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Fundamentals of Optimization

Homework 3 – Solutions

Instructions

1. You should attempt all questions.
2. The total marks for this assignment are 10.
3. The assignment consists of STACK questions (5/10 marks) and *open-ended* questions (5/10 marks).
4. All STACK questions are duly marked and are available in the STACK quiz. You **must solve those by completing the STACK quiz**.
5. For the open-ended questions, please write down your solutions in a concise and reproducible way and remember to justify every step using appropriate references when necessary. Failing to do so may result in deductions.
6. The strict deadline for completing the quiz and handing-in your solutions for the open-ended questions is **noon (12:00) on Friday, 11 November 2022**.
7. For the open-ended questions, please upload a **single PDF**. For some useful suggestions, please see Course Information → Tips for Creating a PDF File for Submission on the Learn page.

STACK Problems

1 Basic Solutions of Polyhedra in Standard Form (2 marks)

(1.1) *STACK question*

By using the enumeration algorithm presented in Section 13.2.5 of the lecture notes, determine the set of all basic solutions and basic feasible solutions of the following polyhedron: $\mathcal{P} = \{x \in \mathbb{R}^4 : x_2 + x_3 + x_4 = 1;\ x_1 + x_2 - x_3 + 2x_4 = 1;\ x \geq \mathbf{0}\}$. For each basic solution and basic feasible solution, determine whether it is degenerate or nondegenerate. You can assume that the coefficient matrix A has full row rank.

[1 mark]

Solution

We have $n = 4$ and $m = 2$. Since A has full row rank, we have $|B| = m = 2$. Moreover,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To solve this question, we consider all possible subsets $B \subset \{1, 2, 3, 4\}$ such that $|B| = 2$, and check whether the conditions for a basic (feasible) solution hold.

- $B = \{1,2\}$: The columns A^1 and A^2 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, the basic variables are $\hat{x}_1 = 0$ and $\hat{x}_2 = 1$, and the nonbasic variables are $\hat{x}_3 = 0$ and $\hat{x}_4 = 0$. Therefore, $\hat{x} = [0,1,0,0]^T$ is a basic feasible solution as $\hat{x} \geq \mathbf{0}$. Note that $\hat{x}$ is degenerate since there is at least one basic variable that is equal to zero (i.e., $\hat{B} \subset B$ or $1 = |\hat{B}| < |B| = 2$).

- $B = \{1,3\}$: The columns A^1 and A^3 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Hence, the basic variables are $\hat{x}_1 = 2$ and $\hat{x}_3 = 1$, and the nonbasic variables are $\hat{x}_2 = 0$ and $\hat{x}_4 = 0$. Therefore, $\hat{x} = [2,0,1,0]^T$ is a basic feasible solution as $\hat{x} \geq \mathbf{0}$. Note that $\hat{x}$ is nondegenerate since both basic variables are different from zero (i.e., $\hat{B} = B$).

- $B = \{1,4\}$: The columns A^1 and A^4 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Hence, the basic variables are $\hat{x}_1 = -1$ and $\hat{x}_4 = 1$, and the nonbasic variables are $\hat{x}_2 = 0$ and $\hat{x}_3 = 0$. Therefore, $\hat{x} = [-1,0,0,1]^T$ is a is a basic solution but not feasible since $\hat{x} \ngeq \mathbf{0}$. Note that $\hat{x}$ is nondegenerate since both basic variables are different from zero (i.e., $\hat{B} = B$).

- $B = \{2,3\}$: The columns A^2 and A^3 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence, the basic variables are $\hat{x}_2 = 1$ and $\hat{x}_3 = 0$, and the nonbasic variables are $\hat{x}_1 = 0$ and $\hat{x}_4 = 0$. Therefore, $\hat{x} = [0,1,0,0]^T$ is a basic feasible solution as $\hat{x} \geq \mathbf{0}$. Note that $\hat{x}$ is degenerate since there is at least one basic variable that is equal to zero (i.e., $\hat{B} \subset B$ or $1 = |\hat{B}| < |B| = 2$). Note that this is the same basic feasible solution as the one given by $B = \{1,2\}$.

