29.1 Outline

- Sensitivity Analysis and Reoptimization: Changes in b
- Sensitivity Analysis and Reoptimization: Changes in c
- Review Problems

29.2 Motivation and Setup

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- (P) is the primal problem and (D) is the dual problem.
- In this lecture, we study how the optimal value and the optimal solutions of (P) and (D) change with respect to the changes in the problem parameters $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.
- This is an important problem due to
  (i) uncertainty or incompleteness of the parameters
  (ii) scenario analysis (what-if)
- Suppose that the pair of problems (P) and (D) are solved either using the usual (primal) simplex method or the dual simplex method.
- Suppose that $A \in \mathbb{R}^{m \times n}$ has full row rank and that $x^* \in \mathbb{R}^n$ is an optimal vertex of (P) with corresponding index sets $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$, and $y^* = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ is the corresponding dual optimal solution.
- There are two main questions of interest:
  (i) **Sensitivity Analysis:** Under what conditions on the changes in the problem parameters b and c would the dictionary corresponding to the index sets B and N still remain optimal for the modified primal-dual pair of problems?
  (ii) **Reoptimization:** If the dictionary corresponding to the index sets B and N is no longer optimal for the modified primal-dual pair of problems, how do we reoptimize effectively?

29.2.1 Observations

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- Changes in $b \in \mathbb{R}^m$ affect the feasible region of (P) but not the objective function of (P).
- Changes in $b \in \mathbb{R}^m$ affect the objective function of (D) but not the feasible region of (D).
- Changes in $c \in \mathbb{R}^n$ affect the objective function of (P) but not the feasible region of (P).
- Changes in $c \in \mathbb{R}^n$ affect the feasible region of (D) but not the objective function of (D).

29.2.2 Main Idea

- Suppose that the optimal dictionary is given by

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b + \sum_{j \in N} \left(-(A_B)^{-1} A^j\right) x_j \end{aligned}$$

- Suppose that some of the parameters given by $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are modified.
  (i) Identify the parts of the optimal dictionary that would be affected by such a change.
  (ii) Identify whether the primal and/or dual feasibility would be affected by such a change.
  (iii) Derive conditions on the parameters such that primal and dual feasibility are maintained.
  (iv) If the primal or dual feasibility is violated after the modification, determine which variant of the simplex method (primal or dual) would be appropriate to reoptimize the modified pair of primal and dual problems.

29.3 Changes in b

Recall the optimal dictionary for the original pair of primal-dual linear programming problems:

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b + \sum_{j \in N} \left(-(A_B)^{-1} A^j\right) x_j \end{aligned}$$

- Let us first consider changes in $b \in \mathbb{R}^m$.

- Suppose that b_i is replaced by $b_i + \delta$, where $\delta \in \mathbb{R}$ and $i \in \{1, \ldots, m\}$.

- Therefore, $b \in \mathbb{R}^m$ is replaced by $b + \delta e^i$, where $e^i \in \mathbb{R}^m$ denotes the ith unit vector, $i \in \{1, \ldots, m\}$.

- Changes in $b \in \mathbb{R}^m$ only affect the values of the current basic variables and the objective function value at the current dictionary (i.e., only primal feasibility is affected).

- The current dictionary remains optimal for the modified primal-dual pair of problems if and only if

$$(A_B)^{-1}\left(b + \delta e^i\right) \geq \mathbf{0}.$$

- Note that

$$(A_B)^{-1}\left(b + \delta e^i\right) = x_B^* + \delta (A_B)^{-1} e^i \geq \mathbf{0} \Longleftrightarrow \delta (A_B)^{-1} e^i \geq -x_B^*.$$

- Therefore, we obtain m inequalities on $\delta \in \mathbb{R}$ such that the current dictionary remains optimal if and only if δ satisfies each of these inequalities simultaneously.

- As a function of $\delta \in \mathbb{R}$,

  (i) the values of basic variables are given by $x_B^*(\delta) = (A_B)^{-1}\left(b + \delta e^i\right) = x_B^* + \delta (A_B)^{-1} e^i$;

  (ii) the values of nonbasic variables are given by $x_N^*(\delta) = \mathbf{0}$;

  (iii) the corresponding dual solution is given by $y^*(\delta) = \left((A_B)^{-1}\right)^T c_B$;

  (iv) the objective function value is given by $z^*(\delta) = c_B^T (A_B)^{-1}\left(b + \delta e^i\right) = z^* + \delta c_B^T (A_B)^{-1} e^i$.

