2 Linear Programs with Bounded Variables

2.1 Simplex method for linear programs in standard form

Consider the standard linear programming problem

$$\begin{array}{lrl} \min & z = & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} & & Ax = b, \\ & & x \geq 0, \end{array} \tag{2.1}$$

where $A \in \Re^{m \times n}$ $(m \leq n)$ is of full row rank.

Let x be a basic feasible solution (bfs) to (2.1). Then, by the definition of bfs, we can prove that there exists an index set $\mathcal{N}$ with $|\mathcal{N}| = n - m$ such that

$$x_i = 0 \quad \forall\, i \in \mathcal{N}$$

and the formed matrix

$$M := \begin{bmatrix} A \\ \{e_i^T\}_{i \in \mathcal{N}} \end{bmatrix}$$

is nonsingular.

Suppose that

$$\mathcal{N} = \{\mathcal{N}(1), \ldots, \mathcal{N}(n-m)\}.$$

Define the complement of $\mathcal{N}$ in $\{1, \ldots, n\}$ by

$$\mathcal{B} = \{1, \ldots, n\} \backslash \mathcal{N}.$$

There are m indices in $\mathcal{B}$. Suppose that these m elements are

$$\{\mathcal{B}(1), \ldots, \mathcal{B}(m)\}.$$

We call variables x_i, $i \in \mathcal{B}$ basic variables and otherwise nonbasic variables.

Let

$$B = [A_{\mathcal{B}(1)} \; \ldots \; A_{\mathcal{B}(m)}]$$

the corresponding basis matrix and

$$N = [A_{\mathcal{N}(1)} \; \ldots \; A_{\mathcal{N}(n-m)}]$$

be the corresponding nonbasis matrix. Therfore, by the nonsingularity of matrix M, the basis matrix B is nonsingular. The vector

$$x_{\mathcal{B}} = (x_{\mathcal{B}(1)}, \ldots, x_{\mathcal{B}(m)})^T$$

of the basic variables is given by

$$x_B = B^{-1}b - B^{-1}N \times \mathbf{0} = B^{-1}b\,.$$

Define

$$\bar{A} = B^{-1}A, \quad \bar{b} = B^{-1}b.$$

Thus, (2.1) can be equivalently written as

$$\begin{array}{lll} \min & z = & c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ \text{s.t.} & & x_{\mathcal{B}} + \bar{A}_{\mathcal{N}} x_{\mathcal{N}} = \bar{b}, \\ & & x_j \geq 0, \quad j = 1, \ldots, n, \end{array} \tag{2.2}$$

i.e.,

$$\begin{array}{lll} \min & z = & c_{\mathcal{B}}^T \bar{b} + \sum_{j \in \mathcal{N}} \bar{c}_j x_j \\ \text{s.t.} & & x_{\mathcal{B}} + \sum_{j \in \mathcal{N}} \bar{A}_j x_j = \bar{b}, \\ & & x_j \geq 0, \quad j = 1, \ldots, n, \end{array} \tag{2.3}$$

where

$$\bar{c}_j = c_j - c_{\mathcal{B}}^T \bar{A}_j.$$

Note that for each $j \in \mathcal{B}$,

$$\bar{c}_j = c_j - c_{\mathcal{B}}^T B^{-1} A_j = c_j - c_j = 0.$$

Therfore, we obtain from (2.3) the following **optimality condition**: In a minimization problem in standard form, if every nonbasic variable x_j, $j \in \mathcal{N}$, has a nonnegative objective coefficient $\bar{c}_j$, then the basic feasible solution given by that standard form minimizes the objective function over the feasible region.

The full tableau implementation of the simplex method at the very beginning is:

$$\begin{array}{|c|cccc|} \hline 0 & c_1 & c_2 & \ldots & c_n \\ \hline b_1 & | & | & \ldots & | \\ \vdots & A_1 & A_2 & \ldots & A_n \\ b_m & | & | & \ldots & | \\ \hline \end{array}$$

The next step is to use elementary row operations to change $c_{\mathcal{B}(1)}$, $c_{\mathcal{B}(2)}$, $\ldots$ and $c_{\mathcal{B}(m)}$ to be zero to get

$$\begin{array}{|c|cccc|} \hline -c_{\mathcal{B}}^T x_{\mathcal{B}} & \bar{c}_1 & \bar{c}_2 & \ldots & \bar{c}_n \\ \hline x_{\mathcal{B}(1)} & | & | & \ldots & | \\ \vdots & \bar{A}_1 & \bar{A}_2 & \ldots & \bar{A}_n \\ x_{\mathcal{B}(m)} & | & | & \ldots & | \\ \hline \end{array}$$

