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Fundamentals of Optimization

Homework 1 – Solutions

Instructions

1. You should attempt all questions.
2. The total marks for this assignment are 10.
3. The assignment consists of STACK questions (5/10 marks) and *open-ended* questions (5/10 marks).
4. All STACK questions are duly marked and are available in the STACK quiz. You **must solve those by completing the STACK quiz**.
5. For the open-ended questions, please write down your solutions in a concise and reproducible way and remember to justify every step using appropriate references when necessary. Failing to do so may result in deductions.
6. The strict deadline for completing the quiz and handing-in your solutions for the open-ended questions is **noon (12:00) on Friday, 14 October 2022**.
7. For the open-ended questions, please upload a **single PDF**. For some useful suggestions, please see Course Information → Tips for Creating a PDF File for Submission on the Learn page.

STACK Problems

1 Basic Concepts (3 marks)

STACK question

Decide, for each of the following three optimization problems, whether

(i) the feasible region is *empty*; or *nonempty and bounded*; or *nonempty and unbounded*;

(ii) the feasible region is a *convex set*; or a *nonconvex set*;

(iii) the objective function is *a convex function*; *a concave function*; *both convex and concave*; or *neither convex nor concave*;

(iv) the optimization problem is a *convex optimization problem*; or a *nonconvex optimization problem*;

(v) the optimization problem *is infeasible*, *is unbounded*, or *has a finite optimal value*;

(vi) write down the optimal value using the convention in the lectures (use `+inf` for $+\infty$ and `-inf` for $-\infty$);

(vii) the set of optimal solutions is *empty*; or *nonempty*;

(viii) the set of optimal solutions is a *convex set*; or a *nonconvex set*.

(1.1) $\min\{x^3 - 2x^2 + x - 2 : x^2 - 2x - 8 \geq 0, \quad x \in \mathbb{R}\}$.

(1.2) $\min\{2x^2 - 12x - 6 : x^2 - 6x \geq -5, \quad x \in \mathbb{R}\}$.

[3 marks]

Solution

(1.1) Note that $x^2 - 2x - 8 \geq 0$ if and only if $(x-1)^2 - 9 \geq 0$ if and only if $|x - 1| \geq 3$ if and only if $x \in (-\infty, -2] \cup [4, \infty)$. The feasible region is therefore given by $\mathcal{S} = (-\infty, -2] \cup [4, \infty)$. The feasible region is nonempty and unbounded since there does not exist any finite number $K \in \mathbb{R}$ such that $\mathcal{S} \subseteq [-K, K]$. $\mathcal{S}$ is a nonconvex set since $-2 \in \mathcal{S}$, $4 \in \mathcal{S}$ but $(1/2)(-2) + (1/2)(4) = 1 \notin \mathcal{S}$. The objective function is given by $f(x) = x^3 - 2x^2 + x - 2$. Let $x = 1$, $y = -1$, and $\lambda = 1/2$. Then,

$$f(\lambda x + (1-\lambda)y) = f(0) = -2 > \lambda f(x) + (1-\lambda) f(y) = (1/2)(-2) + (1/2)(-6) = -4,$$

which implies that f is not a convex function. Similarly, if $x = 0$, $y = 2$, and $\lambda = 1/2$. Then,

$$f(\lambda x + (1-\lambda)y) = f(1) = -2 < \lambda f(x) + (1-\lambda) f(y) = (1/2)(-2) + (1/2)(0) = -1,$$

which implies that f is not a concave function. Therefore, f is neither convex nor concave. Since f is not a convex function, the optimization problem is a nonconvex optimization problem. By computing the first derivative of the objective function given by

$$f'(x) = 3x^2 - 4x + 1,$$

you can easily see that f is strictly increasing on $(-\infty, -2]$. Therefore, define a sequence of feasible solutions given by $x^k = -1 - k \in \mathcal{S}$, $k = 1, 2, \ldots$. Then, $f(x^k) \to -\infty$ as $k \to \infty$. Therefore, the optimization problem is unbounded and the optimal value is given by $z^* = -\infty$. In this example, no feasible solution attains the optimal value, i.e., $\mathcal{S}^* = \emptyset$. Therefore, the set of optimal solutions is empty. Finally, $\mathcal{S}^*$ is a convex set by Remark 1 in Section 3.2 in the lecture notes.

