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Fundamentals of Optimization

Exercise 4 – Solutions

Remarks

- All questions that are available in the STACK quiz are duly marked. Please solve those using STACK.
- We have added marks for each question. Please note that those are purely for illustrative purposes. The exercise set will not be marked.
- We can derive the inverse of a nonsingular matrix $A \in \mathbb{R}^{2\times 2}$ in closed form:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

STACK Problems

1 Simplex Method for Nondegenerate LPs (2.5 marks)

STACK question

You are given three linear programming problems below such that the matrix A has full row rank. For each problem, a starting dictionary is given. For each given dictionary, write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether the given dictionary is optimal or not. If applicable, perform only **one** iteration of the simplex method starting from the given dictionary. For the next dictionary (if applicable), write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether this dictionary is optimal or not.

(1.1) LP

$$\begin{array}{lrcrcrcrcrcl}
\min & -4x_1 & - & 3x_2 & & & & & & & & \\
\text{s.t.} & & & x_2 & + & x_3 & & & & & = & 6 \\
& x_1 & + & x_2 & & & + & x_4 & & & = & 7 \\
& 3x_1 & + & 2x_2 & & & & & + & x_5 & = & 18 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}$$

and the starting dictionary is given by

$$\begin{array}{rcrcrcr}
z & = & -22 & - & x_3 & + & 4x_4 \\
x_1 & = & 1 & + & x_3 & - & x_4 \\
x_2 & = & 6 & - & x_3 & & \\
x_5 & = & 3 & - & x_3 & + & 3x_4
\end{array}$$

Solution

Dictionary 1

We have $B = \{1, 2, 5\}$, $N = \{3, 4\}$, $\hat{x} = [1, 6, 0, 0, 3]^T$, $\bar{c} = [0, 0, -1, 4, 0]^T$, and $\hat{z} = -22$.

As $\bar{c} \not\geq \mathbf{0}$ and the current vertex is nondegenerate, the current dictionary is not optimal by Proposition 17.1.

Since there is only one nonbasic variable with a negative reduced cost, the entering variable is x_3, i.e., $j^* = 3$.

The min-ratio test yields

$$\lambda^* = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} = \min \left\{ \frac{-6}{-1}, \frac{-3}{-1} \right\} = 3\,.$$

Hence, $k^* = 5$ and x_5 will leave the basis.

Thus, we move x_3 to the left-hand side and x_5 to the right-hand side of Row 3. Afterwards, we substitute x_3 in the right-hand sides of Rows 0, 1, and 2. We performed one iteration and the next dictionary is shown below.

Dictionary 2

$$\begin{array}{rcrcrcr} z & = & -25 & + & x_4 & + & x_5 \\ x_1 & = & 4 & + & 2x_4 & - & x_5 \\ x_2 & = & 3 & - & 3x_4 & + & x_5 \\ x_3 & = & 3 & + & 3x_4 & - & x_5 \end{array}$$

We have $B = \{1, 2, 3\}$, $N = \{4, 5\}$, $\hat{x} = [4, 3, 3, 0, 0]^T$, $\bar{c} = [0, 0, 0, 1, 1]^T$, and $\hat{z} = -25$.

As all reduced costs are non-negative, $\bar{c} \geq \mathbf{0}$, the current vertex is optimal by Corollary 15.4. Therefore, we found an optimal solution after performing one iteration of the simplex method.

(1.2) LP

$$\begin{array}{lrcrcrcrcrcr} \min & 5x_1 & + & 2x_2 & - & 6x_3 & & & & & & \\ \text{s.t.} & -3x_1 & + & x_2 & + & 3x_3 & + & x_4 & & & = & 31 \\ & 4x_1 & - & 2x_2 & + & 2x_3 & & & + & x_5 & = & 18 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

and the starting dictionary is given by

$$\begin{array}{rcrcrcrcr} z & = & 0 & + & 5x_1 & + & 2x_2 & - & 6x_3 \\ x_4 & = & 31 & + & 3x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 18 & - & 4x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

Solution

Dictionary 1

We have $B = \{4, 5\}$, $N = \{1, 2, 3\}$, $\hat{x} = [0, 0, 0, 31, 18]^T$, $\bar{c} = [5, 2, -6, 0, 0]^T$, and $\hat{z} = 0$.

