Integer Programming Models for MA4260 Model Building in Operations Research

Defeng Sun
Department of Mathematics
National University of Singapore
Republic of Singapore
Office: S14-04-25; Phone: 6516-3343
Email: matsundf@nus.edu.sg

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Refer to Chapter 9, H. P. Williams' Book:

I. The Use of Discrete Variables

Integer variables may serve a number of purposes

- Indivisible (Discrete) Variables: number of aeroplanes, cars, houses or people.
- Decision variables: $\delta = 0, 1$;

$$\gamma = \begin{cases} 0 & \text{not built} \\ 1 & \text{depot of type A built} \\ 2 & \text{depot of type B built} \end{cases}$$

- Indicator variables

  δ — — $\{0, 1\}$ variable, linked to continuous variables:

  x — — the quantity of an ingredient to be included in a blend.

  δ — — indicator variable to distinguish the state where $x = 0$ and the state where $x > 0$.

  By introducing the following constraint we can force δ to take the value 1 when $x > 0$:

$$x - M\delta \leq 0, \tag{1}$$

  where M is a constant coefficient representing a known upper bound for x.

Logically, we have achieved the condition

$$x > 0 \rightarrow \delta = 1. \tag{2}$$

In some cases, we may also wish to impose the condition

$$x = 0 \rightarrow \delta = 0, \tag{3}$$

or equivalently

$$(3) \Longleftrightarrow \delta = 1 \rightarrow x > 0. \tag{4}$$

(2) + (3) [or (4)] impose the condition

$$\delta = 1 \leftrightarrow x > 0. \tag{5}$$

Realistically, a threshold imposed:

$$(4) \Longrightarrow \delta = 1 \rightarrow x \geq m > 0. \tag{6}$$

The above condition can be imposed by the constraint

$$x - m\delta \geq 0. \tag{7}$$

Example 1. The Fixed Charge Problem

x – – quantity of a product at a marginal cost per unit of c_1.

c_2 – – set-up cost if the product is manufactured at all.

$x = 0$, the total cost $= 0$.

$x > 0$, the total cost $= c_1 x + c_2$. [Not continuous at $x = 0$.]

Introduce an indicator variable δ so that if any of the product is manufactured $\delta = 1$:

$$\text{total cost} = c_1 x + c_2 \delta$$

with constraints

$$x - M\delta \leq 0, \quad \delta \in \{0, 1\}.$$

[By using (1)]

Example 2. Blending

x_A −− the proportion of ingredient A to be included in a blend.

x_B −− the proportion of ingredient B to be included in a blend.

Extra condition

A is included $\Longrightarrow$ B must be included.

Introduce a $0-1$ indicator variable: $\delta = 1$ if $x_A > 0$:

$$\boxed{x_A - \delta \leq 0. \qquad (8)}$$

[A special case of (8).]

Use δ to impose the condition

$$\boxed{\delta = 1 \rightarrow x_B > 0. \qquad (9)}$$

Choose some proportionate level m (say $1/100$) below which we will regard B as out of the blend. This gives

$$x_B - m\delta \geq 0. \tag{10}$$

Hence, two extra conditions (8) and (10) are imposed by introducing the $0-1$ variable δ.

An extension of the extra condition:

A is included $\Longleftrightarrow$ B is also included.

In addition to (8) and (10), we need to impose two more constraints:

$$x_B - \delta' \leq 0$$

and

$$x_A - m\delta' \geq 0,$$

where δ' is another $0-1$ indicator variable: $\delta' = 1$ if $x_B > 0$:

- Use indicator variables to show whether an inequality holds or does not hold.

Indicate

$$\sum_j a_j x_j \leq b$$

holds or does not hold.

First, model

$$\delta = 1 \rightarrow \sum_j a_j x_j \leq b. \tag{11}$$

(11) is equivalent to

$$\boxed{(1-\delta) = 0 \to \sum_j a_j x_j - b \le 0.}$$

Thus, it can be represented by the constraint

$$\sum_j a_j x_j - b \le M(1-\delta)$$

i.e.,

$$\boxed{\sum_j a_j x_j + M\delta \le M + b, \qquad (12)}$$

where M is an upper bound for the expression $\sum_j a_j x_j - b$. [When $\delta = 0$, no constraint imposed because $M \ge \sum_j a_j x_j - b$.]

