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Fundamentals of Optimization

Exercise Set 0

Numerical Problems

1 Simple Optimization Problems

For each of the following three optimization problems, identify the feasible region and the objective function. Determine whether the given optimization problem is *infeasible*, *unbounded*, or *has a finite optimal value*. Find the optimal value using the convention in the lectures and determine the set of all optimal solutions.

(1.1) $\max\{(x-1)^2 : x^2 \leq -1, \quad x \in \mathbb{R}\}$.

(1.2) $\min\{1/(x+1)^2 : x \geq 0, \quad x \in \mathbb{R}\}$.

(1.3) $\min\{x^2 - 4x + 6 : |x-2| \geq 1, \quad x \in \mathbb{R}\}$.

(1.4) $\max\{x^2 - 3x + 6 : x^2 \geq 1, \quad x \in \mathbb{R}\}$.

2 Convex Sets and Convex Functions

(2.1) Decide for each of the following sets whether they are *convex* or not.

(a) $\mathcal{C} = \{x \in \mathbb{R} : x^2 \geq 3\}$.
(b) $\mathcal{C} = \{x \in \mathbb{R}^2 : (x_1+1)^2 + (x_2-1)^2 \leq -3\}$.
(c) $\mathcal{C} = \{x \in \mathbb{R}^2 : |x_1| + |x_2| \leq 7\}$.

(2.2) Decide for each of the following three functions whether they are *convex*, *concave*, *both*, or *neither*.

(a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = 3x_1 - 2x_2$.
(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$.
(c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 4x$.

3 Level Sets, Sublevel Sets, and Superlevel Sets

(3.1) For each of the functions $f : \mathbb{R}^2 \to \mathbb{R}$ given below, and for each $\alpha \in \mathbb{R}$, give an algebraic description of the level set $\mathcal{L}_\alpha(f)$, the sublevel set $\mathcal{L}_\alpha^-(f)$, and the superlevel set $\mathcal{L}_\alpha^+(f)$.

(a) $f(x) = x_1^2 + x_2^2$.
(b) $f(x) = \max\{|x_1|, |x_2|\}$.
(c) $f(x) = x_1^2 - x_2^2$.

Open Ended Problems

4 Epigraphs and Convex Functions

Let $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$ be two functions. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function given by $h(x) = \max\{f_1(x), f_2(x)\}$.

(4.1) Prove the following proposition:

$$\mathrm{epi}(h) = \mathrm{epi}(f_1) \cap \mathrm{epi}(f_2).$$

(Hint: One way of proving that two sets A and B are equal to one another is to show that $A \subseteq B$ and $B \subseteq A$.)

(4.2) Suppose that f_1 and f_2 are convex functions. Show, by using (5.1), that h is a convex function.

5 Level Sets and Sublevel Sets

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function.

(5.1) By Proposition 4.1, if f is a linear function, then the level set $\mathcal{L}_\alpha(f)$ is a convex set for each $\alpha \in \mathbb{R}$. Consider the converse proposition given below:

If $\mathcal{L}_\alpha(f)$ is a convex set for each $\alpha \in \mathbb{R}$, then f is a linear function.

Either prove this proposition or give a counterexample (i.e., find an example $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies the hypothesis but does not satisfy the conclusion).

(5.2) By Proposition 4.2, if f is a convex function, then the sublevel set $\mathcal{L}_\alpha^-(f)$ is a convex set for each $\alpha \in \mathbb{R}$. Consider the converse proposition given below:

If $\mathcal{L}_\alpha^-(f)$ is a convex set for each $\alpha \in \mathbb{R}$, then f is a convex function.

Either prove this proposition or give a counterexample (i.e., find an example $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies the hypothesis but does not satisfy the conclusion).