Applications of SOC Programming

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Lecture Notes for
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Consider the second order cone programming (SOCP)

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & \|A_i x + b_i\| \le c_i^T x + d_i, \\ & i = 1, \ldots, N, \end{array}$$

where $f, x \in \Re^n$, $A_i \in \Re^{(n_i-1)\times n}$, $b_i \in \Re^{n_i-1}$, $c_i \in \Re^n$, $d_i \in \Re$.

The standard second orde cone (SOC) in $\Re^k$ is

$$\mathcal{C}_k = \left\{ \begin{bmatrix} u \\ t \end{bmatrix} \mid u \in \Re^{k-1}, t \in \Re, \|u\| \leq t \right\}.$$

More regularly,

$$\mathcal{K}^k = \left\{ \begin{bmatrix} t \\ u \end{bmatrix} \mid u \in \Re^{k-1}, t \in \Re, t \geq \|u\| \right\}.$$

$$\|A_i x + b_i\| \leq c_i^T x + d_i \Longleftrightarrow \begin{bmatrix} A_i \\ c_i^T \end{bmatrix} x + \begin{bmatrix} b_i \\ d_i \end{bmatrix} \in \mathcal{C}_{n_i}\,.$$

The SOCP becomes

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & \begin{bmatrix} A_i \\ c_i^T \end{bmatrix} x + \begin{bmatrix} b_i \\ d_i \end{bmatrix} \in \mathcal{C}_{n_i}\,, \\ & i = 1, \ldots, N \end{array}$$

Since

$$\mathcal{C}_1 = \{t \,|\, t \in \Re, t \geq 0\},$$

- when $n_i = 1$ for $i = 1, \ldots, N$, the SOCP reduces to

$$\begin{aligned} \min \quad & f^T x \\ \text{s.t.} \quad & 0 \leq c_i^T x + d_i\,, \\ & i = 1, \ldots, N\,. \end{aligned}$$

—- Linear Programming in the dual form.

- When $c_i \equiv 0$ for $i = 1, \ldots, N$, the SOCP reduces to

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & \|A_i x + b_i\|^2 \le d_i^2\,, \\ & i = 1, \ldots, N\,, \end{array}$$

$$\begin{array}{lll} & \min & f^T x \\ \text{i.e.,} & \text{s.t.} & x^T (A_i^T A_i) x + 2 b_i^T A_i x + b_i^T b_i - d_i^2 \le 0\,, \\ & & i = 1, \ldots, N\,. \end{array}$$

— A special case of convex quadratically constrained quadratic programming (QCQP).

Let us consider the general convex QCQP

$$\begin{array}{ll} \min & x^T P_0 x + 2q_0^T x + r_0 \\ \text{s.t.} & x^T P_i x + 2q_i^T x + r_i \le 0\,, \\ & i = 1, \ldots, p\,, \end{array}$$

where $P_0, P_1, \ldots, P_p \in \Re^{n\times n}$, symmetric, positive definite. There exists $P_i^{1/2} \succeq 0$ such that

$$(P_i^{1/2})^2 = P_i, \quad i = 0, 1, \ldots, p\,.$$

Suppose that $P_i \succ 0$, $i = 0, 1, \ldots, p$. Then the convex QCQP becomes

$$\begin{array}{ll} \min & (P_0^{1/2}x)^T(P_0^{1/2}x) + 2(P_0^{-1/2}q_0)^T(P_0^{1/2}x) + r_0 \\ \text{s.t.} & (P_i^{1/2}x)^T(P_i^{1/2}x) + 2(P_i^{-1/2}q_i)^T(P_i^{1/2}x) \\ & \qquad\qquad\qquad +r_i \leq 0\,,\ i = 1, \ldots, p\,. \end{array}$$

