NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
Semester I (2009/2010) MA4254 Discrete Optimization Tutorial 4

Q1. Let A be the node-arc incidence matrix of a digraph $G = (V, E)$. Can we claim that all extreme points to $\{x \in \Re^n \mid Ax = b, x \geq 0\}$ are necessarily $\{0, 1\}$ vectors? where $b = [-1, 0 \ldots, 0, +1]^T$. Why? [Note that a vector is called a $\{0, 1\}$ vector if each of its components is either 0 or 1. For example, $(1, 0, 1, 0, 0, 0)^T$ and $(1, 1, 1)^T$ are $\{0, 1\}$ vectors.]

Q2. Consider the standard linear programming

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \geq 0 \\ & x \geq 0 \end{array}$$

and its dual

$$(D) \quad \begin{array}{ll} \max & \pi^T b \\ \text{s.t.} & \pi^T A \leq c^T \,. \end{array}$$

Suppose that we have a current π which is feasible to the dual problem (D). Define the index set J by

$$J = \{j : \pi^T A_j = c_j\} \,,$$

where A_j is the jth column of A. Then for any $j \notin J$, we have $\pi^T A_j < c_j$. We call J the set of *admissible columns.* In order to search for an x such that it is not only feasible to the primal problem (P) but also it, togther with π, satisfies the complementary condition of (P) and (D), we invent a new LP, called the *restricted primal* (RP), as follows

$$(RP) \quad \begin{array}{rcl} \xi^* = & \min & \displaystyle\sum_{i=1}^m x_i^a \\ & \text{s.t.} & Ax + x^a = b \\ & & x_j \geq 0 \,, \text{ for all } j \,, \\ & & x_j = 0 \,, \, j \notin J \,, \\ & & x_i^a \geq 0 \,, \, i = 1, \ldots, m \,, \end{array}$$

i.e.,

$$(RP) \quad \begin{array}{rcl} \xi^* = & \min & 0^T x_J + \displaystyle\sum_{i=1}^m x_i^a \\ & \text{s.t.} & A_J x_J + x^a = b \\ & & x_J \geq 0, x^a \geq 0 \,. \end{array}$$

The dual of (RP) is

$$
(DRP) \qquad \begin{aligned} w^* = \quad & \max \quad \pi^T b \\ & \text{s.t.} \quad \pi^T A_j \le 0,\ j \in J \\ & \qquad\ \ \pi_i \le 1,\ i = 1, \ldots, m\,. \end{aligned}
$$

Let $(\bar{x}_J, \bar{x}^a)$ be an optimal basic feasible solution to (RP) and $\bar{\pi}$ be an optimal basic feasible solution to (DRP) obtained from $(\bar{x}_J, \bar{x}^a)$. If $w^* = 0$, then $\xi^* = 0$. Such an x is found. Otherwise, $w^* > 0$ and we can update π to

$$\pi^{\text{new}} = \pi + \theta\bar{\pi}\,.$$

The new cost to (D) is

$$(\pi^{\text{new}})^T b = \pi^T b + \theta\bar{\pi}^T b = \pi^T b + \theta w^*,$$

which means that we shall get a better π if we can take $\theta > 0$. On the other hand, π^{new} should be feasible to (D), i.e.,

$$(\pi^{\text{new}})^T A_j = \pi^T A_j + \theta\bar{\pi}^T A_j \le c_j\,.$$

Since for every $j \in J$, $\bar{\pi}^T A_j \le 0$, we only need to consider those $\bar{\pi}^T A_j > 0$, $j \notin J$. Therefore, we can take

$$\theta = \min_{\substack{j \notin J \\ \text{such that} \\ \bar{\pi}^T A_j > 0}} \frac{c_j - \pi^T A_j}{\bar{\pi}^T A_j}$$

a) Show that every admissible column in the optimal basis of (RP) remains admissble at the start of the next iteration for π^{new}. [Hint: consider the reduced costs and note that the coefficients of x_J in the objective function in (RP) are zeros]

b) Show that in the primal-dual simplex method for solving the shortest path problem every admissible column (columns correspond to every adimissible index) in the optimal basis of the restricted primal problem remails admissible at the start of the next iteration. [Note that this means that we can fully make use of the information from the last iteration to start the next iteration.]

Q3. A digraph $G = (V, E)$ has arc lengths given by the following matrix

$$
C = \begin{bmatrix} 0 & 2 & 6 & \infty & \infty & \infty \\ \infty & 0 & \infty & 1 & 3 & \infty \\ \infty & \infty & 0 & \infty & \infty & 2 \\ \infty & \infty & 1 & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & 2 & \infty & 0 \end{bmatrix}.
$$

Let A be the node-arc incidence matrix of digraph $G = (V, E)$. Then the problem of finding the shortest path from node 1 to node 6 can be reformulated as the following integer linear programming (ILP):

$$\begin{aligned} \min \quad & \sum_{j=1}^{9} c_j x_j \\ \text{s.t.} \quad & Ax = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ +1 \end{bmatrix} \\ & x_i \in \{0, 1\}, \ i = 1 \ldots, 9 \,, \end{aligned}$$

where c_i is the length of the ith arc in G.

(1) Show that the solving the ILP is equivalent to solving the following linear programming (LP):

$$\begin{aligned} \min \quad & \sum_{j=1}^{9} c_j x_j \\ \text{s.t.} \quad & Ax = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ +1 \end{bmatrix} \\ & 0 \le x_i \le 1 \,, \ i = 1 \ldots, 9 \,. \end{aligned}$$

(2) Use the primal-dual simplex method to solve the above LP.

(3) Give an optimal solution to the ILP.