NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
Semester 1 (2003/2004) MA4253 Mathematical Programming Tutorial 1

Solution to selected questions

Q1. Let y, z be any two points of S. Then for any $\lambda \in [0,1]$, we have

$$\begin{aligned} &\|A(\lambda y + (1-\lambda)z) - b\| \\ &= \|\lambda(Ay - b) + (1-\lambda)(Az - b)\| \\ &\leq \lambda\|Ay - b\| + (1-\lambda)\|Az - b\| \\ &\leq \lambda + (1-\lambda) = 1\,. \end{aligned}$$

Therefore, $\lambda y + (1-\lambda)z \in S$ for all $\lambda \in [0,1]$. This shows that S is a convex set.

Q2. (a) S is a polyhedral set. Assume that a_1 and a_2 are linearly independent (it is much simpler if they are linearly dependent).

Case 1): $n = 1$. Trivial.

Case 2): $n = 2$. The set S is the enclosed part of the following four line segments

$$[a_1 + a_2, a_1 - a_2]\,,\ [a_2 + a_1, a_2 - a_1]\,,\ [-a_1 + a_2, -a_1 - a_2] \text{ and } [-a_2 - a_1, -a_2 + a_1]\,.$$

We need to find four hyperplanes containing these four line segments respectively. We take $L_1 := [a_1 + a_2, a_1 - a_2]$ as an example. We choose $u_1 = a_1 + \theta_1 a_2$ such that u_1 is on L_1 and $u_1^T a_2 = 0$, where θ_1 is a scalar defined by

$$\theta_1 = -\frac{a_1^T a_2}{\|a_2\|^2}.$$

Then, u_1 is on L_1 and is perpendicular to L_1. Define the following hyperplane

$$h_1 = \{x \in \Re^n \,|\, u_1^T x = \|u_1\|^2 = \|a_1\|^2 - \frac{(a_1^T a_2)^2}{\|a_2\|^2}\}.$$

Obviously, L_1 is contained in h_1.

Similarly, let

$$u_2 = a_2 - \frac{a_2^T a_1}{\|a_1\|^2} a_1\,,\ u_3 = -a_1 + \frac{a_1^T a_2}{\|a_2\|^2} a_2\,,\ u_4 = -a_2 + \frac{a_2^T a_1}{\|a_1\|^2} a_1$$

and define three more hyperplanes

$$h_i = \{x \in \Re^n \,|\, u_i^T x = \|u_i\|^2\},\ i = 2, 3, 4,$$

respectively. The set S is the intersection of the four halfspaces $\{x \in \Re^n \,|\, u_i^T x \le \|u_i\|^2\}$, $i = 1, 2, 3, 4$, i.e.,

$$S = \{x \in \Re^n \,|\, u_i^T x \le \|u_i\|^2 \,,\, i = 1, 2, 3, 4\,\}\,.$$

Case 3): $n \ge 3$. This case is similar to Case 2) except that S must stay in the intersetion of hyperplanes

$$h_d = \{x \in \Re^n \,|\, d^T x = 0\,\}$$

for any nonzero vector $d \in \Re^n$ such that $d^T a_1 = d^T a_2 = 0$ (for $n = 2$, no such d exists). Let

$$\mathcal{N} = \{d \in \Re^n \,|\, d^T a_1 = d^T a_2 = 0\,\}.$$

Let $d_1, \ldots, d_{n-2}$ be $n-2$ linearly independent vectors in $\mathcal{N}$. Then

$$\bigcap_{d \in \mathcal{N}} h_d = \bigcap_{i=1,\ldots,n-2} h_{d_i}\,.$$

Therefore,

$$S = \{x \in \Re^n \,|\, d_j^T x = 0\,,\, j = 1, \ldots, n-2\,,\, u_i^T x \le \|u_i\|^2\,,\, i = 1, 2, 3, 4\,\}\,,$$

where h_i, $i = 1, 2, 3, 4$ are defined as in Case 2).

