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Fundamentals of Optimization

Exercise 5

Remarks

- All questions that are available in the STACK quiz are duly marked. Please solve those using STACK.
- We have added marks for each question. Please note that those are purely for illustrative purposes. The exercise set will not be marked.
- We can derive the inverse of a nonsingular matrix $A \in \mathbb{R}^{2\times 2}$ in closed form:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

1 The Two-Phase Method

Below, you are given two linear programming problems in which the matrix A has full row rank. You are asked to solve the problems using the Two-Phase Method.

Instructions

General: For each dictionary, and before carrying out the steps of Case 2b (if applicable), write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether the dictionary is optimal or not.

When entering the indices in the set B, pay attention to the row numbers in the corresponding dictionary, i.e., if x_3, x_1, and x_4 are the basic variables in Rows 1, 2, and 3, respectively, then $B = \{3, 1, 4\}$. For the index set N, simply enter the indices in increasing order.

Use the nonbasic variable with the most negative reduced cost as the entering variable and break ties in favour of the nonbasic variable with the smallest index.

Phase 1: The first dictionary is the one that starts with $\hat{x} = \mathbf{0}$ and $\hat{a} = b$.

To avoid any ambiguities when writing down the indices in the sets B and N, identify $x_{n+i} = a_i$, $i = 1, \ldots, m$, and please use x_{n+i} instead of a_i for your calculations. For example for $n = 4$ and $m = 2$, the artificial variables are $x_5 = a_1$ and $x_6 = a_2$ and sets B and N for the basis variables x_2, x_3, and a_1 are given as $B = \{2, 3, 5\}$ and $N = \{1, 4, 6\}$.

For the first dictionary, order the basic variables in increasing order of their indices, i.e. the basic variable with the smallest index should the first basic variable, the basic variable with the second smallest index should the second basic variable, and so on.

Phase 2: If the optimal value of the Phase 1 problem is zero, proceed to Phase 2, leaving aside all artificial variables. Just set up the first dictionary for the original linear programming problem. Please retain the order in which the basis variables appeared in the rows of the final dictionary of Phase 1, i.e. don't rearrange the rows. For example for $n = 5$ and $m = 3$, if the basis variables in Rows 1, 2, and 3 of the final dictionary of Phase 1 are x_5, x_2, and x_4, respectively, then this is also the order in which the basis variables appear in the rows of the first dictionary in Phase 2. Write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether this dictionary is optimal or not. **In Phase 2, you do not need to perform additional simplex iterations if this dictionary is not optimal.**

Determine whether the original linear programming problem has an empty or nonempty feasible region.

(1.1) LP

$$\begin{array}{lrcrcrcrcr}
\min & -x_1 & - & 3x_2 & - & 4x_3 & + & x_4 & & \\
\text{s.t.} & x_1 & + & x_2 & + & 2x_3 & & & = & 4 \\
 & 2x_1 & + & 3x_2 & & & - & x_4 & = & 18 \\
 & & & & & \multicolumn{3}{r}{x_1,\, x_2,\, x_3,\, x_4} & \geq & 0
\end{array}$$

(1.2) LP

$$\begin{array}{lrcrcrcrcr}
\min & -x_1 & - & x_2 & + & x_3 & + & 2x_4 & & \\
\text{s.t.} & x_1 & - & x_2 & - & x_3 & & & = & 1 \\
 & -x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 1 \\
 & & & & & \multicolumn{3}{r}{x_1,\, x_2,\, x_3,\, x_4} & \geq & 0
\end{array}$$

2 The Dual Simplex Method - Part I

Below, you are given two linear programming problems such that the matrix A has full row rank. Perform only one iteration of the dual simplex method from the given starting solution. Determine whether Dictionary 2 is (i) primal feasible, (ii) dual feasible, and (iii) optimal.

Instructions

- For each dictionary, and before carrying out Step 3 (if applicable), write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$.

  When entering the indices in the set B, pay attention to the row numbers in the corresponding dictionary, i.e., if x_3, x_1, and x_4 are the basic variables in Rows 1, 2, and 3, respectively, then $B = \{3, 1, 4\}$. For the index set N, simply enter the indices in increasing order.
- Use the nonbasic variable with the most negative value as the leaving variable and break ties in favour of the nonbasic variable with the smallest index.
- For the first dictionary, order the basic variables in increasing order of their indices, i.e., the basic variable with the smallest index should the first basic variable, the basic variable with the second smallest index should the second basic variable, and so on.
- If there is a tie between two or more nonbasic variables in the minimum ratio test, then break the tie in favour of the nonbasic variable with the smallest index.

Questions

(2.1) Linear program

$$\begin{array}{llrcrcrcrcrcr}
\min & & x_1 & + & x_2 & & & & & & & & \\
\text{s.t.} & & 2x_1 & + & x_2 & + & x_3 & & & & & = & 7 \\
 & - & 2x_1 & - & x_2 & & & + & x_4 & & & = & -4 \\
 & & x_1 & - & x_2 & & & & & + & x_5 & = & -1 \\
 & & & & & & \multicolumn{5}{r}{x_1,\, x_2,\, x_3,\, x_4,\, x_5} & \geq & 0
\end{array}$$

with starting solution $\hat{x} = [0, 0, 7, -4, -1]^T$.

