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Fundamentals of Optimization

Exercise 2 – Solutions

Remarks

- All questions that are available in the STACK quiz are duly marked. Please solve those using STACK.
- We have added marks for each question. Please note that those are purely for illustrative purposes. The exercise set will not be marked.

STACK Problems

1 Basic Solutions and Basic Feasible Solutions (3 marks)

STACK question

Consider the following polyhedron

$$\mathcal{P} = \{x \in \mathbb{R}^3 : -4x_1+x_2-x_3 \leq -4,\ -2x_1+x_2-x_3 \leq -2,\ -4x_2+x_3 \leq 4,\ 4x_2+x_3 = 0,\ x_3 \leq 5,\ x_3 \geq 0\}\,.$$

Decide, for each of the points $\hat{x}$ given below, whether $\hat{x}$ is *infeasible and not a basic solution*, *feasible but not a basic feasible solution*, *a basic solution but infeasible*, or *a basic feasible solution*.

(1.1) $\hat{x} = [0, 0, 0]^T$.

(1.2) $\hat{x} = [1, 0, 0]^T$.

(1.3) $\hat{x} = [1, -4/3, -4/3]^T$.

(1.4) $\hat{x} = [3/8, -1/2, 2]^T$.

(1.5) $\hat{x} = [11/16, -1/4, 1]^T$.

Solution

By labelling the constraints $1, \ldots, 6$, respectively, we have $M_1 = \{6\}$, $M_2 = \{1, 2, 3, 5\}$, and $M_3 = \{4\}$.

(1.1) We have $I(\hat{x}) = \{4, 6\}$. Note that $M_3 \subseteq I(\hat{x})$. However, $|I(\hat{x})| = 2 < 3 = n$, the set $\{a^i : i \in I(\hat{x})\}$ cannot span $\mathbb{R}^3$. As $\hat{x}$ violates the first constraint given by $-4x_1+x_2-x_3 \leq -4$, $\hat{x}$ is infeasible and not a basic solution.

(1.2) We have $I(\hat{x}) = \{1, 2, 4, 6\}$. Note that $M_3 \subseteq I(\hat{x})$. The set $\{a^i : i \in I(\hat{x})\}$ contains 4 vectors, and may or may not span $\mathbb{R}^3$. We need to check whether this set contains $n = 3$ linearly independent vectors. Consider the vectors a^1, a^2, and a^4. We can show that they are linearly independent:

$$\det \begin{pmatrix} -4 & -2 & 0 \\ 1 & 1 & 4 \\ -1 & -1 & 1 \end{pmatrix} = -10 \neq 0\,.$$

Alternatively, you can use the Gaussian elimination procedure to show that these three vectors are linearly independent. Therefore, $\hat{x}$ is a basic solution. You can easily check that $\hat{x}$ satisfies the remaining two constraints. Therefore, $\hat{x}$ is a basic feasible solution.

(1.3) We have $I(\hat{x}) = \{1, 2, 3\}$. Note that $M_3 = \{4\}$ and $M_3 \not\subseteq I(\hat{x})$. It follows that $\hat{x}$ is not a basic solution. Furthermore, you can easily check that $\hat{x}$ violates the fourth constraint. Therefore, $\hat{x}$ is infeasible and not a basic solution.

(1.4) We have $I(\hat{x}) = \{1, 3, 4\}$. Note that $M_3 \subseteq I(\hat{x})$. The set $\{a^i : i \in I(\hat{x})\}$ contains exactly 3 vectors, and may or may not span $\mathbb{R}^3$. We just need to check if these vectors are linearly independent:

$$\det \begin{pmatrix} -4 & 0 & 0 \\ 1 & -4 & 4 \\ -1 & 1 & 1 \end{pmatrix} = 32 \neq 0\,.$$

Alternatively, you can use the Gaussian elimination procedure to show that these three vectors are linearly independent. Therefore, $\hat{x}$ is a basic solution. You can easily check that $\hat{x}$ satisfies the remaining three constraints. Therefore, $\hat{x}$ is a basic feasible solution.

(1.5) We have $I(\hat{x}) = \{1, 4\}$. Note that $M_3 \subseteq I(\hat{x})$. However, the set $\{a^i : i \in I(\hat{x})\}$ contains only two elements. Therefore, the set $\{a^i : i \in I(\hat{x})\}$ cannot span $\mathbb{R}^3$. As $\hat{x}$ satisfies each of the remaining four constraints, $\hat{x}$ is feasible but not a basic solution.