We use the smallest ratio rule to determine at what value t_1 for the incoming variable, a basic variable first reaches its lower bound, i.e., zero in this case. We must perform such a check in every constraint in which the incoming variable x_s (in this case $\bar{c}_s < 0$) has a positive coefficient; thus when increasing nonbasic variable x_s, we require that $x_s \leq t_1$ where

$$t_1 = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} \mid \bar{a}_{is} > 0 \right\},$$

Anticyclying rules: Lexicography rule and Bland's rule ("first-first" rule).

2.2 Linear programs with bounded variables

Consider the following linear programming problem with bounded variables

$$\begin{array}{lll} \max & z = & \displaystyle\sum_{j=1}^{n} c_j x_j \\ \text{s.t.} & & \displaystyle\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m \\ & & l_j \leq x_j \leq u_j, \quad j = 1, \ldots, n, \end{array} \tag{2.4}$$

where $l_j \leq u_j$, $j = 1, \ldots, n$ and $A = (a_{ij}) \in \Re^{m \times n}$ is of full row rank. The lower bounds may be $-\infty$ and the upper bounds u_j may be $+\infty$, indicating respectively that the decision variable x_j is unbounded from below or from above. Note that when each $l_j = 0$ and each $u_j = +\infty$, the above problem reduces to the one we have studied before.

The bounded-variable problem can be solved by the simplex method, by adding variables to the upper-bound constraints and surplus variables to the lower-bound constraints, thereby converting them to equalities. This approach handles the bounding constraints explicitly. In contrast, the approach proposed in this chapter modifies the simplex method to consider the bounded-variable constraints implicitly.

2.3 Bounded variable optimality condition

Let x be a basic feasible solution to (2.4). Then, by the definition of bfs, we can prove that there exists an index set $\mathcal{N}$ with $|\mathcal{N}| = n - m$ such that for all

$$x_i = l_i \text{ (or } u_i) \quad \forall\, i \in \mathcal{N}$$

and the formed matrix

$$M := \begin{bmatrix} A \\ \{e_i^T\}_{i\in\mathcal{N}} \end{bmatrix}$$

is nonsingular. Define the complement of $\mathcal{N}$ in $\{1, \dots, n\}$ by

$$\mathcal{B} = \{1, \dots, n\} \backslash \mathcal{N}.$$

There are m indices in $\mathcal{B}$. Again, We call variables x_i, $i \in \mathcal{B}$ basic variables and otherwise nonbasic variables.

Let

$$B = [A_{\mathcal{B}(1)} \ \dots \ A_{\mathcal{B}(m)}]$$

be the corresponding basis matrix and

$$N = [A_{\mathcal{N}(1)} \ \dots \ A_{\mathcal{N}(n-m)}]$$

be the corresponding nonbasis matrix. Therfore, by the nonsingularity of matrix M, the basis matrix B is nonsingular. Define

$$\bar{A} = B^{-1}A, \quad \bar{b} = B^{-1}b.$$

Thus, (2.4) can be equivalently written as

$$\begin{array}{lll} \max & z = & c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ \text{s.t.} & & x_{\mathcal{B}} + \bar{A}_{\mathcal{N}} x_{\mathcal{N}} = \bar{b}, \\ & & l_j \le x_j \le u_j, \quad j = 1, \dots, n, \end{array} \tag{2.5}$$

i.e.,

$$\begin{array}{lll} \max & z = & c_{\mathcal{B}}^T \bar{b} + \sum_{j\in\mathcal{N}} \bar{c}_j x_j \\ \text{s.t.} & & x_{\mathcal{B}} + \sum_{j\in\mathcal{N}} \bar{A}_j x_j = \bar{b}, \\ & & l_j \le x_j \le u_j, \quad j = 1, \dots, n, \end{array} \tag{2.6}$$

where

$$\bar{c}_j = c_j - c_{\mathcal{B}}^T \bar{A}_j.$$

Note that for each $j \in \mathcal{B}$, $\bar{c}_j = 0$.

From (2.6), we obtain the following optimality condition: In a maximization problem in canonical form, if every nonbasic variable at its lower bound has a nonpositive objective coefficient, and every nonbasic variable at its upper bound has a nonnegative objective coefficient, then the basic feasible solution given by that canonical form maximizes the objective function over the feasible region.

