MA4254: Discrete Optimization*

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Aims/Objectives: Discrete optimization deals with problems of maximizing or minimizing a function over a feasible region of discrete structure. These problems come from many fields like operations research, management science, and computer science. The primary objective of this course is twofold: a) to study key techniques to separate easy problems from difficult ones and b) to use typical methods to deal with difficult problems.

Mode of Evaluation: Tutorial class performance (10%); Mid-Term test (20%) and Final examination (70%)

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References:

1) D. Bertsimas and J. N. Tsitsiklis, Introduction to Linear Optimization. Athena Scientific, 1997.

2) G. L. Nemhauser and L. A. Wolsey, Integer and Combinatorial Optimization. John Wiley and Sons, 1999.

3) C. H. Papadimitriou and K. Steiglitz, Combinatorial Optimization: Algorithms and Complexity. Prentice-Hall, 1982. Second edition by Dover, 1998.

PARTIAL lecture notes will be made available in my webpage
http://www.math.nus.edu.sg/~matsundf/

1 Introduction

In this Chapter we will briefly discuss the problems we are going to study; give a short review about simplex methods for solving linear programming problems and introduce some basic concepts in graphs and digraphs.

1.1 Linear Programming (LP): a short review

Consider the following linear programming

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & \quad x \geq 0 \end{array}$$

and its dual

$$(D) \quad \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y \leq c \\ & \quad y \geq 0\,. \end{array}$$

- **Simplex Method**
  - Dantzig (1947)
  - Very efficient
  - Not polynomial time algorithm. Klee and Minty (1972) gave an counterexample.
  - Average analysis versus worst-case analysis
- **Russian's Ellipsoid Method**
  - Polynomial time algorithm (Khachiyan, 1979)
  - Less efficient
- **Interior-Point Algorithms**
  - Karmarkar (1984)
  - Polynomial times algorithm
  - Efficient for some large-scale sparse LPs
- **Others**

1.2 Discrete Optimization (DO)

Also Combinatorial Optimization (CO)

Mathematical formula in general:

$$\begin{array}{ll} \min & \alpha(x) \\ \text{s.t.} & x \in F \end{array}$$

$--$ x decision policy

$--$ F is the collection of feasible decision policies

$--$ $\alpha(x)$ measures the value of members of F.

A typical DO (CO) problem:

$$(IP) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & \quad x \geq 0 \\ & x_j \text{ integer for } j \in I \subseteq N := \{1, \cdots, n\}. \end{array}$$

where $c \in \Re^n$, $x \in \Re^n$, $b \in \Re^m$ and $A \in \Re^{m \times n}$.

- $I = \emptyset$, (IP) $\Longrightarrow$ (LP)
- $I = N$, (IP) $\Longrightarrow$ pure IP

1.3 Specific Forms

1. The $0-1$ Knapsack Problem

Suppose there are n projects.

$--$ jth project has a cost a_j and a value c_j

$--$ each project either done or not

$--$ A budget of b available to fund the projects

Then the $0-1$ *Knapsack Problem* can be formulated as

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & a^T x \leq b \\ & x \in B^n, \end{array}$$

where $a = (a_1, \cdots, a_n)^T$ and B^n is the set of n-dimensional binary vector.

2. The Assignment Problem

−− n people and m jobs, where $n \geq m$

−− Each job must be assigned to exactly one person, and each person can do at most one job

−− The cost of person j doing job i is c_{ij}.

Then the *Assignment Problem* can be formulated as

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \cdots, m \\ & \sum_{i=1}^{m} x_{ij} \leq 1, \quad j = 1, \cdots, n \\ & x \in B^{mn}\,. \end{aligned}$$

Extensions — *Three-Index Assignment Problem*

3. Set-Covering, Set-Packing, and Set-Partitioning Problems

The *Set-Covering Problem* is

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq \mathbf{1} \\ & x \in B^n\,. \end{aligned}$$

The *Set-Packing Problem* is

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq \mathbf{1} \\ & x \in B^n\,. \end{aligned}$$

4. Traveling Salesman Problem (TSP)

We are given a set of nodes $V = \{1, \cdots, n\}$ and a set of arcs $\mathcal{A}$. The nodes represent cities, and the arcs represent ordered pairs of cities between which direct travel is possible.

