Week 2

More linear programming models

Methodology, Modelling, and Consulting Skills
The University of Edinburgh

What is new?

- Multiple time periods
- Multiple inputs and outputs

Multi-period problems

- The blending and diet models are *single-period*.
- Production plans usually have a **multi-period** horizon (for example, year divided in months).
- **The decisions at one period have consequences on later decisions.**
- Usual goal: **maximize net profit/minimize net cost**.

Problem 1 – Multi-period

Sailco Corporation must decide how many sailboats to produce in each quarter. The demand is different in each quarter depending on the season. Sailco has no warehouse and therefore all the sailboats must be sold in the same quarter they are produced. Find a production plan that meets the demand of the whole year at minimum cost.

Solution

Collect data

Quarter	Spring	Summer	Autumn	Winter
Demand	40	60	75	25

- The production cost per boat is £400

Solution

- Notation is easier if we let spring denote quarter 1, summer denote quarter 2, etc.
- We define the variables by letting

$$x_t \text{ be number of sailboats produced during quarter } t,\ t = 1, 2, 3, 4$$

- The objective function is to minimise the total production cost

$$400x_1 + 400x_2 + 400x_3 + 400x_4$$

- We introduce the constraints

$$x_1 \geq 40, \quad x_2 \geq 60, \quad x_3 \geq 75, \quad x_4 \geq 25$$

  to ensure that we satisfy all demand

Solution

$$
\begin{aligned}
\min \quad & 400x_1 + 400x_2 + 400x_3 + 400x_4 \\
\text{s.t.} \quad & x_1 \geq 40, \\
& x_2 \geq 60, \\
& x_3 \geq 75, \\
& x_4 \geq 25, \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

- **Note that it is not optimal to produce more than the existing demand. So, we could write the corresponding constraints as equalities**
- The optimal solution is trivial

$$x_1^* = 40,\ x_2^* = 60,\ x_3^* = 75,\ x_4^* = 25, \quad z^* = 80\,000$$

Problem 2 – Multi-period with inventory

Sailco Corporation must decide how many sailboats to produce in each quarter. The demand is different in each quarter depending on the season. **Due to workload limitations, Sailco cannot produce more than 60 sailboats per quarter. Sailco has developed a sufficiently large warehouse where they can store as many sailboats as required at no additional cost.**

Find a production plan that meets the demand of the whole year at the minimum possible cost

Assume, for simplicity, that the sailboats produced in one quarter can be used to meet the demand in that quarter.

Quarter	Spring	Summer	Autumn	Winter
Demand	40	60	75	25

Solution

- Let
  1. x_t be the number of sailboats **produced** in quarter t, $t = 1, 2, 3, 4$ and
  2. y_t be the number of sailboats **available for sale** in quarter t, $t = 1, 2, 3, 4$.

- The objective function is

$$400 \sum_{t=1}^{4} x_t$$

Solution

- We introduce the constraints

$$y_1 \geq 40, \quad y_2 \geq 60, \quad y_3 \geq 75, \quad y_4 \geq 25$$

  to ensure that the number of available sailboats meets the demand for each quarter
- We also introduce the constraints

$$x_t \leq 60, \; t = 1, 2, 3, 4$$

  to ensure that we do not exceed our production capacity

Solution

- **Equation for inventory balance:**

$$\begin{aligned}\text{Inventory at end of quarter } t \; &= \text{Inventory at end of the previous quarter } t-1\\ &+ \text{ production in quarter } t\\ &- \text{supply in quarter } t\end{aligned}$$

- If we let

  i_t be the inventory level at end of quarter t, $t = 1, 2, 3, 4$

  then we can introduce the constraints
  - $i_1 = (i_0) + x_1 - y_1,$
  - $i_t = i_{t-1} + x_t - y_t, \; t = 2, 3, 4$

Solution

$$
\begin{array}{lll}
\min & 400 \sum_{t=1}^{4} x_t & \\
\text{s.t.} & x_t \le 60, & t = 1, 2, 3, 4, \\
& i_1 = x_1 - y_1, & \\
& i_t = i_{t-1} + x_t - y_t, & t = 2, 3, 4, \\
& y_1 \ge 40, & \\
& y_2 \ge 60, & \\
& y_3 \ge 75, & \\
& y_4 \ge 25, & \\
& x_t, y_t, i_t \ge 0, & t = 1, 2, 3, 4.
\end{array}
$$

Solution

- An optimal solution is

$$x^* = (60, 60, 60, 20), \qquad y^* = (40, 60, 75, 25),$$
$$i^* = (20, 20, 5, 0),$$

  with optimal value $z^* = 80\,000$.