For example, suppose at some step that x_2 and x_4 are nonbasic variables constrained by

$$4 \le x_2 \le 15,$$
$$2 \le x_4 \le 5;$$

and that

$$z = 4 - \frac{1}{4}x_2 + \frac{1}{2}x_4,$$
$$x_2 = 4,$$
$$x_4 = 5,$$

in the current canonical form. In any feasible solution, $x_2 \ge 4$, so $-\frac{1}{4}x_2 \le -1$, also, $x_4 \le 5$, so that $\frac{1}{2}x_4 \le \frac{1}{2}(5) = 2\frac{1}{2}$. Consequently,

$$z = 4 - \frac{1}{4}x_2 + \frac{1}{2}x_4 \le 4 - 1 + 2\frac{1}{2} = 5\frac{1}{2}$$

for any feasible solution. Since the current solution with $x_2 = 4$ and $X_4 = 5$ attains this bound, it must be optimal.

2.4 Improving a nonoptimal solution.

If the objective coefficient $\bar{c}_j$ of nonbasic variable x_j is positive and $x_j = l_j$, then we increase x_j; if $\bar{c}_j < 0$ and $x_j = u_j$, we decrease x_j. In either case, the objective value is improving.

Feasibility. When changing the value of a nonbasic variable, we wish to maintain feasibility. As we have known, for problems with only nonnegative variables, we have to test, via the ratio rule, to see when a basic variable first becomes zero. Here we must consider the following contingencies:

i) the nonbasic variable being changed reaches its upper bound or lower bound; or

ii) some basic variables reaches either its upper bound or lower bound.

Implementation. In the first case, no pivoting is required. The nonbasic variable simply changes from lower bound to upper bound, or upper bound to lower bound and remains nonbasic. In the second case, pivoting is used to remove the basic variable reaching either its lower or upper bound from the basis.

These ideas can be implemented in a number of ways.

• For example, we can keep track of the lower bounds throughout the algorithm; or every lower bound

$$x_j \geq l_j$$

can be converted to zero by defining a new variable

$$x_j'' = x_j - l_j \geq 0,$$

and substituting $x_j'' + l_j$ for x_j everywhere throughout the model. So we may assume that all the finite lower bounds are zeros.

• Also, we can always redefine variables so that every nonbasic variable is at its lower bound. Let x_j' denote the slack variable for the upper bound constraint $x_j \leq u_j$; that is

$$x_j + x_j' = u_j.$$

Whenever x_j is nonbasic at its upper bound u_j, the slack variable $x_j' = 0$. Consequently, substituting $u_j - x_j'$ for x_j in the model makes x_j' nonbasic at value zero in place of x_j. If, subsequently in the algorithm, x_j' becomes nonbasic at its upper bound, which is also u_j, we can make the substitution for x_j', replacing it with $u_j - x_j$, and x_j will appear nonbasic at value zero. These transformations are usually referred to as **upper-bounding substitution**.

The computational steps of the upper-bounding algorithm are very simple to describe if both of the above transformations are used. Since all nonbasic variables (either x_j or x_j') are at value zero, we increase a variable for maximization as in the usual simplex method if its objective coefficient ($\bar{c}_j$) is positive. We use the usual ratio rule to determine at what value t_1 for the incoming variable, a basic variable first reaches zero. We also ensure that variables do not exceed their upper bounds. We must perform such a check in every constraint in which the incoming variable x_s has a negative coefficient; thus when increasing nonbasic variable x_s, we require that $x_s \leq t_2$ where

$$t_2 = \min_i \left\{ \frac{u_i - \bar{b}_i}{-\bar{a}_{is}} \mid \bar{a}_{is} < 0 \right\},$$

and u_i is the upper bound for the basic variable x_i in the ith constraint, $\bar{b}_i$ is the current value for this variable, and $\bar{a}_{is}$ are the constraint coefficients for variable x_s. This test may be called **upper-bounding ratio test**.

