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Fundamentals of Optimization

Exercise Set 0 – Solutions

Numerical Problems

1 Simple Optimization Problems

For each of the following three optimization problems, identify the feasible region and the objective function. Determine whether the given optimization problem is *infeasible*, *unbounded*, or *has a finite optimal value*. Find the optimal value using the convention in the lectures and determine the set of all optimal solutions.

(1.1) $\max\{(x-1)^2 : x^2 \leq -1, \quad x \in \mathbb{R}\}$.

(1.2) $\min\{1/(x+1)^2 : x \geq 0, \quad x \in \mathbb{R}\}$.

(1.3) $\min\{x^2 - 4x + 6 : |x-2| \geq 1, \quad x \in \mathbb{R}\}$.

(1.4) $\max\{x^2 - 3x + 6 : x^2 \geq 1, \quad x \in \mathbb{R}\}$.

Solution

(1.1) The feasible region is given by $\mathcal{S} = \emptyset$ since $x^2 \geq 0$ for each $x \in \mathbb{R}$. The objective function is given by $f(x) = (x-1)^2$. Since the feasible region is empty, the problem is clearly infeasible. By our convention, we define the optimal value $z^* = -\infty$ since this is an infeasible maximization problem. Since there is no feasible solution, clearly there is no optimal solution (recall that the set of optimal solutions is a subset of the feasible region), i.e., $\mathcal{S}^* = \emptyset$.

(1.2) The feasible region is given by $\mathcal{S} = [0, \infty)$. The objective function is given by $f(x) = 1/(x+1)^2$. The problem is obviously feasible since, for example, $0 \in \mathcal{S}$. Note that $(x+1)^2$ is an increasing function whenever $x \geq 0$. Therefore, the function $f(x) = 1/(x+1)^2$ is a decreasing function for $x \geq 0$. If you consider the sequence of feasible solutions given by $x^k = k$, where $k = 1, 2, \ldots$, it is easy to see that the objective function value given by $f(x^k)$ tends to 0 as $k \to \infty$. Therefore, the optimal value is finite and is given by $z^* = 0$ since $f(x) \geq 0$ for each feasible solution and $f(x^k) \to 0$ as $k \to \infty$ (i.e., 0 is the largest lower bound on the objective function values of all feasible solutions). There is no optimal solution, i.e., $\mathcal{S}^* = \emptyset$, since no feasible solution actually attains the optimal value. Recall that $+\infty$ or $-\infty$ are **not** real numbers! Note also that the optimal value in this example is finite (i.e., a real number) even though the feasible region itself is not bounded.

(1.3) The feasible region is given by $\mathcal{S} = (-\infty, 1] \cup [3, \infty)$ since $|x-2| \geq 1$ if and only if $x \in \mathcal{S}$ (you may think about the complement of the feasible region given by $|x-2| < 1$ and then take the complement of that set again). The objective function is given by $f(x) = x^2 - 4x + 6$. The problem is feasible, as, for example, $3 \in \mathcal{S}$. You can easily verify that the function $f(x) = x^2 - 4x + 6$ is increasing on $[3, \infty)$ and decreasing on $(-\infty, 1]$. Therefore, we simply need to consider the endpoints of these half lines given by 1 and 3. We see that $f(1) = f(3) = 3$. Therefore, the optimal value is finite and is given by $z^* = 3$, and there are two optimal solutions, i.e., $\mathcal{S}^* = \{1, 3\}$.

(1.4) The feasible region is given by $\mathcal{S} = (-\infty, -1] \cup [1, \infty)$ since $x^2 \geq 1$ if and only if $|x| \geq 1$. The objective function is given by $f(x) = x^2 - 3x + 6$. The problem is feasible, as, for example, $1 \in \mathcal{S}$. You can easily verify that the function $f(x) = x^2 - 3x + 6$ is decreasing on $(-\infty, 3/2]$ and increasing on $[3/2, \infty)$. Therefore, if you consider the sequence of feasible solutions given by $x^k = k + 1$, where $k = 1, 2, \ldots$, it is easy to see that the objective function value given by $f(x^k)$ tends to $+\infty$ as $k \to \infty$. It follows that the problem is unbounded and we define the optimal value as $z^* = +\infty$. Note that there are no optimal solutions, i.e., $\mathcal{S}^* = \emptyset$, since no feasible solution attains an objective function value of $+\infty$.

2 Convex Sets and Convex Functions

(2.1) Decide for each of the following sets whether they are *convex* or not.

(a) $\mathcal{C} = \{x \in \mathbb{R} : x^2 \geq 3\}$.

(b) $\mathcal{C} = \{x \in \mathbb{R}^2 : (x_1 + 1)^2 + (x_2 - 1)^2 \leq -3\}$.

(c) $\mathcal{C} = \{x \in \mathbb{R}^2 : |x_1| + |x_2| \leq 7\}$.

Solution

(a) The set is not convex. To prove this claim, we just need to find one counterexample to the definition of a convex set. Let $x = -2$, $y = 2$, and $\lambda = 1/2$. Note that $x \in \mathcal{C}$ and $y \in \mathcal{C}$, however, $\lambda x + (1 - \lambda) y = 0 \notin \mathcal{C}$, proving our claim.

(b) The set is convex. In fact, $\mathcal{C}$ is empty and, thus, by definition convex (see Remark 1 in Section 3.2 in the lecture notes).