(1.2) Note that $x^2 - 6x \geq -5$ if and only if $(x-3)^2 - 4 \geq 0$ if and only if $|x - 3| \geq 2$ if and only if $x \in (-\infty, 1] \cup [5, \infty)$. The feasible region is therefore given by $\mathcal{S} = (-\infty, 1] \cup [5, \infty)$. The feasible region is nonempty and unbounded since there does not exist any finite number $K \in \mathbb{R}$ such that $\mathcal{S} \subseteq [-K, K]$. $\mathcal{S}$ is a nonconvex set since $1 \in \mathcal{S}$, $5 \in \mathcal{S}$ but $(1/2)(1) + (1/2)(5) = 3 \notin \mathcal{S}$. The objective function is given by $f(x) = 2x^2 - 12x - 6$. We claim that f is a convex function. Let $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $\lambda \in [0, 1]$. Then,

$$\begin{aligned}
f(\lambda x + (1-\lambda)y) &= 2(\lambda x + (1-\lambda)y)^2 - 12(\lambda x + (1-\lambda)y)) - 6 \\
&= \lambda(2x^2 - 12x - 6) + (1-\lambda)(2y^2 - 12y - 6) + (\lambda^2 - \lambda)(2x^2 - 4xy + 2y^2) \\
&= \lambda f(x) + (1-\lambda) f(y) - 2\lambda(1-\lambda)(x-y)^2 \\
&\leq \lambda f(x) + (1-\lambda) f(y),
\end{aligned}$$

where we used $\lambda \in [0, 1]$ and $(x - y)^2 \geq 0$ to derive the inequality in the last line. It follows that f is a convex function. To determine if this is a convex optimization problem, we need to first check if f is a convex function, which we just established. In addition, we need to check the constraints, i.e., we need to check whether the function on the left-hand side of the single $\geq$-type constraint is a concave function. We claim that $g(x) = x^2 - 6x$ is not a concave function. To see this, let $x = 0$, $y = 6$, and $\lambda = 1/2$. Then,

$$g(\lambda x + (1-\lambda)y) = g(3) = -9 < \lambda g(x) + (1-\lambda) g(y) = (1/2)(0) + (1/2)(0) = 0,$$

which implies that g is not a concave function. Since g is not a concave function, the optimization problem is a nonconvex optimization problem. By computing the first derivative of the objective function given by

$$f'(x) = 4x - 12,$$

you can easily see that f is strictly increasing on $(3, +\infty)$ and strictly decreasing on $(-\infty, 3)$. Therefore, the best feasible solution is given by $\min\{f(1), f(5)\} = -16$. Therefore, the optimal value is given by $z^* = -16$. Note that this value is attained by each of $x^1 = 1$ and $x^2 = 5$. Therefore, $\mathcal{S}^* = \{1, 5\}$. Therefore, the set of optimal solutions is nonempty. Finally, $\mathcal{S}^*$ is a nonconvex set since $1 \in \mathcal{S}^*$, $5 \in \mathcal{S}^*$, but for $\lambda = 1/2$, $\lambda x + (1-\lambda)y = 3 \notin \mathcal{S}^*$.

2 Level Sets, Sublevel Sets, Superlevel Sets, and Epigraphs (2 marks)

STACK question

Decide, for each of the two functions,

(i) whether $\mathrm{epi}(f)$ is a *convex set* or *nonconvex set*;

(ii) whether the sublevel set $\mathcal{L}_\alpha^-(f)$, where $\alpha = 0$, is a *convex set* or *nonconvex set*;

(iii) whether the level set $\mathcal{L}_\alpha(f)$, where $\alpha = 1$, is a *convex set* or *nonconvex set*;

(iv) whether the superlevel set $\mathcal{L}_\alpha^+(f)$, where $\alpha = 1$, is a *convex set* or *nonconvex set.*

(2.1) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = \min\{|x_1|, |x_2|\}$.