- For $(i, j) \in \mathcal{A}$, c_{ij} is the direct travel time from city i to city j.

The *TSP* is to find a tour, starting at city 1, that

(a) visits each other city exactly once and then returns to city 1, and

(b) takes the least total travel time.

5. Facility Location Problem, Network Flow Problem, and many more

1.4 Why DO (CO) difficult

Arrangements grow exponentially is the superficial reason.

Total Unimodularity (TU) Theory; Shortest Path; Matroids and Greedy Algorithm; Complexity ($\mathcal{P} \neq \mathcal{NP}$ conjecture); Interior-Point Algorithms; Cutting Plane; Branch and Bound; Decomposition; Flowshop Scheduling, etc.

1.5 Convex sets

In linear programming and nonlinear programming, we have already met many convex sets. For examples, the *line segment* between two points in $\Re^n$ is a convex set; a unit ball in $\Re^n$ is a convex set; and more importantly a polyhedral set is a convex set (a formal definition of a polyhedral set is to be given shortly). But, what is a convex set?

Definition 1.1 *A set $S \subseteq \Re^n$ is* **convex** *if for any $x, y \in S$, and any $\lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \in S$, i.e., the whole line segment between x and y is in S.*

Exercise: Give two more sets which are convex and two sets which are not convex.

1.6 Hyperplanes and half spaces

Definition 1.2 *Let a be a nonzero vector in $\Re^n$ and b be a scalar. Then the set*

$$\{x \in \Re^n \mid a^T x = b\}$$

is called a **hyperplane**, *where a^T is the transpose of the (column) vector a.*

Geometrically, the hyperplane $\{x \in \Re^n \mid a^T x = b\}$ can be understood by expressing it in the form

$$\{x \in \Re^n \mid a^T(x - x^0) = 0\}$$

where x^0 is any point in the hyperplane, i.e., $a^T x^0 = b$. This representation can then be interpreted as

$$\{x \in \Re^n \mid a^T(x - x^0) = 0\} = x^0 + a^\perp,$$

where $a^\perp$ denotes the orthogonal complement of a, i.e., the set of all vectors orthogonal to it:

$$a^\perp = \{d \in \Re^n \mid a^T d = 0\}.$$

This shows that the hyperplane consists an "offset" of the hyperplane from the origin (i.e., x^0), plus all vectors orthogonal to the (normal) vector.

A hyperplane divides $\Re^n$ into two parts, which are called half spaces.

Definition 1.3 *Let a be a nonzero vector in $\Re^n$ and b be a scalar. Then the set*

$$\{x \in \Re^n \mid a^T x \geq b\}$$

is called a **halfspace**.

Obviously, a halfspace is a convex set and $\{x \in \Re^n \mid a^T x \leq b\}$ is the other halfspace.

Let x^0 be any point on the hyperplane $\{x \in \Re^n \mid a^T(x - x^0) = 0\ \}$. Then the halfspace $\{x \in \Re^n \mid a^T x \geq b\ \}$ can be expressed as

$$\{x \in \Re^n \mid a^T(x - x^0) \geq 0\ \}.$$

This suggests a simple geometric interpretation: the half space consists of x^0 plus any vector that makes an acute angle with the normal vector a. See the figure below.

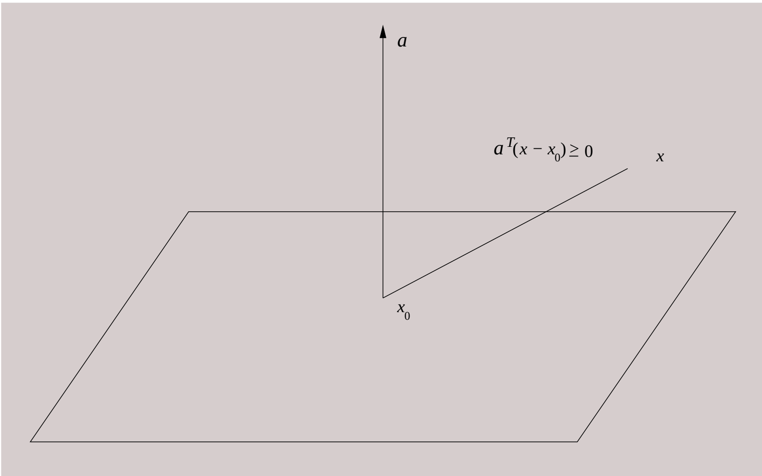

Figure 1.1: The half space $\{x \in \Re^n \mid a^T x \geq b\ \}$ consists of x^0 plus any vector that makes an acute angle with the normal vector a

