MATH11111: Fundamentals of Optimization **Fall 2022**

Lecture 9 Graphical Solution Method

Lecturer: E. Alper Yıldırım *Week: 3*

9.1 Outline

- Linear Programming with One Variable
- Linear Programming with Two Variables
- Review Problems

9.2 Introduction

In this lecture, we will try to start addressing the following question.

Question 1. *How can we solve a linear programming problem?*

Remark 9.1. *Solving an optimization problem means*

- *finding the optimal value and an optimal solution (if any); or*
- *detecting that the problem is unbounded; or*
- *verifying that the problem is infeasible.*

Recall the general linear programming problem:

$$
\begin{array}{lllll}
\text{(P)} & \min & c^T x & & \\
& \text{s.t.} & & & \\
& & (a^i)^T x & \geq & b_i, \quad i \in M_1, \\
& & (a^i)^T x & \leq & b_i, \quad i \in M_2, \\
& & (a^i)^T x & = & b_i, \quad i \in M_3,
\end{array}
$$

where $c \in \mathbb{R}^n$; $a^i \in \mathbb{R}^n$ for each $i \in M_1 \cup M_2 \cup M_3$; $b_i \in \mathbb{R}$ for each $i \in M_1 \cup M_2 \cup M_3$; and M_1, M_2, and M_3 are finite index sets.

9.2.1 One Decision Variable ($n = 1$)

1. If $n = 1$ (i.e., there is only one decision variable $x_1 \in \mathbb{R}$), then each constraint is in the form of $(a_1^i)x_1 \geq b_i$, or $(a_1^i)x_1 \leq b_i$, or $(a_1^i)x_1 = b_i$, where $a_1^i \in \mathbb{R}$ and $b_i \in \mathbb{R}$ for each $i \in M_1 \cup M_2 \cup M_3$.
2. The feasible region is either the empty set, or a single point, or a line segment, or a half line (assuming $a_1^i \neq 0, \ i \in M_1 \cup M_2 \cup M_3$).
3. The objective function is given by $c_1 x_1$, where $c_1 \in \mathbb{R}$.

4. If $c_1 > 0$, the optimal solution is the smallest feasible solution (if any); or the problem is unbounded and the optimal value is given by $z^* = -\infty$; or the problem is infeasible and the optimal value is given by $z^* = +\infty$.

5. If $c_1 < 0$, the optimal solution is the largest feasible solution (if any); or the problem is unbounded and the optimal value is given by $z^* = -\infty$; or the problem is infeasible and the optimal value is given by $z^* = +\infty$.

6. If $c_1 = 0$, then any feasible solution is an optimal solution and $z^* = 0$; or the problem is infeasible and the optimal value is given by $z^* = +\infty$. (Such an optimization problem is called a *feasibility problem.*)

Observations

1. If $c_1 < 0$, we keep moving in the feasible region towards $+\infty$ (i.e., in the direction of $-c_1 > 0$).
2. If $c_1 > 0$, we keep moving in the feasible region towards $-\infty$ (i.e., in the direction of $-c_1 < 0$).
3. If $n = 1$, the set of optimal solutions, denoted by $\mathcal{P}^*$, is either the empty set, a single point (i.e., a vertex or an end point), or equal to the feasible region.
4. Therefore, if $\mathcal{P}^*$ is nonempty and $\mathcal{P}$ contains at least one vertex, then $\mathcal{P}^*$ contains at least one vertex.

9.2.2 Two Decision Variables ($n = 2$)

Let us now assume $n = 2$ (i.e., there are two decision variables x_1 and x_2).

Example 9.1. *Consider the following two-variable linear programming problem:*

$$\begin{array}{lrcrcl} \min & c_1x_1 & + & c_2x_2 & & \\ s.t. & & & & & \\ & x_1 & + & 2x_2 & \leq & 4 \\ & 3x_1 & + & x_2 & \leq & 6 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

We will consider three different objective functions given by (i) $c = [1, -2]^T$, (ii) $c = [-2, -4]^T$, and (iii) $c = [0, 0]^T$.

Let us first ignore the objective function and draw the feasible region. Note that the feasible region does not depend on the objective function. In this example, the feasible region, which is depicted in Figure 9.1, is a nonempty polytope (i.e., bounded polyhedron).

(i) Let $c = [1, -2]^T$. Since this is a minimization problem, the improving direction is given by $-c = [-1, 2]^T$, which points in the northwest direction (see Figure 9.2). Therefore, the unique optimal solution is given by the intersections of the boundaries of the two constraints $x_1 + 2x_2 \leq 4$ and $x_1 \geq 0$. Therefore, we can solve for the following system simultaneously:

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_1 \qquad &= 0 \end{aligned}$$

We obtain $x_1 = 0$ and $x_2 = 2$. Therefore, $\mathcal{P}^* = \{[0, 2]^T\}$, i.e., it is given by a single vertex. Substituting this solution into the objective function, we obtain $z^* = 1 \cdot (0) - 2 \cdot 2 = -4$.

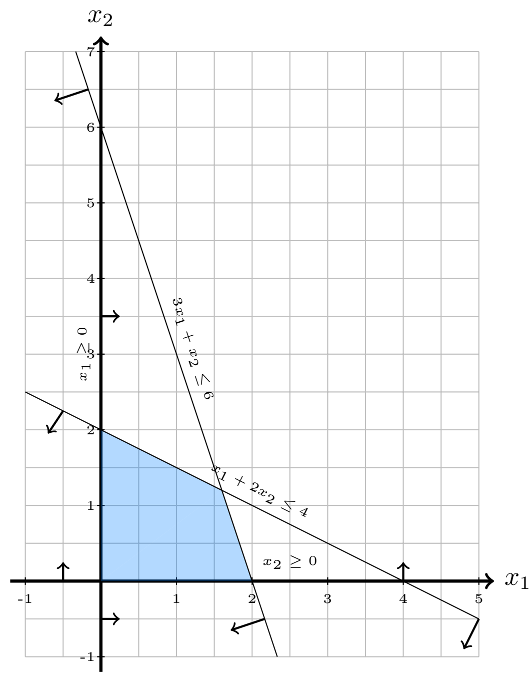

Figure 9.1: Feasible region of Example 9.1 (blue region)