As $\bar{c} \not\geq \mathbf{0}$ and the current vertex is nondegenerate, it is not optimal by Proposition 17.1.

As there is only one nonbasic variable with a negative reduced cost, the entering variable is x_3, i.e., $j^* = 3$.

The min-ratio test yields

$$\lambda^* = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} = \min \left\{ \frac{-31}{-3}, \frac{-18}{-2} \right\} = \frac{18}{2}\,.$$

Hence, $k^* = 5$ and x_5 will leave the basis.

Thus, we move x_3 to the left-hand side and x_5 to the right-hand side of Row 2. Afterwards, we substitute x_3 in the right-hand sides of Rows 0 and 1. We performed one iteration and the new dictionary is shown below.

Dictionary 2

$$\begin{array}{rcrcrcrcr} z & = & -54 & + & 17x_1 & - & 4x_2 & + & 3x_5 \\ x_4 & = & 4 & + & 9x_1 & - & 4x_2 & + & \frac{3}{2}x_5 \\ x_3 & = & 9 & - & 2x_1 & + & x_2 & - & \frac{1}{2}x_5 \end{array}$$

We have $B = \{4, 3\}$, $N = \{1, 2, 5\}$, $\hat{x} = [0, 0, 9, 4, 0]^T$, $\bar{c} = [17, -4, 0, 0, 3]^T$, and $\hat{z} = -54$.

As there is one nonbasic variable with a negative reduced cost and the current vertex is nondegenerate, it is not optimal by Proposition 17.1. (If we were to continue with the simplex method, we would choose x_2 as the entering variable and perform the min-ratio test to determine the leaving variable.)

(1.3) LP

$$\begin{array}{lrcrcrcrcrcr} \min & -x_1 & - & 2x_2 & & & & & & & & \\ \text{s.t.} & -6x_1 & + & 3x_2 & + & 3x_3 & & & & & = & 15 \\ & x_1 & - & 3x_2 & & & + & x_4 & & & = & 1 \\ & -2x_1 & & & & & & & + & x_5 & = & 8 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

and the starting dictionary is given by

$$\begin{array}{rcrcrcr} z & = & -10 & - & 5x_1 & + & 2x_3 \\ x_2 & = & 5 & + & 2x_1 & - & x_3 \\ x_4 & = & 16 & + & 5x_1 & - & 3x_3 \\ x_5 & = & 8 & + & 2x_1 & & \end{array}$$

Solution

Dictionary 1

We have $B = \{2, 4, 5\}$, $N = \{1, 3\}$, $\hat{x} = [0, 5, 0, 16, 8]^T$, $\bar{c} = [-5, 0, 2, 0, 0]^T$, and $\hat{z} = -10$.

As there is one nonbasic variable with a negative reduced cost and the current vertex is nondegenerate, it is not optimal by Proposition 17.1. The entering variable is x_1, i.e., $j^* = 1$.

When we try to do the min-ratio test, we realise that there are no negative entries in the column corresponding to x_1 in Rows 1, 2, and 3. Therefore, none of the three rows is eligible for the minimum ratio test. Hence, we can increase x_1 indefinitely without ever leaving the feasible region. Thus, the problem is unbounded. We can obtain a direction $d \in \mathbb{R}^5$ of unboundedness by defining $d_1 = 1$, $d_3 = 0$, and

$$d_B = -(A_B)^{-1} A_N d_N = -(A_B)^{-1} A^1 = -\begin{bmatrix} 3 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -6 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

Note that these entries are precisely the ones in the column corresponding to x_1 in Rows 1, 2, and 3. We obtain $d = [1, 2, 0, 5, 2]^T$. You can verify that d is a feasible direction at $\hat{x}$, i.e., $\hat{x} + \lambda d \in \mathcal{P}$ for any real number $\lambda \geq 0$, and $c^T(\hat{x} + \lambda d) = c^T\hat{x} + \lambda c^T d = -10 - 5\lambda \to -\infty$ as $\lambda \to \infty$. Therefore the original problem is unbounded and its optimal value is given by $z^* = -\infty$.

