MATH11111: Fundamentals of Optimization **Fall 2022**

Lecture 22 — The Efficiency of the Simplex Method

Lecturer: E. Alper Yıldırım *Week: 8*

22.1 Outline

- The Klee-Minty Example
- Exponential Complexity
- Review Problems

22.2 Overview

- In this lecture, we will discuss the efficiency of the simplex method, i.e., how fast the simplex method "solves" a linear programming problem.
- Note that the simplex method is an algorithm.
- Every iteration consists of simple arithmetic operations.
- Under appropriate assumptions and using Bland's rule in the presence of degeneracy, we know that the simplex method always terminates after a finite number of iterations.
- **Question:** Can we estimate the number of iterations on average? How about the worst case?

22.3 Number of Vertices

- Consider a linear programming problem in standard form, with m equality constraints and n variables:

$$\min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\}.$$

- Since the simplex method only visits vertices, and the maximum number of vertices is given by $\binom{n}{m}$, we conclude that the number of iterations is at most $\binom{n}{m}$.
- However, this upper bound is probably loose because not every choice of index sets B and N necessarily yields a basic feasible solution.
- Furthermore, it is not clear if one can construct a polyhedron in standard form that has exactly $\binom{n}{m}$ vertices.
- Even if such a polyhedron exists, it is not clear if one can construct an objective function such that the simplex method visits each and every vertex.

22.4 An Example

Consider the following linear programming problem:

$$
\begin{array}{lrcrcrcr}
\min & -100x_1 & - & 10x_2 & - & x_3 & & \\
\text{s.t.} & & & & & & & \\
 & x_1 & & & & & \leq & 1 \\
 & 20x_1 & + & x_2 & & & \leq & 100 \\
 & 200x_1 & + & 20x_2 & + & x_3 & \leq & 10000 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

- Define slack variables x_4, x_5, and x_6 for each of the first three inequality constraints, respectively, and convert the problem into an equivalent problem in standard from.
- Note that the two-phase method is not needed since there is an easily identifiable starting vertex.
- Use $\hat{x} = [0, 0, 0, 1, 100, 10000]^T$ as the initial vertex to start the simplex method.

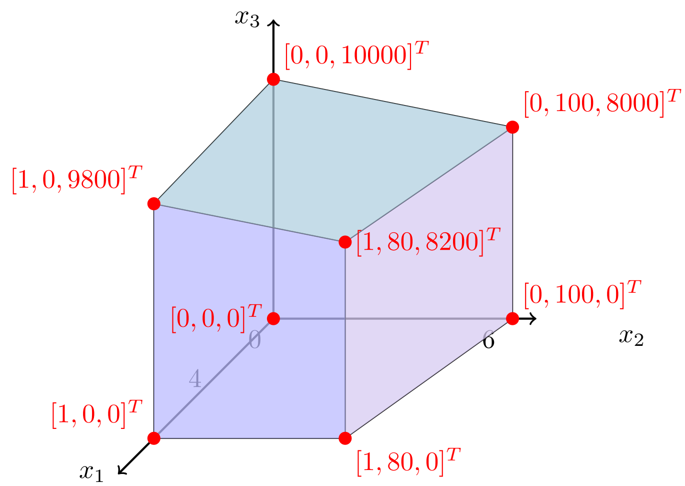

Figure 22.1: The feasible region of the problem in Section 22.4 (Courtesy of Prof John E. Mitchell)

22.4.1 Summary of Iterations

If you choose the nonbasic variable with the most negative reduced cost at each iteration, you will obtain the sequence of vertices outlined in Table 22.1.

The vertices visited by the simplex method on this example is illustrated in Figure 22.2.

