NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
Semester 1 (2003/2004) MA4253 Mathematical Programming Tutorial 6

Solution to Selected Questions

Q1. [**Outline of proof:** From the assumptions, there exists a basic feasible solution (BFS) to the primal problem. Based on the BFS, we reformulate the primal problem in terms of the "reduced costs" form. The uniqueness of the solution to primal problem is then obtained from this form. We then use the complementary slackness to get the dual uniqueness.]

Proof. Since there exists an optimal solution to the primal problem, there is an extreme point $\hat{x}$ of the constraint set such that $\hat{x}$ is an optimal solution to the primal problem [you may use Minkowski's Theorem to check this].

Without loss of generality (remove redundant constraints if necessary), we assume that A is of full row rank. Again, without loss of generality, we assume that $A = [B\ N]$ and B is the basis matrix to problem

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0\,. \end{aligned} \tag{1}$$

Then problem (1) becomes

$$\begin{aligned} \min \quad & 0^T x_{\mathcal{B}} + \sum_{j \in \mathcal{N}} (c_j - c_{\mathcal{B}}^T B^{-1} A_j) x_j + c_{\mathcal{B}}^T \bar{b} \\ \text{s.t.} \quad & x_{\mathcal{B}} + \sum_{j \in \mathcal{N}} B^{-1} A_j x_j = B^{-1} b = \bar{b} \\ & x_{\mathcal{B}}\,, x_{\mathcal{N}} \geq 0 \end{aligned} \tag{$*$}$$

where $\mathcal{B}$ and $\mathcal{N}$ denote the index set of basic variables and the index set of nonbasic variables, respectively. Let

$$\bar{c}_j = c_j - c_{\mathcal{B}}^T B^{-1} A_j\,.$$

Then we have

a) $\bar{c}_j \geq 0,\ j \in \mathcal{N}$ because otherwise there exists some $k \in \mathcal{N}$ such that $\bar{c}_k < 0$, we can increase x_j sufficiently small and keep all other nonbasic variables zero to get a new x such that

$$\begin{cases} x_i > 0\,, i \in \mathcal{B} \text{ (since } \hat{x}_{\mathcal{B}} > 0) \\ x_k > 0 \\ x_j = 0\,, j \in \mathcal{N}\,, j \neq k \end{cases}$$

and

$$c^T x = c_{\mathcal{B}}^T \bar{b} + \bar{c}_k x_k < c_{\mathcal{B}}^T \bar{b} = c^T \hat{x},$$

which contradicts the fact that $\hat{x}$ is an optimal solution.

b) $\bar{c}_j > 0,\ j \in \mathcal{N}$ by part (ii) of the assumption and part a) in the proof that $\bar{c}_j \geq 0, j \in \mathcal{N}$.

For any feasible solution $\bar{x}$, if $\bar{x}_k > 0$ for some $k \in \mathcal{N}$, then

$$c^T \bar{x} = c_{\mathcal{B}}^T \bar{b} + \bar{c}_k \bar{x}_k > c_{\mathcal{B}}^T \bar{b}\,.$$

This means that if $\bar{x}$ is any optimal feasible solution, then we must have

$$\bar{x}_j = 0\,, \forall\, j \in \mathcal{N}\,.$$

By $(*)$, this further implies that

$$\bar{x}_{\mathcal{B}} = \bar{b} = \hat{x}_{\mathcal{B}}\,.$$

Therfore, the primal problem has a unique solution.