Then we get (SOCP − 1)

$$
\begin{aligned}
\min \quad & \|P_0^{1/2}x + P_0^{-1/2}q_0\|^2 + r_0 - q_0^T P_0^{-1} q_0 \\
\text{s.t.} \quad & \|P_i^{1/2}x + P_i^{-1/2}q_i\|^2 + r_i - q_i^T P_i^{-1} q_i \leq 0\,, \\
& i = 1, \ldots, p
\end{aligned}
$$

and (SOCP − 2)

$$
\begin{aligned}
\min \quad & t \\
\text{s.t.} \quad & \|P_0^{1/2}x + P_0^{-1/2}q_0\| \le t \\
& \|P_0^{1/2}x + P_i^{-1/2}q_i\| \le (q_i^T P_i^{-1} q_i - r_i)^{1/2}, \\
& i = 1, \ldots, p\,.
\end{aligned}
$$

The optimal value of (SOCP − 1) is

$$(t^*)^2 + r_0 - q_0^T P_0^{-1} q_0,$$

where t^* is the optimal value of (SOCP − 2) .

EXERCISE. How about the case that some $P_i \succeq 0$, but $P_i \succ 0$ does not hold?

A special case: Convex QP

$$
\begin{aligned}
\min \quad & x^T P_0 x + 2q_0^T x + r_0 \qquad (P_0 \succ 0) \\
\text{s.t.} \quad & a_i^T x \le b_i\,, \\
& i = 1, \ldots, p\,,
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
\min \quad & t \\
\text{s.t.} \quad & \|P_0^{1/2} x + P_0^{-1/2} q_0\| \le t \\
& a_i^T x \le b_i\,, \\
& i = 1, \ldots, p\,.
\end{aligned}
$$

Sum of Norms: Let $F_i \in \Re^{n_i \times n}$, $g_i \in \Re^{n_i}$, $i = 1, \ldots, p$. Consider

$$\min \quad \sum_{i=1}^{p} \|F_i x + g_i\|$$

$$\Updownarrow$$

$$\begin{array}{ll} \min & \displaystyle\sum_{i=1}^{p} t_i \\ \text{s.t.} & \|F_j x + g_j\| \le t_j \,, \\ & j = 1, \ldots, p\,. \end{array}$$

Maxima of Norms: Consider

$$\min \quad \max_{1 \le i \le p} \|F_i x + g_i\|$$

$$\Updownarrow$$

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \|F_j x + g_j\| \le t\,, \\ & j = 1, \ldots, p\,. \end{aligned}$$

A special case of Sum of Norms: Consider the l_1-norm approximation problem

$$\boxed{\min \quad \|Ax - b\|_1}$$

where $x \in \mathbb{C}^q$, $A \in \mathbb{C}^{p \times q}$, $b \in \mathbb{C}^q$. The l_1 norm on $\mathbb{C}^p$ is defined by

$$\|v\|_1 = \sum_{i=1}^{p} |v_i| \,.$$

$$\Updownarrow$$

$$\begin{aligned} \min \quad & \sum_{i=1}^{p} t_i \\ \text{s.t.} \quad & \left\| \begin{bmatrix} \Re a_i^T & -\Im a_i^T \\ \Im a_i^T & \Re a_i^T \end{bmatrix} z - \begin{bmatrix} \Re b_i \\ \Im b_i \end{bmatrix} \right\| \le t_i \, , \\ & i = 1, \ldots, p \, . \end{aligned}$$

in the variables $z = [\Re x^T \ \Im x^T]^T \in \Re^{2q}$ and t_i, $i = 1, \ldots, p$.

EXERCISE. How about the l_∞ norm approximation problem?

An extension: Consider

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{k} y_{[i]} \\
\text{s.t.} \quad & \|F_i x + g_i\| = y_i \,, \\
& i = 1, \ldots, p \,,
\end{aligned}
$$

where $y_{[1]}, y_{[2]}, \ldots, y_{[p]}$ are the numbers of $y_1, y_2, \ldots, y_p$ arranged in the decreasing order

$$y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[p]} \,.$$

$$\Updownarrow$$

$$\begin{aligned} \min \quad & kt + \sum_{i=1}^{p} y_i \\ \text{s.t.} \quad & \|F_i x + g_i\| \le t + y_i\,,\ i = 1, \ldots, p\,, \\ & y_i \ge 0\,,\ i = 1, \ldots, p\,, \end{aligned}$$

in variables $x, y \in \Re^p$ and $t \in \Re$.

