# Sub-Quadratic Convergence of a Smoothing Newton Algorithm for the $P_0-$ and Monotone LCP

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August 17, 2001

**Abstract**

Given $M \in \Re^{n\times n}$ and $q \in \Re^n$, the linear complementarity problem (LCP) is to find $(x, s) \in \Re^n \times \Re^n$ such that $(x, s) \geq 0$, $s = Mx + q$, $x^T s = 0$. By using the Chen-Harker-Kanzow-Smale (CHKS) smoothing function, the LCP is reformulated as a system of parameterized smooth-nonsmooth equations. As a result, a smoothing Newton algorithm, which is a modified version of Qi-Sun-Zhou's algorithm [Mathematical Programming, Vol. 87, 2000, pp. 1–35], is proposed to solve the LCP with $M$ being assumed to be a $P_0$-matrix ($P_0-$LCP). The proposed algorithm needs only to solve one linear system of equations and to do one line search at each iteration. It is proved in this paper that the proposed algorithm has the following convergence results: (i) it is well-defined and any accumulation point of the iteration sequence is a solution of the $P_0-$LCP; (ii) it generates a bounded sequence if the $P_0-$LCP has a nonempty and bounded solution set; (iii) if an accumulation point of the iteration sequence satisfies a nonsingularity condition, which implies the $P_0-$LCP has a unique solution, then the whole iteration sequence converges to this accumulation point sub-quadratically with a $Q$-rate $2-t$, where $t \in (0, 1)$ is a parameter; and (iv) if $M$ is positive semidefinite and an accumulation point of the iteration sequence satisfies a strict complementarity condition, then the whole sequence converges to the accumulation point quadratically. To the best of our knowledge, this is the first smoothing (non-interior continuation) method to have the above local convergence properties both (iii) and (iv).

**Keywords** Linear complementarity problem, smoothing Newton method, global convergence, sub-quadratic convergence.

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# 1 Introduction

Recently there has been much interest in smoothing (non-interior continuation) Newton methods for solving a few mathematical programming problems, such as linear programming problems [37, 6, 17, 18], linear complementarity problems (LCPs) [1, 2, 3, 7, 10, 24], nonlinear complementarity problems (NCPs) [4, 5, 8, 9, 11, 14, 15, 21, 23, 25, 27, 29, 30, 32, 34, 38, 41, 42, 45, 43], variational inequality problems [33, 39], semidefinite complementarity problems [12, 13, 26, 40], and so on. The main idea of this class of methods is to reformulate the problem concerned as a family of parameterized smooth equations and then to solve the smooth equations approximately by using Newton methods at each iteration. By driving the parameter to zero, it is hopeful that a solution to the original problem can be found.

Among these authors, it is Smale [37] who initiated the study on smoothing (non-interior continuation) Newton methods for solving linear programming problems and LCPs. Independent of Smale's work [37], Chen and Harker [7] introduced a non-interior continuation method for solving the LCP with a $P_0$ and $R_0$ matrix. They concentrated on establishing properties of smoothing paths. Later, the smoothing function used in [7], was refined and generalized by Kanzow [24], Chen and Mangasarian [11], and Gabrial and Moré [20]. Burke and Xu [1] introduced the concept of neighborhood of smoothing paths into their continuation method. This allowed them to establish a global linear convergence result for LCPs. Chen and Xiu [9] improved Burke and Xu's method by simplifying the definition of neighborhood and adding an approximate Newton step to obtain a local quadratic convergence result. In [2, 3], Burke and Xu further proposed two predictor-corrector-type non-interior continuation methods for LCPs. They also obtained a local quadratic convergence result. Qi and Sun [32] analyzed the local superlinear convergence of the Hotta and Yoshise's non-interior point method for NCPs [23]. It should be noted that in order to obtain the local superlinear convergence the papers [2, 3, 9, 32] need to assume that the strict complementarity condition holds at the solution point and that the iteration matrices are uniformly nonsingular. It is well-known that the latter assumption implies the solution set is a singleton. To relax this relatively restrictive assumption, Tseng [42] developed a new approach to the analysis of local quadratic convergence of general predictor-corrector-type path-following methods for solving monotone complementarity problems. By using the error bound theory Tseng discussed the local quadratic convergence under the strict complementarity condition. The assumptions made in [42] do not imply (explicitly or implicitly) that the solution set is a singleton. Very recently, Engelke and Kanzow [17, 18] further investigated the methods developed in [42] and proposed two specific predictor-corrector smoothing methods for solving linear programming problems. Under the assumption that the iteration sequence converges to a strict complementary solution, they proved the local quadratic convergence of their algorithms without assuming the uniqueness of the solution. Very encouraging numerical results were also reported in [17, 18]. Just as the algorithms developed in [2, 3, 5, 9], the algorithms given in [42, 17, 18] usually need to solve two linear systems of equations and to do two or three line searches at each iteration. It should also be pointed out that the methods given in [42, 17, 18] depend strongly on the strict complementarity condition.

By exploiting a so-called Jacobian consistency property for smoothing functions, Chen, Qi, and Sun [14] designed a class of globally and locally superlinearly convergent smoothing Newton methods for NCPs with a nonsingularity condition, but without the strict complementarity condition. Some modifications were made in [15] and [27]. Different to [14], a class of new smoothing methods were proposed by Qi, Sun, and Zhou [34] for solving NCPs. The Qi-Sun-Zhou (QSZ) method treated the smoothing parameter as a free variable and needs to solve one linear system of equations at each iteration. By making use of the semismoothness property of smoothing functions, the QSZ method was proved to possess fast local convergence under a nonsingularity assumption [34]. Very encouraging numerical results of this class of methods were reported in [45]. Due to its simplicity and weaker assumptions made on smoothing functions, the QSZ method has also been used to solve other problems [29, 30, 38, 40, 45]. It is worth mentioning that the assumptions used in [14, 15, 29, 30, 34, 38, 45] imply the solution set is a singleton, but do not imply the strict complementarity condition.

In this paper, we focus on LCPs. Given $M \in \Re^{n\times n}$ and $q \in \Re^n$, the LCP is to find a vector $(x, s) \in \Re^{2n}$ such that

$$(x, s) \geq 0, \quad s = Mx + q, \quad x^T s = 0. \tag{1.1}$$

We shall present a smoothing Newton method for solving (1.1) by assuming $M$ to be a $P_0$-matrix, i.e., all of its principal minors are nonnegative. This smoothing Newton method is a modified version of the QSZ method [34]. Just as the QSZ method, the new method needs only to solve one linear system of equations and to do one line search at each iteration. Based on the regularization technique [19, 29, 38], by using the upper semicontinuity property of the inverse of a weakly univalent function [36, Theorem 2.5], we investigate the boundedness of the generated iteration sequence under the assumption that the solution set of (1.1) is nonempty and bounded. Compared to previous smoothing (non-interior continuation) Newton methods, the method presented in this paper possesses the following stronger local convergence properties:

- If an accumulation point of the iteration sequence satisfies a nonsingularity condition, which implies the $P_0-$LCP has a unique solution, then the whole iteration sequence converges to this accumulation point sub-quadratically with a $Q$-rate $2 - t$, where $t \in (0, 1)$ is a parameter and can be close to zero as much as wanted.

- If $M$ is a positive semidefinite matrix and an accumulation point of the iteration sequence satisfies a strict complementarity condition, then the whole iteration sequence converges to this accumulation point quadratically. It is worth noting that here only one accumulation point is assumed to satisfy the strict complementarity condition.

To the best of our knowledge, this is the first smoothing (non-interior continuation) method to have the above local convergence properties simultaneously.

The rest of this paper is organized as follows. In the next section, we present a modified smoothing Newton algorithm. We prove its global convergence in Section 3. In Section 4, we show the local sub-quadratic convergence of the algorithm with a nonsingularity condition, but without the strict complementarity condition. In Section 5, we prove the local quadratic convergence of the algorithm without the nonsingularity condition, but with the strict complementarity. Conclusions are given in Section 6.

To help the later discussion, we introduce some notation here. All vectors are column vectors, the supscript $T$ denotes transpose, $\Re^n$ denotes the space of $n-$dimensional real column vectors, and $\Re^n_+$ (respectively, $\Re^n_{++}$) denotes the nonnegative (respectively, positive) orthant in $\Re^n$. We denote $\mathcal{I} = \{1, 2, \ldots, n\}$. For any vector $u$, we denote by $u_i$ the $i$th component of $u$ and, for any $\mathcal{K} \subset \mathcal{I}$, by $u_\mathcal{K}$ the vector obtained after removing from $u$ those $u_i$ with $i \notin \mathcal{K}$. We also write $u$ as $\text{vec}\{u_i : i \in \mathcal{I}\}$. We denote by $\text{diag}\{u_i : i \in \mathcal{I}\}$ the diagonal matrix whose $i$th diagonal element is $u_i$. We denote by $\|u\|$ the 2-norm of $u$. For any vectors $u, v \in \Re^n$, we write $(u^T, v^T)^T$ as $(u, v)$ for simplicity, and denote by $\min\{u, v\}$ the vector whose $i$th component is $\min\{u_i, v_i\}$. We denote by $\Re^{n\times n}$ the space of $n \times n$ real matrices. For any $A \in \Re^{n\times n}$ and $\mathcal{K}, \mathcal{L} \in \mathcal{I}$, we denote $A_{\mathcal{KL}}$ the submatrix of $A$ obtained by removing all rows of $A$ with indices outside of $\mathcal{K}$ and removing all columns of $A$ with indices outside of $\mathcal{L}$. Also, we denote $\|A\| = \max_{u\in\Re^n, \|u\|=1} \|Mu\|$. For any continuously differentiable function $g = (g_1, g_2, \ldots, g_n)^T : \Re^m \to \Re^m$, we denote its Jacobian by $g' = (\bigtriangledown g_1, \bigtriangledown g_2, \ldots, \bigtriangledown g_n)^T$, where $\bigtriangledown g_i$ denotes the gradient of $g_i$ for $i = 1, 2, \ldots, m$. We denote by $\mathcal{F}$ and $\mathcal{S}$ the feasible set and the solution set of (1.1), respectively, i.e.,

$$\mathcal{F} := \{(x, s) \in \Re^{2n} : s = Mx + q\}, \quad \mathcal{S} := \{(x, s) \in \mathcal{F} : (x, s) \geq 0, x^T s = 0\}.$$

