5 The Potential Reduction Algorithm

In this chapter, we introduce another interior point algorithm – potential reduction algorithm to solve the linear programming problem

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

and its dual

$$\begin{aligned} \max \quad & b^T p \\ \text{s.t.} \quad & A^T p + s = c \\ & s \geq 0\,, \end{aligned}$$

where A is an $m \times n$ matrix. This algorithm only requires a polynomial number of iterations.

We need the following assumption in this chapter.

Assumption 5.1 *The matrix A has linearly independent rows and there exist $x > 0$ and (p, s) with $s > 0$, which are feasible for the primal and dual problem, respectively. That is, (x, p, s) satisfies*

$$Ax = b,\ x > 0,\ A^T p + s = 0,\ s > 0\,.$$

The potential reduction algorithm makes progress towards optimality by decreasing the objective function value while staying away from the boundary of the feasible region through so called a **potential function**. For a given constant q larger than n, the potential reduction function $G(x, s)$ is defined by

$$G(x, s) = q \log s^T x - \sum_{j=1}^{n} \log x_j - \sum_{j=1}^{n} \log s_j\,.$$

If x and (p, s) are primal and dual feasible solutions, respectively, then

$$c^T x - b^T p = (A^T p + s)^T x - x^T A^T p = s^T x\,.$$

This implies that the first term of $G(x, s)$ is a measure of the duality gap. The second and third terms penalize proximity to the boundary of the feasible sets for the primal and dual, respectively.

In the sequel, for any $x > 0$ we denote

$$X = \operatorname{diag}(x_1, \ldots, x_n)$$

and $e = (1, \ldots, 1)^T$.

Lemma 5.1 *The following function*

$$f(x, s) := n \log s^T x - \sum_{j=1}^{n} \log x_j - \sum_{j=1}^{n} \log s_j\,, \quad (x, s) > 0$$

attains its minimum $n \log n$ at $(x, s) > 0$ with

$$x_j s_j = s^T x / n\,,\ j = 1, \ldots, n\,.$$

Proof. Define

$$g(x,s) \;=\; e^{f(x,s)/n} \quad (x,s) \;>\; 0\,.$$

Then,

$$\begin{aligned} g(x,s) \;&=\; e^{[\log x^T s \,-\, (\sum_{j=1}^n \log x_j s_j)/n]} \\ &=\; \frac{e^{[\log x^T s]}}{e^{[(\sum_{j=1}^n \log x_j s_j)/n]}} \\ &=\; \frac{x_1 s_1 \,+\, \ldots \,+\, x_n s_n}{(x_1 s_1 \ldots x_n s_n)^{1/n}} \\ &\geq\; n\,, \end{aligned}$$

where the minimum value is attained at $x_1 s_1 = \ldots = x_n s_n$. Therefore, $f(x,s)$ attains its minimum $n \log n$ at (x,s) with

$$x_j s_j \;=\; s^T x/n\,,\; j \;=\; 1,\ldots,n.$$

QED

The next theorem shows that if we can decrease the potential function $G(x,s)$ at each step by a certain amount, we can get an ε-optimal solution to the linear programming problem after a small number of iterations.

Theorem 5.1 *Let $x^0 > 0$ and (p^0, s^0) with $s^0 > 0$, be feasible solutions to the primal and dual problem, respectively. Let $\varepsilon > 0$ be the optimality tolerance. Any algorithm that maintains primal and dual feasibility and reduces $G(x,s)$ by an amount greater than or equal to $\delta > 0$ at each iteration, finds a solution to the primal and dual problems with duality gap*

$$(s^K)^T x^K \;\leq\; \varepsilon\,,$$

after

$$K \;=\; \left\lceil \frac{G(x^0,s^0) \,+\, (q-n)\log(1/\varepsilon) \,-\, n\log n}{\delta} \right\rceil$$

iterations.

