Semismooth Homeomorphisms and Strong Stability of Semidefinite and Lorentz Complementarity Problems

Jong-Shi Pang[*], Defeng Sun[†], and Jie Sun[‡]

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Abstract

Based on an inverse function theorem for a system of semismooth equations, this paper establishes several necessary and sufficient conditions for an isolated solution of a complementarity problem defined on the cone of symmetric positive semidefinite matrices to be strongly regular/stable. We show further that for a parametric complementarity problem of this kind, if a solution corresponding to a base parameter is strongly stable, then a semismooth implicit

[*]Department of Mathematical Sciences, The Johns Hopkins University, Baltimore, Maryland 21218-2682, U.S.A. Email: pang@mts.jhu.edu. The work of this author's research was partially supported by the National Science Foundation under grant CCR-0098013.

[†]Department of Mathematics, National University of Singapore, Singapore. Email: matsundf@nus.edu.sg. This author's research was partially supported by Grant R146-000-035-101 of the National University of Singapore.

[‡]Department of Decision Sciences and Singapore-MIT Alliance, National University of Singapore, Singapore. Email: jsun@nus.edu.sg. This author's research was partially supported by Grants R314000-026/028/042-112 of the National University of Singapore and Singapore-MIT Alliance.

solution function exists whose directional derivatives can be computed by solving certain affine problems on the critical cone at the base solution. Similar results are also derived for a complementarity problem defined on the Lorentz cone. The analysis relies on some new properties of the directional derivatives of the projector onto the semidefinite cone and the Lorentz cone.

1 Introduction

The concept of a strongly regular solution to a generalized equation introduced by Robinson [39] and the related concept of a strongly stable stationary point of a differentiable nonlinear program (NLP) introduced by Kojima [18] are two of the most important ideas in contemporary perturbation analysis of mathematical programming problems. Beginning with Jongen, Mobert, Rückermann, and Tammer [12], many authors have established the equivalence between these two concepts for the Karush-Kuhn-Tucker (KKT) system of a nonlinear program with finitely many twice differentiable functions. The article by Klatte and Kummer [17] presents a unified framework that handles both concepts simultaneously and contains a brief bibliographical note. For an excellent survey of perturbation analysis of optimization problems, see the review by Bonnans and Shapiro [3] and their comprehensive monograph [4].

Extending the seminal work of Robinson and Kojima mentioned above, many authors have investigated the solution stability of variational inequalities (VIs); see, e.g., [9], which characterizes strong stability in linearly constrained VIs in terms of the "Aubin property". For a comprehensive treatment of the subject of solution stability of VIs, we refer the reader to Chapter 4 in [10]. As explained in the paper [27], there are substantial differences between the sensitivity and stability analysis of an NLP and that of a VI. Most importantly, the lack of symmetry in the defining function of a VI invalidates a straightforward optimization approach for such analysis. Focusing on an NLP, Kojima was the first person to utilize degree theory on a nondifferentiable system of equations to derive stability results in mathematical programming. Kojima's equation approach turns out to be very fruitful for the sensitivity and stability study of the VI and the related complementarity problem (CP). The forthcoming monograph [10] contains a long chapter on this subject, which is developed based on the equation approach and degree theory; there are many references in the bibliography therein. Among its many applications, the strong stability of a solution to a VI plays a very important role in the derivation of optimality conditions for mathematical programs with equilibrium constraints; see [29, 34].

As evidenced in Kojima's classic paper, the strong stability and strong regularity concepts are intimately related to inverse and implicit function theorems for systems of nonsmooth equations. This connection was further illustrated in the work of Robinson [41] who obtained an implicit function theorem of a B-differentiable equation under a crucial strong B-differentiability assumption. Prior to Robinson, Clarke [8] established an implicit function theorem for a locally Lipschitz continuous function under a nonsingularity assumption on the generalized Jacobian matrices that he championed. Kummer [22, 23, 24] obtained a complete characterization of a locally Lipschitz homeomorphism in terms of a set of directional-derivative-like vectors and applied the results to nonsmooth parametric optimization. To date, the application of Kummer's implicit function theorem to the VI has not been fruitfully explored.

Nonsmooth implicit/inverse function theorems

While the assumption in Clarke's implicit function theorem is very restrictive, the orignal application of Robinson's implicit function theorem for a strongly B-differentiable function to the VI was essentially restricted to a linearly constrained problem [39]. The restrictiveness of the strong B-differentiability was first noted by Kuntz and Scholtes [26] who wrote that "strong B-differentiability is a rather restrictive requirement for piecewise differentiable functions"; in the same paper, Kuntz and Scholtes also showed that "generically a piecewise differentiable function can be locally transformed into a B-differentiable function by means of a piecewise differentiable homeomorphism". These results allow these two authors to obtain "structural inverse function theorems for piecewise differentiable functions". When applied to a VI defined by finitely many differentiable convex functions, Kuntz and Scholtes assumed that "every collection of at most n of the (active constraint gradients) are linearly independent".

About the same time as the above work of Kuntz and Scholtes, Pang and Ralph [36] employed degree theory to obtain an implicit function theorem for a piecewise smooth function and applied it to the parametric analysis of normal maps defined on non-polyhedral sets satisfying Janin's constant-rank constraint qualification (CRCQ). The latter CQ is much broader than the assumption used by Kuntz and Scholtes (e.g., linear constraints naturally satisfy the CRCQ but not necessarily the Kuntz-Scholtes condition). Subsequently, Ralph and Scholtes [38] extended the Pang-Ralph theorem to a composite piecewise smooth function.

Ideally, a *complete* inverse (or implicit) function theorem should contain the following two ingredients: (a) conditions on a "first-order approximation" of the base function that are necessary and sufficient for the existence and uniqueness of the inverse (or implicit) function, and (b) inheritance of continuity and differentiability properties of the inverse (or implicit) function from the given function. The Pang-Ralph implicit/inverse function theorem for piecewise differentiable functions is complete in this sense. Our main contribution in this paper is twofold. One, to provide such a theorem for the class of vector *semismooth* functions [30, 37]; and two, more importantly, to apply the theorem to complementarity problems defined on the cone of symmetric positive semidefinite matrices and on the Lorentz cone, thereby obtaining necessary and sufficient conditions for the strong stability/regularity of a solution to such a complementarity problem in terms of a *canonically linearized* complementarity subproblem of the same kind. The latter application is made possible by recent results that establish the semismoothness of metric projections onto these cones: For the cone of symmetric positive semidefinite matrices, see Sun and Sun [49]; for the Lorentz cone, see Chen, Sun, and Sun [5].

The inverse function Theorem 6 and the implicit function Corollary 8 for semismooth functions that we establish in this paper are a synthesis of various known results in the literature, which by themselves are not complete in the aforementioned sense. Specifically, in his habilitation thesis, Scholtes [45, part 1 of Theorem 3.2.3] showed that if a

"B-differentiable" vector function Φ is a "locally Lipschitz homeomorphism" at a point x, then its "B-derivative" is a Lipschitz homeomorphism"; moreover the local inverse of Φ is B-differentiable at $\Phi(x)$ and its B-derivative is the inverse of the directional derivative $\Phi'(x;\cdot)$. (The latter result is contained in the earlier paper by Kummer [25, Lemma 2].) Scholtes' result is not a complete inverse function theorem because it does not provide sufficient conditions on the B-derivative $\Phi'(x;\cdot)$ (which is a pointwise first-order approximation of a B-differentiable function) for Φ to be a locally Lipschitz homeomorphism. To be fair, the converse of Scholtes' result (i.e., the Lipschitz homeomorphism of $\Phi'(x;\cdot)$ implying the locally Lipschitz homeomorphism of Φ at x) does not hold in general. In fact, a major contribution of our work is to show that if Φ is semismooth in the sense of Definition 4, then the converse in question is valid under a certain technical assumption relating the B-subdifferential $\partial_B \Phi(x)$ of Φ at x to that of $\Phi'(x;\cdot)$ at the origin; see (8) in Theorem 6. This assumption first appears in [36] for the class of piecewise differentiable functions.

Within the class of "H-differentiable" functions introduced in [50], Gowda [13] obtained inverse and implicit functions theorems that are in the spirit of [36] but not quite complete in the aforementioned sense. Specifically, for the subclass of "G-semismooth" functions Φ considered in [13], (see the discussion immediately following Theorem 5 for the definition of such a function) the author shows that a G-semismooth local inverse exists at a point x if and only if $\partial_B \Phi(x)$ consists of positively (negatively) oriented matrices and the index of Φ at x is equal to 1 (-1, respectively). While Gowda has not published his manuscript, a recent short note of Sun [48] establishes the same G-semismoothness of the local inverse function, assuming that the latter exists. It should be noted that the G-semismoothness property used by Gowda and Sun deviates from the original definition of [37] in that they do not impose directional differentiability on the function. For semismooth functions in the original sense of Qi and Sun [37], which are directionally differentiable, one has to combine the previous results of Scholtes with those of Gowda and Sun in order to deduce the directional differentiability of the implicit/inverse function. Yet, such a combined result is still not complete without the technical assumption (8) on the B-subdifferentials $\partial_B \Phi(x)$ and $\partial_B \Phi'(x;\cdot)(0)$.

2 The Finite-Dimensional VI/CP

We begin with a brief review of the VI/CP, followed by the formal definition of strong stability and strong regularity. The section ends with a result that establishes several equivalent ways of describing these two solution concepts. We refer the reader to the monograph [10] for a comprehensive study of the finite-dimensional variational inequality and complementarity problem.

Given a closed convex set $K \subseteq \Re^n$, a mapping $F : \Re^n \to \Re^n$, the VI (K, F) is to find a vector $x \in K$ such that

$$(y - x)^T F(x) \, \geq \, 0, \quad \forall \, y \, \in \, K.$$

The solution set of this problem is denoted $\mathrm{SOL}(K, F)$. Of fundamental importance to the VI is its normal map [42, 43, 44]:

$$\mathbf{F}^{\mathrm{nor}}_K(z) \, \equiv \, F(\Pi_K(z)) + z - \Pi_K(z), \quad \forall \, z \, \in \, \Re^n,$$

where Π_K denotes the Euclidean projector onto K. It is well known that if $x \in \mathrm{SOL}(K,F)$, then $z \equiv x - F(x)$ is a zero of $\mathbf{F}_K^{\mathrm{nor}}$; conversely, if z is a zero of $\mathbf{F}_K^{\mathrm{nor}}$, then $x \equiv \Pi_K(z)$ solves the VI (K,F). When K is in addition a cone, the VI (K,F) is equivalent to the CP (K,F):

$$K \ni x \perp F(x) \in K^*,$$

where K^* is the dual cone of C; i.e., $K^* \equiv \{y \in \Re^n : y^T x \geq 0, \ \forall x \in K\}$. For a positive scalar ε, we let $\mathbb{B}(0,\varepsilon)$ denote the open Euclidean ball with center at the origin and radius ε. For any subset S of $\Re^n$, we write $\mathrm{cl}\, S$ to denote the closure of S. We formally define strong stability and strong regularity as follows.

Definition 1 A solution x^* of the VI (K,F) is said to be

(a) *strongly regular* if for every open neighborhood $\mathcal{N}$ of x^* satisfying

$$\mathrm{SOL}(K,F) \cap \mathrm{cl}\mathcal{N} = \{ x^* \}, \tag{1}$$

there exist a positive scalar ε and a Lipschitz continuous function $x_{\mathcal{N}} : \mathbb{B}(0,\varepsilon) \to K$ such that, for every $q \in \mathbb{B}(0,\varepsilon)$, $x_{\mathcal{N}}(q)$ is the unique solution of the VI $(K, q+F)$ that belongs to $\mathcal{N}$;

(b) *strongly stable* if for every open neighborhood $\mathcal{N}$ of x^* satisfying (1), there exist two positive scalars c and ε such that for every continuous function G satisfying

$$\sup_{x \in K \cap \mathrm{cl}\mathcal{N}} \| G(x) - F(x) \| \leq \varepsilon,$$

the set $\mathrm{SOL}(K,G) \cap \mathcal{N}$ is a singleton; moreover, for another continuous function $\tilde{G}$ satisfying the same condition as G, it holds that

$$\| x - x' \| \leq c \| [F(x) - G(x)] - [F(x') - \tilde{G}(x')] \|,$$

where x and x' are the unique elements in the sets $\mathrm{SOL}(K, G) \cap \mathcal{N}$ and $\mathrm{SOL}(K, \tilde{G}) \cap \mathcal{N}$, respectively.