1.7 Polyhedra

Definition 1.4 *A* **polyhedron** *is a set that can be described in the form* $\{x \in \Re^n \mid Ax \geq b\}$*, where* A *is an* $m \times n$ *matrix and* b *is a vector in* $\Re^m$*.*

Let $A \in \Re^{m \times n}$ and $b \in \Re^m$ be defined as follows

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

Then the polyhedron defined in Definition 1.4 is the intersection of the following halfspaces

$$\{x \in \Re^n \mid a_i^T x \geq b_i\ \}, \quad i = 1, \ldots, m.$$

It is noted that these halfspaces are finite in number. The intersection of two polyhedrons is again a polyhedron. So $\{x \in \Re^n \mid Cx \geq d,\ Ax = b\}$ is also a polyhedron, where $C \in \Re^{p \times n}, d \in \Re^p$.

A polyhedron may have different representations. For example

$$\begin{aligned} &\{x \in \Re^2 \mid x_1 + x_2 = 0,\ x_1 \geq 0\} \\ = &\{x \in \Re^2 \mid 2x_1 + 2x_2 \geq 0,\ x_1 + x_2 \leq 0, x_1 \geq 0\}. \end{aligned}$$

A bounded polyhedron is sometimes called a **polytope**.

Let e_i be the ith unit vector in $\Re^n$. Then by noting that $x_i = e_i^T x$ we know that the **nonnegative orthant**

$$\Re^n_+ = \{x \in \Re^n \mid x_i \geq 0,\ i = 1, \ldots, n\}.$$

is a polyhedron.

Definition 1.5 *Let $x^1, \ldots, x^k$ be vectors in $\Re^n$ and let $\lambda_1, \ldots, \lambda_k$ be nonnegative scalars whose sum is one.*

(a) The vector $\sum_{i=1}^{n} \lambda_i x^i$ is said to be a **convex combination** *of the vectors $x^1, \ldots, x^k$.*

(b) The **convex hull** *(*conv *in short) of the vectors $x^1, \ldots, x^k$ is the set of all convex combinations of these vectors.*

It is easy to see by Definition 1.5 that

$$\begin{aligned} &\mathrm{conv}\{e_1, \ldots, e_n\} \\ = &\left\{x \in \Re^n \mid \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0,\ i = 1, \ldots, n\right\} \end{aligned}$$

is a polytope.

1.8 Basic feasible solutions

We have already known that an optimal solution to a linear programming (assume the existence of an optimal solution) can be found at a "corner" of the polyhedron over which we are optimizing. There are quite a number of different but equivalent ways to define the concept of a "corner". Here we introduce two of them – exreme points and basic feasible solutions.

Our first definition defines an *extreme point* of a polyhedron as a point that can not be expressed as a convex combination of two other points of the polyhedron.

Definition 1.6 *Let $P \subseteq \Re^n$ be a polyhedron. A vector $x \in P$ is an* **extreme point** *of P if we cannot find two vectors $y, z \in P$, both different from x, and a scalar $\lambda \in [0, 1]$, such that $x = \lambda y + (1 - \lambda)z$.*

It can be checked easily that the extreme points of

$$P = \text{conv}\{e_1, e_2, e_3\}$$

are e_1, e_2 and e_3.

Clearly, Definition 1.6 is entirely geometric and does not refer to a specific representation of a polyhedron in terms of linear constraints.

Next, we give a definition that relies on a representation of a polyhedron in terms of linear constraints. Some terminology is necessary for this purpose.