- $B = \{2,4\}$: The columns A^2 and A^4 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence, the basic variables are $\hat{x}_2 = 1$ and $\hat{x}_4 = 0$, and the nonbasic variables are $\hat{x}_1 = 0$ and $\hat{x}_3 = 0$. Therefore, $\hat{x} = [0,1,0,0]^T$ is a basic feasible solution as $\hat{x} \geq \mathbf{0}$. Note that $\hat{x}$ is degenerate since there is at least one basic variable that is equal to zero (i.e., $\hat{B} \subset B$ or $1 = |\hat{B}| < |B| = 2$). Note that this is the same basic feasible solution as the one given by $B = \{1,2\}$.

- $B = \{3,4\}$: The columns A^3 and A^4 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}.$$

Hence, the basic variables are $\hat{x}_3 = 1/3$ and $\hat{x}_4 = 2/3$, and the nonbasic variables are $\hat{x}_1 = 0$ and $\hat{x}_2 = 0$. Therefore, $\hat{x} = [0,0,1/3,2/3]^T$ is a basic feasible solution as $\hat{x} \geq \mathbf{0}$. Note that $\hat{x}$ is nondegenerate since both basic variables are different from zero (i.e., $\hat{B} = B$).

Therefore, $\mathcal{P}$ has three basic feasible solutions (i.e., vertices), two of which are nondegenerate and one is degenerate, and one basic solution that is not feasible, which is nondegenerate.

(1.2) *STACK question*

By using the enumeration algorithm presented in Section 13.2.5 of the lecture notes, determine the set of all basic solutions and basic feasible solutions of the following polyhedron: $\mathcal{P} = \{x \in \mathbb{R}^3 : x_1 + 3x_2 = 4; -x_1 - 3x_2 + 2x_3 = -2; x \geq \mathbf{0}\}$. For each basic solution and basic feasible solution, determine whether it is degenerate or nondegenerate. You can assume that the coefficient matrix A has full row rank.

[1 mark]

Solution

We have $n = 3$ and $m = 2$. Since A has full row rank, we have $|B| = m = 2$. Moreover,

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -3 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

To solve this question, we consider all possible subsets $B \subset \{1, 2, 3\}$ such that $|B| = 2$ and check whether the conditions for a basic (feasible) solution hold.

- $B = \{1, 2\}$: The columns A^1 and A^2 are linearly dependent since $(-3)A^1 + A^3 = \mathbf{0} \in \mathbb{R}^2$. Therefore, this choice does not give rise to a basic solution.
- $B = \{1, 3\}$: The columns A^1 and A^3 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

  Hence, the basic variables are $\hat{x}_1 = 4$ and $\hat{x}_3 = 1$, and the only nonbasic variable is $\hat{x}_2 = 0$. Therefore, $\hat{x} = [4, 0, 1]^T$ is a basic feasible solution as $\hat{x} \geq \mathbf{0}$. Note that $\hat{x}$ is nondegenerate since both basic variables are different from zero (i.e., $\hat{B} = B$).
- $B = \{2, 3\}$: The columns A^2 and A^3 are linearly independent and we get

$$(A_B)^{-1} = \begin{bmatrix} 3 & 0 \\ -3 & 2 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix} \quad \text{and} \quad \hat{x}_B = (A_B)^{-1}b = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}.$$

  Hence, the basic variables are $\hat{x}_2 = 4/3$ and $\hat{x}_3 = 1$, and the only nonbasic variable is $\hat{x}_1 = 0$. Therefore, $\hat{x} = [0, 4/3, 1]^T$ is a basic feasible solution as $\hat{x} \geq \mathbf{0}$. Note that $\hat{x}$ is nondegenerate since both basic variables are different from zero (i.e., $\hat{B} = B$).

