University of Edinburgh
School of Mathematics
E. Alper Yıldırım

Fundamentals of Optimization

Exercise 3

Remarks

- All questions that are available in the STACK quiz are duly marked. Please solve those using STACK.
- We have added marks for each question. Please note that those are purely for illustrative purposes. The exercise set will not be marked.
- We can derive the inverse of a nonsingular matrix $A \in \mathbb{R}^{2\times 2}$ in closed form:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

STACK Problems

1 Basic Solutions of Polyhedra in Standard Form (2 marks)

(1.1) *STACK question*

By using the enumeration algorithm presented in Section 13.2.5 of the lecture notes, determine the set of all basic solutions and basic feasible solutions of the following polyhedron: $\mathcal{P} = \{x \in \mathbb{R}^4 : x_1 + 2x_2 - x_3 + 4x_4 = 10;\ 2x_1 + 3x_2 - 2x_3 - 2x_4 = 16;\ x \geq \mathbf{0}\}$. For each basic solution and basic feasible solution, determine whether it is degenerate or nondegenerate. You can assume that the coefficient matrix A has full row rank.

[1 mark]

(1.2) *STACK question*

By using the enumeration algorithm presented in Section 13.2.5 of the lecture notes, determine the set of all basic solutions and basic feasible solutions of the following polyhedron: $\mathcal{P} = \{x \in \mathbb{R}^3 : 3x_1 + 2x_2 + 4x_3 = 4;\ -x_1 + x_2 - 2x_3 = 2;\ x \geq \mathbf{0}\}$. For each basic solution and basic feasible solution, determine whether it is degenerate or nondegenerate. You can assume that the coefficient matrix A has full row rank.

[1 mark]

2 Optimality Conditions and Degeneracy (3 marks)

(2.1) *STACK question*

Consider the following linear program in standard form

$$\min\{-x_1 - 4x_2 - x_3 + 2x_4 : x_1 + 4x_2 + x_3 = 8;\ x_1 + 2x_2 + x_4 = 4;\ x \geq \mathbf{0}\}$$

and the vertices

(a) $\hat{x} = [4, 0, 4, 0]^T$.

(b) $\hat{x} = [0, 2, 0, 0]^T$,

(c) $\hat{x} = [0, 0, 8, 4]^T$.

You can assume that the coefficient matrix A has full row rank. For each vertex, decide whether the vertex is optimal or not, and whether it is degenerate or not.

[3 marks]

Write down a valid choice for the index set B, the reduced costs $\bar{c}_j$, $j \in \{1, \ldots, n\}$, for that basis, and a "candidate" improving direction $d \in \mathbb{R}^n$ if one exists. For the latter, if $\bar{c} \not\geq \mathbf{0}$, use $d_{j^*} = 1$ and $d_j = 0$, $j \in N \setminus \{j^*\}$ to derive the direction, where $j^* \in N$ is the index with the smallest reduced cost $\bar{c}_j$. Verify whether the candidate improving direction d is indeed an improving feasible direction at that vertex. If $\bar{c} \geq \mathbf{0}$, then enter $d = \mathbf{0}$.

If the vertex is degenerate, write down all possible choices of the indices for the index set B, together with the corresponding reduced costs $\bar{c}$ and candidate improving directions.

Open Ended Problems

3 Feasible Directions and Optimality Conditions (2.5 marks)

Consider the following linear programming problem (P) in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Let $\bar{x} \in \mathbb{R}^n$ be an optimal solution of (P) such that $\bar{x}$ is not a vertex.

(3.1) Prove that there exists a feasible direction $\bar{d} \in \mathbb{R}^n$ at $\bar{x}$ such that $\bar{d} \neq \mathbf{0}$ and $c^T \bar{d} = 0$.

[1.5 marks]

(3.2) Prove that (P) has an infinite number of optimal solutions.

[1 mark]

4 Reduced Costs and Optimality Conditions (2.5 marks)

Consider the following linear programming problem in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Assume that A has full row rank. Let $x^* \in \mathbb{R}^n$ be a vertex with the corresponding index sets B and N.

(4.1) Suppose that $\bar{x} \in \mathbb{R}^n$ is a feasible solution of (P) such that $\bar{x} \neq x^*$. Prove that there exists an index $k \in N$ such that $\bar{x}_k > 0$.

[1 mark]

(4.2) Consider the vertex x^* again. Suppose that reduced costs of all nonbasic variables are strictly positive, i.e.,

$$\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j > 0, \quad j \in N.$$

Prove, using (4.1), that x^* is the unique optimal solution.

[1.5 marks]