[**Alternatively**, by assuming that a_1 and a_2 are linearly independent you may work in this way: let

$$B = [a_1\ a_2\ d_1\ \ldots\ d_{n-2}]\,.$$

Then $B \in \Re^{n\times n}$ is nonsingular (check it!). By letting $C = B^{-1}$, we have

$$\begin{aligned}
S &= \{x \in \Re^n \,|\, x = a_1 \times y_1 + a_2 \times y_2 + d_1 \times 0 + \ldots + d_{n-2} \times 0\,, -1 \le y_1, y_2 \le 1\,\} \\
&= \{x \in \Re^n \,|\, x = B[y_1, y_2, 0, \ldots, 0]^T\,, -1 \le y_1, y_2 \le 1\,\} \\
&= \{x \in \Re^n \,|\, Cx = [y_1, y_2, 0, \ldots, 0]^T\,, -1 \le y_1, y_2 \le 1\} \\
&= \{x \in \Re^n \,|\, -1 \le c_1^T x \le 1\,,\, -1 \le c_2^T x \le 1\,,\, c_i^T x = 0\,,\, i = 3, \ldots, n\}\,,
\end{aligned}$$

where c_i^T is the ith row of C, $i = 1, \ldots, n$.]

(b) Obviously, S is a polyhedral set.

(c) S is not a polyhedral set except $n = 1$. For $n = 1$, $S = \{x \in \Re \,|\, 0 \le x \le 1\}$. For $n \ge 2$, we have

$$\begin{aligned}
S &= \{x \in \Re^n \,|\, x^T y \le 1\ \forall\ y \text{ with } \|y\| = 1\} \cap \{x \in \Re^n \,|\, x \ge 0\} \\
&= \{x \in \Re^n \,|\, \|x\| \le 1\} \cap \{x \in \Re^n \,|\, x \ge 0\} \\
&= \{x \in \Re^n \,|\, \|x\| \le 1\,,\, x \ge 0\}\,.
\end{aligned}$$

(d) S is not a polyhedral set because $S = \emptyset$. For $n \geq 2$, we have

$$\begin{aligned} S &= \{x \in \Re^n \,|\, x^T y \geq 1 \ \forall\ y \text{ with } \|y\| = 1\} \cap \{x \in \Re^n \,|\, x \geq 0\} \\ &= \emptyset \cap \{x \in \Re^n \,|\, x \geq 0\} = \emptyset\,. \end{aligned}$$

(e) S is a polyhedral set and

$$\begin{aligned} S &= \{x \in \Re^n \,|\, x \geq 0, x^T y \leq 1 \ \forall\ y \text{ with } \textstyle\sum_i |y_i| = 1\} \\ &= \{x \in \Re^n \,|\, x^T y \leq 1 \ \forall\ y \text{ with } \textstyle\sum_i |y_i| = 1\} \cap \{x \in \Re^n \,|\, x \geq 0\,\} \\ &= \{x \in \Re^n \,|\, -1 \leq x_i \ \leq\ 1, i =\ 1, \ldots, n\} \cap \{x \in \Re^n \,|\, x \geq 0\,\} \\ &= \{x \in \Re^n \,|\, 0 \leq e_i^T x \ \leq\ 1\,,\ i =\ 1, \ldots, n\}\,. \end{aligned}$$

(f) $S = \emptyset$, which is not a polyhedral set.

Q3. There are five inequality constraints. For $x \in \Re^3$ to be an extreme point we need x to satisfy all these five constraints and there are three linearly independent active constraints. These points are

$$(5/2, 0, 0)^T, \quad (0, 0, 8/5)^T, \quad (23/14, 0, 2/7)^T, \quad (19/8, 1/4, 0)^T\,.$$

Q4.

$$(1/2, 0, 1/2)^T, \quad (0, 0, 1)^T\,.$$

Q5. Suppose tha x^* is a basic feasble solution (bfs) of $P = \{x \in \Re^n \mid Ax = b,\ x \geq 0\}$. Then, by the definition of bfs, $x^* \in P$ and there are n linearly independent active constraints. These n linearly independent active constraints come from two parts. The first part has $p(\leq m)$ active constraints coming from $Ax^* = b$ and the rest $q(= n - p)$ active constraints coming from those constraints $x \geq 0$ such that $x_i^* = e_i^T x^* = 0$, $i = 1, \ldots, n$. Therefore, we obtain

$$\begin{aligned} a_{p(j)}^T x^* &= b_{p(j)},\ j\ =\ 1, \ldots, p\,, \\ e_{q(j)}^T x^* &= 0,\ j\ =\ 1, \ldots, q\,, \end{aligned}$$

and that the matrix

$$M = \begin{bmatrix} a_{p(1)}^T \\ \vdots \\ a_{p(p)}^T \\ e_{q(1)}^T \\ \vdots \\ e_{q(q)}^T \end{bmatrix}$$

is nonsingular.