We denote by $\text{dist}((u, v), \mathcal{S})$ the Euclidean distance of the vector $(u, v) \in \Re^{2n}$ to the solution set $\mathcal{S}$ of (1.1), i.e., $\text{dist}((u, v), \mathcal{S}) = \inf_{(x,s)\in\mathcal{S}} \|(u, v) - (x, s)\|$. For any $\alpha, \beta \in \Re_{++}$, we write $\alpha = O(\beta)$ (respectively, $\alpha = o(\beta)$) to mean $\alpha/\beta$ is uniformly bounded (respectively, tends to zero) as $\beta \to 0$. Let $k \geq 0$ denote the iteration index. For any $(\mu, x, s), (\mu_k, x^k, s^k) \in \Re_+ \times \Re^{2n}$, we always use the following notation throughout this paper unless stated otherwise:

$$w := (x, s), \quad w^k := (x^k, s^k), \quad z := (\mu, w) := (\mu, x, s), \quad z^k := (\mu_k, w^k) := (\mu_k, x^k, s^k).$$

# 2 A Smoothing Newton Algorithm

Let $\phi : \Re^3 \to \Re$ denote the Chen-Harker-Kanzow-Smale (CHKS) [7, 24, 37] smoothing function

$$\phi(\mu, a, b) = a + b - \sqrt{(a - b)^2 + 4\mu^2} \tag{2.1}$$

and let $\Phi : \Re^{2n+1} \to \Re^n$ be defined by

$$\Phi(z) := \begin{pmatrix} \phi(\mu, x_1, s_1) \\ \vdots \\ \phi(\mu, x_n, s_n) \end{pmatrix}. \tag{2.2}$$

Then, to find a solution of (1.1) is equivalent to find a root of the following (nonsmooth) equation:

$$H(z) := \begin{pmatrix} \mu \\ s - Mx - q \\ \Phi(z) + p(\mu)x \end{pmatrix} = 0, \tag{2.3}$$

where $p : \Re \to \Re_+$ is a twice continuously differentiable function which satisfies $p(\mu) > 0$ for $\mu \neq 0$, and

$$p(0) = 0, \quad |p(\mu)| = O(\mu^3), \quad \text{and} \quad |p'(\mu)| = O(\mu^2). \tag{2.4}$$

From (2.3), for any $\mu \neq 0$ a straightforward calculation yields

$$H'(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -M & I \\ d(z) + p'(\mu)x & D(z) + p(\mu)I & E(z) \end{bmatrix}, \tag{2.5}$$

where $I$ denotes the $n \times n$ identity matrix,

$$d(z) := \text{vec}\left\{ -\frac{4\mu}{\sqrt{(x_i - s_i)^2 + 4\mu^2}} : i \in \mathcal{I} \right\}, \tag{2.6}$$

$$D(z) := \text{diag}\left\{ 1 - \frac{x_i - s_i}{\sqrt{(x_i - s_i)^2 + 4\mu^2}} : i \in \mathcal{I} \right\}, \tag{2.7}$$

$$E(z) := \text{diag}\left\{ 1 + \frac{x_i - s_i}{\sqrt{(x_i - s_i)^2 + 4\mu^2}} : i \in \mathcal{I} \right\}. \tag{2.8}$$

For any $z := (\mu, w) = (\mu, x, s) \in \Re \times \Re^{2n}$, let

$$\Phi_0(w) := 2\min\{x, s\} \tag{2.9}$$

and

$$\xi(w) := \min\{|x_i - s_i| : i \in \mathcal{I}\}. \tag{2.10}$$

The following lemma is useful in our later analysis.

**Lemma 2.1** *For any $z = (\mu, w) = (\mu, x, s) \in \Re_+ \times \Re^{2n}$, we have*

$$\|\Phi_0(w) - \Phi(z)\| \leq 2\sqrt{n}\mu. \tag{2.11}$$

*Moreover, if $\xi(w) \geq \varepsilon$ for some constant $\varepsilon > 0$, then there exists a constant $C_1(\varepsilon) > 0$ such that*

$$\|\Phi_0(w) - \Phi(z)\| \leq C_1(\varepsilon)\mu^2; \tag{2.12}$$

*and if $\xi(w) \geq \kappa\mu^t$ for two constants $t \in (0, 1)$ and $\kappa > 0$, then*

$$\|\Phi_0(w) - \Phi(z)\| \leq \frac{2n}{\kappa}\mu^{2-t}. \tag{2.13}$$

**Proof.** It follows from [25] that (2.11) holds. We need only to show that (2.12) is satisfied since (2.13) can be shown similarly. For any $i \in \mathcal{I}$, by (2.2), (2.1), and

$$(\Phi_0(w))_i = 2\min\{x_i, s_i\} = x_i + s_i - \sqrt{(x_i - s_i)^2},$$

we have

$$\begin{aligned}(\Phi_0(w))_i - (\Phi(z))_i &= \sqrt{(x_i - s_i)^2 + 4\mu^2} - \sqrt{(x_i - s_i)^2} \\ &= \frac{4\mu^2}{\sqrt{(x_i - s_i)^2 + 4\mu^2} + \sqrt{(x_i - s_i)^2}}.\end{aligned}$$

It is not difficult to see that $\xi(w) \geq \varepsilon$ implies that

$$4/(\sqrt{(x_i - s_i)^2 + 4\mu^2} + \sqrt{(x_i - s_i)^2})$$

is bounded from above, so there exists a constant $C_1(\varepsilon)$ such that (2.12) holds. $\square$

Let $\bar{\mu} \in \Re_{++}$ and $\gamma \in (0, 1)$. Define $\theta : \Re^{2n+1} \to \Re_+$ and $\beta : \Re^{2n+1} \to \Re_+$ by

$$\theta(z) := \|H(z)\| \quad \text{and} \quad \beta(z) := \gamma\theta(z)\min\{1, \theta(z)\}, \tag{2.14}$$

respectively. Suppose that $\tau \in (0, 1)$ and $d(\cdot)$ is defined by (2.6). Denote $u : \Re^{2n+1} \to \Re^n$ by

$$u(z) := \begin{cases} \Phi_0(w) - \Phi(z) + \bar{\mu}\beta(z)d(z) & \text{if } \mu \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.15}$$

and $v : \Re^{2n+1} \to \Re^n$ by

$$v(z) := \begin{cases} \tau\mu e & \text{if } \tau\sqrt{n}\mu \leq \|u(z)\| \\ u(z) & \text{otherwise,} \end{cases} \tag{2.16}$$

where $e$ denotes the $n$-vector of all ones. Suppose that $t \in (0, 1)$ and $\kappa > 0$ are two constants, and that $\xi : \Re^{2n} \to \Re_+$ is defined by (2.10). We now define the function $\Upsilon$ : $\Re^{2n+1} \to \Re \times \Re^n \times \Re^n$ as follows:

- if $\xi(w) > \kappa\mu^t$, then the function $\Upsilon$ is defined by

$$\Upsilon(z) := \begin{pmatrix} \bar{\mu}\beta(z) \\ 0 \\ v(z) \end{pmatrix}; \tag{2.17}$$

- and if $\xi(w) \leq \kappa\mu^t$, then the function $\Upsilon$ is defined by

$$\Upsilon(z) := \begin{pmatrix} \bar{\mu}\beta(z) \\ 0 \\ 0 \end{pmatrix}. \tag{2.18}$$

**Algorithm 2.1** *(A Smoothing Newton Algorithm)*

**Step 0** *Choose* $t, \delta, \sigma \in (0,1)$ *and* $\kappa, \bar{\mu} \in (0, \infty)$. *Let* $x^0 \in \Re^n$ *be an arbitrary vector. Set* $\mu_0 := \bar{\mu}$, $s^0 := Mx^0 + q$, *and* $z^0 := (\mu_0, x^0, s^0)$. *Choose* $\gamma \in (0,1)$ *and* $\tau \in (0,1)$ *such that* $\gamma\theta(z^0) + \tau\sqrt{n} < 1$. *Set* $\eta := \gamma\mu_0 + \tau\sqrt{n}$ *and* $k := 0$.