2 The Simplex Method for Degenerate LPs (2.5 marks)

STACK question

You are given the following linear programming problem that has full row rank:

$$\begin{array}{lrcrcrcrcrcl}
\min & -x_1 & - & 2x_2 & + & x_3 & & & & & & \\
\text{s.t.} & x_1 & & & + & 2x_3 & + & x_4 & & & = & 2 \\
& & & x_2 & + & 2x_3 & & & + & x_5 & = & 2 \\
& & & & & & & & & x_1,\, x_2,\, x_3,\, x_4,\, x_5 & \geq & 0
\end{array}$$

You are given three different starting dictionaries. For each given dictionary, write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether the given dictionary is optimal or not. If applicable, perform only **one** iteration of the simplex method starting from the given dictionary. If the current dictionary is nondegenerate, use the most negative reduced cost to determine the entering variable. If it is degenerate, use Bland's rule to determine the entering and leaving variables (whenever applicable). For the next dictionary (if applicable), write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether this dictionary is optimal or not.

(2.1) The starting dictionary is given by

$$\begin{array}{rclcrcrcr}
z & = & 1 & - & x_1 & - & \frac{5}{2}x_2 & - & \frac{1}{2}x_5 \\
x_3 & = & 1 & & & - & \frac{1}{2}x_2 & - & \frac{1}{2}x_5 \\
x_4 & = & 0 & - & x_1 & + & x_2 & + & x_5
\end{array}$$

Solution

Dictionary 1

We have $B = \{3,4\}$, $N = \{1,2,5\}$, $\hat{x} = [0,0,1,0,0]^T$, $\bar{c} = [-1, -\frac{5}{2}, 0, 0, -\frac{1}{2}]^T$, and $\hat{z} = 1$.

As $\bar{c} \not\geq \mathbf{0}$, the current dictionary is not optimal. Since the current vertex is degenerate (there is one basic variable with a value of zero), we cannot conclude whether this vertex is optimal. Due to the degeneracy of the corresponding vertex, we apply Bland's rule. There are three nonbasic variables with a negative reduced cost. By Bland's rule, we choose the variable with the smallest index (as opposed to choosing the nonbasic variable with the most negative reduced cost as the entering variable). Therefore, the entering variable is x_1, i.e., $j^* = 1$.

The min-ratio test yields

$$\lambda^* = \min_{j \in B: d_j < 0} \frac{-\hat{x}_j}{d_j} = \min \left\{ \frac{0}{-1} \right\} = 0\,.$$

Hence, $k^* = 4$ and x_4 will leave the basis.

Thus, we move x_1 to the left-hand side and x_4 to the right-hand side of Row 2. Afterwards, we substitute x_1 in the right-hand sides of Rows 0 and 1. We performed one iteration and the next dictionary is shown below.

Dictionary 2

$$\begin{array}{rclcrcrcr}
z & = & 1 & - & \frac{7}{2}x_2 & + & x_4 & - & \frac{3}{2}x_5 \\
x_3 & = & 1 & - & \frac{1}{2}x_2 & & & - & \frac{1}{2}x_5 \\
x_1 & = & 0 & + & x_2 & - & x_4 & + & x_5
\end{array}$$

We have $B = \{3,1\}$, $N = \{2,4,5\}$, $\hat{x} = [0,0,1,0,0]^T$, $\bar{c} = [0, -\frac{7}{2}, 0, 1, -\frac{3}{2}]^T$, and $\hat{z} = 1$. Note that the objective function value has not decreased. In fact, the vertex is the same as the previous one. We only obtained another representation using a different choice of basic and nonbasic columns. As $\bar{c} \not\geq \mathbf{0}$, the current dictionary is not optimal. Since the current vertex is degenerate, we cannot conclude whether this vertex is optimal. If we were to continue with the simplex method, we would again use Bland's rule to determine the entering and leaving variables. You can verify that the objective function will indeed improve in the next dictionary.