Next, we shall show that $p = m$, i.e, we can take all equations from $Ax^* = b$. Suppose on the contrary that that $p < m$ and that there is some $i \notin \{p(1), \ldots, p(p)\}$ $(1 \leq i \leq m)$. Then, since

$$\mathrm{span}\{(a_{p(j)}^T)_{j=1}^p,\ (e_{q(j)}^T)_{j=1}^q\} \;=\; \Re^n,$$

a_i^T is a linear combination of $\{(a_{p(j)}^T)_{j=1}^p,\ (e_{q(j)}^T)_{j=1}^q\}$. Hence, there is a nonzero vector $\alpha \in \Re^n$ such that

$$a_i^T \;=\; \alpha_1 a_{p(1)}^T \;+\; \ldots \;+\; \alpha_p a_{p(p)}^T \;+\; \alpha_{p+1} e_{q(1)}^T \;+\; \ldots \;+\; \alpha_n e_{q(q)}^T\,. \tag{1}$$

On the other hand, from the assumption that A is of full row rank, we know that a_i^T is not a linear combination of $\{(a_{p(j)}^T)_{j=1}^p\}$. Therefore, there exists some $\alpha_j \neq 0$ for some $j \geq p+1$. Without loss of generality, we assume that

$$\alpha_{p+1} \neq 0\,.$$

Then, from (1), we get

$$-\alpha_{p+1} e_{q(1)}^T \;=\; \alpha_1 a_{p(1)}^T \;+\; \ldots \;+\; \alpha_p a_{p(p)}^T \;+\; \alpha_{p+2} e_{q(2)}^T \;+\; \ldots \;+\; \alpha_n e_{q(q)}^T \;-\; a_i^T\;,$$

i.e.,

$$e_{q(1)}^T \;=\; -\alpha_{p+1}^{-1}[\alpha_1 a_{p(1)}^T \;+\; \ldots \;+\; \alpha_p a_{p(p)}^T \;+\; \alpha_{p+2} e_{q(2)}^T \;+\; \ldots \;+\; \alpha_n e_{q(q)}^T \;-\; a_i^T]\,.$$

This, together with the fact that $\mathrm{span}\{(a_{p(j)}^T)_{j=1}^p,\ (e_{q(j)}^T)_{j=1}^q\} \;=\; \Re^n$, implies that

$$\mathrm{span}\{(a_{p(j)}^T)_{j=1}^p,\ (e_{q(j)}^T)_{j=2}^q,\ a_i^T\} \;=\; \Re^n.$$

Thus, $\{(a_{p(j)}^T)_{j=1}^p,\ (e_{q(j)}^T)_{j=2}^q,\ a_i^T\}$ are linearly independent.

By repeating the above process we can conclude that

$$\{(a_j^T)_{j=1}^m,\ (e_{q(j)}^T)_{j=m-p+1}^q\}$$

are linearly independent.

Let

$$\mathcal{N} \;=\; \{q(m-p+1), \ldots, q(q)\}$$

and

$$\mathcal{B} \;=\; \{1, \ldots, n\} \backslash \mathcal{N}.$$

Then, from $Ax^* = b$ we get

$$Bx_{\mathcal{B}}^* \;+\; Nx_{\mathcal{N}}^* \;=\; 0\,,$$

where

$$B \;=\; [A_{\mathcal{B}(1)}\ \ldots\ A_{\mathcal{B}(m)}], \quad N \;=\; [A_{\mathcal{N}(1)}\ \ldots\ A_{\mathcal{N}(n-m)}]\,.$$

By noting that for any $i \in \mathcal{N}$, $x_i^* = 0$, we obtain

$$Bx_{\mathcal{B}}^* \;=\; b.$$

Since the matrix

$$\begin{bmatrix} A \\ e_{\mathcal{N}(1)}^T \\ \vdots \\ e_{\mathcal{N}(n-m)}^T \end{bmatrix}$$

is nonsingular, it follows that B is nonsingular. Therefore,

$$x_{\mathcal{B}}^* = B^{-1}b\,.$$

It is obvious $B^{-1}b \geq 0$ because $x^* \geq 0$. The proof is complete. $\square$