Iteration	B	N	Vertex	$\hat{z}$
0	$\{\mathbf{4}, 5, 6\}$	$\{\mathbf{1}, 2, 3\}$	$\hat{x} = [0, 0, 0, 1, 100, 10000]^T$	0
1	$\{1, \mathbf{5}, 6\}$	$\{\mathbf{2}, 3, 4\}$	$\hat{x} = [1, 0, 0, 0, 80, 9800]^T$	-100
2	$\{\mathbf{1}, 2, 6\}$	$\{3, \mathbf{4}, 5\}$	$\hat{x} = [1, 80, 0, 0, 0, 8200]^T$	-900
3	$\{4, 2, \mathbf{6}\}$	$\{1, \mathbf{3}, 5\}$	$\hat{x} = [0, 100, 0, 1, 0, 8000]^T$	-1000
4	$\{\mathbf{4}, 2, 3\}$	$\{\mathbf{1}, 5, 6\}$	$\hat{x} = [0, 100, 8000, 1, 0, 0]^T$	-9000
5	$\{1, \mathbf{2}, 3\}$	$\{4, \mathbf{5}, 6\}$	$\hat{x} = [1, 80, 8200, 0, 0, 0]^T$	-9100
6	$\{\mathbf{1}, 5, 3\}$	$\{2, \mathbf{4}, 6\}$	$\hat{x} = [1, 0, 9800, 0, 80, 0]^T$	-9900
7	$\{4, 5, 3\}$	$\{1, 2, 6\}$	$\hat{x} = [0, 0, 10000, 1, 100, 0]^T$	-10000

Table 22.1: Summary of the simplex iterations on the example in Section 22.4

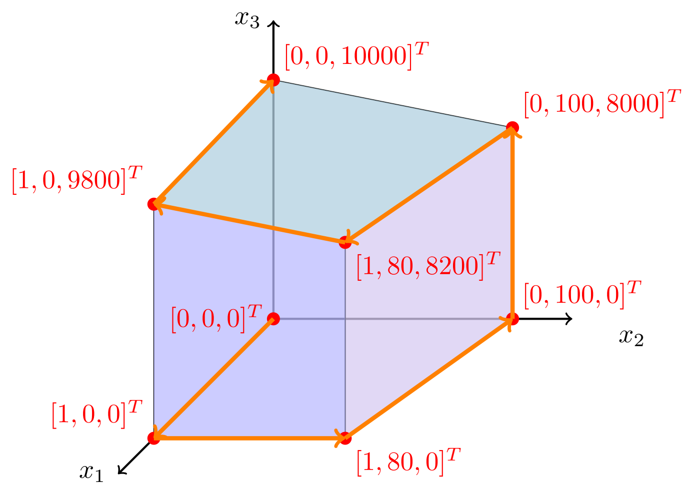

Figure 22.2: The vertices visited by the simplex method on the example in Section 22.4 (Courtesy of Prof John E. Mitchell)

22.4.2 Discussion

- In the original problem, $n = 3$ and the polyhedron has $2^n = 8$ vertices.
- Starting from the first vertex (i.e., the origin), the simplex method visits each vertex before finding the optimal vertex, i.e., it needs $2^n - 1$ iterations.
- Note that each vertex is nondegenerate and the objective function strictly improves at each iteration.
- This example can generalised to any value of n.

$$\begin{array}{lrcll} \min & -\sum_{j=1}^{n} 10^{n-j} x_j & & & \\ \text{s.t.} & & & & \\ & 2\sum_{j=1}^{i-1} 10^{i-j} x_j + x_i & \leq & 100^{i-1}, & i = 1, \ldots, n \\ & x_j & \geq & 0, & j = 1, \ldots, n \end{array}$$

22.5 Main Result

Proposition 22.1 (Klee and Minty, 1972)**.** *For each positive integer n, there is a linear programming problem with n variables such that the simplex method with the most reduced cost performs $2^n - 1$ iterations.*

Remark 22.1. *If $n = 50$ and you can perform 1,000,000 simplex iterations in a second, it would take you more than 35 years to solve the problem!*

22.6 Concluding Remarks

- This result shows that the worst-case performance of the simplex method can be quite poor.
- However, such examples seem to be rare.
- On average, the simplex method performs quite well, with the number of iterations increasing roughly linearly with $m + n$.