- In spring, we make 60 sailboats and sell 40. 20 remain in stock
- In summer, we make 60 sailboats and sell 60. 20 remain in stock
- In autumn, we make 60 sailboats and sell 75. 5 remain in stock
- In winter, we make 20 sailboats and sell 25
- Final stock is 0. Keeping unsold sailboats is not profitable

Beyond the Solution

- The production cost is the same in any season
- The only reason for producing more than required in a season is the limit on the production in each season
- We need to produce in advance for the periods with higher demand

Problem 3 – Multi-period with inventory and holding costs

Sailco Corporation must decide how many sailboats to produce in each quarter. The demand is different in each quarter depending on the season. Due to workload limitations, Sailco cannot produce more than 60 sailboats per quarter. Sailco has developed a sufficiently large warehouse where they can store as many sailboats as required **at a cost of £50 per unit.**

Find a production plan that meets the demand of the whole year at the minimum possible cost

Assume, for simplicity, that the sailboats produced in one quarter can be used to meet the demand in that quarter.

Quarter	Spring	Summer	Autumn	Winter
Demand	40	60	75	25

Solution

- We have the same problem than before PLUS a holding cost
- The holding cost is

$$50i_1 + 50i_2 + 50i_3 + 50i_4 = 50\sum_{t=1}^{4} i_t$$

- We add it to the objective function
- No constraint needs to be changed

Solution

$$
\begin{array}{lll}
\min & \sum_{t=1}^{4} 400x_t + 50i_t & \\
\text{s.t.} & x_t \leq 60, & t = 1, 2, 3, 4, \\
& i_1 = x_1 - y_1, & \\
& i_t = i_{t-1} + x_t - y_t, & t = 2, 3, 4, \\
& y_1 \geq 40, & \\
& y_2 \geq 60, & \\
& y_3 \geq 75, & \\
& y_4 \geq 25, & \\
& x_t, y_t, i_t \geq 0, & t = 1, 2, 3, 4.
\end{array}
$$

Solution

- An optimal solution is

$$x^* = (55, 60, 60, 25), \quad y^* = (40, 60, 75, 25),$$
$$i^* = (15, 15, 0, 0),$$

  with optimal value $z^* = 81\,500$.

- The production $x = (60, 60, 60, 20)$ is no longer optimal because of the holding cost.

Problem 4 – Multi-product inventory

Sailco Corporation must decide how many *sailboats and surfboards* to produce at each quarter. The company has a sufficiently large warehouse to store sailboats and surfboards at a cost. Sailco also has a finite number of labour hours available.
What is the production plan that meets the demand of the whole year at minimum cost?

Solution

Collect data

Demand

	Spring	Summer	Autumn	Winter
Sailboat	40	60	75	25
Surfboard	190	350	130	20

Resources

	Work (h)	Storage cost (£)	Production cost (£)
Sailboat	20	50	400
Surfboard	3	2	35

- Sailco have 1860 hours of work available per quarter

Solution

- We have now two products
- We use two-index variables to differentiate between products
- Let $p = 1$ for sailboats and $p = 2$ for surfboards

Solution

Let

- x_{pt} be the number of units of product p produced in quarter t, $p = 1, 2$, $t = 1, 2, 3, 4$.
- y_{pt} be the number of units of product p available for supply in quarter t, $p = 1, 2$, $t = 1, 2, 3, 4$.
- i_{pt} = be the number units of product p in inventory at the end of quarter t, $p = 1, 2$, $t = 1, 2, 3, 4$.