**Step 1** *If* $\|H(z^k)\| = 0$, *stop.*

**Step 2** *Compute* $\Delta z^k := (\Delta\mu_k, \Delta x^k, \Delta y^k) \in \Re^{2n+1}$ *by*

$$H(z^k) + H'(z^k)\Delta z^k = \Upsilon(z^k). \tag{2.19}$$

**Step 3** *Let* $\lambda_k$ *be the maximum of the values* $1, \delta, \delta^2, \cdots$ *such that*

$$\theta(z^k + \lambda_k \Delta z^k) \leq [1 - \sigma(1-\eta)\lambda_k]\theta(z^k). \tag{2.20}$$

**Step 4** *Set* $z^{k+1} := z^k + \lambda_k \Delta z^k$ *and* $k := k+1$. *Go to Step 1.*

**Remark 2.1** Algorithm 2.1 is a modified version of the QSZ smoothing Newton algorithm developed in [34]. The main feature of Algorithm 2.1 is that we add an item (i.e., $v(z^k)$) into the perturbed Newton equation (see (2.19) and (2.17)). This allows us to obtain the local quadratic convergence of the algorithm under the strict complementarity condition, but without assuming the uniqueness of the solution to (1.1). In addition, just as the QSZ algorithm, Algorithm 2.1 needs only to solve one linear system of equations and to do one line search at each iteration.

**Lemma 2.2** *Suppose that* $M$ *is a* $P_0$*-matrix. If* $\mu_k > 0$ *for some* $k$*, then Algorithm 2.1 is well-defined at the* $k$*-th step.*

**Proof.** Since $M$ is a $P_0$-matrix, it is not difficult to show from (2.5) that the Jacobian matrix $H'(z^k)$ is nonsingular if $\mu_k > 0$. This implies that the equation (2.19) is solvable. Thus, to show Algorithm 2.1 is well-defined at the $k$-th step, it suffices to verify that Step 3 of Algorithm 2.1 is well-defined. For $\alpha \in \Re$, let

$$R^k(\alpha) := H(z^k + \alpha\Delta z^k) - H(z^k) - \alpha H'(z^k)\Delta z^k. \tag{2.21}$$

It follows from (2.19) that $\Delta\mu_k = -\mu_k + \beta(z^k)\mu_0$. Hence, for any $\alpha \in [0,1]$,

$$\mu_k + \alpha\Delta\mu_k = (1-\alpha)\mu_k + \alpha\beta(z^k)\mu_0 > 0,$$

which implies that $\Phi(\cdot)$ is continuously differentiable around $z^k$. Therefore, by (2.3), $H(\cdot)$ is continuously differentiable at $z^k$. This, together with (2.21), implies that

$$\|R^k(\alpha)\| = o(\alpha). \tag{2.22}$$

Since the definition of $\beta(\cdot)$ (see (2.14)) implies $\beta(z^k) \leq \gamma\theta(z^k)$ and the definition of $v(\cdot)$ (see (2.16)) implies $\|v(z^k)\| \leq \tau\sqrt{n}\mu_k$, from (2.17) and (2.18) we have

either

$$\|\Upsilon(z^k)\| \leq \mu_0\gamma\theta(z^k) + \tau\sqrt{n}\mu_k \leq \eta\theta(z^k)$$

or

$$\|\Upsilon(z^k)\| \leq \mu_0\gamma\theta(z^k) < \eta\theta(z^k).$$

Hence, by (2.19) and (2.21), we obtain

$$\begin{aligned} \theta(z^k + \alpha\Delta z^k) &= \|R^k(\alpha) + H(z^k) + \alpha[-H(z^k) + \Upsilon(z^k)]\| \\ &\leq \|R^k(\alpha)\| + [1 - \alpha(1-\eta)]\theta(z^k), \end{aligned} \tag{2.23}$$

which, together with (2.22), implies that there exists a constant $\bar{\alpha} \in (0, 1)$ such that

$$\theta(z^k + \alpha\Delta z^k) \leq [1 - \sigma(1-\eta)\alpha]\theta(z^k) \tag{2.24}$$

holds for any $\alpha \in (0, \bar{\alpha}]$. This demonstrates that (2.20) is well-defined. The proof is completed. $\square$

# 3 Global Convergence

Denote $\Omega := \{z \in \Re \times \Re^{2n} : w \in \mathcal{F}, \mu \geq \mu_0\beta(z)\}$.

**Lemma 3.1** *Suppose that $M$ is a $P_0$-matrix. Then Algorithm 2.1 generates an infinite iteration sequence $\{z^k\}$ with $\mu_k > 0$ and $z^k \in \Omega$ for any $k \geq 0$.*

**Proof**. We prove this result by induction on $k$. Obviously, $\mu_0 > 0$ and $z^0 \in \Omega$ by the choice of the initial point. Assume that $\mu_{k-1} > 0$ and $z^{k-1} \in \Omega$ for some $k \geq 1$. Then it is easy to obtain that $\mu_k > 0$ from the proof of Lemma 2.2. Thus, it suffices to show that $z^k \in \Omega$ for some $k \geq 1$. From the assumption that $z^{k-1} \in \Omega$ for some $k \geq 1$, we have

$$s^{k-1} = Mx^{k-1} + q, \quad \mu_{k-1} \geq \mu_0\beta(z^{k-1}). \tag{3.1}$$

On one hand, (2.19) implies that

$$-M\Delta x^{k-1} + \Delta s^{k-1} = -(s^{k-1} - Mx^{k-1} - q),$$

which yields from the first formula in (3.1) that $\Delta s^{k-1} = M\Delta x^{k-1}$. Hence,

$$\begin{aligned} s^k &= s^{k-1} + \lambda_{k-1}\Delta s^{k-1} \\ &= Mx^{k-1} + q + \lambda_{k-1}M\Delta x^{k-1} \\ &= M(x^{k-1} + \lambda_{k-1}\Delta x^{k-1}) + q \\ &= Mx^k + q. \end{aligned} \tag{3.2}$$

On the other hand, similar to the proof in Lemma 2.2, we have

$$\mu_k = \mu_{k-1} + \lambda_{k-1}\Delta\mu_{k-1} = (1-\lambda_{k-1})\mu_{k-1} + \lambda_{k-1}\beta(z^{k-1})\mu_0. \tag{3.3}$$

If $\theta(z^{k-1}) \geq 1$, then $\beta(z^{k-1}) = \gamma\theta(z^{k-1})$ (by (2.14)). Thus,

$$\begin{array}{rcl} \mu_k - \mu_0\beta(z^k) & = & (1-\lambda_{k-1})\mu_{k-1} + \lambda_{k-1}\mu_0\beta(z^{k-1}) - \mu_0\beta(z^k) \\ & \geq & (1-\lambda_{k-1})\mu_{k-1} + \lambda_{k-1}\mu_0\beta(z^{k-1}) - \mu_0\gamma\theta(z^{k-1}) \\ & \geq & (1-\lambda_{k-1})\mu_0\beta(z^{k-1}) + \lambda_{k-1}\mu_0\beta(z^{k-1}) - \mu_0\gamma\theta(z^{k-1}) \\ & = & 0, \end{array}$$

where the first equality is due to (3.3), the first inequality is due to (2.14) and (2.20), and the second inequality is due to the second formula in (3.1). If $\theta(z^{k-1}) < 1$, then $\beta(z^{k-1}) = \gamma\theta(z^{k-1})^2$ (by (2.14)). Thus,

$$\begin{aligned} &\mu_k - \mu_0\beta(z^k) \\ &\quad = (1-\lambda_{k-1})\mu^{k-1} + \lambda_{k-1}\mu_0\beta(z^{k-1}) - \mu_0\beta(z^k) \\ &\quad \geq (1-\lambda_{k-1})\mu_0\beta(z^{k-1}) + \lambda_{k-1}\mu_0\beta(z^{k-1}) - \gamma\mu_0[1-\sigma(1-\eta)\lambda_{k-1}]^2\theta(z^{k-1})^2 \\ &\quad = \gamma\mu_0\theta(z^{k-1})^2 - \gamma\mu_0[1-\sigma(1-\eta)\lambda_{k-1}]^2\theta(z^{k-1})^2 \\ &\quad = \gamma\mu_0\{1-[1-\sigma(1-\eta)\lambda_{k-1}]^2\}\theta(z^{k-1})^2 \\ &\quad \geq 0. \end{aligned}$$

Hence, we obtain that $\mu_k \geq \mu_0\beta(z^k)$. This, together with (3.2), implies $z^k \in \Omega$. □

**Theorem 3.1** *Suppose that $M$ is a $P_0$-matrix. Then Algorithm 2.1 generates an infinite iteration sequence $\{z^k\}$ with*

$$\lim_{k\to\infty} \theta(z^k) = 0.$$

*In particular, any accumulation point of $\{z^k\}$ is a solution of $H(z) = 0$.*

**Proof.** It follows from Lemma 3.1 that an infinite sequence $\{z^k\}$ is generated by Algorithm 2.1 such that $\{z^k\} \subseteq \Omega$. It is easy to see that the sequence $\{\theta(z^k)\}$ is monotonically decreasing by Algorithm 2.1, and that $\theta(z^k) \geq 0$ for any $k \geq 0$. Thus there exist two nonnegative numbers $\tilde{\theta}$ and $\tilde{\beta}$ such that $\lim_{k\to\infty}\theta(z^k) = \tilde{\theta}$ and $\lim_{k\to\infty}\beta(z^k) = \tilde{\beta}$. If $\tilde{\theta} = 0$, then we have obtained the desired result. Suppose that $\tilde{\theta} > 0$. Then $\tilde{\beta} > 0$ by the definition of $\beta(z)$. Since $z^k \in \Omega$ implies that $\mu_k \geq \mu_0\beta(z^k)$ and

$$\mu_k = \mu_{k-1} + \lambda_{k-1}\Delta\mu_{k-1} = (1-\lambda_{k-1})\mu_{k-1} + \lambda_{k-1}\beta(z^{k-1})\mu_0 \leq \mu_{k-1}$$

for any $k \geq 0$, it follows that $\mu_0 \geq \mu_k \geq \mu_0\tilde{\beta}$ for any $k \geq 0$. In this case, similar to the proof in [38, Proposition 2.1] we can show that $\theta(z^k) \to \infty$ as $\|w^k\| \to \infty$. Thus, the infinite iteration sequence $\{z^k\}$ is bounded because otherwise $\{\theta(z^k)\}$ must be unbounded, which is impossible by the monotonicity decreasing property of $\{\theta(z^k)\}$. Hence, there exists at least one accumulation point $z^* = (\mu_*, x^*, y^*) \in \Re^{2n+1}$ of $\{z^k\}$ such that $\mu^* \in [\beta(z^*)\mu_0, \mu_0]$.

