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Fundamentals of Optimization

Exercise 2

Remarks

- All questions that are available in the STACK quiz are duly marked. Please solve those using STACK.
- We have added marks for each question. Please note that those are purely for illustrative purposes. The exercise set will not be marked.

STACK Problems

1 Basic Solutions and Basic Feasible Solutions (3 marks)

STACK question

Consider the following polyhedron

$$\mathcal{P} = \{x \in \mathbb{R}^3 : -4x_1 + x_2 - x_3 \leq -4,\ -2x_1 + x_2 - x_3 \leq -2,\ -4x_2 + x_3 \leq 4,\ 4x_2 + x_3 = 0,\ x_3 \leq 5,\ x_3 \geq 0\}.$$

Decide, for each of the points $\hat{x}$ given below, whether $\hat{x}$ is *infeasible and not a basic solution*, *feasible but not a basic feasible solution*, *a basic solution but infeasible*, or *a basic feasible solution*.

(1.1) $\hat{x} = [0, 0, 0]^T$.

(1.2) $\hat{x} = [1, 0, 0]^T$.

(1.3) $\hat{x} = [1, -4/3, -4/3]^T$.

(1.4) $\hat{x} = [3/8, -1/2, 2]^T$.

(1.5) $\hat{x} = [11/16, -1/4, 1]^T$.

2 Graphical Method (2 marks)

STACK question

Consider the following polyhedron:

$$\mathcal{P} = \{[x_1, x_2]^T \in \mathbb{R}^2 : -x_1 + x_2 \leq 2,\ x_1 - 2x_2 \geq -6,\ x_1 + 2x_2 \geq 1,\ x_2 \geq 0\}.$$

Using the graphical method, determine, for each of the following objective functions, the optimal value denoted by z^* (use `+inf` for $+\infty$ and `-inf` for $-\infty$), and whether the set of optimal solutions, denoted by $\mathcal{P}^*$, is either *empty*, a *single vertex*, a *line segment*, a *half line*, or $\mathcal{P}^* = \mathcal{P}$.

(2.1) $\min\{3x_1 : x \in \mathcal{P}\}$.

(2.2) $\max\{-2x_1 - 4x_2 : x \in \mathcal{P}\}$.

(2.3) $\min\{0 : x \in \mathcal{P}\}$.

(2.4) $\min\{2x_1 - 4x_2 : x \in \mathcal{P}\}$.

Open Ended Problems

3 Polyhedra in Standard Form (1 mark)

(3.1) Convert the following general linear programming problem into standard form:

$$
\begin{array}{llllllllll}
\max & -x_1 & + & 3x_2 & - & 5x_3 & + & x_4 & & \\
\text{s.t.} & & & & & & & & & \\
& -x_1 & + & 3x_2 & - & 2x_3 & - & 3x_4 & \geq & 6 \\
& 3x_1 & - & x_2 & + & 5x_3 & - & 2x_4 & = & -3 \\
& -x_1 & - & 3x_2 & + & 2x_3 & - & x_4 & \leq & 2 \\
& x_1 \leq 0 & , & x_2 \geq 0 & , & & & x_4 \geq 0. & &
\end{array}
$$

Remark It is irrelevant whether the problem is actually feasible or not.

4 Polytopes vs Polyhedra (2 marks)

(4.1) Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron. Prove the following result:

$\mathcal{P} \subseteq \mathbb{R}^n$ is a polytope if and only if, for every $c \in \mathbb{R}^n$, each of the two linear programming problems given by

$$\text{(P1)} \quad \min\{c^T x : x \in \mathcal{P}\} \quad \text{and} \quad \text{(P2)} \quad \max\{c^T x : x \in \mathcal{P}\}$$

has a finite optimal value.

5 Existence of Vertices in Polyhedra (2 marks)

(5.1) Either prove the following result or give a counterexample:

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron. If $\mathcal{P}$ has no vertices, then there exists a vector $c \in \mathbb{R}^n$ such that each of the two linear programming problems given by

$$\text{(P1)} \quad \min\{c^T x : x \in \mathcal{P}\} \quad \text{and} \quad \text{(P2)} \quad \max\{c^T x : x \in \mathcal{P}\}$$

is unbounded.

(5.2) Either prove the following result or give a counterexample:

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron. If there exists a vector $c \in \mathbb{R}^n$ such that each of the two linear programming problems given by

$$\text{(P1)} \quad \min\{c^T x : x \in \mathcal{P}\} \quad \text{and} \quad \text{(P2)} \quad \max\{c^T x : x \in \mathcal{P}\}$$

is unbounded, then $\mathcal{P}$ has no vertices.