1.9 Bayesian Theory Workshop 1

1. Algebra of expectations. There are 3 random variables, X, Y, and Z, that have the following properties and relationships:

$$E[X] = 3, V[X] = 2; E[Y] = 4, V[Y] = 3; E[Z] = -1, V[Z] = 1$$
$$Cov[X, Y] = 0; Cov[X, Z] = 0; Cov[Y, Z] = 1$$

   Let a=2, b= -3.

   (a) What is $E[aX + bY]$?
   (b) What is $V[aX + bY]$?
   (c) What is $Cov[aY, bZ]$?
   (d) What is $V[aY + bZ]$?

2. Probability distribution mean and variance

   (a) Let $X \sim$ Bernoulli(θ). Show that

$$E[X] = \theta \qquad\qquad V[X] = \theta(1-\theta)$$

   (b) Let $X \sim$ Uniform(α, β). Show that

$$E[X] = \frac{\alpha+\beta}{2} \qquad\qquad V[X] = \frac{(\beta-\alpha)^2}{12}$$

   (c) Let $X \sim$ Poisson(μ). Show that

$$E[X] = \mu$$

   Hints: (a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; (b) $\frac{n\theta^{n-1}}{n!} = \frac{d}{d\theta}\frac{\theta^n}{n!}$

   (d) Let $X \sim$ Exponential(λ). Show that

$$E[X] = \frac{1}{\lambda}$$

   Hint: Integration by parts: $\int f(x)g'(x)dx = f(x)g(x)$ - $\int f'(x)g(x)dx$.

3. Joint probability distribution exercise (Reich & Ghosh). X_1 and X_2 have the joint pmf:

x_1	x_2	$\Pr(X_1 = x_1, X_2 = x_2)$
0	0	0.15
1	0	0.15
2	0	0.15
0	1	0.15
1	1	0.20
2	1	0.20

   (a) Compute the marginal distributions of X_1 and X_2
   (b) Compute the conditional distributions of $X_1|X_2$ and $X_2|X_1$.
   (c) Are X_1 and X_2 independent? Justify your answer.

4. Bayes Theorem exercise (Reich & Ghosh). According to `insurance.com`, the 2017 auto theft rate was 135 per 10,000 residents in Raleigh, North Carolina compared to 214 per 10,000 residents in Durham/Chapel Hill, North Carolina. Assuming Raleigh's population is twice as large as Durham/Chapel Hill and a car has been stolen in one of these two areas, what is the probability it was stolen in Raleigh?