- Recall that

$$(A_B)^{-1}\left(b + \delta e^i\right) = x_B^* + \delta (A_B)^{-1} e^i \geq \mathbf{0} \Longleftrightarrow \delta (A_B)^{-1} e^i \geq -x_B^*.$$

- If we choose $\delta^* \in \mathbb{R}$ that does not satisfy each of these m inequalities simultaneously, then we can reoptimize using the dual simplex method since the updated current dictionary is primal infeasible but dual feasible.

29.3.1 More General Changes

- More generally, a similar analysis applies if b is replaced by $b + \delta b'$, where $\delta \in \mathbb{R}$ and $b' \in \mathbb{R}^m$, i.e., the current dictionary remains optimal if and only if

$$(A_B)^{-1}\left(b + \delta b'\right) = x_B^* + \delta (A_B)^{-1} b' \geq \mathbf{0} \Longleftrightarrow \delta (A_B)^{-1} b' \geq -x_B^*.$$

- If we choose $\delta^* \in \mathbb{R}$ that does not satisfy each of these m inequalities simultaneously, then we can reoptimize using the dual simplex method since the updated current dictionary is primal infeasible but dual feasible.

29.4 Changes in c

Recall the optimal dictionary for the original pair of primal-dual linear programming problems:

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j\in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j\in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

- We now focus on changes in $c \in \mathbb{R}^n$.
- Suppose that c_j is replaced by $c_j + \delta$, where $\delta \in \mathbb{R}$ and $j \in \{1, \ldots, n\}$.
- Changes in $c \in \mathbb{R}^n$ only affect the right-hand side of Row 0 (i.e., only dual feasibility is affected).
- We will identify two different cases:
  - (i) Changing the objective function coefficient of a nonbasic variable ($j \in N$)
  - (ii) Changing the objective function coefficient of a basic variable ($j \in B$)

29.4.1 Changes in the Objective Function Coefficient of a Nonbasic Variable

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j\in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j\in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

- Suppose that c_j is replaced by $c_j + \delta$, where $\delta \in \mathbb{R}$ and $j \in N$.
- A change in $c_j,\ j \in N$ only affects the reduced cost $\bar{c}_j$.
- Therefore, the current dictionary remains optimal for the modified primal-dual pair of problems if and only if
$$c_j + \delta - c_B^T(A_B)^{-1}A^j = \bar{c}_j + \delta \geq 0 \Longleftrightarrow \delta \geq -\bar{c}_j.$$
- We have $\bar{c}_j(\delta) = \bar{c}_j + \delta$ and everything else remains the same.
- If we choose $\delta^* \in \mathbb{R}$ that does not satisfy this inequality, then we can reoptimize using the primal simplex method since the updated current dictionary is primal feasible but dual infeasible.

29.4.2 Changes in the Objective Function Coefficient of a Basic Variable

$$
\begin{aligned}
z &= c_B^T(A_B)^{-1}b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\
x_B &= (A_B)^{-1}b + \sum_{j \in N} \left(-(A_B)^{-1}A^j\right) x_j
\end{aligned}
$$

- Suppose now that c_j is replaced by $c_j + \delta$, where $\delta \in \mathbb{R}$ and $j \in B$.
- A change in c_B affects all the right-hand side entries in Row 0.
- Suppose that $j \in B$ is the ℓth basic variable, where $\ell \in \{1, \ldots, m\}$ (i.e., x_j is the basic variable in Row ℓ).
- Therefore, $c_B \in \mathbb{R}^m$ is replaced by $c_B(\delta) = c_B + \delta e^\ell$, where $e^\ell \in \mathbb{R}^m$.
- Since $c_B \in \mathbb{R}^m$ is replaced by $c_B(\delta) = c_B + \delta e^\ell$, where $e^\ell \in \mathbb{R}^m$, the current dictionary remains optimal if and only if

$$c_k - \left(c_B + \delta e^\ell\right)^T (A_B)^{-1}A^k = \bar{c}_k - \delta(e^\ell)^T(A_B)^{-1}A^k \geq 0 \Longleftrightarrow \delta(e^\ell)^T(A_B)^{-1}A^k \leq \bar{c}_k, \quad k \in N.$$

- Therefore, we obtain $\bar{c}_k(\delta) = \bar{c}_k - \delta(e^\ell)^T(A_B)^{-1}A^k,\ k \in N$. The primal solution is given by $x^*(\delta) = x^*$ and the corresponding dual solution by $y^*(\delta) = \left((A_B)^{-1}\right)^T(c_B + \delta e^\ell) = y^* + \delta\left((A_B)^{-1}\right)^T e^\ell$. The objective function value is given by $z^*(\delta) = \left(c_B + \delta e^\ell\right)^T(A_B)^{-1}b = z^* + \delta(e^\ell)^T(A_B)^{-1}b$.
- If we choose $\delta^* \in \mathbb{R}$ that does not satisfy all of these inequalities simultaneously, then we can reoptimize using the primal simplex method since the updated current dictionary is primal feasible but dual infeasible.