2 Graphical Method (2 marks)

STACK question

Consider the following polyhedron:

$$\mathcal{P} \;=\; \{[x_1, x_2]^T \in \mathbb{R}^2 : -x_1 + x_2 \leq 2,\, x_1 - 2x_2 \geq -6,\, x_1 + 2x_2 \geq 1,\, x_2 \geq 0\}\,.$$

Using the graphical method, determine, for each of the following objective functions, the optimal value denoted by z^* (use `+inf` for $+\infty$ and `-inf` for $-\infty$), and whether the set of optimal solutions, denoted by $\mathcal{P}^*$, is either *empty*, a *single vertex*, a *line segment*, a *half line*, or $\mathcal{P}^* = \mathcal{P}$.

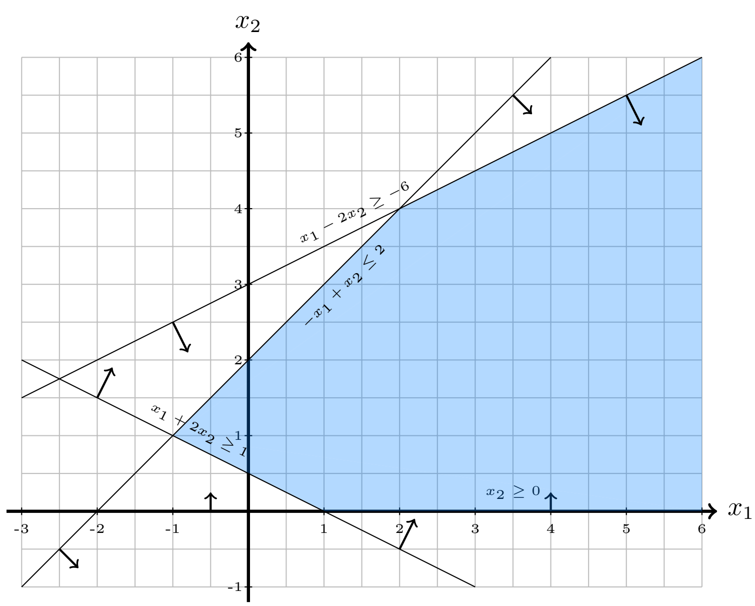

Figure 1: Feasible region for Question 2.

(2.1) $\min\{3x_1 : x \in \mathcal{P}\}$.

Solution

Since this is a minimization problem, the improving direction is given by $-c = [-3, 0]^T$, which is parallel to the x_1 axis. Therefore, the unique optimal solution is given by the intersections of the boundaries of the two constraints $-x_1 + x_2 \leq 2$ and $x_1 + 2x_2 \geq 1$. Therefore, we can solve for the following system simultaneously:

$$\begin{aligned} -x_1 + x_2 &= 2 \\ x_1 + 2x_2 &= 1 \end{aligned}$$

We obtain $x_1 = -1$ and $x_2 = 1$. Therefore, $\mathcal{P}^* = \{[-1, 1]^T\}$, i.e., it is given by a single vertex. Substituting this solution into the objective function, we obtain $z^* = 3 \cdot (-1) = -3$.

(2.2) $\max\{-2x_1 - 4x_2 : x \in \mathcal{P}\}$.

Solution

Since this is a maximization problem, the improving direction is given by $c = [-2, -4]^T$, which points in the southwest direction. Note that the improving direction is perpendicular to the boundary of the constraint $x_1 + 2x_2 \geq 1$. Therefore, any feasible solution on the line segment between the points $[-1, 1]^T$ and $[1, 0]^T$ is an optimal solution, i.e., $\mathcal{P}^* = \{\lambda[-1, 1]^T + (1 - \lambda)[1, 0]^T : \lambda \in [0, 1]\}$. Therefore, it is a line segment. Substituting any point on this line segment into the objective function, we obtain $z^* = (-2) \cdot (1) + (-4) \cdot 0 = -2$.

(2.3) $\min\{0 : x \in \mathcal{P}\}$.

Solution

The objective function value is a constant given by 0, i.e., it does not depend on the feasible solution. Therefore, any feasible solution yields an objective function value of 0. This is a *feasibility* problem, i.e., we only need to check whether $\mathcal{P}$ is nonempty. Since it is nonempty (see Figure 1), we obtain that any feasible solution is optimal, i.e., $\mathcal{P}^* = \mathcal{P}$. The optimal value is clearly $z^* = 0$.

(2.4) $\min\{2x_1 - 4x_2 : x \in \mathcal{P}\}$.

Solution

Since this is a minimization problem, the improving direction is given by $-c = [-2, 4]^T$, which points in the northeast direction. Note that the improving direction is perpendicular to the boundary of the constraint $x_1 - 2x_2 \geq -6$. Therefore, any feasible solution on the half line starting at $[2, 4]^T$ towards the direction $[2, 1]^T$ is an optimal solution, i.e., $\mathcal{P}^* = \{[2, 4]^T + \lambda[2, 1]^T : \lambda \geq 0\}$. Therefore, it is a half line. Substituting any point on this half line into the objective function, we obtain $z^* = 2 \cdot 2 + (-4) \cdot 4 = -12$.

