MATH11111: Fundamentals of Optimization **Fall 2022**

Lecture 18 Development of the Simplex Method

Lecturer: E. Alper Yıldırım *Week: 5*

18.1 Outline

- Development of the Simplex Method
- Review Problems

18.2 Review and Setup

Let (P) denote a linear programming problem in standard form.

$$
\begin{array}{llrcl}
\text{(P)} & \min & c^T x & & \\
& \text{s.t.} & & & \\
& & Ax & = & b, \\
& & x & \geq & \mathbf{0},
\end{array}
$$

where $A \in \mathbb{R}^{m \times n}$ has full row rank, $b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$.

- Let $\hat{x}$ be a vertex of $\mathcal{P}$ with corresponding index sets B and N.
- For each index $j \in N$, the reduced cost of the variable x_j, denoted by $\bar{c}_j$, is given by $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j$.
- If $\bar{c}_j \geq 0$ for each $j \in N$, then $\hat{x}$ is an optimal solution of (P).
- If $\hat{x}$ is nondegenerate, then $\bar{c}_j \geq 0$ for each $j \in N$ if and only if $\hat{x}$ is an optimal solution of (P).

18.3 Nondegeneracy Assumption

In this lecture, we will first assume that $\hat{x}$ is a nondegenerate vertex of $\mathcal{P}$ with corresponding index sets B and N (we will revisit this assumption later on). Compute reduced costs $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j$ for each $j \in N$.

Case 1

If $\bar{c}_j \geq 0$ for each $j \in N$, then $\hat{x}$ is an optimal solution of (P) by Proposition 17.1. Stop.

Case 2

Suppose that there exists a $j^* \in N$ such that $\bar{c}_{j^*} < 0$ (i.e., $\hat{x}$ is not an optimal solution by Proposition 17.1).

Let $d \in \mathbb{R}^n$ be such that $d_{j^*} = 1$ and $d_j = 0$ for each $j \in N \backslash \{j^*\}$. Let $d_B = -(A_B)^{-1} A_N d_N \in \mathbb{R}^m$. Note that $d \in \mathbb{R}^n$ is a feasible direction at $\hat{x}$ such that $c^T d = \bar{c}_{j^*} < 0$ (i.e., it is an *improving* feasible direction).

Case 2a

If $d_B \geq \mathbf{0}$ (i.e., $d_j \geq 0$ for each $j \in B$), then $\hat{x} + \lambda d \in \mathcal{P}$ for each $\lambda \geq 0$ and $c^T(\hat{x} + \lambda d) = c^T \hat{x} + \lambda \bar{c}_{j^*} \to -\infty$ as $\lambda \to \infty$. We have discovered a feasible direction d at $\hat{x}$ along which the objective function can be decreased indefinitely. The problem is unbounded. Stop.

Case 2b

If d_B has at least one negative component, then define $\lambda^* = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j}$ (recall that $\hat{x}_j > 0$ for each $j \in B$ due to nondegeneracy and $\lambda^* > 0$ is a finite number).

Let $k^* \in B$ such that $\lambda^* = \frac{-\hat{x}_{k^*}}{d_{k^*}} = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j}$. Note that $d_{k^*} < 0$.

Let $\bar{x} = \hat{x} + \lambda^* d$. Note that $\bar{x} \in \mathcal{P}$ and $c^T \bar{x} = c^T(\hat{x} + \lambda^* d) = c^T \hat{x} + \lambda^* c^T d = c^T \hat{x} + \lambda^* \bar{c}_{j^*} < c^T \hat{x}$ (i.e., $\bar{x}$ is a better feasible solution than $\hat{x}$).