Proof. By Lemma 5.1, we have

$$\begin{aligned} G(x,s) \;&=\; q\log x^T s \,-\, \sum_{j=1}^n \log x_j \,-\, \sum_{j=1}^n \log s_j \\ &=\; n\log x^T s \,-\, \sum_{j=1}^n \log x_j \,-\, \sum_{j=1}^n \log s_j \,+\, (q-n)\log x^T s \\ &\geq\; n\log n \,+\, (q-n)\log x^T s\,. \end{aligned} \tag{5.1}$$

Fix some $\delta > 0$ and suppose that we have an algorithm with the property

$$G(x^{k+1}, s^{k+1}) \,-\, G(x^k, s^k) \;\leq\; -\delta\,, \quad \forall\, k\,.$$

After K steps we have

$$G(x^K, s^K) \; - \; G(x^0, s^0) \; \leq \; -K\delta \, .$$

For the value of K stated in the theorem, we obtain

$$G(x^K, s^K) \; \leq \; -(q-n) \log \frac{1}{\varepsilon} \; + \; n \log n \, .$$

Using the definition of $G(x, s)$ and equation (5.1), we obtain

$$G(x^K, s^K) \; \geq \; n \log n \; + \; (q-n) \log \, (s^K)^T x^K \, .$$

Therfore,

$$(s^K)^T x^K \; \leq \varepsilon \, ,$$

i.e., we can bring the duality gap below the desired tolerance ε with K iterations. QED

The **potential reduction algorithm** is based on Theorem 5.1. We start with a primal feasible solution $x > 0$ and a dual feasible solution with $s > 0$ by trying to find a direction d such that

$$G(x + d, s) \; < \; G(x, s) \, .$$

The direction d should satisfy

$$Ad \; = \; 0 \, , \qquad \| \, X^{-1} d \, \| \; \leq \; \beta \; < \; 1 \, ,$$

so that the new point $x + d$ is feasible, as we have shown in Lemma 4.1. The problem of minimizing $G(x+d, s)$ subject to the above constraints is a difficult nonlinear optimization problem. However, we can approximate the nonlinear potential function $G(x + d, s)$ by its first order Taylor series expansion in d, and solve the following problem:

$$\begin{aligned} \min \quad & \nabla_x G(x, s)^T d \\ \text{s.t.} \quad & Ad \; = \; 0 \\ & \| \, X^{-1} d \, \| \; \leq \; \beta \, , \end{aligned}$$

for some $\beta < 1$. Note that the above problem is the same as the one encountered in the affine scaling algorithm, except the objective function vector is $\hat{c} = \nabla G(x, s)$ instead of c. In particular, the ith component of the cost vector is

$$\hat{c}_i \; = \; \frac{\partial G(x, s)}{\partial x_i} \; = \; \frac{q s_i}{s^T x} \; - \; \frac{1}{x_i} \, .$$

Applying Lemma 4.2 with $Y = X$ and $c = \hat{c}$, we obtain that the optimal direction is

$$d^* \; = \; -\beta X \frac{u}{\|u\|} \, , \tag{5.2}$$

where

$$u \; = \; X \left(\hat{c} \; - \; A^T (AX^2 A^T)^{-1} AX^2 \hat{c} \right) \, .$$

Since

$$X\hat{c} \,=\, \frac{q}{s^T x} X s \,-\, e\,,$$

we obtain

$$u \,=\, \left(I \,-\, XA^T(AX^2A^T)^{-1}AX\right)\left(\frac{q}{s^T x} X s \,-\, e\right)\,.$$

Moreover, $G(x, s)$ decreases by $\beta\|u\| + O(\beta^2)$, where the first term comes from Lemma 4.2 and the second term is due to the omitted higher order terms in the Taylor series expansion of $G(x, s)$.

The potential reduction algorithm uses the following inputs:

(a) the data of the problem (A, b, c); the matrix A is assumed to have full row rank;

(b) an initial primal and dual feasible solutions $x^0 > 0$, $s^0 > 0$, p^0;

(c) the optimality tolerance $\varepsilon > 0$;

(d) the parameter $0 < \beta < 1, 0 < \gamma < 1, q > n$.