It follows that $\mathcal{P}$ has two nondegenerate basic feasible solutions (i.e., vertices) and no basic solution that is not feasible.

2 Optimality Conditions and Degeneracy (3 marks)

(2.1) *STACK question*

Consider the following linear program in standard form

$$\min\{-x_1 + 2x_2 + x_3 : 2x_1 - 3x_2 + 2x_3 + x_4 = 1;\ 2x_1 + x_2 + 10x_3 + 2x_4 = 5;\ x \geq \mathbf{0}\}$$

and the vertices

(a) $\hat{x} = [2, 1, 0, 0]^T$.
(b) $\hat{x} = [0, 0, 1/2, 0]^T$,
(c) $\hat{x} = [0, 3/7, 0, 16/7]^T$.

You can assume that the coefficient matrix A has full row rank. For each vertex, decide whether the vertex is optimal or not, and whether it is degenerate or not.

[3 marks]

Write down a valid choice for the index set B, the reduced costs $\bar{c}_j$, $j \in \{1, \ldots, n\}$, for that basis, and a "candidate" improving direction $d \in \mathbb{R}^n$ if one exists. For the latter, if $\bar{c} \ngeq \mathbf{0}$, use $d_{j^*} = 1$ and $d_j = 0$, $j \in N \setminus \{j^*\}$ to derive the direction, where $j^* \in N$ is the index with the smallest reduced cost $\bar{c}_j$. Verify whether the candidate improving direction d is indeed an improving feasible direction at that vertex. If $\bar{c} \geq \mathbf{0}$, then enter $d = \mathbf{0}$.

If the vertex is degenerate, write down all possible choices of the indices for the index set B, together with the corresponding reduced costs $\bar{c}$ and candidate improving directions.

Solution

First observe that $n = 4$ and $m = 2$. Since A has full row rank, $|B| = 2$. Moreover,

$$A = \begin{bmatrix} 2 & -3 & 2 & 1 \\ 2 & 1 & 10 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad c = [-1, 2, 1, 0]^T .$$

(a) We have $\hat{x}_1 > 0$ and $\hat{x}_2 > 0$ and, hence, $\hat{B} = \{1, 2\}$. As $|\hat{B}| = m = 2$, $\hat{x}$ is nondegenerate and we obtain $B = \hat{B}$ and $N = \{3, 4\}$. Furthermore, we obtain

$$(A_B)^{-1} = \begin{bmatrix} 2 & -3 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{8} \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$$

and

$$\begin{aligned} \bar{c}_3 &= c_3 - c_B^T (A_B)^{-1} A^3 = 1 - [-1, 2] \frac{1}{8} \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 10 \end{bmatrix} = 1 \\ \bar{c}_4 &= c_4 - c_B^T (A_B)^{-1} A^4 = 0 - [-1, 2] \frac{1}{8} \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3/8 \end{aligned}$$

Since $\bar{c} \geq \mathbf{0}$, we conclude that $\hat{x}$ is an optimal nondegenerate vertex by Corollary 15.4 and $\hat{d} = \mathbf{0}$.

(b) We only have $\hat{x}_3 > 0$ and, hence, $\hat{B} = \{3\}$. As $|\hat{B}| = 1 < m = 2$, $\hat{x}$ is a degenerate vertex. There are now three possibilities to extend $\hat{B}$ to a basis: $B = \{1, 3\}$, $B = \{2, 3\}$, and $B = \{3, 4\}$.

