NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
Semester I (2009/2010) MA4254 Discrete Optimization Tutorial 3

Q1. Show that a square matrix U is integer and unimodular if and only if its inverse U^{-1} is integer and unimodular. $[UU^{-1} = U^{-1}U = I]$

Q2. Show that

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

is totally unimodular.

Q3. Is the following matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

totally unimodular? Why?

Q4. Suppose that $A \in \Re^{m \times n}$ is of full row rank ($m \leq n$), $c, d \in Z^m$ and $c < d$. Show that any extreme point to the polyhedron

$$S = \{x \in \Re^n \,|\, Ax = b\,, c \leq x \leq d\}$$

is an integer if A is totally unimodular and $b \in Z^m$.

Q5. Suppose that there are n people and m jobs, where $n \geq m$. Each job must be assigned to exactly one person, and each person can do at most one job. The cost of person j doing job i is c_{ij}. Then *the Assignment Problem* can be formulated as

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = 1\,, i = 1, \ldots, m \\ & \sum_{i=1}^{m} x_{ij} \leq 1\,, j = 1, \ldots, n \\ & x \in B^{mn}\,. \end{aligned}$$

Show that the above problem can be reformulated as

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = 1\,, i = 1\,, \ldots, m \\
& \sum_{i=1}^{m} x_{ij} \le 1\,, j = 1\,, \ldots, n \\
& x \ge 0\,.
\end{aligned}$$

Q6. Let $V = \{1, \ldots, m\}$. Suppose that $V_1 = \{1\}$, $V_3 = \{m\}$, $V'(1) = \emptyset$, $V(m) = \emptyset$. Then *the maximum flow* problem is to maximize the total flow into vertex m under the capacity constraints

$$\begin{aligned}
\text{maximize} \quad & v \\
\text{subject to} \quad & \sum_{i \in V(1)} x_{1i} = v \\
& \sum_{j \in V(i)} x_{ij} - \sum_{j \in V'(i)} x_{ji} = 0\,, i \in V_2 = \{2, \ldots, m-1\} \\
& \sum_{i \in V'(m)} x_{im} = v \\
& 0 \le x_{ij} \le d_{ij}\,,\ (i,j) \in E\,.
\end{aligned}$$

(i) Write down the dual of the maximum flow problem.

(ii) Show that every basic feasible solution to the dual problem is an integer provided that all d_{ij} are integer.