Exercise: Work out the details.

A Facility Location Problem:

Suppose there are N towns in a city with positions given by $y^{(j)} \in \mathbb{R}^2, j = 1, \ldots, N$. The city council is planning to build a new facility (fire station, hospital, public library, and etc.) to serve all the residents in the N towns. Where should the new facility built?

Let us use $x \in \mathbb{R}^2$ to denote the location of the new facility. One known strategy is to

$$\begin{array}{ll} \min & \displaystyle\sum_{j=1}^{N} \|x - y^{(j)}\|^2 \\ \text{s.t.} & x \in \Omega, \end{array}$$

where Ω is a closed convex set (a polyhedral set in our case).

A Facility Location Problem (continued):

Is the quadratic programming approach a good one? How about

$$\begin{array}{ll} \min & \sum_{j=1}^{N} \|x - y^{(j)}\| \\ \text{s.t.} & x \in \Omega, \end{array}$$

$$\Updownarrow$$

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{N} t_i \\
\text{s.t.} \quad & \|x - y^{(j)}\| \le t_j \, , j = 1, \ldots, p \\
& x \in \Omega.
\end{aligned}
$$

A Facility Location Problem (continued):

It seems a good choice. But, is it fair? Think about the case $N = 2$. Then how about

$$\begin{array}{ll} \min & \max\limits_{j=1,\ldots,N} \|x - y^{(j)}\| \\ \text{s.t.} & x \in \Omega, \end{array}$$

$$\Updownarrow$$

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \|x - y^{(j)}\| \leq t\,, j = 1, \ldots, p \\ & x \in \Omega. \end{aligned}$$

Which strategy is better?

Hyperbolic Constraints:

$$w^2 \leq xy,\ x \geq 0\,,\ y \geq 0$$

$$\Updownarrow$$

$$\left\| \begin{bmatrix} 2w \\ x-y \end{bmatrix} \right\| \leq x+y\,,$$

And more generally when w is a vector

$$w^T w \le xy, \;\; x \ge 0\,, \;\; y \ge 0$$

$$\Updownarrow$$

$$w^T w \le \frac{(x+y)^2 - (x-y)^2}{4}\,, \;\; x \ge 0\,, \;\; y \ge 0$$

$$\Updownarrow$$

$$\left\| \begin{bmatrix} 2w \\ x - y \end{bmatrix} \right\|^2 \leq (x+y)^2\,, \; x \geq 0\,, \; y \geq 0\,,$$

$$\Updownarrow$$

$$\left\| \begin{bmatrix} 2w \\ x - y \end{bmatrix} \right\| \leq x + y\,.$$

An example: Consider

$$
(A) \qquad
\begin{array}{ll}
\min & \displaystyle\sum_{i=1}^{p} 1/(a_i^T x + b_i) \\
\text{s.t.} & a_i^T x + b_i \geq 0\,, i = 1, \ldots, p\,, \\
 & c_j^T x + d_j \geq 0\,, j = 1, \ldots, q\,.
\end{array}
$$

$$\Updownarrow$$

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{p} t_i \\
\text{s.t.} \quad & t_i(a_i^T x + b_i) \geq 1\,,\; t_i \geq 0\,, i = 1, \ldots, p\,, \\
& c_j^T x + d_j \geq 0\,, j = 1, \ldots, q\,.
\end{aligned}
$$

$$\Updownarrow$$

$$\begin{aligned}
\min \quad & \sum_{i=1}^{p} t_i \\
\text{s.t.} \quad & \left\| \begin{bmatrix} 2 \\ a_i^T x + b_i - t_i \end{bmatrix} \right\| \le a_i^T x + b_i + t_i \\
& \qquad i = 1, \ldots, p\,, \\
& c_j^T x + d_j \ge 0\,, j = 1, \ldots, q\,.
\end{aligned}$$

Exercise: Can we convert (A) into an LP?

