Fundamentals of Operational Research
Tutorial 3
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1. A government agency is conducting research on a certain engineering project that must be solved before university lecturers can be replaced by teaching robots. Three research teams are currently trying three different approaches for solving this problem doing independent work. It is estimated that the probability that each team does not succeed is 0.4, 0.6, and 0.8, respectively. Thus, the probability that all three teams will fail is $0.4 \times 0.6 \times 0.8 = 0.192$.

With the objective of minimizing the probability of failure as much as possible, two more scientists join the project. The table below gives the new probability of failure when new scientists join. How should the two scientists be allocated to minimize the probability that all three teams fail? Use dynamic programming to solve this problem.

	Team		
New Scientists	1	2	3
0	0.40	0.60	0.80
1	0.20	0.40	0.50
2	0.15	0.20	0.30

Solution:
Let us start by defining the elements of this dynamic programming problem.

Each stage n is a research team, $n = 1, 2, 3$.

State s_n is the number of new scientists still available at the beginning of stage n, that is, when we are deciding how many scientist we allocate to team n.

If $p_i(x_i)$ represents the probability of failure for team i when it is assigned x_i new scientists, the objective function that we seek to minimize is

$$p_1(x_1)p_2(x_2)p_3(x_3)$$

subject to the condition $x_1 + x_2 + x_3 = 2$, with x_i nonnegative integers, $i = 1, 2, 3$.

Thus, we define $f_n(s_n, x_n)$ as the minimum probability of failure when we start stage n with s_n new scientists still available, we allocate x_n to team n, and optimal decisions are taken thereafter (from stage $n+1$ onwards). That is,

$$f_n(s_n, x_n) = p_n(x_n) \cdot \min\left\{ \prod_{i=n+1}^{3} p_i(x_i) \ / \ x_n + \ldots + x_3 = s_n \text{ and } x_{n+1}, \ldots, x_3 \in \mathbb{Z}^+ \right\}.$$

Therefore,

$$f_n^*(s_n) = \min\{f_n(s_n, x_n) \ / \ x_n \in \{0, 1, \ldots, s_n\}\}$$

and the recursive expression is

$$f_n^*(s_n) = \min\left\{p_n(x_n)f_{n+1}^*(s_n - x_n)\ /\ x_n\{0, 1, \ldots, s_n\}\right\},$$

with $f_3^*(s_3) = \min\{p_3(x_3)\ /\ x_3 \in \{0, 1, \ldots, s_3\}\}$.

We need to compute $f_1^*(2)$. Let us solve the problem stage by stage.

- Stage $n = 3$.

In this stage, we simply look at function p_3. Since it is a decreasing function, we save some calculations by noticing that $f_3(s_3) = p_3(s_3)$. Therefore:

– If $s_3 = 0$, then $f_3^*(0) = p_3(0) = 0.8$. This is associated to $x_3^* = 0$.
– If $s_3 = 1$, then $f_3^*(1) = p_3(1) = 0.5$. This is associated to $x_3^* = 1$.
– If $s_3 = 2$, then $f_3^*(2) = p_3(2) = 0.3$. This is associated to $x_3^* = 2$.

- Stage $n = 2$.

Remember that

$$f_2^*(s_2) = \min\{p_2(x_2)f_3^*(s_2 - x_2)\ /\ x_2 \in \{0, 1, \ldots, s_2\}\}.$$

Thus:

– If $s_2 = 0$, then $f_2^*(0) = p_2(0)f_3^*(0) = 0.6 \times 0.8 = 0.48$. This is associated to $x_2^* = 0$.
– If $s_2 = 1$, then $f_2^*(1) = \min\{p_2(x_2)f_3^*(1-x_2)\ /\ x_2 \in \{0, 1\}\} = \min\{p_2(0)f_3^*(1), p_2(1)f_3^*(0)\} = \min\{0.6 \times 0.5, 0.4 \times 0.8\} = \min\{\underline{0.3}, 0.32\} = 0.3$. This is associated to $x_2^* = 0$.
– If $s_2 = 2$, then $f_2^*(2) = \min\{p_2(x_2)f_3^*(2 - x_2)\ /\ x_2\{0, 1, 2\}\} = \min\{p_2(0)f_3^*(2), p_2(1)f_3^*(1), p_2(2)f_3^*(0)\} = \min\{0.6 \times 0.3, 0.4 \times 0.5, 0.2 \times 0.8\} = \min\{0.18, 0.2, \underline{0.16}\} = 0.16$. This is associated to $x_2^* = 2$.