(2.2) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x_1^2 + x_2^2$.

[2 marks]

Solution

(2.1) We have

$$\mathrm{epi}(f) = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : z \geq \min\{|x_1|, |x_2|\}\}.$$

We claim that $\mathrm{epi}(f)$ is a nonconvex set. To see this, let $(x, z_1) = ([0, 1]^T, 0) \in \mathrm{epi}(f)$, $(y, z_2) = ([1, 0]^T, 0) \in \mathrm{epi}(f)$, but for $\lambda = 1/2$, we have $\lambda(x, z_1)+(1-\lambda)(y, z_2) = ([1/2, 1/2]^T, 0) \notin \mathrm{epi}(f)$, which implies that $\mathrm{epi}(f)$ is a nonconvex set. Note that this implies that f is a nonconvex function by Proposition 3.1.

Note that $f(x) \geq 0$ for each $x \in \mathbb{R}^2$, which implies that $\mathcal{L}_\alpha^-(f) = \emptyset$ for each $\alpha < 0$, which is a convex set by Remark 1 in Section 3.2 in the lecture notes. For each $\alpha \geq 0$, $x \in \mathcal{L}_\alpha^-(f)$ if and only if $\min\{|x_1|, |x_2|\} \leq \alpha$ if and only if $|x_1| \leq \alpha$ or $|x_2| \leq \alpha$. Therefore, for $\alpha \geq 0$, we obtain

$$\mathcal{L}_\alpha^-(f) = \{x \in \mathbb{R}^2 : x_1 \in [-\alpha, \alpha], x_2 \in \mathbb{R}\} \cup \{x \in \mathbb{R}^2 : x_1 \in \mathbb{R}, x_2 \in [-\alpha, \alpha]\}.$$

Therefore, for $\alpha = 0$, we get

$$\mathcal{L}_\alpha^-(f) = \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \in \mathbb{R}\} \cup \{x \in \mathbb{R}^2 : x_1 \in \mathbb{R}, x_2 = 0\},$$

which is the union of x_1- and x_2-axes. You can easily see that this is a nonconvex set since $x = [0, 1]^T \in \mathcal{L}_\alpha^-(f)$, $y = [1, 0]^T \in \mathcal{L}_\alpha^-(f)$, but for $\lambda = 1/2$, we have $\lambda x + (1-\lambda)z_2 = [1/2, 1/2]^T \notin \mathcal{L}_\alpha^-(f)$. Similarly, you can show that $\mathcal{L}_\alpha^-(f)$ is a nonconvex set for each $\alpha > 0$.

Similarly, $\mathcal{L}_\alpha(f) = \emptyset$ for each $\alpha < 0$, which is a convex set by Remark 1 in Section 3.2 in the lecture notes. For $\alpha \geq 0$, note that $f(x) = \alpha$ if and only if $|x_1| = \alpha$ and $|x_2| \geq \alpha$, or $|x_2| = \alpha$ and $|x_1| \geq \alpha$. Therefore, for $\alpha \geq 0$, we obtain

$$\begin{aligned}\mathcal{L}_\alpha(f) = &\{x \in \mathbb{R}^2 : x_1 \in \{-\alpha, \alpha\}, x_2 \in (-\infty, -\alpha] \cup [\alpha, \infty)\} \\ &\cup \{x \in \mathbb{R}^2 : x_1 \in (-\infty, -\alpha] \cup [\alpha, \infty), x_2 \in \{-\alpha, \alpha\}\}.\end{aligned}$$

