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Fundamentals of Optimization

Exercise 5 – Solutions

Remarks

- All questions that are available in the STACK quiz are duly marked. Please solve those using STACK.
- We have added marks for each question. Please note that those are purely for illustrative purposes. The exercise set will not be marked.
- We can derive the inverse of a nonsingular matrix $A \in \mathbb{R}^{2\times 2}$ in closed form:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

1 The Two-Phase Method

Below, you are given two linear programming problems in which the matrix A has full row rank. You are asked to solve the problems using the Two-Phase Method.

Instructions

General: For each dictionary, and before carrying out the steps of Case 2b (if applicable), write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$. Determine whether the dictionary is optimal or not.

When entering the indices in the set B, pay attention to the row numbers in the corresponding dictionary, i.e., if x_3, x_1, and x_4 are the basic variables in Rows 1, 2, and 3, respectively, then $B = \{3, 1, 4\}$. For the index set N, simply enter the indices in increasing order.

Use the nonbasic variable with the most negative reduced cost as the entering variable and break ties in favour of the nonbasic variable with the smallest index.

Phase 1: The first dictionary is the one that starts with $\hat{x} = \mathbf{0}$ and $\hat{a} = b$.

To avoid any ambiguities when writing down the indices in the sets B and N, identify $x_{n+i} = a_i$, $i = 1, \ldots, m$, and please use x_{n+i} instead of a_i for your calculations. For example for $n = 4$ and $m = 2$, the artificial variables are $x_5 = a_1$ and $x_6 = a_2$ and sets B and N for the basis variables x_2, x_3, and a_1 are given as $B = \{2, 3, 5\}$ and $N = \{1, 4, 6\}$.

For the first dictionary, order the basic variables in increasing order of their indices, i.e. the basic variable with the smallest index should the first basic variable, the basic variable with the second smallest index should the second basic variable, and so on.

Phase 2: If the optimal value of the Phase 1 problem is zero, proceed to Phase 2, leaving aside all artificial variables. Just set up the first dictionary for the original linear programming problem. Please retain the order in which the basis variables appeared in the rows of the final dictionary of Phase 1, i.e. don't rearrange the rows. For example for $n = 5$ and $m = 3$, if the basis variables in Rows 1, 2, and 3 of the final dictionary of Phase 1 are x_5, x_2, and x_4, respectively, then this is also the order in which the basis variables appear in the rows of the first dictionary in Phase 2. Write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \dots, n$, and the objective function value $\hat{z}$. Determine whether this dictionary is optimal or not. **In Phase 2, you do not need to perform additional simplex iterations if this dictionary is not optimal.**

Determine whether the original linear programming problem has an empty or nonempty feasible region.

(1.1) LP

$$\begin{array}{lrcrcrcrcr} \min & -x_1 & - & 3x_2 & - & 4x_3 & + & x_4 & & \\ \text{s.t.} & x_1 & + & x_2 & + & 2x_3 & & & = & 4 \\ & 2x_1 & + & 3x_2 & & & - & x_4 & = & 18 \\ & & & & & \multicolumn{3}{r}{x_1,\, x_2,\, x_3,\, x_4} & \geq & 0 \end{array}$$

Solution

Phase 1

The auxiliary problem is given as

$$\begin{array}{lrcrcrcrcrcrcr} \min & & & & & & & & & a_1 & + & a_2 & & \\ \text{s.t.} & x_1 & + & x_2 & + & 2x_3 & & & + & a_1 & & & = & 4 \\ & 2x_1 & + & 3x_2 & & & - & x_4 & & & + & a_2 & = & 18 \\ & & & & & & \multicolumn{6}{r}{x_1,\, x_2,\, x_3,\, x_4,\, a_1,\, a_2} & \geq & 0 \end{array}$$

Starting with $\hat{x} = \mathbf{0}$, $\hat{a} = b$, and substituting the artificial variables in Row 0, we obtain the following initial dictionary, where we replace a_1 and a_2 with x_5 and x_6, respectively.

Dictionary 1

$$\begin{array}{rcrcrcrcrcr} z & = & 22 & - & 3x_1 & - & 4x_2 & - & 2x_3 & + & x_4 \\ x_5 & = & 4 & - & x_1 & - & x_2 & - & 2x_3 & & \\ x_6 & = & 18 & - & 2x_1 & - & 3x_2 & & & + & x_4 \end{array}$$

We have $B = \{5, 6\}$, $N = \{1, 2, 3, 4\}$, $\hat{x} = [0, 0, 0, 0, 4, 18]^T$, $\bar{c} = [-3, -4, -2, 1, 0, 0]^T$, and $\hat{z} = 22$.

As $\bar{c} \ngeq \mathbf{0}$, the current dictionary is not optimal. Since the corresponding vertex is nondegenerate, we know that this vertex is not optimal by Proposition 17.1.

As the vertex is nondegenerate and there are three variables with negative reduced costs, we pick the one with the most negative reduced cost, i.e. $j^* = 2$. Hence, x_2 will enter the basis.

The min-ratio test yields

$$\lambda^* = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} = \min \left\{ \frac{-4}{-1}, \frac{-18}{-3} \right\} = 4\,.$$

Hence, $k^* = 5$ and $x_5 = a_1$ will leave the basis.

Thus, we move x_2 to the left-hand side and x_5 to the right-hand side of Row 1. Afterwards, we substitute x_2 in the right-hand sides of Rows 0 and 2. We performed the first iteration and the new dictionary is shown below.

