

NATIONAL UNIVERSITY OF SINGAPORE
 Department of Mathematics
 Semester 1 (2003/2004) MA4253 Mathematical Programming Tutorial 4

Solution to selected questions

Q1. Similar as the one given in lecture notes.

Q2. Let

$$P = \{x \mid 4x_1 + 9x_2 \leq 18, -2x_1 + 4x_2 \leq 4, x_1, x_2, x_3 \geq 0\}.$$

Then any $x \in P$ can be written

$$x = \sum_{i=1}^4 z_i x^i + z_5 x^5, \quad \sum_{i=1}^4 z_i = 1, z_i \geq 0, i = 1, \dots, 5,$$

where

$$x^1 = \begin{pmatrix} 4.5 \\ 0 \\ 0 \end{pmatrix}, x^2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, x^3 = \begin{pmatrix} \frac{18}{17} \\ \frac{26}{17} \\ 0 \end{pmatrix}, x^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, x^5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let

$$c = (-1 \ -1 \ 0)^T, \quad b = 2, \quad \text{and} \quad A = [1 \ -1 \ 1].$$

Then the Dantzig-Wolfe master program becomes

$$\begin{aligned} \min \quad & \sum_{i=1}^5 f_i z_i \\ \text{s.t.} \quad & \sum_{i=1}^4 p^i z_i + p^5 z_5 = b \\ & \sum_{i=1}^4 z_i + 0z_5 = 1 \\ & z_1, z_2, z_3, z_4, z_5 \geq 0, \end{aligned}$$

where

$$f_i = c^T x^i, \quad p^i = Ax^i, \quad i = 1, \dots, 5.$$

Take $\Lambda = \{1, 2\}$. Then the restricted Dantzig-Wolfe master program becomes

$$\begin{aligned} \min \quad & f_1 z_1 + f_2 z_2 \\ \text{s.t.} \quad & p^1 z_1 + p^2 z_2 = b \\ & z_1 + z_2 = 1 \\ & z_1, z_2 \geq 0, \end{aligned}$$

i.e.,

$$\begin{aligned} \min \quad & -4.5z_1 - 2z_2 \\ \text{s.t.} \quad & 4.5z_1 + (-2)z_2 = 2 \\ & z_1 + z_2 = 1 \\ & z_1, z_2 \geq 0. \end{aligned}$$

The unique solution to the above problem is

$$z_1 = \frac{8}{13}, \quad z_2 = \frac{5}{13}.$$

The optimal dual multipliers are

$$\begin{bmatrix} u \\ u_0 \end{bmatrix} = \begin{bmatrix} -5/11 \\ -27/11 \end{bmatrix}.$$

The next step is to solve the subproblem

$$\begin{aligned} \min \quad & (c - A^T u)^T x - u_0 \\ \text{s.t.} \quad & x \in P. \end{aligned}$$

i.e.,

$$\begin{aligned} \min \quad & (-6/11)x_1 + (-16/11)x_2 + (5/11)x_3 - (-27/11) \\ \text{s.t.} \quad & 4x_1 + 9x_2 \leq 18 \\ & -2x_1 + 4x_2 \leq 4 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

The optimal solution to the subproblem is

$$x = x^3.$$

Since the optimal cost of the subproblem is

$$-\frac{524}{187} - (-27/11) < 0,$$

it follows that the reduced cost of the variable z_3 is negative, and this variable can enter the basis.

Let $\Lambda = \{1, 2, 3\} \dots$.

Q3. (a) The linear programming problem

$$\begin{aligned} \min \quad & 10y_1 - 2y_2 + 4y_3 + 8y_4 + y_5 \\ \text{s.t.} \quad & y_1 - 4y_2 - y_3 \geq 8 \\ & 2y_1 - y_2 + y_3 \geq 2 \\ & 3y_1 + y_4 + y_5 \geq 4 \\ & y_1 + 2y_4 - y_5 \geq 10 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0, \end{aligned}$$

can be written equivalently as

$$\begin{aligned} & -\max \quad b^T y \\ & \text{s.t.} \quad W^T y \leq c \\ & \quad \quad -Iy \leq 0. \end{aligned}$$

So its dual problem is

$$\begin{aligned} & -\min \quad c^T x + 0^T s \\ & \text{s.t.} \quad Wx - s = b \text{ (i.e., } -Wx + s = -b) \\ & \quad \quad x, s \geq 0, \end{aligned}$$

i.e.,

$$\begin{aligned} & -\min \quad (-8)x_1 + (-2)x_2 + (-4)x_3 + (-10)x_4 + 0^T s \\ & \text{s.t.} \quad x_1 + 2x_2 + 3x_3 + x_4 + s_1 = 10 \\ & \quad \quad -4x_1 - x_2 \quad \quad \quad + s_2 = -2 \\ & \quad \quad -x_1 + x_2 \quad \quad \quad + s_3 = 4 \\ & \quad \quad \quad \quad \quad x_3 + 2x_4 + s_4 = 8 \\ & \quad \quad \quad \quad \quad x_3 - x_4 + s_5 = 1 \\ & \quad \quad x, s \geq 0. \end{aligned} \tag{1}$$

This dual problem can be simplified as

$$\begin{aligned} & -\min \quad (-8)x_1 + (-2)x_2 + (-4)x_3 + (-10)x_4 + (0)x_5 \\ & \text{s.t.} \quad x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 10 \\ & \quad \quad -4x_1 - x_2 \quad \quad \quad \leq -2 \\ & \quad \quad -x_1 + x_2 \quad \quad \quad \leq 4 \\ & \quad \quad \quad \quad \quad x_3 + 2x_4 \leq 8 \\ & \quad \quad \quad \quad \quad x_3 - x_4 \leq 1 \\ & \quad \quad x \geq 0. \end{aligned} \tag{2}$$

Problem (2) is a block angular linear problem. Therefore, the Dantzig-Wolfe decomposition method can be applied to solve problem (2), and thus problem (1).

(b) Use the optimal basis matrix obtained in (a) for problem (1).