

Applications of SOC Programming

Defeng Sun

Department of Mathematics

National University of Singapore

Republic of Singapore

September 20, 2006

Lecture Notes for

MA4260 Model Building in Operations Research

Consider the second order cone programming (SOCP)

$$\begin{aligned} \min \quad & f^T x \\ \text{s.t.} \quad & \|A_i x + b_i\| \leq c_i^T x + d_i, \\ & i = 1, \dots, N, \end{aligned}$$

where $f, x \in \Re^n$, $A_i \in \Re^{(n_i-1) \times n}$, $b_i \in \Re^{n_i-1}$, $c_i \in \Re^n$, $d_i \in \Re$.

The standard second order cone (SOC) in \Re^k is

$$\mathcal{C}_k = \left\{ \begin{bmatrix} u \\ t \end{bmatrix} \mid u \in \Re^{k-1}, t \in \Re, \|u\| \leq t \right\} .$$

More regularly,

$$\mathcal{K}^k = \left\{ \begin{bmatrix} t \\ u \end{bmatrix} \mid u \in \Re^{k-1}, t \in \Re, t \geq \|u\| \right\} .$$

$$\|A_i x + b_i\| \leq c_i^T x + d_i \iff \begin{bmatrix} A_i \\ c_i^T \end{bmatrix} x + \begin{bmatrix} b_i \\ d_i \end{bmatrix} \in \mathcal{C}_{n_i}.$$

The SOCP becomes

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & \begin{bmatrix} A_i \\ c_i^T \end{bmatrix} x + \begin{bmatrix} b_i \\ d_i \end{bmatrix} \in \mathcal{C}_{n_i}, \\ & i = 1, \dots, N \end{array}$$

Since

$$\mathcal{C}_1 = \{t \mid t \in \mathbb{R}, t \geq 0\},$$

- when $n_i = 1$ for $i = 1, \dots, N$, the SOCP reduces to

$$\begin{aligned} \min \quad & f^T x \\ \text{s.t.} \quad & 0 \leq c_i^T x + d_i, \\ & i = 1, \dots, N. \end{aligned}$$

— Linear Programming in the dual form.

- When $c_i \equiv 0$ for $i = 1, \dots, N$, the SOCP reduces to

$$\begin{aligned} \min \quad & f^T x \\ \text{s.t.} \quad & \|A_i x + b_i\|^2 \leq d_i^2, \\ & i = 1, \dots, N, \end{aligned}$$

$$\begin{aligned} \min \quad & f^T x \\ \text{i.e., s.t.} \quad & x^T (A_i^T A_i) x + 2b_i^T A_i x + b_i^T b_i - d_i^2 \leq 0, \\ & i = 1, \dots, N. \end{aligned}$$

— A special case of convex quadratically constrained quadratic programming (QCQP).

Let us consider the general convex QCQP

$$\begin{aligned} \min \quad & x^T P_0 x + 2q_0^T x + r_0 \\ \text{s.t.} \quad & x^T P_i x + 2q_i^T x + r_i \leq 0, \\ & i = 1, \dots, p, \end{aligned}$$

where $P_0, P_1, \dots, P_p \in \Re^{n \times n}$, symmetric, positive definite. There exists $P_i^{1/2} \succeq 0$ such that

$$(P_i^{1/2})^2 = P_i, \quad i = 0, 1, \dots, p.$$

Suppose that $P_i \succ 0$, $i = 0, 1, \dots, p$. Then the convex QCQP becomes

$$\begin{aligned} \min \quad & (P_0^{1/2}x)^T(P_0^{1/2}x) + 2(P_0^{-1/2}q_0)^T(P_0^{1/2}x) + r_0 \\ \text{s.t.} \quad & (P_i^{1/2}x)^T(P_i^{1/2}x) + 2(P_i^{-1/2}q_i)^T(P_i^{1/2}x) \\ & + r_i \leq 0, \quad i = 1, \dots, p. \end{aligned}$$