Consider a polyhedron $P \subseteq \Re^n$, defined in terms of the linear equality and inequality constraints

$$a_i^T x \geq b_i, \ i \in M_1$$

$$a_i^T x \leq b_i, \ i \in M_2$$

$$a_i^T x = b_i, \ i \in M_3$$

where M_1, M_2 and M_3 are finite index sets, each a_i is a vector in $\Re^n$ and each b_i is a scalar.

For example, let

$$P \ = \ \{x \in \Re^3 \mid a_1^T x \geq 1, \ a_2^T x \leq 3, \ a_3^T x = 1, \ x \geq 0\} \tag{1.1}$$

where $a_1 = (0,0,2)^T, a_2 = (4,0,0)^T$ and $a_3 = (1,1,1)^T$. Let $a_4 = e_1, a_5 = e_2$ and $a_6 = e_3$. Then

$$M_1 = \{1,4,5,6\},\ M_2 = \{2\},\ M_3 = \{3\}.$$

Definition 1.7 *If a vector x^* satisfies $a_i^T x^* = b_i$ for some $i \in M_1, M_2$ or M_3, we say that the corresponding constraint is* **active** *or* **binding** *at x^*. The* **active set** *of P at x^* is defined as*

$$I(x^*) = \{i \in M_1 \cup M_2 \cup M_3 \mid a_i^T x^* = b_i\},$$

i.e., $I(x^*)$ is the set of indices of constraints that are active at x^*.

For example, suppose that P is defined by (1.1). Let $x^* = (0.5, 0, 0.5)^T$. All active constraints at x^* are

$$a_1^T x \geq 1,\ a_3^T x = 1,\ a_5^T x (= x_2) \geq 0$$

and

$$I(x^*) = \{1,3,5\}.$$

Recall that vectors $x^1, \ldots, x^k \in \Re^n$ are said to be linearly independent if

$$\alpha_1 x^1 + \ldots + \alpha_k x^k = 0 \quad \Longrightarrow \quad \alpha_1 = \ldots = \alpha_k = 0.$$

The maximal number of linearly independent vectors in $\Re^n$ is exactly n. Thus $k \leq n$ if $x^1, \ldots, x^k \in \Re^n$ are linearly independent. Note that $x^1, \ldots, x^n \in \Re^n$ are linearly independent if and only if the matrix $M = [x^1 \ \ldots \ x^n]$ is nonsingular, i.e., the determinant of M is not zero.

If there are n constraints of $P \subseteq \Re^n$ that are active at a vector x^*, then x^* satisfies a certain system of n linear equations in n unknowns. This system has a unique solution if any only if the n vectors a_i of these n equations are linear independent. This is stated precisely in the following proposition.

Proposition 1.1 *Let $x^* \in \Re^n$. The following are equivalent.*

(a) There exist n *vectors in the set* $\{a_i \mid i \in I(x^*)\}$*, which are linearly independent.*

(b) The span of the vectors a_i*,* $i \in I(x^*)$*, is all of* $\Re^n$*, that is, every element of* $\Re^n$ *can be expressed as a linear combination of the vectors* $a_i, i \in I(x^*)$*.*

(c) The system of equations $a_i^T x = b_i, i \in I(x^*)$*, has a unique solution.*

[**Observations:** If $I(x^*)$ contains exactly n elements, the proof of the proposition is trivial. $I(x^*)$ may contain more than n elements.]

Proof. (a) $\Longleftrightarrow$ (b)

Suppose that the vectors $a_i, i \in I(x^*)$, span $\Re^n$. Then, the span of these vectors has dimension n. This clearly implies that exist n vectors in the set $\{a_i \mid i \in I(x^*)\}$, which are linearly independent because otherwise if the maximal number of linearly independent vectors is $k \leq n-1$ the the span of these vectors would have dimension k.

Conversely, suppose that n of the vectors $a_i, i \in I(x^*)$, are linearly independent. Then, the subspace spanned by these n vectors is n-dimensional and must be equal to $\Re^n$. Hence, every element of $\Re^n$ is a linear combination of the vectors $a_i, i \in I(x^*)$.

