Fundamentals of Operational Research
Tutorial 1
School of Mathematics
The University of Edinburgh
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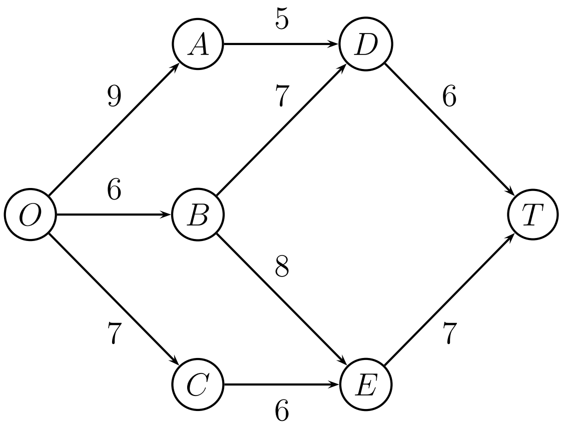
1. Consider the following network. Each number along a link represents the actual distance between the pair of nodes that connects. Find a shortest path from O to T with dynamic programming.



Solution:
This problem is exactly the same as the hiker problem seen in class but on a smaller network with fewer states. Particularly, there is one less stage.

Thus, we will not repeat here the definition of the different elements of the problem and the algorithm that we will use. Refer to the lecture notes for that. The only somehow meaningful difference is that we seek to minimize the length of the route instead of the total level of danger, but it is still a minimization problem.

We look for a path $O \to x_1 \to x_2 \to x_3$ where we know that $x_3 = T$.

- Let us solve $n = 3$.

No matter whether we start on D or on E, the next node will be T.
Thus, for each of these two states there is one single possible choice, which is obviously optimal for its corresponding state.
– If $s = D$, then $f_3^*(D) = 6$ and $x_3^* = T$.
– If $s = E$, then $f_3^*(E) = 7$ and $x_3^* = T$.

- Let us solve $n = 2$.

Now, depending on where we start this stage, we have different possible decisions.

(a) If $s = A$, then there is one single possible decision, which is optimal for this state.
So, if $s = A$, then $f_2^*(A) = d_{AD} + f_3^*(D) = 5 + 6 = 11$, associated to $x_2^* = D$.

(b) If $s = B$, we have two possible decisions:
– $x_2 = D,\ f_2(B, D) = d_{BD} + f_3^*(D) = 7 + 6 = 13.$

– $x_2 = E,\ f_2(B, E) = d_{BE} + f_3^*(E) = 8 + 7 = 15.$

Thus, $f_2^*(B) = \min\{f_2(B, D), f_2(B, E)\} = 13$, associated to $x_2^* = D$.

(c) If $s = C$, we have again one single possible decision, which is optimal for this state. So, if $s = C$, then $f_2^*(C) = d_{CE} + f_3^*(E) = 6 + 7 = 13$, associated to $x_2^* = E$.

- Finally, let us solve $n = 1$.

There are three possible decisions for the single state of this stage:

– $x_1 = A,\ f_1(O, A) = d_{OA} + f_2^*(A) = 9 + 11 = 20.$
– $x_1 = B,\ f_1(O, B) = d_{OB} + f_2^*(B) = 6 + 13 = 19.$
– $x_1 = C,\ f_1(O, C) = d_{OC} + f_2^*(C) = 7 + 13 = 20.$

Therefore, the minimum is $f_1^*(O) = 19$ and $x_1^* = B$.

As a consequence, we can conclude that the shortest path has length 19 and that the single optimal route is $O \to B \to D \to T$.

2. A new railway line needs to be built between locations 1 and 6. The table below shows the distances for the different possible legs of the route:

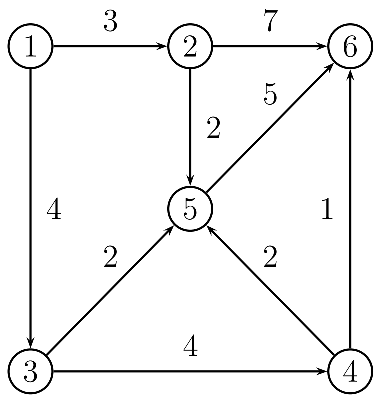
From	1	1	2	2	3	3	4	4	5
To	2	3	5	6	4	5	5	6	6
Distance	3	4	2	7	4	2	2	1	5

(a) Represent the data using a directed graph.