29.4.3 More General Changes

- More generally, if c is replaced by $c + \delta c'$, where $\delta \in \mathbb{R}$ and $c' \in \mathbb{R}^n$, we obtain

$$
\begin{aligned}
c_B(\delta) &= c_B + \delta c'_B \\
c_N(\delta) &= c_N + \delta c'_N
\end{aligned}
$$

- Row 0 coefficients can be updated accordingly and one can check conditions on δ for the nonnegativity of all reduced costs.
- Therefore, we obtain $\bar{c}_k(\delta) = \bar{c}_k + \delta c'_k - \delta(c'_B)^T(A_B)^{-1}A^k,\ k \in N$. The primal solution is given by $x^*(\delta) = x^*$ and the corresponding dual solution by $y^*(\delta) = \left((A_B)^{-1}\right)^T(c_B + \delta c'_B) = y^* + \delta\left((A_B)^{-1}\right)^T c'_B$. The objective function value is given by $z^*(\delta) = (c_B + \delta c'_B)^T(A_B)^{-1}b = z^* + \delta(c'_B)^T(A_B)^{-1}b$.
- We can use the primal simplex method to reoptimize if there is any negative reduced cost.

29.5 Concluding Remarks

- We discussed the effects of changes in $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ on the primal-dual optimal solutions and the optimal value.
- The same analysis can be extended to the case of simultaneously replacing b by $b + \delta b'$, and c by $c + \delta c'$, where $\delta \in \mathbb{R}$, $b' \in \mathbb{R}^m$, and $c' \in \mathbb{R}^n$.
- However, if δ^* does not satisfy all of the resulting inequalities for primal and dual feasibility simultaneously, we may lose both of them.
- In such a case, we need to solve the modified problem from scratch.
- In the next two lectures, we will discuss changes in $A \in \mathbb{R}^{m \times n}$, adding new variables to (P), and adding new constraints to (P).

Exercises

Question 29.1.

$$\begin{array}{ll} (P) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ (D) & \max\{b^T y : A^T y \leq c\} \end{array}$$

Consider the instance of (P) and (D) with the following parameters:

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 1 \\ 5 & 3 & 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 16 \end{bmatrix}, \quad c = \begin{bmatrix} -5 & -1 & 12 & 0 & -1 \end{bmatrix}^T.$$

Here is the optimal dictionary:

$$\begin{array}{rcrcrcrcr} z & = & -16 & + & 2x_2 & + & 12x_3 & + & x_4 \\ x_1 & = & 3 & - & 0.5x_2 & + & 0.5x_3 & - & 0.5x_4 \\ x_5 & = & 1 & - & 0.5x_2 & - & 2.5x_3 & + & 1.5x_4 \end{array}$$

For each change below, find the possible values of $\delta \in \mathbb{R}$ for which the dictionary remains optimal if (i) b_1 is replaced by $10 + \delta$, (ii) c_2 is replaced by $-1 + \delta$, and (iii) c_1 is replaced by $-5 + \delta$.

MATH11111: Fundamentals of Optimization		**Fall 2022**
Lecture 30	Sensitivity Analysis and Reoptimization – Part II	
Lecturer: E. Alper Yıldırım		*Week: 10*

30.1 Outline

- Sensitivity Analysis and Reoptimization: Changes in A
- Sensitivity Analysis and Reoptimization: Adding a new variable
- Review Problems

30.2 Motivation and Setup

Consider the following pair of linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- (P) is the primal problem and (D) is the dual problem.
- In the last lecture, we studied how the optimal value and the optimal solutions of (P) and (D) change with respect to the changes in the problem parameters $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, and how the modified problem can be reoptimized efficiently.
- In this lecture, we will study the same problem for the following additional cases:
  - (i) Changes in $A \in \mathbb{R}^{m \times n}$
  - (ii) Adding a new variable
- Suppose that the pair of problems (P) and (D) are solved either using the usual (primal) simplex method or the dual simplex method.
- Suppose that $A \in \mathbb{R}^{m \times n}$ has full row rank and that $x^* \in \mathbb{R}^n$ is an optimal vertex of (P) with corresponding index sets $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$, and $y^* = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ is the corresponding dual optimal solution.
- There are two main questions of interest:
  - (i) **Sensitivity Analysis:** Under what conditions on the changes in the problem parameters would the dictionary corresponding to the index sets B and N still remain optimal for the modified primal-dual pair of problems?
  - (ii) **Reoptimization:** If the dictionary corresponding to the index sets B and N is no longer optimal for the modified primal-dual pair of problems, how do we reoptimize effectively?