Therefore, for $\alpha = 1$, we get

$$\begin{aligned}\mathcal{L}_\alpha(f) = &\{x \in \mathbb{R}^2 : x_1 \in \{-1, 1\}, x_2 \in (-\infty, -1] \cup [1, \infty)\} \\ &\cup \{x \in \mathbb{R}^2 : x_1 \in (-\infty, -1] \cup [1, \infty), x_2 \in \{-1, 1\}\},\end{aligned}$$

which is a nonconvex set since $x = [1, 1] \in \mathcal{L}_\alpha(f)$, $y = [-1, -1]^T \in \mathcal{L}_\alpha(f)$, but for $\lambda = 1/2$, we have $\lambda x + (1 - \lambda)z_2 = [0, 0]^T \notin \mathcal{L}_\alpha(f)$. Similarly, you can show that $\mathcal{L}_\alpha(f)$ is a nonconvex set for each $\alpha > 0$.

Finally, we obtain $\mathcal{L}^+_\alpha(f) = \mathbb{R}^2$ for each $\alpha \leq 0$ since $f(x) \geq 0$ for each $x \in \mathbb{R}^2$, which is obviously a convex set. For each $\alpha > 0$, $x \in \mathcal{L}^+_\alpha(f)$ if and only if $\min\{|x_1|, |x_2|\} \geq \alpha$ if and only if $|x_1| \geq \alpha$ and $|x_2| \geq \alpha$. Therefore, for $\alpha \geq 0$, we obtain

$$\mathcal{L}^+_\alpha(f) = \{x \in \mathbb{R}^2 : x_1 \in (-\infty, -\alpha] \cup [\alpha, \infty), x_2 \in (-\infty, -\alpha] \cup [\alpha, \infty)\}.$$

Therefore, for $\alpha = 1$, we get

$$\mathcal{L}^+_\alpha(f) = \{x \in \mathbb{R}^2 : |x_1| \geq 1\} \cap \{x \in \mathbb{R}^2 : |x_2| \geq 1\}.$$

This is a nonconvex set since $[1, 1]^T \in \mathcal{L}^+_\alpha(f)$ and $[-1, -1]^T \in \mathcal{L}^+_\alpha(f)$ but the midpoint $[0, 0]^T \notin \mathcal{L}^+_\alpha(f)$. Similarly, you can show that $\mathcal{L}^+_\alpha(f)$ is a nonconvex set for each $\alpha > 0$.

(2.2) We have

$$\text{epi}(f) = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : z \geq x_1^2 + x_2^2\}.$$

Let $(x, z_1) \in \text{epi}(f)$, $(y, z_2) \in \text{epi}(f)$, and let $\lambda \in [0, 1]$. We need to show that $\lambda(x, z_1) + (1 - \lambda)(y, z_2) = (\lambda x + (1 - \lambda)y, \lambda z_1 + (1 - \lambda)z_2) \in \text{epi}(f)$, i.e.,

$$\lambda z_1 + (1 - \lambda)z_2 \geq (\lambda x_1 + (1 - \lambda)y_1)^2 + (\lambda x_2 + (1 - \lambda)y_2)^2. \tag{1}$$

Since $(x, z_1) \in \text{epi}(f)$ and $(y, z_2) \in \text{epi}(f)$, we have

$$z_1 \geq x_1^2 + x_2^2, \quad z_2 \geq y_1^2 + y_2^2.$$

Since $\lambda \in [0, 1]$, by multiplying the first inequality by $\lambda \geq 0$ and the second one by $1 - \lambda \geq 0$, we obtain

$$\begin{aligned}\lambda z_1 + (1 - \lambda)z_2 &\geq \lambda(x_1^2 + x_2^2) + (1 - \lambda)(y_1^2 + y_2^2) \\ &= \lambda^2 x_1^2 + 2\lambda(1 - \lambda)x_1 y_1 + (1 - \lambda)^2 y_1^2 + \lambda(1 - \lambda)(x_1^2 - 2x_1 y_1 + y_1)^2 \\ &\quad + \lambda^2 x_2^2 + 2\lambda(1 - \lambda)x_2 y_2 + (1 - \lambda)^2 y_2^2 + \lambda(1 - \lambda)(x_2^2 - 2x_2 y_2 + y_2)^2 \\ &= (\lambda x_1 + (1 - \lambda)y_1)^2 + (\lambda x_2 + (1 - \lambda)y_2)^2 \\ &\quad + \lambda(1 - \lambda)[(x_1 - y_1)^2 + (x_2 - y_2)^2] \\ &\geq (\lambda x_1 + (1 - \lambda)y_1)^2 + (\lambda x_2 + (1 - \lambda)y_2)^2,\end{aligned}$$