Dictionary 2

$$\begin{array}{rcrcrcrcrcr} z & = & 6 & + & x_1 & + & 6x_3 & + & x_4 & + & 4x_5 \\ x_2 & = & 4 & - & x_1 & - & 2x_3 & & & - & x_5 \\ x_6 & = & 6 & + & x_1 & + & 6x_3 & + & x_4 & + & 3x_5 \end{array}$$

We have $B = \{2,6\}$, $N = \{1,3,4,5\}$, $\hat{x} = [0,4,0,0,0,6]^T$, $\bar{c} = [1,0,6,1,4,0]^T$, and $\hat{z} = 6$. As $\bar{c} \geq \mathbf{0}$, the current dictionary is optimal by Corollary 15.4. Therefore, this is the end of Phase 1. However, note that the optimal value at the end of Phase 1 is given by $z^* = 6 > 0$. Therefore, this is the end of the Two-Phase Method and we conclude that the original problem is infeasible, i.e., it contains no feasible solutions.

Indeed, if you multiply the first contraint by -3 and add it to the second constraint in the original problem, you obtain the implied constraint given by

$$-x_1 - 6x_3 - x_4 = 6.$$

Clearly, this implied equality contradicts the nonnegativity of each decision variable.

(1.2) LP

$$\begin{array}{lrcrcrcrcl}
\min & -x_1 & - & x_2 & + & x_3 & + & 2x_4 & & \\
\text{s.t.} & x_1 & - & x_2 & - & x_3 & & & = & 1 \\
 & -x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 1 \\
 & & & & & \multicolumn{3}{r}{x_1,\, x_2,\, x_3,\, x_4} & \geq & 0
\end{array}$$

Solution

Phase 1

The auxiliary problem is given as

$$\begin{array}{lrcrcrcrcrcrcl}
\min & & & & & & & & & a_1 & + & a_2 & & \\
\text{s.t.} & x_1 & - & x_2 & - & x_3 & & & + & a_1 & & & = & 1 \\
 & -x_1 & + & x_2 & + & 2x_3 & - & x_4 & & & + & a_2 & = & 1 \\
 & & & & & & & \multicolumn{5}{r}{x_1,\, x_2,\, x_3,\, x_4,\, a_1,\, a_2} & \geq & 0
\end{array}$$

Starting with $\hat{x} = \mathbf{0}$, $\hat{a} = b$, and substituting the artificial variables in Row 0, we obtain the following initial dictionary, where we replace a_1 and a_2 with x_5 and x_6, respectively.

Dictionary 1

$$\begin{array}{rcrcrcrcrcr}
z & = & 2 & & & & & - & x_3 & + & x_4 \\
x_5 & = & 1 & - & x_1 & + & x_2 & + & x_3 & & \\
x_6 & = & 1 & + & x_1 & - & x_2 & - & 2x_3 & + & x_4
\end{array}$$

We have $B = \{5,6\}$, $N = \{1,2,3,4\}$, $\hat{x} = [0,0,0,0,1,1]^T$, $\bar{c} = [0,0,-1,1,0,0]^T$, and $\hat{z} = 2$.

As $\bar{c} \not\geq \mathbf{0}$, the current dictionary is not optimal. Since the corresponding vertex is nondegenerate, we know that this vertex is not optimal by Proposition 17.1.

There is only one nonbasic variable with a negative reduced cost, i.e. $j^* = 3$. Hence, x_3 will enter the basis.

The min-ratio test yields

$$\lambda^* = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} = \min \left\{ \frac{-1}{-2} \right\} = \frac{1}{2}.$$

Hence, $k^* = 6$ and $x_6 = a_2$ will leave the basis.

Thus, we move x_3 to the left-hand side and x_6 to the right-hand side of Row 2. Afterwards, we substitute x_3 in the right-hand sides of Rows 0 and 1. We performed the first iteration and the new dictionary is shown below.

Dictionary 2

$$\begin{array}{rcrcrcrcrcr}
z & = & \frac{3}{2} & - & \frac{1}{2}x_1 & + & \frac{1}{2}x_2 & + & \frac{1}{2}x_4 & + & \frac{1}{2}x_6 \\
x_5 & = & \frac{3}{2} & - & \frac{1}{2}x_1 & + & \frac{1}{2}x_2 & + & \frac{1}{2}x_4 & - & \frac{1}{2}x_6 \\
x_3 & = & \frac{1}{2} & + & \frac{1}{2}x_1 & - & \frac{1}{2}x_2 & + & \frac{1}{2}x_4 & - & \frac{1}{2}x_6
\end{array}$$

We have $B = \{5, 3\}$, $N = \{1, 2, 4, 6\}$, $\hat{x} = [0, 0, \frac{1}{2}, 0, \frac{3}{2}, 0]^T$, $\bar{c} = [-\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}]^T$, and $\hat{z} = \frac{3}{2}$.

As $\bar{c} \not\geq \mathbf{0}$, the current dictionary is not optimal. Since the corresponding vertex is nondegenerate, we know that this vertex is not optimal by Proposition 17.1.

There is only one nonbasic variable with a negative reduced cost, i.e. $j^* = 1$. Hence, x_1 will enter the basis.

The min-ratio test yields

$$\lambda^* = \min_{j \in B: d_j < 0} \frac{-\hat{x}_j}{d_j} = \min \left\{ \frac{-\frac{3}{2}}{-\frac{1}{2}} \right\} = 3\,.$$

Hence, $k^* = 5$ and $x_5 = a_1$ will leave the basis.

Thus, we move x_1 to the left-hand side and x_5 to the right-hand side of Row 1. Afterwards, we substitute x_1 in the right-hand sides of Rows 0 and 2. We performed the second iteration and the new dictionary is shown below.

