3 The Dantzig-Wolfe Decomposition Algorithm

Consider the linear programming (LP)

$$\text{(LP)} \qquad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & Cx \geq d \\ & x \geq 0\,, \end{aligned}$$

where c, x, b, and d are vectors, A and C are matrices, all of conformable dimension.

To take full advantage of this decomposition procedure one should partition the constraint matrix into A and C such that the matrix C

1) contains the **vast majority** of the constraints, and
2) contains **special structure** that makes the linear program over the $Cx \geq d$ constraints significantly easier to optimize than the original problem.

Key Idea One - An Equivalent Master Problem.

For simplicity, assume that $\{x \in \Re^n \,|\, Cx \geq d, x \geq 0\}$ is a polytope. Then any point in this polytope can be written as a convex combination of the extreme points of the polytope. Let $x^1, \ldots, x^q$ be the extreme points of this polytope. If $\bar{x}$ is a point in the polytope

$$\{x \in \Re^n \,|\, Cx \geq d,\ x \geq 0\}\,,$$

then there are $\bar{z}_1, \ldots, \bar{z}_q$ such that

$$\bar{x} = \sum_{i=1}^{q} \bar{z}_i x^i,\ \sum_{i=1}^{q} \bar{z}_i = 1,\ \bar{z}_i \geq 0,\ i = 1, \ldots, q\,.$$

Therefore, an equivalent problem to the original problem (LP) is

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & \sum_{i=1}^{q} z_i x^i = x \\ & \sum_{i=1}^{q} z_i = 1 \\ & z \geq 0\,. \end{aligned}$$

This is a problem with both x and z variables. Next substitute

$$x = \sum_{i=1}^{q} z_i x^i$$

into the set $Ax = b$ and into the objective function $c^T x$. The resulting linear program is

$$\begin{aligned} \min \quad & c^T(\sum_{i=1}^{q} z_i x^i) \\ \text{s.t.} \quad & A(\sum_{i=1}^{q} z_i x^i) = b \\ & \sum_{i=1}^{q} z_i = 1 \\ & z \geq 0\,. \end{aligned}$$

By observing

$$c^T(\sum_{i=1}^{q} z_i x^i) = \sum_{i=1}^{q}(c^T x^i) z_i$$

and

$$A(\sum_{i=1}^{q} z_i x^i) = \sum_{i=1}^{q}(Ax^i) z_i\,,$$

and defining

$$f_i = c^T x^i, \quad p^i = Ax^i,\ i = 1, \ldots, q\,,$$

one can see that the original problem (LP) is equivalent to

$$\text{(DW)} \qquad \begin{aligned} \min \quad & \sum_{i=1}^{q} f_i z_i \\ \text{s.t.} \quad & \sum_{i=1}^{q} p^i z_i = b \\ & \sum_{i=1}^{q} z_i = 1 \\ & z \geq 0\,. \end{aligned}$$

The linear program (DW) is the **Dantzig-Wolfe master program**. In (DW) constraints in

$$\sum_{i=1}^{q} p^i z_i = b$$

are the **coupling constraints** and the constraint

$$\sum_{i=1}^{q} z_i = 1$$

is a **convexity row**. If A and C have m_1 and m_2 rows respectively, then the original problem formulation has $m_1 + m_2$ rows. However, the Dantzig-Wolfe master has only $m_1 + 1$ constraints. On the other hand, there are **many more variables** in the Dantzig-Wolfe master than in the original formulation since we create a variable for every extreme point in the **subproblem polytope**

$$\{x \,|\, Cx \geq d,\ x \geq 0\}\,.$$

It is not practical to store the coefficients for all the extreme points of a realistic problem. This motivates the second key idea.

Key Idea Two - Column Generation.