Remark 22.2. *It is still an open question whether there is a variant of the simplex method that does not require an exponential number of iterations in the worst case.*

Exercises

Question 22.1. *In the given example with 3 decision variables, the optimal vertex is in fact adjacent to the starting vertex. However, the simplex method with the most negative reduced cost rule fails to detect this and visits every other vertex before finding the optimal vertex.*

1. *What could have caused this behaviour of the simplex method?*
2. *Can you think of a remedy?*

MATH11111: Fundamentals of Optimization **Fall 2022**

Lecture 23 Introduction to Duality Theory and Weak Duality Theorem

Lecturer: E. Alper Yıldırım *Week: 8*

23.1 Outline

- Relaxations
- Weak Duality Theorem
- Review Problems

23.2 Overview and Motivation

Consider a general optimization problem:

$$(\text{P}) \quad \min\{f(x) : x \in \mathcal{S}\},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{S} \subseteq \mathbb{R}^n$.

Question: How can you find a lower bound on the optimal value of (P), denoted by z^*, without solving it?

We can replace (P) by a simpler but related problem:

1. Replace $\mathcal{S}$ by a set $\mathcal{S}_R \subseteq \mathbb{R}^n$ such that $\mathcal{S} \subseteq \mathcal{S}_R$ (e.g., by removing a subset of the constraints in (P))
2. Replace $f : \mathbb{R}^n \to \mathbb{R}$ by $f_R : \mathbb{R}^n \to \mathbb{R}$, where $f_R(x) \leq f(x)$ for each $x \in \mathcal{S}$.
3. Do both of the above.

23.3 Relaxation

Consider the following optimization problems:

$$\begin{array}{ll} (\text{P}) & \min\{f(x) : x \in \mathcal{S}\}, \\ (\text{R}) & \min\{f_R(x) : x \in \mathcal{S}_R\}. \end{array}$$

Definition 23.1. *For an optimization problem (P), the optimization problem (R) is called a* relaxation *of (P) if*

1. $\mathcal{S} \subseteq \mathcal{S}_R$, *and*
2. $f_R(x) \leq f(x)$ *for each* $x \in \mathcal{S}$.

23.3.1 Properties of Relaxations

Lemma 23.2. *Let (R) be a relaxation of (P). Then, $z_R^* \leq z^*$, where z_R^* and z^* denote the optimal values of (R) and (P), respectively.*

Proof. **Case 1:** If (P) is infeasible, then $z^* = +\infty$ by definition. We clearly have $z_R^* \leq z^*$, regardless of the value of z_R^*.

Case 2: Suppose that (P) has a nonempty feasible region. Let $\hat{x} \in \mathcal{S}$ be an arbitrary feasible solution of (P). Since $\mathcal{S} \subseteq \mathcal{S}_R$, we obtain $\hat{x} \in \mathcal{S}_R$. Since $\hat{x} \in \mathcal{S}$, we have $f_R(\hat{x}) \leq f(\hat{x})$. Therefore, for each feasible solution $\hat{x} \in \mathcal{S}$ of (P), we have $\hat{x} \in \mathcal{S}_R$ and $f_R(\hat{x}) \leq f(\hat{x})$. It follows that $z_R^* \leq z^*$. $\square$

23.4 Back to Linear Programming

Consider a linear programming problem in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ are given, and $x \in \mathbb{R}^n$ denotes the decision variables.

- Note that $Ax = b$ if and only if $(a^i)^T x = b_i$ if and only if $b_i - (a^i)^T x = 0$ for each $i = 1, \ldots, m$, where $(a^i)^T$ is the ith row of A.

- For a given $x \in \mathbb{R}^n$, $b_i - (a^i)^T x$ measures the violation of the ith equality constraint, which can be negative, zero, or positive.

- We will allow the violation of equality constraints (i.e., remove them from (P)).

- However, we will introduce a "price" $y_i \in \mathbb{R}$ for each unit of violation of the ith equality constraint, where $i = 1, \ldots, m$.