30.2.1 Main Idea Revisited

Suppose the optimal dictionary of the original pair of primal-dual linear programming problems is given by the following:

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j \in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

(i) Identify the parts of the optimal dictionary that would be affected by such a change.

(ii) Identify whether the primal and/or dual feasibility would be affected by such a change.

(iii) Derive conditions on the parameters such that primal and dual feasibility are maintained.

(iv) If the primal or dual feasibility is violated after the modification, determine which variant of the simplex method (primal or dual) would be appropriate to reoptimize the modified pair of primal and dual problems.

30.3 Changes in A

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- First, we will focus on changes in the entries of the coefficient matrix $A \in \mathbb{R}^{m \times n}$.
- Changes in A affect the feasible region of (P) and the feasible region of (D) simultaneously.
- Therefore, we may lose both primal feasibility and dual feasibility.
- Let $A^j \in \mathbb{R}^m$ denote the $j$the column of A, $j = 1, \ldots, n$. We will consider two cases:
  (i) Changes in $A^j, \ j \in N$
  (ii) Changes in $A^j, \ j \in B$

30.3.1 Changes in $A^j, \ j \in N$

Recall the optimal dictionary of the original pair of primal-dual linear programming problems:

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j \in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

- Let us first consider changes in $A^j,\ j \in N$.
- Suppose that A_{ij} is replaced by $A_{ij} + \delta$, where $\delta \in \mathbb{R}$, $i \in \{1, \ldots, m\}$, and $j \in N$.
- Therefore, $A^j \in \mathbb{R}^m$ is replaced by $A^j + \delta e^i$, where $e^i \in \mathbb{R}^m$ denotes the ith unit vector, $i \in \{1, \ldots, m\}$.
- Changes in $A^j,\ j \in N$ only affect the values of the reduced cost of the nonbasic variable x_j and the coefficients of x_j on the right-hand side of Rows 1 through m (i.e., only dual feasibility is affected).
- The current dictionary remains optimal for the modified primal-dual pair of problems if and only if

$$c_j - c_B^T (A_B)^{-1}(A^j + \delta e^i) \geq 0.$$

- Note that

$$c_j - c_B^T (A_B)^{-1}(A^j + \delta e^i) = \bar{c}_j - \delta c_B^T (A_B)^{-1} e^i \geq 0 \Longleftrightarrow \delta c_B^T (A_B)^{-1} e^i \leq \bar{c}_j.$$

- Therefore, the current dictionary remains optimal if and only if δ satisfies the inequality above. Note that primal-dual optimal solutions and the optimal value remain the same.
- If we choose $\delta^* \in \mathbb{R}$ that violates this inequality, we update the Row 0 coefficient of x_j using the above relation. We update the coefficients of x_j on the right-hand side of Rows 1 through m using $-(A_B)^{-1}(A^j + \delta e^i)$ and continue with the primal simplex method.
- More generally, a similar analysis applies if A^j is replaced by $A^j + \delta a'$, where $\delta \in \mathbb{R}$ and $a' \in \mathbb{R}^m$.

30.3.2 Changes in $A^j,\ j \in B$

Recall the optimal dictionary of the original pair of primal-dual linear programming problems:

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b + \sum_{j \in N} \left(-(A_B)^{-1} A^j\right) x_j \end{aligned}$$

- Let us now consider changes in $A^j,\ j \in B$.
- Suppose that A_{ij} is replaced by $A_{ij} + \delta$, where $\delta \in \mathbb{R}$, $i \in \{1, \ldots, m\}$, and $j \in B$.
- Changes in $A^j,\ j \in B$ affect the basis matrix $A_B \in \mathbb{R}^{m \times m}$. Note that the entire dictionary is affected by such a change.
- The updated basis matrix may no longer be invertible.
- Even if it is invertible, we may lose both primal and dual feasibility.
- Solve the modified problem from scratch using the Two-Phase Method.