where we used $\lambda \in [0, 1]$, $(x_1 - y_1)^2 + (x_2 - y_2)^2 \geq 0$ to derive the last inequality. This proves (1). It follows that epi(f) is a convex set. Note that this implies that f is a convex function by Proposition 3.1.

Note that $f(x) \geq 0$ for each $x \in \mathbb{R}^2$, which implies that $\mathcal{L}^-_\alpha(f) = \emptyset$ for each $\alpha < 0$, which is a convex set by Remark 1 in Section 3.2 in the lecture notes. For $\alpha \geq 0$, note that $f(x) \leq \alpha$ if and only if $x_1^2 + x_2^2 \leq \alpha$. If $x_2 = \beta$, then we obtain $x_1^2 \leq \alpha - \beta^2$, i.e., $|x_1| \leq \sqrt{\alpha - \beta^2}$. Therefore, for $\alpha \geq 0$,

$$\mathcal{L}^-_\alpha(f) = \bigcup_{\beta \in [-\sqrt{\alpha}, \sqrt{\alpha}]} \left\{[x_1, \beta]^T : x_1 \in [-\sqrt{\alpha - \beta^2}, \sqrt{\alpha - \beta^2}]\right\}.$$

For each $\alpha \geq 0$, we obtain all the points in the interior and on the boundary of the circle of radius $\sqrt{\alpha}$ centred at the origin. You can easily verify that this is a convex set.

Therefore, for $\alpha = 0$, we simply get $\mathcal{L}^-_\alpha(f) = [0, 0]^T$, which is obviously a convex set since it is a singleton.

Similarly, $f(x) \geq 0$ for each $x \in \mathbb{R}^2$, which implies that $\mathcal{L}_\alpha(f) = \emptyset$ for each $\alpha < 0$, which is a convex set by Remark 1 in Section 3.2 in the lecture notes. For $\alpha \geq 0$, note that $f(x) = \alpha$ if and only if $x_1^2 + x_2^2 = \alpha$. If $x_2 = \beta$, then we obtain $x_1^2 = \alpha - \beta^2$, i.e., $|x_1| = \sqrt{\alpha - \beta^2}$. Therefore, for $\alpha \geq 0$,

$$\mathcal{L}_\alpha(f) = \bigcup_{\beta \in [-\sqrt{\alpha}, \sqrt{\alpha}]} \left\{ [x_1, \beta]^T : x_1 \in \{-\sqrt{\alpha - \beta^2}, \sqrt{\alpha - \beta^2}\} \right\}.$$

For each $\alpha \geq 0$, we actually obtain the boundary of the circle of radius $\sqrt{\alpha}$ centred at the origin. For $\alpha = 1$, we get

$$\mathcal{L}_\alpha(f) = \bigcup_{\beta \in [-1,1]} \left\{ [x_1, \beta]^T : x_1 \in \{-\sqrt{1 - \beta^2}, \sqrt{1 - \beta^2}\} \right\},$$

which is clearly a nonconvex set since $[0,1]^T \in \mathcal{L}_\alpha(f)$ and $[0,-1]^T \in \mathcal{L}_\alpha(f)$, but the midpoint $[0,0] \notin \mathcal{L}_\alpha(f)$. You can easily verify that $\mathcal{L}_\alpha(f)$ is a nonconvex set for each $\alpha > 0$.

