 Dictionary 2

$$
\begin{array}{rcrcrcrcrcrcr}
z & = & 4 & - & 2x_1 & - & 5x_2 & - & x_4 & - & 2x_5 & + & a_3 \\
a_1 & = & 2 & - & x_1 & - & 3x_2 & - & 4x_4 & - & x_5 & & \\
a_2 & = & 2 & - & x_1 & - & 2x_2 & + & 3x_4 & - & x_5 & & \\
x_3 & = & \frac{1}{3} & + & \frac{1}{3}x_1 & + & \frac{4}{3}x_2 & & & & & - & \frac{1}{3}a_3
\end{array}
$$

- By Row 0, x_2 is the entering variable due to the most negative reduced cost.
- By Row 1, a_1 is the leaving variable.

Phase 1 Dictionary 3

$$
\begin{array}{rcrcrcrcrcrcr}
z & = & \frac{2}{3} & - & \frac{1}{3}x_1 & + & \frac{17}{3}x_4 & - & \frac{1}{3}x_5 & + & \frac{5}{3}a_1 & + & a_3 \\
x_2 & = & \frac{2}{3} & - & \frac{1}{3}x_1 & - & \frac{4}{3}x_4 & - & \frac{1}{3}x_5 & - & \frac{1}{3}a_1 & & \\
a_2 & = & \frac{2}{3} & - & \frac{1}{3}x_1 & + & \frac{17}{3}x_4 & - & \frac{1}{3}x_5 & + & \frac{2}{3}a_1 & & \\
x_3 & = & \frac{11}{9} & - & \frac{1}{9}x_1 & - & \frac{16}{9}x_4 & - & \frac{4}{9}x_5 & - & \frac{4}{9}a_1 & - & \frac{1}{3}a_3
\end{array}
$$

- By Row 0, x_1 and x_5 have the same negative reduced cost. We arbitrarily pick x_1 as the entering variable.
- By Row 1 and Row 2, each of x_2 and a_2 achieves the same minimum ratio. We arbitrarily pick x_2 as the leaving variable.

Phase 1 Dictionary 4

$$
\begin{array}{rcrcrcrcrcrcr}
z & = & 0 & + & x_2 & + & 7x_4 & & & + & 2a_1 & + & a_3 \\
x_1 & = & 2 & - & 3x_2 & - & 4x_4 & - & x_5 & - & a_1 & & \\
a_2 & = & 0 & + & x_2 & + & 7x_4 & & & + & a_1 & & \\
x_3 & = & 1 & + & \frac{1}{3}x_2 & - & \frac{4}{3}x_4 & - & \frac{1}{3}x_5 & - & \frac{1}{3}a_1 & - & \frac{1}{3}a_3
\end{array}
$$

- By Row 0, since all reduced costs are nonnegative, this solution is optimal.
- Since the optimal value of the Phase 1 Problem is $z^* = 0$, the original problem has a nonempty feasible region.
- However, we still have an artificial variable a_2 as a basic variable, with a value of zero.
- Note that this vertex is degenerate (see Case 2 of Proposition 21.1).
- We will drive the artificial variable a_2 out of the set of basic variables by interchanging it with one of the non-artificial nonbasic variables.

- By Row 2, each of the non-artificial nonbasic variables x_2 and x_4 has a nonzero coefficient.
- Let us arbitrarily pick x_2 and move it to the left-hand side of Row 2, while moving a_2 to the right-hand side.
- We will then substitute this expression for x_2 into Rows 1 and 3.

Phase 1 Dictionary 5

$$\begin{array}{rclclclclclcl} z &=& 0 & & & & & + & a_1 & + & a_2 & + & a_3 \\ x_1 &=& 2 &+& 17x_4 &-& x_5 &+& 2a_1 &-& 3a_2 & & \\ x_2 &=& 0 &-& 7x_4 & & &-& a_1 &+& a_2 & & \\ x_3 &=& 1 &-& \frac{11}{3}x_4 &-& \frac{1}{3}x_5 &-& \frac{2}{3}a_1 &+& \frac{1}{3}a_2 &-& \frac{1}{3}a_3 \end{array}$$

- Note that this dictionary is still optimal and there are no artificial basic variables.
- Since the optimal solution computed in the previous iteration is degenerate, we simply changed the labels of basic and nonbasic variables without actually changing the vertex.
- We will delete Row 0 and all artificial variables a_1, a_2, and a_3 as they are no longer needed.
- We are now ready to move to Phase 2. We will use the original objective function $z = 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5$.
- Since x_1, x_2, and x_3 are basic, we will substitute the right-hand sides of Rows 1, 2, and 3, respectively in Row 0.