In essence, strong regularity pertains to small, constant perturbations of F whereas strong stability pertains to small, continuous perturbations of F. Thus it is clear that strong stability implies strong regularity. Interestingly, the converse turns out to be also true. Before formally stating this result (see Theorem 3), we note that Definition 1 is certainly applicable to a VI $(\Re^n, H)$, which corresponds to the system of equations $H(x) = 0$ where H is a mapping from $\Re^n$ into itself. Thus we say that $x \in H^{-1}(0)$ is strongly regular if for every open neighborhood $\mathcal{N}$ of x satisfying $H^{-1}(0) \cap \mathrm{cl}\,\mathcal{N} = \{x\}$, a scalar $\varepsilon > 0$ and a Lipschitz continuous function $x_{\mathcal{N}} : \mathbb{B}(0, \varepsilon) \to \Re^n$ exist such that, for every $q \in \mathbb{B}(0, \varepsilon)$, $x_{\mathcal{N}}(q)$ is the unique zero of $q + H(x) = 0$ in $\mathcal{N}$. A similar statement can be made for a strongly stable zero of H. In particular, we can speak about the strong regularity and strong stability of a zero of the normal map $\mathbf{F}_K^{\mathrm{nor}}$. Since the latter map involves a change of variables (from x to z) and since the domain of the original VI (K, F), which is the set K, is different from the domain of the equation $\mathbf{F}_K^{\mathrm{nor}}(z) = 0$, which is the entire space $\Re^n$, it is not immediately obvious how the strong regularity (stability) of a solution $x^* \in \mathrm{SOL}(K, F)$ is related to the strong regularity (stability) of the zero $z^* \equiv x^* - F(x^*)$ of the normal map $\mathbf{F}_K^{\mathrm{nor}}$. Again, it can be shown that the two descriptions are equivalent. Let us consider another concept.

Definition 2 A function $H : \Re^n \to \Re^n$ is said to be a *locally Lipschitz homeomorphism* near a vector x if there exists an open neighborhood $\mathcal{N}$ of x such that the restricted map $H|_{\mathcal{N}} : \mathcal{N} \to H(\mathcal{N})$ is Lipchitz continuous and bijective, and its inverse is also Lipschitz continuous.

We can now state the following result, whose proof can be found in [10]. The significance of this result is that the strong stability/regularity of a solution to a VI can be deduced from an inverse function theorem for the normal map.

Theorem 3 Let $F : K \subseteq \Re^n \to \Re^n$ be locally Lipschitz continuous on the closed convex set K. Let $x^* \in \mathrm{SOL}(K, F)$ be given. Let $z^* \equiv x^* - F(x^*)$. The following statements are equivalent:

(a) x^* is a strongly stable solution of the VI (K, F);

(b) x^* is a strongly regular solution of the VI (K, F);

(c) z^* is a strongly regular zero of $\mathbf{F}_K^{\mathrm{nor}}$;

(d) z^* is a strongly stable zero of $\mathbf{F}_K^{\mathrm{nor}}$;

(e) $\mathbf{F}_K^{\mathrm{nor}}$ is a locally Lipschitz homeomorphism near z^*

(f) There exist an open neighborhood $\mathcal{Z}$ of z^* and a constant $c > 0$ such that

$$\| \, \mathbf{F}_K^{\mathrm{nor}}(z) - \mathbf{F}_K^{\mathrm{nor}}(z') \, \| \, \geq \, c \, \| \, z - z' \, \|, \quad \forall \, z, z' \, \in \, \mathcal{Z}.$$

The equivalence of statements (d), (e) and (f) in the above theorem remains valid for all locally Lipschitz continuous functions, of which the normal map $\mathbf{F}_K^{\text{nor}}$ is a special instance; see [10].

3 Semismooth Homeomorphisms

Extending Mifflin's definition for a scalar function [30], Qi and Sun [37] introduced the semismoothness property for a vector function. There are several equivalent ways to define this property. We first give a definition and then summarize the equivalent conditions in Theorem 5 below.

Definition 4 Let $G : \mathcal{D} \subseteq \Re^n \to \Re^m$ be a locally Lipschitz continuous function on the open set $\mathcal{D}$. We say that G is *semismooth* at a point $\bar{x} \in \mathcal{D}$ if G is directionally differentiable near $\bar{x}$ (thus G is B-differentiable near $\bar{x}$) and

$$\lim_{\bar{x} \neq x \to \bar{x}} \frac{\| G'(x; x - \bar{x}) - G'(\bar{x}; x - \bar{x}) \|}{\| x - \bar{x} \|} = 0.$$

If the above requirement is strengthened to

$$\limsup_{\bar{x} \neq x \to \bar{x}} \frac{\| G'(x; x - \bar{x}) - G'(\bar{x}; x - \bar{x}) \|}{\| x - \bar{x} \|^2} < \infty, \tag{2}$$

we say that G is *strongly semismooth* at $\bar{x}$.

For a locally Lipschitz continuous function $G : \mathcal{D} \subseteq \Re^n \to \Re^m$, with $\mathcal{D}$ open, the B-subdifferential of G at $\bar{x} \in \mathcal{D}$, denoted $\partial_B G(\bar{x})$, is the set of all $m \times n$ matrices V such that

$$V = \lim_{k \to \infty} JG(x^k),$$

where $\{x^k\} \subset \mathcal{D}$ is a sequence of F(réchet)-differentiable points of G converging to $\bar{x}$ and $JG(x^k)$ denotes the F-derivative of G at x^k, which is a linear operator mapping $\Re^n$ into $\Re^m$. The convex hull of $\partial_B G(\bar{x})$ yields Clarke's generalized Jacobian of G at $\bar{x}$, denoted $\partial G(\bar{x})$ [8]. For a piecewise smooth function G, $\partial_B G(\bar{x})$ is a finite set, see [36, 38]. Nevertheless, if G is semismooth but not piecewise smooth, $\partial_B G(\bar{x})$ generally can have infinitely many elements, but it must be a nonempty compact set; moreover, as a set-valued map, $\partial_B G$ is upper semicontinuous.

In terms of the elements in the B-subdifferential, we have the following result whose proof can be found in [37, 10].

Theorem 5 Let $G : \mathcal{D} \subseteq \Re^n \to \Re^m$, with $\mathcal{D}$ open, be B-differentiable; i.e., G is locally Lipschitz continuous and directionally differentiable near $\bar{x} \in \mathcal{D}$. The following three statements are equivalent:

(a) G is semismooth at $\bar{x}$;

(b) the following limit holds:

$$\lim_{\substack{\bar{x} \neq x \to \bar{x} \\ \forall\, V \in \partial G(x)}} \frac{\| G'(\bar{x}; x - \bar{x}) - V(x - \bar{x}) \|}{\| x - \bar{x} \|} = 0; \tag{3}$$

(c) the following limit holds:

$$\lim_{\substack{\bar{x} \neq x \to \bar{x} \\ \forall\, V \in \partial G(x)}} \frac{\| G(x) + V(\bar{x} - x) - G(\bar{x}) \|}{\| x - \bar{x} \|} = 0. \tag{4}$$

If G is strongly semismooth at $\bar{x}$, then

$$\limsup_{\bar{x} \neq x \to \bar{x}} \frac{\| G(x) - G(\bar{x}) - G'(\bar{x}, x - \bar{x}) \|}{\| x - \bar{x} \|^2} < \infty, \tag{5}$$

$$\limsup_{\substack{\bar{x} \neq x \to \bar{x} \\ \forall\, V \in \partial G(x)}} \frac{\| G(x) + V(\bar{x} - x) - G(\bar{x}) \|}{\| x - \bar{x} \|^2} \quad < \quad \infty. \tag{6}$$

Gowda [13] called a locally Lipschitz continuous function Φ that satisfies (4) "semismooth" at $\bar{x}$. In order to distinguish this kind of semismoothness, which does not require Φ to be directionally differentiable, we attach the letter G (for Gowda) and say that Φ is G-semismooth at $\bar{x}$ if (4) holds.

Theorem 3 has reduced the strong stability/regularity of a solution to the VI to the locally Lipschitz homeomorphism property of the normal map near a zero. By the next result, Theorem 6, we obtain several necessary and sufficient conditions for the latter property to hold. Most important among these conditions is a globally Lipschitz homeomorphism property of the directional derivative of the normal map. It is the latter property that enables us to obtain the ultimate necessary and sufficient conditions for the strong stability/regularity of a solution to a CP on two special non-polyhedral, self-dual cones that will be discussed in the next section.

The following result is the promised inverse function theorem for semismooth functions. It uses degree theory and the index of a continuous function Φ at its zero x^*, denoted $\mathrm{ind}(\Phi, x^*)$. The reader who is unfamiliar with this theory can consult many excellent references, e.g. [28, 33]. As mentioned in the Introduction, the theorem is a synthesis of various existing results in the literature; as such, we only give the sources of the proofs.

Theorem 6 Let $\Phi : \Re^n \to \Re^n$ be Lipschitz continuous in an open neighborhood $\mathcal{D}$ of a vector $x^* \in \Phi^{-1}(0)$. Consider the following three statements:

(a) every matrix in $\partial\Phi(x^*)$ is nonsingular;

(b) Φ is a locally Lipschitz homeomorphism near x^*;

(c) for every $V \in \partial_B \Phi(x^*)$, $\operatorname{sgn} \det V = \operatorname{ind}(\Phi, x^*) = \pm 1$.

It holds that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Assume in addition that Φ is directionally differentiable at x^*. Consider the following two additional statements:

(d) $\Psi \equiv \Phi'(x^*; \cdot)$ is a globally Lipschitz homeomorphism;

(e) for every $V \in \partial_B \Psi(0)$, $\operatorname{sgn} \det V = \operatorname{ind}(\Psi, 0) = \operatorname{ind}(\Phi, x^*) = \pm 1$.

It holds that (b) $\Rightarrow$ (d) $\Rightarrow$ (e). Moreover, if (b) holds and Φ is directionally differentiable at x^*, then the local inverse of Φ near x^*, denoted Φ^{-1}, is directionally differentiable at the origin; and

$$(\Phi^{-1})'(0; h) \;=\; \Psi^{-1}(h), \quad \forall\, h \,\in\, \Re^n. \tag{7}$$

If Φ is semismooth on $\mathcal{D}$ then (b) $\Leftrightarrow$ (c); in this case, the local inverse of Φ near x^* is semismooth near the origin. Finally, if Φ is semismooth on $\mathcal{D}$ and

$$\partial_B \Phi(x^*) \;\subseteq\; \partial_B \Psi(0), \tag{8}$$

then the four statements (b), (c), (d), and (e) are equivalent.

Proof. (a) $\Rightarrow$ (b) is proved by Clarke [8]. (b) $\Rightarrow$ (c) is proved by Gowda [13, Theorem 3]. (b) $\Rightarrow$ (d) is proved by Kuntz and Scholtes [26]. (d) $\Rightarrow$ (e) is a special case of (b) $\Rightarrow$ (c). Suppose that Φ is directionally differentiable at x^* and (b) holds. The formula (7) can be found in [25]. If Φ is semismooth on $\mathcal{D}$ and (c) holds, then by [13, Corollary 4], it follows that the local inverse of Φ at x^* exists and is G-semismooth, hence locally Lipschitz continuous, in a neighborhood of the origin. Therefore, (b) $\Leftrightarrow$ (c) if Φ is semismooth on $\mathcal{D}$. In this case, the semismoothness of Φ^{-1} follows from results of Scholtes and Gowda. Finally, if Φ is semismooth on $\mathcal{D}$ and (8) holds, then clearly (e) implies (c). Hence, we have established (b) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (c) $\Rightarrow$ (b); the four statements (b), (c), (d), and (e) are therefore equivalent. Q.E.D.

The inclusion (8) plays an essential role for the statements (b) and (c) in Theorem 6, which pertain to the original function Φ, to be equivalent to the corresponding statements (d) and (e), which pertain to the directional derivative Ψ. This inclusion is not used by either Sun or Gowda in their papers. In what follows, we state and prove a result pertaining to this inclusion for a composite function.