Then we get (SOCP – 1)

$$\begin{aligned} \min \quad & \|P_0^{1/2}x + P_0^{-1/2}q_0\|^2 + r_0 - q_0^T P_0^{-1}q_0 \\ \text{s.t.} \quad & \|P_i^{1/2}x + P_i^{-1/2}q_i\|^2 + r_i - q_i^T P_i^{-1}q_i \leq 0, \\ & i = 1, \dots, p \end{aligned}$$

and (SOCP – 2)

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \|P_0^{1/2}x + P_0^{-1/2}q_0\| \leq t \\ & \|P_0^{1/2}x + P_i^{-1/2}q_i\| \leq (q_i^T P_i^{-1}q_i - r_i)^{1/2}, \\ & i = 1, \dots, p. \end{aligned}$$

The optimal value of (SOCP – 1) is

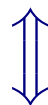
$$(t^*)^2 + r_0 - q_0^T P_0^{-1}q_0,$$

where t^* is the optimal value of (SOCP – 2) .

EXERCISE. How about the case that some $P_i \succeq 0$, but $P_i \succ 0$ does not hold?

A special case: Convex QP

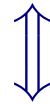
$$\begin{aligned} \min \quad & x^T P_0 x + 2q_0^T x + r_0 \quad (P_0 \succ 0) \\ \text{s.t.} \quad & a_i^T x \leq b_i, \\ & i = 1, \dots, p, \end{aligned}$$



$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \|P_0^{1/2}x + P_0^{-1/2}q_0\| \leq t \\ & a_i^T x \leq b_i, \\ & i = 1, \dots, p. \end{aligned}$$

Sum of Norms: Let $F_i \in \Re^{n_i \times n}$, $g_i \in \Re^{n_i}$, $i = 1, \dots, p$. Consider

$$\min \sum_{i=1}^p \|F_i x + g_i\|$$



$$\begin{aligned} \min \quad & \sum_{i=1}^p t_i \\ \text{s.t.} \quad & \|F_j x + g_j\| \leq t_j, \\ & j = 1, \dots, p. \end{aligned}$$

Maxima of Norms: Consider

$$\min \max_{1 \leq i \leq p} \|F_i x + g_i\|$$



$$\min \quad t$$

$$\text{s.t.} \quad \|F_j x + g_j\| \leq t,$$

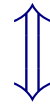
$$j = 1, \dots, p.$$

A special case of Sum of Norms: Consider the l_1 -norm approximation problem

$$\min \|Ax - b\|_1$$

where $x \in \mathbb{C}^q$, $A \in \mathbb{C}^{p \times q}$, $b \in \mathbb{C}^p$. The l_1 norm on \mathbb{C}^p is defined by

$$\|v\|_1 = \sum_{i=1}^p |v_i|.$$



$$\begin{aligned}
 & \min \quad \sum_{i=1}^p t_i \\
 & \text{s.t.} \quad \left\| \begin{bmatrix} \Re a_i^T & -\Im a_i^T \\ \Im a_i^T & \Re a_i^T \end{bmatrix} z - \begin{bmatrix} \Re b_i \\ \Im b_i \end{bmatrix} \right\| \leq t_i, \\
 & \quad i = 1, \dots, p.
 \end{aligned}$$

in the variables $z = [\Re x^T \ \Im x^T]^T \in \mathbb{R}^{2q}$ and $t_i, i = 1, \dots, p$.

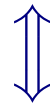
EXERCISE. How about the l_∞ norm approximation problem?

An extension: Consider

$$\begin{aligned} \min \quad & \sum_{i=1}^k y_{[i]} \\ \text{s.t.} \quad & \|F_i x + g_i\| = y_i, \\ & i = 1, \dots, p, \end{aligned}$$

where $y_{[1]}, y_{[2]}, \dots, y_{[p]}$ are the numbers of y_1, y_2, \dots, y_p arranged in the decreasing order

$$y_{[1]} \geq y_{[2]} \geq \dots \geq y_{[p]} .$$



$$\begin{array}{ll} \min & kt + \sum_{i=1}^p y_i \\ \text{s.t.} & \|F_i x + g_i\| \leq t + y_i, \quad i = 1, \dots, p, \\ & y_i \geq 0, \quad i = 1, \dots, p, \end{array}$$

in variables $x, y \in \Re^p$ and $t \in \Re$.