The Potential Reduction Algorithm

1. (Initialization) Start with some feasible solution $x^0 > 0$, $s^0 > 0$, p^0; let $k = 0$.

2. (Optimality check) If $(s_k)^T x^k < \varepsilon$ stop; else go to Step 3.

3. (Computation of update direction) Let

$$X_k \,=\, \mathrm{diag}(x_1^k, \ldots, x_n^k)\,,$$

$$\overline{A}^k \,=\, (AX_k)^T(\,AX_k^2A^T\,)^{-1}AX_k\,,$$

$$u^k \,=\, (I \,-\, \overline{A}^k)\left(\frac{q}{(s^k)^T x^k} X_k s^k \,-\, e\right)\,,$$

$$d^k \,=\, -\beta X_k u^k / \|u^k\|\,.$$

4. (Primal step) If $\|u^k\| \geq \gamma$, then let

$$x^{k+1} \,=\, x^k \,+\, d^k\,,$$

$$s^{k+1} \,=\, s^k\,,$$

$$p^{k+1} \,=\, p^k\,.$$

5. (Primal step) If $\|u^k\| < \gamma$, then let

$$
\begin{aligned}
x^{k+1} &= x^k\,, \\
s^{k+1} &= \frac{(s^k)^T x^k}{q}(X_k)^{-1}(u^k + e)\,, \\
p^{k+1} &= p^k + (AX_k^2A^T)^{-1}AX_k\left(X_k s^k - \frac{(s^k)^T x^k}{q} e\right).
\end{aligned}
$$

6. Let $k := k+1$ and go to step 2.

It is easily shown that for every k, the vector x^k and (p^k, s^k) are primal and dual feasible solutions, respectively. If $\|u^k\| \geq \gamma$ we say that the algorithm performs a **primal step**, while if $\|u^k\| < \gamma$, we say that the algorithm performs a **dual step**.

Theorem 5.2 *The potential reduction algorithm with $\beta < 1$ and $\gamma < 1$, has the following properties.*

(a) If $\|u^k\| \geq \gamma$ (primal step), then

$$G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \leq -\beta\gamma + \frac{\beta^2}{2(1-\beta)}\,.$$

(b) If $\|u^k\| < \gamma$ (dual step), then

$$G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \leq -(q-n) + n\log\frac{q}{n} + \frac{\gamma^2}{2(1-\gamma)}\,.$$

(c) If $q = n + \sqrt{n}$, $\beta \approx 0.285$ and $\gamma \approx 0.479$, then the potential reduction algorithm reduces $G(x,s)$ by at least $\delta = 0.079$ at each iteration.

Proof. Let x and (p, s) be the current solutions to the primal and dual problems respectively. If the potential reduction algorithm takes a primal step, the new point is $(x+d, s)$, where

$$d = -\beta X u / \|u\|\,.$$

Then,

$$
\begin{aligned}
& G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \\
&= q\log\left(1 + \frac{s^T d}{s^T x}\right) - \sum_{j=1}^{n} \log\left(1 + \frac{d_j}{x_j}\right).
\end{aligned}
\tag{5.3}
$$

Since $1 + y \leq e^y$ for all y, we obtain

$$\log(1+y) \leq y\,, \quad \forall y > -1\,. \tag{5.4}$$

Moreover, from the Taylor series expansion of $\log(1 + y)$, we obtain for $|y| \le \beta < 1$ that

$$\begin{aligned}
\log(1 + y) &= y - \frac{y^2}{2} + \frac{y^3}{3} - \cdots \\
&\ge y - \frac{|y|^2}{2} - \frac{|y|^3}{3} - \cdots \\
&\ge y - \frac{|y|^2}{2}\Big(1 + |y| + |y|^2 + \cdots\Big) \\
&= y - \frac{|y|^2}{2(1 - |y|)} \\
&\ge y - \frac{|y|^2}{2(1 - \beta)} .
\end{aligned} \tag{5.5}$$