Dictionary 3

$$\begin{array}{rcrcrcrcrcr} z & = & 0 & & & & & + & x_5 & + & x_6 \\ x_1 & = & 3 & + & x_2 & + & x_4 & - & 2x_5 & - & x_6 \\ x_3 & = & 2 & & & + & x_4 & - & x_5 & - & x_6 \end{array}$$

We have $B = \{1, 3\}$, $N = \{2, 4, 5, 6\}$, $\hat{x} = [3, 0, 2, 0, 0, 0]^T$, $\bar{c} = [0, 0, 0, 0, 1, 1]^T$, and $\hat{z} = 0$.

As all reduced costs are nonnegative, i.e., $\bar{c} \geq \mathbf{0}$, the current vertex is optimal by Corollary 15.4. This is the end of Phase 1. As the Phase 1 problem has an optimal value of 0, the original LP problem has a nonempty feasible region and $\hat{x} = [3, 0, 2, 0]^T$ is a (nondegenerate) vertex of the original problem.

Since all artificial variables are nonbasic, we can remove them from the Phase 1 problem and proceed with the Phase 2 problem starting with the vertex of the original problem computed at the end of Phase 1, making sure that we now use the original objective function.

Phase 2

Delete Row 0 and all artificial variables. Introduce the original objective function as the new Row 0 and substitute x_1 and x_3 in the objective function by Rows 1 and 2, respectively, to put the dictionary in proper form.

Dictionary 1

$$\begin{array}{rcrcrcr} z & = & -1 & - & 2x_2 & + & 2x_4 \\ x_1 & = & 3 & + & x_2 & + & x_4 \\ x_3 & = & 2 & & & + & x_4 \end{array}$$

We have $B = \{1, 3\}$, $N = \{2, 4\}$, $\hat{x} = [3, 0, 2, 0]^T$, $\bar{c} = [0, -2, 0, 2]^T$, and $\hat{z} = -1$.

As $\bar{c} \not\geq \mathbf{0}$, the current dictionary is not optimal. Since the corresponding vertex is nondegenerate, we know that this vertex is not optimal by Proposition 17.1. If we were to continue with the simplex method, then we would choose the nonbasic variable with the most negative reduced cost (i.e., x_2). However, if you try to apply the min-ratio test, you would see that none of the Rows 1 and 2 would be eligible as both entries in the column corresponding to x_2 are nonnegative. We therefore conclude that the original LP problem is unbounded. By setting $d_2 = 1$ and $d_4 = 0$, you can see that

$$d_B = \begin{bmatrix} d_1 \\ d_3 \end{bmatrix} = -(A_B)^{-1} A_N d_N = -(A_B)^{-1} A^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We obtain $d = [1, 1, 0, 0]^T$. You can verify that d is a feasible direction at $\hat{x}$, i.e., $\hat{x} + \lambda d \in \mathcal{P}$ for any real number $\lambda \geq 0$, and $c^T(\hat{x} + \lambda d) = c^T\hat{x} + \lambda c^T d = -1 - 2\lambda \to -\infty$ as $\lambda \to \infty$. Therefore the original problem is unbounded and its optimal value is given by $z^* = -\infty$.

2 The Dual Simplex Method - Part I

Below, you are given two linear programming problems such that the matrix A has full row rank. Perform only one iteration of the dual simplex method from the given starting solution. Determine whether Dictionary 2 is (i) primal feasible, (ii) dual feasible, and (iii) optimal.

Instructions

- For each dictionary, and before carrying out Step 3 (if applicable), write down the current index sets B and N, the values of the decision variables $\hat{x}_j$ and the reduced costs $\bar{c}_j$, $j = 1, \ldots, n$, and the objective function value $\hat{z}$.

  When entering the indices in the set B, pay attention to the row numbers in the corresponding dictionary, i.e., if x_3, x_1, and x_4 are the basic variables in Rows 1, 2, and 3, respectively, then $B = \{3, 1, 4\}$. For the index set N, simply enter the indices in increasing order.

- Use the nonbasic variable with the most negative value as the leaving variable and break ties in favour of the nonbasic variable with the smallest index.

- For the first dictionary, order the basic variables in increasing order of their indices, i.e., the basic variable with the smallest index should the first basic variable, the basic variable with the second smallest index should the second basic variable, and so on.

- If there is a tie between two or more nonbasic variables in the minimum ratio test, then break the tie in favour of the nonbasic variable with the smallest index.

Questions

(2.1) Linear program

$$
\begin{array}{lrcrcrcrcrcr}
\min & x_1 & + & x_2 & & & & & & & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & x_3 & & & & & = & 7 \\
- & 2x_1 & - & x_2 & & & + & x_4 & & & = & -4 \\
 & x_1 & - & x_2 & & & & & + & x_5 & = & -1 \\
 & & & & & & \multicolumn{3}{r}{x_1,\, x_2,\, x_3,\, x_4,\, x_5} & \geq & 0 &
\end{array}
$$

with starting solution $\hat{x} = [0, 0, 7, -4, -1]^T$.

Solution

Dictionary 1

$$
\begin{array}{rcrcrcr}
z & = & 0 & + & x_1 & + & x_2 \\
x_3 & = & 7 & - & 2x_1 & - & x_2 \\
x_4 & = & -4 & + & 2x_1 & + & x_2 \\
x_5 & = & -1 & - & x_1 & + & x_2
\end{array}
$$

We have $B = \{3, 4, 5\}$, $N = \{1, 2\}$, $\hat{x} = [0, 0, 7, -4, -1]^T$, $\bar{c} = [1, 1, 0, 0, 0]^T$, and $\hat{z} = 0$.

As $\hat{x}_B \not\geq \mathbf{0}$, the current dictionary is not primal feasible. Since $\bar{c} \geq \mathbf{0}$, the dictionary is dual feasible. It is not optimal since primal feasibility is violated.

As there are two negative basic variables, we pick the one with the most negative value, i.e., $p = 4$. Hence, x_4 will leave the basis.

