NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
Semester 1 (2003/2004) MA4253 Mathematical Programming Tutorial 10

Q1. Consider the function $\theta : \Re^2 \to \Re$ defined by the following optimization problem

$$\begin{aligned} \theta(u_1, u_2) \quad = \quad \max \quad & x_1(1 + 2u_1^2) + x_2\left(1 + \sqrt{2 + u_1^2 + u_2^2}\right) \\ \text{s.t.} \quad & x_1 + x_2 \le 1 \\ & x_1\,,\, x_2 \ge 0\,. \end{aligned}$$

(i) Show that θ is a convex function.

(ii) Find the subdifferential of θ at $(0,0)^T$ and $(1,1)^T$, respectively.

Q2. Suppose that $c \in \Re^n$, $b \in \Re^m$, $A \in \Re^{m\times n}$, and $H \in \Re^{n\times n}$ a symmetric positive definite matrix. Consider the following two problems:

$$\begin{aligned} \min \quad & c^T x + \tfrac{1}{2} x^T H x \\ \text{s.t.} \quad & Ax \le b \end{aligned}$$

and

$$\begin{aligned} \min \quad & h^T v + \tfrac{1}{2} v^T G v \\ \text{s.t.} \quad & v \ge 0\,, \end{aligned}$$

where $G = AH^{-1}A^T$ and $h = AH^{-1}c + b$. Investigate the relationship between the KKT conditions of these two problems.

Q3. Consider the following problem

$$\begin{aligned} \min \quad & x_1^2 - x_1x_2 + 2x_2^2 - 4x_1 - 5x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 6 \\ & x_1 \le 2 \\ & x_1, x_2 \ge 0\,. \end{aligned}$$

Find a solution to the Lagrangian dual of the above problem.

Q4. Consider the following problem

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = 0 \\ & \|x\|^2 \le 1\,, \end{aligned}$$

where $A \in \Re^{m\times n}$ is of full row rank and $c \in \Re^n$ is not a linear combination of the rows of A.

(i) Give the Lagrangian dual to the above problem.

(ii) Find an optimal solution to the Lagrangian dual problem obtained in part (i).

Q5. Consider the linear programming problem

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \le c\,. \end{aligned}$$

(i) Write down the Lagrangian dual problem.

(ii) What's the dual of the Lagrangian dual problem?

Q6. A new facility is to be placed such that the sum of its squared distance from four existing facilities is minimized. The four facilities are located at points $(1, 2)$, $(-2, 4)$, $(2, 6)$, and $(-6, -3)$. If the coordinate of the new facility is (x_1, x_2), then x_1 and x_2 must satisfy the constraints $x_1 + x_2 = 2$, $x_1 \ge 0$, and $x_2 \ge 0$.

(i) Formulate the above problem as an optimization problem.

(ii) Find an optimal solution by making use of the KKT conditions.

(iii) Give the Lagrangian dual problem.

(iv) Identify an optimal solution to the Lagrangian dual problem.

Q7. Suppose that $f : \Re^n \to \Re$ is a concave function, $A \in \Re^{m\times n}$, $b \in \Re^m$, and X is a nonempty bounded polyhedral set with extreme points $\{x^1, \ldots, x^p\}$. Let

$$\theta(u) = \min_{x\in X}\{f(x) + u^T(Ax - b)\}\,.$$

(i) Show that θ is a concave function.

(ii) Characterize the subdifferential of θ.