- $B = \{1, 3\}$ and $N = \{2, 4\}$: We obtain

$$(A_B)^{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 10 \end{bmatrix}^{-1} = \frac{1}{8} \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}$$

and

$$\begin{aligned} \bar{c}_2 &= c_2 - c_B^T (A_B)^{-1} A^2 = 2 - [-1, 1] \frac{1}{8} \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = -1/2 \\ \bar{c}_4 &= c_4 - c_B^T (A_B)^{-1} A^4 = 0 - [-1, 1] \frac{1}{8} \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1/4 \end{aligned}$$

Since $\bar{c} \ngeq \mathbf{0}$ and $\hat{x}$ is a degenerate vertex, we cannot infer any information about the optimality of $\hat{x}$. The candidate improving direction is obtained by setting $d_2 = 1$ and $d_4 = 0$, and

$$d_B = -(A_B)^{-1} A_N d_N = -\frac{1}{8} \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1/2 \end{bmatrix}.$$

Therefore,

$$d \;=\; [2, 1, -1/2, 0]^T \,.$$

Note that d is an improving direction since $c^T d = [-1, 2, 1, 0]^T [2, 1, -1/2, 0] = -1/2 = \bar{c}_2 < 0$. Let us check if d is indeed a feasible direction at $\hat{x}$. Note that $A(\hat{x} + \lambda d) = b$ for any $\lambda \in \mathbb{R}$ since $Ad = \mathbf{0}$. Consider $\hat{x} + \lambda d = [0, 0, 1/2, 0]^T + \lambda[2, 1, -1/2, 0]^T = [2\lambda, \lambda, 1/2 - (1/2)\lambda, 0]^T$. Note that letting $\lambda^* = 1 > 0$, we obtain $\hat{x} + \lambda d \geq \mathbf{0}$ if $\lambda \in [0, \lambda^*]$. Therefore, d is a feasible direction at $\hat{x}$. It follows that we have $d \in \hat{\mathcal{D}}$, i.e., the candidate improving direction is indeed an improving feasible direction at $\hat{x}$. By Proposition 15.1, we conclude that $\hat{x}$ is not an optimal solution.

- $B = \{2, 3\}$ and $N = \{1, 4\}$: We obtain

$$(A_B)^{-1} \;=\; \begin{bmatrix} -3 & 2 \\ 1 & 10 \end{bmatrix}^{-1} = \frac{1}{32} \begin{bmatrix} -10 & 2 \\ 1 & 3 \end{bmatrix}$$

and

$$\begin{aligned} \bar{c}_1 \;&=\; c_1 - c_B^T (A_B)^{-1} A^1 \;=\; -1 - [2, 1] \frac{1}{32} \begin{bmatrix} -10 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \;=\; -1/4 \\ \bar{c}_4 \;&=\; c_4 - c_B^T (A_B)^{-1} A^4 \;=\; 0 - [2, 1] \frac{1}{32} \begin{bmatrix} -10 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;=\; 5/32 \end{aligned}$$

Since $\bar{c} \ngeq \mathbf{0}$ and $\hat{x}$ is a degenerate vertex, we cannot infer any information about the optimality of $\hat{x}$. The candidate improving direction is obtained by setting $d_1 = 1$ and $d_4 = 0$, and

$$d_B \;=\; -(A_B)^{-1} A_N d_N \;=\; -\frac{1}{32} \begin{bmatrix} -10 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} 1/2 \\ -1/4 \end{bmatrix}.$$

Therefore,

$$d \;=\; [1, 1/2, -1/4, 0]^T \,.$$

Note that d is an improving direction since $c^T d = [-1, 2, 1, 0]^T [1, 1/2, -1/4, 0] = -1/4 = \bar{c}_1 < 0$. Let us check if d is indeed a feasible direction at $\hat{x}$. Note that $A(\hat{x} + \lambda d) = b$ for any $\lambda \in \mathbb{R}$ since $Ad = \mathbf{0}$. Consider $\hat{x} + \lambda d = [0, 0, 1/2, 0]^T + \lambda[1, 1/2, -1/4, 0]^T = [\lambda, (1/2)\lambda, 1/2 - (1/4)\lambda, 0]^T$. Note that letting $\lambda^* = 2 > 0$, we obtain $\hat{x} + \lambda d \geq \mathbf{0}$ if $\lambda \in [0, \lambda^*]$. Therefore, d is a feasible direction at $\hat{x}$. It follows that we have $d \in \hat{\mathcal{D}}$, i.e., the candidate improving direction is indeed an improving feasible direction at $\hat{x}$. By Proposition 15.1, we conclude that $\hat{x}$ is not an optimal solution.