- Stage $n = 1$.

We are only interested in one state for this stage: $s_1 = 2$.

So, for $s_1 = 2$ we have that $f_1^*(2) = \min\{p_1(x_1)f_2^*(2-x_1)\ /\ x_1 \in \{0, 1, 2\}\} = \min\{p_1(0)f_2^*(2), p_1(1)f_1^*(1), p_1(2)f_2^*(0)\} = \min\{0.4 \times 0.16, 0.2 \times 0.3, 0.15 \times 0.48\} = \min\{0.064, \underline{0.06}, 0.072\} = 0.06$. This is associated to $x_1^* = 1$.

Therefore, the minimum probability of failure is 0.06. We can achieve this outcome if we allocating one extra scientist to Team 1 and another to Team 3.

2. A gambler believes that he has found a system to win a popular game in a casino in North Berwick. He starts with 3 chips and he must end with at least 5 chips after 3 plays of the game in order to win. Each play of the game involves betting any desired number of available chips and then winning than number with probability $\frac{2}{3}$ or losing it with probability $\frac{1}{3}$. What is the maximum probability of winning? Can you describe an optimal policy for the three plays?

Solution:

We solve this problem with dynamic programming.

Stage n is the n-th play of the game, $n = 1, 2, 3$.

The state s_n is the number of chips that the gambler has when he starts stage n.

The decision variable x_n is the number of chips to bet at stage n, $x_n \in \{0, 1, \ldots, s_n\}$.

The objective function to be maximized is the probability of ending with at least 5 chips.

Thus, we define $f_n(s_n, x_n)$ as the probability of finishing with at least 5 chips at the end of the three plays given that he starts play n with s_n chips, he bets x_n chips at stage n and optimal decisions are taken thereafter.

This means that

$$f_n^*(s_n) = \max\{f_n(s_n, x_n) \ / \ x_n \in \{0, 1, \dots, s_n\}\}.$$

In order to determine the exact form of $f_n(s_n, x_n)$ we need to take into account the following:

- There is a probability $\frac{1}{3}$ of losing x_n chips. From having s_n on stage n, he passes to have $s_n - x_n$ chips on stage $n+1$.
- There is a probability $\frac{2}{3}$ of winning x_n chips. From having s_n on stage n, he passes to have $s_n + x_n$ chips on stage $n+1$.

Therefore,

$$f_n(s_n, x_n) = \frac{1}{3} f_{n+1}^*(s_n - x_n) + \frac{2}{3} f_{n+1}^*(s_n + x_n).$$

In addition,

$$f_4^*(s_4) = \begin{cases} 0, & \text{if } s_4 < 5 \text{ (the gambler loses)}, \\ 1, & \text{if } s_4 \geq 5 \text{ (the gambler wins)}. \end{cases}$$

As a consequence, the recursive expression is

$$f_n^*(s_n) = \max\left\{\frac{1}{3} f_{n+1}^*(s_n - x_n) + \frac{2}{3} f_{n+1}^*(s_n + x_n) \ / \ x_n \in \{0, 1, \dots, s_n\}\right\}.$$

If any stage $s_n \geq 5$, the gambler has won and will not continue betting.

Let us use the recursive expression to solve each of the 3 stages.

(a) Stage $n = 3$.

$$f_3^*(s_3) = \max\left\{\frac{1}{3} f_4^*(s_3 - x_3) + \frac{2}{3} f_4^*(s_3 + x_3) \ / \ x_3 \in \{0, 1, \dots, s_3\}\right\}.$$

If $s_3 \leq 2$, then he cannot win because he can at most double his number of chips, and he needs to end up with at least 5.