By the assumption, the dual problem also has a solution. Suppose that $\hat{p}$ is any optimal solution to the dual problem. Then $A^T \hat{p} \leq c$ and

$$\hat{x}_i (A_i^T \hat{p} - c_i) = 0\,,$$

where A_i is the ith column of A. In particular, because $\hat{x}_i > 0, i \in \mathcal{B}$, we have

$$A_i^T \hat{p} = c_i\,,\ i \in \mathcal{B}\,.$$

That is, $B^T \hat{p} = c_{\mathcal{B}}$, i.e., $\hat{p} = (B^T)^{-1} c_{\mathcal{B}}$. Therefore, $\hat{p}$ is the unique solution of the dual problem in light of the uniqueness of B. $\square$

Q2. Proof. By Minkowski's theorem and substituting

$$x = \sum_{i=1}^{r} \lambda_i x^i + \sum_{j=r+1}^{q} \mu_j d^j, \quad \sum_{i=1}^{r} \lambda_i = 1\,,$$

$$\lambda_i \geq 0\,,\ i = 1, \ldots, r\,,\ \mu_j \geq 0\,,\ j = r+1, \ldots, q$$

into problem (1), we have

$$\begin{aligned} \min \quad & \sum_{i=1}^{r} \lambda_i (c^T x^i) + \sum_{j=r+1}^{q} \mu_j (c^T d^j) \\ \text{s.t.} \quad & \sum_{i=1}^{r} \lambda_i = 1 \\ & \lambda_i \geq 0\,,\ i = 1, \ldots, r\,,\ \mu_j \geq 0\,,\ j = r+1, \ldots, q\,. \end{aligned}$$

Therefore, problem (1) has an optimal cost $-\infty$ if and only if there exists some d^j such that $c^T d^j < 0$. $\square$

Q3. [**Outline of Proof:** Apply Lemma 4.4 to show that x has exactly m positive components. Such an x must be a basic feasible solution in terms of Lemma 4.4.]

Proof. Let

$$M^+ = \{i \,|\, x_i > 0\}.$$

By Lemma 4.4, $|M^+| \geq m$ and there exist indices

$$\mathcal{B}(1), \ldots, \mathcal{B}(m) \in M^+$$

such that

$$B := [A_{\mathcal{B}}(1) \ \cdots \ A_{\mathcal{B}}(m)]$$

is nonsingular and the corresponding basic feasible solution $\bar{x}$ satisfies

$$\bar{x}_{\mathcal{B}} := B^{-1}b \geq 0, \quad \bar{x}_{\mathcal{N}} = 0.$$

If $|M^+| = m$, then

$$x = \bar{x} = \begin{pmatrix} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{pmatrix}$$

because in this case

$$Bx_{\mathcal{B}} = \sum_{i \in M^+} A_i x_i = b = B\bar{x}_{\mathcal{B}} + N\bar{x}_{\mathcal{N}} = B\bar{x}_{\mathcal{B}}.$$

If $|M^+| > m$, then from

$$x_i(c_i - A_i^T p) = 0, \quad \forall\, i,$$

we know that

$$c_i - A_i^T p = 0, \quad \forall\, i \in M^+. \tag{$**$}$$

In particular,

$$c_{\mathcal{B}(i)} = A_{\mathcal{B}(i)}^T p, \quad i = 1, \ldots, m.$$

That is

$$p = (B^T)^{-1} c_{\mathcal{B}}.$$

The reduced costs for $\bar{x}$ are

$$\begin{cases} \bar{c}_j = c_j - c_{\mathcal{B}}^T B^{-1} A_j = 0, & j = \mathcal{B}(1), \ldots, \mathcal{B}(m) \\ \bar{c}_j = c_j - c_{\mathcal{B}}^T B^{-1} A_j, & j \neq \mathcal{B}(1), \ldots, \mathcal{B}(m). \end{cases}$$

By the assumption,

$$\bar{c}_j = c_j - p^T A_j = c_j - c_{\mathcal{B}}^T B^{-1} A_j \neq 0, \quad j \neq \mathcal{B}(1), \ldots, \mathcal{B}(m).$$

By $(**)$ and the fact that

$$\{\mathcal{B}(1), \ldots, \mathcal{B}(m)\} \in M^+,$$

we get

$$\{\mathcal{B}(1), \ldots, \mathcal{B}(m)\} = M^+.$$

The proof is complete. $\square$