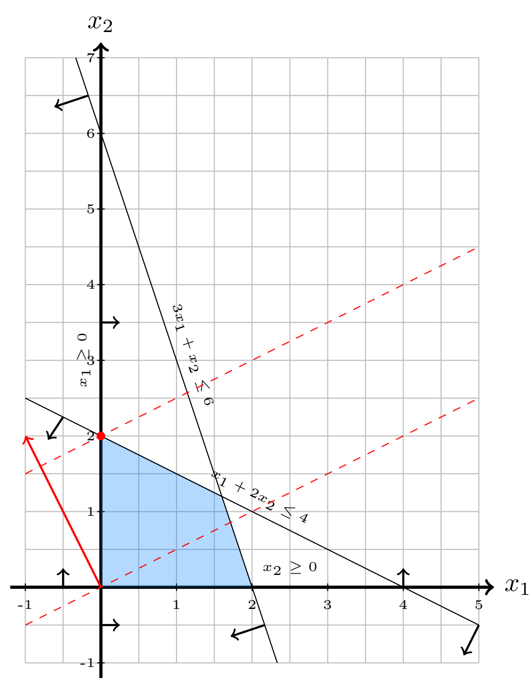

Figure 9.2: Feasible region of Example 9.1(i) (blue region), improving direction (solid red arrow), contour lines of the objective function (dashed red lines), and the optimal solution (red circle)

(ii) *Let* $c = [-2, -4]^T$. *Since this is a minimization problem, the improving direction is given by* $-c = [2, 4]^T$, *which points in the northeast direction (see Figure 9.3). Therefore, any feasible solution on the boundary of the constraint* $x_1 + 2x_2 \leq 4$ *is an optimal solution.*

To compute the coordinates of the end points of this line segment, we first consider the end point at the top left. Note that this point is given by the intersections of the boundaries of the two constraints $x_1 + 2x_2 \leq 4$ *and* $x_1 \geq 0$. *By part (i), we know that* $x^1 = [0, 2]^T$.

Considering now the end point at the bottom right, this point is given by the intersections of the boundaries of the two constraints $x_1 + 2x_2 \leq 4$ *and* $3x_1 + x_2 \leq 6$. *Therefore, we can solve for the following system simultaneously:*

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 + x_2 &= 6 \end{aligned}$$

We obtain $x_1^2 = 8/5$ *and* $x_2^2 = 6/5$. *Therefore,* $\mathcal{P}^* = \{\lambda x^1 + (1-\lambda)x^2 : \lambda \in [0, 1]\}$, *i.e., it is given by a line segment. Note that any point on this line segment is an optimal solution. Substituting any point on this line segment into the objective function (say,* $[0, 2]^T$*), we obtain* $z^* = -2 \cdot (0) - 4 \cdot 2 = -8$.

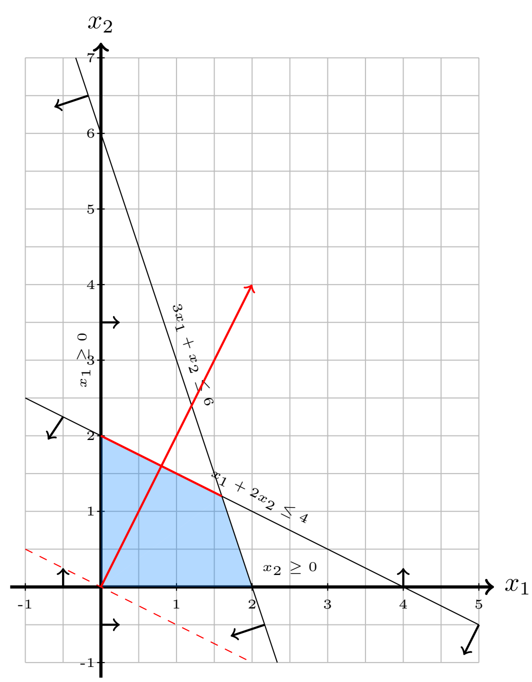

Figure 9.3: Feasible region of Example 9.1(ii) (blue region), improving direction (solid red arrow), contour lines of the objective function (dashed red line), and the set of optimal solutions (red line segment)

(iii) *Let* $c = [0, 0]^T$. *Note that, in this case, the objective function does not depend on* x_1 *and* x_2, *i.e., the objective function value of* any *feasible solution is equal to* 0. *Therefore, in this case any feasible solution is an optimal solution, i.e.,* $\mathcal{P}^* = \mathcal{P}$, *where* $\mathcal{P}$ *denotes the feasible region. The optimal value is given by* $z^* = 0$ *(see Figure 9.4).*

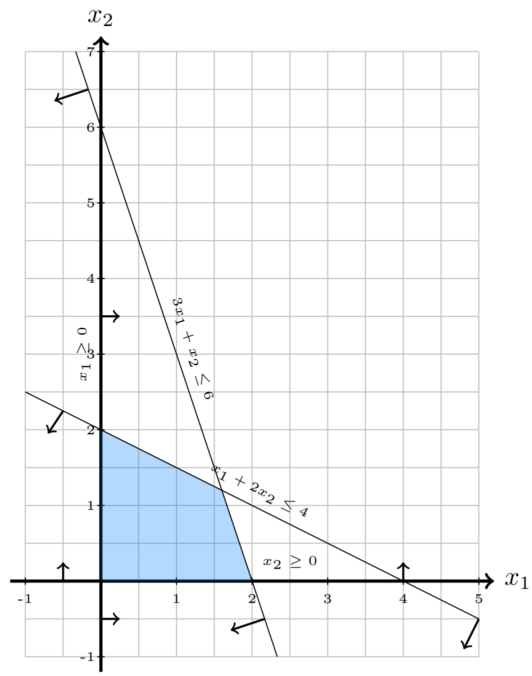

Figure 9.4: Feasible region of Example 9.1(iii) (blue region) and the set of optimal solutions (blue region)

Suppose now that we change the second constraint in Example 9.1 as follows:

$$
\begin{array}{lrcrcr}
\min & c_1x_1 & + & c_2x_2 & & \\
\text{s.t.} & & & & & \\
& x_1 & + & 2x_2 & \leq & 4 \\
& x_1 & + & x_2 & \leq & -1 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0
\end{array}
$$

We will consider three different objective functions given by (i) $c = [1, -2]^T$, (ii) $c = [-2, -4]^T$, and (iii) $c = [0, 0]^T$.