(b) $\Longleftrightarrow$ (c)

If the system of equations $a_i^T x^* = b_i, i \in I(x^*)$, has multiple solutions, say x^1 and x^2, then the nonzero vector $d = x^1 - x^2$ satisfies $a_i^T d = 0, i \in I(x^*)$. Then for any linear combination $\sum_{i \in I(x^*)} \alpha_i a_i$ of vectors $a_i, i \in I(x^*)$, one has

$$d^T \left(\sum_{i \in I(x^*)} \alpha_i a_i \right) = \sum_{i \in I(x^*)} \alpha_i d^T a_i = 0.$$

This, together with the fact that $d^T d > 0$, shows that d is not a linear combination of these vectors. Thus, $a_i, i \in I(x^*)$ do not span $\Re^n$.

Conversely, if the vectors $a_i, i \in I(x^*)$, do not span $\Re^n$, choose a nonzero vector d

which is orthogonal to the subspace spanned by these vectors. If x satisfies $a_i^T x = b_i$ for all $i \in I(x^*)$, we also have $a_i^T(x+d) = b_i$ for all $i \in I(x^*)$, thus obtaining multiple solutions. We have therefore established that (b) and (c) are equivalent. **Q.E.D.**

With a slight abuse of language, we will often say that certain *constraints* are *linearly independent*, meaning that the corresponding vectors a_i are linearly independent. We are now ready to provide an algebraic definition of a corner point of the polyhedron P.

Definition 1.8 *Let $x^* \in \Re^n$.*

(a) The vector x^* is called a **basic solution** *if*

(i) $a_i^T x^* = b_i$, $i \in M_3$.

(ii) Out of $\{a_i\}_{i \in I(x^*)}$, there are n of them that are linearly independent.

(b) If x^* is a basic solution that satisfies all of the constraints, we say that it is a **basic feasible solution**.

Let P be defined by (1.1). Then $x^* = (0.5, 0, 0.5)^T$ is a basic feasible solution because $x^* \in P$ and a_1, a_3, a_5 are linearly independent.

Let us take another example by assuming that $P = \{y \in \Re^m \mid A^T y \le c\}$, where $A \in \Re^{m \times n}$ and $c \in \Re^n$ (surprisingly familiar? Think about the dual form of a linear programming problem). Then $y \in \Re^m$ is a basic solution if $A_i^T y = c_i$, $i \in J \subseteq \{1, \ldots, n\}$ and there exist m vectors in $\{A_i\}_{i \in J}$ such that they are linearly independent. Here A_i denotes the ith column of A. y is a basic feasible solution if it is a basic solution and $A^T y \le c$.

Exercise: What is a basic (feasible) solution to $P = \{x \in \Re^n | \; Ax = b, \; x \ge 0\}$?

Note that if the number m of constraints used to define a polyhedron $P \subseteq \Re^n$ is less than n, the number of active constraints at any given point must also be less than n, and there are no basic or basic feasible solutions.

1.9 Finite basis theorem for polyhedra

Definition 1.9 *A set $C \subseteq \Re^n$ is a* **cone** *if $\lambda x \in C$ for all $\lambda \geq 0$ and all $x \in C$.*

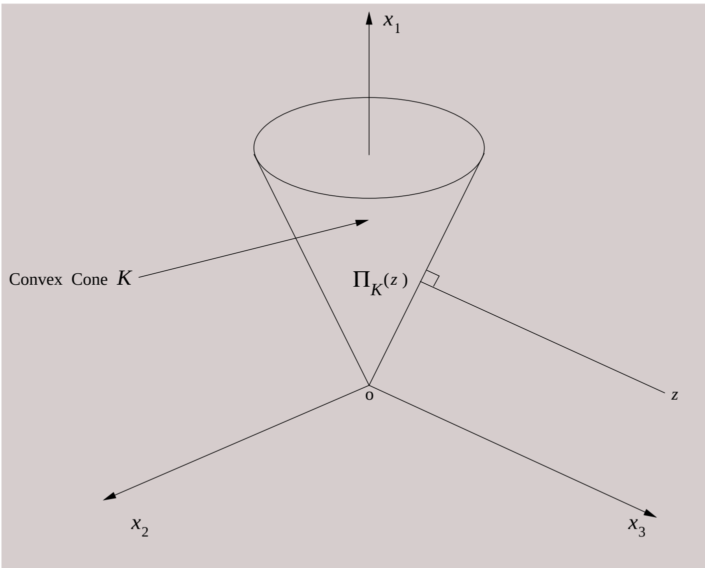

Figure 1.2: A closed convex cone K

From the definition we can see that $0 \in C$. For vectors $x^1, \ldots, x^k \in \Re^n$, let