Subsequencing if necessary, we may assume that $\{z^k\}$ converges to $z^*$. It is easy to see that $\tilde{\theta} = \theta(z^*)$, $\tilde{\beta} = \beta(z^*)$, and $z^* \in \Omega$. Furthermore, it follows that $H'(z^*)$ is invertible since $\mu_* > 0$. By taking limit on (2.19) we know that $\Delta z^*$ in the following equation exists:

$$H(z^*) + H'(z^*)\Delta z^* = \Upsilon(z^*). \tag{3.4}$$

Since $\|\Upsilon(z^k)\| \leq \eta\theta(z^k)$ implies that $\|\Upsilon(z^*)\| \leq \eta\theta(z^*)$, it follows from (3.4) that

$$\begin{aligned} H(z^*)^T H'(z^*)\Delta z^* &= -\theta(z^*)^2 + H(z^*)^T\Upsilon(z^*) \\ &\leq -\theta(z^*)^2 + \theta(z^*)\|\Upsilon(z^*)\| \\ &\leq (-1+\eta)\theta(z^*)^2. \end{aligned} \tag{3.5}$$

On the other hand, from $\theta(z^*) > 0$ we have $\lim_{k\to\infty} \lambda_k = 0$. Hence, the stepsize $\hat{\lambda}_k := \frac{\lambda_k}{\delta}$ does not satisfy the line search criterion (2.20) for any sufficiently large $k$, i.e.,

$$\theta(z^k + \hat{\lambda}_k \Delta z^k) > [1 - \sigma(1-\eta)\hat{\lambda}_k]\theta(z^k)$$

holds for any sufficiently large $k$, which implies that

$$\frac{\theta(z^k + \hat{\lambda}_k \Delta z^k) - \theta(z^k)}{\hat{\lambda}_k} > -\sigma(1-\eta)\theta(z^k).$$

By letting $k \to \infty$, the above inequality gives

$$\frac{H(z^*)^T}{\theta(z^*)} H'(z^*)\Delta z^* \geq -\sigma(1-\eta)\theta(z^*). \tag{3.6}$$

Combining (3.5) with (3.6) we have

$$-1 + \eta + \sigma(1-\eta) \geq 0,$$

which contradicts the fact that $\eta < 1$ and $\sigma \in (0,1)$. This proves $\lim_{k\to\infty} \theta(z^k) = 0$. In particular, if there exists an accumulation point $z^*$ of $\{z^k\}$, then, by the continuity of $\theta$, $\theta(z^*) = 0$, and so $H(z^*) = 0$. $\square$

Theorem 3.1 shows that if there exists an accumulation point $z^*$ of $\{z^k\}$, then $z^*$ is a solution of (1.1). This does not mean that $\{z^k\}$ has an accumulation point. In order to assure that $\{z^k\}$ has an accumulation point, we need the following assumption and lemma.

**Assumption 3.1** *The solution set of* (1.1) *is nonempty and bounded.*

**Lemma 3.2** *[36, Theorem 2.5] Let $g : \Re^n \to \Re^n$ be weakly univalent, that is, $g$ is continuous, and there exist one-to-one continuous functions $g_k : \Re^n \to \Re^n$ such that $g_k \to g$ uniformly on every bounded subset of $R^n$. Suppose that $q^* \in \Re^n$ such that $g^{-1}(q^*)$ is nonempty and compact. Then for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any weakly univalent function $h$ and for any $q$ with $\sup_{\bar{\Omega}_0} \|h(x) - g(x)\| < \delta$ and $\|q - q^*\| < \delta$, we have $\emptyset \neq h^{-1}(q) \subseteq g^{-1}(q^*) + \varepsilon\mathcal{B}$ where $\mathcal{B}$ denotes the open unit ball in $R^n$ and $\Omega_0 := g^{-1}(q^*) + \varepsilon\mathcal{B}$. In particular, $h^{-1}(q)$ and $g^{-1}(q)$ are nonempty, connected, and uniformly bounded for $q$ in a neighborhood of $q^*$.*

**Theorem 3.2** *Suppose that $M$ is a $P_0$-matrix and Assumption 3.1 is satisfied. Then the infinite sequence $\{z^k\}$ generated by Algorithm 2.1 is bounded and any accumulation point of $\{z^k\}$ is a solution of $H(z) = 0$.*

**Proof**. It is not difficult to show that the function $H : \Re^{2n+1} \to \Re^{2n+1}$ defined by (2.3) is a weakly univalent function (see [21]). Since Assumption 3.1 implies that the inverse image $H^{-1}(0)$ is nonempty and bounded, by using Lemma 3.2 we obtain that the sequence $\{z^k\}$ is bounded, and hence, by Theorem 3.1, any accumulation point of $\{z^k\}$ is a solution of $H(z) = 0$. □

# 4 Sub-Quadratic Convergence under Nonsingularity

Let $z^* := (\mu_*, w^*) := (\mu_*, x^*, s^*) \in \Re_+ \times \Re^{2n}$ be an accumulation point of the iteration sequence generated by Algorithm 2.1. Then Theorem 3.1 implies that $\mu_* = 0$ and $(x^*, s^*)$ is a solution of (1.1). In this section, we consider the case that $(x^*, s^*)$ satisfies a nonsingularity condition but may not satisfy the strict complementarity condition. In order to discuss the local superlinear convergence of the algorithm, we need the concept of semismoothness, which was originally introduced by Mifflin [28] for functionals. Qi and Sun [35] extended the definition of semismoothness to vector valued functions. A locally Lipschitz function $F : \Re^{m_1} \to \Re^{m_2}$, which has the generalized Jacobian $\partial F(x)$ in the sense of Clarke [16], is said to be semismooth at $x \in \Re^{m_1}$, if

$$\lim_{\substack{V \in \partial F(x+th') \\ h' \to h, t \downarrow 0}} \{Vh'\}$$

exists for any $h \in \Re^{m_1}$. $F$ is said to be strongly semismooth at $x$ if $F$ is semismooth at $x$ and for any $V \in \partial F(x + h), h \to 0$, it follows that

$$F(x + h) - F(x) - Vh = O(\|h\|^2). \tag{4.1}$$

**Lemma 4.1** *Suppose that $M$ is a $P_0$-matrix. Let $t \in (0, 1)$ be given as in Algorithm 2.1 and the infinite sequence $\{z^k\}$ be generated by Algorithm 2.1. Then, for all $k$ sufficiently large,*

$$\|\Upsilon(z^k)\| = O(\theta(z^k)^{2-t}).$$

**Proof.** Since the infinite sequence $\{z^k\}$ is generated by Algorithm 2.1, it follows from Theorem 3.1 that $\lim_{k\to\infty} \theta(z^k) = 0$. This, together with the definition of $\beta(z^k)$, implies that for all $k$ sufficiently large,

$$\beta(z^k) = \gamma\theta(z^k)^2.$$

For any $k$, we have either $\xi(w^k) \le \kappa(\mu_k)^t$ or $\xi(w^k) > \kappa(\mu_k)^t$. For the former case, the result of the lemma is satisfied trivially since for all $k$ sufficiently large,

$$\|\Upsilon(z^k)\| = \mu_0\beta(z^k) = \mu_0\gamma\theta(z^k)^2 = O(\theta(z^k)^2).$$

Hence, we only need to consider the latter case. For all $k$ sufficiently large, it follows that

$$\mu_0\beta(z^k) = O(\theta(z^k)^2)$$

and

$$\begin{aligned}
\|u(z^k)\| &\leq \|\Phi_0(w^k) - \Phi(z^k)\| + \mu_0\beta(z^k)\|d(z^k)\| \\
&\leq \frac{2n}{\kappa}(\mu_k)^{2-t} + \mu_0\beta(z^k)\|d(z^k)\| \\
&= \frac{2n}{\kappa}(\mu_k)^{2-t} + \left\|\mathrm{vec}\left\{\frac{-4\mu_k}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2}} : i \in \mathcal{I}\right\}\right\| \mu_0\beta(z^k) \\
&\leq \left(\frac{2n}{\kappa}(\mu_k)^{1-t} + \left\|\mathrm{vec}\left\{\frac{-4\mu_k}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2}} : i \in \mathcal{I}\right\}\right\|\right) \mu_k \\
&= O((\mu_k)^{2-t}), \qquad (4.2)
\end{aligned}$$

where the second inequality is due to (2.13), the first equality is due to (2.6), and the third inequality is due to $z^k \in \Omega$ (by Lemma 3.1). By the definition of the function $v(\cdot)$ (see (2.16)), it is easy to see that (4.2) implies that $v(z^k) = u(z^k)$ for all $k$ sufficiently large. Hence, for all $k$ sufficiently large,

$$\|\Upsilon(z^k)\| = \sqrt{(\mu_0\beta(z^k))^2 + \|u(z^k)\|^2} = O(\theta(z^k)^{2-t}).$$