30.4 Adding a New Variable

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll}(\mathrm{P}) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ (\mathrm{D}) & \max\{b^T y : A^T y \leq c\}\end{array}$$

- Suppose now that we wish to add a new variable $x_{n+1} \in \mathbb{R}$ with a cost coefficient $c_{n+1} \in \mathbb{R}$ and the column $A^{n+1} \in \mathbb{R}^m$ in the coefficient matrix.
- The new coefficient matrix is given by $\tilde{A} = \begin{bmatrix} A^1 & \cdots & A^n & A^{n+1} \end{bmatrix} \in \mathbb{R}^{m \times (n+1)}$.
- We have $\operatorname{rank}(\tilde{A}) \geq \operatorname{rank}(A) = m$ and $\operatorname{rank}(\tilde{A}) \leq m$. Therefore, $\tilde{A} \in \mathbb{R}^{m \times (n+1)}$ has full row rank.
- Defining $\tilde{x} \in \mathbb{R}^{n+1}$ by $\tilde{x}_j = x_j^*,\ j = 1, \ldots, n$, and $\tilde{x}_{n+1} = 0$, we obtain $\tilde{A}\tilde{x} = Ax^* + \mathbf{0} = b$ and $\tilde{x} \geq \mathbf{0}$.
- $\tilde{x} \in \mathbb{R}^{n+1}$ is a basic feasible solution of the new primal problem with index sets $\tilde{B} = B$ and $\tilde{N} = N \cup \{n+1\}$. Therefore, the new primal problem has a nonempty feasible region.
- On the dual side, we need to add a new constraint given by $(A^{n+1})^T y \leq c_{n+1}$. The new dual problem may or may not be infeasible.
- We will use $\tilde{x} \in \mathbb{R}^{n+1}$ as the starting primal vertex with corresponding index sets $\tilde{B} = B$ and $\tilde{N} = N \cup \{n+1\}$.
- We need to add a new column on the right-hand side of Rows $0, 1, 2, \ldots, m$ corresponding to the new nonbasic variable x_{n+1}.
- The reduced cost of x_{n+1} is $\bar{c}_{n+1} = c_{n+1} - c_B^T (A_B)^{-1} A^{n+1}$.
- The current dictionary remains optimal if and only if $\bar{c}_{n+1} \geq 0$.
- Otherwise, we compute the coefficients of x_{n+1} on the right-hand sides of Rows 1 through m using $-(A_B)^{-1} A^{n+1}$ and continue with the primal simplex method.

30.5 Concluding Remarks

- We discussed the effects of changes in $A \in \mathbb{R}^{m \times n}$ and adding a new variable on the primal-dual optimal solutions and the optimal value.
- Note that reoptimization of the modified problem can be performed efficiently if one of primal and dual feasibility is maintained.
- If the changes violate both primal and dual feasibility, we need to solve the modified problem from scratch.
- In the next lecture, we will discuss the effects of adding a new constraint on the primal-dual optimal solutions and the optimal value.

Exercises

Question 30.1.

$$\begin{array}{ll} (P) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ (D) & \max\{b^T y : A^T y \leq c\} \end{array}$$

Consider the instance of (P) and (D) with the following parameters:

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 1 \\ 5 & 3 & 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 16 \end{bmatrix}, \quad c = \begin{bmatrix} -5 & -1 & 12 & 0 & -1 \end{bmatrix}^T.$$

Here is the optimal dictionary:

$$\begin{array}{rcrcrcrcr} z & = & -16 & + & 2x_2 & + & 12x_3 & + & x_4 \\ x_1 & = & 3 & - & 0.5x_2 & + & 0.5x_3 & - & 0.5x_4 \\ x_5 & = & 1 & - & 0.5x_2 & - & 2.5x_3 & + & 1.5x_4 \end{array}$$

(i) Find the possible values of $\delta \in \mathbb{R}$ for which the dictionary remains optimal if A_{22} is replaced by $3 + \delta$. (ii) What if A_{11} is replaced by 5? (iii) What if we add a new variable x_6 with $c_6 = -2$ and $A_6 = [1, -1]^T$?