The minimum ratio test yields

$$
\min_{j \in N: \bar{a}_{pj} > 0} \frac{\bar{c}_j}{\bar{a}_{pj}} = \min\left\{\frac{1}{2}, \frac{1}{1}\right\} = \frac{1}{2}.
$$

Hence, $q = 1$ and x_1 will enter the basis.

Thus, we move x_1 to the left-hand side and x_4 to the right-hand side of Row 2. Afterwards, we substitute x_1 in the right-hand sides of Rows 0, 1, and 3. We performed the first iteration and the new dictionary is shown below.

Dictionary 2

$$\begin{array}{rcrcrcr} z & = & 2 & + & \frac{1}{2}x_2 & + & \frac{1}{2}x_4 \\ x_3 & = & 3 & & & - & x_4 \\ x_1 & = & 2 & - & \frac{1}{2}x_2 & + & \frac{1}{2}x_4 \\ x_5 & = & -3 & + & \frac{3}{2}x_2 & - & \frac{1}{2}x_4 \end{array}$$

We have $B = \{3, 1, 5\}$, $N = \{2, 4\}$, $\hat{x} = [2, 0, 3, 0, -3]^T$, $\bar{c} = [0, \frac{1}{2}, 0, \frac{1}{2}, 0]^T$, and $\hat{z} = 2$.

As $\hat{x}_B \not\geq \mathbf{0}$, the current dictionary is not primal feasible. Since $\bar{c} \geq \mathbf{0}$, the dictionary is dual feasible. It is not optimal since primal feasibility is violated.

If we were to continue with the dual simplex method, then we would choose x_5 as the leaving variable since it is the only one with a negative value. Then, we would perform the minimum-ratio test and continue.

(2.2) Linear program

$$\begin{array}{lrrrrrrrrrrrr} \min & & 2x_1 & + & x_2 & & & & & & & & \\ \text{s.t.} & - & x_1 & - & 2x_2 & + & x_3 & & & & & = & -6 \\ & & 2x_1 & + & x_2 & & & + & x_4 & & & = & 8 \\ & - & 3x_1 & - & x_2 & & & & & + & x_5 & = & -4 \\ & & & & & & x_1,\, x_2,\, x_3,\, x_4,\, x_5 & & \geq & & 0 & & \end{array}$$

with starting solution $\hat{x} = [0, 0, -6, 8, -4]^T$.

Solution

Dictionary 1

$$\begin{array}{rcrcrcr} z & = & 0 & + & 2x_1 & + & x_2 \\ x_3 & = & -6 & + & x_1 & + & 2x_2 \\ x_4 & = & 8 & - & 2x_1 & - & x_2 \\ x_5 & = & -4 & + & 3x_1 & + & x_2 \end{array}$$

We have $B = \{3, 4, 5\}$, $N = \{1, 2\}$, $\hat{x} = [0, 0, -6, 8, -4]^T$, $\bar{c} = [2, 1, 0, 0, 0]^T$, and $\hat{z} = 0$.

As $\hat{x}_B \not\geq \mathbf{0}$, the current dictionary is not primal feasible. Since $\bar{c} \geq \mathbf{0}$, the dictionary is dual feasible. It is not optimal since primal feasibility is violated.

As there are two negative basic variables, we pick the one with the most negative value, i.e., $p = 3$. Hence, x_3 will leave the basis.

The minimum ratio test yields

$$\min_{j \in N : \bar{a}_{pj} > 0} \frac{\bar{c}_j}{\bar{a}_{pj}} = \min \left\{ \frac{2}{1}, \frac{1}{2} \right\} = \frac{1}{2}.$$

Hence, $q = 2$ and x_2 will enter the basis.

Thus, we move x_2 to the left-hand side and x_3 to the right-hand side of Row 1. Afterwards, we substitute x_2 in the right-hand sides of Rows 0, 2, and 3. We performed the first iteration and the new dictionary is shown below.

Dictionary 2

$$\begin{array}{rcrcrcr} z & = & 3 & + & \frac{3}{2}x_1 & + & \frac{1}{2}x_3 \\ x_2 & = & 3 & - & \frac{1}{2}x_1 & + & \frac{1}{2}x_3 \\ x_4 & = & 5 & - & \frac{3}{2}x_1 & - & \frac{1}{2}x_3 \\ x_5 & = & -1 & + & \frac{5}{2}x_1 & + & \frac{1}{2}x_3 \end{array}$$

We have $B = \{2, 4, 5\}$, $N = \{1, 3\}$, $\hat{x} = [0, 3, 0, 5, -1]^T$, $\bar{c} = [\frac{3}{2}, 0, \frac{1}{2}, 0, 0]^T$, and $\hat{z} = 3$.

As $\hat{x}_B \ngeq \mathbf{0}$, the current vertex is not primal feasible. Since $\bar{c} \geq \mathbf{0}$, the dictionary is dual feasible. It is not optimal since primal feasibility is violated.

If we were to continue with the dual simplex method, then we would choose x_5 as the leaving variable since it is the only one with a negative value. Then, we would perform the minimum-ratio test and continue.