This completes the proof. □

**Theorem 4.1** *Suppose that $M$ is a $P_0$-matrix and Assumption 3.1 is satisfied. Suppose that $z^*$ is an accumulation point of the sequence $\{z^k\}$ generated by Algorithm 2.1. If all $V \in \partial H(z^*)$ are nonsingular, then the whole iteration sequence $\{z^k\}$ converges to $z^*$,*

$$\|z^{k+1} - z^*\| = (\|z^k - z^*\|^{2-t}) \qquad (4.3)$$

*and*

$$\mu_{k+1} = O((\mu_k)^{2-t}). \qquad (4.4)$$

**Proof.** By Theorem 3.2, the generated sequence $\{z^k\}$ is bounded and any accumulation point is a solution of $H(z) = 0$. Hence, $H(z^*) = 0$. Since all $V \in \partial H(z^*)$ are nonsingular, from [35, Proposition 3.1], for all $z^k$ sufficiently close to $z^*$, we have

$$\|[H'(z^k)]^{-1}\| = O(1). \qquad (4.5)$$

From [34, Theorem 3], we know that the function $H$ is strongly semismooth at $z^*$. Hence, by (4.1) and Lemma 4.1, for all $z^k$ sufficiently close to $z^*$,

$$\begin{array}{rcl} \|z^k + \Delta z^k - z^*\| & = & \|z^k + [H'(z^k)]^{-1}[-H(z^k) + \Upsilon(z^k)] - z^*\| \\ & = & O(\|H(z^k) - H(z^*) - H'(z^k)(z^k - z^*)\| + \|\Upsilon(z^k)\|) \\ & = & O(\|z^k - z^*\|^2) + O(\theta(z^k)^{2-t}) \\ & = & O(\|z^k - z^*\|^{2-t}). \end{array} \tag{4.6}$$

By the proof of Theorem 3.1 of [31], for all $z^k$ sufficiently close to $z^*$, we have

$$\|z^k - z^*\| = O(\|H(z^k) - H(z^*)\|). \tag{4.7}$$

Hence, by using the Lipschitz continuity of $H$, (4.6) and (4.7), we have

$$\begin{array}{rcl} \theta(z^k + \Delta z^k) & = & \|H(z^k + \Delta z^k)\| \\ & = & O(\|z^k + \Delta z^k - z^*\|) \\ & = & O(\|z^k - z^*\|^{2-t}) \\ & = & O(\|H(z^k) - H(z^*)\|^{2-t}). \end{array} \tag{4.8}$$

Therefore, for all $z^k$ sufficiently close to $z^*$ we have

$$z^{k+1} = z^k + \Delta z^k,$$

which, together with (4.6), proves that $\{z^k\}$ converges to $z^*$ and (4.3) holds.

Next, from the definition of $\beta(z^k)$ and the fact that $z^k \to z^*$, for all $k$ sufficiently large,

$$\beta(z^k) = \gamma\|H(z^k)\|^2.$$

Hence, because for all $k$ sufficiently large we have $z^{k+1} = z^k + \Delta z^k$, it holds that

$$\mu_{k+1} = \mu_k + \Delta\mu_k = \mu_0\gamma\|H(z^k)\|^2$$

for all $k$ sufficiently large. This, together with (4.8), proves (4.4). The proof is completed. $\square$

In Theorem 4.1, for the sub-quadratic convergence, all $V \in \partial H(z^*)$ are assumed to be nonsingular. It is noticed here that this nonsingularity implies the solution set is a singleton, but not necessarily a strict complementary solution. For conditions to guarantee this nonsingularity assumption, see [34]. In the next section, under a strict complementarity condition, we shall discuss the quadratic convergence of our method with multiple solutions.

# 5 Quadratic Convergence under Strict Complementarity

In this section, we shall discuss the rate of convergence of Algorithm 2.1 without assuming the nonsingularity condition, but with a strict complementarity condition and $M$ being a positive semidefinite matrix. Let $z^* := (\mu_*, w^*) := (\mu_*, x^*, s^*) \in \Re_+ \times \Re^{2n}$ be an accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm 2.1. Then Theorem 3.1 says that $\mu_* = 0$ and $(x^*, s^*)$ is a solution of (1.1). In this section, we consider the case that $(x^*, s^*)$ satisfies the strict complementarity condition $x^* + s^* > 0$. Let

$$\mathcal{B} := \{i \in \mathcal{I} : x_i^* > 0\}, \quad \mathcal{N} := \{i \in \mathcal{I} : s_i^* > 0\}.$$

Since $(x^*, s^*)$ is a strict complementary solution of (1.1), it follows that $\mathcal{B} \cup \mathcal{N} = \mathcal{I}$ and $\mathcal{B} \cap \mathcal{N} = \emptyset$.

For any $w = (x, s) \in \Re^n \times \Re^n$, let

$$G(w) := \begin{pmatrix} s - (Mx + q) \\ 2s_{\mathcal{B}} \\ 2x_{\mathcal{N}} \end{pmatrix} \tag{5.1}$$

and

$$\mathcal{S}_0 := \{w \in \Re^{2n} : G(w) = 0\}. \tag{5.2}$$

**Lemma 5.1** *Denote*

$$\varepsilon := \min\{\min_{i \in \mathcal{B}} x_i^*, \min_{i \in \mathcal{N}} s_i^*\}$$

*and*

$$\Delta(w^*) := \{w = (x, s) \in \Re^n \times \Re^n : \ |x_i - x_i^*| \le \varepsilon/3, \ |s_i - s_i^*| \le \varepsilon/3, \quad i \in \mathcal{I}\}.$$

*Then, for any $w \in \Delta(w^*) \cap \mathcal{F}$, there exists a constant $\lambda > 0$ such that*

$$\|\Phi_0(w)\| = \|G(w)\| \ge \lambda \text{dist}(w, \mathcal{S}_0), \tag{5.3}$$

*where $\Phi_0(\cdot)$ is defined by (2.9).*

**Proof.** It is obvious that $G(w) = 0$ is solvable since $G(w^*) = 0$. Hence $\mathcal{S}_0 \neq \emptyset$. By Hoffman's result on error bound of the linear system [22], there exists a positive number $\lambda > 0$ such that for any $w \in \Re^{2n}$,

$$||G(w)|| \ge \lambda \text{dist}(w, \mathcal{S}_0). \tag{5.4}$$

For any $w \in \Delta(w^*)$, we have

$$\begin{aligned} x_i &= x_i^* + x_i - x_i^* \ge \varepsilon - \frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon, \quad |s_i| \le \frac{1}{3}\varepsilon \quad \forall\, i \in \mathcal{B}, \\ s_i &= s_i^* + s_i - s_i^* \ge \varepsilon - \frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon, \quad |x_i| \le \frac{1}{3}\varepsilon \quad \forall\, i \in \mathcal{N}. \end{aligned}$$

Hence, it holds that

$$\|\Phi_0(w)\| = \|G(w)\| \quad \text{for all } w \in \Delta(w^*) \cap \mathcal{F}. \tag{5.5}$$

By using (5.5) and (5.4), we obtain (5.3). $\square$

Lemma 5.1 indicates that if $w \in \mathcal{S}_0$ with $w$ sufficiently close to $w^* := (x^*, s^*)$, then $w$ solves (1.1), i.e., $w \in \mathcal{S}$. Furthermore, from (5.3), there exists a constant $\bar{\rho} > 0$ such that

$$\mathrm{dist}(w, \mathcal{S}_0) \leq \bar{\rho} \|\Phi_0(w)\| \tag{5.6}$$

for all $w \in \mathcal{F}$ sufficiently close to $w^*$. Noting that $\max\{\mu_k, \|\Phi(z^k)\|\} \leq \theta(z^k)$, we can obtain from (2.11), (5.6) and Lemma 3.1 that for all $z^k$ sufficiently close to $z^*$,

$$\begin{aligned} \mathrm{dist}(w^k, \mathcal{S}_0) &\leq \bar{\rho}(\|\Phi(z^k)\| + \|\Phi_0(w^k) - \Phi(z^k)\|) \\ &\leq \bar{\rho}(\theta(z^k) + 2\sqrt{n}\mu_k) \\ &\leq C_2 \theta(z^k), \end{aligned} \tag{5.7}$$

where $C_2 := \bar{\rho}(1 + 2\sqrt{n})$.