Once again, let us first ignore the objective function and draw the feasible region. Recall that the feasible region does not depend on the objective function. As illustrated by Figure 9.5, the halfspaces corresponding to the four constraints do not have a common point. In fact, it is easy to see that the three constraints $x_1 \geq 0$, $x_2 \geq 0$, and $x_1 + x_2 \leq -1$ cannot be satisfied simultaneously. Therefore, the feasible region of this modified problem is the empty set, i.e., $\mathcal{P} = \emptyset$. As a result, regardless of the objective function, this linear programming problem is infeasible and we define the optimal value $z^* = +\infty$ since it is a minimization problem. Clearly, $\mathcal{P}^* = \emptyset$, i.e., there is no optimal solution since there is no feasible solution.

Observations

1. If $n = 2$ and $\mathcal{P}$ is a nonempty polytope, then $\mathcal{P}^*$ is always nonempty, and $\mathcal{P}^*$ is either a single vertex, or a line segment, or $\mathcal{P}^* = \mathcal{P}$.

2. Note that $\mathcal{P}$ always contains at least one vertex (i.e., a corner point).

3. Note that $\mathcal{P}^*$ always contains at least one vertex of $\mathcal{P}$.

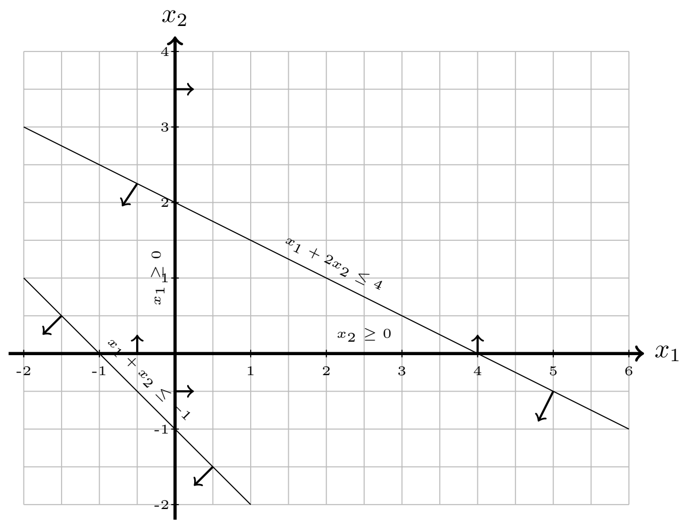

Figure 9.5: Feasible region of the modified Example 9.1

4. Such a linear programming problem cannot be unbounded.

5. If $\mathcal{P} = \emptyset$ (i.e., the problem is infeasible), then $\mathcal{P}^* = \emptyset$. Recall that, in this case, we define $z^* = +\infty$ for a minimization problem.

Example 9.2. *Consider the following linear programming problem:*

$$\begin{array}{lrcrcl} \min & c_1x_1 & + & c_2x_2 & & \\ s.t. & & & & & \\ & -x_1 & + & 2x_2 & \leq & 2 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

We will consider five different objective functions given by (i) $c = [1, -1]^T$, (ii) $c = [1, 0]^T$, (iii) $c = [2, -4]^T$, (iv) $c = [-1, 1]^T$, and (v) $c = [0, 0]^T$.

Once again, let us first ignore the objective function and draw the feasible region. Recall that the feasible region does not depend on the objective function. In this example, the feasible region, which is depicted in Figure 9.6, is a nonempty polyhedron. Note that, in contrast with Example 9.1, it is unbounded, i.e., it is a polyhedron but not a polytope.

(i) Let $c = [1, -1]^T$. Since this is a minimization problem, the improving direction is given by $-c = [-1, 1]^T$, which points in the northwest direction (see Figure 9.7). Therefore, the unique optimal solution is given by the intersections of the boundaries of the two constraints $x_1 + 2x_2 \leq 2$ and $x_1 \geq 0$. Therefore, we can solve for the following system simultaneously:

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ x_1 &= 0 \end{aligned}$$

We obtain $x_1 = 0$ and $x_2 = 1$. Therefore, $\mathcal{P}^* = \{[0, 1]^T\}$, i.e., it is given by a single vertex. Substituting this solution into the objective function, we obtain $z^* = 1 \cdot (0) - 1 \cdot 1 = -1$.

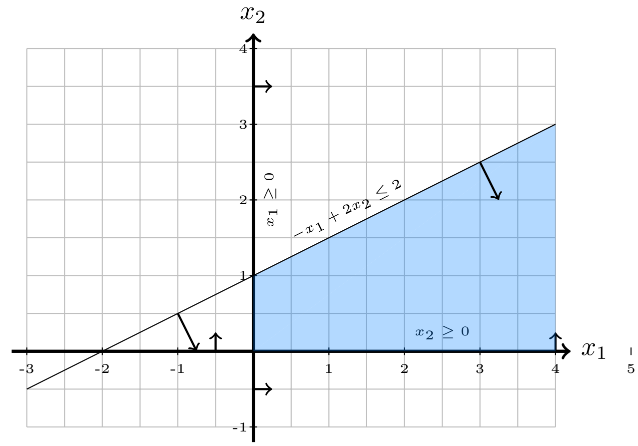

Figure 9.6: Feasible region of Example 9.2 (blue region)