We therefore obtain the following relations:

$$
\begin{aligned}
\lambda^* &= \frac{-\hat{x}_{k^*}}{d_{k^*}} = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} \\
\bar{x}_j &= \hat{x}_j + \lambda^* d_j \geq 0, \quad j \in B \backslash \{k^*\} \\
\bar{x}_{k^*} &= \hat{x}_{k^*} + \lambda^* d_{k^*} = \hat{x}_{k^*} + \left(\frac{-\hat{x}_{k^*}}{d_{k^*}} \right) d_{k^*} = 0 \\
\bar{x}_{j^*} &= \hat{x}_{j^*} + \lambda^* d_{j^*} = \lambda^* > 0 \\
\bar{x}_j &= \hat{x}_j + \lambda^* d_j = 0, \quad j \in N \backslash \{j^*\}
\end{aligned}
$$

Table 18.1 summarises the similarities and differences between $\bar{x}$ and $\hat{x}$.

Index	$\hat{x}$	$\bar{x}$
$j \in B \backslash \{k^*\}$	$\hat{x}_j > 0$	$\bar{x}_j \geq 0$
k^*	$\hat{x}_{k^*} > 0$	$\bar{x}_{k^*} = 0$
j^*	$\hat{x}_{j^*} = 0$	$\bar{x}_{j^*} > 0$
$j \in N \backslash \{j^*\}$	$\hat{x}_j = 0$	$\bar{x}_j = 0$

Table 18.1: Comparison of $\hat{x}$ and $\bar{x}$

As illustrated by Table 18.1, we can make the following observations:

- Only one of the nonbasic variables at $\hat{x}$ becomes positive at $\bar{x}$.

- All the remaining nonbasic variables remain at zero.
- At least one of the basic variables at $\hat{x}$ becomes zero at $\bar{x}$.
- All the remaining basic variables at $\hat{x}$ remain nonnegative at $\bar{x}$.
- Let $\bar{B} = (B\backslash\{k^*\}) \cup \{j^*\}$ and $\bar{N} = (N\backslash\{j^*\}) \cup \{k^*\}$.
- Note that $|\bar{B}| = m$, $|\bar{N}| = n - m$, and $\bar{B} \cup \bar{N} = \{1, \dots, n\}$.

The following proposition outlines a very important property of the new feasible solution $\bar{x}$ and forms the backbone of the *simplex method*.

Proposition 18.1. *$\bar{x}$ is a vertex of $\mathcal{P}$.*

Proof. Since $\bar{x} \in \mathcal{P}$, $\bar{x}_j = 0$ for each $j \in \bar{N}$, and $|\bar{B}| = m$, it suffices to show that $A_{\bar{B}} \in \mathbb{R}^{m\times m}$ is an invertible matrix, i.e., its columns are linearly independent. Since $A_B \in \mathbb{R}^{m\times m}$ is an invertible matrix and $\bar{B} = (B\backslash\{k^*\}) \cup \{j^*\}$, the columns of $A_{\bar{B}}$ corresponding to indices $j \in B\backslash\{k^*\}$ are clearly linearly independent. Suppose, for a contradiction, that the columns of $A_{\bar{B}}$ are not linearly independent. Then, the column A^{j^*} should be in the span of the columns of A_B corresponding to indices $j \in B\backslash\{k^*\}$. Suppose that A^{k^*} is the ℓth column of A_B, where $\ell \in \{1, \dots, m\}$. Then, there exist real numbers $\mu_j,\ j = 1, \dots, m,\ j \neq \ell$, such that $A^{j^*} = \sum\limits_{j\in\{1,\dots,m\}\backslash\{\ell\}} \mu_j A_B e^j$. Recall that

$$d_B = -(A_B)^{-1}A_N d_N = -(A_B)^{-1}\left(\sum_{j\in N} A^j d_j\right) = -(A_B)^{-1}A^{j^*}.$$

Therefore,

$$d_B = -(A_B)^{-1}A^{j^*} = -\sum_{j\in\{1,\dots,m\}\backslash\{\ell\}} \mu_j (A_B)^{-1} A_B e^j = -\sum_{j\in\{1,\dots,m\}\backslash\{\ell\}} \mu_j e^j,$$

which implies that $(d_B)_\ell = d_{k^*} = 0$ since k^* is the ℓth index in the set B. This contradicts $d_{k^*} < 0$. □

The next definition relates the starting vertex $\hat{x}$ and the new vertex $\bar{x}$.