3 Sensitivity Analysis

Consider the following linear program

$$\begin{array}{lrrrrrrrrrrrr} \min & - & 3x_1 & - & x_2 & - & 4x_3 & + & 2x_4 & & & & \\ \text{s.t.} & & 4x_1 & + & 6x_2 & + & 5x_3 & + & x_4 & + & x_5 & = & 6 \\ & & 3x_1 & + & 5x_2 & + & 4x_3 & & & + & x_5 & = & 5 \\ & & & & & & & & \multicolumn{3}{r}{x_1,\, x_2,\, x_3,\, x_4,\, x_5} & \geq & 0 \end{array}$$

and the optimal dictionary

$$\begin{array}{rcrcrcrcr} z & = & -4 & + & x_1 & + & 3x_2 & + & 6x_4 \\ x_3 & = & 1 & - & x_1 & - & x_2 & - & x_4 \\ x_5 & = & 1 & + & x_1 & - & x_2 & + & 4x_4 \end{array}$$

Instructions

- For the primal simplex, if there is more than one nonbasic variable with a negative reduced cost, choose the one with the most negative reduced cost as the entering variable; break ties in favour of the basic variable with the smallest index. In the minimum ratio test, if there is a tie between two or more rows, choose the row whose basic variable has the smallest index to determine the leaving variable.
- For the dual simplex, if there is more than one negative basic variable, choose the one with the most negative value as the leaving variable; break ties in favour of the basic variable with the smallest index. In the minimum ratio test, if there is a tie between two or more nonbasic variables, then break the tie in favour of the nonbasic variable with the smallest index.

Questions

(3.1) By how much can we increase the value of $b_2 = 5$ in the original problem such that current dictionary remains optimal? Justify your solution.

Solution

Note that changes in the right-hand side vector only affect the values of the basic variables and the current objective function value. Therefore, dual feasibility is maintained. We only need to check for primal feasibility.

Defining $b_2' = b_2 + \delta$, to retain primal feasibility, we have to ensure that $\delta A_B^{-1} e^2 \geq -x_B^*$. Thus

$$\delta \begin{bmatrix} 1 & -1 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \delta \begin{bmatrix} -1 \\ 5 \end{bmatrix} \geq \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{rcl} \delta & \leq & 1 \\ \delta & \geq & -\frac{1}{5} \end{array}$$

Hence, we can increase b_2 by at most 1.

(3.2) If we increase the value of $b_1 = 6$ by $\delta = 1$ in the original problem, does the current dictionary remain optimal? Justify your solution.

If not, perform **one iteration** of the appropriate simplex method and derive a new dictionary with its corresponding solution. For the new dictionary, write down the index sets B and N, values of all variables, reduced costs of all variables, and the current objective function value. State whether the corresponding solution is optimal or not. State whether it is primal feasible or dual feasible or both. Justify your solution.

Solution

Note that changes in the right-hand side vector only affect the values of the basic variables and the current objective function value. Therefore, dual feasibility is maintained. We only need to check for primal feasibility.

First, we have to check the corresponding condition for b_1. Defining $b_1' = b_1 + \delta$, we obtain

$$\delta \begin{bmatrix} 1 & -1 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \delta \begin{bmatrix} 1 \\ -4 \end{bmatrix} \geq \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{ccc} \delta & \geq & -1 \\ \delta & \leq & \frac{1}{4} \end{array}$$

As we can increase b_1 by at most $\frac{1}{4}$, the dictionary will lose primal feasibility if we increase it by 1, and, hence, will no longer be optimal. To re-optimize, we have to use the dual simplex method.

Next, we perform one iteration of the dual simplex method. To that end, we first compute the updated dictionary. A change of the right-hand side will only affect the values for the basic variables and the objective function value. Concerning the values of the basic variables, substituting $\delta = 1$, we get

$$x_B^*(\delta) = (A_B)^{-1}(b + \delta e^1) = \begin{bmatrix} 1 & -1 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Concerning the objective function value of the solution, substituting $\delta = 1$, we get

$$z(\delta) = c_B^T x_B^*(\delta) = \begin{bmatrix} -4 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = -8.$$

Hence, we have

$$\begin{array}{rcrcrcrcr} z & = & -8 & + & x_1 & + & 3x_2 & + & 6x_4 \\ x_3 & = & 2 & - & x_1 & - & x_2 & - & x_4 \\ x_5 & = & -3 & + & x_1 & - & x_2 & + & 4x_4 \end{array}$$

There is only one negative basic variable, so we get $p = 5$ and x_5 will leave the basis.

To obtain the index $q \in N$ of the entering variable, the minimum ratio test immediately yields

$$\min_{j \in N: \bar{a}_{pj} > 0} \frac{-\bar{c}_j}{\bar{a}_{pj}} = \min \left\{ \frac{1}{1}, \frac{6}{4} \right\} = \frac{1}{1}.$$

Hence, $q = 1$ and x_1 will enter the basis.

Thus, we move x_1 to the left-hand side and x_5 to the right-hand side of Row 2. Afterwards, we substitute x_1 in the right-hand sides of Rows 0 and 1. We performed one iteration and the new dictionary is shown below.

Dictionary

$$\begin{array}{rcrcrcrcr} z & = & -5 & + & 4x_2 & + & 2x_4 & + & x_5 \\ x_3 & = & -1 & - & 2x_2 & + & 3x_4 & - & x_5 \\ x_1 & = & 3 & + & x_2 & - & 4x_4 & + & x_5 \end{array}$$

We have $B = \{3, 1\}$, $N = \{2, 4, 5\}$, $\hat{x} = [3, 0, -1, 0, 0]^T$, $\bar{c} = [0, 4, 0, 2, 1]^T$, and $\hat{z} = -5$.

We observe that this solution is still primal infeasible, and therefore not optimal (we need one more iteration to get a primal feasible dictionary).

(3.3) By how much can we decrease the value of $c_2 = -1$ in the original problem such that the current dictionary remains optimal? Justify your solution.

Solution

Note that x_2 is a nonbasic variable. Note also that changes in the cost coefficient of a nonbasic variable only affect the reduced cost of that particular nonbasic variable. Therefore, primal feasibility is maintained. We only need to check for dual feasibility.

Define $c_2' = c_2 + \delta$. So, to retain dual feasibility, we have to ensure that $\delta \geq -\bar{c}_2 = -3$. Thus, we can decrease c_2 by at most 3.