Exercise: Work out the details.

A Facility Location Problem:

Suppose there are N towns in a city with positions given by $y^{(j)} \in \mathbb{R}^2, j = 1, \dots, N$. The city council is planning to build a new facility (fire station, hospital, public library, and etc.) to serve all the residents in the N towns. Where should the new facility built?

Let us use $x \in \mathbb{R}^2$ to denote the location of the new facility. One known strategy is to

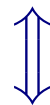
$$\begin{aligned} \min \quad & \sum_{j=1}^N \|x - y^{(j)}\|^2 \\ \text{s.t.} \quad & x \in \Omega, \end{aligned}$$

where Ω is a closed convex set (a polyhedral set in our case).

A Facility Location Problem (continued):

Is the quadratic programming approach a good one? How about

$$\begin{array}{ll} \min & \sum_{j=1}^N \|x - y^{(j)}\|^2 \\ \text{s.t.} & x \in \Omega, \end{array}$$

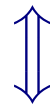


$$\begin{aligned} \min \quad & \sum_{i=1}^N t_i \\ \text{s.t.} \quad & \|x - y^{(j)}\| \leq t_j, j = 1, \dots, p \\ & x \in \Omega. \end{aligned}$$

A Facility Location Problem (continued):

It seems a good choice. But, is it fair? Think about the case $N = 2$. Then how about

$$\begin{array}{ll} \min & \max_{j=1,\dots,N} \|x - y^{(j)}\| \\ \text{s.t.} & x \in \Omega, \end{array}$$

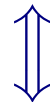


$$\begin{array}{ll}\min & t \\ \text{s.t.} & \|x - y^{(j)}\| \leq t, j = 1, \dots, p \\ & x \in \Omega.\end{array}$$

Which strategy is better?

Hyperbolic Constraints:

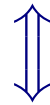
$$w^2 \leq xy, \quad x \geq 0, \quad y \geq 0$$



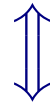
$$\left\| \begin{bmatrix} 2w \\ x - y \end{bmatrix} \right\| \leq x + y,$$

And more generally when w is a vector

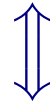
$$w^T w \leq xy, \quad x \geq 0, \quad y \geq 0$$



$$w^T w \leq \frac{(x+y)^2 - (x-y)^2}{4}, \quad x \geq 0, \quad y \geq 0$$



$$\left\| \begin{bmatrix} 2w \\ x - y \end{bmatrix} \right\|^2 \leq (x + y)^2, \quad x \geq 0, \quad y \geq 0,$$



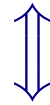
$$\left\| \begin{bmatrix} 2w \\ x - y \end{bmatrix} \right\| \leq x + y.$$

An example: Consider

$$\begin{array}{ll} (A) & \min \quad \sum_{i=1}^p 1/(a_i^T x + b_i) \\ & \text{s.t.} \quad a_i^T x + b_i \geq 0, i = 1, \dots, p, \\ & \quad \quad c_j^T x + d_j \geq 0, j = 1, \dots, q. \end{array}$$



$$\begin{array}{ll}\min & \sum_{i=1}^p t_i \\ \text{s.t.} & t_i(a_i^T x + b_i) \geq 1, \quad t_i \geq 0, i = 1, \dots, p, \\ & c_j^T x + d_j \geq 0, j = 1, \dots, q.\end{array}$$

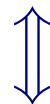


$$\begin{aligned} \min \quad & \sum_{i=1}^p t_i \\ \text{s.t.} \quad & \left\| \begin{bmatrix} 2 \\ a_i^T x + b_i - t_i \end{bmatrix} \right\| \leq a_i^T x + b_i + t_i \\ & i = 1, \dots, p, \\ & c_j^T x + d_j \geq 0, j = 1, \dots, q. \end{aligned}$$

Exercise: Can we convert (A) into an LP?