MATH11111: Fundamentals of Optimization **Fall 2022**

Lecture 31 Sensitivity Analysis and Reoptimization – Part III

Lecturer: E. Alper Yıldırım *Week: 10*

31.1 Outline

- Sensitivity Analysis and Reoptimization: Adding a new equality constraint
- Sensitivity Analysis and Reoptimization: Adding a new inequality constraint
- Review Problems

31.2 Motivation and Setup

Consider the following pair of linear programming problems:

$$\begin{array}{ll} (\text{P}) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ (\text{D}) & \max\{b^T y : A^T y \leq c\} \end{array}$$

- (P) is the primal problem and (D) is the dual problem.
- In the last lecture, we studied how the optimal value and the optimal solutions of (P) and (D) change with respect to the changes in $A \in \mathbb{R}^{m \times n}$ and after adding a new variable, as well as how the modified problem can be reoptimized efficiently.
- In this lecture, we will study the same problem for the following additional cases:
  - (i) Adding a new equality constraint
  - (ii) Adding a new inequality constraint
- Suppose that the pair of problems (P) and (D) are solved either using the usual (primal) simplex method or the dual simplex method.
- Suppose that $A \in \mathbb{R}^{m \times n}$ has full row rank and that $x^* \in \mathbb{R}^n$ is an optimal vertex of (P) with corresponding index sets $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$, and $y^* = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ is the corresponding dual optimal solution.
- There are two main questions of interest:
  - (i) **Sensitivity Analysis:** Under what conditions on the changes in the problem parameters would the dictionary corresponding to the index sets B and N still remain optimal for the modified primal-dual pair of problems?
  - (ii) **Reoptimization:** If the dictionary corresponding to the index sets B and N is no longer optimal for the modified primal-dual pair of problems, how do we reoptimize effectively?

31.2.1 Main Idea Revisited

Suppose that the optimal dictionary for the original primal-dual pair of linear programming problems is given by

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b + \sum_{j \in N} \left(-(A_B)^{-1} A^j\right) x_j \end{aligned}$$

(i) Identify the parts of the optimal dictionary that would be affected by such a change.

(ii) Identify whether the primal and/or dual feasibility would be affected by such a change.

(iii) Derive conditions on the parameters such that primal and dual feasibility are maintained.

(iv) If the primal or dual feasibility is violated after the modification, determine which variant of the simplex method (primal or dual) would be appropriate to reoptimize the modified pair of primal and dual problems.

31.3 Adding a New Equality Constraint

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- Suppose now that we wish to add a new equality constraint $(a^{m+1})^T x = b_{m+1}$, where $a^{m+1} \in \mathbb{R}^n$ and $b_{m+1} \in \mathbb{R}$.
- If the optimal solution of the original primal problem $x^* \in \mathbb{R}^n$ satisfies $(a^{m+1})^T x^* = b_{m+1}$, then x^* is an optimal solution of the modified problem.
- Otherwise, x^* is not a feasible solution of the modified problem.
- The modified primal problem may not be feasible.
- Note that we need an additional basic variable corresponding to the new primal equality constraint.
- Let

$$\tilde{A} = \begin{bmatrix} (a^1)^T \\ \vdots \\ (a^m)^T \\ (a^{m+1})^T \end{bmatrix} \in \mathbb{R}^{(m+1) \times n}$$

- The new matrix $\tilde{A}$ may not have full row rank.

- Even if it has full row rank, it is not clear how to correctly identify the new basic variable.
- In the new dual problem, we need to define a new dual variable y_{m+1} corresponding to the new primal constraint.
- Let $\tilde{y} \in \mathbb{R}^{m+1}$ be given by $\tilde{y}_j = y_j^*$ for $j = 1, \ldots, m$, and $\tilde{y}_{m+1} = 0$, where $y^* \in \mathbb{R}^m$ is the optimal solution of the original dual problem.
- Since $\tilde{A}^T \tilde{y} = A^T y^* + \tilde{y}_{m+1} a^{m+1} = A^T y^* \leq c$, $\tilde{y} \in \mathbb{R}^{m+1}$ is a feasible solution of the modified dual problem.
- However, $\tilde{y} \in \mathbb{R}^{m+1}$ may not necessarily be a basic feasible solution.
- Solve the modified primal problem from scratch using the Two-Phase Method.

