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Fundamentals of Optimization

Exercise 1 – Solutions

Remarks

- All questions that are available in the STACK quiz are duly marked. Please solve those using STACK.
- We have added marks for each question. Please note that those are purely for illustrative purposes. The exercise set will not be marked.

STACK Problems

1 Basic Concepts (3 marks)

STACK question

Decide, for each of the following three optimization problems, whether

(i) the feasible region is *empty*; or *nonempty and bounded*; or *nonempty and unbounded*;

(ii) the feasible region is a *convex set*; or a *nonconvex set*;

(iii) the objective function is *a convex function*; *a concave function*; *both convex and concave*; or *neither convex nor concave*;

(iv) the optimization problem is a *convex optimization problem*; or a *nonconvex optimization problem*;

(v) the optimization problem *is infeasible*, *is unbounded*, or *has a finite optimal value*;

(vi) write down the optimal value using the convention in the lectures (use `+inf` for $+\infty$ and `-inf` for $-\infty$);

(vii) the set of optimal solutions is *empty*; or *nonempty*;

(viii) the set of optimal solutions is a *convex set*; or a *nonconvex set*.

(1.1) $\max\{1/(x^2+1) : |x-1| \geq 2, \quad x \in \mathbb{R}\}$.

(1.2) $\min\{x^2 - 4x + 3 : -x^2 - 10x \geq 24, \quad x \in \mathbb{R}\}$.

[3 marks]

2 Level Sets, Sublevel Sets, Superlevel Sets, and Epigraphs (2 marks)

(2.1) *STACK question*

Decide, for each of the two functions,

(i) whether $\text{epi}(f)$ is a *convex set* or *nonconvex set*;

(ii) whether the sublevel set $\mathcal{L}_\alpha^-(f)$, where $\alpha = 0$, is a *convex set* or *nonconvex set*;

(iii) whether the level set $\mathcal{L}_\alpha(f)$, where $\alpha = 1$, is a *convex set* or *nonconvex set*;

(iv) whether the superlevel set $\mathcal{L}_\alpha^+(f)$, where $\alpha = 1$, is a *convex set* or *nonconvex set*.

(a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = \max\{|x_1|, |x_2|\}$.

(b) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x_1^2 - x_2^2$.

[2 marks]

Open Ended Problems

3 Level Sets and Sublevel Sets (2.5 marks)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function.

(3.1) By Proposition 4.1, if f is a linear function, then the level set $\mathcal{L}_\alpha(f)$ is a convex set for each $\alpha \in \mathbb{R}$. Consider the converse proposition given below:

If $\mathcal{L}_\alpha(f)$ is a convex set for each $\alpha \in \mathbb{R}$, then f is a linear function.

Either prove this proposition or give a counterexample (i.e., find an example $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies the hypothesis but does not satisfy the conclusion).

[1.5 marks]

(3.2) By Proposition 4.2, if f is a convex function, then the sublevel set $\mathcal{L}_\alpha^-(f)$ is a convex set for each $\alpha \in \mathbb{R}$. Consider the converse proposition given below:

If $\mathcal{L}_\alpha^-(f)$ is a convex set for each $\alpha \in \mathbb{R}$, then f is a convex function.

Either prove this proposition or give a counterexample (i.e., find an example $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies the hypothesis but does not satisfy the conclusion).

[1 marks]

4 Vertices of Convex Sets (2.5 marks)

(4.1) Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a nonempty convex set and let $\hat{x} \in \mathcal{C}$. Prove the following result:

If $\hat{x}$ is a vertex of $\mathcal{C}$, then there does not exist a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $\hat{x} - d \in \mathcal{C}$ and $\hat{x} + d \in \mathcal{C}$.

[1.5 marks]

(4.2) Consider the following set

$$\mathcal{C} = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 1\}.$$

By relying on (4.1), show that $\mathcal{C}$ does not contain any vertices.

[1 mark]