Quadratic/linear fractional problem:

$$\begin{aligned} \min \quad & \sum_{i=1}^p \frac{\|F_i x + g_i\|^2}{(a_i^T x + b_i)} \\ \text{s.t.} \quad & a_i^T x + b_i > 0, i = 1, \dots, p. \end{aligned}$$



$$\begin{aligned} \min \quad & \sum_{i=1}^p t_i \\ \text{s.t.} \quad & (F_i x + g_i)^T (F_i x + g_i) \leq t_i (a_i^T x + b_i) \\ & t_i \geq 0, \quad a_i^T x + b_i > 0, i = 1, \dots, p. \end{aligned}$$

Since for each $i = 1, \dots, p$,

$$(F_i x + g_i)^T (F_i x + g_i) \leq t_i (a_i^T x + b_i), t_i \geq 0, a_i^T x + b_i \geq 0$$



$$\left\| \begin{bmatrix} 2(F_i x + g_i) \\ a_i^T x + b_i - t_i \end{bmatrix} \right\| \leq a_i^T x + b_i + t_i,$$

we get an SOC programming again

$$\begin{array}{ll} \min & \sum_{i=1}^p t_i \\ \text{s.t.} & \left\| \begin{bmatrix} 2(F_i x + g_i) \\ a_i^T x + b_i - t_i \end{bmatrix} \right\| \leq a_i^T x + b_i + t_i, \\ & i = 1, \dots, p. \end{array}$$

Logarithmic Chebyshev approximation problem:

$$\min \max_{i=1,\dots,p} |\log(a_i^T x) - \log(b_i)|$$

where $A = [a_1 \cdots a_p]^T \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^p$. Assume $b > 0$ and interpret $\log(a_i^T x)$ as $-\infty$ when $a_i^T x \leq 0$. This can be interpreted as an approximation of an overdetermined linear system $Ax \approx b$.

First note that when $a_i^T x > 0$,

$$|\log(a_i^T x) - \log(b_i)| = \log \max(a_i^T x / b_i, b_i / a_i^T x).$$

Then we get (removing log)

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & 1/t \leq a_i^T x / b_i \leq t, \quad i = 1, \dots, p, \\ & t > 0, a_i^T x / b_i \geq 0, \quad i = 1, \dots, p. \end{aligned}$$



$$\min \quad t$$

$$\text{s.t.} \quad a_i^T x / b_i \leq t,$$

$$\left\| \begin{bmatrix} 2 \\ t - a_i^T x / b_i \end{bmatrix} \right\| \leq t + a_i^T x + b_i,$$
$$i = 1, \dots, p.$$

Geometric mean problem:

$$\begin{array}{ll} \max & \prod_{i=1}^p (a_i^T x + b_i)^{1/p} \\ \text{s.t.} & a_i^T x + b_i \geq 0, i = 1, \dots, p. \end{array}$$

We consider the case that $p = 4$. By using variables substitution, we get

$$\begin{aligned} \max \quad & t_3 \\ \text{s.t.} \quad & (a_1^T x + b_1)(a_2^T x + b_2) = t_1^2, \\ & a_1^T x + b_1 \geq 0, \quad a_2^T x + b_2 \geq 0, \\ & (a_3^T x + b_3)(a_4^T x + b_4) = t_2^2, \\ & a_3^T x + b_3 \geq 0, \quad a_4^T x + b_4 \geq 0, \\ & t_1 t_2 = t_3^2, \quad t_1 \geq 0, \quad t_2 \geq 0. \end{aligned}$$

Because of “maximization”, we get

$$\begin{aligned} \max \quad & t_3 \\ \text{s.t.} \quad & (a_1^T x + b_1)(a_2^T x + b_2) \geq t_1^2, \\ & a_1^T x + b_1 \geq 0, \quad a_2^T x + b_2 \geq 0, \\ & (a_3^T x + b_3)(a_4^T x + b_4) \geq t_2^2, \\ & a_3^T x + b_3 \geq 0, \quad a_4^T x + b_4 \geq 0, \\ & t_1 t_2 = t_3^2, \quad t_1 \geq 0, \quad t_2 \geq 0. \end{aligned}$$