Proposition 7 Let $\Phi : \mathcal{D} \subseteq \Re^n \to \Re^m$, with $\mathcal{D}$ open, be B-differentiable on $\mathcal{D}$. Suppose that for every $x \in \mathcal{D}$,

$$\partial_B \Phi(x) \;=\; \partial_B \Psi(0), \tag{9}$$

where $\Psi \equiv \Phi'(x; \cdot)$. Let $F : \Re^m \to \Re^\ell$ be continuously differentiable in an open neighborhood of $\Phi(\bar{x})$, where $\bar{x} \in \mathcal{D}$. With $\Xi \equiv (F \circ \Phi)'(\bar{x}; 0)$ it holds that

$$\partial_B (\, F \circ \Phi \,)(\bar{x}) \;=\; JF(\Phi(\bar{x})) \circ \partial_B \Phi(\bar{x}) \;=\; \partial_B \Xi(0). \tag{10}$$

Proof. To prove the first equality in (10), observe that if $y \in \mathcal{D}$ is a F-differentiable point of Φ that is sufficiently close to $\bar{x}$, then y is also a F-differentiable point of $F \circ \Phi$; moreover, we have $J(F \circ \Phi)(y) = JF(\Phi(y)) \circ J\Phi(y)$. Consequently, $JF(\Phi(\bar{x})) \circ \partial_B \Phi(\bar{x}) \subseteq \partial_B (\, F \circ \Phi \,)(\bar{x})$. Conversely, let $V \in \partial_B (F \circ \Phi)(\bar{x})$. There exists a sequence of F-differentiable points $\{x^k\} \subset \mathcal{D}$ of $F \circ \Phi$ converging to $\bar{x}$ such that $V = \lim_{k \to \infty} J(F \circ \Phi)(x^k)$. For each fixed k, $L \equiv J(F \circ \Phi)(x^k)$ is a linear operator from $\Re^n$ into $\Re^\ell$. We have, for each $y \in \Re^n$, $L(y) = JF(\Phi(x^k)) \circ \Phi'(x^k; y)$. As a linear operator, we have $L = JL(y)$ for any $y \in \Re^n$; i.e., L is the F-derivative of itself at every vector in the whole space $\Re^n$. In particular, taking any sequence $\{y^\nu\}$ of F-differentiable points of $\Psi^k \equiv \Phi'(x^k; \cdot)$ that converges to zero, we then have $J(F \circ \Phi)(x^k) = JL(y^\nu) = JF(\Phi(x^k)) \circ J\Psi^k(y^\nu)$. Since the sequence $\{J\Psi^k(y^\nu)\}$ (indexed by ν with k fixed) is bounded and every accumulation point of this sequence belongs to $\partial_B \Psi^k(0)$, it follows that $J(F \circ \Phi)(x^k) \in JF(\Phi(x^k)) \circ \partial_B \Psi^k(0)$. By (9), we deduce $J(F \circ \Phi)(x^k) \in JF(\Phi(x^k)) \circ \partial_B \Phi(x^k)$. Passing to the limit $k \to \infty$, using the continuous differentiability of F and the upper semicontinuity of ∂_B, we deduce $V \in JF(\Phi(\bar{x})) \circ \partial_B \Phi(\bar{x})$. Consequently $\partial_B(F \circ \Phi)(\bar{x}) = JF(\Phi(\bar{x})) \circ \partial_B \Phi(\bar{x})$. Since $\Xi(y) = JF(\Phi(\bar{x})) \circ \Phi'(\bar{x}; y)$, it follows that Ξ is the composition of the linear transformation $JF(\Phi(\bar{x})) : \Re^m \to \Re^\ell$ and the directional derivative $\Psi = \Phi'(\bar{x}; \cdot) : \Re^n \to \Re^m$. We can therefore apply the previous proof to Ξ and deduce that $\partial_B \Xi(0) = JF(\phi(\bar{x})) \circ \partial_B \Psi(0)$, provided that $\partial_B \Psi(0) = \partial_B \Upsilon(0)$, where $\Upsilon \equiv \Psi'(0; \cdot)$. Since Ψ is a positively homogeneous function, its directional derivative at the origin is equal to Ψ itself; i.e., $\Psi = \Upsilon$. Consequently, the last displayed equality holds. By (9), (10) follows readily. Q.E.D.

We make two remarks regarding the above proposition. First, without assuming the directional differentiability of Φ, Clarke [8, page 75] showed that for any $v \in \Re^n$,

$$\partial(F \circ \Phi)(\bar{x})v \, = \, JF(\Phi(\bar{x}))(\, \partial\Phi(\bar{x})v \,).$$

This, however, does not imply either

$$\partial(F \circ \Phi)(\bar{x}) \,=\, JF(\Phi(\bar{x})) \circ \partial\Phi(\bar{x}), \quad \text{or} \quad \partial_B(F \circ \Phi)(\bar{x}) \,=\, JF(\Phi(\bar{x})) \circ \partial_B\Phi(\bar{x})\,.$$

Second, it is natural for the reader to wonder why it is necessary to give the detailed proof for the first equality in (10) as we did above, because after all it seems that the set of F-differentiable points of the composite function $F \circ \Phi$ would naturally coincide with that of the function Φ. A moment's thought reveals that this is false in general; an easy counterexample is to let F be the zero function. Therefore, our proof, while not difficult, is needed.

We apply Theorem 6 to the following situation. Let $G : \Re^{N+m} \to \Re^n$ be a function of two arguments $(w,p) \in \Re^{N+m}$, and let $\Phi : \Re^n \to \Re^N$ be a nonsmooth function. Let $G(\Phi(x^*),p^*) = 0$; suppose that Φ is semismooth at x^* and G is continuously differentiable in an open neighborhood of $(\Phi(x^*),p^*)$. Consider the mapping $\Xi : \Re^{n+m} \to \Re^{n+m}$ defined by

$$\Xi(x,p) \,\equiv\, \begin{bmatrix} G(\Phi(x),p) \\ \\ p - p^* \end{bmatrix}, \quad (\,x,p\,) \,\in\, \Re^{n+m},$$

which vanishes at (x^*,p^*). We have

$$\begin{aligned} \Xi'((x^*,p^*);(dx,dp)) \;&=\; \begin{bmatrix} J_wG(\Phi(x^*),p^*)\Phi'(x^*;dx) + J_pG(\Phi(x^*),p^*)dp \\ \\ dp \end{bmatrix} \\ &=\; \begin{bmatrix} J_wG(\Phi(x^*),p^*) & J_pG(\Phi(x^*),p^*) \\ \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi'(x^*;dx) \\ \\ dp \end{bmatrix}, \end{aligned}$$

where J_wG and J_pG denote the partial F-derivative of G with respect to the w and p argument respectively. We have the following implicit function theorem for the parametric composite equation $G(\Phi(x), p) = 0$, which does not require a proof.

Corollary 8 Assume that $\partial_B\Phi(x^*) \subseteq \partial_B\Phi'(x^*;\cdot)(0)$ and that $J_wG(\Phi(x^*), p^*) \circ \Phi'(x^*;\cdot)$ is a globally Lipschitz homeomorphism. There exist a neighborhood $\mathcal{U}$ of p^*, a neighborhood $\mathcal{V}$ of x^*, and a Lipschitz continuous function $x : \mathcal{U} \to \mathcal{V}$ that is semismooth at p^* such that for every $p \in \mathcal{U}$, $x(p)$ is the unique vector in $\mathcal{V}$ satisfying $G(\Phi(x(p)), p) = 0$. Moreover, for every vector $dp \in \Re^m$, $x'(p^*, dp)$ is the unique solution dx of the following equation:

$$J_wG(\Phi(x^*), p^*)\Phi'(x^*; dx) + J_pG(\Phi(x^*), p^*)dp \;=\; 0.$$

The normal map: general discussion

Consider the VI (K, F), where K is a closed convex set in $\Re^n$ and $F : \Re^n \to \Re^n$ is continuously differentiable in an open neighborhood of a solution x^* of the problem. We wish to apply Theorem 6 to the normal map:

$$\mathbf{F}_K^{\text{nor}}(z) \;=\; F(\Pi_K(z)) + z - \Pi_K(z), \quad z \;\in\; \Re^n,$$

at the zero $z^* \equiv x^* - F(x^*)$. For this purpose, we need to establish the semismoothness of $\mathbf{F}_K^{\text{nor}}$ at z^* and to verify the key equality:

$$\partial_B\mathbf{F}_K^{\text{nor}}(z^*) \;\equiv\; \partial_B(\,(\mathbf{F}_K^{\text{nor}})'(z^*;\cdot)\,)(0). \tag{11}$$

If F is continuously differentiable, the semismoothness of $\mathbf{F}_K^{\rm nor}$ follows easily from that of the projector Π_K. The verification of (11) is easy provided that we can establish

$$\partial_B \Pi_K(u) \,=\, \partial_B \Pi_K'(u;\cdot)(0), \quad \forall\, u \,\in\, \Re^n. \tag{12}$$

This is due to the fact that $\mathbf{F}_K^{\rm nor}$ is the composite map $G \circ \Phi$, where $G : \Re^{2n} \to \Re^n$ is given by $G(u,v) \,\equiv\, F(u) + v - u$ and $\Phi(u) \,\equiv\, (\Pi_K(u), u)$. Assuming that Π_K is directionally differentiable, we have $\Phi'(u;du) = (\Pi_K'(u;du), du)$. Moreover, Φ is F-differentiable at u if and only if Π_K is F-differentiable at u; at such a vector, we have

$$J\Phi(u) \,=\, \left[\begin{array}{c} J\Pi_K(u) \\ \\ I \end{array} \right].$$

Consequently, $\partial_B \Phi(u) = \partial_B \Pi_K(u) \times \{I\}$. Hence $\partial_B \Phi(u) = \partial_B \Phi'(u;\cdot)(0)$ if and only if (12) holds.

In summary, we see that Theorem 6 is applicable to the normal map $\mathbf{F}_K^{\rm nor}$ provided that we can establish two things: (i) the projector Π_K is semismooth, and (ii) (12) holds. For the special cones K we are interested in, the semismoothness of Π_K follows from existing results. So the main task in the next section is the verification of (12) for these cones. This turns out to be not an easy task. After completing this technical task, we then study the strong stability/regularity of a solution to the associated CPs in Section 5.

4 Projections on Two Self-Dual Cones

In this section, we focus on two special self-dual cones: the cone of symmetric positive semidefinite matrices and the Lorentz cone. For the purpose of verifying (12) for these cones, we first establish several new properties of the projections onto them, extending some recent results in [1, 49] for the positive semidefinite cone and [5] for the Lorentz cone.

4.1 The semidefinite cone

Let $\mathcal{S}^n$ denote the space of $n \times n$ symmetric matrices; let $\mathcal{S}^n_+$ and $\mathcal{S}^n_{++}$ denote the cone of $n \times n$ symmetric positive semidefinite and positive definite matrices, respectively. We write $A \succeq 0$ to mean that A is a symmetric positive semidefinite matrix. For any two matrices A and B in $\mathcal{S}^n$, we write

$$A \bullet B \equiv \sum_{i,j=1}^{n} a_{ij} b_{ij} = \operatorname{tr}(AB)$$

for the *Frobenius inner product* between A and B, where "tr" denotes the trace of a matrix. We note that for any orthogonal matrix Q,

$$(QAQ^T) \bullet (QBQ^T) = A \bullet B.$$

The *Frobenius norm* on $\mathcal{S}^n$ is the norm induced by the above inner product:

$$\| A \| \equiv \sqrt{A \bullet A} = \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}.$$

Under the Frobenius norm, the projection $\Pi_{\mathcal{S}^n_+}(A)$ of a matrix $A \in \mathcal{S}^n$ onto the cone $\mathcal{S}^n_+$ is the unique minimizer of the following convex program in the matrix variable B:

$$\begin{array}{ll} \text{minimize} & \| A - B \| \\ \text{subject to} & B \in \mathcal{S}^n_+. \end{array}$$

Throughout the following discussion, we let A_+ denote the (Frobenius) projection of $A \in \mathcal{S}^n$ onto $\mathcal{S}^n_+$. This projection satisfies the following complementarity condition:

$$\mathcal{S}^n_+ \ni A_+ \perp A_+ - A \in \mathcal{S}^n_+, \tag{13}$$

where the $\perp$ notation means "perpendicular under the above matrix inner product"; i.e., $C \perp D \Leftrightarrow C \bullet D = 0$ for any two matrices C and D in $\mathcal{S}^n$. The projection A_+ has an explicit representation. Namely, if

$$A = P \Lambda P^T, \tag{14}$$

where Λ is the diagonal matrix of eigenvalues of A and P is a corresponding orthogonal matrix of orthonormal eigenvectors, then

$$A_+ = P \Lambda_+ P^T,$$

where Λ_+ is the diagonal matrix whose diagonal entries are the nonnegative parts of the respective diagonal entries of Λ. Define three fundamental index sets associated with the matrix A:

$$\alpha \equiv \{ i : \lambda_i > 0 \}, \quad \beta \equiv \{ i : \lambda_i = 0 \}, \quad \gamma \equiv \{ i : \lambda_i < 0 \};$$

these are the index sets of positive, zero, and negative eigenvalues of A, respectively. Write

$$\Lambda = \begin{bmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & \Lambda_\gamma & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = [\ W_\alpha \quad W_\gamma \quad Z\ ]$$

with $W_\alpha \in \Re^{n\times|\alpha|}$, $W_\gamma \in \Re^{n\times|\gamma|}$, and $Z \in \Re^{n\times|\beta|}$. Thus the columns of W_α, W_γ, and Z are the orthonormal eigenvectors corresponding to the positive, negative, and zero eigenvalues of A, respectively. Let $\kappa \equiv \alpha \cup \gamma$ and define three diagonal matrices of order $|\kappa|$:

$$D \equiv \begin{bmatrix} \Lambda_\alpha & 0 \\ 0 & \Lambda_\gamma \end{bmatrix} \quad D_+ \equiv \begin{bmatrix} \Lambda_\alpha & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad |\,D\,| \equiv \begin{bmatrix} \Lambda_\alpha & 0 \\ 0 & |\,\Lambda_\gamma\,| \end{bmatrix}.$$

Define the matrix $U \in \mathcal{S}^n$ with entries

$$u_{ij} \equiv \frac{\max\{\lambda_i, 0\} + \max\{\lambda_j, 0\}}{|\,\lambda_i\,| + |\,\lambda_j\,|}, \quad i, j = 1, \ldots, n,$$

where $0/0$ is defined to be 1. Define the linear transformation $\mathcal{L}_A : \mathcal{S}^n \to \mathcal{S}^n$ as follows: for $H \in \mathcal{S}^n$,

$$\mathcal{L}_A(H) \equiv P \begin{bmatrix} W_\alpha^T H W_\alpha & U_{\alpha\gamma} \circ W_\alpha^T H W_\gamma & W_\alpha^T H Z \\ W_\gamma^T H W_\alpha \circ U_{\alpha\gamma}^T & W_\gamma^T H W_\gamma & W_\gamma^T H Z \\ Z^T H W_\alpha & Z^T H W_\gamma & Z^T H Z \end{bmatrix} P^T,$$

where $\circ$ denotes the Hadamard product.