Definition 18.1. *Let $\hat{x} \in \mathcal{P}$ be a vertex and let $B \subseteq \{1, \dots, n\}$ and $N \subseteq \{1, \dots, n\}$ denote the indices of the basic and nonbasic variables, respectively. Let $\bar{x} \in \mathcal{P}$ be a vertex such that $\bar{x} \neq \hat{x}$ and let $\bar{B} \subseteq \{1, \dots, n\}$ and $\bar{N} \subseteq \{1, \dots, n\}$ denote the indices of the basic and nonbasic variables, respectively. Then, $\hat{x}$ and $\bar{x}$ are called* adjacent *if B and $\bar{B}$ have exactly $m - 1$ common indices. The line segment that joins $\hat{x}$ and $\bar{x}$ is called an* edge *of the feasible region $\mathcal{P}$.*

Remark 18.1. *Note that the vertices $\hat{x}$ and $\bar{x}$ in Case 2b are adjacent.*

Proposition 18.1 reveals that the new solution $\bar{x}$ computed in Case 2b is also a vertex of $\mathcal{P}$ that is adjacent to $\hat{x}$. Recall that $\bar{x}$ is a better feasible solution than $\hat{x}$, i.e., $c^T\bar{x} < c^T\hat{x}$.

If we assume that $\bar{x}$ is also a nondegenerate vertex, then we can repeat the same procedure, starting from $\bar{x}$. This procedure can be turned into an algorithm, called the *simplex method*.

18.4 The Simplex Method

Let (P) be a linear programming problem in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\}$$

Assume that $\mathcal{P} \neq \emptyset$, A has full row rank, and we have a vertex $\hat{x}$ of $\mathcal{P}$. Let $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ denote the indices corresponding to basic and nonbasic variables, respectively. Here is the outline of the simplex method:

1. Compute the reduced costs $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j, \ j \in N$.
2. **Case 1:** If $\bar{c}_j \geq 0$ for each $j \in N$, then stop. $\hat{x}$ is an optimal solution of (P) and $z^* = c^T \hat{x}$.
3. **Case 2:** If there exists a $j^* \in N$ such that $\bar{c}_{j^*} < 0$, then let $d \in \mathbb{R}^n$ be such that $d_{j^*} = 1$ and $d_j = 0$ for each $j \in N \backslash \{j^*\}$ and $d_B = -(A_B)^{-1} A_N d_N = -(A_B)^{-1} A^{j^*} \in \mathbb{R}^m$.
   (a) **Case 2a:** If $d_B \geq \mathbf{0}$, then stop. (P) is unbounded along the direction $d \in \mathbb{R}^n$.
   (b) **Case 2b:** If d_B has at least one negative component, then define $\lambda^* = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} = \frac{-\hat{x}_{k^*}}{d_{k^*}}$. Set $\bar{x} = \hat{x} + \lambda^* d$, $\bar{B} = (B \backslash \{k^*\}) \cup \{j^*\}$ and $\bar{N} = (N \backslash \{j^*\}) \cup \{k^*\}$. Set $\hat{x} \leftarrow \bar{x}$, $B \leftarrow \bar{B}$, and $N \leftarrow \bar{N}$. Go to Step 1.

Remarks

1. In Case 2, if there is more than one negative reduced cost, we can pick any one arbitrarily.
2. However, for any $j \in N$ such that $\bar{c}_j < 0$, recall that $c^T(\hat{x} + \lambda d) = c^T \hat{x} + \lambda \bar{c}_j$.
3. Therefore, $\bar{c}_j$ gives us the rate of decrease of the objective function and it makes sense to choose the most negative reduced cost.
4. Reduced costs are only computed for nonbasic variables. Suppose that $j \in B$ is the ℓth index in B. We have $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j = c_j - c_B^T (A_B)^{-1} A_B e^\ell = c_j - (c_B)_\ell = c_j - c_j = 0$. **Therefore, the reduced cost of any basic variable is equal to zero.**
5. The computation of λ^* in Case 2b is referred to as the *minimum ratio test.*
6. The computation of reduced costs at a vertex and the case checks is referred to as an *iteration of the simplex method.*

18.5 Finite Convergence of the Simplex Method

An interesting and relevant question is whether the simplex method always terminates after a finite number of iterations. The following proposition addresses this question under the nondegeneracy assumption.