$$\begin{aligned} &\text{cone}\{x^1, \ldots, x^k\} \\ = &\left\{ x \in \Re^n \mid x = \sum_{i=1}^{k} \lambda_i x^i,\ \lambda_i \geq 0,\ i = 1, \ldots, k \right\}. \end{aligned}$$

Then, cone$\{x^1, \ldots, x^k\}$ is a cone and convex set, which is called the convex cone generated by $x^1, \ldots, x^k$.

The set $P = \{x \in \Re^n \mid Ax \geq 0\}$ is called a **polyhedral cone**.

Excercise: Is cone$\{x^1, \ldots, x^k\}$ a polyhedral cone?

Given $A \in \Re^{m \times n}$ and $b \in \Re^m$, consider

$$P = \{x \in \Re^n \mid Ax \geq b\}$$

and $y \in P$.

Definition 1.10 *The* **recession cone** *of P at y is defined as the set*

$$\{d \in \Re^n |\ A(y + \lambda d) \geq b, \text{ for all } \lambda \geq 0\}.$$

Roughly speaking, the recession cone of P at y is the set of all directions d along which we can move indefinitely away from y, without leaving the set P. It can be easily seen that the recession cone of P at y is the same as

$$\{d \in \Re^n |\ Ad \geq 0\},$$

and is the polyhedral cone. This means that the recession cone is independent of the starting point y. For $P = \{x \in \Re^n | Ax = b, x \geq 0\}$, where $A \in \Re^{m \times n}$ and $b \in \Re^m$, the recession cone is

$$\{d \in \Re^n |\ Ad = 0, d \geq 0\}.$$

Definition 1.11

(a) A nonzero element d of a polyhedral cone $C \subseteq \Re^n$ is called an **extreme ray** *if there are $n-1$ linearly independent constraints that are active at d.*

(b) An extreme ray of the recession cone associated with a nonempty polyhedron P is also called an extreme ray of P.

For example, consider the simple polyhedral cone $\Re^n_+$. Then the extreme rays of $\Re^n_+$ are

$$\{e_1, \ldots, e_n\}.$$

Since there are n linearly independent constraints that are active at the zero vector, by the definition the zero vector is an extreme point of $\Re^n_+$ (actually the only extreme point).

The following theorem is called the finite basis theorem for polyhedra, obtained by Minkowski.

Theorem 1.1 *Let $P = \{x \in \Re^n \mid Ax \leq b\}$, where $A \in \Re^{m \times n}$ and rank (A)=n. Then there exist $x^1, \ldots, x^q$ and $d^1, \ldots, d^r$ in $\Re^n$ such that*

$$P = \text{conv}\{x^1, \ldots, x^q\} + \text{cone}\{d^1, \ldots, d^r\},$$

where $\{x^1, \ldots, x^q\}$ is the set of extreme points (basic feasible solutions) of P and $\{d^1, \ldots, d^r\}$ is the set of extreme rays of P.

Note that Theorem 1.1 implies that P can only have finitely many extreme points. It can also be seen easily that P is bounded if and only if the recession cone of P contains the zero vector only. There are several ways to prove the above theorem. For example, one may use Farkas' lemma and the duality theory of linear programming to prove the above theorem.