Since $|d_j/x_j| = |\beta u_j|/\|u\| \le \beta < 1$, we applying inequalities (5.4) and (5.5) to (5.3), and obtain

$$\begin{aligned}
& G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \\
\le\ & q\frac{s^T d}{s^T x} - \sum_{j=1}^{n}\left(\frac{d_j}{x_j} - \frac{d_j^2}{2(1 - \beta)x_j^2}\right) \\
=\ & \left(q\frac{s}{s^T x} - X^{-1}e\right)^T d + \frac{\|X^{-1}d\|^2}{2(1 - \beta)} \\
=\ & \left(q\frac{s}{s^T x} - X^{-1}e\right)^T d + \frac{\beta^2}{2(1 - \beta)} \\
=\ & \hat{c}^T d + \frac{\beta^2}{2(1 - \beta)} \\
=\ & -\beta\|u\| + \frac{\beta^2}{2(1 - \beta)} .
\end{aligned}$$

Therefore, if $\|u\| \ge \gamma$,

$$G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \le -\beta\gamma + \frac{\beta^2}{2(1 - \beta)} .$$

We next address the case $\|u\| < \gamma$ (dual step). Since

$$u = (I - \overline{A})\left(\frac{q}{s^T x}Xs - e\right) ,$$

we obtain by rearranging,

$$\overline{A}\left(\frac{q}{s^T x}Xs - e\right) + u + e - \frac{q}{s^T x}Xs = 0 .$$

This leads to

$$A^T(AX^2A^T)^{-1}AX\left(Xs - \frac{s^T x}{q}e\right) + \frac{s^T x}{q}X^{-1}(u + e) - s = 0 .$$

If we let

$$\bar{s} = \frac{s^T x}{q}X^{-1}(u + e) ,$$

$$\bar{p} = p + (AX^2A^T)^{-1}AX\left(Xs - \frac{s^Tx}{q}e\right),$$

we obtain

$$A^T(\bar{p} - P) + \bar{s} - s = 0,$$

and thus

$$A^T\bar{p} + \bar{s} = A^Tp + s = c.$$

Notice that since $\|u\| < \gamma < 1$, we have $u + e > 0$, and therfore, $\bar{s} > 0$. Hence, the solution $(\bar{p}, \bar{s})$ is dual feasible.

The difference in the potential function after a dual step becomes

$$\begin{aligned} &G(x, \bar{s}) - G(x, s) \\ &= q\log\left(\frac{\bar{s}^Tx}{s^Tx}\right) - \sum_{j=1}^{n}\log\bar{s}_j + \sum_{j=1}^{n}\log s_j\,. \end{aligned} \tag{5.6}$$

We next bound the various terms appearing in (5.6). Note that $x^TX^{-1} = e^T$ and therefore,

$$\bar{s}^Tx = x^T\bar{s}^T = \frac{s^Tx}{q}x^TX^{-1}(u + e) = \frac{s^Tx}{q}(e^Tu + n)\,. \tag{5.7}$$

Moreover,

$$\begin{aligned} &\sum_{j=1}^{n}\log\bar{s}_j \\ &= \sum_{j=1}^{n}\log\left(\frac{(s^Tx)(1 + u_j)}{qx_j}\right) \\ &= n\log\frac{s^Tx}{q} + \sum_{j=1}^{n}\log(1 + u_j) - \sum_{j=1}^{n}\log x_j \\ &\geq n\log\frac{s^Tx}{q} + \sum_{j=1}^{n}\left(u_j - \frac{u_j^2}{2(1 - \gamma)}\right) - \sum_{j=1}^{n}\log x_j \\ &\geq n\log\frac{s^Tx}{q} + e^Tu - \sum_{j=1}^{n}\log x_j - \frac{\gamma^2}{2(1 - \gamma)}\,, \end{aligned} \tag{5.8}$$