Proposition 18.2. *Let (P) be a linear programming problem in standard form. Suppose that the feasible region $\mathcal{P}$ is nonempty, A has full row rank, and every vertex of $\mathcal{P}$ is nondegenerate. Then, the simplex method terminates after a finite number of iterations with exactly one of the following two possible outcomes:*

1. *An optimal vertex $\hat{x}$ with a finite optimal value, or*

2. *A vertex $\hat{x}$ and a direction d along which the objective function is unbounded below.*

Proof. Recall that every linear programming problem in standard form with a nonempty feasible region has at least one vertex. Therefore, we can always find a starting vertex (for example by using complete enumeration). Since every vertex of $\mathcal{P}$ is nondegenerate, Case 1 is a necessary and sufficient condition of optimality. Furthermore, we have $\lambda^* > 0$ whenever we perform Case 2b, therefore we always move to a different vertex. Since the objective function improves at the next vertex, we cannot go back to an earlier vertex that was already visited. Since every linear programming problem has at most a finite number of vertices, the result follows. □

18.6 Discussion and Questions

Proposition 18.2 shows that the simplex method terminates after a finite number of iterations under the stated assumptions. It can either find an optimal vertex or detect that the problem is unbounded.

This development gives rise to a number of questions:

- **Question 1:** Is there a simple implementation of the simplex method?
- **Question 2:** What happens if there are degenerate vertices?
- **Question 3:** What if there are no feasible solutions (i.e., $\mathcal{P} = \emptyset$)?
- **Question 4:** Is there a simple procedure to obtain a starting vertex (without using complete enumeration)?

We will address each of these questions in the following lectures.

Exercises

Question 18.1. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a nonempty polyhedron in standard form and let $\hat{x}$ be a nondegenerate vertex of $\mathcal{P}$. Suppose that there exists a $j^* \in N$ such that $\bar{c}_{j^*} < 0$ and d_B has at least one negative component. Can the next vertex $\bar{x} = \hat{x} + \lambda^* d$ be a degenerate vertex?*

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Lecture 19 — Simplex Method in Dictionary Form

Lecturer: E. Alper Yıldırım — *Week: 5*

19.1 Outline

- Simplex Method in Dictionary Form
- Review Problems

19.2 Overview

In this lecture, we will discuss a simple implementation of the simplex method, an algorithm for solving linear programming problems. We will illustrate this idea on two examples.

19.3 Example 1

Consider the following linear programming problem:

$$
\begin{array}{lrcrcl}
\min & 3x_1 & - & 2x_2 & & \\
\text{s.t.} & & & & & \\
& -3x_1 & + & 3x_2 & \leq & 6 \\
& -4x_1 & + & 2x_2 & \leq & 2 \\
& x_1 & - & 2x_2 & \leq & 2 \\
& x_1 & , & x_2 & \geq & 0
\end{array}
$$

Since the problem is not in standard form, we introduce nonnegative slack variables x_3, x_4, and x_5 for the first, second, and the third inequality constraints, respectively.

We therefore obtain the following equivalent linear programming problem in standard form:

$$
\begin{array}{lrcrcrcrcrcl}
\min & 3x_1 & - & 2x_2 & & & & & & & & \\
\text{s.t.} & & & & & & & & & & & \\
& -3x_1 & + & 3x_2 & + & x_3 & & & & & = & 6 \\
& -4x_1 & + & 2x_2 & & & + & x_4 & & & = & 2 \\
& x_1 & - & 2x_2 & & & & & + & x_5 & = & 2 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}
$$

Note that we have $m = 3$ and $n = 5$, and

$$A = \begin{bmatrix} -3 & 3 & 1 & 0 & 0 \\ -4 & 2 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

In order to solve this problem using the simplex method, we need to have a vertex as a starting solution.