1.10 Simplex Method Revisited

Consider the standard linear programming problem

$$(P) \qquad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array} \tag{1.2}$$

where $A \in \Re^{m \times n}$ $(m \leq n)$ is of full row rank, and its dual form

$$(D) \qquad \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y \leq c. \end{array} \tag{1.3}$$

Let x be a basic feasible solution to the standard form problem, let $B(1), \dots, B(m)$ be the indices of the basic variables, and let $B = [A_{B(1)} \dots A_{B(m)}]$ the corresponding basis matrix. In particular, we have $x_i = 0$ for every nonbasic variable, while the vector $x_B = (x_{B(1)}, \dots, x_{B(m)})^T$ of basic variables is given by

$$x_B = B^{-1} b \geq 0.$$

The full tableau implementation of the simplex method at the very beginning is:

$$\begin{array}{|c|ccc|} \hline 0 & c_1 & \dots & c_n \\ \hline b_1 & | & & | \\ \vdots & A_1 & \dots & A_n \\ b_m & | & & | \\ \hline \end{array}$$

Let $\bar{c}^T = c^T - c_B^T B^{-1} A$. By using elementary row operations to change $c_{B(1)}, \dots, c_{B(m)}$ to be zero, we have the following tableau:

$$\begin{array}{|c|ccc|} \hline -c_B^T x_B & \bar{c}_1 & \dots & \bar{c}_n \\ \hline x_{B(1)} & | & & | \\ \vdots & B^{-1}A_1 & \dots & B^{-1}A_n \\ x_{B(m)} & | & & | \\ \hline \end{array}$$

Anticycling rules: Lexicography and Bland's rule.

Finding an initial basic feasible solution: The artificial variables method and the big$-M$ method.

For the **dual simplex method**, we have

0	c_1	$\ldots$	c_n
b_1	$\vert$		$\vert$
$\vdots$	A_1	$\ldots$	A_n
b_m	$\vert$		$\vert$

and

$-c_B^T x_B$	$\bar{c}_1$	$\ldots$	$\bar{c}_n$
$x_{B(1)}$	$\vert$		$\vert$
$\vdots$	$B^{-1}A_1$	$\ldots$	$B^{-1}A_n$
$x_{B(m)}$	$\vert$		$\vert$

We do not require $B^{-1}b$ to be nonnegative, which means that we have a basic, but not necessarily feasible solution to the primal problem. However, we assume that $\bar{c} \geq 0$; equivalently, the vector $y^T = c_B^T B^{-1}$ satisfies $y^T A \leq c^T$, and we have a feasible solution to the dual problem. The cost of this dual feasible solution is $y^T b = c_B^T B^{-1} b = c_B^T x_B$, which is the negative of the entry at the upper left corner of the tableau.

1.11 Graphs and Digraphs

1.11.1 Graphs

Definition 1.12 *A graph G is a pair (V, E), where V is a finite set and E is a set of unordered pairs of elements of V. Elements of V are called vertices and elements of E edges. We say that a pair of distinct vertices are adjacent if they define an edge, and that the edge is said to be incident to its defining vertices. The degree of a vertex v (denoted $\deg(v)$) is the number f edges incident to that vertex.*

An Example.

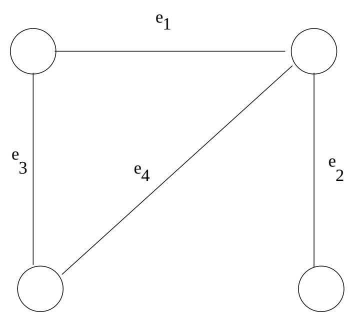

Figure 1.3: A Graph

Definition 1.13 *An v_1v_k-path (or path connecting v_1 and v_k) is a sequence of edges*

$$v_1v_2, \ldots, v_{i-1}v_i, \ldots, v_{k-1}v_k.$$

A cycle is a sequence of edges

$$v_1v_2, \ldots, v_{i-1}v_i, \ldots, v_{k-1}v_k, v_kv_1.$$

In both cases vertices are all distinct. A graph is acyclic if it has no cycle.

Proposition 1.2 *If every vertex of G has degree of at least two then G has a cycle.*

Proof. Let $P = v_1v_2, \ldots, v_{k-1}v_k$ be a path of G with a maximum number of edges. Since $\deg(v_k) \geq 2$, there is an edge v_kw where $w \neq v_{k-1}$. It follows from the choice of P that w is a vertex of P, i.e., $w = v_i$ for some $i \in \{1, \ldots, k-2\}$. Then $v_iv_{i+1}, \ldots, v_{k-1}v_k, v_kv_i$ is a cycle. Q.E.D.