(b) Use dynamic programming to find a shortest route and the length of that route.

Solution:

(a) We represent each location with a node and we represent each possible connection of the route with an arc.



(b) Observe that, unlike in the hiker problem seen in class, here we have different routes with different number of legs. For example, $1 \to 2 \to 6$ has 3 nodes and 2 legs, whereas $1 \to 3 \to 5 \to 6$ has 4 nodes and 3 legs.

Thus, we cannot see the stages as in the hiker problem, where each stage was one part of the journey and we knew exactly the number of parts that the journey would have.

But we can try a different approach: for each node n we calculate the length of the shortest route between location n and location 6, which is the final destination.

If each node is one stage of the problem, going backwards we will finish when we have solved $n = 1$.

Let $f_n(s, x_n)$ be the length of the partial route from node s to node 6 when the first leg of this partial route is arc (s, x_n). That is, x_n is the node where we go at the end of stage n.

Using the same notation seen in class, let x_n^* be the decision that minimizes $f_n(s, x_n)$. That is:

$$f_n^*(s) = f_n(s, x_n^*) = \min \{f_n(s, x_n)\}_{(s,x_n)\in A},$$

where A is the set of arcs of our directed graph.

Besides, it is easy to see that

$$\begin{aligned} f_n(s, x_n) =& \text{immediate length} + \text{minimum future length (stages } n+1 \text{ onwards)} \\ =& d_{sx_n} + f_{n+1}^*(x_n), \end{aligned}$$

where d_{ij} is the length or arc (i, j).

Now that we have defined the elements of our algorithm to solve this problem, let us go step by step.

- We start with $n = 6$.

It is obvious that $f_6^* = 0$, as we are already on node 6, which is our destination.
Note that, as in this problem at each stage we have one single state (the node where we are), we can simplify slightly the notation (and we have done it on the previous paragraph). As $s = n$, we can write f_n^* instead of $f_n^*(s)$ and $f_n(x_n)$ instead of $f_n(s, x_n)$.

- Let us solve stage $n = 5$.

From node 5 there is only one possibility: to go to node 6. Thus:
– $x_5 = 6,\ f_5(6) = d_{56} + f_6^* = 5 + 0 = 5$.
So, $f_5^* = \min\{f_5(x_n)\}_{(5,x_n)\in A} = 5$, associated to $x_5^* = 6$.

- Let us solve stage $n = 4$.

Now we have two possible choices: from node 4 we can go to either node 5 or to node 6. Thus:
– $x_4 = 5,\ f_4(5) = d_{45} + f_5^* = 2 + 5 = 7$.
– $x_4 = 6,\ f_4(6) = d_{46} + f_6^* = 1 + 0 = 1$.
So, $f_4^* = \min f_4(5), f_4(6)\} = \min\{7, 1\} = 1$, associated to $x_4^* = 6$.

- Let us solve stage $n = 3$.

The possible decisions on where to go next are:
– $x_3 = 5,\ f_3(5) = d_{35} + f_5^* = 2 + 5 = 7$.
– $x_3 = 4,\ f_3(4) = d_{34} + f_4^* = 4 + 1 = 5$.
Then $f_3^* = 5$, associated to $x_3^* = 4$.

- Let us solve stage $n = 2$.

The possible decisions are:
– $x_2 = 5,\ f_2(5) = d_{25} + f_5^* = 2 + 5 = 7$.
– $x_2 = 6,\ f_2(6) = d_{26} + f_6^* = 7 + 0 = 7$.
Then $f_2^* = 7$, associated to either $x_2^* = 5$ or $x_2^* = 6$ because there is a tie.

- Finally, we solve stage $n = 1$.

The possible decisions are:

– $x_1 = 2,\ f_1(2) = d_{12} + f_2^* = 3 + 7 = 10.$
– $x_1 = 3,\ f_1(3) = d_{13} + f_3^* = 4 + 5 = 9.$

Therefore $f_1^* = 9$, associated to $x_1^* = 3$.

We can now conclude that the length of the shortest route is 9. A route of this length (actually, "the" route because there is only one) $1 \to 3 \to 4 \to 6$.