Phase 2 Problem

Phase 2 Dictionary 1

$$\begin{array}{rclclcl} z &=& 7 &+& 3x_4 &-& 5x_5 \\ x_1 &=& 2 &+& 17x_4 &-& x_5 \\ x_2 &=& 0 &-& 7x_4 & & \\ x_3 &=& 1 &-& \frac{11}{3}x_4 &-& \frac{1}{3}x_5 \end{array}$$

- This is the starting dictionary for Phase 2.
- $B = \{1, 2, 3\}$ and $N = \{4, 5\}$.
- Note that this basic feasible solution is degenerate.
- By Row 0, x_5 is the entering variable.
- By Row 1, x_1 is the leaving variable.

Phase 2 Dictionary 2

$$
\begin{array}{rcrcrcr}
z & = & -3 & + & 5x_1 & - & 82x_4 \\
x_5 & = & 2 & - & x_1 & + & 17x_4 \\
x_2 & = & 0 & & & - & 7x_4 \\
x_3 & = & \frac{1}{3} & + & \frac{1}{3}x_1 & - & \frac{28}{3}x_4
\end{array}
$$

- $B = \{5, 2, 3\}$ and $N = \{1, 4\}$.
- Note that this is also a degenerate basic feasible solution.
- By Row 0, x_4 is the entering variable.
- By Row 2, x_2 is the leaving variable.

Phase 2 Dictionary 3

$$
\begin{array}{rcrcrcr}
z & = & -3 & + & 5x_1 & + & \frac{82}{7}x_2 \\
x_5 & = & 2 & - & x_1 & - & \frac{17}{7}x_2 \\
x_4 & = & 0 & & & - & \frac{1}{7}x_2 \\
x_3 & = & \frac{1}{3} & + & \frac{1}{3}x_1 & + & \frac{4}{3}x_2
\end{array}
$$

- $B = \{5, 4, 3\}$ and $N = \{1, 2\}$.
- Note that this vertex is optimal since there are no negative reduced costs in Row 0.
- Note that this is the same solution as in the previous dictionary and only the index sets B and N have changed.
- Therefore, $x^* = [0, 0, 1/3, 0, 2]^T$ is an optimal solution and the optimal value of the original linear programming problem is $z^* = -3$.
- This is the end of Phase 2.

21.6 Outline of the Two-Phase Method

1. Define an artificial variable for each equality constraint and minimize the sum of the artificial variables in Phase 1.
2. Set all original variables to zero and all artificial variables to their corresponding right-hand side values and use this as an initial vertex in Phase 1.
3. **Make sure to put the initial dictionary in proper form by substituting each artificial variable in Row 0 by using their corresponding right-hand side expressions.**
4. If the optimal value of Phase 1 problem is positive, stop. The original problem is infeasible.

5. If the optimal value of Phase 1 problem is zero and there are no artificial basic variables, delete Row 0 and all artificial variables. Introduce the original objective function as the new Row 0 and **ensure that the initial dictionary in proper form (i.e., only nonbasic variables should appear on the right-hand side of Row 0 in the first dictionary of the Phase 2 problem**. Proceed to Phase 2.

6. If the optimal value of Phase 1 problem is zero and there is at least one artificial variable in the basis, drive all artificial variables out of the basis (i.e., replace each of them by a nonbasic and non-artificial variable) and perform Step 5.

Exercises

Question 21.1. *Suppose that you solve a linear programming problem in standard form with m equality constraints by using the two-phase method. Suppose that the optimal value at the end of Phase 1 is zero and that none of the artificial variables $a_1, \ldots, a_m$ is in the basis. Prove that Row 0 should read $z = 0 + \sum_{i=1}^{m} a_i$ in the final dictionary of Phase 1 (please see Phase 1 Dictionary 5 in the example above).*