**Lemma 5.2** *Suppose that $M$ is a $P_0$-matrix and Assumption 3.1 is satisfied. Let $z^*$ be an accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm 2.1. If $(x^*, s^*)$ satisfies the strict complementarity condition $x^* + s^* > 0$, then for all $z^k$ sufficiently close to $z^*$,*

$$v(z^k) = u(z^k),$$

*where $v(\cdot)$ and $u(\cdot)$ are defined by* (2.16) *and* (2.15)*, respectively.*

**Proof.** Since the strict complementarity condition holds, i.e., $x^* + s^* > 0$, it is easy to see that there exists a constant $\varepsilon > 0$ such that

$$|x_i^k - s_i^k| \geq \varepsilon, \quad \text{for all } i \in \mathcal{I} \text{ and } z^k \text{ sufficiently close to } z^*. \tag{5.8}$$

By the definition of $u(\cdot)$ (see (2.15)) and (2.12), there exists a constant $C_1(\varepsilon) > 0$ such that for all $z^k$ sufficiently close to $z^*$,

$$\begin{aligned} \|u(z^k)\| &\leq \|\Phi_0(w^k) - \Phi(z^k)\| + \|d(z^k)\mu_0\beta(z^k)\| \\ &\leq C_1(\varepsilon)(\mu_k)^2 + \|d(z^k)\|\mu_0\beta(z^k) \\ &\leq C_1(\varepsilon)(\mu_k)^2 + \left\| \mathrm{vec}\left\{ \frac{-4\mu_k}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2}} : i \in \mathcal{I} \right\} \right\| \mu_0 \beta(z^k) \\ &\leq \left( C_1(\varepsilon)\mu_k + \left\| \mathrm{vec}\left\{ \frac{-4\mu_k}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2}} : i \in \mathcal{I} \right\} \right\| \right) \mu_k \\ &\leq \gamma\sqrt{n}\mu_k, \end{aligned} \tag{5.9}$$

where the last inequality due to (5.8) and $\mu_k$ sufficiently close to 0. Therefore, by noting the definition of $v(\cdot)$ (see (2.16)) and (5.9), we have $v(z^k) = u(z^k)$ for all $z^k$ sufficiently close to $z^*$. $\square$

In the following lemma, by using Lemma 5.1 and the proofs in [42, 17], we shall prove for monotone LCPs that the term $\|\Delta z^k\|$ is of the same order as $\|\theta(z^k)\|$ for all $z^k$ sufficiently close to a strict complementary solution.

**Lemma 5.3** *Suppose that $M$ is a positive semidefinite matrix and Assumption 3.1 is satisfied. Let $z^*$ be an accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm 2.1. If $(x^*, s^*)$ satisfies the strict complementarity condition $x^* + s^* > 0$, then there exists a constant $C_3 > 0$ such that*

$$\|\Delta z^k\| \leq C_3\theta(z^k) \tag{5.10}$$

*holds for all $z^k$ sufficiently close to $z^*$.*

**Proof.** By making use of the fact that $v(z^k) = u(z^k)$ (by Lemma 5.2) and $w^k \in \mathcal{F}$ ($z^k \in \Omega$ by Lemma 3.1), we obtain from (2.19) that for all $z^k$ sufficiently close to $z^*$,

$$\Delta\mu_k = -\mu_k + \mu_0\beta(z^k), \tag{5.11}$$

$$-M\Delta x^k + \Delta s^k = 0, \tag{5.12}$$

and

$$(d(z^k) + p'(\mu_k)x^k)\Delta\mu_k + (D(z^k) + p(\mu_k)I)\Delta x^k + E(z^k)\Delta s^k = -\Phi(z^k) + u(z^k), \tag{5.13}$$

where the functions $d(\cdot)$, $D(\cdot)$, and $E(\cdot)$ are defined by (2.6), (2.7), and (2.8), respectively and the function $u(\cdot)$ is defined by (2.15), i.e.,

$$u(z^k) := \Phi_0(w^k) - \Phi(z^k) + \mu_0\beta(z^k)d(z^k). \tag{5.14}$$

We prove (5.10) by investigating the following three issues:

Firstly, we consider the estimation of $|\Delta\mu_k|$. From (5.11), for all $z^k$ sufficiently close to $z^*$,

$$|\Delta\mu_k| \leq \mu_k + \mu_0\beta(z^k) \leq (1 + \mu_0\gamma)\theta(z^k). \tag{5.15}$$

Secondly, we consider the estimation of $\|\Delta x^k\|$. For any $z^k$ sufficiently close to $z^*$, there exists a $w^{k*} := (x^{k*}, s^{k*}) \in \mathcal{S}_0$ (and hence $w^{k*} := (x^{k*}, s^{k*}) \in \mathcal{S}$ by Lemma 5.1) such that

$$\mathrm{dist}(w^k, \mathcal{S}_0) = \|w^k - w^{k*}\|. \tag{5.16}$$

By using the fact that $w^k, w^{k*} \in \mathcal{F}$, we get from (5.11)-(5.14) that

$$s^k + \Delta s^k - s^{k*} = M(x^k + \Delta x^k - x^{k*}) \tag{5.17}$$

and

$$(D(z^k)+p(\mu_k)I)(x^k+\Delta x^k-x^{k*})+E(z^k)(s^k+\Delta s^k-s^{k*})=\alpha(z^k), \tag{5.18}$$

where $\alpha(z^k):=\varpi(z^k)+\varrho(z^k)$ with

$$\varpi(z^k):=(D(z^k)+p(\mu_k)I)(x^k-x^{k*})+E(z^k)(s^k-s^{k*})-\Phi_0(w^k) \tag{5.19}$$

and

$$\begin{aligned}\varrho(z^k): &= \Phi_0(w^k)-\Phi(z^k)+u(z^k)-d(z^k)\Delta\mu_k-p'(\mu_k)\Delta\mu_k x^k\\ &= 2(\Phi_0(w^k)-\Phi(z^k))+d(z^k)\mu_0\beta(z^k)-d(z^k)\Delta\mu_k-p'(\mu_k)\Delta\mu_k x^k\\ &= 2(\Phi_0(w^k)-\Phi(z^k))+d(z^k)\mu_k-p'(\mu_k)\Delta\mu_k x^k.\end{aligned} \tag{5.20}$$

From the positive semidefiniteness of $M$ and (5.17), it follows that $(x^k+\Delta x^k-x^{k*})^T(s^k+\Delta s^k-s^{k*})\geq 0$. Hence, multiplying (5.18) on the left side by $(x^k+\Delta x^k-x^{k*})^TE(z^k)^{-1}$ yields

$$\begin{aligned}&\min_{i\in\mathcal{I}}(D(z^k)_{ii}+p(\mu_k))E(z^k)_{ii}\|E(z^k)^{-1}(x^k+\Delta x^k-x^{k*})\|^2\\ &\quad\leq (x^k+\Delta x^k-x^{k*})^TE(z^k)^{-1}(D(z^k)+p(\mu_k)I)E(z^k)E(z^k)^{-1}(x^k+\Delta x^k-x^{k*})\\ &\quad\leq (x^k+\Delta x^k-x^{k*})^TE(z^k)^{-1}\alpha(z^k)\\ &\quad\leq \|E(z^k)^{-1}(x^k+\Delta x^k-x^{k*})\|\|\alpha(z^k)\|,\end{aligned}$$

and so,

$$\min_{i\in\mathcal{I}}(D(z^k)_{ii}+p(\mu_k))E(z^k)_{ii}\|E(z^k)^{-1}(x^k+\Delta x^k-x^{k*})\| \;\leq\; \|\alpha(z^k)\|. \tag{5.21}$$

It can be seen easily that there exists a constant $\varepsilon>0$ such that

$$x_i^k-s_i^k>\varepsilon \text{ for all } i\in\mathcal{B} \quad\text{and}\quad s_i^k-x_i^k>\varepsilon \text{ for all } i\in\mathcal{N} \tag{5.22}$$

hold for all $z^k$ sufficiently close to $z^*$. Since $w^{k*}\in\mathcal{S}_0$, we have

$$s_i^{k*}=0 \text{ for all } i\in\mathcal{B} \quad\text{and}\quad x_i^{k*}=0 \text{ for all } i\in\mathcal{N}. \tag{5.23}$$

By using (5.22), (5.23), and the fact that $D(z^k)+E(z^k)=2I$ for all $k$ (by (2.7) and (2.8)), we obtain from (5.19) that for all $z^k$ sufficiently close to $z^*$,

$$\begin{aligned}\varpi(z^k)_{\mathcal{B}} &= (D(z^k)_{\mathcal{BB}}+p(\mu_k)I_{\mathcal{BB}})(x^k_{\mathcal{B}}-x^{k*}_{\mathcal{B}})+E(z^k)_{\mathcal{BB}}(s^k_{\mathcal{B}}-s^{k*}_{\mathcal{B}})-2s^k_{\mathcal{B}}\\ &= D(z^k)_{\mathcal{BB}}(x^k_{\mathcal{B}}-x^{k*}_{\mathcal{B}}-s^k_{\mathcal{B}}+s^{k*}_{\mathcal{B}})+p(\mu_k)(x^k_{\mathcal{B}}-x^{k*}_{\mathcal{B}}),\end{aligned}$$

and similarly,

$$\varpi(z^k)_{\mathcal{N}}=E(z^k)_{\mathcal{NN}}(s^k_{\mathcal{N}}-s^{k*}_{\mathcal{N}}-x^k_{\mathcal{N}}+x^{k*}_{\mathcal{N}})+p(\mu_k)(x^k_{\mathcal{N}}-x^{k*}_{\mathcal{N}}).$$

Hence, for all $z^k$ sufficiently close to $z^*$,

$$\|\varpi(z^k)\| \leq [\max\{\|D(z^k)_{\mathcal{BB}}\|, \|E(z^k)_{\mathcal{NN}}\|\} + p(\mu_k)](\|x^k - x^{k*}\| + \|s^k - s^{k*}\|).$$

For any $k \geq 0$, let

$$r_k = \min_{i \in \mathcal{I}} (D(z^k)_{ii} + p(\mu_k))E(z^k)_{ii}. \tag{5.24}$$

Using the property of $p(\cdot)$ and (2.4), a straightforward calculation yields that for $i = 1, \cdots, n$ and all $z^k$ sufficiently close to $z^*$,

$$p(\mu_k)E(z^k)_{ii} = p(\mu_k)\left(1 + \frac{x_i^k - s_i^k}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2}}\right) = O((\mu_k)^3).$$

