Conic Programming Duality

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Lecture Supplementary Material for
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Consider

$$
\begin{array}{ll}
\min & \langle c, x\rangle \\
\text{s.t.} & Ax = b, \\
 & x \in K,
\end{array}
$$

where $c, x \in X$, $A : X \to \Re^m$ a linear operator, $b \in \Re^m$, $K \subseteq X$ a closed convex cone, X is a finite-dimensional real Hilbert space, $\langle , \rangle$ is the inner product of X.

Define $\mathcal{L}(x, y, v) := \langle c, x\rangle + \langle b - Ax, y\rangle + \langle -x, v\rangle$. Then the dual problem is

$$\begin{array}{ll} \max & \theta(y, v) \\ \text{s.t.} & v \in K^*, \end{array}$$

where $K^* = \{d \in X \mid \langle d, z\rangle \geq 0 \;\; \forall\, z \in K\}$ is the dual cone of K and

$$\theta(y, v) = \min\{\mathcal{L}(x, y, v) \mid x \in X\}\,.$$

Fix (y, v). Since $\mathcal{L}$ is a convex function (linear) on x

$$\mathcal{L}(x, y, v) = \langle c - A^* y - v, x \rangle + \langle b, y \rangle \,,$$

we have

$$\theta(y, v) = \begin{cases} \langle b, y \rangle & \text{if } c - A^* y - v = 0 \\ -\infty & \text{if } c - A^* y - v \neq 0 \,, \end{cases}$$

where A^* is the adjoint operator of A, i.e.,

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \forall x \in X, \; y \in \Re^m \,.$$

Then the dual problem becomes

$$\begin{aligned} \max \quad & \langle b, y \rangle \\ \text{s.t.} \quad & c - A^* y - v = 0 \\ & v \in K^*. \end{aligned}$$

If $K = K^*$ (self-dual), we get

$$\begin{aligned} \max \quad & \langle b, y \rangle \\ \text{s.t.} \quad & c - A^* y \in K\,. \end{aligned}$$

If $K = \Re^n_+$, we get the linear programming

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array}$$

and its dual

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & c - A^T y \geq 0 \\ & (\textbf{or } s + A^T y - c = 0,\ s \geq 0\,.) \end{array}$$

If $K = \mathcal{K}^n$ (SOC), we get the SOC programming

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x_1 \geq \|(x_2, \cdots, x_n)\|, \end{array}$$

and its dual

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & c - A^T y \in \mathcal{K}^n\,. \\ & (\textbf{or } s + A^T y - c = 0,\ s \in \mathcal{K}^n\,.) \end{array}$$

If $K = \mathcal{S}^n_+$ (SDP cone), we get the SDP programming

$$\begin{array}{lll} \min & \langle c, x\rangle & (\mathbf{c}, \mathbf{x}\ \textbf{matrices}) \\ \text{s.t.} & Ax = b, & (\textbf{A linear operator}) \\ & x \succeq 0, & \end{array}$$

where $Ax = [\langle A_1, x\rangle, \cdots, \langle A_m, x\rangle]^T$, $A_1, \ldots, A_m$ are symmetric matrices, and its dual

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & c - (A_1 y_1 + \cdots + A_m y_m) \succeq 0 \\ & (\textbf{or } s + A^T y - c = 0,\ s \succeq 0\,.) \end{array}$$