(2.2) The starting dictionary is given by

$$\begin{array}{rcrcrcrcr} z & = & -2 & - & 2x_2 & + & 3x_3 & + & x_4 \\ x_1 & = & 2 & & & - & 2x_3 & - & x_4 \\ x_5 & = & 2 & - & x_2 & - & 2x_3 & & \end{array}$$

Solution

We have $B = \{1, 5\}$, $N = \{2, 3, 4\}$, $\hat{x} = [2, 0, 0, 0, 2]^T$, $\bar{c} = [0, -2, 3, 1, 0]^T$, and $\hat{z} = -2$.

As $\bar{c} \not\geq \mathbf{0}$, the current dictionary is not optimal. Since the current vertex is nondegenerate (both basic variables are nonzero), we conclude that this vertex is not optimal by Proposition 17.1. Due to the nondegeneracy of the corresponding vertex, we choose the nonbasic variable with the most negative reduced cost as the entering variable (as opposed to Bland's rule). As there is only one such nonbasic variable, the entering variable is x_2, i.e., $j^* = 2$.

The min-ratio test yields

$$\lambda^* = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} = \min \left\{ \frac{-2}{-1} \right\} = 2 \, .$$

Hence, $k^* = 5$ and x_5 will leave the basis.

Thus, we move x_2 to the left-hand side and x_5 to the right-hand side of Row 2. Afterwards, we substitute x_2 in the right-hand sides of Rows 0 and 1. We performed one iteration and the next dictionary is shown below.

Dictionary 2

$$\begin{array}{rcrcrcrcr} z & = & -6 & + & 7x_3 & + & x_4 & + & 2x_5 \\ x_1 & = & 2 & - & 2x_3 & - & x_4 & & \\ x_2 & = & 2 & - & 2x_3 & & & - & x_5 \end{array}$$

We have $B = \{1, 2\}$, $N = \{3, 4, 5\}$, $\hat{x} = [2, 2, 0, 0, 0]^T$, $\bar{c} = [0, 0, 7, 1, 2]^T$, and $\hat{z} = -6$. Note that the objective function value has decreased and the new vertex is nondegenerate. As $\bar{c} \geq \mathbf{0}$, the current dictionary is optimal by Corollary 15.4. Therefore, $\hat{x}$ is a nondegenerate optimal vertex and the simplex method will be terminated at this dictionary with an optimal solution.

(2.3) The starting dictionary is given by

$$\begin{array}{rcrcrcrcr} z & = & 1 & - & \frac{3}{2}x_1 & - & 2x_2 & - & \frac{1}{2}x_4 \\ x_3 & = & 1 & - & \frac{1}{2}x_1 & & & - & \frac{1}{2}x_4 \\ x_5 & = & 0 & + & x_1 & - & x_2 & + & x_4 \end{array}$$

Solution

We have $B = \{3, 5\}$, $N = \{1, 2, 4\}$, $\hat{x} = [0, 0, 1, 0, 0]^T$, $\bar{c} = [-\frac{3}{2}, -2, 0, -\frac{1}{2}, 0]^T$, and $\hat{z} = 1$.

As $\bar{c} \ngeq \mathbf{0}$, the current dictionary is not optimal. Since the current vertex is degenerate, we cannot conclude whether this vertex is optimal. Due to the degeneracy of the corresponding vertex, we apply Bland's rule. There are three nonbasic variables with a negative reduced cost. By Bland's rule, we choose the variable with the smallest index (as opposed to choosing the nonbasic variable with the most negative reduced cost as the entering variable). Therefore, the entering variable is x_1, i.e., $j^* = 1$.

The min-ratio test yields

$$\lambda^* = \min_{j \in B: d_j < 0} \frac{-\hat{x}_j}{d_j} = \min \left\{ \frac{-1}{-\frac{1}{2}} \right\} = 2\,.$$

Hence, $k^* = 3$ and x_3 will leave the basis.