Quadratic/linear fractional problem:

$$
\begin{array}{ll}
\min & \displaystyle\sum_{i=1}^{p} \frac{\|F_i x + g_i\|^2}{(a_i^T x + b_i)} \\
\text{s.t.} & a_i^T x + b_i > 0\,, i = 1, \ldots, p\,.
\end{array}
$$

$$\Updownarrow$$

$$
\begin{array}{ll}
\min & \displaystyle\sum_{i=1}^{p} t_i \\
\text{s.t.} & (F_i x + g_i)^T (F_i x + g_i) \le t_i (a_i^T x + b_i) \\
 & t_i \ge 0,\ a_i^T x + b_i > 0\,, i = 1, \ldots, p\,.
\end{array}
$$

Since for each $i = 1, \ldots, p$,

$$(F_i x + g_i)^T (F_i x + g_i) \le t_i(a_i^T x + b_i)\,, t_i \ge 0,\ a_i^T x + b_i \ge 0$$

$$\Updownarrow$$

$$\boxed{\left\| \begin{bmatrix} 2(F_i x + g_i) \\ a_i^T x + b_i - t_i \end{bmatrix} \right\| \le a_i^T x + b_i + t_i\,,}$$

we get an SOC programming again

$$
\begin{array}{ll}
\min & \displaystyle\sum_{i=1}^{p} t_i \\
\text{s.t.} & \left\| \begin{bmatrix} 2(F_i x + g_i) \\ a_i^T x + b_i - t_i \end{bmatrix} \right\| \leq a_i^T x + b_i + t_i \,, \\
& \qquad i = 1, \ldots, p \,.
\end{array}
$$

Logarithmic Chebyshev approximation problem:

$$\min \max_{i=1,\ldots,p} |\log(a_i^T x) - \log(b_i)|$$

where $A = [a_1 \cdots a_p]^T \in \Re^{p\times n}$, $b \in \Re^p$. Assume $b > 0$ and interpret $\log(a_i^T x)$ as $-\infty$ when $a_i^T x \leq 0$. This can be interpreted as an approximation of an overdetermined linear system $Ax \approx b$.

First note that when $a_i^T x > 0$,

$$|\log(a_i^T x) - \log(b_i)| = \log\ \max(a_i^T x/b_i, b_i/a_i^T x)\,.$$

Then we get (removing log)

$$
\begin{aligned}
\min \quad & t \\
\text{s.t.} \quad & 1/t \le a_i^T x/b_i \le t\,, \; i = 1, \ldots, p\,, \\
& t > 0, a_i^T x/b_i \ge 0\,, \; i = 1, \ldots, p\,.
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
\min \quad & t \\
\text{s.t.} \quad & a_i^T x / b_i \le t\,, \\
& \left\| \begin{bmatrix} 2 \\ t - a_i^T x / b_i \end{bmatrix} \right\| \le t + a_i^T x + b_i\,, \\
& \qquad i = 1, \ldots, p\,.
\end{aligned}
$$

Geometric mean problem:

$$\begin{array}{ll} \max & \prod_{i=1}^{p}(a_i^T x + b_i)^{1/p} \\ \text{s.t.} & a_i^T x + b_i \geq 0\,, i = 1, \ldots, p\,. \end{array}$$

We consider the case that $p = 4$. By using variables substitution, we get

$$
\begin{array}{ll}
\max & t_3 \\
\text{s.t.} & (a_1^T x + b_1)(a_2^T x + b_2) = t_1^2\,, \\
& a_1^T x + b_1 \ge 0,\ \ a_2^T x + b_2 \ge 0\,, \\
& (a_3^T x + b_3)(a_4^T x + b_4) = t_2^2\,, \\
& a_3^T x + b_3 \ge 0,\ \ a_4^T x + b_4 \ge 0\,, \\
& t_1 t_2 = t_3^2\,,\ \ t_1 \ge 0,\ \ t_2 \ge 0\,.
\end{array}
$$

Because of "maximization", we get

$$
\begin{aligned}
\max \quad & t_3 \\
\text{s.t.} \quad & (a_1^T x + b_1)(a_2^T x + b_2) \geq t_1^2\,, \\
& a_1^T x + b_1 \geq 0,\ \ a_2^T x + b_2 \geq 0\,, \\
& (a_3^T x + b_3)(a_4^T x + b_4) \geq t_2^2\,, \\
& a_3^T x + b_3 \geq 0,\ \ a_4^T x + b_4 \geq 0\,, \\
& t_1 t_2 = t_3^2,\ \ t_1 \geq 0,\ \ t_2 \geq 0\,.
\end{aligned}
$$