where we have used equation (5.5) and the fact that $|u_j| \leq \|u\| \leq \gamma$. Substituting equations (5.7) and (5.8) into equation (5.6), we obtain

$$\begin{aligned} &G(x, \bar{s}) - G(x, s) \\ &\leq q\log\frac{e^Tu + n}{q} + n\log q - e^Tu + \frac{\gamma^2}{2(1 - \gamma)} \\ &\quad - \left(n\log(s^Tx) - \sum_{j=1}^{n}\log x_j - \sum_{j=1}^{n}\log s_j\right). \end{aligned}$$

Using Lemma 5.1 to bound the last part of the above expression, we obtain

$$
\begin{aligned}
& G(x,\bar{s}) - G(x,s) \\
\le\ & q \log\left(1 - \frac{q - n - e^T u}{q}\right) + n \log\frac{q}{n} - e^T u + \frac{\gamma^2}{2(1-\gamma)} \\
\le\ & q\left(-\frac{q - n - e^T u}{q}\right) + n \log\frac{q}{n} - e^T u + \frac{\gamma^2}{2(1-\gamma)} \\
=\ & -(q-n) + n\log\frac{q}{n} + \frac{\gamma^2}{2(1-\gamma)}\,.
\end{aligned}
$$

For $q = n + \sqrt{n}$, the potential change after a dual step is less than or equal to

$$
-\sqrt{n} + n\log\left(1 + \frac{1}{\sqrt{n}}\right) + \frac{\gamma^2}{2(1-\gamma)} \le -0.3 + \frac{\gamma^2}{2(1-\gamma)}\,.
$$

Using $\beta = 0.28537$ and $\gamma = 0.479056$, and substituting in the bounds we have obtained, we find that both the primal and dual step reduce $G(x,s)$ by at least $\delta = 0.079$ at each iteration. QED

Initialization: Next, we show how to select an initial solution that can be used to start the algorithm. Consider the following pair of artificial primal and dual problems:

$$
\begin{array}{llllll}
\min & c^T x & + & M_1 x_{n+1} & & \\
\text{s.t.} & Ax & + & (b - Ae)x_{n+1} & & = b \\
& (e-c)^T x & & & + x_{n+2} & = M_2 \\
& x_1, \ldots, x_{n+2} \ge 0\,, & & & &
\end{array}
$$

and

$$
\begin{array}{llllll}
\max & b^T p & + & M_2 p_{m+1} & & \\
\text{s.t.} & A^T p & + & (e-c)p_{m+1} & + s & = c \\
& (b - Ae)^T p & & & + s_{n+1} & = M_1 \\
& & & p_{m+1} & + s_{n+2} & = 0 \\
& s_1, \ldots, s_{n+2} \ge 0\,. & & & &
\end{array}
$$

Here, x_{n+1}, x_{n+2} are artificial primal variables, $p_{m+1}, s_{n+1}, s_{n+2}$ are artificial dual variables, and M_1, M_2 are large positive numbers to be specified later. The coefficient M_2 must satisfy

$$
M_2 > (e - c)^T e\,.
$$

The vectors

$$
(x^0, x^0_{n+1}, x^0_{n+2}) = (e, 1, M_2 - (e-c)^T c)\,,
$$

$$(p^0, p^0_{m+1}, s^0, s^0_{n+1}, s^0_{n+2}) \;=\; (0, -1, e, M_1, 1)\,,$$

are feasible solutions to be the artificial primal and dual problems, respectively, and can be used to start the potential reduction algorithm. The relation between the artificial and the original problems is described next.