Thus, we move x_1 to the left-hand side and x_3 to the right-hand side of Row 1. Afterwards, we substitute x_1 in the right-hand sides of Rows 0 and 2. We performed one iteration and the next dictionary is shown below.

Dictionary 2

$$\begin{array}{rcrcrcrcr} z & = & -2 & - & 2x_2 & + & 3x_3 & + & x_4 \\ x_1 & = & 2 & & & - & 2x_3 & - & x_4 \\ x_5 & = & 2 & - & x_2 & - & 2x_3 & & \end{array}$$

We have $B = \{1, 5\}$, $N = \{2, 3, 4\}$, $\hat{x} = [2, 0, 0, 0, 2]^T$, $\bar{c} = [0, -2, 3, 1, 0]^T$, and $\hat{z} = -2$.

As $\bar{c} \ngeq \mathbf{0}$, the current dictionary is not optimal. Since the current vertex is nondegenerate (both basic variables are nonzero), we conclude that this vertex is not optimal by Proposition 17.1. Due to the nondegeneracy of the corresponding vertex, if we were to continue with the simplex method, we would choose the nonbasic variable with the most negative reduced cost as the entering variable (as opposed to Bland's rule). You may refer to (2.2) for the next simplex iteration.

Open Ended Problems

3 Puzzle (1 mark)

Consider the following linear program

$$\begin{array}{lrcrcrcrcrcr} \min & c_1 x_1 & + & c_2 x_2 & - & 4x_3 & & & & & & \\ \text{s.t} & 3x_1 & + & a_{12} x_2 & + & 3x_3 & + & x_4 & & & = & 30 \\ & 2x_1 & + & a_{22} x_2 & + & a_{23} x_3 & & & + & x_5 & = & 18 \\ & & & & & & & & x_1, x_2, x_3, x_4, x_5 & & \geq & 0 \end{array}$$

and the final dictionary with the optimal solution

$$\begin{array}{rcrcrcrcr} z & = & -27 & + & 5x_1 & + & \frac{1}{2}x_3 & + & \bar{c}_5 x_5 \\ x_2 & = & 9 & - & x_1 & - & \frac{3}{2}x_3 & - & \frac{1}{2}x_5 \\ x_4 & = & 21 & - & 2x_1 & - & \frac{3}{2}x_3 & + & \frac{1}{2}x_5 \end{array}$$

Determine the unknown values for a_{12}, a_{22}, a_{23}, c_1, c_2, and $\bar{c}_5$. Justify your solution.

[1 mark]

Solution

From the dictionary, we can observe that x_2 and x_4 are basic variables, so we have $B = \{2,4\}$ and $N = \{1,3,5\}$.

Starting with the right-hand side, we know that $\hat{x}_B = (A_B)^{-1}b$, or equivalently, $A_B\hat{x}_B = b$, which is easier, i.e.,

$$\begin{pmatrix} a_{12} & 1 \\ a_{22} & 0 \end{pmatrix}\begin{pmatrix} 9 \\ 21 \end{pmatrix} = \begin{pmatrix} 30 \\ 18 \end{pmatrix}.$$

Hence, we get

$$9a_{12} + 21 = 30 \quad \text{and} \quad 9a_{22} = 18$$

and thus $a_{12} = 1$ and $a_{22} = 2$. Moreover, from that

$$A_B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad (A_B)^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

Looking at the formula for the reduced costs

$$\bar{c}_j = c_j - c_B^T(A_B)^{-1}A^j, \quad j \in N,$$

while we have deduced $(A_B)^{-1}$, we also need $c_B = [c_2, c_4]^T = [c_2, 0]^T$. To obtain c_2, we remember that

$$-27 = c_B^T\hat{x}_B = 9\,c_2 + 0 \cdot 21 \quad \Rightarrow \quad c_2 = -3\,,$$

which gives us

$$c_B^T(A_B)^{-1} = [-3, 0]\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{2} \end{bmatrix}^T.$$

Finally, we can start using the equations for the reduced costs. We have three unknowns left, c_1, a_{23}, and $\bar{c}_5$, and three equations, one for each nonbasic variable. As we have one unknown per equation, we readily obtain $c_1 = 2$, $a_{23} = 3$, and $\bar{c}_5 = \frac{3}{2}$.