Let $\hat{x} = [0, 0, 6, 2, 2]^T \in \mathbb{R}^5$. Note that $\hat{x}$ is a feasible solution since it satisfies all of the constraints. Furthermore, $\hat{B} = \{j \in \{1, \ldots, 5\} : \hat{x}_j \neq 0\} = \{3, 4, 5\}$ and $\hat{N} = \{j \in \{1, \ldots, 5\} : \hat{x}_j = 0\} = \{1, 2\}$.

Since $|\hat{B}| = m = 3$ and

$$A_{\hat{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is clearly invertible, $\hat{x}$ is a vertex with $B = \hat{B}$ and $N = \hat{N}$.

In this solution, x_3, x_4, and x_5 are the basic variables, and x_1 and x_2 are nonbasic variables. The current objective function value at $\hat{x}$ is equal to 0.

Remark: We use x to denote a general solution and $\hat{x}$ to denote a specific solution. A similar convention will be used in general for other notation as well.

Dictionary 1

Let us define a new variable z to denote the objective function and express each basic variable as a function of nonbasic variables:

$$\begin{array}{rcrcrcr} z & = & 0 & + & 3x_1 & - & 2x_2 \\ x_3 & = & 6 & + & 3x_1 & - & 3x_2 \\ x_4 & = & 2 & + & 4x_1 & - & 2x_2 \\ x_5 & = & 2 & - & x_1 & + & 2x_2 \end{array}$$

This is called a *dictionary*. The first row, referred to as Row 0, corresponds to the objective function. The remaining three rows, denoted by Row 1, Row 2, and Row 3, correspond to each of the three basic variables.

- Since x_1 and x_2 are nonbasic, $\hat{x}_1 = \hat{x}_2 = 0$.
- x_3, x_4, and x_5 are basic variables, and $\hat{x}_3 = 6$, $\hat{x}_4 = 2$, and $\hat{x}_5 = 2$.
- The objective function value is $\hat{z} = 0$.

Looking at Row 0, we see that we can reduce z by increasing x_2 since it has a negative coefficient. We will therefore try to increase the value of x_2 while keeping $x_1 = 0$. By Rows 1, 2, and 3, this change will affect the values of each of x_3, x_4, and x_5.

- By Row 1, $x_3 = 6 - 3x_2 \geq 0$ if and only if $x_2 \leq 2$.

- By Row 2, $x_4 = 2 - 2x_2 \geq 0$ if and only if $x_2 \leq 1$.

- By Row 3, $x_5 = 2 + 2x_2$. We can therefore increase x_2 indefinitely and continue to have $x_5 \geq 0$.

The smallest bound is given by Row 2. This step is called the *minimum ratio test*. If $x_2 = 1$, we obtain $x_4 = 0$. Therefore, we will move x_2 to the left-hand side and x_4 to the left-hand side in Row 2.

Therefore, x_2 is the new basic variable, called the *entering variable*, and x_4 is the new nonbasic variable, called the *leaving variable*.

We now move x_2 to the left-hand side of Row 2 and move x_4 to the right-hand side of Row 2.

By Row 2, we obtain $x_2 = 1 + 2x_1 - (1/2)x_4$.

We now substitute this for x_2 in the right-hand sides of Rows 0, 1, and 3.

Dictionary 2

After the steps above, we arrive at the following dictionary:

$$\begin{array}{rcrcrcr} z & = & -2 & - & x_1 & + & x_4 \\ x_3 & = & 3 & - & 3x_1 & + & \frac{3}{2}x_4 \\ x_2 & = & 1 & + & 2x_1 & - & \frac{1}{2}x_4 \\ x_5 & = & 4 & + & 3x_1 & - & x_4 \end{array}$$

- We have $B = \{3, 2, 5\}$ and $N = \{1, 4\}$.

- The basic variables are given by $\hat{x}_3 = 3$, $\hat{x}_2 = 1$, and $\hat{x}_5 = 4$.