Associated with the projection problem (13) is the critical cone of $\mathcal{S}^n_+$ at $A \in \mathcal{S}^n$ defined as:

$$\mathcal{C}(A; \mathcal{S}^n_+) \equiv \mathcal{T}(A_+; \mathcal{S}^n_+) \cap (\,A_+ - A\,)^\perp,$$

where $\mathcal{T}(A_+; \mathcal{S}^n_+)$ is the tangent cone of $\mathcal{S}^n_+$ at A_+ and $(A_+ - A)^\perp$ is the subset of matrices in $\mathcal{S}^n$ that are orthogonal to $(A_+ - A)$ under the matrix inner product. The importance of the critical cone in the local analysis of constrained optimization is well known. In the present context, this cone can be completely described [4, 10]:

$$\mathcal{C}(A; \mathcal{S}^n_+) = \{\, C \in \mathcal{S}^n \,:\, W_\gamma^T C W_\gamma = 0,\, W_\gamma^T C Z = 0,\, Z^T C Z \succeq 0 \,\}. \tag{15}$$

The affine hull of $\mathcal{C}(A; \mathcal{S}^n_+)$, which we denote $\mathcal{L}(A; \mathcal{S}^n_+)$, is easily seen to be the linear subspace:

$$\{ C \in \mathcal{S}^n : W_\gamma^T C W_\gamma = 0, \, W_\gamma^T C Z = 0 \}.$$

Directional derivatives of A_+

Based on the theory of second order regular sets [2], Bonnans, Cominetti, and Shapiro [1] has shown that $\Pi_{\mathcal{S}^n_+}$ is directionally differentiable and that for any $H \in \mathcal{S}^n$, $\Pi'_{\mathcal{S}^n_+}(A; H)$ is the unique minimizer of the following convex program in the matrix variable X:

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2} (X - H) \bullet (X - H) + \operatorname{tr}(B_- X B_+ X) \\ &\text{subject to} \quad X \in \mathcal{C}(A; \mathcal{S}^n_+), \end{aligned} \tag{16}$$

where

$$B_- \equiv W_\gamma \, | \, \Lambda_\gamma \, | \, W_\gamma^T \quad \text{and} \quad B_+ \equiv W_\alpha \, \Lambda_\alpha^{-1} \, W_\alpha^T$$

with B_+ taken to be the vacuous matrix if α is empty. Sun and Sun show in the recent paper [49] that $\Pi_{\mathcal{S}^n_+}$ is a strongly semismooth matrix-valued function and give an explicit formula for the directional derivative of the absolute value function

$$| \, A \, |_{\mathcal{S}^n_+} \equiv \Pi_{\mathcal{S}^n_+}(A) + \Pi_{\mathcal{S}^n_+}(-A).$$

(See [6] for some extended results on more general matrix-valued functions). Such a formula immediately yields a corresponding formula for the directional derivative $\Pi'_{\mathcal{S}^n_+}(A; H)$; see (17). The convex program (16) suggests that $\Pi'_{\mathcal{S}^n_+}(A; H)$ can be viewed as a "skewed projection" of a certain matrix onto the critical cone $\mathcal{C}(A; \mathcal{S}^n_+)$. In (18), we make this view precise by showing that the directional derivative of the projector $\Pi_{\mathcal{S}^n_+}$ at a matrix $A \in \mathcal{S}^n$

along the direction $H \in \mathcal{S}^n$ is equal to the projection of the image of the direction H under the linear transformation $\mathcal{L}_A$ onto the critical cone $\mathcal{C}(A;\mathcal{S}^n)$. This interpretation generalizes a similar but much simpler result for the Euclidean projector onto a polyhedral set [16, 35] whose directional derivative is equal to the projection onto the critical cone.

Proposition 9 For any two matrices A and H in $\mathcal{S}^n$,

$$\Pi'_{\mathcal{S}^n_+}(A;H) \;=\; P \begin{bmatrix} W_\alpha^T H W_\alpha & U_{\alpha\gamma} \circ W_\alpha^T H W_\gamma & W_\alpha^T H Z \\ W_\gamma^T H W_\alpha \circ (U_{\alpha\gamma})^T & 0 & 0 \\ Z^T H W_\alpha & 0 & \Pi_{\mathcal{S}^{|\beta|}_+}(Z^T H Z) \end{bmatrix} P^T, \tag{17}$$

and

$$\Pi'_{\mathcal{S}^n_+}(A;H) \;=\; \Pi_{\mathcal{C}(A;\mathcal{S}^n_+)}(\mathcal{L}_A(H)). \tag{18}$$

Proof. Write $f(A) \equiv |A|_{\mathcal{S}^n_+}$ for $A \in \mathcal{S}^n$. By a result in [49], we have

$$f'(A;H) \;=\; P \begin{bmatrix} L^{-1}_{|D|}(D\tilde{H}_{\kappa\kappa} + \tilde{H}_{\kappa\kappa}D) & |D|^{-1}D\tilde{H}_{\kappa\beta} \\ \tilde{H}^T_{\kappa\beta}D|D|^{-1} & |\tilde{H}_{\beta\beta}| \end{bmatrix} P^T,$$

for any $H \in \mathcal{S}^n$, where $\tilde{H} \equiv P^T H P$ and $L_X : \mathcal{S}^n \to \mathcal{S}^n$ is the Lyapunov operator:

$$L_X(Y) \;\equiv\; XY + YX, \quad \forall\, Y \;\in\; \mathcal{S}^n.$$

Since $\Pi_{\mathcal{S}^n_+} = (I + |\cdot|_{\mathcal{S}^n_+})/2$, the identitity (17) follows easily from the above expression for $f'(A;H)$ and a simple manipulation. Since $\nabla(\mathrm{tr}(B_- X B_+ X)) = B_- X B_+ + B_+ X B_-$, the unique minimizer of (16), denoted $\bar{X}$, satisfies

$$\mathcal{C}(A;\mathcal{S}^n_+) \;\ni\; \bar{X} \;\perp\; \bar{X} - H + B_-\bar{X}B_+ + B_+\bar{X}\bar{B}_- \;\in\; \mathcal{C}(A;\mathcal{S}^n_+)^*.$$

Using (17) for $\bar{X} = \Pi'_{\mathcal{S}^n_+}(A;H)$, we can easily verify that $H - B_-\bar{X}B_+ - B_+\bar{X}\bar{B}_- = \mathcal{L}_A(H)$. This establishes the desired second equality in (18). Q.E.D.

The following corollary is stated for ease of reference later.

Corollary 10 The following two statements hold.

(a) The functions $|\cdot|_{\mathcal{S}^n_+}$ and $\Pi_{\mathcal{S}^n_+}$ are F-differentiable at $A \in \mathcal{S}^n$ if and only if A is nonsingular. In this case, $\Pi'_{\mathcal{S}^n_+}(A;\cdot) = L^{-1}_{|A|_{\mathcal{S}^n_+}} \circ L_{A_+}$.

(b) For any $A \in \mathcal{S}^n$, the directional derivative $\Pi'_{\mathcal{S}^n_+}(A;\cdot)$ is F-differentiable at $H \in \mathcal{S}^n$ if and only if $\tilde{H}_{\beta\beta}$ is nonsingular.

(c) For any $A, H \in \mathcal{S}^n$

$$\Pi'_{\mathcal{S}^n_+}(A;H) = P \begin{bmatrix} L^{-1}_{|D|} L_{D_+} \tilde{H}_{\kappa\kappa} & |D|^{-1} D_+ \tilde{H}_{\kappa\beta} \\ (\tilde{H}_{\kappa\beta})^T D_+ |D|^{-1} & \Pi_{\mathcal{S}^{|\beta|}_+}(\tilde{H}_{\beta\beta}) \end{bmatrix} P^T. \tag{19}$$

The next technical result establishes the equality (9) that paves the way for the application of Proposition 7.

Lemma 11 Let $A \in \mathcal{S}^n$ be arbitrary. Let $\Psi \equiv \Pi'_{\mathcal{S}^n_+}(A;\cdot)$. It holds that

$$\partial_B \Pi_{\mathcal{S}^n_+}(A) \;=\; \partial_B \Psi(0). \tag{20}$$

Moreover, for any $V \in \partial_B \Pi_{\mathcal{S}^n_+}(A)$, there exist two index sets α' and γ' that partition β and a matrix $\Gamma_{\alpha'\gamma'}$ with entries in $[0,1]$ such that for any $H \in \mathcal{S}^n$,

$$V(H) \;=\; P \begin{bmatrix} W_\alpha^T H W_\alpha & U_{\alpha\gamma} \circ W_\alpha^T H W_\gamma & W_\alpha^T H Z \\ \\ W_\gamma^T H W_\alpha \circ (\,U_{\alpha\gamma}\,)^T & 0 & 0 \\ \\ Z^T H W_\alpha & 0 & S(Z^T H Z) \end{bmatrix} P^T,$$

where

$$S(Z^T H Z) \;\equiv\; \begin{bmatrix} (\,Z^T H Z\,)_{\alpha'\alpha'} & \Gamma_{\alpha'\gamma'} \circ (\,Z^T H Z\,)_{\alpha'\gamma'} \\ \\ (\,Z^T H Z\,)_{\gamma'\alpha'} \circ (\,\Gamma_{\alpha'\gamma'}\,)^T & 0 \end{bmatrix}.$$

Thus $V(H)$ belongs to the linear subspace

$$\mathcal{L}_{\gamma'}(A;\mathcal{S}^n_+) \;\equiv\; \{\, C \;\in\; \mathcal{L}(A;\mathcal{S}^n_+) \;:\; Z^T_{\gamma'} C Z_{\gamma'} \;=\; 0\,\},$$

where $\mathcal{L}(A;\mathcal{S}^n_+)$ is the affine hull of the critical cone $\mathcal{C}(A;\mathcal{S}^n_+)$.

