Nash Equilibrium Problems

Defeng Sun
Department of Mathematics
National University of Singapore
Republic of Singapore

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Lecture Notes for
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Given a closed set $K \subseteq \Re^n$ and a mapping $F : K \to \Re^n$, the *variational inequality*, denoted $\mathrm{VI}(K, F)$, is to find a vector $x \in K$ such that

$$\boxed{(y - x)^T F(x) \geq 0\,, \quad \forall\, y \in K\,.}$$

The set of solutions to this problem is denoted $\mathrm{SOL}(K, F)$.

Given a mapping $F : \Re^n_+ \to \Re^n$, the *nonlinear complementarity problem*, denoted NCP(F), is to find a vector $x \in \Re^n_+$ such that

$$x \geq 0\,, \quad F(x) \geq 0\,, \quad \text{and} \quad x^T F(x) = 0\,.$$

The relationship of VI($\Re^n_+, F$) and NCP(F):

Thm. x solves NCP(F) $\Longleftrightarrow x \in \mathrm{SOL}(\Re^n_+, F)$.

Nash Equilibrium Problems:

Suppose that there are N players in a noncooperative game. Each player has a cost function and strategy set that may depend on the other players' actions.

For simplicity, we assume that player i's strategy set is $K_i \subseteq \Re^{n_i}$ which is independent of the other players' actions. Player i's cost function $\theta(\mathrm{x})$ depends on all players' strategies, where

$$\mathrm{x} = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^N \end{bmatrix}, \quad x^i \in \Re^{n_i}.$$

Player i's strategy is to minimize his cost function for any given tuple

$$\tilde{\mathrm{x}}^i = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^{i-1} \\ x^{i+1} \\ \vdots \\ x^N \end{bmatrix}, \text{ deleting } x^i \text{ from x}$$

of other players' strategies. That is,

$$\begin{aligned} \min \quad & \theta_i(y^i, \tilde{\mathrm{x}}^i) \\ \text{s.t.} \quad & y^i \in K_i\,. \end{aligned}$$

Denote the solution set of the above optimization problem by $S_i(\tilde{\mathrm{x}}^i)$. This solution set depends on $\tilde{\mathrm{x}}^i$, the other players' strategies.

A Nash equilibrium is a tuple of strategies $\mathrm{x} = (x^i : i = 1, 2, \ldots N)$ with the property that

$$\boxed{\text{for each } i,\ x^i \in S_i(\tilde{\mathrm{x}}^i)\,.}$$

In words, a Nash equilibrium is a tuple of strategies, one for each player, such that no player can lower the cost by unilaterally deviating his action from his designated strategy.

We have the following theorem.

Theorem. Let each $K_i \subseteq \Re^{n_i}$ be a closed convex set. Suppose that for each fixed tuple $\tilde{\mathrm{x}}^i$, the function $\theta_i(y^i, \tilde{\mathrm{x}}^i)$ is convex and continuously differentiable in y^i. Then a tuple $\mathrm{x} \equiv (x^i \,:\, i = 1, 2, \ldots N)$ is a Nash equilibrium if and only if $x \in \mathrm{SOL}(K, F)$, where

$$K = \prod_{i=1}^{N} K_i \quad \text{and} \quad F(\mathrm{x}) = \left(\nabla_{x^i} \theta_i(\mathrm{x})\right)_{i=1}^{N} .$$

Proof: " $\Longrightarrow$ " By convexity, we know that x is a Nash equilibrium if and only if for each $i = 1, 2, \ldots, N$, x^i solves the following individual $\mathrm{VI}(K_i, \nabla_{x^i}\theta_i(\mathrm{x}))$:

$$(y^i - x^i)^T \nabla_{x^i}\theta_i(\mathrm{x}) \geq 0\,, \quad \forall\, y^i \in K_i\,.$$

$$\Downarrow$$

$$x \in \mathrm{SOL}(K, F)\,.$$

" $\Longleftarrow$ " Conversely, if $\mathrm{x} \equiv (x^i \,:\, i = 1, 2, \ldots N)$ solves the VI(K, F), then

$$(\mathrm{y} - \mathrm{x})^T F(\mathrm{x}) \geq 0\,, \quad \forall\, \mathrm{y} \in K\,.$$

In particular, by taking

$$\mathrm{y} = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^{i-1} \\ y^i \\ x^{i+1} \\ \vdots \\ x^N \end{bmatrix}, \text{ replacing } x^i \text{ by } y^i \,,$$

we know that the above VI becomes

$$(y^i - x^i)^T \nabla_{x^i} \theta_i(\mathrm{x}) \geq 0 \,, \quad \forall\, y^i \in K_i \,.$$

Hence, by convexity, x^i solves

$$\begin{array}{ll} \min & \theta(y^i, \tilde{\mathrm{x}}^i) \\ \text{s.t.} & y^i \in K_i \,. \end{array}$$

Sometimes, the Nash equilibrium problem is called an "N-person nonzero-sum game".

An Example in Nash Equilibrium

Define $L : \Re^n \times \Re^m \to \Re$ as follows

$$\boxed{L(x,y) = p^T x + q^T y + \frac{1}{2}x^T P x + x^T R y - \frac{1}{2} y^T Q y\,, \quad x \in \Re^n,\ y \in \Re^m\,,}$$

where $p \in \Re^n$, $q \in \Re^m$, $P = P^T \succ 0$, $P \in \Re^{n\times n}$, $Q = Q^T \succ 0$, $Q \in \Re^{m\times m}$, and $R \in \Re^{n\times m}$. Let X be a closed convex set in $\Re^n$ and Y be a closed convex set in $\Re^m$. For each $x \in \Re^n$, let

$$\varphi(x) = \max_{v \in \Re^m} L(x,v)$$

and for each $y \in \Re^m$, let

$$\phi(y) = \min_{u \in \Re^n} L(u,y)\,.$$

Suppose that there are two players in a noncooperative game. Player 1's strategy, for each fixed but arbitrary player 2's strategy $y \in Y$, is to

$$\begin{array}{ll} \min & \theta_1(x,y) \\ \text{s.t.} & x \in X\,, \end{array}$$

where $\theta_1(x,y) = \varphi(x) + x^T S_1 y$ and $S_1 \in \Re^{n\times m}$. Player 2's strategy, for each fixed but arbitrary player 1's strategy $x \in X$, is to

$$\begin{array}{ll} \min & \theta_2(x,y) \\ \text{s.t.} & y \in Y\,, \end{array}$$

where $\theta_2(x,y) = -\phi(y) + x^T S_2 y$ and $S_2 \in \Re^{n\times m}$.

(i) Give explicit formulas of $\varphi(x)$ and $\phi(y)$ for each $x \in \Re^n$ and $y \in \Re^m$.

(ii) Show that $\theta_1(x, y)$ is a convex function in x and $\theta_2(x, y)$ is a convex function in y.

(iii) Model the problem of finding a Nash equilibrium of the above two players' noncooperative game as a variational inequality problem.