The revised simplex algorithm is applied to a **restricted master program (DWR)** which is

$$
\text{(DWR)} \quad \begin{array}{lrcl}
\min & \sum_{i \in \Lambda} f_i z_i & & \\
\text{s.t.} & \sum_{i \in \Lambda} p^i z_i & = & b \\
& \sum_{i \in \Lambda} z_i & = & 1 \\
& z & \geq & 0\,,
\end{array}
$$

where Λ is a very small subset of the columns in the full master problem. We should have at least $m_1 + 1$ columns in Λ in order to find a basis. Assume that the restricted master is feasible.

Let (u, u_0) be a set of optimal dual multiplies associated with the constraints

$$\sum_{i=1}^{q} p^i z_i \;=\; b$$

and

$$\sum_{i=1}^{q} z_i \;=\; 1\,,$$

respectively, in the restricted master. The reduced cost of variable z_i is

$$\bar{f}_i \;=\; f_i \;-\; \begin{bmatrix} u \\ u_0 \end{bmatrix}^T \begin{bmatrix} p^i \\ 1 \end{bmatrix} \;=\; f_i \;-\; u^T p^i \;-\; u_0\,.$$

If the reduced costs $\bar{f}_i \geq 0$ for all $i = 1, \ldots, q$, the optimal solution to the restricted master is the optimal solution to the full master. If at least one column prices out negative for $i \notin \Lambda$, then this column is added to the restricted master and the column that leaves the basis can be deleted from the restricted master.

Calculating the reduced cost explicitly for all the columns in the full master is not practical. But,

$$\begin{aligned}&\min\{f_i \,-\, u^T p^i \,-\, u_0 \,|\, i = 1, \ldots, q\}\\ &= \min\{c^T x^i \,-\, u^T A x^i \,-\, u_0 \,|\, i = 1, \ldots, q\}\\ &= \min\{(c \,-\, A^T u)^T x^i \,|\, i = 1, \ldots, q\} \,-\, u_0\,.\end{aligned}$$

Since there is always an optimal extreme point solution to a linear program when an optimum exists, finding the minimum reduced cost is equivalent to solving the following linear program

$$(\mathrm{DWS(u,u_0)}) \qquad \begin{array}{ll} \min & (c \,-\, A^T u)^T x \,-\, u_0 \\ \text{s.t.} & Cx \,\geq\, d \\ & x \,\geq\, 0\,. \end{array}$$

Problem $(\mathrm{DWS(u,u_0)})$ is the Dantzig-Wolfe subproblem. After optimizing the restricted master, all the columns in the full master are priced out by solving the subproblem $(\mathrm{DWS(u,u_0)})$.

By assumption, the subproblem polyhedron

$$\{x \in \Re^n \,|\, Cx \,\geq\, d,\ x \,\geq\, 0\}$$

is bounded, i.e., a polytope, so $(\mathrm{DWS(u,u_0)})$ cannot be unbounded. If $(\mathrm{DWS(u,u_0)})$ is infeasible, then the original linear program (LP) is infeasible. Any such scheme of pricing where all columns are explicitly priced out is called **column generation**.

The linear program used to generate the columns is called the **subproblem**. This is why all of the rows in the original problem are not included in the matrix C. If this were the case, then the pricing problem would be equivalent to the original linear program. The objective is to put as many rows as possible into the C matrix, while leaving the constraint set $Cx \geq d$ with as much special structure as possible in order to admit a fast solution.

Dantzig-Wolfe Algorithm

Step 1. Find an initial set of columns Λ and form the Dantzig-Wolfe restricted master program (DWR). Assume the restricted master is feasible.

Step 2. Optimize the restricted master program (DWR) and obtain an optimal set of dual multipliers u, u_0.

Step 3. Solve the Dantzig-Wolfe subproblem $(\mathrm{DWS(u,u_0)})$. Let x^j be an optimal basic feasible solution with objective function value

$$(c \,-\, A^T u)^T x^j \,-\, u_0 \,=\, f_j \,-\, u^T p^j \,-\, u_0\,.$$

If $f_j - u^T p^j - u_0 = 0$ stop, the basic feasible solution corresponding to the current restricted master is optimal, else go to **Step 4**.