- The nonbasic variables are given by $\hat{x}_1 = 0$ and $\hat{x}_4 = 0$.

- The objective function value is $\hat{z} = -2 < 0$.

Since A_B is invertible, this is a vertex. We have therefore obtained a new vertex with a strictly better objective function value than the previous vertex.

By Row 0, x_1 is the entering variable since it has a negative coefficient on the right-hand side of Row 0.

- Note that Rows 2 and 3 do not give any restrictions (or upper bounds) on x_1.

- Row 1 is the only row that gives a finite upper bound of 1 on the value of x_1.

Therefore, x_1 is the entering variable and x_3 is the leaving variable.

We now move x_1 to the left-hand side of Row 1 and x_3 to the right-hand side of Row 1.

By Row 1, $x_1 = 1 - \frac{1}{3}x_3 + \frac{1}{2}x_4$.

We now substitute this expression for x_1 in Rows 0, 2, and 3.

Dictionary 3

After the steps above, we arrive at the following dictionary:

$$\begin{array}{rcrcrcr} z & = & -3 & + & \frac{1}{3}x_3 & + & \frac{1}{2}x_4 \\ x_1 & = & 1 & - & \frac{1}{3}x_3 & + & \frac{1}{2}x_4 \\ x_2 & = & 3 & - & \frac{2}{3}x_3 & + & \frac{1}{2}x_4 \\ x_5 & = & 7 & - & x_3 & + & \frac{1}{2}x_4 \end{array}$$

- We have $B = \{1, 2, 5\}$ and $N = \{3, 4\}$.
- The basic variables are given by $\hat{x}_1 = 1$, $\hat{x}_2 = 3$, and $\hat{x}_5 = 7$.
- The nonbasic variables are given by $\hat{x}_3 = 0$ and $\hat{x}_4 = 0$.
- The objective function value is $\hat{z} = -3 < -2$.

Since A_B is invertible, this is a vertex.

By Row 0, the coefficients of both nonbasic variables on the right-hand side of Row 0 are nonnegative. Therefore, we can no longer improve the objective function (i.e., there are no eligible entering variables).

Termination and Discussion

- We have found an optimal solution given by $x^* = [1, 3, 0, 0, 7]^T$ and the optimal value is $z^* = c^T x^* = -3$ after performing two iterations.
- Note that we started with a vertex having an objective function value of zero, and each vertex was strictly better than the previous one.
- Each of the three vertices was nondegenerate since the basic variables were always strictly positive.
- In each dictionary, Row 0 expressed the objective function in terms of nonbasic variables in that dictionary, and Rows 1, 2, and 3 expressed each basic variable in terms of nonbasic variables in that dictionary.
- We used Row 0 to determine the entering variable and used Rows 1, 2, and 3 to determine the leaving variable.

19.4 Relation with the Simplex Method

Question 1. *How is the above procedure related to the simplex method?*

Let (P) denote a linear programming problem in standard form.

$$\begin{array}{lll} \text{(P)} & \min & c^T x \\ & \text{s.t.} & \\ & & \begin{array}{rcl} Ax & = & b, \\ x & \geq & \mathbf{0}, \end{array} \end{array}$$

where $A \in \mathbb{R}^{m \times n}$ has full row rank, $b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$. Let $\mathcal{P}$ denote the (nonempty) feasible region.

- Let $\hat{x} \in \mathcal{P}$ be a vertex with index sets B and N.
- Since $Ax = b$ for each $x \in \mathcal{P}$, we obtain $A_B x_B + A_N x_N = b$.
- Multiplying both sides by $(A_B)^{-1}$ from the left, we obtain $x_B + (A_B)^{-1} A_N x_N = (A_B)^{-1} b$, i.e.,

$$x_B = (A_B)^{-1} b - (A_B)^{-1} A_N x_N.$$

- Since $z = c^T x = c_B^T x_B + c_N^T x_N$, we can substitute $x_B = (A_B)^{-1} b - (A_B)^{-1} A_N x_N$ and obtain

$$z = c_B^T (A_B)^{-1} b - c_B^T (A_B)^{-1} A_N x_N + c_N^T x_N = c_B^T (A_B)^{-1} b + c_N^T x_N - c_B^T (A_B)^{-1} A_N x_N.$$

- Note that $c_N^T x_N = \sum\limits_{j \in N} c_j x_j$ and $A_N x_N = \sum\limits_{j \in N} A^j x_j$.