(3.4) If we increase the value of $c_5 = 0$ by $\delta = 2$ in the original problem, does the current dictionary remain optimal? Justify your solution.

If not, perform **one iteration** of the appropriate simplex method and derive a new dictionary with its corresponding solution. For the new dictionary, write down the index sets B and N, values of all variables, reduced costs of all variables, and the current objective function value. State whether the corresponding solution is optimal or not. State whether it is primal feasible or dual feasible or both. Justify your solution.

Solution

Note that x_5 is a basic variable. Note also that changes in the cost coefficient of a basic variable affect the reduced costs of all nonbasic variables and the current objective function value. Therefore, primal feasibility is maintained. We only need to check for dual feasibility.

Define $c_5' = c_5 + \delta$. x_5 is the basic variable in Row 2, so $\ell = 2$. To retain dual feasibility we have to ensure for every $j \in N$ that

$$\delta (e^2)^T (A_B)^{-1} A^j \leq \bar{c}_j \,.$$

With

$$(e^2)^T (A_B)^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -4 & 5 \end{bmatrix} = [-4, 5]$$

we get

$$\begin{aligned}
\delta[-4, 5] A^1 &= \delta[-4, 5] \begin{bmatrix} 4 \\ 3 \end{bmatrix} = -\delta \leq \bar{c}_1 = 1 \quad \Leftrightarrow \quad \delta \geq -1 \\
\delta[-4, 5] A^2 &= \delta[-4, 5] \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \delta \leq \bar{c}_2 = 3 \quad \Leftrightarrow \quad \delta \leq 3 \\
\delta[-4, 5] A^4 &= \delta[-4, 5] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -4\delta \leq \bar{c}_4 = 6 \quad \Leftrightarrow \quad \delta \geq -\frac{3}{2}
\end{aligned}$$

Hence, the current dictionary remains optimal if and only if $-1 \leq \delta \leq 3$. So, if we increase c_5 by 2, we maintain dual feasibility and, hence, the dictionary will remain optimal.

On the other hand, if we decrease c_5 by 2, for instance, we lose dual feasibility and, hence, the dictionary will no longer be optimal. To re-optimize, we will have to use the primal simplex method.

Next, we perform one iteration of the primal simplex method. To that end, we first compute the updated dictionary. A change in the cost coefficient of a basic variable will affect all entries in Row 0, but none in the other rows of the dictionary.

Concerning the reduced costs of the nonbasic variables x_j, $j \in N$, we have

$$\bar{c}_j(\delta) \;=\; \bar{c}_j - \delta (e^2)^T (A_B)^{-1} A^j$$

Using the above calculations, we obtain for $\delta = -2$

$$\begin{aligned}
\bar{c}_1(\delta) \;&=\; \bar{c}_1 - (-\delta) \;=\; 1 - 2 \;=\; -1 \\
\bar{c}_2(\delta) \;&=\; \bar{c}_2 - (\delta) \;=\; 3 + 2 \;=\; 5 \\
\bar{c}_4(\delta) \;&=\; \bar{c}_4 - (-4\delta) \;=\; 6 - 8 \;=\; -2
\end{aligned}$$

Concerning the objective function value of the solution, we get

$$z(\delta) \;=\; z^* + \delta (e^2)^T x_B^* \;=\; -4 + \begin{bmatrix} 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;=\; -6.$$

Hence, we have

$$\begin{array}{rcrcrcrcr}
z & = & -6 & - & x_1 & + & 5x_2 & - & 2x_4 \\
x_3 & = & 1 & - & x_1 & - & x_2 & - & x_4 \\
x_5 & = & 1 & + & x_1 & - & x_2 & + & 4x_4
\end{array}$$

We have two variables with negative reduced costs and we choose the one with the smaller value, i.e., $j^* = 4$. Hence, x_4 will enter the basis.

The minimum ratio test immediately yields $k^* = 3$ and x_3 will leave the basis.

Thus, we move x_4 to the left-hand side and x_3 to the right-hand side of Row 1. Afterwards, we substitute x_4 in the right-hand sides of Rows 0 and 2. We performed one iteration and the new dictionary is shown below.

Dictionary

$$\begin{array}{rcrcrcrcr}
z & = & -8 & + & x_1 & + & 7x_2 & + & 2x_3 \\
x_4 & = & 1 & - & x_1 & - & x_2 & - & x_3 \\
x_5 & = & 5 & - & 3x_1 & - & 5x_2 & - & 4x_3
\end{array}$$

We have $B = \{4, 5\}$, $N = \{1, 2, 3\}$, $\hat{x} = [0, 0, 0, 1, 5]^T$, $\bar{c} = [1, 7, 2, 0, 0]^T$, and $\hat{z} = -8$.

Observe that this solution is now dual feasible and, hence, optimal (as we retained primal feasibility).

(3.5) Consider the variable x_2. If we change A^2 from $[6, 5]^T$ to $[5, 5]^T$ in the original problem, i.e. $A^2(\delta) = A^2 + \delta e^1$ with $\delta = -1$, does the current dictionary remain optimal? Justify your solution.

If not, perform **one iteration** of the appropriate simplex method and derive a new dictionary with its corresponding solution. For the new dictionary, write down the index sets B and N, values of all variables, reduced costs of all variables, and the current objective function value. State whether the corresponding solution is optimal or not. State whether it is primal feasible or dual feasible or both. Justify your solution.

Solution

Note that x_2 is a nonbasic variable. Such a change only affects the reduced cost of x_2 and the coefficients of x_2 in Rows 1 and 2. The primal feasibility is therefore maintained. We only need to check for dual feasibility.