Solution

- The production cost for sailboats is

$$400x_{11} + 400x_{12} + 400x_{13} + 400x_{14}$$

- The production cost for surfboards is

$$35x_{21} + 35x_{22} + 35x_{23} + 35x_{24}$$

- The holding cost for sailboats is

$$50i_{11} + 50i_{12} + 50i_{13} + 50i_{14}$$

- The holding cost for surfboards is

$$2i_{21} + 2i_{22} + 2i_{23} + 2i_{14}$$

Solution

- The constraints that limit the hours of work are

$$20x_{1t} + 3x_{2t} \leq 1860, \; t = 1, 2, 3, 4$$

- We introduce the constraints

$$y_{11} \geq 40, \quad y_{12} \geq 60, \quad y_{13} \geq 75, \quad y_{14} \geq 25,$$

$$y_{21} \geq 190, \quad y_{22} \geq 350, \quad y_{23} \geq 130, \quad y_{24} \geq 20$$

  to ensure that the number of available sailboats and surfboards meets the demand for each quarter

Solution

The inventory balance constraints are therefore

- Sailboats, quarter 1:

$$i_{11} = x_{11} - y_{11}$$

- Sailboats, quarter t, $t = 2, 3, 4$

$$i_{1t} = i_{1,t-1} + x_{1t} - y_{1t}$$

- Surfboards, quarter 1

$$i_{21} = x_{21} - y_{21}$$

- Surfboards, quarter t, $t = 2, 3, 4$

$$i_{2t} = i_{2,t-1} + x_{2t} - y_{2t}$$

Solution

$$
\begin{aligned}
\min \quad & \sum_{t=1}^{4} 400x_{1t} + \sum_{t=1}^{4} 35x_{2t} + \sum_{t=1}^{4} 50i_{1t} + \sum_{t=1}^{4} 2i_{2t} \\
\text{s.t.} \quad & 20x_{1t} + 3x_{2t} \le 1860, && t = 1, 2, 3, 4, \\
& y_{11} \ge 40, \;\; y_{12} \ge 60, \\
& y_{13} \ge 75, \;\; y_{14} \ge 25, \\
& y_{21} \ge 190, \;\; y_{22} \ge 350, \\
& y_{23} \ge 130, \;\; y_{24} \ge 20, \\
& i_{p1} = x_{p1} - y_{p1}, && p = 1, 2, \\
& i_{pt} = i_{p,t-1} + x_{pt} - y_{pt}, && p = 1, 2, \;\; t = 2, 3, 4, \\
& i_{pt}, x_{pt}, y_{pt} \ge 0, && p = 1, 2, \;\; t = 1, 2, 3, 4
\end{aligned}
$$

Solution

- An optimal solution is

$$
\begin{aligned}
x_{1\cdot}^* &= (40, 60, 75, 25), \quad y_{1\cdot}^* = (40, 60, 75, 25), \\
i_{1\cdot}^* &= (0, 0, 0, 0), \quad x_{2\cdot}^* = (330, 220, 120, 20), \\
y_{2\cdot}^* &= (190, 350, 130, 20), \quad i_{2\cdot}^* = (140, 10, 0, 0),
\end{aligned}
$$

with optimal value $z^* = 104\,450$

Solution

- In spring:
  - Sailboats: make 40 and sell 40.
  - Surfboards: make 330 and sell 190. Inventory level at the end of the quarter: 140.
- In summer:
  - Sailboats: make 60 and sell 60.
  - Surfboards: make 220 and sell 350. Inventory level at the end of the quarter: 10.
- In autumn:
  - Sailboats: make 75 and sell 75.
  - Surfboards: make 120 and sell 130.
- In winter:
  - Sailboats: make 25 and sell 25.
  - Surfboards: make 20 and sell 20.

A general model?

Index sets

P	set of products
T	set of time periods

Variables

x_{pt}	units of product p made in time period t ($p \in P$, $t \in T$)
y_{pt}	units of product p sold in time period t ($p \in P$, $t \in T$)
i_{pt}	units of product p in inventory at the end of time period t ($p \in P$, $t \in T$)

Parameters

D_{pt}	demand of product p in time period t ($p \in P$, $t \in T$)
C_p	production cost for product p ($p \in P$)
S_p	storage cost for product p ($p \in P$)
W_p	work hours required for product p ($p \in P$)
W_{Total}	The total number of available work hours

Can you write a general formulation for this problem?

Summary

Basic model building blocks

- Diet problem
- Blending problem
- Multi-period
- Multi-period with inventory
- Multi-product