Open Ended Problems

3 Polyhedra in Standard Form (1 mark)

(3.1) Convert the following general linear programming problem into standard form:

$$\begin{array}{lccccccccc} \max & -x_1 & + & 3x_2 & - & 5x_3 & + & x_4 & & \\ \text{s.t.} & & & & & & & & & \\ & -x_1 & + & 3x_2 & - & 2x_3 & - & 3x_4 & \geq & 6 \\ & 3x_1 & - & x_2 & + & 5x_3 & - & 2x_4 & = & -3 \\ & -x_1 & - & 3x_2 & + & 2x_3 & - & x_4 & \leq & 2 \\ & x_1 \leq 0 & , & x_2 \geq 0 & , & & & x_4 \geq 0. & & \end{array}$$

Remark It is irrelevant whether the problem is actually feasible or not.

Solution

- Since it is a maximization problem, we need to negate the objective function to turn it into a minimization problem (i.e., min $x_1 - 3x_2 + 5x_3 - x_4$).
- For the first and third constraints, we need to define two new nonnegative variables s_1 and s_2 that we subtract and add, respectively.
- Concerning the nonpositive variable x_1, we can replace x_1 by the new nonnegative variable $x_1^- = -x_1$.
- Concerning the free variable x_3, we can replace x_3 by the difference of two new nonnegative variables x_3^+ and x_3^- through $x_3 = x_3^+ - x_3^-$.

This results in the following equivalent linear programming problem in standard form:

$$
\begin{array}{lcccccccccccccc}
\min & -x_1^- & - & 3x_2 & + & 5x_3^+ & - & 5x_3^- & - & x_4 & & & & & \\
\text{s.t.} & & & & & & & & & & & & & & \\
& x_1^- & + & 3x_2 & - & 2x_3^+ & + & 2x_3^- & - & 3x_4 & - & s_1 & & & = & 6 \\
& -3x_1^- & - & x_2 & + & 5x_3^+ & - & 5x_3^- & - & 2x_4 & & & & & = & -3 \\
& x_1^- & - & 3x_2 & + & 2x_3^+ & - & 2x_3^- & - & x_4 & & & + & s_2 & = & 2 \\
& x_1^- \geq 0 & , & x_2 \geq 0 & , & x_3^+ \geq 0 & , & x_3^- \geq 0 & , & x_4 \geq 0 & , & s_1 \geq 0 & , & s_2 \geq 0. & &
\end{array}
$$

Note that this linear programming problem, which is in standard form, is equivalent to the original one in the sense that there is a one-to-one correspondence between their feasible solutions. Therefore, there is also a one-to-one correspondence between their optimal solutions (if any). However, since we negate the objective function of the original problem, we need to exercise care about their optimal values. In particular, if z^* denotes the optimal value of the problem in minimization form, then the optimal value of the original maximization problem is $-z^*$.

4 Polytopes vs Polyhedra (2 marks)

(4.1) Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron. Prove the following result:

> $\mathcal{P} \subseteq \mathbb{R}^n$ is a polytope if and only if, for every $c \in \mathbb{R}^n$, each of the two linear programming problems given by
>
> $$\text{(P1)} \quad \min\{c^T x : x \in \mathcal{P}\} \quad \text{and} \quad \text{(P2)} \quad \max\{c^T x : x \in \mathcal{P}\}$$
>
> has a finite optimal value.

Solution

Since this is an if and only if statement, we need to prove both implications:

$\Rightarrow$: Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polytope (i.e., a bounded polyhedron). Then, there exists a real number $K \in \mathbb{R}$ such that

$$x \in \mathcal{P} \Rightarrow |x_j| \leq K, \quad j = 1, \ldots, n.$$

Let $c \in \mathbb{R}^n$ be an arbitrary vector. Then, for each $x \in \mathcal{P}$, we have

$$
\begin{aligned}
|c^T x| &= \left| \sum_{j=1}^n c_j x_j \right| \\
&\leq \sum_{j=1}^n |c_j||x_j| \\
&\leq K \left(\sum_{j=1}^n |c_j| \right) \\
&\leq KL(c),
\end{aligned}
$$

where $L(c) = \sum_{j=1}^{n} |c_j| \in \mathbb{R}$ and we used the triangle inequality in the second line. It follows that, for each $c \in \mathbb{R}^n$, there exists a real number $M(c) = KL(c)$ that depends on c such that

$$-M(c) \leq c^T x \leq M(c), \quad \text{for all } x \in \mathcal{P}.$$

It follows that the objective function value of any feasible solution in (P1) is bounded below by $-M(c)$ and the objective function value of any feasible solution in (P2) is bounded above by $M(c)$. Therefore, neither of the two problems (P1) and (P2) is unbounded. Since $\mathcal{P} \subseteq \mathbb{R}^n$ is nonempty, it follows that each of (P1) and (P2) has a finite optimal value for every $c \in \mathbb{R}^n$.