Theorem 5.3 *Let x^* and (p^*, s^*) be optimal solutions to the original primal and dual problems, respectively, whose existence is assumed. If*

$$M_1 \;\geq\; \max\{(b \,-\, Ae)^T p^*\,, 0\} \,+\, 1$$

and

$$M_2 \geq \max\{((e \,-\, c)^T x^*\,, (e \,-\, c)^T e\,, 0\} \,+\, 1\,,$$

then the following hold:

(a) A feasible solution $(\bar{x}, \bar{x}_{n+1}, \bar{x}_{n+2})$ to the artificial primal problem is optimal if and only if $\bar{x}$ is an optimal solution to the original primal problem and $\bar{x}_{n+1} = 0$.

(b) A feasible solution $(\bar{p}, \bar{p}_{m+1}, \bar{s}, \bar{s}_{n+1}, \bar{s}_{n+2})$ to the artificial dual problem is optimal if and only if $(\bar{p}, \bar{s})$ is an optimal solution to the original dual problem and $\bar{p}_{m+1} = 0$.

Proof. (a) Let $(\bar{x}, \bar{x}_{n+1}, \bar{x}_{n+2})$ be an optimal solution to the artificial primal problem. We will first show that $\bar{x}_{n+1} = 0$. Assume on the contrary that $\bar{x}_{n+1} > 0$.

Since x^* is feasible for the original problem, we can define $x^*_{n+1} = 0$ and $x^*_{n+2} = M_2 - (e-c)^T x^*$, so that the solution $(x^*, x^*_{n+1}, x^*_{n+2})$ is feasible for the artificial primal problem. Then

$$c^T x^* \,+\, M_1 x^*_{n+1} \;=\; (p^*)^T b \;=\; (p^*)^T (A\bar{x} \,+\, (b \,-\, Ae)\bar{x}_{n+1})\,.$$

Since $A^T p^* + s^* = c$, $\bar{x}_{n+1} > 0$, and $M_1 > (b - Ae)^T p^*$, we obtain

$$c^T x^* \,+\, M_1 x^*_{n+1} \;<\; (c \,-\, s^*)^T \bar{x} \,+\, M_1 \bar{x}_{n+1} \;\leq\; c^T \bar{x} \,+\, M_1 \bar{x}_{n+1}\,,$$

because $(s^*)^T \bar{x} \geq 0$. This contradicts the optimality of $(x^*, x^*_{n+1}, x^*_{n+2})$. Therfore, $(s^*)^T \bar{x} \geq 0$. In addition, the previous inequality shows that the solution $(x^*, x^*_{n+1}, x^*_{n+2})$ is optimal for the artificial primal problem and the optimal cost of the artificial primal problem is $c^T \bar{x} = c^T x^*$. Since $\bar{x}$ satisfies all the constraints of the original primal problem, it must be an optimal solution.

Conversely, let x^* be an optimal solution to the original primal problem. Then $(\bar{x}, \bar{x}_{n+1}, \bar{x}_{n+2})$ with $\bar{x} = x^*$, $\bar{x}_{n+1} = 0$, $\bar{x}_{n+2} = M_2 - (e-c)^T x^*$, is feasible for the artificial primal problem. The objective value $c^T \bar{x} + M_1 \bar{x}_{n+1}$ coincides with the optimal cost $c^T x^* + M_1 x^*_{n+1}$, and therefore we have an optimal solution to the artificial optimal problem.

(b) Let $(\bar{p}, \bar{p}_{m+1}, \bar{s}, \bar{s}_{n+1}, \bar{s}_{n+2})$ be an optimal solution to the artificial dual problem. We will show that $\bar{p}_{m+1} = 0$. Assume on the contrary that $\bar{p}_{m+1} < 0$.