Finally, we obtain $\mathcal{L}_\alpha^+(f) = \mathbb{R}^2$ for each $\alpha \geq 0$ since $f(x) \geq 0$ for each $x \in \mathbb{R}^2$, which is obviously a convex set. For each $\alpha > 0$, $x \in \mathcal{L}_\alpha^+(f)$ if and only if $x_1^2 + x_2^2 \geq \alpha$. Using a similar argument, if $x_2 = \beta$, then $x_1^2 \geq \alpha - \beta^2$. If $\beta \in [-\sqrt{\alpha}, \sqrt{\alpha}]$, then we have $|x_1| \geq \sqrt{\alpha - \beta^2}$. Otherwise, if $|\beta| > \sqrt{\alpha}$, then $\alpha - \beta^2 < 0$, which implies that $x_1 \in \mathbb{R}$. Therefore, for each $\alpha > 0$, we get

$$\mathcal{L}_\alpha^+(f) = \left(\bigcup_{\beta \in [-\sqrt{\alpha}, \sqrt{\alpha}]} \left\{ [x_1, \beta]^T : x_1 \in (-\infty, -\sqrt{\alpha - \beta^2}] \cup [\sqrt{\alpha - \beta^2}, \infty) \right\} \right) \cup \left(\bigcup_{|\beta| > \sqrt{\alpha}} \left\{ [x_1, \beta]^T : x_1 \in \mathbb{R} \right\} \right).$$

For each $\alpha > 0$, we obtain all the points outside of the circle of radius $\sqrt{\alpha}$ centred at the origin, including the points on the boundary of the circle. For each $\alpha > 0$, this is a nonconvex set since $[-\sqrt{\alpha}, 0]^T \in \mathcal{L}_\alpha^+(f)$ and $[\sqrt{\alpha}, 0]^T \in \mathcal{L}_\alpha^+(f)$ but the midpoint $[0,0]^T \notin \mathcal{L}_\alpha^+(f)$. For $\alpha = 1$, we get

$$\mathcal{L}_\alpha^+(f) = \left(\bigcup_{\beta \in [-1,1]} \left\{ [x_1, \beta]^T : x_1 \in (-\infty, -\sqrt{1 - \beta^2}] \cup [\sqrt{1 - \beta^2}, \infty) \right\} \right) \cup \left(\bigcup_{|\beta| > 1} \left\{ [x_1, \beta]^T : x_1 \in \mathbb{R} \right\} \right),$$

which again is a nonconvex set.

Open Ended Problems

3 Level Sets and Sublevel Sets (2.5 marks)

Consider the following optimization problem:

$$\text{(P)} \quad \min_x \{ f(x) : x \in \mathcal{S} \},$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$, and $\mathcal{S} \subseteq \mathbb{R}^n$. Suppose that the optimal value of (P) is denoted by $z^* \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

(3.1) Prove the following proposition:

(P) is an unbounded optimization problem if and only if

$$\mathcal{S} \cap \mathcal{L}_\alpha^-(f) \neq \emptyset, \quad \text{for all } \alpha \in \mathbb{R},$$

where $\mathcal{L}_\alpha^-(f)$ denotes the sublevel set of f at level $\alpha \in \mathbb{R}$,

[1.5 marks]

Solution

Since this is an if only if statement, we need to establish both implications.

$\Rightarrow$: Suppose that (P) is an unbounded optimization problem. Then, there exists a sequence $x^k \in \mathcal{S},\ k = 1, 2, \ldots$ of feasible solutions such that $\lim\limits_{k\to\infty} f(x^k) = -\infty$. Therefore, for every $\alpha \in \mathbb{R}$, there exists some positive integer $k^* \in \mathbb{Z}$ such that $f(x^{k^*}) \leq \alpha$ (i.e., for every real number α, we can find a feasible solution whose objective function value is less than or equal to α since (P) is an unbounded optimization problem). It follows that $x^{k^*} \in \mathcal{S}$ and $x^{k^*} \in \mathcal{L}_\alpha^-(f)$, which implies that $\mathcal{S} \cap \mathcal{L}_\alpha^-(f) \neq \emptyset$.