Therefore:

- If $s_3 \leq 2$, then $f_3^*(s_3) = 0$ and there is not decision to be taken.
- If $s_3 = 3$, then $f_3^*(3) = \max\left\{\frac{1}{3} f_4^*(3 - x_3) + \frac{2}{3} f_4^*(3 + x_3) \ / \ x_3 \in \{0, 1, 2, 3\}\right\} =$ $\max\left\{\frac{1}{3} f_4^*(3) + \frac{2}{3} f_4^*(3), \frac{1}{3} f_4^*(2) + \frac{2}{3} f_4^*(4), \frac{1}{3} f_4^*(1) + \frac{2}{3} f_4^*(5), \frac{1}{3} f_4^*(0) + \frac{2}{3} f_4^*(6)\right\} =$ $\max\left\{0, 0, \frac{2}{3}, \frac{2}{3}\right\} = \frac{2}{3}$. This is associated to $x_3^* = 2$.
- If $s_3 = 4$, then $f_3^*(4) = \max\left\{\frac{1}{3} f_4^*(4 - x_3) + \frac{2}{3} f_4^*(4 + x_3) \ / \ x_3 \in \{0, 1, 2, 3, 4\}\right\} =$ $\max\left\{\frac{1}{3} f_4^*(4) + \frac{2}{3} f_4^*(4), \frac{1}{3} f_4^*(3) + \frac{2}{3} f_4^*(5), \frac{1}{3} f_4^*(2) + \frac{2}{3} f_4^*(6), \frac{1}{3} f_4^*(1) + \frac{2}{3} f_4^*(7), \frac{1}{3} f_4^*(0) + \frac{2}{3} f_4^*(8)\right\} =$ $\max\left\{0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right\} = \frac{2}{3}$. This is associated to $x_3^* \in \{1, 2, 3, 4\}$.
- If $s_3 \geq 5$, then $f_3^*(s_3) = 1$. It is associated to $x_3^* = 0$ (or any values less than or equal to $s_3 - 5$).

(b) Stage $n = 2$.

$$f_2^*(s_2) = \max\left\{\frac{1}{3}f_3^*(s_2 - x_2) + \frac{2}{3}f_3^*(s_2 + x_2) \;/\; x_2 \in \{0, 1, \ldots, s_2\}\right\}.$$

• If $s_2 = 0$, then $f_2^*(0) = 0$. The gambler has no chips to bet. There is no decision to be taken.

• If $s_2 = 1$, then $f_2^*(1) = 0$. There is no need to do any calculation. In the best scenario, the gambler will win this play and will have 2 chips when starting stage 3. But that is not enough for him to win the game, as $f_3^*(2) = 0$. There is no decision to be taken.

• If $s_2 = 2$, then $f_2^*(2) = \max\left\{\frac{1}{3}f_3^*(2 - x_2) + \frac{2}{3}f_3^*(2 + x_2) \;/\; x_2 \in \{0, 1, 2\}\right\} =$
$\max\left\{\frac{1}{3}f_3^*(2) + \frac{2}{3}f_3^*(2), \frac{1}{3}f_3^*(1) + \frac{2}{3}f_3^*(3), \frac{1}{3}f_3^*(0) + \frac{2}{3}f_3^*(4)\right\} = \max\left\{0, \frac{2}{3} \times \frac{2}{3}, \frac{2}{3} \times \frac{2}{3}\right\} =$
$\max\left\{0, \frac{4}{9}, \frac{4}{9}\right\} = \frac{4}{9}$. It is associated to $x_2^* = 1$ or $x_2^* = 2$.

• If $s_2 = 3$, then $f_2^*(3) = \max\left\{\frac{1}{3}f_3^*(3 - x_2) + \frac{2}{3}f_3^*(3 + x_2) \;/\; x_2 \in \{0, 1, 2, 3\}\right\} =$
$\max\left\{\frac{1}{3}f_3^*(3) + \frac{2}{3}f_3^*(3), \frac{1}{3}f_3^*(2) + \frac{2}{3}f_3^*(4), \frac{1}{3}f_3^*(1) + \frac{2}{3}f_3^*(5), \frac{1}{3}f_3^*(0) + \frac{2}{3}f_3^*(6)\right\} =$
$\max\left\{\frac{2}{3}, \frac{2}{3} \times \frac{2}{3}, \frac{2}{3} \times 1, \frac{2}{3} \times 1\right\} = \max\left\{\frac{2}{3}, \frac{4}{9}, \frac{2}{3}, \frac{2}{3}\right\} = \frac{2}{3}$. It is associated to $x_2^* \in \{0, 2, 3\}$.