(c) The set is convex. To see that, let $x^1 = [x_1^1, x_2^1]^T$, $x^2 = [x_1^2, x_2^2]^T \in \mathcal{C}$ and $\lambda \in [0, 1]$. For $\lambda x^1 + (1 - \lambda) x^2$ we obtain

$$\begin{aligned}
\left|\lambda x_1^1 + (1-\lambda) x_1^2\right| + \left|\lambda x_2^1 + (1-\lambda) x_2^2\right| &\leq \lambda |x_1^1| + (1-\lambda)|x_1^2| + \lambda |x_2^1| + (1-\lambda)|x_1^2| \\
&= \lambda \left(|x_1^1| + |x_2^1|\right) + (1-\lambda)\left(|x_1^2| + |x_2^2|\right) \\
&\leq \lambda \cdot 7 + (1-\lambda) \cdot 7 = 7,
\end{aligned}$$

where the first inequality follows from the triangle inequality and from $\lambda \in [0, 1]$.

(2.2) Decide for each of the following three functions whether they are *convex*, *concave*, *both*, or *neither*.

(a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = 3x_1 - 2x_2$.

(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$.

(c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 4x$.

Solution

(a) This function is linear in $x \in \mathbb{R}^2$. Let $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$ be two arbitrary vectors. Then, for any $\lambda \in [0, 1]$,

$$\begin{aligned}
f(\lambda x + (1-\lambda) y) &= 3(\lambda x_1 + (1-\lambda) y_1) - 2(\lambda x_2 + (1-\lambda) y_2 \\
&= \lambda(3x_1 - 2x_2) + (1-\lambda)(3y_1 - 2y_2),
\end{aligned}$$

which implies that f is both convex and concave due to the equality above.

(b) This function is convex, as the triangle inequality gives us

$$|\lambda x + (1-\lambda)y| \leq \lambda|x| + (1-\lambda)|y|$$

for any $x, y \in \mathbb{R}$.

It is not concave since for $x = -1$, $y = 1$, and $\lambda = 1/2$, we have

$$|\lambda x + (1-\lambda)y| = 0 \ngeq \lambda|x| + (1-\lambda)|y| = 1.$$

(c) This function is neither concave nor convex. We can see that this function factorizes into

$$f(x) \;=\; x(x-2)(x+2)$$

Picking the two zeros $x = -2$ and $y = 2$, we have $\lambda f(x) + (1-\lambda)f(y) = 0$, $\forall \lambda \in [0,1]$. But for $\lambda_1 = \frac{1}{4}$ and $\lambda_2 = \frac{3}{4}$ we get

$$f(\lambda_1 x + (1-\lambda_1)y) \;=\; -3 \;<\; 0 \quad \text{and} \quad f(\lambda_2 x + (1-\lambda_2)y) \;=\; 3 \;>\; 0$$

Open Ended Problems

3 Epigraphs and Convex Functions

Let $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$ be two functions. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function given by $h(x) = \max\{f_1(x), f_2(x)\}$.

(3.1) Prove the following proposition:

$$\text{epi}(h) = \text{epi}(f_1) \cap \text{epi}(f_2).$$

(Hint: One way of proving that two sets A and B are equal to one another is to show that $A \subseteq B$ and $B \subseteq A$.)

Solution

Using the given hint, we will establish the claim by showing that $\text{epi}(h) \subseteq \text{epi}(f_1) \cap \text{epi}(f_2)$ and $\text{epi}(f_1) \cap \text{epi}(f_2) \subseteq \text{epi}(h)$.

$\text{epi}(h) \subseteq \text{epi}(f_1) \cap \text{epi}(f_2)$: Let $(x, z) \in \text{epi}(h)$, where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Then, $z \geq h(x)$. Since $h(x) = \max\{f_1(x), f_2(x)\}$, it follows that $h(x) \geq f_1(x)$ and $h(x) \geq f_2(x)$. Combining each of these inequalities with $z \geq h(x)$, we obtain $z \geq f_1(x)$ and $z \geq f_2(x)$. Therefore, $(x, z) \in \text{epi}(f_1)$ and $(x, z) \in \text{epi}(f_2)$, and we conclude that $(x, z) \in \text{epi}(f_1) \cap \text{epi}(f_2)$. Therefore, we obtain $\text{epi}(h) \subseteq \text{epi}(f_1) \cap \text{epi}(f_2)$.

$\text{epi}(f_1) \cap \text{epi}(f_2) \subseteq \text{epi}(h)$: Let $(x, z) \in \text{epi}(f_1) \cap \text{epi}(f_2)$, where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Therefore, $(x, z) \in \text{epi}(f_1)$ and $(x, z) \in \text{epi}(f_2)$. It follows that $z \geq f_1(x)$ and $z \geq f_2(x)$. Therefore, we obtain $z \geq \max\{f_1(x), f_2(x)\} = h(x)$. We conclude that $(x, z) \in \text{epi}(h)$, i.e., $\text{epi}(f_1) \cap \text{epi}(f_2) \subseteq \text{epi}(h)$.

(3.2) Suppose that f_1 and f_2 are convex functions. Show, by using (3.1), that h is a convex function.

Solution

Since f_1 and f_2 are convex functions, we obtain that each of $\text{epi}(f_1)$ and $\text{epi}(f_2)$ is a convex set by Proposition 3.1. Since convexity is preserved under taking intersections (see Remark 3 in Section 3.2 in the lecture notes), it follows that $\text{epi}(f_1) \cap \text{epi}(f_2)$ is a convex set. By (3.1), we have $\text{epi}(h) = \text{epi}(f_1) \cap \text{epi}(f_2)$, where $h(x) = \max\{f_1(x), f_2(x)\}$. Using Proposition 3.1 once more again (in the other direction), we obtain that h is a convex function.