We therefore arrive at the following system of equations:

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b + \sum_{j \in N} \left(-(A_B)^{-1} A^j\right) x_j \end{aligned}$$

- Since $\hat{x} \in \mathcal{P}$ is a vertex with index sets B and N, we obtain $\hat{x}_N = \mathbf{0}$ and $\hat{x}_B = (A_B)^{-1} b$. Therefore, $\hat{z} = c^T \hat{x} = c_B^T \hat{x}_B + c_N^T \hat{x}_N = c_B^T (A_B)^{-1} b$.
- The expressions $c_B^T (A_B)^{-1} b$ and $(A_B)^{-1} b$ are the current values of the objective function value and the basic variables at $\hat{x}$, respectively.
- The coefficients of the nonbasic variables on the right-hand side of Row 0 are the reduced costs.
- If $\bar{c}_{j^*} < 0$, then the column $-(A_B)^{-1} A^{j^*}$ yields d_B obtained by setting $d_{j^*} = 1$ and $d_j = 0$ for each $j \in N \backslash \{j^*\}$.

19.5 Example 2

Consider the following linear programming problem:

$$\begin{array}{lrcrcl} \min & -x_1 & + & x_2 & & \\ \text{s.t.} & & & & & \\ & -3x_1 & + & 3x_2 & \leq & 6 \\ & -4x_1 & + & 2x_2 & \leq & 2 \\ & x_1 & - & 2x_2 & \leq & 2 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

The equivalent problem in standard form is given by

$$
\begin{array}{lrcrcrcrcrcl}
\min & -x_1 & + & x_2 & & & & & & & & \\
\text{s.t.} & & & & & & & & & & & \\
& -3x_1 & + & 3x_2 & + & x_3 & & & & & = & 6 \\
& -4x_1 & + & 2x_2 & & & + & x_4 & & & = & 2 \\
& x_1 & - & 2x_2 & & & & & + & x_5 & = & 2 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0.
\end{array}
$$

Let $\hat{x} = [0, 0, 6, 2, 2]^T$.

Dictionary 1

$$
\begin{array}{rcrcrcr}
z & = & 0 & - & x_1 & + & x_2 \\
x_3 & = & 6 & + & 3x_1 & - & 3x_2 \\
x_4 & = & 2 & + & 4x_1 & - & 2x_2 \\
x_5 & = & 2 & - & x_1 & + & 2x_2
\end{array}
$$

- We have $B = \{3, 4, 5\}$ and $N = \{1, 2\}$.
- The values of basic variables are given by $\hat{x}_3 = 6$, $\hat{x}_4 = 2$, and $\hat{x}_5 = 2$.
- The values of nonbasic variables are given by $\hat{x}_1 = 0$ and $\hat{x}_2 = 0$.
- The objective function value is $\hat{z} = 0$.
- Since A_B is invertible, this is a vertex.
- By Row 0, x_1 is the entering variable and x_5 is the leaving variable by Row 3.
- By Row 3, $x_1 = 2 + 2x_2 - x_5$. Substitute this expression for x_1 in Rows 0, 1, and 2.

Dictionary 2

$$
\begin{array}{rcrcrcr}
z & = & -2 & - & x_2 & + & x_5 \\
x_3 & = & 12 & + & 3x_2 & - & 3x_5 \\
x_4 & = & 10 & + & 6x_2 & - & 4x_5 \\
x_1 & = & 2 & + & 2x_2 & - & x_5
\end{array}
$$

- We have $B = \{3, 4, 1\}$ and $N = \{2, 5\}$.
- The values of basic variables are given by $\hat{x}_3 = 12$, $\hat{x}_4 = 10$, and $\hat{x}_1 = 2$.
- The values of nonbasic variables are given by $\hat{x}_2 = 0$ and $\hat{x}_5 = 0$.
- The objective function value is $\hat{z} = -2 < 0$.
- Since A_B is invertible, this is a vertex.