- For a given vector of prices $y \in \mathbb{R}^m$, the total violation cost is given by

$$\sum_{i=1}^m y_i \left(b_i - (a^i)^T x\right) = \underbrace{\sum_{i=1}^m y_i b_i}_{y^T b} - \underbrace{\sum_{i=1}^m y_i \left((a^i)^T x\right)}_{y^T A x} = y^T b - y^T A x = y^T (b - Ax).$$

- We will (i) remove the equality constraints $Ax = b$ from (P) and (ii) add the total violation cost to the objective function:

$$\text{(D}(y)) \quad \min\{c^T x + y^T (b - Ax) : x \geq \mathbf{0}\}, \quad y \in \mathbb{R}^m.$$

- For each fixed $y \in \mathbb{R}^m$, (D(y)) is a linear programming problem since

$$c^T x + y^T (b - Ax) = \underbrace{b^T y}_{constant} + (\underbrace{c - A^T y}_{fixed})^T x.$$

23.4.1 Relations Between Two Problems

Consider a linear programming problem in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ are given, and $x \in \mathbb{R}^n$ denotes the decision variables.

For a fixed $y \in \mathbb{R}^m$, consider the following linear programming problem:

$$\text{(D}(y)\text{)} \quad \min\{c^T x + y^T(b - Ax) : x \geq \mathbf{0}\}$$

Lemma 23.3. *For each $y \in \mathbb{R}^m$, (D(y)) is a relaxation of (P). Therefore, $z^*(y) \leq z^*$, where $z^*(y)$ and z^* denote the optimal values of (D(y)) and (P), respectively, i.e., the optimal value of (D(y)) is a lower bound on the optimal value of (P).*

Proof. Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ and $\mathcal{P}_R = \{x \in \mathbb{R}^n : x \geq \mathbf{0}\}$ denote the feasible regions of (P) and (D(y)), respectively. We clearly have $\mathcal{P} \subseteq \mathcal{P}_R$.

For a given $y \in \mathbb{R}^m$, let $\hat{x} \in \mathcal{P}$ be an arbitrary feasible solution of (P). Then, $A\hat{x} = b$ and $\hat{x} \geq \mathbf{0}$. The objective function value of (P) evaluated at $\hat{x}$ is given by $c^T\hat{x}$. Since $\mathcal{P} \subseteq \mathcal{P}_R$, we have $\hat{x} \in \mathcal{P}_R$. Consider the objective function value of (D(y)) evaluated at $\hat{x}$. We obtain

$$c^T\hat{x} + y^T\underbrace{(b - A\hat{x})}_{\mathbf{0}} = c^T\hat{x} \leq c^T\hat{x}.$$

Therefore, (D(y)) is a relaxation of (P). The last assertion follows from Lemma 23.2. □

- (D(y)) is called the *Lagrangian relaxation* of (P).
- By Lemma 23.3, for each price vector $y \in \mathbb{R}^m$, we have $z^*(y) \leq z^*$.
- For each $y \in \mathbb{R}^m$, recall that the objective function of (D(y)) is

$$c^T x + y^T(b - Ax) = \underbrace{b^T y}_{constant} + (c - A^T y)^T x.$$

- We therefore obtain

$$z^*(y) = \begin{cases} b^T y & \text{if } c - A^T y \geq \mathbf{0}, \\ -\infty & \text{otherwise.} \end{cases}$$

23.4.2 The Dual Problem

Consider the following linear programming problems:

$$\begin{aligned} &\text{(P)} && \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\}, \\ &\text{(D}(y)\text{)} && \min\{c^T x + y^T(b - Ax) : x \geq \mathbf{0}\}, \end{aligned}$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$ are given, and $x \in \mathbb{R}^n$ denotes the decision variables.

- For any price vector $y \in \mathbb{R}^m$:
  (i) If $c - A^T y \geq \mathbf{0}$, then $z^*(y) = b^T y \leq z^*$.
  (ii) If $c - A^T y \not\geq \mathbf{0}$ (i.e., $c - A^T y$ has at least one negative component), then $z^*(y) = -\infty \leq z^*$ (i.e., we obtain a trivial lower bound on z^*).

- It is reasonable to ask for the best (i.e., the largest) lower bound on z^*:

$$\text{(D)} \quad \max\{b^T y : c - A^T y \geq \mathbf{0}\} = \max\{b^T y : A^T y \leq c\}$$

- (D) is a linear programming problem and is called the *dual problem*, whereas (P) is called the *primal problem*.