$\Leftarrow$: Suppose that

$$\mathcal{S} \cap \mathcal{L}_\alpha^-(f) \neq \emptyset, \quad \text{for all } \alpha \in \mathbb{R}.$$

Choose $\alpha = -1 \in \mathbb{R}$. Since $\mathcal{S} \cap \mathcal{L}_\alpha^-(f) \neq \emptyset$, there exists $x^1 \in \mathcal{S}$ such that $f(x^1) \leq -1$. Now choose $\alpha = -2 \in \mathbb{R}$. By a similar reasoning, there exists $x^2 \in \mathcal{S}$ such that $f(x^2) \leq -2$. Repeating this process for $\alpha \in \{-3, -4, \ldots\}$, we obtain a sequence $x^k \in \mathcal{S},\ k = 1, 2, \ldots$ of feasible solutions such that

$$f(x^k) \leq -k, \quad k = 1, 2, \ldots.$$

It follows that $\lim\limits_{k\to\infty} f(x^k) = -\infty$, which implies that (P) is an unbounded optimization problem.

(3.2) Suppose that $z^* \in \mathbb{R}$ (i.e., the optimal value is finite). Let $\mathcal{S}^* \subseteq \mathbb{R}^n$ denote the set of optimal solutions of (P). Prove the following identity:

$$\mathcal{S}^* = \mathcal{L}_{z^*}(f) \cap \mathcal{S},$$

where $\mathcal{L}_{z^*}(f)$ denotes the level set of f for $\alpha = z^*$. (Hint: One way of showing that the two sets are equal is to show that each set is a subset of the other one as done in Problem 4.1 in Exercise Set 0.) **[1 marks]**

Solution

Following the given hint, we will try to show that each set is a subset of the other one:

$\mathcal{S}^* \subseteq \mathcal{L}_{z^*}(f) \cap \mathcal{S}$: If $\mathcal{S}^* = \emptyset$, then this is clearly true. Otherwise, let $\hat{x} \in \mathcal{S}^*$. Then, $\hat{x} \in \mathcal{S}$ since $\mathcal{S}^* \subseteq \mathcal{S}$ and $f(\hat{x}) = z^*$, i.e., $\hat{x} \in \mathcal{L}_{z^*}(f)$ by definition of the optimal value since $\hat{x}$ is an optimal solution. Therefore, $\hat{x} \in \mathcal{L}_{z^*}(f) \cap \mathcal{S}$.

$\mathcal{L}_{z^*}(f) \cap \mathcal{S} \subseteq \mathcal{S}^*$: If $\mathcal{L}_{z^*}(f) \cap \mathcal{S} = \emptyset$, then this is clearly true. Otherwise, let $\hat{x} \in \mathcal{L}_{z^*}(f) \cap \mathcal{S}$. It follows that $\hat{x} \in \mathcal{S}$ and $f(\hat{x}) = z^*$. Since z^* denotes the optimal value, we have $f(\bar{x}) \geq z^*$ for each $\bar{x} \in \mathcal{S}$. Since $f(\hat{x}) = z^*$, it follows that $\hat{x} \in \mathcal{S}^*$.

4 Vertices of Convex Sets (2.5 marks)

Let $\mathcal{C}_1 \subseteq \mathbb{R}^n$ and $\mathcal{C}_2 \subseteq \mathbb{R}^n$ be two nonempty convex sets and let $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$. Suppose that $\mathcal{C} \neq \emptyset$.

(4.1) Prove the following result:

If $\hat{x} \in \mathcal{C}$ and $\hat{x}$ is a vertex of at least one of $\mathcal{C}_1$ and $\mathcal{C}_2$, then $\hat{x}$ is a vertex of $\mathcal{C}$.