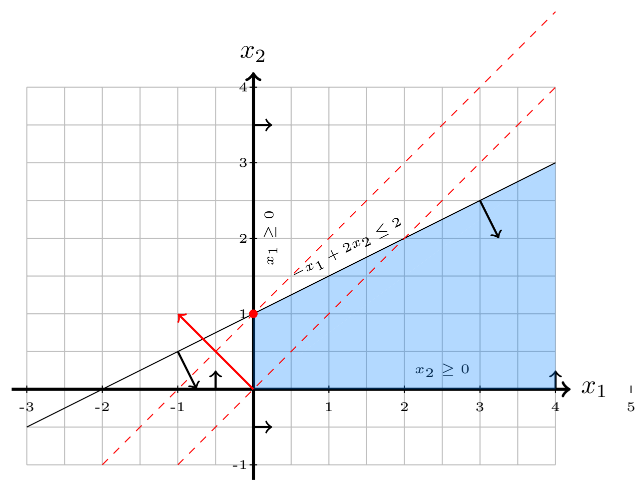

Figure 9.7: Feasible region of Example 9.2(i) (blue region), improving direction (solid red arrow), contour lines of the objective function (dashed red lines), and the optimal solution (red circle)

(ii) *Let $c = [1, 0]^T$. Since this is a minimization problem, the improving direction is given by $-c = [-1, 0]^T$, which points to the west parallel to the x_1 axis (see Figure 9.8). Therefore, any feasible solution on the boundary of the constraint $x_1 \geq 0$ is an optimal solution.*

To compute the coordinates of the end points of this line segment, we first consider the end point at the bottom. Note that this point is given by the intersections of the boundaries of the two constraints $x_1 \geq 0$ and $x_2 \geq 0$. Solving the equation system simultaneously, we obtain $x^1 = [0, 0]^T$.

Considering now the end point at the top, this point is given by the intersections of the boundaries of the two constraints $-x_1 + 2x_2 \leq 2$ and $x_1 \geq 0$. Solving the equation system simultaneously, we obtain $x^2 = [0, 1]^T$.

Therefore, $\mathcal{P}^* = \{\lambda x^1 + (1-\lambda)x^2 : \lambda \in [0, 1]\}$, i.e., it is given by a line segment. Note that any point on this line segment is an optimal solution. Substituting any point on this line segment into the objective function (say, $[0, 0]^T$), we obtain $z^* = 1 \cdot (0) - 0 \cdot (0) = 0$.

(iii) *Let $c = [2, -4]^T$. Since this is a minimization problem, the improving direction is given by $-c =$*

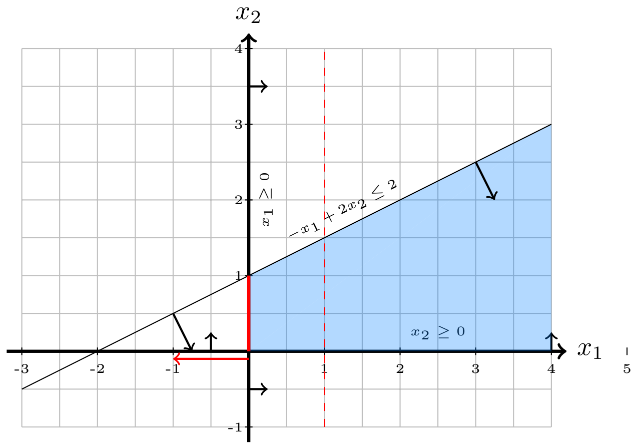

Figure 9.8: Feasible region of Example 9.2(ii) (blue region), improving direction (solid red arrow), contour lines of the objective function (dashed red lines), and the set of optimal solutions (red line segment)

$[-2, 4]^T$, which points in the northwest direction (see Figure 9.9). Therefore, any feasible solution on the boundary of the constraint $-x_1 + 2x_2 \leq 2$ is an optimal solution.

To compute the end point of this half line, we note that it is given by the intersection of the boundary of the constraints $-x_1 + 2x_2 \leq 2$ and $x_1 \geq 0$. Solving the equation system simultaneously, we obtain $x^1 = [0, 1]^T$.

A half line is given by a starting point and a direction. To find the direction, simply pick any point on this half line different from x^1, say $x^2 = [2, 2]^T$. The direction can be computed by $d = x^2 - x^1 = [2, 1]^T$. Therefore, $\mathcal{P}^* = \{x^1 + \lambda d : \lambda \geq 0\}$, i.e., it is given by a half line. Note that any point on this half line is an optimal solution. Substituting any point on this line segment into the objective function (say, $[0, 1]^T$), we obtain $z^* = 2 \cdot (0) - 4 \cdot (1) = -4$.

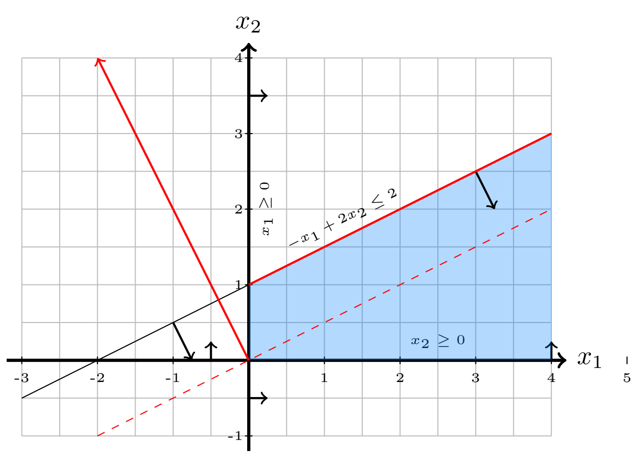

Figure 9.9: Feasible region of Example 9.2(iii) (blue region), improving direction (solid red arrow), contour lines of the objective function (dashed red lines), and the set of optimal solutions (red half line)

(iv) *Let $c = [-1, 1]^T$. Since this is a minimization problem, the improving direction is given by $-c = [1, -1]^T$, which points in the southeast direction (see Figure 9.10). Note that the contour lines of the objective function will always have a nonempty intersection with the feasible region in the improving* *direction. Therefore, this problem is unbounded, i.e.,* $z^* = -\infty$ *and* $\mathcal{P}^* = \emptyset$ *since no feasible solution achieves this objective function value (i.e., there is no best feasible solution).*

You can verify that starting at $x^1 = [0,0]^T$ *and moving in the direction* $d = [1,0]^T$*, if you consider the half line* $\{x^1 + \lambda d : \lambda \geq 0\}$*, which is contained in the feasible region, the objective function value along this half line is given by* $z(\lambda) = (-1) \cdot (0 + \lambda) + 1 \cdot (0) = -\lambda$*, which tends to* $-\infty$ *as* λ *tends to* $+\infty$*. Such a direction is called* a direction of unboundedness.