Proof. Let $V \in \partial_B \Pi_{\mathcal{S}^n_+}(A)$. By Corollary 10 and the definition of the elements in $\partial_B \Pi_{\mathcal{S}^n_+}(A)$, it follows that there exists a sequence of nonsingular matrices $\{A^k\}$ in $\mathcal{S}^n$ converging to A such that $V = \lim_{k\to\infty} J\Pi_{\mathcal{S}^n_+}(A^k)$, where $J\Pi_{\mathcal{S}^n_+}(A^k)$ denotes the F-derivative of $\Pi_{\mathcal{S}^n_+}$ at A^k. Let $A^k \equiv P^k \Lambda^k (P^k)^T$ be the orthogonal decomposition of A^k, where Λ^k is the diagonal matrix of eigenvalues of A^k and P^k is a corresponding matrix of orthonormal eigenvectors. Writing each Λ^k in the same form as Λ:

$$\Lambda^k \;=\; \begin{bmatrix} \Lambda^k_\alpha & 0 & 0 \\ \\ 0 & \Lambda^k_\gamma & 0 \\ \\ 0 & 0 & \Lambda^k_\beta \end{bmatrix},$$

we have $\Lambda = \lim_{k\to\infty} \Lambda^k$, which implies that Λ^k_κ is a nonsingular matrix for all k sufficiently large and $\lim_{k\to\infty} \Lambda^k_\beta = 0$. For any $H \in \mathcal{S}^n$ with $\tilde{H}^k = (P^k)^T H P^k$, we have

$$J\Pi_{\mathcal{S}^n_+}(A^k)(H) \;=\; P^k \left[L^{-1}_{|\Lambda^k|}\, L_{(\Lambda^k)_+}(\tilde{H}^k) \right] (P^k)^T.$$

Without loss of generality, by taking a subsequence if necessary, we may assume that $\{P^k\}$ is a convergent sequence with limit $P^\infty \equiv \lim_{k\to\infty} P^k$, which implies that

$$A \;=\; \lim_{k\to\infty} A^k \;=\; \lim_{k\to\infty} P^k \Lambda^k (P^k)^T \;=\; P^\infty \Lambda (P^\infty)^T.$$

Therefore, P^∞ can be identified with the matrix P that we have been using all along for diagonalizing A. We will simply use P, rather than P^∞, in the remainder of the proof. Let $Z^k \equiv J\Pi_{\mathcal{S}^n_+}(A^k)(H)$. We have

$$\tilde{Z}^k \;\equiv\; (P^k)^T \left[\, J\Pi_{\mathcal{S}^n_+}(A^k)(H) \,\right] P^k \;=\; L^{-1}_{|\Lambda^k|}\, L_{(\Lambda^k)_+}(\tilde{H}^k),$$

which implies that $|\Lambda^k|\tilde{Z}^k + \tilde{Z}^k|\Lambda^k| = (\Lambda^k)_+\tilde{H}^k + \tilde{H}^k(\Lambda^k)_+$. Writing this out, we have

$$\begin{aligned} &\begin{bmatrix} |\Lambda^k_\kappa|\tilde{Z}^k_{\kappa\kappa} + \tilde{Z}^k_{\kappa\kappa}|\Lambda^k_\kappa| & |\Lambda^k_\kappa|\tilde{Z}^k_{\kappa\beta} + \tilde{Z}^k_{\kappa\beta}|\Lambda^k_\beta| \\[2ex] |\Lambda^k_\beta|\tilde{Z}^k_{\beta\kappa} + \tilde{Z}^k_{\beta\kappa}|\Lambda^k_\kappa| & |\Lambda^k_\beta|\tilde{Z}^k_{\beta\beta} + \tilde{Z}^k_{\beta\beta}|\Lambda^k_\beta| \end{bmatrix} \\ &= \begin{bmatrix} (\Lambda^k_\kappa)_+\tilde{H}^k_{\kappa\kappa} + \tilde{H}^k_{\kappa\kappa}(\Lambda^k_\kappa)_+ & (\Lambda^k_\kappa)_+\tilde{H}^k_{\kappa\beta} + \tilde{H}^k_{\kappa\beta}(\Lambda^k_\beta)_+ \\[2ex] (\Lambda^k_\beta)_+\tilde{H}^k_{\beta\kappa} + \tilde{H}^k_{\beta\kappa}(\Lambda^k_\kappa)_+ & (\Lambda^k_\beta)_+\tilde{H}^k_{\beta\beta} + \tilde{H}^k_{\beta\beta}(\Lambda^k_\beta)_+ \end{bmatrix}. \end{aligned}$$

Hence, it follows that $\tilde{Z}^k_{\kappa\kappa} = L^{-1}_{|\Lambda^k_\kappa|} L_{(\Lambda^k_\kappa)_+}(\tilde{H}^k_{\kappa\kappa})$, $\tilde{Z}^k_{\beta\beta} = L^{-1}_{|\Lambda^k_\beta|} L_{(\Lambda^k_\beta)_+}(\tilde{H}^k_{\beta\beta})$ and

$$\lim_{k\to\infty} \left[\, \tilde{Z}^k_{\kappa\beta} - |\Lambda^k_\kappa|^{-1}(\Lambda^k_\kappa)_+(\tilde{H}^k_{\kappa\beta}) \,\right] \;=\; 0.$$

Again, by taking a subsequence if necessary, we may assume that $\{\tilde{Z}^k_{\beta\beta}\}$ is a convergent sequence. Hence, for any $H \in \mathcal{S}^n$, it holds that

$$P^T V(H) P \;=\; \begin{bmatrix} L^{-1}_{|D|} L_{D_+}(\tilde{H}_{\kappa\kappa}) & |D|^{-1} D_+ \tilde{H}_{\kappa\beta} \\ \\ (\tilde{H}_{\kappa\beta})^T D_+ |D|^{-1} & \lim\limits_{k\to\infty} \left\{ L^{-1}_{|\Lambda^k_\beta|} L_{(\Lambda^k_\beta)_+}(\tilde{H}^k_{\beta\beta}) \right\} \end{bmatrix}, \tag{21}$$

where $\tilde{H} = P^T H P$. For each k, define

$$M^k \;\equiv\; P \begin{bmatrix} 0 & 0 \\ \\ 0 & \Lambda^k_\beta \end{bmatrix} P^T.$$

Let $\tilde{M}^k \equiv P^T M^k P$. Then

$$\tilde{M}^k = \begin{bmatrix} 0 & 0 \\ \\ 0 & \Lambda^k_\beta \end{bmatrix}.$$

Since $\tilde{M}^k_{\beta\beta}$ is nonsingular, it follows that Ψ is F-differentiable at M^k and for any $H \in \mathcal{S}^n$,

$$\begin{aligned} J\Psi(M^k)(H) \;&=\; \lim_{\tau\downarrow 0} \left\{ \frac{\Psi(M^k + \tau H) - \Psi(M^k)}{\tau} \right\} \\ &=\; P \begin{bmatrix} L^{-1}_{|D|}\, L_{D_+}(\tilde{H}_{\kappa\kappa}) & |\,D\,|^{-1}\, D_+\, \tilde{H}_{\kappa\beta} \\ \\ (\tilde{H}_{\kappa\beta})^T D_+\,|\,D\,|^{-1} & \lim\limits_{\tau\downarrow 0} \dfrac{\Pi_{\mathcal{S}^{|\beta|}_+}(\,\Lambda^k_\beta + \tau\,\tilde{H}_{\beta\beta}\,) - \Pi_{\mathcal{S}^{|\beta|}_+}(\Lambda^k_\beta)}{\tau} \end{bmatrix} P^T \\ &=\; P \begin{bmatrix} L^{-1}_{|D|}\, L_{D_+}(\tilde{H}_{\kappa\kappa}) & |\,D\,|^{-1}\, D_+\, \tilde{H}_{\kappa\beta} \\ \\ (\tilde{H}_{\kappa\beta})^T D_+|\,D\,|^{-1} & L^{-1}_{|\Lambda^k_\beta|}\, L_{(\Lambda^k_\beta)_+}(\tilde{H}_{\beta\beta}) \end{bmatrix} P^T, \end{aligned}$$

where we have used (19) and part (a) of Corollary 10 applied to $\Pi_{\mathcal{S}^{|\beta|}_+}$ at the F-differentiable point Λ^k_β in the bottom right block in the last equality. Comparing with (21), we conclude that $V(H) = \lim\limits_{k\to\infty} J\Psi(M^k)(H)$. Since H is arbitrary in $\mathcal{S}^n$, it follows that $V \in \partial_B \Psi(0)$.

Conversely, let $V \in \partial_B \Psi(0)$. We know that Ψ is F-differentiable at $M \in \mathcal{S}^n$ if and only if $\tilde{M}_{\beta\beta}$ is nonsingular, where $\tilde{M} = P^T M P$. Hence there exists a sequence of matrices $\{M^k\} \subset \mathcal{S}^n$ converging to 0 such that $\tilde{M}^k_{\beta\beta}$ is nonsingular for every k and $V = \lim_{k\to\infty} J\Psi(M^k)$, where $\tilde{M}^k = P^T M^k P$. For any $H \in \mathcal{S}^n$, we have

$$\begin{aligned}
J\Psi(M^k)(H) &= \lim_{\tau \downarrow 0} \frac{\Psi(M^k + \tau H) - \Psi(M^k)}{\tau} \\
&= P \begin{bmatrix} L^{-1}_{|D|} L_{D_+}(\tilde{H}_{\kappa\kappa}) & |\,D\,|^{-1}\, D_+\, \tilde{H}_{\kappa\beta} \\ (\tilde{H}_{\kappa\beta})^T D_+ \,|\,D\,|^{-1} & \displaystyle\lim_{\tau\downarrow 0} \frac{\Pi_{\mathcal{S}^{|\beta|}_+}(\,\tilde{M}^k_{\beta\beta} + \tau\,\tilde{H}_{\beta\beta}\,) - \Pi_{\mathcal{S}^{|\beta|}_+}(\tilde{M}^k_{\beta\beta})}{\tau} \end{bmatrix} P^T \\
&= P \begin{bmatrix} L^{-1}_{|D|}\, L_{D_+}(\tilde{H}_{\kappa\kappa}) & |\,D\,|^{-1}\, D_+\, \tilde{H}_{\kappa\beta} \\ (\tilde{H}_{\kappa\beta})^T\, D_+ \,|\,D\,|^{-1} & L^{-1}_{|\tilde{M}^k_{\beta\beta}|_{\mathcal{S}^{|\beta|}_+}}\; L_{\Pi_{\mathcal{S}^{|\beta|}_+}(\tilde{M}^k_{\beta\beta})}(\tilde{H}_{\beta\beta}) \end{bmatrix} P^T,
\end{aligned}$$

where $\tilde{H} = P^T H P$. Define

$$A^k \equiv A + P \begin{bmatrix} 0 & 0 \\ 0 & \tilde{M}^k_{\beta\beta} \end{bmatrix} P^T$$

and $\tilde{A}^k \equiv P^T A^k P$. We have,

$$\tilde{A}^k = \begin{bmatrix} D & 0 \\ 0 & \tilde{M}^k_{\beta\beta} \end{bmatrix},$$

which is nonsingular. It is easy to see that

$$|\,A^k\,|_{\mathcal{S}^n_+} = P \begin{bmatrix} |\,D\,| & 0 \\ 0 & |\,\tilde{M}^k_{\beta\beta}\,|_{\mathcal{S}^{|\beta|}_+} \end{bmatrix} P^T \quad \text{and} \quad \Pi_{\mathcal{S}^n_+}(A^k) = P \begin{bmatrix} D_+ & 0 \\ 0 & \Pi_{\mathcal{S}^{|\beta|}_+}(\tilde{M}^k_{\beta\beta}) \end{bmatrix} P^T.$$

The nonsingularity of $\tilde{A}^k$ implies the nonsingularity of A^k. Thus, $\Pi_{\mathcal{S}^n_+}$ is differentiable at A^k and

$$Z^k \equiv J\Pi_{\mathcal{S}^n_+}(A^k)(H) = L^{-1}_{|A^k|_{\mathcal{S}^n_+}} L_{\Pi_{\mathcal{S}^n_+}(A^k)}(H),$$

which implies that

$$|\, A^k \,|_{\mathcal{S}^n_+} Z^k + Z^k \,|\, A^k \,|_{\mathcal{S}^n_+} = \Pi_{\mathcal{S}^n_+}(A^k)H + H\Pi_{\mathcal{S}^n_+}(A^k).$$

Therefore,

$$P^T |\, A^k \,|_{\mathcal{S}^n_+} P \,\tilde{Z}^k + \tilde{Z}^k \, P^T |\, A^k \,|_{\mathcal{S}^n_+} P = P^T \,\Pi_{\mathcal{S}^n_+}(A^k) P \,\tilde{H} + \tilde{H}\, P^T \Pi_{\mathcal{S}^n_+}(A^k) P,$$

where $\tilde{Z}^k \equiv P^T Z^k P$ and $\tilde{H} \equiv P^T H P$. Hence,

$$\begin{bmatrix} |\, D \,|\, \tilde{Z}^k_{\kappa\kappa} + \tilde{Z}^k_{\kappa\kappa} \,|\, D \,| & |\, D \,|\, \tilde{Z}^k_{\kappa\beta} + \tilde{Z}^k_{\kappa\beta} \,|\, \tilde{M}^k_{\beta\beta} \,|_{\mathcal{S}^{|\beta|}_+} \\ \\ |\, \tilde{M}^k_{\beta\beta} \,|_{\mathcal{S}^{|\beta|}_+} \tilde{Z}^k_{\beta\kappa} + \tilde{Z}^k_{\beta\kappa} \,|\, D \,| & |\, \tilde{M}^k_{\beta\beta} \,|_{\mathcal{S}^{|\beta|}_+} \tilde{Z}^k_{\beta\beta} + \tilde{Z}^k_{\beta\beta} \,|\, \tilde{M}^k_{\beta\beta} \,|_{\mathcal{S}^{|\beta|}_+} \end{bmatrix}$$

$$= \begin{bmatrix} D_+ \,\tilde{H}^k_{\kappa\kappa} + \tilde{H}^k_{\kappa\kappa}\, D_+ & D_+ \,\tilde{H}^k_{\kappa\beta} + \tilde{H}^k_{\kappa\beta}\, \tilde{M}^k_{\beta\beta} \\ \\ \Pi_{\mathcal{S}^{|\beta|}_+}(\tilde{M}^k_{\beta\beta})\, \tilde{H}^k_{\beta\kappa} + \tilde{H}^k_{\beta\kappa}\, D_+ & \Pi_{\mathcal{S}^{|\beta|}_+}(\tilde{M}^k_{\beta\beta})\, \tilde{H}^k_{\beta\beta} + \tilde{H}^k_{\beta\beta}\, \Pi_{\mathcal{S}^{|\beta|}_+}(\tilde{M}^k_{\beta\beta}) \end{bmatrix},$$

which implies

$$\tilde{Z}^k_{\kappa\kappa} = L^{-1}_{|D|} L_{D_+}(\tilde{H}^k_{\kappa\kappa}), \quad \tilde{Z}^k_{\beta\beta} = L^{-1}_{|\tilde{M}^k_{\beta\beta}|_{\mathcal{S}^{|\beta|}_+}} L_{\Pi_{\mathcal{S}^{|\beta|}_+}(\tilde{M}^k_{\beta\beta})}(\tilde{H}^k_{\beta\beta}).$$

and

$$\lim_{k\to\infty} [\, \tilde{Z}^k_{\kappa\beta} - |\, D \,|^{-1} D_+ \,\tilde{H}^k_{\kappa\beta} \,] = 0.$$

Consequently, $V(H) = \lim_{k\to\infty} J\Pi_{\mathcal{S}^n_+}(A^k)(H)$, which implies $V \in \partial_B \Pi_{\mathcal{S}^n_+}(A)$. Hence (20) follows.