Definition 1.14 *G is connected if each pair of vertices is connected by a path.*

Proposition 1.3 *Let G be a connected graph with a cycle C and let e be an edge of C. Then $G - e$ is connected.*

Proof. Let v_1, v_2 be vertices of $G - e$. We need to show there exists a v_1v_2-path P' of $G - e$. Since G is connected there exists a v_1v_2-path P of G. If P does not use e then we are done. Otherwise P implies there exists a v_1w_1-path P_1 and a w_2v_2-path P_2, where w_1, w_2 are endpoints of e. Moreover, $C - w_1w_2$ is a w_1w_2-path. The result now follows. Q.E.D.

Definition 1.15 *H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It is a spanning subgraph if in addition $V(H) = V(G)$.*

Definition 1.16 *A tree is a connected acyclic graph.*

Theorem 1.2 *If $T = (V, E)$ is a tree, then $|E| = |V| - 1$.*

Proof. Let us proceed by induction of the number of vertices of V. The base case $|V| = 1$ is trivial since then $|E| = 0$. Assume now $|V| \geq 2$ and suppose the theorem holds for all trees with $|V| - 1$ vertices. Since T is acyclic, it follows form Proposition 1.2 that there is a vertex v with $\deg(v) \leq 1$. Since T is connected and $|V| \geq 2$, $\deg(v) \neq 0$. Thus, there is a unique uv incident to v. Let T' be defined as follows $V(T') = V - \{v\}$ and $E(T') = E - \{uv\}$. Observe that T' is a tree. Hence by induction $|E(T')| = |V(T')| - 1$ and it follows $|E| = |V| - 1$. Q.E.D.

Proposition 1.4 *Let $G = (V, E)$ be a connected graph. Then $|E| \geq |V| - 1$. Moreover, if equality holds then G is a tree.*

Proof. If G has a cycle then remove from G any edge on the cycle. Repeat until the resulting graph T is acyclic. It follows from Proposition 1.3 that T is connected. Hence T is a tree and by Theorem 1.2,

$$|E(G)| \geq |E(T)| = |V(G)| - 1.$$

Q.E.D.

1.11.2 Bipartite Graph

$G = (S, T, E)$: For any edge in E with one vertex in S and the other in T.

1.11.3 Vertex-Edge Incidence Matrix

Definition 1.17 *The* **vertex-edge incidence matrix** *of a graph $G = (V, E)$ is a matrix A with $|V|$ rows and $|E|$ columns whose entries are either 0 or 1 such that*

- *The rows correspond to the vertices of G,*
- *The columns correspond to the edges of G, and the entry $A_{v,ij}$ for vertex v and edge ij is given by*

$$A_{v,ij} = \begin{cases} 0 & \text{if } v \neq i \text{ and } v \neq j \\ 1 & \text{if } v = i \text{ or } j. \end{cases}$$

1.11.4 Digraphs (Directed Graphs)

Definition 1.18 *A directed graph (or digraph) D is a pair (N, A) where N is a finite set and A is a set of ordered pairs of elements of N. Elements of N are called nodes and elements of A arcs. Node i is the tail (resp. head) of arc ij. The in-degree (resp. out-degree) of node v (denoted $\deg^+(v)$ (resp. $\deg^-(v)$)) is the number of arcs with head (resp. tail) v.*

1.11.5 Bipartite Digraph

$D = (S, T, A)$

1.11.6 Node-Arc Incidence Matrix

Definition 1.19 *The* **node-arc incidence matrix** *of a graph $D = (N, A)$ is a matrix M with $|V|$ rows and $|A|$ columns whose entries are either 0, +1, or -1 such that*

- *The rows correspond to the nodes of D,*
- *The columns correspond to the arcs of D, and the entry $M_{v,ij}$ for node v and arc*

ij is given by

$$M_{v,ij} = \begin{cases} 0 & \text{if } v \neq i \text{ and } v \neq j \\ +1 & \text{if} v = j, \text{ and} \\ -1 & \text{if } v = i. \end{cases}$$