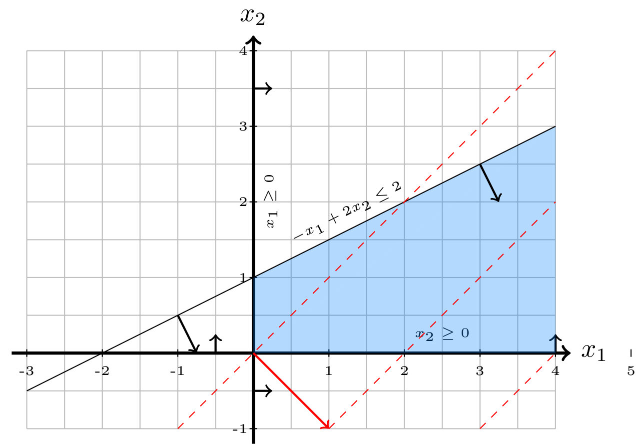

Figure 9.10: Feasible region of Example 9.2(iv) (blue region), improving direction (solid red arrow), and contour lines of the objective function (dashed red lines)

(v) $c = [0,0]^T$. *As in Example 9.1(iii), the objective function does not depend on* x_1 *and* x_2*, i.e., the objective function value of* any *feasible solution is equal to* 0*. Therefore, in this case any feasible solution is an optimal solution, i.e.,* $\mathcal{P}^* = \mathcal{P}$*, where* $\mathcal{P}$ *denotes the feasible region. The optimal value is given by* $z^* = 0$ *(see Figure 9.11).*

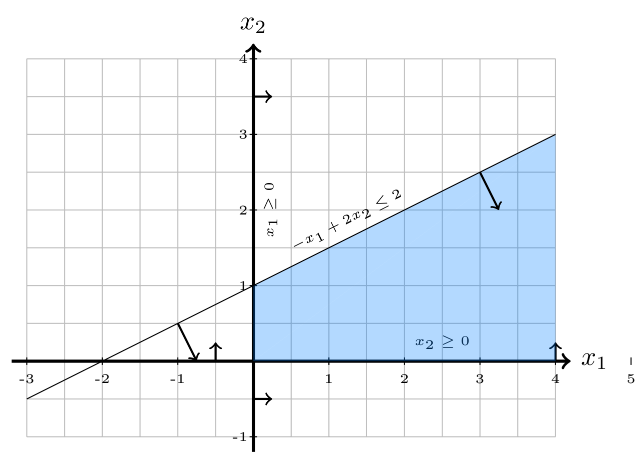

Figure 9.11: Feasible region of Example 9.2(v) (blue region) and the set of optimal solutions (blue region)

Observations

1. If $n = 2$ and $\mathcal{P}$ is a nonempty and unbounded polyhedron that has at least one vertex, then $\mathcal{P}^*$ is either the empty set, or a single vertex, or a line segment, or a half line, or $\mathcal{P}^* = \mathcal{P}$.
2. If $\mathcal{P}$ contains at least one vertex and $\mathcal{P}^* \neq \emptyset$, then $\mathcal{P}^*$ always contains at least one vertex of $\mathcal{P}$.
3. Such a linear programming problem may have a finite optimal value or may be unbounded (even though $\mathcal{P}$ itself is unbounded).
4. If $\mathcal{P} = \emptyset$, then $\mathcal{P}^* = \emptyset$.
5. Recall that a nonempty and unbounded polyhedron may not have any vertices (if it contains a line).

Remarks

1. Linear programming problems with one or two decision variables can be easily solved by using the graphical method.
2. We keep moving in the feasible region towards the improving direction, given by $-c$ for a minimization problem, until the last point of contact (if any).
3. The graphical method can be extended to linear programming problems with three decision variables but drawing three-dimensional objects is much more tricky.
4. For linear programming problems with at least four decision variables, we need to develop a different algorithm.

Exercises

Question 9.1. *Can you construct a linear programming problem with exactly two optimal solutions?*

MATH11111: Fundamentals of Optimization	**Fall 2022**
Lecture 10	Optimality of Vertices and Vertex Enumeration
Lecturer: E. Alper Yıldırım	*Week: 3*

10.1 Outline

- Optimality of Vertices
- Vertex Enumeration
- Review Problems

10.2 Optimality of Vertices

In Lecture 9, we discussed the graphical solution method for solving linear programming problems with one or two decision variables. In each of the examples, if the feasible region $\mathcal{P}$ contains at least one vertex and the set of optimal solutions $\mathcal{P}^*$ is nonempty, then $\mathcal{P}^*$ contains at least one vertex of $\mathcal{P}$. In this lecture, we will show that this property is satisfied by every polyhedron.

Proposition 10.1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x \geq b_i,\ i \in M_1; (a^i)^T x \leq b_i,\ i \in M_2; (a^i)^T x = b_i,\ i \in M_3\}$ be a nonempty polyhedron that contains at least one vertex and let $c \in \mathbb{R}^n$. Consider the linear programming problem given by*

$$\min\{c^T x : x \in \mathcal{P}\}.$$

Suppose that the set of optimal solutions, denoted by $\mathcal{P}^*$, is nonempty. Then, $\mathcal{P}^*$ contains at least one vertex of $\mathcal{P}$ (i.e., there exists at least one optimal solution which is a vertex of $\mathcal{P}$).

Proof. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron and let $\mathcal{P}^* \subseteq \mathcal{P}$ denote the set of optimal solutions. Let $z^* \in \mathbb{R}$ denote the optimal value of the linear programming problem $\min\{c^T x : x \in \mathcal{P}\}$. Then, $c^T x^* = z^*$ for any $x^* \in \mathcal{P}^*$. Therefore, $\mathcal{P}^* = \mathcal{P} \cap \{x \in \mathbb{R}^n : c^T x = z^*\}$. It follows that $\mathcal{P}^*$ itself is a polyhedron. Since $\mathcal{P}^* \subseteq \mathcal{P}$ and $\mathcal{P}$ does not contain a line, then $\mathcal{P}^*$ does not contain a line. Therefore, $\mathcal{P}^*$ has at least one vertex by Proposition 8.1.