• If $s_=4$, then $f_2^*(4) = \max\left\{\frac{1}{3}f_3^*(4 - x_2) + \frac{2}{3}f_3^*(4 + x_2) \;/\; x_2 \in \{0, 1, 2, 3, 4\}\right\} =$
$\max\left\{\frac{1}{3}f_3^*(4) + \frac{2}{3}f_3^*(4), \frac{1}{3}f_3^*(3) + \frac{2}{3}f_3^*(5), \frac{1}{3}f_3^*(2) + \frac{2}{3}f_3^*(6), \frac{1}{3}f_3^*(1) + \frac{2}{3}f_3^*(7), \frac{1}{3}f_3^*(0) + \frac{2}{3}f_3^*(8)\right\} =$

$\max\left\{\frac{2}{3}, \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times 1, \frac{2}{3} \times 1, \frac{2}{3} \times 1, \frac{2}{3} \times 1\right\} = \max\left\{\frac{2}{3}, \frac{8}{9}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right\} = \frac{8}{9}$. It is associated to $x_2^* = 1$.

• If $s_2 \geq 5$, then $f_2^*(s_2) = 1$. It is associated to $x_2^* = 0$ (or to any value less than or equal to $s_2 - 5$).

(c) Stage $n = 1$.

Now we are only interested in $s_1 = 3$, the initial number of chips.
$f_1^*(3) = \max\left\{\frac{1}{3}f_2^*(3 - x_1) + \frac{2}{3}f_2^*(3 + x_1) \;/\; x_1 \in \{0, 1, 2, 3\}\right\} =$
$\max\left\{\frac{1}{3}f_2^*(3) + \frac{2}{3}f_2^*(3), \frac{1}{3}f_2^*(2) + \frac{2}{3}f_2^*(4), \frac{1}{3}f_2^*(1) + \frac{2}{3}f_2^*(5), \frac{1}{3}f_2^*(0) + \frac{2}{3}f_2^*(6)\right\} =$
$\max\left\{\frac{2}{3}, \frac{1}{3} \times \frac{4}{9} + \frac{2}{3} \times \frac{8}{9}, \frac{2}{3} \times 1, \frac{2}{3} \times 1\right\} = \max\left\{\frac{2}{3}, \frac{20}{27}, \frac{2}{3}, \frac{2}{3}\right\} = \frac{2}{3}$. It is associated to $x_1^* = 1$.

The maximum probability of winning is $\frac{20}{27}$.

There are several optimal policies. One is:

1. The gambler starts with 3 chips and bets 1 chip.
2. If the gambler wins play 1, he starts play 2 with 4 chips and bets 1 chip.