Therefore, the current dictionary remains optimal if

$$\bar{c}_2(\delta) = c_2 - c_B^T (A_B)^{-1} (A^2 + \delta e^1) = \bar{c}_2 - \delta c_B^T (A_B)^{-1} e^1 \geq 0 \quad \Leftrightarrow \quad \delta c_B^T (A_B)^{-1} e^1 \;\leq\; \bar{c}_2 \,.$$

Substituting $\delta = -1$, we get

$$(-1)\begin{bmatrix}-4 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ -4 & 5\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = (-1)\begin{bmatrix}-4 & 0\end{bmatrix}\begin{bmatrix}1\\-4\end{bmatrix} = 4 \not\leq 3 = \bar{c}_2.$$

Hence, the dictionary loses dual feasibility and, hence, does not remain optimal. To re-establish optimality, we use the primal simplex method (as the dictionary is still primal feasible).

Next, we perform one iteration of the primal simplex method. To that end, we first compute the updated dictionary. Concerning the reduced cost of x_2, using $\delta = -1$, we get

$$\bar{c}_2(\delta) = \bar{c}_2 - \delta c_B^T (A_B)^{-1} e^1 = 3 - 4 = -1\,.$$

Concerning the new coefficients of x_2 in Rows 1 and 2 of the dictionary, using $\delta = -1$, we get

$$-(A_B)^{-1}(A^2 + \delta e^1) = -\begin{bmatrix}1 & -1\\ -4 & 5\end{bmatrix}\begin{bmatrix}5\\5\end{bmatrix} = \begin{bmatrix}0\\-5\end{bmatrix}$$

Hence, we have

$$\begin{array}{rcrcrcrcr} z & = & -4 & + & x_1 & - & x_2 & + & 6x_4 \\ x_3 & = & 1 & - & x_1 & & & - & x_4 \\ x_5 & = & 1 & + & x_1 & - & 5x_2 & + & 4x_4 \end{array}$$

The only variable with a negative reduced cost is x_2, i.e., $j^* = 2$. Hence, x_2 will enter the basis.

The minimum ratio test immediately yields $k^* = 5$ and x_5 will leave the basis.

Thus, we move x_2 to the left-hand side and x_5 to the right-hand side of Row 2. Afterwards, we substitute x_2 in the right-hand sides of Rows 0 and 2. We performed one iteration of the primal simplex method and the new dictionary is shown below.

Dictionary

$$\begin{array}{rcrcrcrcr} z & = & -4\frac{1}{5} & + & \frac{4}{5}x_1 & + & 5\frac{1}{5}x_4 & + & \frac{1}{5}x_5 \\ x_3 & = & 1 & - & x_1 & - & x_4 & & \\ x_2 & = & \frac{1}{5} & + & \frac{1}{5}x_1 & + & \frac{4}{5}x_4 & - & \frac{1}{5}x_5 \end{array}$$

We have $B = \{3, 2\}$, $N = \{1, 4, 5\}$, $\hat{x} = [0, \frac{1}{5}, 1, 0, 0]^T$, $\bar{c} = [\frac{4}{5}, 0, 0, 5\frac{1}{5}, \frac{1}{5}]^T$, and $\hat{z} = -4\frac{1}{5}$.

This dictionary is now dual feasible and, thus, optimal.

(3.6) Next, we want to add a new variable x_6 to the original problem. The corresponding column is given by $A^6 = [2, 3]^T$. Find the range of values for c_6 for which the current dictionary remains optimal. Justify your solution.

Solution

We add x_6 to the set of nonbasic variables, i.e., $N = \{1, 2, 4, 6\}$. As a result, the dictionary will remain primal feasible, but it may no longer be dual feasible. To check that, we have to calculate the reduced cost $\bar{c}_6$ of the variable in the optimal dictionary. Therefore, the current dictionary remains optimal if

$$\bar{c}_6 = c_6 - c_B^T (A_B)^{-1} A^6 \geq 0 \quad \Leftrightarrow \quad c_B^T (A_B)^{-1} A^6 \leq c_6\,.$$

We get

$$\begin{bmatrix}-4 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ -4 & 5\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-4 & 0\end{bmatrix}\begin{bmatrix}-1\\7\end{bmatrix} = 4 \leq c_6.$$

Hence, the dictionary remains optimal if and only if $c_6 \geq 4$.

(3.7) Finally, we want to add the constraint

$$x_1 - 2x_2 + 3x_3 - 3x_4 + x_5 \ \le\ 3$$

to the original problem.

Is the current solution still optimal for the modified problem? If not, find the new optimal solution. Justify your solution.

Solution

Let $m = 3$, $b_3 = 3$ and $(a^3)^T = [1, -2, 3, -3, 1]$.

First, we check whether the optimal solution $x^* = [0, 0, 1, 0, 1]^T$ satisfies new constraint

$$(a^3)^T x^* \ =\ x_1 - 2x_2 + 3x_3 - 3x_4 + x_5 \ =\ 0 - 0 + 3 \cdot 1 - 0 + 1 \ =\ 4 \ \not\le\ 3 \ =\ b_3$$

No, it doesn't. So the current optimal solution x^* is not a feasible solution to the modified problem.

By adding a nonnegative slack variable x_6 to convert the new inequality constraint to an equality constraint, we obtain

$$x_1 - 2x_2 + 3x_3 - 3x_4 + x_5 + x_6 \ =\ 3$$

We can extend the optimal solution for the original problem to a basic solution for the modified problem by defining $\tilde{B} = \{3, 5, 6\}$, $\tilde{N} = \{1, 2, 4\}$ and

$$\hat{x}_j \ =\ \begin{cases} x_j^* & j = 1, \dots, 5 \\ 3 - (a^3)^T x^* & j = 6 \end{cases} \qquad \Rightarrow \qquad \hat{x} \ =\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

As we can see, the solution is no longer primal feasible, but it will remain dual feasible. Hence, we have to apply the dual simplex method to determine an optimal solution.