- By Row 0, x_2 is the entering variable.
- By Rows 1, 2, and 3, there is no leaving variable, i.e., we can increase x_2 indefinitely without ever leaving the feasible region!
- Therefore, the problem is unbounded along the direction $d = [2, 1, 3, 6, 0]^T$.

19.6 Summary

Consider a linear programming problem in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq 0\}$$

19.6.1 Setup and Initialisation

Assume that the feasible region is nonempty, A has full row rank, and we have an initial basic feasible solution $\hat{x}$ with index sets B and N. Define a new variable z corresponding to the objective function. Set up the initial dictionary by

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b - \sum_{j \in N} (A_B)^{-1} A^j x_j \end{aligned}$$

19.6.2 Procedure

1. If $\bar{c}_j \geq 0$ for each $j \in N$ in Row 0, then stop. $\hat{x}$ is optimal and the optimal value is $z^* = c_B^T (A_B)^{-1} b$.
2. If there exists $j^* \in N$ such that $\bar{c}_j < 0$, then
   (a) If the coefficients of x_{j^*} on the right-hand sides of Rows 1 through m are all nonnegative, then stop. The problem is unbounded along the direction d given by $d_{j^*} = 1$, $d_j = 0$ for each $j \in N \backslash \{j^*\}$, and $d_B = -(A_B)^{-1} A^{j^*}$.
   (b) Otherwise, x_{j^*} is the entering variable.
      i. Apply the minimum ratio test to determine the leaving variable x_{k^*}, where $k^* \in B$.
      ii. Use the row corresponding to x_{k^*} to move x_{j^*} to the left-hand side and x_{k^*} to the right-hand side.
      iii. Substitute this expression for x_{j^*} in every other row, including Row 0.
      iv. Update the dictionary, the index sets B and N, and the current vertex $\hat{x}$ accordingly and go to Step 1.

19.7 Dictionary vs Tableau

You may have previously studied the simplex method in tableau form (as opposed to the dictionary form). Both dictionary and tableau forms are in fact equivalent. The tableau form is easier to implement. However, it is somewhat more "cryptic." We adopt dictionaries since they are more explicit and easier to understand.

19.8 Discussion and Questions

We discussed a simple and intuitive implementation of the simplex method. In both examples, all visited vertices were nondegenerate and we had an "easily identifiable" starting vertex (i.e., set all original variables to zero and each slack variables to the corresponding right-hand side value).

- **Question 1:** What happens if there are degenerate vertices?
- **Question 2:** What if there are no feasible solutions (i.e., $\mathcal{P} = \emptyset$)?
- **Question 3:** Is there a simple procedure to obtain a starting vertex if it is not easy to identify one?

We will address these questions in the subsequent lectures.

Exercises

Question 19.1. *Consider Example 1 again:*

$$
\begin{array}{lrcrcl}
\min & 3x_1 & - & 2x_2 & & \\
s.t. & & & & & \\
 & -3x_1 & + & 3x_2 & \leq & 6 \\
 & -4x_1 & + & 2x_2 & \leq & 2 \\
 & x_1 & - & 2x_2 & \leq & 2 \\
 & x_1 & , & x_2 & \geq & 0
\end{array}
$$

After adding the slack variables x_3, x_4, and x_5 for the first, second, and the third inequality constraints, respectively, here are the basic feasible solutions computed by the simplex method: (i) $\hat{x}^1 = [0, 0, 6, 2, 2]^T$; (ii) $\hat{x}^2 = [0, 1, 3, 0, 4]^T$; (iii) $\hat{x}^3 = [1, 3, 0, 0, 7]^T$. Solve the given problem using the graphical method and identify the feasible solutions of the original problem corresponding to $\hat{x}^1, \hat{x}^2$, and $\hat{x}^3$.