Summing up, we have

$$a_{12} = 1,\ a_{22} = 2,\ a_{23} = 3,\ c_1 = 2,\ c_2 = -3,\ \bar{c}_5 = \frac{3}{2}.$$

4 Duality (4 marks)

(4.1) Consider the following polyhedron in standard form:

$$\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given and $x \in \mathbb{R}^n$.

(i) Suppose that $\mathcal{P}$ is nonempty. Prove that there does not exist a vector $\hat{y} \in \mathbb{R}^m$ such that $A^T\hat{y} \leq \mathbf{0}$ and $b^T\hat{y} > 0$.

[1 mark]

Solution

Suppose that $\mathcal{P}$ is nonempty. Suppose, for a contradiction, that there exists a vector $\hat{y} \in \mathbb{R}^m$ such that $A^T\hat{y} \leq \mathbf{0}$ and $b^T\hat{y} > 0$. Let $\hat{x} \in \mathcal{P}$ be an arbitrary element. Then, $\hat{x} \geq \mathbf{0}$. Since $-A^T\hat{y} \geq \mathbf{0}$, $\hat{x} \in \mathbb{R}^n$, and $-A^T\hat{y} \in \mathbb{R}^n$, we obtain

$$0 \leq \hat{x}^T(-A^T\hat{y}) = -\hat{x}^TA^T\hat{y} = -\hat{y}^T(A\hat{x}) = -\hat{y}^Tb = -b^T\hat{y},$$

where we used $A\hat{x} = b$ in the third equality, which implies that $b^T\hat{y} \leq 0$, contradicting our hypothesis. Therefore, we conclude that there does not exist a vector $\hat{y} \in \mathbb{R}^m$ such that $A^T\hat{y} \leq \mathbf{0}$ and $b^T\hat{y} > 0$.

(ii) Conversely, suppose that there does not exist a vector $\hat{y} \in \mathbb{R}^m$ such that $A^T\hat{y} \leq \mathbf{0}$ and $b^T\hat{y} > 0$. Prove that $\mathcal{P}$ is nonempty.

Hint: Try to set up an appropriate pair of primal-dual problems and use the appropriate duality relations.

[1 mark]

Solution

Consider the following linear programming problem:

$$\text{(LP1)} \quad \max\{b^T y : A^T y \leq \mathbf{0}\}$$

Note that (LP1) has a nonempty feasible region since $\bar{y} = \mathbf{0} \in \mathbb{R}^m$ is clearly a feasible solution of (LP1). We therefore conclude that the optimal value of (LP1), denoted by $z^*_{LP1} \geq 0$ since it is a maximization problem.
Suppose that there does not exist a vector $\hat{y} \in \mathbb{R}^m$ such that $A^T \hat{y} \leq \mathbf{0}$ and $b^T \hat{y} > 0$. We claim that the optimal value of (LP1) should satisfy $z^*_{LP1} = 0$. Suppose not. Then, by the previous observation, we should have $z^*_{LP1} > 0$. It follows that there exists a feasible solution of (LP1) whose objective function value is strictly positive, since otherwise zero would be the smallest upper bound on the objective function value of every feasible solution, and therefore we would have $z^*_{LP1} = 0$. Therefore, there exists $\hat{y} \in \mathbb{R}^m$ such that $A^T \hat{y} \leq \mathbf{0}$ and $b^T \hat{y} > 0$. However, this contradicts our hypothesis. Therefore, (LP1) has a finite optimal value given by 0.
Consider the dual of (LP1). Since (LP1) is in the form of the dual of a linear programming problem in standard form, by the primal-dual symmetry, we conclude that the dual of (LP1) is in standard form and given by

$$\text{(LP2)} \quad \min\{\mathbf{0}^T x : Ax = b, \quad x \geq \mathbf{0}\}$$

Note that the feasible region of (LP2) is given by $\mathcal{P}$ and the objective function value is equal to zero, which implies that (LP2) is a feasibility problem. Since (LP1) has a finite optimal value given by $z^*_{LP1} = 0$, it follows from Strong Duality Theorem that (LP2) also has a finite optimal value and $z^*_{LP2} = 0$. This implies that (LP2) has a nonempty feasible region, since otherwise the dual of (LP2), which is given by (LP1), is either infeasible or unbounded by the duality relations in Table 24.1 in the lecture notes. We therefore conclude that $\mathcal{P}$ is nonempty.