- $B = \{3, 4\}$ and $N = \{1, 2\}$: We obtain

$$(A_B)^{-1} \;=\; \begin{bmatrix} 2 & 1 \\ 10 & 2 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} -2 & 1 \\ 10 & -2 \end{bmatrix}$$

and

$$\begin{aligned} \bar{c}_1 \;&=\; c_1 - c_B^T (A_B)^{-1} A^1 \;=\; -1 - [1, 0] \frac{1}{6} \begin{bmatrix} -2 & 1 \\ 10 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \;=\; -2/3 \\ \bar{c}_2 \;&=\; c_2 - c_B^T (A_B)^{-1} A^2 \;=\; 2 - [1, 0] \frac{1}{6} \begin{bmatrix} -2 & 1 \\ 10 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \;=\; 5/6 \end{aligned}$$

Since $\bar{c} \ngeq \mathbf{0}$ and $\hat{x}$ is a degenerate vertex, we cannot infer any information about the optimality of $\hat{x}$. The candidate improving direction is obtained by setting $d_1 = 1$ and $d_2 = 0$, and

$$d_B \;=\; -(A_B)^{-1} A_N d_N \;=\; -\frac{1}{6} \begin{bmatrix} -2 & 1 \\ 10 & -2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} 1/3 \\ -8/3 \end{bmatrix}$$

Therefore,

$$d \;=\; [1, 0, 1/3, -8/3]^T \,.$$

Note that d is an improving direction since $c^T d = [-1, 2, 1, 0]^T [1, 0, 1/3, -8/3] = -2/3 = \bar{c}_1 < 0$. Let us check if d is indeed a feasible direction at $\hat{x}$. Note that $A(\hat{x} + \lambda d) = b$ for any $\lambda \in \mathbb{R}$ since $Ad = \mathbf{0}$. Consider $\hat{x} + \lambda d = [0, 0, 1/2, 0]^T + \lambda[1, 0, 1/3, -8/3]^T = [\lambda, 0, 1/2 + (1/3)\lambda, (-8/3)\lambda]^T$. Note that there does not exist a real number $\lambda^* > 0$ such that $\hat{x} + \lambda d \geq \mathbf{0}$ if $\lambda \in [0, \lambda^*]$ since the fourth component becomes negative whenever $\lambda > 0$. Therefore, d is not a feasible direction at $\hat{x}$. Note that we have $d \in \mathcal{D} \backslash \hat{\mathcal{D}}$. Therefore, with this choice of B and N, we cannot draw any conclusion about the optimality of $\hat{x}$.

(c) We have $\hat{x}_2 > 0$ and $\hat{x}_4 > 0$ and, hence, $\hat{B} = \{2, 4\}$. As $|\hat{B}| = m = 2$, $\hat{x}$ is nondegenerate and we obtain $B = \hat{B}$ and $N = \{1, 3\}$. Furthermore, we obtain

$$(A_B)^{-1} \;=\; \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{7} \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}$$

and

$$\begin{aligned} \bar{c}_1 &= c_1 - c_B^T (A_B)^{-1} A^1 \;=\; -1 - [2, 0] \frac{1}{7} \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \;=\; -3/7 \\ \bar{c}_3 &= c_3 - c_B^T (A_B)^{-1} A^3 \;=\; 1 - [2, 0] \frac{1}{7} \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 10 \end{bmatrix} \;=\; -5/7 \end{aligned}$$