Let $x^* \in \mathcal{P}^*$ denote a vertex of $\mathcal{P}^*$. We claim that x^* is also a vertex of $\mathcal{P}$. Let

$$I(x^*) = \left\{i \in M_1 \cup M_2 \cup M_3 : (a^i)^T x^* = b_i\right\}.$$

If the set $\{a^i : i \in I(x^*)\}$ contains n linearly independent vectors, then we are done since x^* is a vertex of $\mathcal{P}$. Otherwise, there exists a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $(a^i)^T d = 0$ for each $i \in I(x^*)$. By using a similar argument as in the proof of Proposition 7.1, there exists a real number $\epsilon^* > 0$ such that $x^* - \epsilon^* d \in \mathcal{P}$ and $x^* + \epsilon^* d \in \mathcal{P}$. Since $c^T x \geq z^*$ for each $x \in \mathcal{P}$, we obtain $c^T(x^* - \epsilon^* d) = z^* - \epsilon^* c^T d \geq z^*$ and $c^T(x^* + \epsilon^* d) = z^* + \epsilon^* c^T d \geq z^*$. Therefore, $c^T d = 0$. However, x^* is a vertex of $\mathcal{P}^*$, which implies that the set $\{c\} \cup \{a^i : i \in I(x^*)\}$ contains n linearly independent vectors. Therefore, $d = \mathbf{0}$, which is a contradiction. It follows that x^* is also a vertex of $\mathcal{P}$.

□

Let $\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x \geq b_i, \; i \in M_1; (a^i)^T x \leq b_i, \; i \in M_2; (a^i)^T x = b_i, \; i \in M_3\}$ be a nonempty polyhedron that contains at least one vertex. By Proposition 6.1, since $\mathcal{P}$ is a convex set, a vector $\hat{x} \in \mathcal{P}$ is a vertex of $\mathcal{P}$ if and only if **there exists** a vector a, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, such that $\hat{x}$ is the **unique** optimal solution of the linear programming problem $\min\{a^T x : x \in \mathcal{P}\}$. By Proposition 10.1, for **every** vector $c \in \mathbb{R}^n$, there exists at least one vertex of $\mathcal{P}$ which is an optimal solution of $\min\{c^T x : x \in \mathcal{P}\}$, under the assumption that $\mathcal{P}^*$ is nonempty.

10.3 A Naive Enumeration Algorithm

By Proposition 10.1, if $\mathcal{P}$ has at least one vertex and $\mathcal{P}^* \neq \emptyset$, then the set of optimal solutions contains at least one vertex. By Proposition 8.2, every polyhedron has a finite number of vertices. Therefore, by combining these two observations, we may consider a naive enumeration idea for solving general linear programming problems by simply computing all the vertices, evaluating the objective function value at each vertex, and returning the vertex with the best objective function value:

Naive Enumeration Algorithm:

1. Compute all vertices of the polyhedron.
2. Compute the objective function at each vertex.
3. Output the vertex with the best objective function value.

10.3.1 Computing Vertices

Consider a general polyhedron given by

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T x \geq b_i, & i \in M_1, \\ (a^i)^T x \leq b_i, & i \in M_2, \\ (a^i)^T x = b_i, & i \in M_3 \end{array} \right\}.$$

Below, we give an algorithm for enumerating all vertices by using the equivalence between vertices of a polyhedron and basic feasible solutions (see Proposition 7.1).

Algorithm for Enumeration of Vertices of a Polyhedron:

1. For each subset $J \subseteq M_1 \cup M_2$ do
   (a) Let $I = M_3 \cup J$.
   (b) If the set $\{a^i : i \in I\}$ contains n linearly independent vectors, then let $I^* \subseteq I$ be such that the set $\{a^i : i \in I^*\}$ contains exactly n linearly independent vectors.
      i. Solve the system $(a^i)^T x = b_i, \; i \in I^*$ to obtain the unique solution $\hat{x} \in \mathbb{R}^n$. ($\hat{x}$ is a basic solution.)
      ii. Check if $\hat{x} \in \mathcal{P}$ (i.e., if $\hat{x}$ is a basic feasible solution).

Note that the above algorithm terminates after a finite number of iterations by successfully computing all vertices of $\mathcal{P}$ (if any) since there are only a finite number of choices for the set I. Therefore, this algorithm can be used in Step 1 of the Naive Enumeration Algorithm. However, it has a number of shortcomings outlined below:

1. A polyhedron may have an exponential number of vertices.
2. A polyhedron may not contain a vertex.
3. A polyhedron may not contain any vectors at all (i.e., it can be equal to the empty set).
4. The naive enumeration algorithm cannot detect if a linear programming problem is infeasible or unbounded.

Therefore, we need to develop a more clever algorithm for solving linear programming problems that can also correctly detect unboundedness and infeasibility.

Exercises

Question 10.1. *For each value of $k = 0, 1, \ldots$, show that you can construct a linear programming problem with two decision variables such that the set of optimal solutions $\mathcal{P}^*$ contains exactly k vertices.*

MATH11111: Fundamentals of Optimization	**Fall 2022**
Lecture 11 Polyhedra and Linear Programming in Standard Form	
Lecturer: E. Alper Yıldırım	*Week: 3*

11.1 Outline

- Polyhedra in Standard Form
- Linear Programming in Standard Form
- Review Problems

11.2 Introduction and Motivation

Recall that we can solve linear programming problems with up to 3 variables geometrically using the graphical solution method. If there is a larger number of variables, we need an alternative method. The naive vertex enumeration algorithm outlined in Lecture 10 has several drawbacks. In this lecture, we will start to discuss building blocks to develop a more effective algorithm for solving a general linear programming problem. To that end, we will identify a particular convenient form for a polyhedron and a linear programming problem.