(2.2) Linear program

$$\begin{array}{lrcrcrcrcrcl}
\min & & 2x_1 & + & x_2 & & & & & & & \\
\text{s.t.} & - & x_1 & - & 2x_2 & + & x_3 & & & & = & -6 \\
& & 2x_1 & + & x_2 & & & + & x_4 & & = & 8 \\
& - & 3x_1 & - & x_2 & & & & & + \ x_5 & = & -4 \\
& & & & & & & & x_1,\, x_2,\, x_3,\, x_4,\, x_5 & \geq & 0 &
\end{array}$$

with starting solution $\hat{x} = [0, 0, -6, 8, -4]^T$.

3 Sensitivity Analysis

Consider the following linear program

$$\begin{array}{lrcrcrcrcrcl}
\min & - & 3x_1 & - & x_2 & - & 4x_3 & + & 2x_4 & & & \\
\text{s.t.} & & 4x_1 & + & 6x_2 & + & 5x_3 & + & x_4 & + \ x_5 & = & 6 \\
& & 3x_1 & + & 5x_2 & + & 4x_3 & & & + \ x_5 & = & 5 \\
& & & & & & & & x_1,\, x_2,\, x_3,\, x_4,\, x_5 & \geq & 0 &
\end{array}$$

and the optimal dictionary

$$\begin{array}{rcrcrcrcr}
z & = & -4 & + & x_1 & + & 3x_2 & + & 6x_4 \\
x_3 & = & 1 & - & x_1 & - & x_2 & - & x_4 \\
x_5 & = & 1 & + & x_1 & - & x_2 & + & 4x_4
\end{array}$$

Instructions

- For the primal simplex, if there is more than one nonbasic variable with a negative reduced cost, choose the one with the most negative reduced cost as the entering variable; break ties in favour of the basic variable with the smallest index. In the minimum ratio test, if there is a tie between two or more rows, choose the row whose basic variable has the smallest index to determine the leaving variable.
- For the dual simplex, if there is more than one negative basic variable, choose the one with the most negative value as the leaving variable; break ties in favour of the basic variable with the smallest index. In the minimum ratio test, if there is a tie between two or more nonbasic variables, then break the tie in favour of the nonbasic variable with the smallest index.

Questions

(3.1) By how much can we increase the value of $b_2 = 5$ in the original problem such that current dictionary remains optimal? Justify your solution.

(3.2) If we increase the value of $b_1 = 6$ by $\delta = 1$ in the original problem, does the current dictionary remain optimal? Justify your solution.

If not, perform **one iteration** of the appropriate simplex method and derive a new dictionary with its corresponding solution. For the new dictionary, write down the index sets B and N, values of all variables, reduced costs of all variables, and the current objective function value. State whether the corresponding solution is optimal or not. State whether it is primal feasible or dual feasible or both. Justify your solution.

(3.3) By how much can we decrease the value of $c_2 = -1$ in the original problem such that the current dictionary remains optimal? Justify your solution.

(3.4) If we increase the value of $c_5 = 0$ by $\delta = 2$ in the original problem, does the current dictionary remain optimal? Justify your solution.

If not, perform **one iteration** of the appropriate simplex method and derive a new dictionary with its corresponding solution. For the new dictionary, write down the index sets B and N, values of all variables, reduced costs of all variables, and the current objective function value. State whether the corresponding solution is optimal or not. State whether it is primal feasible or dual feasible or both. Justify your solution.

(3.5) Consider the variable x_2. If we change A^2 from $[6,5]^T$ to $[5,5]^T$ in the original problem, i.e. $A^2(\delta) = A^2 + \delta e^1$ with $\delta = -1$, does the current dictionary remain optimal? Justify your solution.

If not, perform **one iteration** of the appropriate simplex method and derive a new dictionary with its corresponding solution. For the new dictionary, write down the index sets B and N, values of all variables, reduced costs of all variables, and the current objective function value. State whether the corresponding solution is optimal or not. State whether it is primal feasible or dual feasible or both. Justify your solution.

(3.6) Next, we want to add a new variable x_6 to the original problem. The corresponding column is given by $A^6 = [2,3]^T$. Find the range of values for c_6 for which the current dictionary remains optimal. Justify your solution.

(3.7) Finally, we want to add the constraint

$$x_1 - 2x_2 + 3x_3 - 3x_4 + x_5 \leq 3$$

to the original problem.

Is the current solution still optimal for the modified problem? If not, find the new optimal solution. Justify your solution.

Open Ended Problems

4 The Dual Simplex Method - Part II

(4.1) Suppose that you are applying the dual simplex method and the current dictionary is given by

$$\begin{aligned} z &= \hat{z} + \sum_{j \in N} \bar{c}_j x_j \\ x_i &= \hat{x}_i + \sum_{j \in N} \bar{a}_{ij} x_j, \quad i \in B. \end{aligned}$$

Suppose that the current dictionary is dual feasible but primal infeasible.

Let $p \in B$ an index with the corresponding row given by

$$x_p = \hat{x}_p + \sum_{j \in N} \bar{a}_{pj} x_j.$$

Suppose that $\hat{x}_p < 0$ and $\bar{a}_{pj} \leq 0$ for each $j \in N$.

(i) What can you conclude about the primal problem? Justify your solution.

(ii) What can you conclude about the dual problem? Justify your solution.