(4.2) Determine the set of all optimal solutions for the following linear program without using the simplex method:

$$\begin{array}{llrcrcrcrcl}
(P) & \min & -2x_1 & + & 6x_2 & + & 6x_3 & - & x_4 & & \\
 & \text{s.t.} & 2x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 1 \\
 & & -3x_1 & + & 2x_2 & + & x_3 & - & x_4 & = & 1 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0.
\end{array}$$

[2 marks]

Solution

We start by formulating the dual problem, which will only have two variables:

$$\begin{array}{llrcrcr}
(D) & \max & y_1 & + & y_2 & & \\
 & \text{s.t.} & 2y_1 & - & 3y_2 & \leq & -2 \\
 & & y_1 & + & 2y_2 & \leq & 6 \\
 & & 2y_1 & + & y_2 & \leq & 6 \\
 & & -y_1 & - & y_2 & \leq & -1 \\
 & & y_1 & , & y_2 & \in & \mathbb{R}.
\end{array}$$

The graphical representation of the dual problem, together with its optimal solution, are shown in Figure 1.

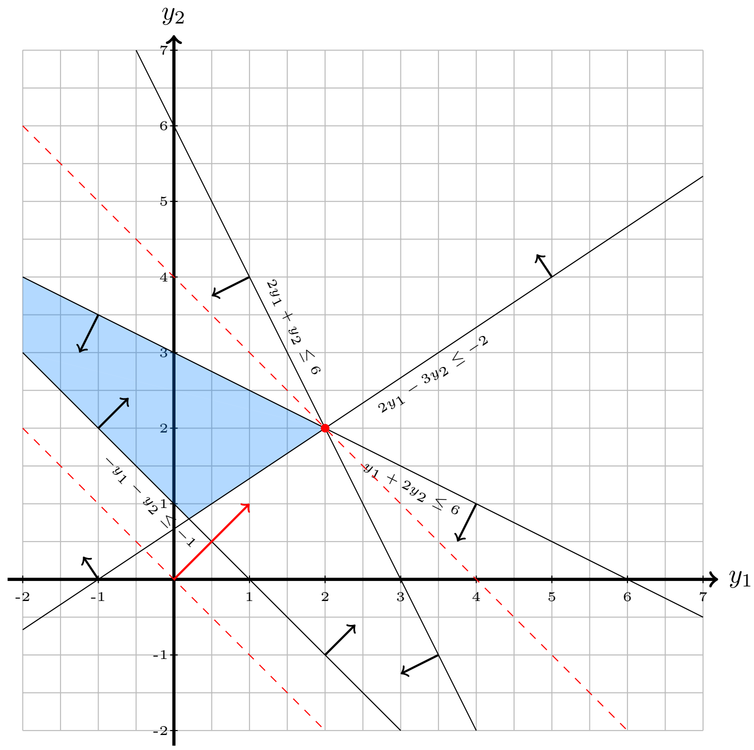

Figure 1: Feasible region of Question 4.1 (blue region), improving direction (solid red arrow), level sets of the objective function (dashed red lines), and the optimal solution (red circle)

The unique optimal solution of (D) is given by $y^* = [2, 2]^T$ with $z_D^* = y_1^* + y_2^* = 4$. Since three of the four constraints of (D) are active at y^*, we conclude that y^* is a degenerate basic feasible solution of (D).

To obtain an optimal primal solution, we write down and solve the complementary slackness conditions. Since y^* is the unique optimal solution of (D), every primal optimal solution of (P) should satisfy primal feasibility and complementary slackness conditions together with the unique dual optimal solution.