11.3 Polyhedra in Standard Form

Definition 11.1. *A polyhedron* $\mathcal{P} \subseteq \mathbb{R}^n$ *is said to be* in standard form *if* $\mathcal{P}$ *is given by*

$$\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x = b_i,\ i = 1, \ldots, m;\ x_j \geq 0,\ j = 1, \ldots, n\}.$$

11.3.1 Relation with General Polyhedra

Consider a general polyhedron given by

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T x \geq b_i, & i \in M_1, \\ (a^i)^T x \leq b_i, & i \in M_2, \\ (a^i)^T x = b_i, & i \in M_3 \end{array} \right\}.$$

Therefore, $\mathcal{P}$ is in standard form if

- each constraint $(a^i)^T x \geq b_i,\ i \in M_1$ is given by $(e^j)^T x = x_j \geq 0$ for some $j \in J_1$, where $J_1 \subseteq \{1, \ldots, n\}$; and
- each constraint $(a^i)^T x \leq b_i,\ i \in M_2$ is given by $-(e^j)^T x = -x_j \leq 0$ for some $j \in J_2$, where $J_2 \subseteq \{1, \ldots, n\}$; and

- $J_1 \cup J_2 = \{1, \ldots, n\}$.

It follows that a polyhedron in standard form is a special case of a general polyhedron.

11.4 Linear Programming in Standard Form

Definition 11.2. *A linear programming is said to be* in standard form *if it is given by*

$$\begin{array}{llrcll} (P) & \min & c^T x & & & \\ & s.t. & & & & \\ & & (a^i)^T x & = & b_i, & i = 1, \ldots, m, \\ & & x_j & \geq & 0, & j = 1, \ldots, n. \end{array}$$

Remark 11.1. *Similar to the relation between a polyhedron in standard form and a general polyhedron, a linear programming in standard form is a special case of general linear programming. Note that the objective function is minimized in a linear programming problem in standard form.*

11.4.1 Transformation to Standard Form

Proposition 11.1. *Any linear programming problem can be transformed into an equivalent linear programming problem in standard form.*

Remark 11.2.

1. *"Equivalent" means there is a one-to-one correspondence between the feasible solutions of two problems and they have the same optimal value.*

2. *Therefore, a linear programming in standard form is not really a special case.*

Proof. If (P) is already in standard form, we are done. Otherwise, for each $i \in M_1$ such that the corresponding constraint is not given by $(a^i)^T x = (e^j)^T x = x_j \geq 0$ for some $j \in \{1, \ldots, n\}$, we define a new nonnegative variable s_i and replace $(a^i)^T x \geq b_i$ by $(a^i)^T x - s_i = b_i$ and $s_i \geq 0$. For each $i \in M_2$ such that the corresponding constraint is not given by $(a^i)^T x = -(e^j)^T x = -x_j \leq 0$ for some $j \in \{1, \ldots, n\}$, we define a new nonnegative variable s_i and replace $(a^i)^T x \leq b_i$ by $(a^i)^T x + s_i = b_i$ and $s_i \geq 0$. Therefore, we replace each inequality constraint by equality constraints. For each $j = 1, \ldots, n$ such that $x_j \geq 0$ is not included in the inequality constraints, if $x_j \leq 0$ is a constraint, then we can define a new variable by $x'_j = -x_j \geq 0$ and replace each occurrence of x_j by $-x'_j$. Otherwise, we can define two new nonnegative variables $x_j^+ \geq 0$ and $x_j^- \geq 0$ and replace each occurrence of x_j by $x_j^+ - x_j^-$. We therefore obtain an equivalent linear programming problem in standard form. □

11.4.2 Advantages of Linear Programming in Standard Form

1. Linear programming problems in standard form have a simpler structure than general linear programming problems.

2. This transformation allows us to develop an algorithm only for linear programming problems in standard form and use it to solve all linear programming problems.

3. A polyhedron in standard form has certain desirable properties.

11.4.3 Existence of Vertices in Polyhedra in Standard Form

Proposition 11.2. *Let*

$$\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x = b_i,\ i = 1, \ldots, m;\ x_j \geq 0,\ j = 1, \ldots, n\}$$

be a nonempty polyhedron in standard form. Then, $\mathcal{P}$ *has at least one vertex.*

Proof. Since $\mathcal{P}$ is a nonempty polyhedron, it suffices to show that $\mathcal{P}$ does not contain a line and then the claim follows directly from Proposition 8.1. Note that $\mathcal{P} \subseteq \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_j \geq 0,\ j = 1, \ldots, n\}$. Since $\mathbb{R}^n_+$ does not contain a line, $\mathcal{P}$ cannot contain a line. Therefore, $\mathcal{P}$ has at least one vertex. □

Remark 11.3. *Note that, in contrast, a general nonempty polyhedron may not have any vertices.*

Exercises

Question 11.1. *Convert the following linear programming problem into standard form:*

$$\begin{array}{lccccccccc}
\max & -x_1 & - & x_2 & + & x_3 & - & 2x_4 & & \\
s.t. & & & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & 6 \\
 & & & x_2 & - & 3x_3 & + & 2x_4 & \leq & 3 \\
 & x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & \geq & 9 \\
 & x_1 \geq 0 & , & x_2 \geq 0 & , & x_3 \leq 0 & . & & &
\end{array}$$

MATH11111: Fundamentals of Optimization	**Fall 2022**
Lecture 12 Full Row Rank Assumption in Standard Form Polyhedra	
Lecturer: E. Alper Yıldırım	*Week: 3*

12.1 Outline

- Polyhedra in Standard Form: Different Cases
- Full Row Rank Assumption
- Review Problems

12.2 A Compact Representation

Recall that a linear programming in standard form is given by

$$\begin{array}{llrcll} \text{(P)} & \min & c^T x & & & \\ & \text{s.t.} & & & & \\ & & (a^i)^T x & = & b_i, & i = 1, \ldots, m, \\ & & x_j & \geq & 0, & j = 1, \ldots, n. \end{array}$$

Let us define $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $x \in \mathbb{R}^n$ as follows:

$$A = \begin{bmatrix} (a^1)^T \\ (a^2)^T \\ \vdots \\ (a^m)^T \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Then, (P) can be represented by

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\}$$

12.3 Polyhedra in Standard Form

Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form. Then, $\mathcal{P} = \mathcal{L} \cap \mathbb{R}^n_+$, where $\mathcal{L} = \{x \in \mathbb{R}^n : Ax = b\}$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq \mathbf{0}\}$. Consider the columns of $A \in \mathbb{R}^{m \times n}$:

$$A = \begin{bmatrix} A^1 & A^2 & \ldots & A^n \end{bmatrix},$$

where $A^1, \ldots, A^n \in \mathbb{R}^m$. Therefore, $Ax = b$ if and only if $b = \sum_{j=1}^n A^j x_j$, i.e., if and only if $b \in \text{span}\{A^1, \ldots, A^n\}$.