By using the “maximization” technique one more time, we get

$$\begin{aligned} \max \quad & t_3 \\ \text{s.t.} \quad & (a_1^T x + b_1)(a_2^T x + b_2) \geq t_1^2, \\ & a_1^T x + b_1 \geq 0, \quad a_2^T x + b_2 \geq 0, \\ & (a_3^T x + b_3)(a_4^T x + b_4) \geq t_2^2, \\ & a_3^T x + b_3 \geq 0, \quad a_4^T x + b_4 \geq 0, \\ & t_1 t_2 \geq t_3^2, \quad t_1 \geq 0, \quad t_2 \geq 0. \end{aligned}$$

Eventually, we obtain

$$\begin{aligned} \max \quad & t_3 \\ \text{s.t.} \quad & (1) + (2) \\ & \left\| \begin{bmatrix} 2t_3 \\ t_1 - t_2 \end{bmatrix} \right\| \leq t_1 + t_2. \end{aligned}$$

$$(1) \left\| \begin{bmatrix} 2t_1 \\ (a_1^T x + b_1) - (a_2^T x + b_2) \end{bmatrix} \right\| \leq a_1^T x + b_1 + a_2^T x + b_2$$

$$(2) \left\| \begin{bmatrix} 2t_2 \\ (a_3^T x + b_3) - (a_4^T x + b_4) \end{bmatrix} \right\| \leq a_3^T x + b_3 + a_4^T x + b_4$$

SOC representation:

A convex set $C \subseteq \Re^n$ is *second-order cone representable* if it can be represented by a number of second-order cone constraints, possibly after introducing auxiliary variables, i.e., there exist $A_i \in \Re^{(n_i-1) \times (n+m)}$, $b_i \in \Re^{n_i-1}$, $c_i \in \Re^{n+m}$, $d_i \in \Re$ such that $x \in C$ if and only if there exists $y \in \Re^m$ such that

$$\left\| A_i \begin{bmatrix} x \\ y \end{bmatrix} + b_i \right\| \leq c_i^T \begin{bmatrix} x \\ y \end{bmatrix} + d_i ,$$

where $i = 1, \dots, N$.

We say that a function f is SOC-representable if its epigraph

$$\{(x, t) \mid f(x) \leq t\}$$

is SOC representable. If f and C are both SOC-representable, then the convex optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in C \end{array}$$

can be cast an SOC programming.

Robust Linear Programming:

Consider

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \leq b_i, \quad i = 1, \dots, m, \end{aligned}$$

in which there is some uncertainty on c, a_i, b_i . For simplicity, suppose that c and b_i are fixed, and a_i are known to lie in given ellipsoids

$$a_i \in \mathcal{E}_i := \{\bar{a}_i + P_i u \mid \|u\| \leq 1\},$$

where $P_i = P_i^T \succeq 0$.

Consider the *robust linear programming*

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & a_i^T x \leq b_i, \forall a_i \in \mathcal{E}_i, i = 1, \dots, m. \end{array}$$

The robust linear constraint

$$a_i^T x \leq b_i, \forall a_i \in \mathcal{E}_i$$

can be expressed as

$$\max\{a_i^T x \mid a_i \in \mathcal{E}_i\} = \bar{a}_i^T x + \|P_i x\| \leq b_i.$$

Hence, the robust LP is

$$\min \quad c^T x$$

$$\text{s.t.} \quad \bar{a}_i^T x + \|P_i x\| \leq b_i, i = 1, \dots, m.$$

Robust linear squares.

Antenna array weight design.

Grasping force optimization.

FIR (finite impulse response) filter design.

Portfolio optimization with loss risk constraints.

Truss design.

Equilibrium of system with piecewise-linear springs ...