31.4 Adding a New Inequality Constraint

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- Suppose now that we wish to add a new inequality constraint $(a^{m+1})^T x \leq b_{m+1}$, where $a_{m+1} \in \mathbb{R}^n$ and $b^{m+1} \in \mathbb{R}$.
- If the optimal solution of the original primal problem $x^* \in \mathbb{R}^n$ satisfies $(a^{m+1})^T x^* \leq b_{m+1}$, then x^* is an optimal solution of the modified problem.
- Otherwise, x^* is not a feasible solution of the modified problem.
- The modified primal problem may be infeasible.
- Suppose that we add a new inequality constraint $(a^{m+1})^T x \leq b_{m+1}$ such that $(a^{m+1})^T x^* > b_{m+1}$.
- We need to convert the modified problem into standard form by defining a nonnegative slack variable x_{n+1} so that $(a^{m+1})^T x + x_{n+1} = b_{m+1}$.
- We also need a new basic variable corresponding to the new equality constraint.
- Let $\hat{x} \in \mathbb{R}^{n+1}$ be given by $\hat{x}_j = x_j^*$ for each $j = 1, \ldots, n$ and $\hat{x}_{n+1} = b_{m+1} - (a^{m+1})^T x^*$, where $x^* \in \mathbb{R}^n$ is the optimal solution of the original primal problem.
- Then, $\hat{x} \in \mathbb{R}^{n+1}$ will satisfy all the equality constraints of the modified primal problem. However, it will be infeasible since $\hat{x}_{n+1} = b_{m+1} - (a^{m+1})^T x^* < 0$.
- Let
$$\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ (a^{m+1})^T & 1 \end{bmatrix} \in \mathbb{R}^{(m+1)\times(n+1)}, \quad \tilde{b} = \begin{bmatrix} b \\ b_{m+1} \end{bmatrix} \in \mathbb{R}^{m+1}, \quad \tilde{c} = \begin{bmatrix} c \\ 0 \end{bmatrix} \in \mathbb{R}^{n+1}$$

Lemma 31.1. *Let $\tilde{A} \in \mathbb{R}^{(m+1)\times(n+1)}$ denote the new coefficient matrix.*

(i) $\tilde{A} \in \mathbb{R}^{(m+1)\times(n+1)}$ has full row rank.

(ii) If we define $\hat{x} \in \mathbb{R}^{n+1}$ such that $\hat{x}_j = x_j^*$ for each $j = 1, \ldots, n$ and $\hat{x}_{n+1} = b_{m+1} - (a^{m+1})^T x^*$, where $x^* \in \mathbb{R}^n$ is the optimal solution of the original primal problem, then $\hat{x} \in \mathbb{R}^{n+1}$ is a basic solution of the modified primal problem.

Proof. (i) Since $A \in \mathbb{R}^{m\times n}$ has full column rank, it follows that the first m rows of $\tilde{A} \in \mathbb{R}^{(m+1)\times(n+1)}$ are linearly independent. The last row of $\tilde{A}$ cannot be written as a linear combination of the first m rows of $\tilde{A}$ since the last component of the last row is 1 but the last components of all other rows are equal to 0. It follows that the rows of $\tilde{A}$ are linearly independent, i.e., $\tilde{A}$ has full row rank.

(ii) Let $\hat{x} \in \mathbb{R}^{n+1}$ be such that $\hat{x}_j = x_j^*$ for each $j = 1, \ldots, n$ and $\hat{x}_{n+1} = b_{m+1} - (a^{m+1})^T x^*$. Then, we have $\tilde{A}\hat{x} = \tilde{b}$, i.e., $\hat{x} \in \mathbb{R}^{n+1}$ will satisfy all of the equality constraints of the modified primal problem. Let us define the index sets $\tilde{B} = B \cup \{n+1\}$ and $\tilde{N} = N$. Note that $|\tilde{B}| = m+1$. By part (1) and Proposition 13.2, it suffices to show that $\tilde{A}_{\tilde{B}} \in \mathbb{R}^{(m+1)\times(m+1)}$ is invertible.

$$\tilde{A}_{\tilde{B}} = \begin{bmatrix} A_B & \mathbf{0} \\ (a_B^{m+1})^T & 1 \end{bmatrix}.$$

Using a similar argument as in part (1), since $A_B \in \mathbb{R}^{m\times m}$ is invertible, it follows that the first m rows of $\tilde{A}_{\tilde{B}}$ are linearly independent. The last row of $\tilde{A}_{\tilde{B}}$ cannot be written as a linear combination of the first m rows of $\tilde{A}_{\tilde{B}}$ since the last component of the last row is 1 but the last components of all other rows are equal to 0. It follows that the rows of $\tilde{A}_{\tilde{B}}$ are linearly independent, i.e., $\tilde{A}_{\tilde{B}}$ is invertible. By Proposition 13.2, $\hat{x} \in \mathbb{R}^{n+1}$ is a basic solution of the modified primal problem.