Then there exists a $r > 0$ such that for all $z^k$ sufficiently close to $z^*$,

$$r_k = \min_{i \in \mathcal{I}} \frac{4(\mu_k)^2}{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} + O((\mu_k)^3) \geq r(\mu_k)^2, \tag{5.25}$$

which implies that for all $z^k$ sufficiently close to $z^*$,

$$\frac{p(\mu_k)}{r_k} = O(\mu_k).$$

For all $z^k$ sufficiently close to $z^*$, from (5.22) and (5.25) we have for $j \in \mathcal{B}$ that

$$\begin{aligned}
\frac{D(z^k)_{jj}}{r_k} &\leq \frac{1 - \frac{x_j^k - s_j^k}{\sqrt{(x_j^k - s_j^k)^2 + 4(\mu_k)^2}}}{r(\mu_k)^2} \\
&= \frac{\sqrt{(x_j^k - s_j^k)^2 + 4(\mu_k)^2} - (x_j^k - s_j^k)}{r(\mu_k)^2\sqrt{(x_j^k - s_j^k)^2 + 4(\mu_k)^2}} \\
&= \frac{4}{r\sqrt{(x_j^k - s_j^k)^2 + 4(\mu_k)^2}\left(\sqrt{(x_j^k - s_j^k)^2 + 4(\mu_k)^2} + (x_j^k - s_j^k)\right)} \\
&= O(1)
\end{aligned}$$

and for any $j \in \mathcal{N}$ that

$$\begin{aligned}
\frac{E(z^k)_{jj}}{r_k} &\leq \frac{4}{r\sqrt{(s_j^k - x_j^k)^2 + 4(\mu_k)^2}\left(\sqrt{(s_j^k - x_j^k)^2 + 4(\mu_k)^2} + (s_j^k - x_j^k)\right)} \\
&= O(1).
\end{aligned}$$

Hence, there exists a constant $C_4 > 0$ such that for all $z^k$ sufficiently close to $z^*$,

$$\frac{\|\varpi(z^k)\|}{r_k} \leq C_4(\|x^k - x^{k*}\| + \|s^k - s^{k*}\|) \leq 2C_2C_4\theta(z^k), \tag{5.26}$$

where the second inequality is due to (5.7) and (5.16). On the other hand, by (5.20) we have

$$\begin{aligned}
& \varrho(z^k)_i \\
= \;& 2\left[\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} - \sqrt{(x_i^k - s_i^k)^2}\right] + d(z^k)_i\mu_k - p'(\mu_k)\Delta\mu_k x^k \\
= \;& \frac{8(\mu_k)^2}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} + \sqrt{(x_i^k - s_i^k)^2}} - \frac{4(\mu_k)^2}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2}} - p'(\mu_k)\Delta\mu_k x^k \\
= \;& \frac{4(\mu_k)^2\left[\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} - \sqrt{(x_i^k - s_i^k)^2}\right]}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2}\left(\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} + \sqrt{(x_i^k - s_i^k)^2}\right)} - p'(\mu_k)\Delta\mu_k x^k \\
= \;& \frac{16(\mu_k)^4}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2}\left(\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} + \sqrt{(x_i^k - s_i^k)^2}\right)^2} - p'(\mu_k)\Delta\mu_k x^k,
\end{aligned}$$

which, together with (5.22), implies that for all $z^k$ sufficiently close to $z^*$,

$$\frac{|\varrho(z^k)_i|}{r_k} \leq \frac{|\varrho(z^k)_i|}{r(\mu_k)^2} = O((\mu_k)^2) + O(|\Delta\mu_k|).$$

Hence, there exist two constants $\bar{C}_5 > 0$ and $\hat{C}_5 > 0$ such that for all $z^k$ sufficiently close to $z^*$,

$$\begin{aligned}
\frac{\|\varrho(z^k)\|}{r_k} &\leq \bar{C}_5\mu_k + \hat{C}_5|\Delta\mu_k| \\
&\leq \bar{C}_5\|H(z^k)\| + \hat{C}_5(1 + \mu_0\gamma)\|H(z^k)\| \\
&= C_5\theta(z^k),
\end{aligned} \tag{5.27}$$

where $C_5 := \bar{C}_5 + \hat{C}_5(1 + \mu_0\gamma)$. By combining (5.21) with (5.26), (5.27) and (5.24), we obtain that for all $z^k$ sufficiently close to $z^*$,

$$\|E(z^k)^{-1}(x^k + \Delta x^k - x^{k*})\| \leq \frac{\|\varpi(z^k)\|}{r_k} + \frac{\|\varrho(z^k)\|}{r_k} \leq (2C_2C_4 + C_5)\theta(z^k). \tag{5.28}$$

Since $\|E(z^k)\| \leq 2$, (5.7) and (5.16) imply that for all $z^k$ sufficiently close to $z^*$,

$$\|E(z^k)^{-1}(x^k + \Delta x^k - x^{k*})\| \geq \frac{\|\Delta x^k\| - \|x^k - x^{k*}\|}{\|E(z^k)\|} \geq \frac{1}{2}(\|\Delta x^k\| - C_2\theta(z^k)).$$

This, together with (5.28), implies that for all $z^k$ sufficiently close to $z^*$,

$$\|\Delta x^k\| = O(\theta(z^k)). \tag{5.29}$$

Thirdly, we consider the estimation of $\|\Delta s^k\|$. From (5.18) and (5.26)–(5.28) we have that for all $z^k$ sufficiently close to $z^*$,

$$
\begin{aligned}
&\|s^k + \Delta s^k - s^{k*}\| \\
&\le \|[D(z^k) + p(\mu_k)I]^{-1}E(z^k)^{-1}\|\|D(z^k) + p(\mu_k)I\|\|\alpha(z^k)\| \\
&\quad + \|D(z^k) + p(\mu_k)I\|\|E(z^k)^{-1}(x^k + \Delta x^k - x^{k*})\| \\
&\le 3\frac{\|\varpi(z^k)\| + \|\varrho(z^k)\|}{\min_{i\in\mathcal{I}}(D(z^k)_{ii} + p(\mu_k))E(z^k)_{ii}} + 3\|E(z^k)^{-1}(x^k + \Delta x^k - x^{k*})\| \\
&\le 6(2C_2C_4 + C_5)\theta(z^k),
\end{aligned} \tag{5.30}
$$

where the second inequality is due to $\|D(z^k)+p(\mu_k)I\| \le 2+p(\mu_k)$ and $\alpha(z^k) = \varpi(z^k)+\varrho(z^k)$ and the last inequality is due to (5.28). Hence, from (5.7) and (5.30) for all $z^k$ sufficiently close to $z^*$ we have

$$\|\Delta s^k\| \le \|s^k + \Delta s^k - s^{k*}\| + \|s^k - s^{k*}\| = O(\theta(z^k)). \tag{5.31}$$

Now, by combining (5.15) with (5.29) and (5.31) we obtain that there exists a constant $C_3 > 0$ such that (5.10) holds for all $z^k$ sufficiently close to $z^*$. □

**Lemma 5.4** *Suppose that all the conditions assumed in Lemma 5.3 are satisfied. Then there exists a constant $C_6 > 0$ such that for all $z^k$ sufficiently close to $z^*$, $z^{k+1} = z^k + \Delta z^k$ and*

$$\theta(z^{k+1}) \le C_6\theta(z^k)^2. \tag{5.32}$$

**Proof.** From Lemma 1.2 in [32] we have

$$\|\phi''(\mu, a, b)\| \le \frac{4}{\sqrt{(a-b)^2 + 4\mu^2}}$$

for any $(\mu, a, b) \in \Re_+ \times \Re^{2n}$ and $(a-b)^2 + 4\mu^2 \neq 0$. For any $k \ge 0$ and $i = 1, \cdots, n$, let $y_i^k := (\mu_k, x_i^k, s_i^k)$ and $\Delta y_i^k := (\Delta\mu_k, \Delta x_i^k, \Delta s_i^k)$. Then by using the strict complementarity condition we can obtain a positive number $\bar{C}$ such that $\|\phi''(y_i)\| \le \bar{C}$ for all $z = (\mu, x, s)$ sufficiently close to $z^*$, where $y_i := (\mu_k, x_i, s_i)$, $i = 1, \cdots, n$. Hence, by Lemma 5.3 for all $z^k$ sufficiently close to $z^*$ we have

$$
\begin{aligned}
|\phi(y_i^k + \Delta y_i^k) - \phi(y_i^k) - \phi'(y_i^k)\Delta y_i^k| &\le \int_0^1 t\int_0^1 \|\phi''(y_i^k + ts\Delta y_i^k)\|dsdt\|\Delta y_i^k\|^2 \\
&\le \bar{C}\int_0^1 t\int_0^1 dsdt\|\Delta y_i^k\|^2 \\
&= \bar{C}\|\Delta y_i^k\|^2,
\end{aligned}
$$

and so,

$$\|\Phi(z^k + \Delta z^k) - \Phi(z^k) - \Phi'(z^k)\Delta z^k\| \le \bar{C}\|\Delta z^k\|^2.$$

Then, from the definition of $H(\cdot)$ (see (2.3)), Lemma 5.3 and the fact that $p(\cdot)$ is twice continuously differentiable, there exists a constant $C_7 > 0$ such that for all $z^k$ sufficiently close to $z^*$,

$$\|H(z^k + \Delta z^k) - H(z^k) - H'(z^k)\Delta z^k\| \le C_7\|\Delta z^k\|^2. \tag{5.33}$$