Since $\bar{c} \not\geq \mathbf{0}$ and $\hat{x}$ is a nondegenerate vertex, we can conclude that $\hat{x}$ is not optimal by Proposition 17.1. The candidate improving direction is obtained by setting $d_1 = 0$ and $d_3 = 1$, and

$$d_B \;=\; -(A_B)^{-1} A_N d_N \;=\; -\frac{1}{7} \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 10 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} -6/7 \\ -32/7 \end{bmatrix}$$

Therefore,

$$d \;=\; [0, -6/7, 1, -32/7]^T \,.$$

Note that d is an improving direction since $c^T d = [-1, 2, 1, 0]^T [0, -6/7, 1, -32/7] = -5/7 = \bar{c}_3 < 0$. Let us check if d is indeed a feasible direction at $\hat{x}$. Note that $A(\hat{x} + \lambda d) = b$ for any $\lambda \in \mathbb{R}$ since $Ad = \mathbf{0}$. Consider $\hat{x} + \lambda d = [0, 3/7, 0, 16/7]^T + \lambda[0, -6/7, 1, -32/7]^T = [0, 3/7 - (6/7)\lambda, \lambda, 16/7 - (32/7)\lambda]^T$. Note that letting $\lambda^* = 1/2 > 0$, we obtain $\hat{x} + \lambda d \geq \mathbf{0}$ if $\lambda \in [0, \lambda^*]$. Therefore, d is a feasible direction at $\hat{x}$. Note that we have $d \in \hat{\mathcal{D}}$ since $\mathcal{D} = \hat{\mathcal{D}}$ for a nondegenerate vertex by Section 17.4.1 in the lecture notes.

Note that this linear programming problem has a unique (why?) optimal solution which is a nondegenerate vertex.

Open Ended Problems

3 Feasible Directions and Optimality Conditions (2.5 marks)

Consider the following linear programming problem (P) in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Let $\bar{x} \in \mathbb{R}^n$ be an arbitrary feasible solution of (P). As we did in class, let us define the following sets:

$$\begin{aligned} \hat{B} &= \{j \in \{1, \ldots, n\} : \bar{x}_j > 0\}, \\ \hat{N} &= \{j \in \{1, \ldots, n\} : \bar{x}_j = 0\}. \end{aligned}$$

Let $\hat{\mathcal{D}} \subseteq \mathbb{R}^n$ denote the set of all feasible directions at $\bar{x}$.

(3.1) Prove the following:
$$\hat{\mathcal{D}} = \{d \in \mathbb{R}^n : Ad = \mathbf{0}, \quad d_j \geq 0, \quad j \in \hat{N}\}.$$

[1.5 marks]

Solution

Note that $\bar{x} \in \mathbb{R}^n$ is an arbitrary feasible solution of (P). Since $\bar{x}$ may not be a vertex, we cannot assume that the submatrix $A_{\hat{B}}$ has full column rank or is invertible. Therefore, we need to give a general argument. Since this is a set equality, we need to establish both set inclusions.

$\subseteq$: Let $\hat{d} \in \hat{\mathcal{D}}$. By Definition 15.1, there exists a real number $\lambda^* > 0$ such that $\bar{x} + \lambda^* \hat{d}$ is a feasible solution of (P). It follows that $A(\bar{x} + \lambda^* \hat{d}) = b$, which implies that $A\hat{d} = \mathbf{0}$ since $A\bar{x} = b$ and $\lambda^* > 0$. In addition, for any $j = 1, \ldots, n$, we should have $\bar{x}_j + \lambda^* \hat{d}_j \geq 0$. In particular, for any index $j \in \hat{N}$, we have $\bar{x}_j + \lambda^* \hat{d}_j = 0 + \lambda^* \hat{d}_j \geq 0$, which implies that $\hat{d}_j \geq 0$ for each $j \in \hat{N}$ since $\lambda^* > 0$. It follows that $A\hat{d} = \mathbf{0}, \quad \hat{d}_j \geq 0, \quad j \in \hat{N}$.