   2.1. If the gambler wins play 2, he starts play 3 with 5 chips and does not bet any chip. He wins the game.

   2.2. If the gambler loses play 2, he starts play 3 with 3 chips and he bets 2 chips.

      2.2.1 If the gambler wins play 3, he wins the game.

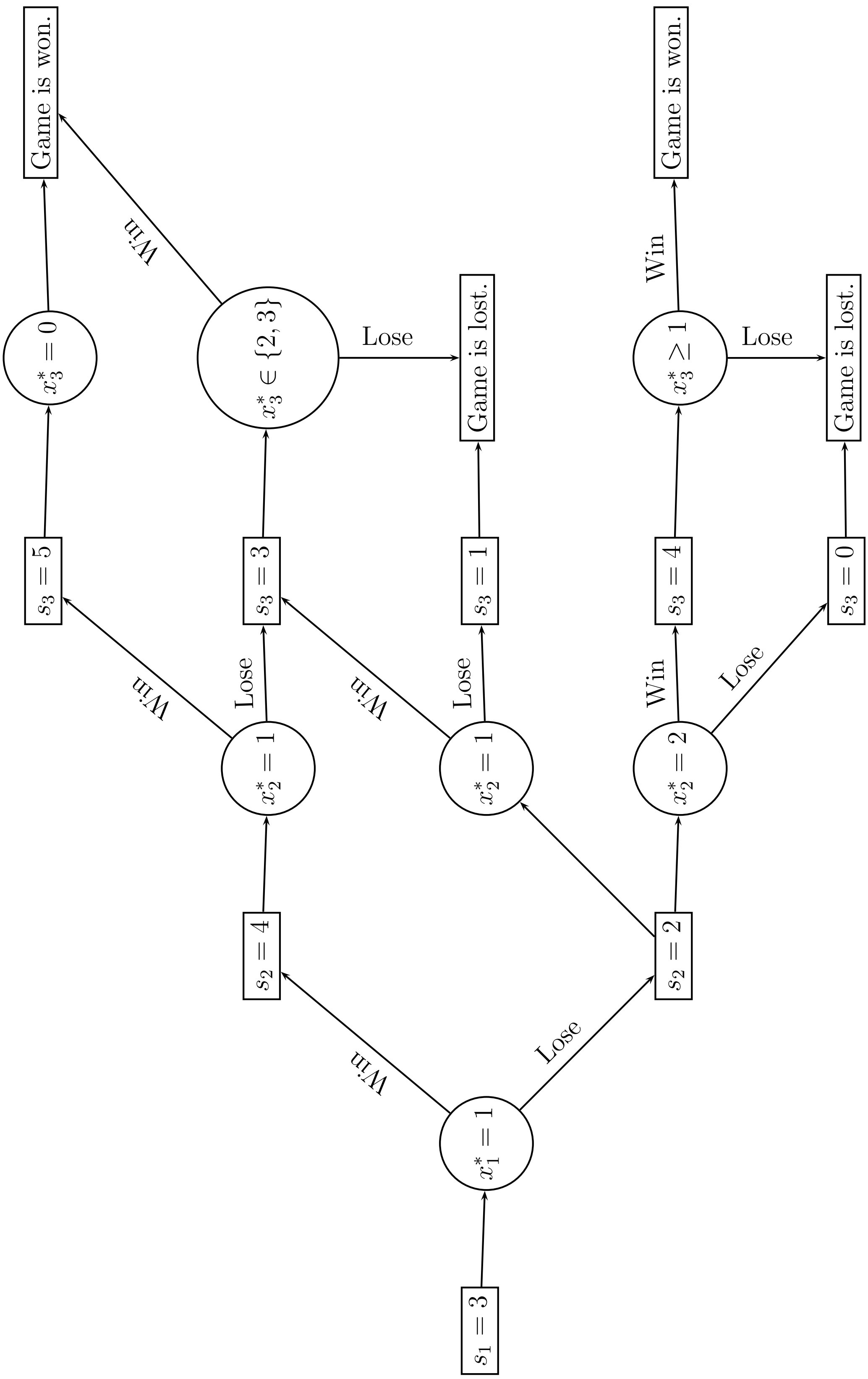
      2.2.2 If the gambler loses play 3, he loses the game.

   2.3 If the gambler loses play 1, he starts play 2 with 2 chips. He bets 1 chip.

      2.3.1. If the gambler wins play 2, he starts play 3 with 3 chips. This is scenario 2.2.

      2.3.2. If the gambler loses play 2, ha starts play 3 with 1 chip and he loses the game.

The decision tree on the following page shows all the optimal policies with their outcomes. Note that, due to the stochastic nature of this problem, it may happen that the gambler loses even if he follows an optimal policy.


$s_1 = 3$
$x_1^* = 1$
Win
Lose
$s_2 = 4$
$s_2 = 2$
$x_2^* = 1$
Win
Lose
$x_2^* = 1$
Win
Lose
$x_2^* = 2$
Win
Lose
$s_3 = 5$
$s_3 = 3$
$s_3 = 1$
$s_3 = 4$
$s_3 = 0$
$x_3^* = 0$
$x_3^* \in \{2, 3\}$
Win
Lose
$x_3^* \geq 1$
Win
Lose
Game is won.
Game is lost.
Game is won.
Game is lost.