In general, two feasible solutions $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ of the primal and the dual problem, respectively, must satisfy

$$\begin{aligned} \bar{x}_1 \left(-2 - 2\bar{y}_1 + 3\bar{y}_2\right) &= 0 \\ \bar{x}_2 \left(6 - \bar{y}_1 - 2\bar{y}_2\right) &= 0 \\ \bar{x}_3 \left(6 - 2\bar{y}_1 - \bar{y}_2\right) &= 0 \\ \bar{x}_4 \left(-1 + \bar{y}_1 + \bar{y}_2\right) &= 0 \end{aligned}$$

$$\begin{aligned} \bar{y}_1 \left(1 - 2\bar{x}_1 - \bar{x}_2 - 2\bar{x}_3 + \bar{x}_4\right) &= 0 \\ \bar{y}_2 \left(1 + 3\bar{x}_1 - 2\bar{x}_2 - \bar{x}_3 + \bar{x}_4\right) &= 0 \end{aligned}$$

Substituting the values of the unique dual optimal solution in the complementary slackness conditions, we get that any primal optimal solution $\bar{x}$ must satisfy the following conditions:

$$\begin{aligned}
\bar{x}_1 \cdot 0 &= 0 & & \\
\bar{x}_2 \cdot 0 &= 0 & & \\
\bar{x}_3 \cdot 0 &= 0 & & \\
\bar{x}_4 \cdot 3 &= 0 & \qquad\qquad & \bar{x}_4 = 0 \\
1 - 2\bar{x}_1 - \bar{x}_2 - 2\bar{x}_3 + \bar{x}_4 &= 0 & \Rightarrow \qquad\qquad & 2\bar{x}_1 + \bar{x}_2 + 2\bar{x}_3 = 1 \\
1 + 3\bar{x}_1 - 2\bar{x}_2 - \bar{x}_3 + \bar{x}_4 &= 0 & & -3\bar{x}_1 + 2\bar{x}_2 + \bar{x}_3 = 1
\end{aligned}$$

Note that we end up with two equations and three unknowns. To find a solution, we will treat $\bar{x}_3$ as a parameter. Let $\bar{x}_3 = \alpha$, where $\alpha \in \mathbb{R}$. Then, we obtain the following system:

$$\begin{aligned}
2\bar{x}_1 + \bar{x}_2 &= 1 - 2\alpha \\
-3\bar{x}_1 + 2\bar{x}_2 &= 1 - \alpha
\end{aligned}$$

Solving this system simultaneously, we obtain

$$\begin{aligned}
\bar{x}_1 &= \frac{1 - 3\alpha}{7} \\
\bar{x}_2 &= \frac{5 - 8\alpha}{7}
\end{aligned}$$

Therefore, we obtain an infinite number of solutions given by

$$\bar{x} = \left[\frac{1 - 3\alpha}{7}, \frac{5 - 8\alpha}{7}, \alpha, 0\right]^T.$$

Note that the equality constraints of (P) are satisfied for any choice of $\alpha \in \mathbb{R}$. In order to find the set of optimal solutions of (P), by using the complementary slackness conditions, we only need to ensure that $\bar{x}$ is feasible for (P), i.e., it also satisfies the nonnegativity constraints. We therefore need to choose α such that $\bar{x} \geq \mathbf{0}$. You can easily verify that each component of $\bar{x}$ is nonnegative if and only if

$$0 \leq \alpha \leq \frac{1}{3}.$$

By the complementary slackness conditions, we conclude that the set of optimal solutions of (P), denoted by $\mathcal{P}^*$, is given by

$$\mathcal{P}^* = \left\{ \bar{x} \in \mathbb{R}^4 : \bar{x} = \left[\frac{1 - 3\alpha}{7}, \frac{5 - 8\alpha}{7}, \alpha, 0\right]^T, \quad 0 \leq \alpha \leq 1/3 \right\}.$$

Note that (P) has an infinite number of optimal solutions. You can easily verify that the primal objective function value of any member of $\mathcal{P}^*$ is given by 4 independently of the value of α, verifying the Strong Duality Theorem since the optimal value of (D) is also equal to 4.