$\Leftarrow$: Suppose now that each of (P1) and (P2) has a finite optimal value for every $c \in \mathbb{R}^n$. Choose $c = e_j \in \mathbb{R}^n$, which is the jth unit vector (i.e., only the jth coordinate is 1 and all other coordinates are equal to zero), where $j = 1, \ldots, n$. Then, $c^T x = x_j$. By the hypothesis, since each of (P1) and (P2) has a finite optimal value for every $c \in \mathbb{R}^n$ and for every $j = 1, \ldots, n$, there exists real numbers u_j and l_j such that

$$l_j \leq x_j \leq u_j, \quad j = 1, \ldots, n \quad \text{for all } x \in \mathcal{P}.$$

Let $K = \max\{\max_{j=1,\ldots,n} |l_j|, \max_{j=1,\ldots,n} |u_j|\}$. Note that $K < +\infty$ since the maximum of a finite number of real numbers is a real number. It follows that there exists a real number $K \in \mathbb{R}$ such that

$$x \in \mathcal{P} \Rightarrow |x_j| \leq K, \quad j = 1, \ldots, n.$$

Therefore, we conclude that $\mathcal{P}$ is bounded, i.e., it is a polytope.

5 Existence of Vertices in Polyhedra (2 marks)

(5.1) Either prove the following result or give a counterexample:

> Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron. If $\mathcal{P}$ has no vertices, then there exists a vector $c \in \mathbb{R}^n$ such that each of the two linear programming problems given by
>
> $$\text{(P1)} \quad \min\{c^T x : x \in \mathcal{P}\} \quad \text{and} \quad \text{(P2)} \quad \max\{c^T x : x \in \mathcal{P}\}$$
>
> is unbounded.

Solution

The proposition is true and we give a proof below. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron that has no vertices. Then, by Proposition 8.1, $\mathcal{P}$ contains a line, i.e., there exists $\tilde{x} \in \mathcal{P}$ and $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $\tilde{x} + \lambda d \in \mathcal{P}$ for every $\lambda \in \mathbb{R}$.

Let us now choose $c = d \in \mathbb{R}^n$. Then, for this choice of c, we claim that each of (P1) and (P2) is unbounded. Indeed, for (P1), consider the sequence of feasible solutions given by $x^k = \tilde{x} - kd$, where $k = 1, 2, \ldots$. If you consider the objective function value along this sequence, we obtain

$$c^T(\tilde{x} - kd) = d^T(\tilde{x} - kd) = d^T\tilde{x} - k\|d\|^2, \quad k = 1, 2, \ldots,$$

which tends to $-\infty$ as k tends to $+\infty$. Therefore, for this choice of c, (P1) is unbounded. One can similarly show that (P2) is unbounded by simply choosing the sequence of feasible solutions $x^k = \tilde{x} + kd$, where $k = 1, 2, \ldots$. This proves the claim.

(5.2) Either prove the following result or give a counterexample:

> Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron. If there exists a vector $c \in \mathbb{R}^n$ such that each of the two linear programming problems given by
>
> $$\text{(P1)} \quad \min\{c^T x : x \in \mathcal{P}\} \quad \text{and} \quad \text{(P2)} \quad \max\{c^T x : x \in \mathcal{P}\}$$
>
> is unbounded, then $\mathcal{P}$ has no vertices.

Solution

While it may be tempting to think that the claim is true, one needs to be a bit more careful. Indeed, one can construct a counterexample.

Let $\mathcal{P} = \{x \in \mathbb{R}^2 : x_1 \geq 0,\, x_2 \geq 0\}$. Clearly, $\mathcal{P}$ is a nonempty polyhedron (it is given by the nonnegative orthant).

Consider $c = [1, -1]^T$. By using the graphical method, you can easily show that each of the two linear programming problems given by (P1) and (P2) is unbounded (the improving direction points in the northwest direction for (P1) and southeast direction for (P2)).

Note that $\mathcal{P}$ has a vertex given by $[0, 0]^T$. By Proposition 8.1, it does not contain a line (since either a line will have no intersection with $\mathcal{P}$, or it will eventually exit the nonnegative orthant). Therefore, the assertion is not true.