UNIVERSITY OF EDINBURGH
SCHOOL OF MATHEMATICS

Generalised Regression Models

GRM: Problem Sheet 2 **Semester 1, 2022–2023**

Work on Questions 1(ii), 2(ii) and 3 in the workshop.

1. Show that the following distributions are members of the exponential family with probability density function of the form

$$\exp\{a(y)b(\theta)+c(\theta)+d(y)\},$$

and use the general results obtained in the lectures to deduce the mean and variance of $a(Y)$ in each case.

(i) Exponential distribution: $f(y;\theta)=\theta e^{-y\theta}$.

(ii) Binomial distribution: $f(y;\theta)=\binom{m}{y}\theta^y(1-\theta)^{m-y}$ (m known).

(iii) Gamma distribution: $f(y;\theta)=\frac{y^{\phi-1}\theta^{\phi}e^{-y\theta}}{\Gamma(\phi)}$ (ϕ known).

(iv) Negative binomial distribution: $f(y;\theta)=\binom{y+r-1}{r-1}\,\theta^r(1-\theta)^y$ (r known).

For those distributions in canonical form, write down the *natural parameter*.

2. For each of the distributions given in Question 1: (i) Exponential, (ii) Binomial, (iii) Gamma, and (iv) Negative binomial.

(a) Determine the maximum likelihood estimate, $\widehat{\theta}$, of θ based on a random sample of observations $y_1,\ldots,y_n$.

(b) Obtain an expression for the asymptotic variance of the maximum likelihood estimator of θ.

3. Suppose that random variables $Y_1,\ldots,Y_n$ are independently and Normally distributed with common variance σ^2 and expectations of the form

$$\mathrm{E}(Y_i|x_i)=\beta_0+\beta_1 x_i \quad (i=1,\ldots,n),$$

where σ, β_0 and β_1 are unknown parameters and the x_i are known values of an explanatory variable.

(a) Show that the joint probability density function of $Y_1,\ldots,Y_n$ is

$$f(y_1,\ldots,y_n;\beta_0,\beta_1,\sigma)=(2\pi)^{-\frac{1}{2}n}\sigma^{-n}\exp(-\frac{1}{2}\sigma^{-2}Q),$$

where $Q=\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)^2$.

(b) Hence show that the maximum likelihood estimates of β_0 and β_1 are the same as their least squares estimates.

(c) Find the maximum likelihood estimate of σ^2.
[NOTE: The MLE of σ^2 is not used in practice; the unbiased estimator of σ^2 is used (see Question 3 on Problem Sheet 1).]

4. A random variable Y has a log normal distribution, i.e.

$$\log(Y) \sim N(\mu, \sigma^2)$$

Let $\nu = E(Y)$ and $\tau = \text{var}(Y)$, then it can be shown that

$$\nu = e^{\mu + \frac{\sigma^2}{2}} \quad \text{and} \quad \tau = \left(e^{\sigma^2} - 1\right) e^{2\mu + \sigma^2}.$$

Obtain MLEs of ν and τ for a sample of size n:

(a) if μ and σ^2 are unknown;
(b) if σ^2 is known, but μ is unknown.

For (b), obtain the expectation of the MLE for ν. Derive Fisher's information for ν in this case, and hence obtain the asymptotic variance for the MLE of ν.

5. Consider the density

$$f(y;\theta) = \exp[y\theta - c(\theta) + d(y)]$$

and let $\mathrm{E}(Y) = \mu$ and $l = \log f(y;\theta)$.

Show that

(a) $\mu = c'(\theta)$ and $\text{var}(Y) = c''(\theta)$
(b) $\frac{dl}{d\theta} = y - \mu$
(c) $\frac{dl}{d\mu} = \frac{y-\mu}{\text{var}(Y)}$

6. A generalization of the 1-parameter exponential family, to allow 2-parameter distributions, is the family defined by,

$$f(y;\theta,\phi) = \exp\left[\frac{y\theta - c(\theta)}{a(\phi)} + d(y,\phi)\right]$$

(a) Show that $\mathrm{E}(Y) = c'(\theta)$ and $\text{var}(Y) = c''(\theta)a(\phi)$
(b) Show that the normal distribution $N(\mu, \sigma^2)$ and the Gamma distribution

$$f(y;\lambda,\alpha) = \frac{\lambda(\lambda y)^{\alpha-1} e^{-\lambda y}}{\Gamma(\alpha)}$$

belong to this family.
(c) The lognormal distribution is defined as the distribution of a random variable Y such that $\log(Y) \sim N(\mu, \sigma^2)$. Find the density of Y and examine whether it belongs to the exponential family defined above.

7. If $l(\theta; \mathbf{y})$ is the log likelihood for a random sample $y_1, \ldots, y_n$ from a density $f(y;\theta)$, where $\theta = (\theta_1, \ldots, \theta_p)'$ is a p-dimensional vector of parameters, show that

$$E\left(\frac{\partial l}{\partial \theta_j}\right) = 0 \qquad \text{and} \qquad \text{cov}\left(\frac{\partial l}{\partial \theta_j}, \frac{\partial l}{\partial \theta_k}\right) = -E\left(\frac{\partial^2 l}{\partial \theta_j \partial \theta_k}\right)$$