Step 4. Add the column

$$\begin{bmatrix} p^j \\ 1 \end{bmatrix} = \begin{bmatrix} Ax^j \\ 1 \end{bmatrix}$$

to the restricted master, update

$$\Lambda \Longleftarrow \Lambda \cup \{j\}$$

and go to **Step 2**.

There are numerous variations on this basic algorithm which affect the efficiency of the algorithm. For example, the column that j replaces in the basis can be deleted from the restricted master so that the restricted master does not grow in size.

An example. Consider the linear programming problem

$$\begin{array}{ll} \min & -4x_1 - x_2 - 6x_3 \\ \text{s.t.} & 3x_1 + 2x_2 + 4x_3 = 17 \\ & 1 \le x_1 \le 2 \\ & 1 \le x_2 \le 2 \\ & 1 \le x_3 \le 2\,. \end{array}$$

Naturally, we divide the constraints into two groups: the first group consists of the constraint

$$Ax = b,$$

where $A = [3\ 2\ 4]$ and $b = 17$; the second group is the constraint

$$P = \{x \in \Re^3 \mid 1 \le x_i \le 2,\ i = 1, 2, 3\}\,.$$

Note that P is bounded and has eight extreme points. The master problem has two equality constraints, namely

$$\sum_{i=1}^{8} (Ax^i) z_i = 17\,,$$

$$\sum_{i=1}^{8} z_i = 1\,,$$

where x^i are the extreme points of P. The columns of the constraint matrix in the master problem are of the form

$$\begin{bmatrix} Ax^i \\ 1 \end{bmatrix}.$$

Let us pick two of the extreme points of P, say, $x^1 = (2,2,2)^T$ and $x^2 = (1,1,2)^T$, and let the corresponding variables z_1 and z_2 be our initial basic variables. We have $Ax^1 = 18$, $Ax^2 = 13$, and therefore, the corresponding basis matrix is

$$B = \begin{bmatrix} 18 & 13 \\ 1 & 1 \end{bmatrix};$$

its inverse is

$$B^{-1} = \begin{bmatrix} 0.2 & -2.6 \\ -0.2 & 3.6 \end{bmatrix}.$$

By calculation,

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = B^{-1}\begin{bmatrix} 17 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}.$$

Since these values are nonnegative, we have a basic feasible solution to the master problem. Let $\Lambda = \{1,2\}$. Then the restricted master program is

$$\begin{array}{ll} \min & f_1 z_1 + f_2 z_2 \\ \text{s.t.} & (Ax^1)z_1 + (Ax^2)z_2 = b \\ & z_1 + z_2 = 1 \\ & z_1, z_2 \geq 0, \end{array}$$

where

$$f_1 = c^T x^1 = [-4 \ -1 \ -6]\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = -22$$

and

$$f_2 = c^T x^2 = [-4 \ -1 \ -6]\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = -17.$$

Therefore, the restricted master problem is

$$\begin{array}{ll} \min & (-22)z_1 + (-17)z_2 \\ \text{s.t.} & 18z_1 + 13z_2 = 17 \\ & z_1 + z_2 = 1 \\ & z_1, z_2 \geq 0. \end{array}$$

The unique solution to the above restricted master problem is exactly $(0.8, 0.2)^T$.

Next, we determine the optimal dual multipliers. Recalling that

$$f_{\mathcal{B}(1)} = f_1 = -22$$

and

$$f_{\mathcal{B}(2)} = f_2 = -17\,,$$

we therefore have

$$\begin{bmatrix} u \\ u_0 \end{bmatrix}^T = f_{\mathcal{B}}^T B^{-1} = [-22 \ -17]B^{-1} = [-1 \ -4]\,.$$

The subproblem to solve is

$$\begin{array}{ll} \min & (c - A^T u)^T x - u_0 \\ \text{s.t.} & x \in P\,. \end{array}$$