Let Ψ be F-differentiable at $E \in \mathcal{S}^n$. Let $Z^T E Z = Q \Theta Q^T$ be the orthogonal decomposition of $Z^T E Z$, where $Q \in \Re^{|\beta| \times |\beta|}$ is an orthogonal matrix of eigenvectors of $Z^T E Z$ and Θ is the diagonal matrix of eigenvalues θ_i of the same matrix; for any $H \in \mathcal{S}^n$,

$$J\Psi(E)(H) \, = \, P \left[\begin{array}{ccc} W_\alpha^T H W_\alpha & U_{\alpha\gamma} \circ W_\alpha^T H W_\gamma & W_\alpha^T H Z \\ & & \\ W_\gamma^T H W_\alpha \circ (\, U_{\alpha\gamma} \,)^T & 0 & 0 \\ & & \\ Z^T H W_\alpha & 0 & G \end{array} \right] P^T,$$

where with $Y \equiv Z^T H Z$,

$$G \, \equiv \, Q \left[\begin{array}{cc} Q_{\alpha'}^T Y Q_{\alpha'} & \Gamma_{\alpha'\gamma'} \circ Q_{\alpha'}^T Y Q_{\gamma'} \\ & \\ Q_{\gamma'}^T Y Q_{\alpha'} \circ \Gamma_{\alpha'\gamma'}^T & 0 \end{array} \right] Q^T \, \in \, \Re^{|\beta| \times |\beta|},$$

$\alpha' \equiv \{ i \in \beta : \theta_i > 0 \}$ and $\gamma' \equiv \{ i \in \beta : \theta_i < 0 \}$ are disjoint subsets of β whose union is β and

$$\Gamma_{ij} \, \equiv \, \frac{\theta_i}{\theta_i + |\, \theta_j \,|}, \quad (\, i, j \,) \, \in \, \alpha' \times \gamma'.$$

Based on such an F-derivative, it follows that for any element V in $\partial_B \Psi(0)$, there exists a sequence $\{E^k\} \subset \mathcal{S}^n$ converging to the zero matrix such that Ψ is F-differentiable at every E^k and for any $H \in \mathcal{S}^n$, $V(H)$ is the limit of the sequence $\{J\Psi(E^k)(H)\}$. Without loss of generality, we may assume that there exists a partition of $\beta = \alpha' \cup \gamma'$ into two disjoint subsets α' and γ' such that

$$\alpha' \, \equiv \, \{\, i \, \in \, \beta \, : \, \theta_i^k(Z^T E^k Z) \, > \, 0 \,\} \quad \text{and} \quad \gamma' \, \equiv \, \{\, i \, \in \, \beta \, : \, \theta_i^k(Z^T E^k Z) \, < \, 0 \,\}$$

for all k, that the sequence $\{Q^k\} \subset \Re^{|\beta| \times |\beta|}$ of orthogonal matrices in the decomposition $Z^T E^k Z = (Q^k)^T \Theta^k Q^k$ converges to the identity matrix of order $|\beta|$, and that the sequence

$\{\Gamma^k_{ij}\}$, where

$$\Gamma^k_{ij} \equiv \frac{\theta_i(Z^T E^k Z)}{\theta_i(Z^T E^k Z) + |\,\theta_j(Z^T E^k Z)\,|}, \quad (\,i,j\,) \in \alpha' \times \gamma'$$

converges to some scalar Γ^∞_{ij} in the interval $[0,1]$ for all $(i,j) \in \alpha' \times \gamma'$. From this description, the desired formula for $V(H)$ follows easily. The last assertion about the membership of $V(H)$ in the linear subspace $\mathcal{L}_{\gamma'}(A;\mathcal{S}^n_+)$ is obvious. Q.E.D.

Remark. By distinguishing the zero and nonzero entries in the matrix $\Gamma_{\alpha'\gamma'}$, it is possible to show that the range of V belongs to a certain linear subspace of $\mathcal{L}_{\gamma'}(A;\mathcal{S}^n_+)$. We omit such fine details.

Combining Proposition 7, Lemma 11 and the discussion at the end of the last section, we immediately obtain the following result, which does not require further proof.

Proposition 12 Let $F : \mathcal{S}^n \to \mathcal{S}^n$ be continuously differentiable. For any $A \in \mathcal{S}^n$,

$$\partial_B \mathbf{F}^{\rm nor}_{\mathcal{S}^n_+}(A) \equiv \partial_B (\mathbf{F}^{\rm nor}_{\mathcal{S}^n_+})'(A;\cdot)(0) = (\,JF(\Pi_{\mathcal{S}^n_+}(A)) - I\,) \circ \partial_B \Pi_{\mathcal{S}^n_+}(A) + I.$$

4.2 The Lorentz cone

We next consider the Lorentz cone, also known as the second-order cone (SOC):

$$\mathcal{K}^n \equiv \left\{ (x,t) \in \Re^n \times \Re \,:\, \sqrt{x^T x} \leq t \right\}.$$

The Euclidean projection $\Pi_{\mathcal{K}^n}(x,t)$ of a vector $(x,t) \in \Re^{n+1}$ is the unique minimizer of the following convex program in the variable $(y,\tau) \in \Re^{n+1}$:

$$\begin{array}{ll} \text{minimize} & (\,y-x\,)^T(\,y-x\,) + (\,\tau - t\,)^2 \\ \text{subject to} & \sqrt{y^T y} \leq \tau. \end{array} \qquad (22)$$

By a direct calculation, it is not difficult to show that

$$\Pi_{\mathcal{K}^n}(x,t) \;\equiv\; \begin{cases} \frac{1}{2}\left(1+\dfrac{t}{\|\,x\,\|_2}\right)(\,x,\|x\|_2\,) & \text{if } |t| < \|\,x\,\|_2 \\[2ex] (\,x,t\,) & \text{if } \|\,x\,\|_2 \;\le\; t \\[2ex] 0 & \text{if } \|\,x\,\|_2 \;\le\; -t. \end{cases}$$

Recently, the strong semismoothness of this projector is established in [5]. In what follows, we show that the directional derivative $\Pi'_{\mathcal{K}^n}((x,t);(dx,dt))$ along the direction $(dx,dt) \in \Re^{n+1}$ can again be interpreted as a certain skewed projection of (dx,dt) onto the critical cone of $\mathcal{K}^n$ at (x,t). There are two cases for which this interpretation is known to be true: one is the classic case where the base point (x,t) belongs to the cone $\mathcal{K}^n$ (see [52]); and the other case is when the first n-components of the projected vector $\Pi_{\mathcal{K}^n}(x,t)$ are not all zero (see [46, 36]). We write $(\bar{x},\bar{t}) \in \Re^{n+1}$ for the projection $\Pi_{\mathcal{K}^n}(x,t)$ and $\mathcal{C}(x,t)$ for the critical cone

$$\mathcal{C}((x,t);\mathcal{K}^n) \;\equiv\; \mathcal{T}((\bar{x},\bar{t});\mathcal{K}^n) \,\cap\, (\bar{x}-x,\bar{t}-t)^{\perp}\,.$$

Also define the symmetric positive definite matrix $A(x,t) \in \Re^{(n+1)\times(n+1)}$ to be I if $t \le -\|x\|_2$ and

$$A(x,t) \;=\; \begin{bmatrix} \left(1+\dfrac{\lambda}{\|\,\bar{x}\,\|}\right)I - \dfrac{\lambda}{\|\,\bar{x}\,\|}\,\dfrac{\bar{x}\bar{x}^T}{\bar{x}^T\bar{x}} & 0 \\[3ex] 0 & 1 \end{bmatrix}$$

otherwise, where $\lambda \equiv \max\left\{0, \frac{1}{2}(\|x\|_2 - t)\right\}$.

Proposition 13 For any (x,t) and (dx,dt) in $\Re^{n+1}$, $\Pi'_{\mathcal{K}^n}((x,t);(dx,dt))$ is the unique minimizer of the convex program in the variable (y, τ):

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} \begin{bmatrix} y \\ \\ \tau \end{bmatrix}^T A(x,t) \begin{bmatrix} y \\ \\ \tau \end{bmatrix} - \begin{bmatrix} y \\ \\ \tau \end{bmatrix}^T \begin{bmatrix} dx \\ \\ dt \end{bmatrix} \\ \\ \text{subject to} & (\, y, \tau \,) \in \mathcal{C}(x,t). \end{array} \tag{23}$$

Proof. If $(x,t) \in \mathcal{K}^n$, then $A(x,t)$ is the identity matrix. This case is due to Zarantonello [52]. If $(x,t) \notin \mathcal{K}^n$ and $\bar{x} \neq 0$, then the square root function $\sqrt{y^T y}$ is continuously differentiable at $y = \bar{x}$. In this case, λ is the unique Karush-Kuhn-Tucker multiplier of the single constraint in the projection program (22) and the matrix $A(x,t) = I + \lambda \nabla^2 g(\bar{x}, \bar{t})$, where $g(y,\tau) \equiv \sqrt{y^T y} - \tau$ is the convex function that defines the Lorentz cone, which can be written as $\mathcal{K}^n = \{(y,\tau) \in \Re^{n+1} : g(y,\tau) \leq 0\}$. The result is proved by Shapiro [46]. Finally, if $(x,t) \notin \mathcal{K}^n$ and $\bar{x} = 0$, then (x,t) must belong to $-\mathcal{K}^n$. In this case, by making use of the fact that

$$\Pi'_{\mathcal{K}^n}((x,t);(dx,dt)) \;=\; (dx,dt) - \Pi'_{(-\mathcal{K}^n)}((x,t);(dx,dt))$$

and the proof for the first case, we have

$$\Pi'_{\mathcal{K}^n}((x,t);(dx,dt)) \;=\; (dx,dt) - \Pi_{\mathcal{C}((x,t);-\mathcal{K}^n)}((dx,dt)),$$

where

$$\mathcal{C}((x,t);-\mathcal{K}^n) \;\equiv\; \mathcal{T}(\Pi_{(-\mathcal{K}^n)}((x,t));-\mathcal{K}^n) \;\cap\; \Big[\Pi_{(-\mathcal{K}^n)}((x,t)) - (dx,dt)\Big]^{\perp}.$$

Since $\mathcal{T}((0,0);\mathcal{K}^n) = \mathcal{K}^n$, we have

$$\mathcal{C}((x,t);\mathcal{K}^n) \;=\; \begin{cases} \{(0,0)\} & \text{if } \|x\| < -t, \\ \\ \{\alpha(x,\|x\|) :\; \alpha \geq 0\} & \text{if } -t = \|x\| \end{cases}$$

and

$$\mathcal{C}((x,t);-\mathcal{K}^n) \;=\; \begin{cases} \Re^{n+1} & \text{if } \|x\| < -t, \\ \\ \{(y,\tau) \in \Re^{n+1} :\; y^T x + \tau\|x\| \leq 0\} & \text{if } -t = \|x\|. \end{cases}$$

Hence, after direct calculations, for $(x,t) \notin \mathcal{K}^n$ and $\bar{x} = 0$, we have

$$\Pi'_{\mathcal{K}^n}((x,t);(dx,dt)) = \Pi_{\mathcal{C}((x,t);\mathcal{K}^n)}((dx,dt)).$$

Thus the claim also holds in this remaining case. Q.E.D.