- $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are referred to as *primal* and *dual variables,* respectively.

23.5 Weak Duality Theorem

Consider the following pair of primal and dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

Proposition 23.1 (Weak Duality Theorem)**.** *Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ be feasible solutions of (P) and (D), respectively. Then,*

(i) $b^T \bar{y} \leq c^T \bar{x}$;

(ii) if $b^T \bar{y} = c^T \bar{x}$, then $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ are optimal solutions of (P) and (D), respectively.

In addition,

(iii) if (P) is unbounded, then (D) is infeasible;

(iv) if (D) is unbounded, then (P) is infeasible;

Proof. (i) Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ be feasible solutions of (P) and (D), respectively. Since $c - A^T \bar{y} \geq \mathbf{0}$, we have $z^*(\bar{y}) = b^T \bar{y}$. It follows from Lemma 23.3 that $z^*(\bar{y}) = b^T \bar{y} \leq z^* \leq c^T \bar{x}$. The claim follows.

(ii) Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ be feasible solutions of (P) and (D), respectively, such that $b^T \bar{y} = c^T \bar{x}$. Then, by part (i), for any feasible solution $\hat{x} \in \mathbb{R}^n$ of (P), we have $b^T \bar{y} = c^T \bar{x} \leq c^T \hat{x}$. Therefore, $z^* = c^T \bar{x}$, where z^* denotes the optimal value of (P). A similar argument shows that $z_D^* = b^T \bar{y}$, where z_D^* denotes the optimal value of (D). It follows that $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ are optimal solutions of (P) and (D), respectively.

(iii) Suppose that (P) is unbounded. Suppose, for a contradiction, that (D) is not infeasible. Then, there exists $\bar{y} \in \mathbb{R}^m$ such that $c - A^T \bar{y} \geq \mathbf{0}$. By part (i), we obtain $z^*(\bar{y}) = b^T \bar{y} \leq z^*$, which contradicts that (P) is unbounded.

(iv) The proof of this assertion is very similar to that of part (iii) and is therefore omitted.

$\square$

23.6 Concluding Remarks and Outlook

- The linear programming problems (P) and (D) are closely related.
- Denoting the optimal values of (P) and (D) by z^* and z_D^*, respectively, we immediately obtain $z_D^* \leq z^*$.
- In the next lecture, we will show that $z_D^* = z^*$, provided that both (P) and (D) have nonempty feasible regions (known as the *strong duality theorem*).

Exercises

Consider the following pair of primal and dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

Question 23.1. *Suppose that we add a new equality constraint $a^T x = \alpha$ to the primal problem (P), where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$. Denote the modified problem by (P′) and its dual by (D′). Compare the optimal value of (D), denoted by z_D^*, and the optimal value of (D′), denoted by $z_{D'}^*$.*

MATH11111: Fundamentals of Optimization	**Fall 2022**
Lecture 24	Primal-Dual Symmetry and Strong Duality Theorem
Lecturer: E. Alper Yıldırım	*Week: 8*

24.1 Outline

- Primal-Dual Symmetry
- Strong Duality Theorem
- Review Problems

24.2 Quick Review

Consider the following pair of linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- (P) is the primal problem and (D) is the dual problem.
- Denoting the optimal values of (P) and (D) by z^* and z_D^*, respectively, we have $z_D^* \leq z^*$.
- In this lecture, we will establish further properties between (P) and (D), including symmetry and strong duality.

24.3 Primal-Dual Symmetry

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

Proposition 24.1. *The dual of (D) is equivalent to (P) (i.e., the dual of the dual is the primal). Therefore, the roles of the primal and the dual problem are symmetric.*

Proof. First, we need to transform (D) into standard form. Since $y \in \mathbb{R}^m$ is unrestricted, we replace it by the difference of $y^+ \in \mathbb{R}^m$ and $y^- \in \mathbb{R}^m$, where $y^+ \geq \mathbf{0}$ and $y^- \geq \mathbf{0}$. We also add nonnegative slack variables

