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THE UNIVERSITY *of* EDINBURGH

Fundamentals of Optimization

Homework 3

Instructions

1. You should attempt all questions.
2. The total marks for this assignment are 10.
3. The assignment consists of STACK questions (5/10 marks) and *open-ended* questions (5/10 marks).
4. All STACK questions are duly marked and are available in the STACK quiz. You **must solve those by completing the STACK quiz**.
5. For the open-ended questions, please write down your solutions in a concise and reproducible way and remember to justify every step using appropriate references when necessary. Failing to do so may result in deductions.
6. The strict deadline for completing the quiz and handing-in your solutions for the open-ended questions is **noon (12:00) on Friday, 11 November 2022**.
7. For the open-ended questions, please upload a **single PDF**. For some useful suggestions, please see Course Information → Tips for Creating a PDF File for Submission on the Learn page.

STACK Problems

1 Basic Solutions of Polyhedra in Standard Form (2 marks)

(1.1) *STACK question*

By using the enumeration algorithm presented in Section 13.2.5 of the lecture notes, determine the set of all basic solutions and basic feasible solutions of the following polyhedron: $\mathcal{P} = \{x \in \mathbb{R}^4 : x_2 + x_3 + x_4 = 1;\ x_1 + x_2 - x_3 + 2x_4 = 1;\ x \geq \mathbf{0}\}$. For each basic solution and basic feasible solution, determine whether it is degenerate or nondegenerate. You can assume that the coefficient matrix A has full row rank.

[1 mark]

(1.2) *STACK question*

By using the enumeration algorithm presented in Section 13.2.5 of the lecture notes, determine the set of all basic solutions and basic feasible solutions of the following polyhedron: $\mathcal{P} = \{x \in \mathbb{R}^3 : x_1 + 3x_2 = 4;\ -x_1 - 3x_2 + 2x_3 = -2;\ x \geq \mathbf{0}\}$. For each basic solution and basic feasible solution, determine whether it is degenerate or nondegenerate. You can assume that the coefficient matrix A has full row rank.

[1 mark]

2 Optimality Conditions and Degeneracy (3 marks)

(2.1) *STACK question*

Consider the following linear program in standard form

$$\min\{-x_1 + 2x_2 + x_3 : 2x_1 - 3x_2 + 2x_3 + x_4 = 1;\ 2x_1 + x_2 + 10x_3 + 2x_4 = 5;\ x \geq \mathbf{0}\}$$

and the vertices

(a) $\hat{x} = [2, 1, 0, 0]^T$.

(b) $\hat{x} = [0, 0, 1/2, 0]^T$,

(c) $\hat{x} = [0, 3/7, 0, 16/7]^T$.

You can assume that the coefficient matrix A has full row rank. For each vertex, decide whether the vertex is optimal or not, and whether it is degenerate or not.

[3 marks]

Write down a valid choice for the index set B, the reduced costs $\bar{c}_j$, $j \in \{1, \ldots, n\}$, for that basis, and a "candidate" improving direction $d \in \mathbb{R}^n$ if one exists. For the latter, if $\bar{c} \not\geq \mathbf{0}$, use $d_{j^*} = 1$ and $d_j = 0$, $j \in N \setminus \{j^*\}$ to derive the direction, where $j^* \in N$ is the index with the smallest reduced cost $\bar{c}_j$. Verify whether the candidate improving direction d is indeed an improving feasible direction at that vertex. If $\bar{c} \geq \mathbf{0}$, then enter $d = \mathbf{0}$.

If the vertex is degenerate, write down all possible choices of the indices for the index set B, together with the corresponding reduced costs $\bar{c}$ and candidate improving directions.

Note that this linear programming problem has a unique (why?) optimal solution which is a nondegenerate vertex.

Open Ended Problems

3 Feasible Directions and Optimality Conditions (2.5 marks)

Consider the following linear programming problem (P) in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Let $\bar{x} \in \mathbb{R}^n$ be an arbitrary feasible solution of (P). As we did in class, let us define the following sets:

$$\begin{aligned} \hat{B} &= \{j \in \{1, \ldots, n\} : \bar{x}_j > 0\}, \\ \hat{N} &= \{j \in \{1, \ldots, n\} : \bar{x}_j = 0\}. \end{aligned}$$

Let $\hat{\mathcal{D}} \subseteq \mathbb{R}^n$ denote the set of all feasible directions at $\bar{x}$.

(3.1) Prove the following:

$$\hat{\mathcal{D}} = \{d \in \mathbb{R}^n : Ad = \mathbf{0}, \quad d_j \geq 0, \quad j \in \hat{N}\}.$$

[1.5 marks]

(3.2) Suppose that $c \in \mathbb{R}^n$ is given by $c_j = 0$ for each $j \in \hat{B}$ and $c_j \geq 0$ for each $j \in \hat{N}$. Prove, relying on (3.1), that $\bar{x}$ is an optimal solution of (P).

[1 mark]

4 Reduced Costs and Optimality Conditions (2.5 marks)

Consider the following linear programming problem in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Let $\bar{x} \in \mathbb{R}^n$ be an arbitrary feasible solution of (P). Let us again define the following sets:

$$\begin{aligned} \hat{B} &= \{j \in \{1, \dots, n\} : \bar{x}_j > 0\}, \\ \hat{N} &= \{j \in \{1, \dots, n\} : \bar{x}_j = 0\}. \end{aligned}$$

Let $\hat{\mathcal{D}} \subseteq \mathbb{R}^n$ denote the set of all feasible directions at $\bar{x}$.

(4.1) Prove the following result: $\bar{x}$ is an optimal solution of (P) if and only if the optimal value of the linear programming problem given by

$$\text{(P2)} \quad \min\{c^T d : d \in \hat{\mathcal{D}}\}$$

is equal to zero.

[1 mark]

(4.2) Assume that A has full row rank. Suppose now that $\bar{x}$ is a nondegenerate vertex with the corresponding index sets B and N. Suppose that there exists an index $j \in N$ such that

$$\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j < 0.$$

Prove that the linear programming problem (P2) is unbounded below.

[1.5 marks]