Workshop 1 Sol'ns: Bayesian Theory.

1. Algebra of expectations. There are 3 random variables, X, Y, and Z, that have the following properties and relationships:

$$E[X] = 3, V[X] = 2; E[Y] = 4, V[Y] = 3; E[Z] = -1, V[Z] = 1$$
$$Cov[X,Y] = 0; Cov[X,Z] = 0; Cov[Y,Z] = 1$$

Let a=2, b= -3.

(a) What is $E[aX + bY]$?

$$E[aX + bY] = E[2X - 3Y] = 2E[X] - 3E[Y] = 2*3 - 3*4 = -6$$

(b) What is $V[aX + bY]$?

$$V[aX + bY] = Var[aX] + Var[bY] + 2Cov[aX, bY] = a^2V[X] + b^2V[Y] + 2*0 = 4*2 + 9*3 = 35$$

(c) What is $Cov[aY, bZ]$?

$$Cov[aY, bZ] = abCov[Y, Z] = 2*(-3)*1 = -6$$

(d) What is $V[aY + bZ]$?

$$\begin{aligned} V[aY + bZ] &= V[aY] + V[bZ] + 2Cov[aY, bZ] \\ &= a^2V[Y] + b^2V[Z] + 2Cov[aY, bZ] = 4*3 + 9*1 + 2*(-6) = 9 \end{aligned}$$

2. Probability distribution exercises

(a) Given $X \sim$ Bernoulli(θ),

$$E[X] = \sum_{x=0}^{1} x \Pr(X = x) = 0*(1-\theta) + 1*\theta = \theta$$
$$V[X] = \sum_{x=0}^{1} (x-\theta)^2 \Pr(X = x) = (0-\theta)^2 * (1-\theta) + (1-\theta)^2 * \theta = \theta(1-\theta)$$

Note: $V[X] = E[X^2]$-$(E[X])^2$ is an alternative calculation.

(b) Given $X \sim$ Uniform(a, b),

$$E[X] = \int_a^b x\frac{1}{b-a}dx = \frac{1}{b-a}\frac{1}{2}x^2\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$
$$E[X^2] = \int_a^b x^2\frac{1}{b-a}dx = \frac{1}{b-a}\frac{1}{3}x^3\Big|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$
$$V[X] = E[X^2] - E[X]^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

(c) Given $X \sim$ Poisson(μ),

$$E[X] = \sum_{x=0}^{\infty} x\frac{e^{-\mu}\mu^x}{x!} = \sum_{x=1}^{\infty} x\frac{e^{-\mu}\mu^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\mu}\mu^x}{x-1!} = e^{-\mu}\mu\sum_{j=0}^{\infty}\frac{\mu^j}{j!} = e^{-\mu}\mu e^{\mu} = \mu$$

(d) Given $X \sim$ Exponential(λ),

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x}dx = \lambda\left[\int_0^\infty f(x)g'(x)dx\right] = \lambda\left[f(x)g(x)|_0^\infty - \int_0^\infty f'(x)g(x)dx\right]$$

where $f(x) = x, g'(x)dx = e^{-\lambda x}dx$. Then $f'(x) = 1, g(x) = -\frac{1}{\lambda}e^{-\lambda x}$

$$= \lambda\left[-\frac{x}{\lambda e^{\lambda x}}|_0^\infty - \frac{1}{\lambda^2}e^{-\lambda x}|_0^\infty\right] = \lambda\left[0 - 0 - 0 + \frac{1}{\lambda^2}\right] = \frac{1}{\lambda}$$

3. Joint probability distribution (with marginals found by summing over rows and columns):

	X_1			
X_2	0	1	2	
0	0.15	0.15	0.15	0.45
1	0.15	0.20	0.20	0.55
	0.30	0.35	0.35	1.00

(a) As can be seen from the table above, the marginal dist'n for X_1 is 0.30, 0.35, and 0.35 for the values 0, 1, and 2, respectively. And the marginal dist'n for X_2 is 0.45 and 0.55 for 0 and 1.

(b) The conditional dist'ns for $(X_1|X_2 = 0)$ is 1/3, 1/3, and 1/3 for X_1=0, 1, and 2, respectively. And for $X_1|X_2 = 1$ 0.15/0.55, 0.20/0.55, and 0.20/0.55.

The conditional dist'ns for $(X_2|X_1 = 0)$ are 0.5 and 0.5 for X_2=0 and 1; $X_2|X_1 = 1$ they are 3/7 and 4/7; and $X_2|X_1 = 2$ they are 3/7 and 4/7.

(c) For X_1 and X_2 to be independent, $\Pr(X_1 = x_1|X_2 = x_2)$ must equal $\Pr(X_1 = x_1)$, and vice versa. They are not independent; e.g., $\Pr(X_1 = 0|X_2 = 0) \neq \Pr(X_1 = 0)$, 0.333 vs 0.30.

4. Let R=Raleigh, D=Durham/Chapel Hill, and S=stolen.

$$\begin{aligned}
\Pr(R|S) &= \frac{\Pr(R,S)}{\Pr(S)} \\
&= \frac{\Pr(S|R)\Pr(R)}{\Pr(S|R)\Pr(R) + \Pr(S|D)\Pr(D)} \\
&= \frac{(135/10000) * (2/3)}{(135/10000) * (2/3) + (214/10000) * (1/3)} \\
&= \frac{135 * 2}{135 * 2 + 214 * 1} \\
&= 0.558
\end{aligned}$$