Next, we perform one iteration of the dual simplex method. To that end, we first compute the corresponding dictionary of the modified problem. We have

$$A_{\tilde{B}} \ =\ \begin{bmatrix} 5 & 1 & 0 \\ 4 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

By simple manipulations (see Lecture 31), we obtain

$$(A_{\tilde{B}})^{-1} \ =\ \begin{bmatrix} 1 & -1 & 0 \\ -4 & 5 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

The new row of the dictionary can be computed as

$$x_6 \ =\ b_3 - (a^3)^T x^* \ +\ \sum_{j \in N} ((a_B^3)^T (A_B)^{-1} A^j - a_j^3) x_j$$

All the other rows remain unchanged and we obtain

$$\begin{array}{rcrcrcrcr} z & = & -4 & + & x_1 & + & 3x_2 & + & 6x_4 \\ x_3 & = & 1 & - & x_1 & - & x_2 & - & x_4 \\ x_5 & = & 1 & + & x_1 & - & x_2 & + & 4x_4 \\ x_6 & = & -1 & + & x_1 & + & 6x_2 & + & 2x_4 \end{array}$$

There is only one negative basic variable, so we get $p = 6$ and x_6 will leave the basis.

To obtain the index $q \in N$ of the entering variable, the minimum ratio test immediately yields

$$\min_{j \in N: \bar{a}_{pj} > 0} \frac{-\bar{c}_j}{\bar{a}_{pj}} = \min \left\{ \frac{1}{1}, \frac{3}{6}, \frac{6}{2} \right\} = \frac{3}{6}.$$

Hence, $q = 2$ and x_2 will enter the basis.

Thus, we move x_2 to the left-hand side and x_6 to the right-hand side of Row 6. Afterwards, we substitute x_2 in the right-hand sides of Rows 0, 1 and 2. We performed one iteration and the new dictionary is shown below.

Dictionary

$$\begin{array}{rclclclcl}
z & = & -3\frac{1}{2} & + & \frac{1}{2}x_1 & + & 5x_4 & + & \frac{1}{2}x_6 \\
x_3 & = & \frac{5}{6} & - & \frac{5}{6}x_1 & - & \frac{2}{3}x_4 & - & \frac{1}{6}x_6 \\
x_5 & = & \frac{5}{6} & + & \frac{7}{6}x_1 & + & \frac{13}{3}x_4 & - & \frac{1}{6}x_6 \\
x_2 & = & \frac{1}{6} & - & \frac{1}{6}x_1 & - & \frac{1}{3}x_4 & + & \frac{1}{6}x_6
\end{array}$$

We have $B = \{3, 5, 2\}$, $N = \{1, 4, 6\}$, $\hat{x} = [0, \frac{1}{6}, \frac{5}{6}, 0, \frac{5}{6}, 0]^T$, $\bar{c} = [\frac{1}{2}, 0, 0, 5, 0, \frac{1}{2}]^T$, and $\hat{z} = -3\frac{1}{2}$.

We observe that this solution is now still primal feasible. As it retained dual feasibility, its is therefore optimal.

Open Ended Problems

4 The Dual Simplex Method - Part II

(4.1) Suppose that you are applying the dual simplex method and the current dictionary is given by

$$\begin{aligned}
z &= \hat{z} + \sum_{j \in N} \bar{c}_j x_j \\
x_i &= \hat{x}_i + \sum_{j \in N} \bar{a}_{ij} x_j, \quad i \in B.
\end{aligned}$$

Suppose that the current dictionary is dual feasible but primal infeasible.

Let $p \in B$ an index with the corresponding row given by

$$x_p = \hat{x}_p + \sum_{j \in N} \bar{a}_{pj} x_j.$$

Suppose that $\hat{x}_p < 0$ and $\bar{a}_{pj} \leq 0$ for each $j \in N$.

(i) What can you conclude about the primal problem? Justify your solution.

Solution

Consider the row corresponding to the basic variable x_p. If we move all the variables to the left-hand side, we obtain:

$$(*) \quad x_p - \sum_{j \in N} \bar{a}_{pj} x_j = \hat{x}_p.$$

Observe that this equation should be satisfied by each primal feasible solution since it is given by one of the equations obtained by multiplying both sides of $Ax = b$ by $(A_B)^{-1}$ from the left. Since the primal problem is in standard form, all of the decision variables are nonnegative. Since $\bar{a}_{pj} \leq 0$ for each $j \in N$, we obtain $-\bar{a}_{pj} \geq 0$ for each $j \in N$. Therefore, for any primal feasible solution, the left-hand side of $(*)$ is always nonnegative. However, $\hat{x}_p < 0$. This implies that the primal problem cannot have any feasible solutions. Therefore, we conclude that the primal problem is infeasible.

Remark: Note that we have a leaving variable but there is no eligible entering variable in this case. In the primal simplex method, if there is a nonbasic variable with a negative reduced cost and if none of the basic variables is eligible for the minimum ratio test, then we have a similar situation, i.e., there is an entering variable but there is no eligible leaving variable. In the primal simplex method, we conclude that the primal problem is unbounded, which implies that the dual problem is infeasible. In the dual simplex method, we conclude that the primal problem is infeasible. See the next part for the conclusion about the dual problem.

(ii) What can you conclude about the dual problem? Justify your solution.

Solution

By part (i), the primal problem is infeasible. By the duality relations in Slide 13 in Lecture 24, the dual problem is either infeasible or unbounded. Since the current dictionary is dual feasible, we conclude that the dual problem should be unbounded.