By calculating, we have

$$(c - A^T u)^T = [-4 \ -1 \ -6] - (-1)[3 \ 2 \ 4] = [-1 \ 1 \ -2]\,.$$

Therefore, the subproblem turns to be

$$\begin{array}{ll} \min & -x_1 + x_2 - 2x_3 - (-4) \\ \text{s.t.} & 1 \le x_1 \le 2 \\ & 1 \le x_2 \le 2 \\ & 1 \le x_3 \le 2\,. \end{array}$$

The optimal solution to be subproblem is $x = (2, 1, 2)^T$. This is a new extreme point of P, which we will denote by x^3. Since the optimal cost of the subproblem is

$$-2 + 1 - 2 \times 2 - (-4) = -5 + 4 = -1 < 0,$$

it follows that the reduced cost of the variable z_3 is negative, and this variable can enter the basis.

Let $\Lambda = \{1, 2, 3\}$. Since $Ax^3 = 16$ and $f_3 = c^T x^3 = -21$, we have the new restricted master problem

$$\begin{array}{ll} \min & f_1 z_1 + f_2 z_2 + f_3 z_3 \\ \text{s.t.} & (Ax^1)z_1 + (Ax^2)z_2 + (Ax^3)z_3 = b \\ & z_1 + z_2 + z_3 = 1 \\ & z_1, z_2, z_3 \ge 0\,, \end{array}$$

i.e.,

$$\begin{array}{ll} \min & (-22)z_1 + (-17)z_2 + (-21)z_3 \\ \text{s.t.} & 18z_1 + 13z_2 + 16z_3 = 17 \\ & z_1 + z_2 + z_3 = 1 \\ & z_1, z_2, z_3 \ge 0\,. \end{array}$$

Note that $(0.8, 0.2, 0)^T$ is an basic feasible solution to the above restricted master problem. Its optimal basic feasible solution is

$$z_1 = 0.5, \quad z_2 = 0, \quad z_3 = 0.5$$

and the basis matrix is

$$B = \begin{bmatrix} 18 & 16 \\ 1 & 1 \end{bmatrix}$$

with its inverse

$$B^{-1} = \begin{bmatrix} 0.5 & -8 \\ -0.5 & 9 \end{bmatrix}.$$

The corresponding optimal dual multipliers are

$$\begin{bmatrix} u \\ u_0 \end{bmatrix}^T = f_{\mathcal{B}}^T B^{-1} = [-22 \ -21]B^{-1} = [-0.5 \ -13].$$

Next, we consider

$$\begin{array}{ll} \min & (c - A^T u)^T x - u_0 \\ \text{s.t.} & x \in P. \end{array}$$

By calculating, we have

$$(c - A^T u)^T = [-4 \ -1 \ -6] - (-0.5)[3 \ 2 \ 4] = [-2.5 \ 0 \ -4],$$

which, implies that the subproblem is

$$\begin{array}{ll} \min & -2.5x_1 + 0x_2 - 4x_3 - (-13) \\ \text{s.t.} & 1 \leq x_1 \leq 2 \\ & 1 \leq x_2 \leq 2 \\ & 1 \leq x_3 \leq 2. \end{array}$$

We find that $(2, 2, 2)^T$ is an optimal solution and the optimal cost is

$$-2.5 \times 2 + 0 - 4 \times 2 + 13 = 0.$$

Therfore, we conclude that the reduced cost of every z_i is nonnegative, and we have an optimal solution to the master problem. In terms of the variables x_i, the optimal solution is

$$x^* = \frac{1}{2}x^1 + \frac{1}{2}x^3 = \begin{bmatrix} 2 \\ 1.5 \\ 2 \end{bmatrix}.$$

Starting the algorithm. In order to start the decomposition algorithm, we need to find a basic feasible solution to the master problem. This is done by observation in the numerical example. In general, this can be done as follows. We first apply **Phase I** of the simplex method to the polytope

$$P = \{x \in \Re^n \mid Cx \geq d,\ x \geq 0\}$$

to find an extreme point x^1 of P. By possibly multiplying both sides of some of the coupling constraints by -1, we can assume that $Ax^1 \leq b$. Let y be a vector of auxiliary variable of dimension m_1. We form the auxiliary master problem

$$\begin{array}{ll} \min & \sum_{i=1}^{m_1} y_t \\ \text{s.t.} & \sum_{i=1}^{q} (Ax^i) z_i + y = b \\ & \sum_{i=1}^{q} z_i = 1 \\ & y, z \geq 0\,. \end{array}$$