$\supseteq$: Conversely, suppose that $\hat{d} \in \mathbb{R}^n$ satisfies $A\hat{d} = \mathbf{0}, \quad \hat{d}_j \geq 0, \quad j \in \hat{N}$. We need to show that $\hat{d} \in \hat{\mathcal{D}}$. First, $A(\bar{x} + \lambda \hat{d}) = A\bar{x} + \lambda A\hat{d} = b + \mathbf{0} = b$ for any $\lambda \in \mathbb{R}$. Second, consider $\bar{x}_j + \lambda \hat{d}_j$, where $j = 1, \ldots, n$. For each $j \in \hat{N}$ and each $\lambda \geq 0$, we have $\bar{x}_j + \lambda \hat{d}_j = 0 + \lambda \hat{d}_j \geq 0$ since $\hat{d}_j \geq 0$. For each $j \in \hat{B}$, we consider two cases: (i) Case 1: If $\hat{d}_j \geq 0$, then we have $\bar{x}_j + \lambda \hat{d}_j > 0$ since $\bar{x}_j > 0$ and $\hat{d}_j \geq 0$. (ii) Case 2: If $\hat{d}_j < 0$, then we have $\bar{x}_j + \lambda \hat{d}_j \geq 0$ if and only if $\lambda \leq (-\bar{x}_j)/\hat{d}_j$. Note that the right-hand side is strictly positive. Therefore, we can pick λ^* to be the smallest of these ratios, which is a positive number since the minimum of a finite number of positive numbers is positive. We conclude that we can always find a real number $\lambda^* > 0$ such that $\bar{x} + \lambda^* \hat{d}$ is a feasible solution of (P). Therefore, $\hat{d}$ is a feasible direction at $\bar{x}$, i.e., $\hat{d} \in \hat{\mathcal{D}}$.

(3.2) Suppose that $c \in \mathbb{R}^n$ is given by $c_j = 0$ for each $j \in \hat{B}$ and $c_j \geq 0$ for each $j \in \hat{N}$. Prove, relying on (3.1), that $\bar{x}$ is an optimal solution of (P).

[1 mark]

Solution

By Proposition 15.1, $\bar{x}$ is an optimal solution of (P) if and only if $c^T d \geq 0$ for each $d \in \hat{\mathcal{D}}$. By (3.1), we have
$$\hat{\mathcal{D}} = \{d \in \mathbb{R}^n : Ad = \mathbf{0}, \quad d_j \geq 0, \quad j \in \hat{N}\}.$$

Therefore, for any $d \in \hat{\mathcal{D}}$, we obtain

$$c^T d = \sum_{j=1}^{n} c_j d_j = \sum_{j \in \hat{B}} \underbrace{c_j}_{=0} d_j + \sum_{j \in \hat{N}} \underbrace{c_j}_{\geq 0} \underbrace{d_j}_{\geq 0} \geq 0,$$

which implies that $\bar{x}$ is an optimal solution of (P) by Proposition 15.1.

4 Reduced Costs and Optimality Conditions (2.5 marks)

Consider the following linear programming problem in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Let $\bar{x} \in \mathbb{R}^n$ be an arbitrary feasible solution of (P). Let us again define the following sets:

$$\begin{aligned}\hat{B} &= \{j \in \{1, \ldots, n\} : \bar{x}_j > 0\}, \\ \hat{N} &= \{j \in \{1, \ldots, n\} : \bar{x}_j = 0\}.\end{aligned}$$

Let $\hat{\mathcal{D}} \subseteq \mathbb{R}^n$ denote the set of all feasible directions at $\bar{x}$.

(4.1) Prove the following result: $\bar{x}$ is an optimal solution of (P) if and only if the optimal value of the linear programming problem given by

$$\text{(P2)} \quad \min\{c^T d : d \in \hat{\mathcal{D}}\}$$

is equal to zero.