Let $r = \text{rank(A)}$. Note that r equals the largest number of linearly independent columns $\{A^1, \ldots, A^n\}$, or equivalently, the largest number of linearly independent rows $\{(a^1)^T, \ldots, (a^m)^T\}$. Therefore, $r \leq \min\{m, n\}$.

Case 1: Suppose that $b \notin \text{span}\{A^1, \ldots, A^n\}$. Then, $\mathcal{L} = \{x \in \mathbb{R}^n : Ax = b\} = \emptyset$, and $\mathcal{P} = \mathcal{L} \cap \mathbb{R}^n_+ = \emptyset$. In this case, we have $r < m$, i.e., A does not have full row rank.

Case 2: Suppose now that $b \in \text{span}\{A^1, \ldots, A^n\}$.

> **Case 2a:** Suppose that $r = n \leq m$. In this case, the set $\{A^1, \ldots, A^n\}$ is linearly independent, i.e., A has full column rank. Then, there is a unique solution $\hat{x} \in \mathbb{R}^n$ such that $A\hat{x} = b$, i.e., $\mathcal{L} = \{\hat{x}\}$. If $\hat{x} \geq \mathbf{0}$, then $\mathcal{P} = \mathcal{L} = \{\hat{x}\}$, otherwise $\mathcal{P} = \emptyset$.
>
> **Case 2b:** Suppose that $r = m \leq n$. In this case, the set $\{A^1, \ldots, A^n\} \subset \mathbb{R}^m$ contains m linearly independent vectors, or A has full row rank, i.e., $\text{span}\{A^1, \ldots, A^n\} = \mathbb{R}^m$. For any $b \in \mathbb{R}^m$, the set $\mathcal{L} = \{x \in \mathbb{R}^n : Ax = b\} \neq \emptyset$. If there exists $\hat{x} \in \mathcal{L}$ such that $\hat{x} \geq \mathbf{0}$, then $\mathcal{P} \neq \emptyset$. Otherwise, $\mathcal{P} = \emptyset$.
>
> **Case 2c:** Suppose now that $r < m$ and $r < n$. In this case, A does not have full row rank and does not have full column rank. Therefore, the set $\{a^1, \ldots, a^m\} \subset \mathbb{R}^n$ contains r linearly independent vectors.
>
> Suppose that $\{a^1, \ldots, a^r\}$ are linearly independent (otherwise we can rearrange the columns). Then, for each $j = r+1, \ldots, m$, there exist real numbers $\lambda_1^j, \ldots, \lambda_r^j$ such that $a^j = \sum_{i=1}^{r} \lambda_i^j a^i$. Since $b \in \text{span}\{A^1, \ldots, A^n\}$, there exists $\hat{x} \in \mathbb{R}^n$ such that $A\hat{x} = b$, i.e., $(a^i)^T\hat{x} = b_i,\ i = 1, \ldots, m$.
>
> For each $j = r+1, \ldots, m$, if we multiply $(a^i)^T\hat{x} = b_i$ by λ_i^j for each $i = 1, \ldots, r$ and add them up, we obtain $\sum_{i=1}^{r} \lambda_i^j (a^i)^T\hat{x} = \sum_{i=1}^{r} \left(\lambda_i^j a^i\right)^T \hat{x} = (a^j)^T\hat{x} = \sum_{i=1}^{r} \lambda_i^j b_i$. Similarly, for each $j = r+1, \ldots, m$, since $(a^j)^T\hat{x} = b_j$, we obtain $b_j = \sum_{i=1}^{r} \lambda_i^j b_i$.
>
> Therefore, for each $j = r+1, \ldots, m$, the equation $(a^j)^T\hat{x} = b_j$ is given by a linear combination of the first r equations and is therefore redundant. We can remove the redundant equations $(a^j)^T x = b_j$ for each $j = r+1, \ldots, m$ without changing the set of solutions. Let
>
> $$\hat{A} = \begin{bmatrix} (a^1)^T \\ (a^2)^T \\ \vdots \\ (a^r)^T \end{bmatrix} \in \mathbb{R}^{r \times n}, \quad \hat{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix} \in \mathbb{R}^r.$$
>
> Then, $Ax = b$ if and only if $\hat{A}x = \hat{b}$. Furthermore, $\text{rank}(\hat{A}) = r$, i.e., $\hat{A}$ has full row rank. Therefore, if $b \in \text{span}\{A^1, \ldots, A^n\}$ and $r < m$ and $r < n$, we can remove the redundant equations from $Ax = b$ and the reduced system has full row rank. Observe that the reduced system is in the form of Case 2b.

Let us now summarise the different cases for the system $Ax = b$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. If A does not have full row rank, then there are two possibilities:

1. The system $Ax = b$ has no solution (see Case 1) and $\mathcal{L} = \mathcal{P} = \emptyset$.

2. The system $Ax = b$ has at least one solution and the system can be reduced to $\hat{A}x = \hat{b}$, where $\hat{A}$ has full row rank (see Case 2c).

Both of these cases can be easily checked using Gaussian elimination.

12.3.1 Full Row Rank Assumption on A

Assumption 12.1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be polyhedron in standard form. Suppose that $\mathcal{L} = \{x \in \mathbb{R}^n : Ax = b\} \neq \emptyset$. Then, we can assume that A has full row rank.*

Exercises

Question 12.1. *Consider the system $Ax = b$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.*

(i) Show that the set of solutions to this system of equations is a convex set.

(ii) Show that the set of solutions to this system of equations is either the empty set, or consists of one solution, or an infinite number of solutions.