A basic feasible solution to the auxiliary problem is obtained by letting

$$z_1 = 1, \quad z_j = 0 \text{ for } j \neq 1$$

and

$$y = b - Ax^1\,.$$

Starting from here, we can use the Dantzig-Wolfe decomposition algorithm to solve the auxiliary master problem. If the optimal cost is positive, then the master problem is infeasible. If the optimal cost is zero, an optimal solution to the auxiliary problem provides us with a basic feasible solution to the master problem.

Block angular linear programs provide a natural partition of the original constraint set into A and C. **A block angular linear program** has the following structure (BAP)

$$\begin{array}{llllllll} \min & (c^1)^T x^1 & + (c^2)^T x^2 & + \ldots + & (c^m)^T x^m & & \\ \text{s.t.} & A_1 x^1 & + A_2 x^2 & + \ldots + & A_m x^m & = & b \\ & C_1 x^1 & & & & \geq & d^1 \\ & & C_2 x^2 & & & \geq & d^2 \\ & & & \ddots & & & \\ & & & & C_m x^m & \geq & d^m \\ & x^1, & x^2, & \ldots & , x^m & \geq & 0\,. \end{array}$$

Partition the constraint matrix of (BAP) into A and C as follows.

$$A = [A_1\ A_2\ \ldots\ A_m]$$

and

$$C = \begin{bmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_m \end{bmatrix}.$$

The matrix C has very special structure. The subproblem can be optimized by solving m **independent** linear programs. There is another interesting aspect to the block angular structure. It is possible to create a master problem exactly as before where each p^j is

$$p^j = [C_1 \; C_2 \; \ldots \; C_m] \begin{bmatrix} x^{1j} \\ x^{2j} \\ \vdots \\ x^{mj} \end{bmatrix},$$

where x^{ij} is the jth extreme point of the polytope defined by the equations

$$C_i x^i \geq d^i, \; x^i \geq 0\,.$$

Assuming these polyhedra are bounded, by the finite basis theorem any solution to $C_i x^i \geq d^i$, $x^i \geq 0$ is expressed as

$$x^i = \sum_{j=1}^{q_i} z_{ij} x^{ij}, \quad \sum_{j=1}^{q_i} z_{ij} = 1, \quad z_{ij} \geq 0\,.$$

Substituting $\sum_{j=1}^{q_i} z_{ij} x^{ij}$ for x^i in (BAP) gives the following master program

$$\begin{array}{ll} \min & \sum_{i=1}^{m} \sum_{j=1}^{q_i} f_{ij} z_{ij} \\ \text{s.t.} & \sum_{i=1}^{m} \sum_{j=1}^{q_i} p^{ij} z_{ij} = b \\ & \sum_{i=1}^{m} z_{ij} = 1, \quad i = 1, \ldots, m \\ & z_{ij} \geq 0, \; j = 1, \ldots, q_i, \; i = 1, \ldots, m\,, \end{array}$$

where

$$f_{ij} = (c^i)^T x^{ij}$$

and

$$p^{ij} = C^i x^{ij}\,.$$

Now, there are m subproblems to solve instead of one. Letting u_{i0} denote the dual variable associated with the ith convexity row, the ith subproblem is

$$\begin{array}{ll} \min & (c_i - (A_i)^T u)^T x^i - u_{i0} \\ \text{s.t.} & C_i x^i \geq d^i\,, \\ & x^i \geq 0\,. \end{array}$$

How can we modify the above Dantzig-Wolfe decomposition algorithm to cover the case that

$$P = \{x \in \Re^n \,|\, Cx \geq d, \ x \geq 0\}$$

is unbounded by using the Minkowski Theorem?