We will now consider how to model the reverse of the constraint (11), i.e.,

$$\sum_j a_j x_j - b \leq 0 \rightarrow \delta = 1. \tag{13}$$

This is conveniently expressed as

$$\delta = 0 \rightarrow \sum_j a_j x_j - b \not\leq 0, \tag{14}$$

i.e.,

$$\delta = 0 \rightarrow \sum_j a_j x_j - b > 0. \tag{15}$$

We must rewrite

$$\sum_j a_j x_j - b > 0 \text{ as } \sum_j a_j x_j - b \geq \varepsilon,$$

where ε is some small tolerance value beyond which we will regard the constraint as having been broken.

[Should be coefficients a_j be integers as well as the variables x_j, as often happens in the type of situation, there is no difficulty as ε can be taken as 1.]

(15) may now be written as

$$\delta = 0 \rightarrow -\sum_j a_j x_j + b + \varepsilon \leq 0, \tag{16}$$

which can be modelled as

$$-\sum_j a_j x_j + b + \varepsilon \leq (-m + \varepsilon)\delta,$$

i.e.,

$$\boxed{\sum_j a_j x_j - (m - \varepsilon)\delta \geq b + \varepsilon, \qquad (17)}$$

where m is a lower bound for expression

$$\sum_j a_j x_j - b.$$

Should we wish to indicate whether a " $\geq$ " inequality such as

$$\sum_j a_j x_j \geq b$$

holds or not by means of an indicator variable δ, the required constraint can easily be obtained by transforming the above constraint into a " $\leq$ " form. The corresponding constraint (12) and (17) above are

$$\boxed{\sum_j a_j x_j + m\delta \geq m + b, \qquad (18)}$$

$$\sum_j a_j x_j - (M + \varepsilon)\delta \leq b - \varepsilon, \tag{19}$$

where m and M are again lower and upper bounds respectively on the expression

$$\sum_j a_j x_j - b.$$

Finally, to use an indicator variable δ for an " $=$ " constraint such as

$$\sum_j a_j x_j = b$$

is slightly more complicated. We can use $\delta = 1$ to indicate the " $\leq$ " and " $\geq$ " cases to hold simultaneously. This is done by stating both (12) and (18) together.

If $\delta = 0$, we want to force either " $\leq$ " or " $\geq$ " constraint to be broken. This may be done by expressing (17) and (19) with two indicator variables δ' and δ'' giving

$$\sum_j a_j x_j - (m - \varepsilon)\delta' \geq b + \varepsilon, \tag{20}$$

$$\sum_j a_j x_j - (M + \varepsilon)\delta'' \leq b - \varepsilon. \tag{21}$$

The indicator variable δ forces the required condition by the extra constraint

$$\delta' + \delta'' - \delta \leq 1. \tag{22}$$

Example 3. Use a $0-1$ variable δ to indicate whether or not the following constraint is satisfied:

$$2x_1 + 3x_2 \leq 1.$$

[$x_1, x_2 \geq 0$ and $x_1, x_2 \leq 1$, continuous.]

$$\delta = 1 \rightarrow 2x_1 + 3x_2 - 1 \leq 0,$$

$$2x_1 + 3x_2 - 1 \leq M(1 - \delta)$$

with $M = 4$.

$$\delta = 0 \rightarrow 2x_1 + 3x_2 - 1 \geq \varepsilon(= 0.01),$$

$$\delta = 0 \rightarrow 2x_1 + 3x_2 - 1 - \varepsilon \geq 0,$$

which can represented by

$$2x_1 + 3x_2 - 1 - \varepsilon \geq \bar{m}\delta,$$

where $\bar{m}$ is an lower bound of $2x_1 + 3x_2 - 1 - \varepsilon$. By simple calculation, $\bar{m} = -1 - \varepsilon = -1.01$. Thus, we obtain

$$2x_1 + 3x_2 - 1.01\delta \geq 1.01.$$