Table 1: A bounded variable linear programming problem

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_1	15	1				4	1
x_2	8		1			6	2
x_3	4			1		-7	-2
x_4	2				1	-1	-1
$(-z)$	0					2	1
Upper Bounds		15	15	15	5	1	8
						↑	

For example, when increasing nonbasic variable x_s to value t, in a constraint with x_1 basic, such as:

$$x_1 - 2x_s = 3,$$

we require that:

$$x_1 = 3 + 2t \le u_1,$$

that is

$$t \le \frac{u_1 - 3}{2}.$$

Note that, in contrast to the usual ratio test, the upper-bounding ratio uses negative coefficients $\bar{a}_{is}$ for the nonbasic variable x_s being increased. In general, the incoming variable x_s (or x'_s) is set to

$$x_s = \min\{u_s, t_1, t_2\}.$$

If the minimum is

i) u_s, then the upper bounding substitution is made for x_s (or x'_s);

ii) t_1, then a usual simplex pivot is made to introduce x_s into the basis;

iii) t_2, then the upper bounding substitution is made for the basic variable x_k (or x'_k) reaching its upper bound and x_s is introduced into the basis in place of x'_k (or x_k) by a usual simplex pivot.

2.5 An example

Consider the following maximization problem with data with by Table 1.

Table 2: continued

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5'	x_6
x_1	11	1				-4	1
x_2	2		1			-6	2^*
x_3	11			1		7	-2
x_4	3				1	1	-1
$(-z)$	-2					-2	1
Upper Bounds		15	15	15	5	1	8
							$\uparrow$

Step 1: From Table 1, we obtain

$$t_1 = \frac{8}{6},$$

$$t_2 = \frac{11}{7},$$

$$u_5 = 1.$$

Thus,

$$\min\{u_5, t_1, t_2\} = u_5.$$

Then, the upper bounding substitution is made for x_5 at the next step (Table 2).

Step 2: This time, from Table 2 we get

$$t_1 = 1,$$

$$t_2 = 2,$$

$$u_6 = 8.$$

Since

$$\min\{u_6, t_1, t_2\} = t_1,$$

the usual simplex pivot is made to introduce x_2 into the basis.

Step 3: By calculation, we get from Table 3 that

$$t_1 = 13,$$

Table 3: continued

Basic variables	Current values	x_1	x_2	x_3	x_4	x'_5	x_6
x_1	10	1	$-\frac{1}{2}$			-1	
x_6	1		$\frac{1}{2}$			-3	1
x_3	13		1	1		1	
x_4	4		$\frac{1}{2}$		1	-2^*	
$(-z)$	-3		$-\frac{1}{2}$			1	
Upper Bounds		15	15	15	5	1	8
						↑	

Table 4: continued

Basic variables	Current values	x_1	x_2	x_3	x'_4	x'_5	x_6
x_1	10	1	$-\frac{1}{2}$			-1	
x_6	1		$\frac{1}{2}$			-3	1
x_3	13		1	1		1	
x'_4	1		$-\frac{1}{2}$		1	2^*	
$(-z)$	-3		$-\frac{1}{2}$			1	
Upper Bounds		15	15	15	5	1	8
						↑	

$$t_2 = \frac{1}{2},$$
$$u_5 = 1.$$

The smallest value of u_5, t_1, t_2 is $t_2 = \frac{1}{2}$. So, at the next step the upper bounding substitution is made for x_4 reaching its upper bound and x'_5 is introduced into the basis in replace of x'_4 (Table 4).

Step 4: Analogously, from Table 4 we have

$$t_1 = \frac{1}{2},$$
$$t_2 = \frac{7}{3},$$
$$u_5 = 1.$$

Since

$$\min\{u_5, t_1, t_2\} = t_1,$$

Table 5: continued

Basic variables	Current values	x_1	x_2	x_3	x'_4	x'_5	x_6
x_1	$10\frac{1}{2}$	1	$-\frac{3}{4}$		$\frac{1}{2}$		
x_6	$2\frac{1}{2}$		$-\frac{1}{4}$		$\frac{3}{2}$		1
x_3	$12\frac{1}{2}$		$\frac{5}{4}$	1	$-\frac{1}{2}$		
x'_5	$\frac{1}{2}$		$-\frac{1}{4}$		$\frac{1}{2}$	1	
$(-z)$	$-3\frac{1}{2}$		$-\frac{1}{4}$		-$\frac{1}{2}$		
Upper Bounds		15	15	15	5	1	8

the usual simplex pivot is made to introduce x'_5 into the basis (Table 5).

Step 6: The found optimal solution is

$$x_1 = 10\frac{1}{2}, \quad x_2 = 0, \quad x_3 = 12\frac{1}{2},$$

$$x_4 = 5 - x'_4 = 5, \quad x_5 = 1 - x'_5 = \frac{1}{2}, \quad x_6 = 2\frac{1}{2}.$$