Hence, from Lemmas 5.2 and 5.3, (5.33) and (2.19) we have that for all $z^k$ sufficiently close to $z^*$,

$$\begin{aligned}
& \theta(z^k + \Delta z^k) \\
= \; & \|(H(z^k + \Delta z^k) - H(z^k) - H'(z^k)\Delta z^k + H(z^k) + H'(z^k)\Delta z^k\| \\
\le \; & \|H(z^k + \Delta z^k) - H(z^k) - H'(z^k)\Delta z^k\| + \|H(z^k) + H'(z^k)\Delta z^k\| \\
\le \; & C_7\|\Delta z^k\|^2 + \sqrt{(\gamma\mu_0\theta(z^k)^2)^2 + \|u(z^k)\|^2} \\
\le \; & C_7(C_3)^2\theta(z^k)^2 + \sqrt{(\gamma\mu_0\theta(z^k)^2)^2 + \|u(z^k)\|^2}.
\end{aligned} \tag{5.34}$$

Lemma 2.1 and the strict complementarity condition imply that there exists a constant $C_8$ such that for all $z^k$ sufficiently close to $z^*$,

$$\|u(z^k)\| \le C_8(\mu_k)^2 + n\gamma\mu_0\theta(z^k)^2,$$

which, together with (5.34), implies that there exists a constant $C_6$ such that for all $z^k$ sufficiently close to $z^*$,

$$\theta(z^k + \Delta z^k) \le C_6\theta(z^k)^2,$$

i.e., (5.32) holds. This further implies that $z^{k+1} = z^k + \Delta z^k$ for all $z^k$ sufficiently close to $z^*$. The proof is completed. $\square$

For any $\varepsilon > 0$, define

$$N(z^*, \varepsilon) := \{z \in \Re_+ \times \Re^{2n} : \|z - z^*\| \le \varepsilon\}.$$

Since $H$ is locally Lipschitz continuous around $z^*$, there exists a constant $\mathcal{L} > 0$ such that

$$\|H(y^1) - H(y^2)\| \le \mathcal{L}\|y^1 - y^2\|$$

holds for any $y^1, y^2 \in N(z^*, \varepsilon)$. Let

$$\bar{\varepsilon} := \min\left\{\frac{\varepsilon}{2 + 4C_3\mathcal{L}}, \quad \frac{1}{2C_6\mathcal{L}}\right\}, \tag{5.35}$$

where $C_3$ and $C_6$ are given by (5.10) and (5.32), respectively.

The following lemma, which to some extent is motivated by a result in Yamashita and Fukushima [44] on the Levenberg-Marquardt method, is on the convergence of the whole iteration sequence $\{z^k\}$.

**Lemma 5.5** *Suppose that all the conditions assumed in Lemma 5.3 are satisfied. Let $\bar{\varepsilon}$ be defined by (5.35). If for some $k$ the iterate $z^k \in N(z^*, \bar{\varepsilon})$ and $\varepsilon$ is sufficiently small, then $z^{k+q} \in N(z^*, \varepsilon/2)$ for all $q = 0, 1, 2, \ldots$ and $\{z^{k+q}\}_{q=1}^{\infty}$ is a convergent sequence.*

**Proof.** Suppose for some $k$, $z^k \in N(z^*, \bar{\varepsilon})$. Then, it is obvious that $z^k \in N(z^*, \bar{\varepsilon}) \subseteq N(z^*, \varepsilon/2)$ by (5.35). By reducing $\varepsilon$ if necessary, we have from Lemmas 5.4 and 5.3 and (5.35) that

$$\begin{aligned}
\|z^{k+1} - z^*\| &= \|z^k + \Delta z^k - z^*\| \\
&\le \|z^k - z^*\| + \|\Delta z^k\| \\
&\le \bar{\varepsilon} + C_3 \theta(z^k) \\
&= \bar{\varepsilon} + C_3 \|H(z^k) - H(z^*)\| \\
&\le \bar{\varepsilon} + C_3 \mathcal{L} \|z^k - z^*\| \\
&\le (1 + C_3 \mathcal{L}) \bar{\varepsilon} \\
&\le \varepsilon/2,
\end{aligned}$$

which implies $z^{k+1} \in \mathcal{N}(z^*, \varepsilon/2)$.

Suppose that for some $q > 1$, $z^{k+1}, \cdots, z^{k+q} \in N(z^*, \varepsilon/2)$. We shall then show $z^{k+q+1} \in \mathcal{N}(z^*, \varepsilon/2)$. From Lemma 5.4, we have

$$\begin{aligned}
\theta(z^{k+q}) &\le C_6 \theta(z^{k+q-1})^2 \le \cdots \le (C_6)^{2^q - 1} \theta(z^k)^{2^q} = (C_6)^{2^q-1} \|H(z^k) - H(z^*)\|^{2^q} \\
&\le (C_6)^{2^q-1} \mathcal{L}^{2^q} \|z^k - z^*\|^{2^q} \le (C_6)^{2^q-1} \mathcal{L}^{2^q} \left(\frac{1}{2C_6 \mathcal{L}}\right)^{2^q-1} \bar{\varepsilon} \\
&= \mathcal{L} \bar{\varepsilon} (1/2)^{2^q-1} \le \mathcal{L} \bar{\varepsilon} (1/2)^{2q-1},
\end{aligned} \tag{5.36}$$

which, together with (5.10), implies that

$$\sum_{i=1}^{q} \|\Delta z^{k+i}\| \le C_3 \sum_{i=1}^{q} \theta(z^{k+i}) \le C_3 \mathcal{L} \bar{\varepsilon} \sum_{i=1}^{q} (1/2)^{2i-1} \le C_3 \mathcal{L} \bar{\varepsilon} \sum_{i=1}^{\infty} (1/2)^i = C_3 \mathcal{L} \bar{\varepsilon}.$$

This further leads to

$$\begin{aligned}
\|z^{k+q+1} - z^*\| &\le \|z^{k+1} - z^*\| + \sum_{i=1}^{q} \|\Delta z^i\| \\
&\le (1 + C_3 \mathcal{L}) \bar{\varepsilon} + C_3 \mathcal{L} \bar{\varepsilon} \\
&= (1 + 2C_3 \mathcal{L}) \bar{\varepsilon} \\
&\le \varepsilon/2.
\end{aligned}$$

Therefore, $z^{k+q+1} \in \mathcal{N}(z^*, \varepsilon/2)$.

From (5.36) and Lemma 5.4 we know that for all $q \ge 0$,

$$\|\Delta z^{k+q}\| \le C_3 \mathcal{L} \bar{\varepsilon} (1/2)^{2q-1}.$$

Hence, for any positive integers $l, m$ with $l \geq m \geq q$,

$$\|z^l - z^m\| \leq \sum_{i=m}^{l-1} \|\Delta z^i\| \leq \sum_{i=m}^{\infty} \|\Delta z^i\| \leq C_3 \mathcal{L} \bar{\varepsilon} \sum_{i=m}^{\infty} (1/2)^{2i-1} = \frac{1}{3} C_3 \mathcal{L} \bar{\varepsilon} (1/2)^{2m-3},$$

which indicates that the sequence $\{z^k\}$ is a Cauchy sequence. This implies that $\{z^k\}$ is a convergent sequence. The proof is completed. □

**Theorem 5.1** *Suppose that $M$ is a positive semidefinite matrix and Assumption 3.1 is satisfied. Let $z^*$ be an accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm 2.1. If $(x^*, s^*)$ satisfies the strict complementarity condition $x^* + s^* > 0$, then*

*(i) The whole sequence $\{z^k\}$ generated by Algorithm 2.1 converges to $z^*$;*

*(ii)*

$$\|H(z^{k+1})\| = O\left(\|H(z^k)\|^2\right);$$

*and*

*(iii)*

$$\mu_{k+1} = O\left((\mu_k)^2\right).$$

**Proof.** By Lemmas 5.4 and 5.5, we know that $\{z^k\}$ is a convergent sequence and for all $k$ sufficiently large, $z^{k+1} = z^k + \Delta z^k$ and

$$\theta(z^{k+1}) \leq C_6 \theta(z^k)^2.$$

Hence, parts (i) and (ii) hold.

Since $z^{k+1} = z^k + \Delta z^k$ for all $k$ sufficiently large,

$$\mu_{k+1} = \mu_k + \Delta_k = \gamma \mu_0 \theta(z^k)^2$$

holds for for all $k$ sufficiently large. This, together with part (ii), proves part (iii). The proof is completed. □

It is worth pointing out that in Theorem 5.1 only one accumulation point of the iteration sequence is assumed to satisfy the strict complementarity condition and the whole sequence is proved to converge to this accumulation point while in [42] all accumulation points are assumed to satisfy the strict complementarity condition uniformly and in [17, 18] the whole sequence is assumed to converge to a strict complementary solution. This indicates that even in this case our results are stronger than those in [42, 17, 18].

## 6 Conclusions

By modifying the QSZ method in [34], we have proposed a smoothing Newton algorithm for solving the $P_0$ and monotone linear complementarity problem. Algorithm 2.1 is simpler than the predictor-corrector-type smoothing algorithms given in [42, 17, 18]. It needs only to solve one linear system of equations and to do one line search at each iteration. We have proved that Algorithm 2.1 converges globally and locally sub-quadratically if the $P_0-$LCP satisfies a nonsingularity condition, and converges quadratically if $M$ is positive semidefinite and there is an accumulation point satisfying a strict complementarity condition. We also have shown the convergence of the whole sequence generated by Algorithm 2.1. One future research topic is to study the local behavior of Algorithm 2.1 or its variants when the LCP has multiple solutions and the strict complementarity condition does not hold.

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