Letting

$$\begin{bmatrix} \hat{dx} \\ \\ \hat{dt} \end{bmatrix} \;\equiv\; A(x,t)^{-1} \begin{bmatrix} dx \\ \\ dt \end{bmatrix},$$

we see that the program (23) is equivalent to

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} \begin{bmatrix} y - \hat{dx} \\ \\ \tau - \hat{dt} \end{bmatrix}^T A(x,t) \begin{bmatrix} y - \hat{dx} \\ \\ \tau - \hat{dt} \end{bmatrix} \\ \\ \text{subject to} & (\, y, \tau\,) \,\in\, \mathcal{C}(x,t), \end{array}$$

which shows that $\Pi'_{\mathcal{K}^n}((x,t);(dx,dt))$ is the projection of $(\hat{dx}, \hat{dt})$ onto the critical cone $\mathcal{C}(x,t)$ under the matrix norm induced by the symmetric positive definite matrix $A(x,t)$. Thus unlike the previous case of the cone $\mathcal{S}^n$ where only the direction is linearly transformed, the directional derivative of the projector onto the Lorentz cone involves both a linear transformation of the direction and a norm change in defining the projection onto the critical cone.

The proof of Proposition 13 enables us to establish the following technical result analogous to Lemma 11.

Lemma 14 Let $(x,t) \in \Re^{n+1}$ be given. Let $\Psi \equiv \Pi'_{\mathcal{K}^n}((x,t);\cdot)$. It holds that $\partial_B \Pi_{\mathcal{K}^n}(x,t) = \partial_B \Psi(0)$.

Proof. There are several cases that we can dispense of easily. These are (i) $|t| < \|x\|_2$; (ii) $t > \|x\|_2$, (iii) $-t > \|x\|_2$, and (iv) $(x,t) = (0,0)$. In the first three cases, $\Pi_{\mathcal{K}^n}$ is a continuously differentiable function in a neighborhood of (x,t); thus the equality between the two B-subdifferentials is immediate. In the fourth case, $\Psi = \Pi_{K^n}$ and the desired equality is obvious. There are two remaining cases: (a) $t = \|x\|_2 > 0$ and (b) $-t = \|x\|_2 > 0$. Since the proof of these two cases are similar, we prove only case (a). In this case, for (x',t') sufficiently close to (x,t), we have

$$\Pi_{\mathcal{K}^n}(x',t') \equiv \begin{cases} \frac{1}{2}\left(1 + \dfrac{t'}{\|x'\|_2}\right)(x', \|x'\|_2) & \text{if } |t'| < \|x'\|_2 \\ \\ (x',t') & \text{if } \|x'\|_2 \le t'. \end{cases}$$

Thus, for $(h,\tau) \in \Re^n \times \Re$,

$$\Psi(h,\tau) = \begin{cases} \begin{bmatrix} \dfrac{\tau x}{2\|x\|} + h - \dfrac{xx^T}{2\|x\|^2}h \\ \\ \dfrac{x^T h}{2\|x\|} + \dfrac{\tau}{2} \end{bmatrix} & \text{if } \begin{cases} t\tau < x^T h \\ \quad \text{or} \\ t\tau = x^T h \text{ and } |\tau| < \|h\|_2 \end{cases} \\ \begin{bmatrix} h \\ \\ \tau \end{bmatrix} & \text{if } \begin{cases} t\tau > x^T h \\ \quad \text{or} \\ t\tau = x^T h \text{ and } |\tau| \ge \|h\|_2. \end{cases} \end{cases}$$

Obviously,

$$\partial_B \Psi(0) \subseteq \left\{ I, \begin{bmatrix} I - \dfrac{xx^T}{2\|x\|^2} & \dfrac{x}{2\|x\|} \\ \\ \dfrac{x^T}{2\|x\|} & \dfrac{1}{2} \end{bmatrix} \right\} = \partial_B \Pi_{\mathcal{K}^n}(x,t).$$

Next, we prove the reverse inclusion: $\partial_B \Psi(0) \supseteq \partial_B \Pi_{\mathcal{K}^n}(x,t)$. Let $h^k \equiv x$ and $\tau_k \equiv (1+1/k)t$. Then $t\tau_k > x^T h^k$. Hence, $\lim_{k\to\infty} J\Psi(h^k, \tau_k) = I$. Similarly, we can construct a sequence $\{(\tilde{h}^k, \tilde{\tau}_k)\}$ also converging to (x,t) such that

$$\lim_{k\to\infty} J\Psi(\tilde{h}^k, \tilde{\tau}_k) = \begin{bmatrix} I - \dfrac{xx^T}{2\|x\|^2} & \dfrac{x}{2\|x\|} \\ \\ \dfrac{x^T}{2\|x\|} & \dfrac{1}{2} \end{bmatrix}.$$

Thus, in this case $\partial_B \Psi(0) = \partial_B \Pi_{\mathcal{K}^n}(x,t)$. Q.E.D.

Let $\mathbf{A}^{\mathrm{nor}}_{\mathcal{C}(x,t)}$ denote the normal map of the pair $(\mathcal{C}(x,t), A(x,t))$; i.e., for all $(z,\tau) \in \Re^{n+1}$,

$$\mathbf{A}^{\mathrm{nor}}_{\mathcal{C}(x,t)}(z,\tau) = A(x,t)\Pi_{\mathcal{C}(x,t)}(z,\tau) + (z,\tau) - \Pi_{\mathcal{C}(x,t)}(z,\tau).$$

We then have

$$\Pi'_{\mathcal{K}^n}((x,t);(dx,dt)) \equiv \Pi_{\mathcal{C}(x,t)} \circ (\, \mathbf{A}^{\mathrm{nor}}_{\mathcal{C}(x,t)}\,)^{-1}(dx,dt).$$

Since the matrix $A(x,t)$ is positive definite, $\mathbf{A}^{\mathrm{nor}}_{\mathcal{C}(x,t)}$ is a globally Lipschitz homeomorphism from $\Re^{n+1}$ onto itself; moreover, its inverse is given by

$$(\, \mathbf{A}^{\mathrm{nor}}_{\mathcal{C}(x,t)}\,)^{-1} = (\, I - A(x,t)\,)\, \Pi^{A(x,t)}_{\mathcal{C}(x,t)} + I,$$

where $\Pi^{A(x,t)}_{\mathcal{C}(x,t)}$ is the operator that maps $(dx,dt) \in \Re^{n+1}$ onto the unique solution of the convex program (23); see [36, Lemma 8].

5 CPs on Two Self-Dual Cones

In this section, we investigate the application of Theorem 6 to CPs on the cone of symmetric positive semidefinite matrices and on the Lorentz cone. As in the last section, we first deal with the former problem and then with the latter problem in the subsequent subsection. Since the treatment of these two problems are rather similar, we omit some final details for the Lorentz CP.

5.1 CPs in SDP matrices

The linear complementarity problem in symmetric positive semidefinite matrices, abbreviated as SDLCP, was introduced by Kojima, Shindo, Hara [21] and further studied in [19, 20] where interior-point methods for solving this problem were investigated. Analytic properties of the SDLCP are derived by Gowda and his collaborators [14, 15]. The nonlinear extension of the SDLCP is considered by Monteiro and Pang [31, 32] who treat the problem as a constrained equation. Computational methods for solving the semidefinite complementarity problem (SDCP) can be found in [6, 7, 51]. Shapiro [47] studied first- and second-order perturbation analysis of nonlinear semidefinite optimization problems.

The SDCP can be identified as a special VI (K, F) where the set K is the cone of symmetric positive semidefinite matrices and the inner product is the Frobenius inner product. We formally define this problem as follows. Let $F : X \in \mathcal{S}^n \mapsto F(X) \in \mathcal{S}^n$ be a mapping from $\mathcal{S}^n$ into itself. The SDCP is to find a matrix X satisfying

$$\mathcal{S}^n_+ \ni X \perp F(X) \in \mathcal{S}^n_+. \tag{24}$$

Let X^* be a solution of (24) and define $Z^* \equiv X^* - F(X^*)$. We have $\Pi_{\mathcal{S}^n_+}(Z^*) = X^*$. We assume that F is continuously differentiable in an open neighborhood of X^*; it follows that the normal map of the problem (24):

$$\mathbf{F}^{\rm nor}_{\mathcal{S}^n_+}(Z) \equiv F(\Pi_{\mathcal{S}^n_+}(Z)) + Z - \Pi_{\mathcal{S}^n_+}(Z), \quad Z \in \mathcal{S}^n_+$$

is semismooth near Z^*. Let $Z^* \equiv P\Lambda P^T$ be the orthogonal decomposition of Z^*. Using the same notation as in Subsection 4.1, we define

$$\mathcal{L}_{Z^*}(H) = P \begin{bmatrix} W_\alpha^T H W_\alpha & U_{\alpha\gamma} \circ W_\alpha^T H W_\gamma & W_\alpha^T H Z \\ W_\gamma^T H W_\alpha \circ U_{\alpha\gamma}^T & W_\gamma^T H W_\gamma & W_\gamma^T H Z \\ Z^T H W_\alpha & Z^T H W_\gamma & Z^T H Z \end{bmatrix} P^T.$$

We then have $\Pi'_{\mathcal{S}^n_+}(Z^*; H) = \Pi_{\mathcal{C}(Z^*;\mathcal{S}^n_+)}(\mathcal{L}_{Z^*}(H))$. Note that the critical cone $\mathcal{C}(Z^*;\mathcal{S}^n_+)$ coincides with the critical cone of $\mathcal{S}^n_+$ at the solution X^* of the CP (24); i.e.,

$$\mathcal{C}(Z^*;\mathcal{S}^n_+) = \mathcal{T}(X^*, \mathcal{S}^n_+) \cap F(X^*)^\perp.$$

Writing $\mathcal{C} \equiv \mathcal{C}(Z^*;\mathcal{S}^n_+)$ and $S \equiv \mathcal{L}_{Z^*}(H)$, we have

$$\begin{aligned} (\,\mathbf{F}^{\rm nor}_{\mathcal{S}^n_+}\,)'(Z^*; H) &= JF(X^*)\Pi_{\mathcal{C}}(S) + H - \Pi_{\mathcal{C}}(S) \\ &= JF(X^*)\Pi_{\mathcal{C}}(S) + (\,H - S\,) + S - \Pi_{\mathcal{C}}(S); \end{aligned}$$

furthermore,

$$H - S = P \begin{bmatrix} 0 & \tilde{U}_{\alpha\gamma} \circ W_\alpha^T H W_\gamma & 0 \\ W_\gamma^T H W_\alpha \circ \tilde{U}_{\alpha\gamma}^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^T,$$

where

$$\tilde{u}_{ij} \,\equiv\, 1 - u_{ij} \,=\, \frac{|\,\lambda_j\,|}{|\,\lambda_i\,| + |\,\lambda_j\,|}, \quad (\,i,j\,) \,\in\, \alpha \times \gamma$$

Recalling that

$$\Pi_{\mathcal{C}}(S) \,=\, P \left[\begin{array}{ccc} W_\alpha^T H W_\alpha & U_{\alpha\gamma} \circ W_\alpha^T H W_\gamma & W_\alpha^T H Z \\ & & \\ W_\gamma^T H W_\alpha \circ (\,U_{\alpha\gamma}\,)^T & 0 & 0 \\ & & \\ Z^T H W_\alpha & 0 & \Pi_{\mathcal{S}_+^{|\beta|}}(\,Z^T H Z\,) \end{array} \right] P^T,$$

we may construct a linear transformation $\mathcal{A}_{Z^*} : \mathcal{S}^n \to \mathcal{S}^n$ that maps $\Pi_{\mathcal{C}}(S)$ onto $H - S$.