$s \in \mathbb{R}^n$ to convert the inequality constraints into equality constraints. Finally, we multiply the objective function by -1 to convert it into minimization. We obtain

$$(\mathrm{D}') \quad \min\{-b^T y^+ + b^T y^- : A^T y^+ - A^T y^- + s = c, \quad y^+ \geq \mathbf{0},\ y^- \geq \mathbf{0},\ s \geq \mathbf{0}\}$$

Using $w \in \mathbb{R}^n$ as the dual variables, the dual of (D') is given by

$$(\mathrm{P}') \quad \max\{c^T w : Aw \leq -b, \quad -Aw \leq b, \quad w \leq \mathbf{0}\}$$

Replace w in (P') by $x = -w$. Note that $Aw = -b$, which implies that $Ax = b$. Since $w \leq \mathbf{0}$, we have $x \geq \mathbf{0}$. Finally, $-c^T w = c^T x$, which implies that maximizing $c^T w$ is equivalent to minimizing $c^T x$. We immediately obtain that (P') is equivalent to (P). □

24.4 Strong Duality Theorem

Consider the following primal-dual pair of linear programming problems:

$$\begin{array}{ll} (\mathrm{P}) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ (\mathrm{D}) & \max\{b^T y : A^T y \leq c\} \end{array}$$

Proposition 24.2 (Strong Duality Theorem)**.** *Suppose that both the primal problem (P) and the dual problem (D) have nonempty feasible regions and that A has full row rank. Then, each of the two problems has a finite optimal value and $z_D^* = z^*$.*

Proof. Since (P) has a nonempty feasible region, it either has a finite optimal value or is unbounded. By weak duality, if (P) is unbounded, then (D) is infeasible, which contradicts the hypothesis. Therefore, (P) has a finite optimal value denoted by z^*. A similar argument shows that (D) has a finite optimal value denoted by z_D^*.

Since (P) is in standard form and (P) has a nonempty feasible region, (P) contains at least one basic feasible solution. Therefore, there is at least one optimal vertex $x^* \in \mathbb{R}^n$ of (P) computed by the simplex method. Let B and N denote the corresponding indices, where $|B| = m$, $|N| = n - m$, and $B \cap N = \emptyset$.

We have $x_B^* = (A_B)^{-1} b \geq \mathbf{0}$ and $x_N^* = \mathbf{0}$. Since x^* is an optimal vertex, the reduced costs of all nonbasic variables should be nonnegative, i.e.,

$$\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j \geq 0, \quad j \in N,$$

where $A^j \in \mathbb{R}^m$ denotes the $j$the column of A.

Consider the constraints of the dual problem given by $A^T y \leq c$, which can be rewritten as $(A^j)^T y \leq c_j$, or equivalently, as $c_j - (A^j)^T y \geq 0, j = 1, \ldots, n$.

Let us define $y^* = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$. Recall that

$$\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j = c_j - (A^j)^T \underbrace{\left((A_B)^{-1}\right)^T c_B}_{y^*} \geq 0, \quad j \in N,$$

which implies that $c_j - (A^j)^T y^* \geq 0$ for each $j \in N$. Recall that the reduced costs of basic variables are equal to zero (see Remark 4 in Section 18.4 in the lecture notes). Therefore, we have $c_j - (A^j)^T y^* = 0$ for each $j \in B$. It follows that y^* satisfies $A^T y^* \leq c$, i.e., y^* is a feasible solution of (D).

Finally, note that

$$b^T y^* = b^T \left((A_B)^{-1}\right)^T c_B = c_B^T \underbrace{(A_B)^{-1} b}_{x_B^*} = c_B^T x_B^* = c_B^T x_B^* + c_N^T \underbrace{x_N^*}_{\mathbf{0}} = c^T x^*.$$

Therefore, $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^m$ are feasible for (P) and (D), respectively, and $b^T y^* = c^T x^*$. By part (ii) of the Weak Duality Theorem, x^* and y^* are optimal solutions of (P) and (D), respectively. It follows that $z^* = c^T x^* = b^T y^* = z_D^*$, where z^* and z_D^* denote the optimal values of (P) and (D), respectively. The claim follows. □

24.4.1 Discussion

- By the proof of Proposition 24.2, the simplex method applied to solve the primal problem (P) computes an optimal solution of the dual problem (D) as a byproduct if (P) has a finite optimal value.
- If x^* is an optimal vertex of the primal problem (P) with index sets B and N, then $y^* = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ is an optimal solution of the dual problem (D).
- Note that $b^T y^* = b^T \left((A_B)^{-1}\right)^T c_B = c_B^T (A_B)^{-1} b = c_B^T x_B^* = c^T x^* = z^*$ since $x_N^* = \mathbf{0}$.

