Fundamentals of Operational Research
Tutorial 4
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1. As part of the decisions on a drug test, a mathematical optimization model is used. Include constraints that represent the following conditions:

(a) If drug A is used, then drug B must also be used.
(b) If drug C is used, neither drug B nor drug D can be used.
(c) If drug D is used, then at least one of drug A or drug E must be used.
(d) If drug A and drug C are used, then drug F and drug G cannot both be used.

Solution:
Let us use variable x_i to represent whether drug i is used (with value 1) or not (with value 0).

(a) This is one of the first logical constraints that we have seen in class:

$$x_A \leq x_B.$$

(b) We can see this as an if-then constraint:

- "Drug C is used" is modelled as $x_C = 1$.

So, we can define $f(x_C) = x_C$.
As x_C is a binary variable, $x_C = 1$ is equivalent to $f(x_C) > 0$.

- "Neither drug B nor drug D are used" is modelled as $x_B = x_D = 0$.

As x_B and x_D are nonnegative, this is equivalent to $x_B + x_D \leq 0$, which can be written as $-x_B - x_D \geq 0$.
So, that is $g(x_B, x_D) \geq 0$ with $g(x_B, x_D) = -x_B - x_D$.
We know that f implies g can be enforced by the following constraints:

$$f(x_C) \leq M_1(1-y),$$
$$-g(x_B, x_D) \leq M_2 y,$$

where y is an auxiliary binary variable, $M_1 = \max\{f(x_C) \ / \ x_C \in \{0,1\}\}$, and $M_2 = \max\{-g(x_B, x_D) \ / \ x_B, x_D \in \{0,1\}\}$.
Thus, $M_1 = \max\{x_C \ / \ x_C \in \{0,1\}\} = 1$ and $M_2 = \max\{x_B + x_D \ / \ x_B, x_D \in \{0,1\}\} = 2$.
The constraints are:

$$x_C \leq 1 - y,$$
$$x_B + x_D \leq 2y.$$

The first constraint can be written as $x_C + y \leq 1$ to see more clearly the incompatibility between these two binary variables.
In this case, there is an alternative way of writing the condition with a single constraint:

$$x_B + x_D \leq 2(1 - x_C).$$

This replaces the two other constraints. Note that this is not always possible.

(c) We can see this as an if-then condition:

- "Drug D is used" is represented as $x_D = 1$.

Using the notation studied in class and using that x_D is binary, we have that $x_D \geq 1$. Thus $f(x_D) > 0$ for $f(x_D) = x_D$.

- "At least one of drug A or drug E are used" is represented as $x_A + x_E \geq 1$.

Thus, $g(x_A, x_E) \geq 0$ with $g(x_A, x_E) = x_A + x_E - 1$.

Therefore, the condition can be modelled through the following constraints:

$$\begin{aligned} f(x_D) &\leq M_1(1-y), \\ -g(x_A, x_E) &\leq M_2 y, \end{aligned}$$

where y is an auxiliary binary variable, $M_1 = \max\{f(x_D) \ / \ x_D \in \{0,1\}\}$, and $M_2 = \max\{-g(x_A, x_E) \ / \ x_A, x_E \in \{0,1\}\}$.

Thus, $M_1 = \max\{x_D \ / \ x_D \in \{0,1\}\} = 1$ and $M_2 = \max\{-x_A - x_E + 1 \ / \ x_A, x_E \in \{0,1\}\} = 1$.

So, the constraints are:

$$\begin{aligned} x_D &\leq 1 - y, \\ -x_A - x_E + 1 &\leq y. \end{aligned}$$

In order to interpret them more easily, we can rewrite them as follows:

$$\begin{aligned} x_D + y &\leq 1, \\ x_A + x_E + y &\geq 1. \end{aligned}$$

Note that in this case we could model the condition with a single constraint:

$$x_A + x_E \geq x_D.$$

This is an alternative way of answering, but both are correct.

(d) This is again an if-then condition:

- "Drug A and drug C are both used" means that $x_A = x_C = 1$.

Thus, this is $f(x_A, x_C) > 0$ with $f(x_A, x_C) = x_A + x_C - 1$ because the two variables are binary.

- "Drug F and drug G cannot both be used" is written as $x_F + x_G \leq 1$.

Thus, this is $1 - x_F - x_G \geq 0$. So, $g(x_F, x_G) \geq 0$ with $g(x_F, x_G) = 1 - x_F - x_G$.

Now, the condition is modelled as

$$\begin{aligned} f(x_A, x_C) &\leq M_1(1-y), \\ -g(x_F, x_G) &\leq M_2 y, \end{aligned}$$

with y an auxiliary binary variable, $M_1 = \max\{f(x_A, x_C) \ / \ x_A, x_C \in \{0,1\}\}$, and $M_2 = \max\{-g(x_F, x_G) \ / \ x_F, x_G \in \{0,1\}\}$.