Since (p^*, s^*) is feasible for the original problem, we can define $p^*_{m+1} = 0$ and $s^*_{n+1} = M_1 - p^T(b - Ae)$ and $s^*_{n+2} = 0$, so that the solution $(p^*, p^*_{m+1}, s^*, s^*_{n+1}, s^*_{n+2})$ is feasible for the artificial dual problem. Then

$$b^T p^* + M_2 p^*_{m+1} = (x^*)^T c = (x^*)^T (A^T \bar{p} + (e - c)\bar{p}_{m+1} + \bar{s}) .$$

Since $Ax^* = b$, $\bar{p}_{m+1} < 0$, and $M_2 > (e - c)^T x^*$, we obtain

$$b^T p^* + M_2 p^*_{m+1} > b^T \bar{p} + M_2 \bar{p}_{m+1} + (x^*)^T \bar{s} \geq b^T \bar{p} + M_2 \bar{p}_{m+1} ,$$

because $(x^*)^T \bar{s} > 0$. This contradicts the optimality of $(\bar{p}, \bar{p}_{m+1}, \bar{s}, \bar{s}_{n+1}, \bar{s}_{n+2})$. Therefore, $\bar{p}_{m+1} = 0$. In addition, the previous inequality shows that the solution $(p^*, p^*_{m+1}, s^*, s^*_{n+1}, s^*_{n+2})$ is optimal for the artificial dual problem and the optimal cost of the artificial dual problem is $b^T \bar{p} = b^T p^*$. Since $(\bar{p}, \bar{s})$ satisfies all the constraints of the original dual problem, it must be an optimal solution.

Conversely, let (p^*, s^*) be an optimal solution to the original dual problem. Then $(\bar{p}, \bar{p}_{m+1}, \bar{s}, \bar{s}_{n+1}, \bar{s}_{n+2})$ with $\bar{p} = p^*$, $\bar{p}_{m+1} = 0$, $\bar{s} = s^*$, $\bar{s}_{n+1} = M_1 - (b - Ae)^T p^*$, $\bar{s}_{n+2} = 0$, is feasible for the artificial dual problem. The objective value $b^T \bar{p} + M_2 \bar{p}_{m+1}$ coincides with the optimal cost $b^T p^* + M_2 p^*_{m+1}$, and therefore we have an optimal solution to the artificial dual problem. QED

On the complexity of the algorithm: Suppose that A, b, and c have integer entries whose magnitude is bounded by U. If x^* and (p^*, s^*) are optimal basic feasible solutions to the primal and dual problem, respectively, the components of x^* and p^* are bounded by $(nU)^n$. From Theorem 5.3, it follows that we can select some M_1, M_2 of the order of $(nU)^{n+2}$, solve the artificial problems, and obtain an optimal solution to the original problems. Furthermore, it is easily checked that the initial potential function satisfies

$$G(x^0, s^0) = O(qn \log(nU)) .$$

Applying Theorems 5.1 and 5.2 with $q = n + \sqrt{n}$ and $G(x^0, s^0)$ given above, we conclude that the potential reduction algorithm finds solutions x^K, s^K, with duality gap

$$(s^K)^T x^K \leq \varepsilon ,$$

after

$$K = O\left(\sqrt{n} \log \frac{1}{\varepsilon} + n^2 \log(nU)\right)$$

iterations. Moreover, the dependence of this bound on ε is tight, since there exist examples that require at least

$$\Omega\left(\sqrt{n} \log \frac{1}{\varepsilon}\right)$$

iterations.

The work per iteration involves the matrix inversion $(AX_k^2 A^T)^{-1}$ and two matrix multiplications to calculate $\overline{A}^k$. Therfore, each iteration requires $O(nm^2 + m^3)$ arithmetic operations. Since $m \leq n$, each iteration requires at most $O(n^3)$ arithmetic operations. As a result, the potential reduction algorithm finds an ε-optimal solution using

$$O\left(n^{3.5}\log\frac{1}{\varepsilon} + n^5\log(nU)\right)$$

arithmetic operations. Notice that this bound grows only polynomially in $n, \log U$, and $\log\varepsilon$. If ε is taken to be sufficiently small, an exact solution can be found by rounding. This results in a polynomial time (in n and $\log U$) algorithm. [The potential reduction algorithm is due to Ye (1991). The algorithm presented here is du to Freund (1991), and is a simplication of Ye's method.]