24.5 Primal-Dual Relations

Consider the following primal-dual pair of linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- By the Strong Duality Theorem, if (P) has a finite optimal value, then so does (D) and their optimal values are the same.
- By primal-dual symmetry, if (D) has a finite optimal value, then so does (P) and their optimal values are the same.
- By the Weak Duality Theorem, if (P) is unbounded, then (D) is infeasible.
- By primal-dual symmetry, if (D) is unbounded, then (P) is infeasible.
- **Question:** If (P) (or (D)) is infeasible, what can we infer about the status of (D) (or (P))?

24.5.1 An Example

Example 24.1. *Consider the following linear programming problem in standard form:*

$$\begin{array}{llrcrcrcrcr} (P) & \min & x_1 & - & 2x_2 & & & & & & \\ & s.t. & & & & & & & & & \\ & & x_1 & - & x_2 & - & x_3 & & & = & 2 \\ & & -x_1 & + & x_2 & & & - & x_4 & = & -1 \\ & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0. \end{array}$$

If you add the two equality constraints in (P), you get the following implied equality:

$$-x_3 - x_4 = 1 \Leftrightarrow x_3 + x_4 = -1.$$

Since $x \geq \mathbf{0}$, we immediately conclude that (P) is infeasible.

Consider now the dual problem:

$$\begin{array}{llrcrcr} (D) & \max & 2y_1 & - & y_2 & & \\ & s.t. & & & & & \\ & & y_1 & - & y_2 & \leq & 1 \\ & & -y_1 & + & y_2 & \leq & -2 \\ & & -y_1 & & & \leq & 0 \\ & & & - & y_2 & \leq & 0 \\ & & y_1 & , & y_2 & \in & \mathbb{R}. \end{array}$$

If you add the first two inequalities in (D), you obtain the following implied inequality:

$$0 \leq -1.$$

We immediately conclude that (D) is also infeasible.

As illustrated by Example 24.1, we may have examples in which both (P) and (D) are simultaneously infeasible.

24.5.2 Primal-Dual Relations

By using Weak Duality Theorem, Strong Duality Theorem, and Example 24.1, we arrive at the set of all possible relations between the primal problem (P) and the dual problem (D) outlined in Table 24.1.

		Dual Problem		
		Finite optimal value	Unbounded	Infeasible
Primal Problem	Finite optimal value	✓	✗	✗
	Unbounded	✗	✗	✓
	Infeasible	✗	✓	✓

Table 24.1: Possible primal-dual relations

24.6 Concluding Remarks

Consider the following primal-dual pair of linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- We established further properties between the primal problem (P) and the dual problem (D).
- We will continue to develop even further properties in the next lecture.

Exercises

Question 24.1. *Consider the following linear programming problems:*

$$\begin{array}{ll} (P1) & \min\{c^T x : Ax \le b, \quad x \ge \mathbf{0}\} \\ (P2) & \min\{c^T x : Ax \ge b, \quad x \ge \mathbf{0}\} \end{array}$$

Convert each problem (P1) and (P2) into standard form. Take the dual of each one and simplify the dual problems.

Question 24.2. *Consider the linear programming problem:*

$$\begin{array}{llccccccccc} (P) & \min & x_1 & + & x_2 & - & x_3 & & & & \\ & s.t. & & & & & & & & & \\ & & x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 2 \\ & & x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1 \\ & & x_1 & , & x_2 & , & x_3 & , & x_4 & \ge & 0. \end{array}$$

Show that (P) has a finite optimal value and find that optimal value without using the simplex method.