Specifically, for a matrix

$$C \,\equiv\, \left[\begin{array}{ccc} C_{\alpha\alpha} & C_{\alpha\gamma} & C_{\alpha\beta} \\ & & \\ C_{\gamma\alpha} & C_{\gamma\gamma} & C_{\gamma\beta} \\ & & \\ C_{\beta\alpha} & C_{\beta\gamma} & C_{\beta\beta} \end{array} \right] \in \mathcal{S}^n,$$

let

$$\mathcal{A}_{Z^*}(PCP^T) \,\equiv\, P \left[\begin{array}{ccc} 0 & \Sigma_{\alpha\gamma} \circ C_{\alpha\gamma} & 0 \\ & & \\ C_{\gamma\alpha} \circ \Sigma_{\alpha\gamma}^T & 0 & 0 \\ & & \\ 0 & 0 & 0 \end{array} \right] P^T,$$

where

$$\sigma_{ij} \,\equiv\, \frac{\tilde{u}_{ij}}{u_{ij}} \,=\, \frac{|\,\lambda_j\,|}{\lambda_i}, \quad (\,i,j\,) \,\in\, \alpha \times \gamma;$$

we then have $\mathcal{A}_{Z^*}(\Pi_{\mathcal{C}}(S)) = H - S$. Hence

$$(\,\mathbf{F}^{\rm nor}_{\mathcal{S}_+^n}\,)'(Z^*; H) \,=\, (\,JF(X^*) + \mathcal{A}_{Z^*}\,)(\Pi_{\mathcal{C}}(S)) + S - \Pi_{\mathcal{C}}(S).$$

Consequently, letting $\mathbf{G}^{\rm nor}_{\mathcal{C}}$ denote the normal map of the CP:

$$\mathcal{C} \ni S^* \perp -Q + (\, JF(X^*) + \mathcal{A}_{Z^*}\,)(S^*) \in \mathcal{C}^*, \tag{25}$$

we may conclude that

$$(\, \mathbf{F}^{\rm nor}_{\mathcal{S}^n_+}\,)'(Z^*; \cdot) \;=\; \mathbf{G}^{\rm nor}_{\mathcal{C}} \circ \mathcal{L}_{Z^*}; \tag{26}$$

in other words, $(\mathbf{F}^{\rm nor}_{\mathcal{S}^n_+})'(Z^*; H) = Q$ if and only if with $S^* = \Pi_{\mathcal{C}}(S)$ being a solution of the CP (25),

$$H \;=\; (\, \mathcal{L}_{Z^*}\,)^{-1}(S) \;=\; (\, \mathcal{L}_{Z^*}\,)^{-1}[\, S^* + Q - (\, JF(X^*) + \mathcal{A}_{Z^*}\,)(S^*)\,)\,].$$

Based on the above derivation, we obtain the following result that connects the globally Lipschitz homeomorphism of the directional derivative $(\mathbf{F}^{\rm nor}_{\mathcal{S}^n_+})'(Z^*; \cdot)$ with the solution of the SDLCP (25). Part of the significance of this result is that the latter CP depends only on the given solution X^* of the original SDCP (24) and is independent of the linear operator $\mathcal{L}_{Z^*}$ that is used in the above derivation and the proof below.

Lemma 15 The directional derivative $(\mathbf{F}^{\rm nor}_{\mathcal{S}^n_+})'(Z^*; \cdot)$ is a globally Lipschitz homeomorphism if and only if for every $Q \in \mathcal{S}^n$, the SDLCP (25) has a unique solution $S^*(Q)$ that is Lipschitz continuous in Q.

Proof. Since $\mathcal{L}_{Z^*}$ is a nonsingular linear transformation, it follows that $(\mathbf{F}^{\rm nor}_{\mathcal{S}^n_+})'(Z^*; \cdot)$ is a globally Lipschitz homeomorphism if and only if $\mathbf{G}^{\rm nor}_{\mathcal{C}} = (\mathbf{F}^{\rm nor}_{\mathcal{S}^n_+})'(Z^*; \cdot) \circ (\mathcal{L}_{Z^*})^{-1}$ is so. In turn, from VI/CP theory, we know that the normal map $\mathbf{G}^{\rm nor}_{\mathcal{C}}$ is a globally Lipschitz homeomorphism if and only if the claimed unique and Lipschitz solvability of the CP (25) is valid. Q.E.D.

The following result gives a further application of the formula (26).

Lemma 16 The following three statements are equivalent.

(a) The directional derivative $(\mathbf{F}^{\rm nor}_{\mathcal{S}^n_+})'(Z^*;\cdot)$ has the origin as the unique zero.

(b) The normal map $\mathbf{G}^{\rm nor}_{\mathcal{C}}$ has the origin as the unique zero.

(c) The SDLCP (25) with $Q = 0$ has zero as the unique solution.

Moreover, if any one of these statements holds, then $\mathrm{ind}\left((\mathbf{F}^{\rm nor}_{\mathcal{S}^n_+})'(Z^*;\cdot),0\right) = \mathrm{ind}(\mathbf{G}^{\rm nor}_{\mathcal{C}},0)$.

Proof. The proof of the equivalence of statements (a), (b), and (c) is similar to the proof of Lemma 15. We prove the index equality. Assume any one of the three statements (a), (b), and (c). By the homotopy invariance of the degree, it suffices to show that for every $t \in (0,1)$,

$$\mathbf{G}^{\rm nor}_{\mathcal{C}} \circ [\, t\,\mathcal{L}_{Z^*} + (\,1-t\,)\,I\,](H) \,=\, 0 \;\Rightarrow\; H \,=\, 0. \tag{27}$$

Clearly,

$$[\, t\,\mathcal{L}_{Z^*} + (\,1-t\,)\,](H) \,=\, P \left[\begin{array}{ccc} W_\alpha^T H W_\alpha & U^t_{\alpha\gamma} \circ W_\alpha^T H W_\gamma & W_\alpha^T H Z \\ W_\gamma^T H W_\alpha \circ (\,U^t_{\alpha\gamma}\,)^T & W_\gamma^T H W_\gamma & W_\gamma^T H Z \\ Z^T H W_\alpha & Z^T H W_\gamma & Z^T H Z \end{array} \right] P^T,$$

where $U^t_{\alpha\gamma}$ is the matrix whose entries are given by

$$u^t_{ij} \,\equiv\, t\,u_{ij} + (\,1-t\,) \,=\, \frac{\lambda_i + (\,1-t\,)\,|\,\lambda_j\,|}{\lambda_i + |\,\lambda_j\,|}, \quad (\,i,j\,) \,\in\, \alpha \times \gamma.$$

Since each u_{ij}^t is positive, it can be proved that $t\mathcal{L}_{Z^*}+(1-t)I$ is a nonsingular transformation on $\mathcal{S}^n$; see the proof for $t = 1$ at the end of Subsection 4.1. By (b), the implication (27) holds readily. Q.E.D.

For a nonsingular linear transformation from $\mathcal{S}^n$ into itself, the sign of the determinant of this transformation is equal to the index of the transformation at the origin. This extended notion of the determinant is used in the theorem below, which provides several necessary and sufficient conditions for a solution of the SDCP (24) to be strongly stable/regular. Its proof follows easily from Lemmas 15 and 16 and Theorems 3 and 6.

Theorem 17 Let $F : \mathcal{S}^n \to \mathcal{S}^n$ be continuously differentiable in a neighborhood of a solution X^* of the SDCP (24). The following three statements are equivalent.

(a) X^* is strongly stable/regular;

(b) for every $Q \in \mathcal{S}^n$, the SDLCP (25) has a unique solution that is Lipschitz continuous in Q;

(c) for every $V \in \partial_B \Pi_{\mathcal{S}^n_+}(Z^*)$, $\operatorname{sgn}\det((JF(X^*) + \mathcal{A}_{Z^*}) \circ V + I - V) = \operatorname{ind}(\mathbf{G}_{\mathcal{C}}^{\mathrm{nor}}, 0) = \pm 1$.

Calculation of directional derivatives

We may apply Corollary 8 to a parametric CP in SPSD matrices:

$$\mathcal{S}^n_+ \ni X \perp F(X,p) \in \mathcal{S}^n_+, \tag{28}$$

where $F : \mathcal{S}^n \times \Re^m \to \mathcal{S}^n$ is a given mapping. In what follows, we show how to calculate the directional derivative of an implicit solution function of the above problem at a base parameter vector $p^* \in \Re^m$. For this purpose, let X^* be a strongly stable solution of the above problem at p^*. Assume that F is continuously differentiable in a neighborhood of the pair (X^*, p^*). It follows that there exist open neighborhoods $\mathcal{V} \subseteq \mathcal{S}^n_+$ of X^* and $\mathcal{P} \subseteq \Re^m$ of p^* and a locally Lipschitz continuous function $X : \mathcal{P} \to \mathcal{V}$ such that for every $p \in \mathcal{P}$, $X(p)$ is the unique matrix in $\mathcal{V}$ that solves (28); moreover, the implicit solution function X is semismooth at p^*. We wish to compute $X'(p^*; dp)$ for $dp \in \Re^m$. For each $p \in \mathcal{P}$, let $Z(p) \equiv X(p) - F(X(p), p)$. We have $X(p) = \Pi_{\mathcal{S}^n_+}(Z(p))$ and

$$F(\Pi_{\mathcal{S}^n_+}(Z(p)), p) + Z(p) - \Pi_{\mathcal{S}^n_+}(Z(p)) \,=\, 0.$$

Taking the directional derivative of the above normal equation at p^* along the direction dp and writing $dZ \equiv Z'(p^*; dp)$, we obtain

$$J_x F(X^*, p^*) \Pi'_{\mathcal{S}^n_+}(Z^*; dZ) + J_p F(X^*, p^*) dp + dZ - \Pi'_{\mathcal{S}^n_+}(Z^*; dZ) \,=\, 0.$$

Note that $X'(p^*; dp) = \Pi'_{\mathcal{S}^n_+}(Z^*; dZ)$. By the previous derivation, we deduce that $X'(p^*; dp)$ is the unique solution S^* of the CP:

$$\mathcal{C} \ni S^* \perp J_p F(X^*, p^*) dp + (\, J_x F(X^*, p^*) + \mathcal{A}_{Z^*}\,)(S^*) \in \mathcal{C}^*,$$

where $\mathcal{C} \equiv \mathcal{T}(X^*; \mathcal{S}^n_+) \cap F(X^*, p^*)^\perp$ is the critical cone of the CP $(\mathcal{S}^n_+, F(\cdot, p^*))$ at the solution X^*.

5.2 The Lorentz CP

Given a function $F : \Re^{n+1} \to \Re^{n+1}$, we call the complementarity problem [5, 11]:

$$\mathcal{K}^n \ni (\, x, t\,) \perp F(x,t) \in \mathcal{K}^n$$

the Lorentz CP. Since $\mathcal{K}^n$ is self-dual, this CP is equivalent to the VI $(\mathcal{K}^n, F)$. Assume that F is continuously differentiable in an open neighborhood of a solution (x^*, t_*) of the Lorentz CP. It follows that the normal map

$$\mathbf{F}^{\rm nor}_{\mathcal{K}^n}(z,\tau) \equiv F(\Pi_{\mathcal{K}^n}(z,\tau)) + (z,\tau) - \Pi_{\mathcal{K}^n}(z,\tau), \quad (z,\tau) \in \Re^{n+1}$$

is semismooth near $(z^*, \tau_*) \equiv (x^*, t_*) - F(x^*, t_*)$. Using the notation in Subsection 4.2, let

$$\left(\hat{dz}, \hat{d\tau}\right) \equiv \left(\mathbf{A}^{\rm nor}_{\mathcal{C}(z^*,\tau_*)}\right)^{-1}(dz, d\tau);$$

we have

$$(dz, d\tau) = A(z^*, \tau_*)\Pi_{\mathcal{C}(z^*,\tau_*)}(\hat{dz}, \hat{d\tau}) + \left(\hat{dz}, \hat{d\tau}\right) - \Pi_{\mathcal{C}(z^*,\tau_*)}(\hat{dz}, \hat{d\tau}).$$

Consequently,

$$\begin{aligned}
&(\mathbf{F}^{\rm nor}_{\mathcal{K}^n})'((z^*, \tau_*); (dz, d\tau)) \\
&= JF(x^*, t_*)\Pi'_{\mathcal{K}^n}((z^*, \tau_*); (dz, d\tau)) + (dz, d\tau) - \Pi'_{\mathcal{K}^n}((z^*, \tau_*); (dz, d\tau)) \\
&= JF(x^*, t_*)\Pi_{\mathcal{C}(z,\tau_*)}(\hat{dz}, \hat{d\tau}) + (dz, d\tau) - \Pi_{\mathcal{C}(z^*,\tau_*)}(\hat{dz}, \hat{d\tau}) \\
&= G\Pi_{\mathcal{C}(z^*,\tau_*)}(\hat{dz}, \hat{d\tau}) + \left(\hat{dz}, \hat{d\tau}\right) - \Pi_{\mathcal{C}(z^*,\tau_*)}(\hat{dz}, \hat{d\tau}),
\end{aligned}$$

where $G \equiv JF(x^*, t_*) + A(z^*, \tau_*) - I$. Hence if we let $\mathbf{G}^{\rm nor}_{\mathcal{C}}$ be the normal map of the pair $(\mathcal{C}(z^*, \tau_*), G)$, it follows that

$$\mathbf{F}^{\rm nor}_{\mathcal{K}^n} = \mathbf{G}^{\rm nor}_{\mathcal{C}} \circ \left(\mathbf{A}^{\rm nor}_{\mathcal{C}(z^*,\tau_*)}\right)^{-1}.$$

From this point on, the analysis of the Lorentz CP is very similar to that of the SDCP. The details are not repeated.

6 Conclusion

In this paper, we have established a complete inverse function theorem for semismooth equations and deduced from the theorem an implicit function theorem for such equations that depend on a parameter. We have shown how the inverse/implicit function theorem can be used to obtain necessary and sufficient conditions for the strong stability/regularity of solutions to CPs on the cone of SPSD matrices and on the Lorentz cone. We have further shown how the directional derivatives of a strongly stable parametric solution can be calculated by differentiating the parametric equation. Our development relies on certain directional derivative formulas for the projections on the cone of SPSD matrices and on the Lorentz cone.

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