[1 mark]

Solution

Note that (P2) is indeed a linear programming problem since $\hat{D}$ is a polyhedron by (3.1). Since this is an if-and-only-if statement, we need to prove both implications:

$\Rightarrow$: Suppose that $\bar{x}$ is an optimal solution of (P). Then, by Proposition 15.1, $c^T d \geq 0$ for each $d \in \hat{\mathcal{D}}$. Therefore, zero is a lower bound on the optimal value of (P2). On the other hand, by (3.1), $\hat{d} = \mathbf{0} \in \mathbb{R}^n$ clearly belongs to $\hat{\mathcal{D}}$. This implies that $c^T \hat{d} = 0$, i.e., there is a feasible solution that achieves this lower bound. It follows that the optimal value of (P2) is equal to zero.

$\Leftarrow$: Conversely, suppose that the optimal value of (P2) is equal to zero. Then, we have $c^T d \geq 0$ for each $d \in \hat{\mathcal{D}}$ by the definition of optimal value. Then, by Proposition 15.1, we conclude that $\bar{x}$ is an optimal solution of (P).

(4.2) Assume that A has full row rank. Suppose now that $\bar{x}$ is a nondegenerate vertex with the corresponding index sets B and N. Suppose that there exists an index $j \in N$ such that

$$\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j < 0.$$

Prove that the linear programming problem (P2) is unbounded below.

[1.5 marks]

Solution

Assume that A has full row rank. Suppose now that $\bar{x}$ is a nondegenerate vertex with the corresponding index sets B and N. Note that we have $\hat{B} = B$ and $\hat{N} = N$ due to nondegeneracy by Definition 16.2.

Suppose that there exists an index $j^* \in N$ such that

$$\bar{c}_{j^*} = c_{j^*} - c_B^T (A_B)^{-1} A^{j^*} < 0.$$

Let us construct a feasible improving direction $\hat{d} \in \mathbb{R}^n$ as follows. Define $\hat{d}_{j^*} = 1$ and $\hat{d}_j = 0$ for each $j \in N \backslash \{j^*\}$. Then, $\hat{d}_j \geq 0$ for each $j \in N$ (or, equivalently $j \in \hat{N}$). By (3.1), $\hat{d} \in \hat{\mathcal{D}}$ if and only if $A\hat{d} = A_{\hat{B}} \hat{d}_{\hat{B}} + A_{\hat{N}} \hat{d}_{\hat{N}} = A_B \hat{d}_B + A_N \hat{d}_N = \mathbf{0}$. Since A_B is an invertible matrix by Proposition 13.2, we obtain $\hat{d}_B = -(A_B)^{-1} A_N \hat{d}_N$. Note that $\hat{d} \neq \mathbf{0}$ since $\hat{d}_{j^*} = 1$. Furthermore,

$$c^T \hat{d} = c_B^T \hat{d}_B + c_N^T \hat{d}_N = (c_N^T - c_B^T (A_B)^{-1} A_N) \hat{d}_N = \sum_{j \in N} \bar{c}_j \hat{d}_j = \bar{c}_{j^*} < 0.$$

We have obtained a feasible solution of (P2) with a negative objective function value. Finally, we claim that $\alpha \hat{d}$ is a feasible solution of (P2) for any real number $\alpha \geq 0$. To see this, note that $A(\alpha\hat{d}) = \alpha A\hat{d} = \mathbf{0}$ and $\alpha\hat{d}_j \geq 0$ for each $j \in N$ since $\alpha \geq 0$ and $\hat{d}_j \geq 0$ for each $j \in N$. This proves our claim. Finally, if we evaluate the objective function at $\alpha\hat{d}$, we obtain $c^T(\alpha\hat{d}) = \alpha(c^T\hat{d}) = \alpha\bar{c}_{j^*} \to -\infty$ as $\alpha \to \infty$ since $\bar{c}_{j^*} < 0$. This implies that the linear programming problem (P2) is unbounded.