[1.5 marks]

Solution

Note that $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$ is a convex set since convexity is preserved under taking intersections by Remark 3 in Section 3.2. Suppose that $\hat{x} \in \mathcal{C}$ and $\hat{x}$ is a vertex of at least one of $\mathcal{C}_1$ and $\mathcal{C}_2$. Without loss of generality, suppose that $\hat{x}$ is a vertex of $\mathcal{C}_1$. Then, there exists a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : a^T x = \alpha\}$, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$, and a halfspace $\mathcal{H}^+ = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}$ such that

(i) $\mathcal{C}_1 \cap \mathcal{H} = \{\hat{x}\}$, and

(ii) $\mathcal{C}_1 \subseteq \mathcal{H}^+$.

Since $\hat{x} \in \mathcal{C}$, $\hat{x} \in \mathcal{H}$ and $\mathcal{C} \cap \mathcal{H} \subseteq \mathcal{C}_1 \cap \mathcal{H} = \{\hat{x}\}$, we obtain $\mathcal{C} \cap \mathcal{H} = \{\hat{x}\}$. Similarly, since $\mathcal{C} \subseteq \mathcal{C}_1 \subseteq \mathcal{H}^+$, we obtain $\mathcal{C} \subseteq \mathcal{H}^+$. It follows that there exists a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : a^T x = \alpha\}$, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$, and a halfspace $\mathcal{H}^+ = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}$ such that (a) $\mathcal{C} \cap \mathcal{H} = \{\hat{x}\}$, and (b) $\mathcal{C} \subseteq \mathcal{H}^+$. By (a) and (b), we conclude that $\hat{x}$ is a vertex of $\mathcal{C}$.

(4.2) Consider the following proposition, which is the converse of the proposition in (4.1):

> If $\hat{x} \in \mathcal{C}$ is a vertex of $\mathcal{C}$, then $\hat{x}$ is a vertex of at least one of $\mathcal{C}_1$ and $\mathcal{C}_2$.

Either prove this proposition or find a counterexample.

[1 mark]

Solution

The proposition is not true as we can find many counterexamples. Suppose, for instance, that

$$\mathcal{C}_1 = \{x \in \mathbb{R}^2 : x_1 \geq 0\}, \quad \mathcal{C}_2 = \{x \in \mathbb{R}^2 : x_2 \geq 0\}.$$

Clearly, $\mathcal{C}_1 \subset \mathbb{R}^2$ and $\mathcal{C}_2 \subset \mathbb{R}^2$ are both convex sets since each of them is a halfspace in $\mathbb{R}^2$ and halfspaces are convex sets by Corollary 4.7. We claim that neither set has a vertex. To see this, suppose, for a contradiction, that $\hat{x} \in \mathcal{C}_1$ is a vertex of $\mathcal{C}_1$. Then, $\hat{x}_1 \geq 0$ and $\hat{x}_2 \in \mathbb{R}$. Let $d = [0, 1]^T \in \mathbb{R}^2$. Then, it easy to see that $\hat{x} - d \in \mathcal{C}_1$ and $\hat{x} + d \in \mathcal{C}_1$. By Problem 4.1 in Exercise Set 1, $\hat{x}$ cannot be a vertex, which is a contradiction. Similarly, one can show that $\mathcal{C}_2$ contains no vertices by defining $d = [1, 0]^T$. On the other hand, consider

$$\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2 = \{x \in \mathbb{R}^2 : x_1 \geq 0, \quad x_2 \geq 0\}.$$

We claim that $[0, 0]^T \in \mathcal{C}$ is a vertex of $\mathcal{C}$. Let $a = [1, 1]^T \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$ and $\alpha = 0$. Let $\mathcal{H} = \{x \in \mathbb{R}^2 : a^T x = \alpha\} = \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$ and $\mathcal{H}^+ = \{x \in \mathbb{R}^n : x_1 + x_2 \geq 0\}$. Then, it is straightforward to show that (a) $\mathcal{C} \cap \mathcal{H} = \{\hat{x}\}$ and (b) $\mathcal{C} \subseteq \mathcal{H}^+$. Therefore, $\hat{x}$ is a vertex of $\mathcal{C}$ but not a vertex of $\mathcal{C}_1$ or $\mathcal{C}_2$. This is a counterexample to the given proposition.