By using the "maximization" technique one more time, we get

$$
\begin{aligned}
\max \quad & t_3 \\
\text{s.t.} \quad & (a_1^T x + b_1)(a_2^T x + b_2) \geq t_1^2\,, \\
& a_1^T x + b_1 \geq 0,\;\; a_2^T x + b_2 \geq 0\,, \\
& (a_3^T x + b_3)(a_4^T x + b_4) \geq t_2^2\,, \\
& a_3^T x + b_3 \geq 0,\;\; a_4^T x + b_4 \geq 0\,, \\
& t_1 t_2 \geq t_3^2,\;\; t_1 \geq 0,\;\; t_2 \geq 0\,.
\end{aligned}
$$

Eventually, we obtain

$$
\begin{aligned}
\max \quad & t_3 \\
\text{s.t.} \quad & (1) + (2) \\
& \left\| \begin{bmatrix} 2t_3 \\ t_1 - t_2 \end{bmatrix} \right\| \le t_1 + t_2 \,.
\end{aligned}
$$

$$
(1) \left\| \begin{bmatrix} 2t_1 \\ (a_1^T x + b_1) - (a_2^T x + b_2) \end{bmatrix} \right\| \le a_1^T x + b_1 + a_2^T x + b_2
$$

$$
(2) \left\| \begin{bmatrix} 2t_2 \\ (a_3^T x + b_3) - (a_4^T x + b_4) \end{bmatrix} \right\| \le a_3^T x + b_3 + a_4^T x + b_4
$$

SOC representation:

A convex set $C \subseteq \Re^n$ is *second-order cone representable* if it can be represented by a number of second-order cone constraints, possibly after introducing auxiliary variables, i.e., there exist $A_i \in \Re^{(n_i-1)\times(n+m)}$, $b_i \in \Re^{n_i-1}$, $c_i \in \Re^{n+m}$, $d_i \in \Re$ such that $x \in C$ if and only if there exists $y \in \Re^m$ such that

$$\left\| A_i \begin{bmatrix} x \\ y \end{bmatrix} + b_i \right\| \leq c_i^T \begin{bmatrix} x \\ y \end{bmatrix} + d_i \,,$$

where $i = 1, \ldots, N$.

We say that a function f is SOC-representable if its epigraph

$$\{(x,t) \mid f(x) \le t\}$$

is SOC representable. If f and C are both SOC-representable, then the convex optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in C \end{array}$$

can be cast an SOC programming.

Robust Linear Programming:

Consider

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \le b_i\,, \; i = 1, \ldots, m\,, \end{aligned}$$

in which there is some uncertainty on c, a_i, b_i. For simplicity, suppose that c and b_i are fixed, and a_i are known to lie in given ellipsoids

$$a_i \in \mathcal{E}_i := \{\bar{a}_i + P_i u \,|\, \|u\| \le 1\}\,,$$

where $P_i = P_i^T \succeq 0$.

Consider the *robust linear programming*

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & a_i^T x \leq b_i \,, \forall\; a_i \in \mathcal{E}_i, \;\; i = 1, \ldots, m \,. \end{array}$$

The robust linear constraint

$$a_i^T x \leq b_i \,, \forall\; a_i \in \mathcal{E}_i$$

can be expressed as

$$\max\{a_i^T x \,|\, a_i \in \mathcal{E}_i \,\} = \bar{a}_i^T x + ||P_i x\| \leq b_i \,.$$

Hence, the robust LP is

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & \bar{a}_i^T x + \|P_i x\| \leq b_i\,, i = 1, \ldots, m\,. \end{array}$$

Robust linear squares.

Antenna array weight design.

Grasping force optimization.

FIR (finite impulse response) filter design.

Portfolio optimization with loss risk constraints.

Truss design.

Equilibrium of system with piecewise-linear springs ...