□

31.4.1 New Dictionary

Recall the optimal dictionary for the original pair of primal-dual linear programming problems:

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j\in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j\in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

- Recall

$$\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ (a^{m+1})^T & 1 \end{bmatrix} \in \mathbb{R}^{(m+1)\times(n+1)}, \quad \tilde{b} = \begin{bmatrix} b \\ b_{m+1} \end{bmatrix} \in \mathbb{R}^{m+1}, \quad \tilde{c} = \begin{bmatrix} c \\ 0 \end{bmatrix} \in \mathbb{R}^{n+1}$$

$$\tilde{B} = B \cup \{n+1\}, \quad \tilde{N} = N, \quad \tilde{A}_{\tilde{B}} = \begin{bmatrix} A_B & \mathbf{0} \\ (a_B^{m+1})^T & 1 \end{bmatrix}, \quad \tilde{c}_{\tilde{B}} = \begin{bmatrix} c_B \\ 0 \end{bmatrix},$$

$\hat{x} \in \mathbb{R}^{n+1}$, $\hat{x}_j = x_j^*$ for each $j = 1, \ldots, n$ and $\hat{x}_{n+1} = b_{m+1} - (a^{m+1})^T x^*$.

- The new dictionary is given by

$$\begin{aligned} z &= \tilde{c}_{\tilde{B}}^T(\tilde{A}_{\tilde{B}})^{-1}\tilde{b} + \sum_{j\in\tilde{N}} (\tilde{c}_j - \tilde{c}_{\tilde{B}}^T(\tilde{A}_{\tilde{B}})^{-1}\tilde{A}^j)x_j \\ x_{\tilde{B}} &= (\tilde{A}_{\tilde{B}})^{-1}\tilde{b} + \sum_{j\in\tilde{N}} \left(-(\tilde{A}_{\tilde{B}})^{-1}\tilde{A}^j\right) x_j \end{aligned}$$

- By simple manipulations, we obtain

$$\left(\tilde{A}_{\tilde{B}}\right)^{-1} = \begin{bmatrix} A_B^{-1} & \mathbf{0} \\ -(a_B^{m+1})^T(A_B)^{-1} & 1 \end{bmatrix}.$$

- We substitute these expressions in the dictionary above:

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j\in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j\in N}\left(-(A_B)^{-1}A^j\right)x_j \\ x_{n+1} &= \underbrace{b_{m+1} - (a^{m+1})^T x^*}_{<0} + \sum_{j\in N}\left((a_B^{m+1})^T(A_B)^{-1}A^j - a_j^{m+1}\right)x_j \end{aligned}$$

- The updated dictionary is dual feasible since $\bar{c}_j \geq 0$ for each $j \in N$.
- It is primal infeasible since the value of the new basic variable x_{n+1} is negative.
- We therefore use the dual simplex method to reoptimize the modified primal-dual pair of problems.

31.5 Concluding Remarks

- We discussed the effects of adding a new constraint on the primal-dual optimal solutions and the optimal value.
- Note that reoptimization of the modified problem can be performed efficiently if one of primal and dual feasibility is maintained.
- If the changes violate both primal and dual feasibility, we need to solve the modified problem from scratch.
- This lecture wraps up our discussion of linear programming. In the remaining lectures, we will discuss unconstrained nonlinear optimization.

Exercises

Question 31.1.

$$\begin{array}{ll} (P) & \min\{c^Tx : Ax = b, \quad x \geq \mathbf{0}\} \\ (D) & \max\{b^Ty : A^Ty \leq c\} \end{array}$$

Consider the instance of (P) and (D) with the following parameters:

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 1 \\ 5 & 3 & 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 16 \end{bmatrix}, \quad c = \begin{bmatrix} -5 & -1 & 12 & 0 & -1 \end{bmatrix}^T.$$

Here is the optimal dictionary:

$$\begin{array}{rcrcrcrcr} z & = & -16 & + & 2x_2 & + & 12x_3 & + & x_4 \\ x_1 & = & 3 & - & 0.5x_2 & + & 0.5x_3 & - & 0.5x_4 \\ x_5 & = & 1 & - & 0.5x_2 & - & 2.5x_3 & + & 1.5x_4 \end{array}$$

For each change below, determine if the optimal solution changes and how we should reoptimize.

(i) *Add a new constraint* $3x_1 - x_2 + x_3 - 2x_4 + x_5 = 10$.

(ii) *Add a new constraint* $3x_1 - x_2 + x_3 - 2x_4 + x_5 = 9$.

(iii) *Add a new constraint* $3x_1 - x_2 + x_3 - 2x_4 + x_5 \leq 12$.

(iv) *Add a new constraint* $3x_1 - x_2 + x_3 - 2x_4 + x_5 \leq 9$.