MATH11177: Bayesian Theory
Solutions to worksheet 2

1. We wish to calculate the probability of having chosen a route, r, given that the journey, j, was completed in 1.5 hours $Pr(r|j \leq 1.5)$. By Bayes' Theorem, we have,

$$\Pr(R=r|j\leq 1.5) = \frac{\Pr(j\leq 1.5|R=r)\Pr(R=r)}{\Pr(j\leq 1.5)} = \frac{\Pr(j\leq 1.5|R=r)0.25}{\sum_{r=1}^{4}\Pr(j\leq 1.5|R=r)0.25} = \frac{\Pr(j\leq 1.5|R=4)}{2.4}$$

Then for each of the 4 routes:

$$\Pr(R=1|j\leq 1.5) = \tfrac{0.2}{2.4} = 0.083 \qquad \Pr(R=2|j\leq 1.5) = \tfrac{0.5}{2.4} = 0.208$$
$$\Pr(R=3|j\leq 1.5) = \tfrac{0.8}{2.4} = 0.333 \qquad \Pr(R=4|j\leq 1.5) = \tfrac{0.9}{2.4} = 0.375$$

2. **a** Let I denote "is infected", P denote Positive test, and N is Negative test. By Bayes' Theorem, we have,

$$\Pr(I|P) = \frac{\Pr(P|I)\Pr(I)}{\Pr(P)} = \frac{\Pr(P|I)\Pr(I)}{\Pr(P|I)\Pr(I)+\Pr(P|N)\Pr(N)}$$
$$= \frac{0.75*0.0222}{0.75*0.0222+(1-0.99)*(1-0.0222)} = 0.63$$

b & c Substituting the corresponding values for $Pr(I)$ in the expression above we obtain:

Moderate	Medium	High
0.89	0.97	0.99

d This exercise highlights highlights the effects of stratification, by Infection Risk Group in this case, which reflects additional prior information.

A more advanced approach, which accounts for how difficult it is to know the real prevalence is a hierarchical model where prevalence is a random variable.

3. Poisson data with a prior, $\pi(\mu) \propto 1/\sqrt{\mu}$. Note: this is an improper prior.

(a) The posterior distribution for μ given $\mathbf{y} = y_1, \cdots, y_n$:

$$\Pr(\mu|\mathbf{y}) \propto \pi(\mu)f(\mathbf{y}|\mu) = \mu^{-0.5}\prod_{i=1}^{n} e^{-\mu}\mu^{\sum_{i=1}^{n} y_i}$$
$$= e^{-n\mu}\mu^{-0.5+n\bar{y}}$$

where $\bar{y}$ is the sample mean. The right-hand term is the kernel of a Gamma$(n\bar{y}+0.5, n)$ pdf.

(b) The posterior distribution for this particular sample is Gamma(52+0.5,10). The posterior mean is the expected value for a Gamma random variable, namely α/β, or in this case, $52.5/10 = 5.25$. The standard deviation is the square root of the variance, $\sqrt{52.5/10^2} = \sqrt{0.525} = 0.725$. Note that as n increases, this posterior mean will converge to the sample mean $\bar{y}$ and the standard deviation will go to 0.

4. The prior predictive distribution for Y, which can take on the values of 0, 1, or 2 is the following.

$$\Pr(Y=0) = \sum_\theta \Pr(Y=0|\theta)\ \pi(\theta) = 0.2*0.5 + 0.4*0.25 + 0.6*0.25 = 0.35$$
$$\Pr(Y=1) = \sum_\theta \Pr(Y=1|\theta)\ \pi(\theta) = 0.2^2*0.5 + 0.4^2*0.25 + 0.6^2*0.25 = 0.15$$
$$\Pr(Y=2) = \sum_\theta \Pr(Y=2|\theta)\ \pi(\theta) = (1-0.2-0.2^2)*0.5 + (1-0.4-0.4^2)*0.25 + (1-0.6-0.6^2)*0.25 = 0.5$$

5. Extra-variation Poisson prior:

   (a) The prior for θ is found by integrating the joint prior for θ and λ over λ.

$$\begin{aligned}
\pi(\theta) &= \int_0^\infty p(\theta,\lambda)d\lambda = \int_0^\infty p(\theta|\lambda)p(\lambda)d\lambda = \int_0^\infty \frac{\exp(-\lambda)\lambda^\theta}{\theta!}\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}\exp(-\beta\lambda)d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)\Gamma(\theta+1)}\int_0^\infty \lambda^{\theta+\alpha-1}\exp(-(\beta+1)\lambda)d\lambda, \\
&\quad \text{where the integrand is the kernel of a Gamma}(\theta+\alpha,\ \beta+1) \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)\Gamma(\theta+1)}\frac{\Gamma(\theta+\alpha)}{(\beta+1)^{(\theta+\alpha)}}\int_0^\infty \frac{(\beta+1)^{(\theta+\alpha)}}{\Gamma(\theta+\alpha)}\lambda^{\theta+\alpha-1}\exp(-(\beta+1)\lambda)d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)\Gamma(\theta+1)}\frac{\Gamma(\theta+\alpha)}{(\beta+1)^{(\theta+\alpha)}}\times 1 \\
&= \frac{\Gamma(\theta+\alpha)}{\Gamma(\alpha)\Gamma(\theta+1)}\left(\frac{\beta}{\beta+1}\right)^\alpha\left(\frac{1}{\beta+1}\right)^\theta, \text{ A Negative Binomial}(\alpha,\beta) \text{ pmf.}
\end{aligned}$$

   (b) The sampling distribution for x is:

$$f(x|\theta) = \frac{\theta!}{x!(\theta-x)!}p^x(1-p)^{\theta-x}$$

Viewing this as a likelihood for θ, where x is now the fixed value, θ must be greater than or equal to x, i.e., $\theta \in \{x, x+1, x+2, \cdots\}$. Thus the posterior for θ is

$$\begin{aligned}
p(\theta|x) \propto \pi(\theta)f(x|\theta) &= \frac{\Gamma(\theta+\alpha)}{\Gamma(\alpha)\Gamma(\theta+1)}\left(\frac{\beta}{\beta+1}\right)^\alpha\left(\frac{1}{\beta+1}\right)^\theta \propto \frac{\Gamma(\theta+\alpha)}{\Gamma(\theta+1)}\left(\frac{1}{\beta+1}\right)^\theta\frac{\theta!}{(\theta-x)!}(1-p)^\theta \\
&= \frac{\Gamma(\theta+\alpha)\Gamma(\theta+1)}{\Gamma(\theta-x+1)\Gamma(\theta+1)}\left(\frac{1-p}{\beta+1}\right)^\theta = \frac{\Gamma(\theta+\alpha)}{\Gamma(\theta-x+1)}\left(\frac{1-p}{\beta+1}\right)^\theta
\end{aligned}$$

for $\theta = x, x+1, x+2, \cdots$.
To make the above final result a valid probability distribution it must sum to 1, i.e., it must be divided by the marginal distribution for the data, $m(x)$:

$$m(x) = \sum_{\theta=x}^\infty \frac{\Gamma(\theta+\alpha)}{\Gamma(\theta-x+1)}\left(\frac{1-p}{\beta+1}\right)^\theta$$

Thus the posterior for θ is:

$$p(\theta|x) = \frac{\frac{\Gamma(\theta+\alpha)}{\Gamma(\theta-x+1)}\left(\frac{1-p}{\beta+1}\right)^\theta}{m(x)}$$

for $\theta = x, x+1, x+2, \cdots$.