Thus, $M_1 = \max\{x_A + x_C - 1 \ / \ x_A, x_C \in \{0,1\}\} = 1$ and $M_2 = \max\{-1 + x_F + x_G \ / \ x_F, x_G \in \{0,1\}\} = 1$.

We write the constraints as:

$$
\begin{aligned}
x_A + x_C - 1 &\leq 1 - y, \\
-1 + x_F + x_G &\leq y.
\end{aligned}
$$

In order to have a more clear interpretation, the constraints can be rewritten as follows:

$$
\begin{aligned}
x_A + x_C + y &\leq 2, \\
x_F + x_G &\leq 1 + y.
\end{aligned}
$$

Note that alternatively we could model the condition with a single constraint:

$$x_A + x_C + x_F + x_G \leq 3.$$

2. Let x_i be the proportion of component C_i in a mixture. Model the following condition: "If the proportion of C_1 exceeds 0.3, then the proportion of C_2 must be at least 0.1 and the proportion of C_3 must not exceed 0.2".

Solution:

First of all, note that $0 \leq x_i \leq 1, \; i = 1, 2, 3$ because we are using these variables to represent proportions.

Next we write each of the parts involved in this condition:

- $p =$ "The proportion of C_1 exceeds 0.3". This is $x_1 > 0.3$.
- $q =$ "The proportion of C_2 must be at least 0.1". This is $x_2 \geq 0.1$.
- $r =$ "The proportion of C_3 must not exceed 0.2". This is $x_3 \leq 0.2$.

The condition that we need to model is "if p holds, then q and r must both hold".

This is the same than "if p holds, then q holds" and "if p holds, then r holds". This is precisely what we are going to model.

We define $f(x_1) = x_1 - 0.3$, $g(x_2) = x_2 - 0.1$, and $h(x_3) = 0.2 - x_3$.

We would like to model the following two conditions simultaneously:

- If $f(x_1) > 0$, then $g(x_2) \geq 0$.
- If $f(x_1) > 0$, then $h(x_3) \geq 0$.

This is done through the following constraints (seen in class, but with the particularity that the first constraint is common to the two conditions written above):

$$
\begin{aligned}
f(x_1) &\leq M_1(1 - y), \\
-g(x_2) &\leq M_2 y, \\
-h(x_3) &\leq M_3 y,
\end{aligned}
$$

where y is an auxiliary binary variable, $M_1 = \max\{f(x_1) \,/\, 0 \leq x_1 \leq 1\}$, $M_2 = \max\{-g(x_2) \,/\, 0 \leq x_2 \leq 1\}$, and $M_3 = \max\{-h(x_3) \,/\, 0 \leq x_3 \leq 1\}$.

Thus, $M_1 = \max\{x_1 - 0.3 \,/\, 0 \leq x_1 \leq 1\} = 0.7$, $M_2 = \max\{-x_2 + 0.1 \,/\, 0 \leq x_2 \leq 1\} = 0.1$, and $M_3 = \max\{-0.2 + x_3 \,/\, 0 \leq x_3 \leq 1\} = 0.8$.

The constraints are:

$$x_1 - 0.3 \le 0.7(1-y),$$
$$-x_2 + 0.1 \le 0.1y,$$
$$-0.2 + x_3 \le 0.8y,$$

with $0 \le x_i \le 1,\ i = 1, 2, 3$.

So that we have an easier interpretation, we can rewrite them as follows:

$$x_1 \le 1 - 0.7y,$$
$$0.1 \le x_2 + 0.1y,$$
$$x_3 \le 0.2 + 0.8y.$$

We can check that the constraints work as desired:

- If $x_1 > 0.3$, the first constraint enforces that $y = 0$. Then the other two constraints become $x_x \ge 0.1$ and $x_3 \le 0.2$, respectively. This is exactly what we want.
- $x_1 \le 0.3$, then $y \in \{0, 1\}$. When $y = 1$, the other two constraints are $x_2 \ge 0$ and $x_3 \le 1$. Thus, no limitation at all. Again, what we need.

3. Model the condition

"$0 \le x_1 \le 1$ and $x_2 = 0$" or "$0 \le x_2 \le 1$ and $x_1 = 0$".

Solution:
Although it seems to be an either-or constraint from the lectures, it is not. The reason is that we cannot represent "$0 \le x_1 \le 1$ and $x_2 = 0$" with a single constraint. The same happens with the second part of the condition.

Sometimes we simply have to think of a way to model a condition, with no "recipe" that we can apply.

We introduce an auxiliary binary variable y and we write the constraints

$$0 \le x_1 \le 1 - y,$$
$$0 \le x_2 \le y.$$