NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
Semester I (2006/2007) MA4260 Model Building in OR
(Supplementary material on eigenvalues and eigenvectors)

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For any $A \in \mathbb{C}^{n\times n}$, the *spectrum* $\sigma(A)$ of A is the set of complex numbers ζ such that $\zeta I - A$ is not one-to-one. The determinant $\det(\zeta I - A)$ of the matrix $\zeta I - A$ is called the *characteristic polynomial* of A. By the definition of $\sigma(A)$, for any $\mu \in \sigma(A)$, there exists a vector $0 \neq v \in \mathbb{C}^n$ such that $(A-\mu I)v = 0$. The number μ is called an *eigenvalue* of A, and any corresponding v is called an eigenvector. The spectrum $\sigma(A)$ is always nonempty and A has at most n distinct eigenvalues as all eigenvalues of A are roots of the characteristic polynomial of A.

For any $A \in \mathbb{R}^{n\times n}$, the spectrum $\sigma(A)$ of A may contain no real numbers, for example,

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

has only two complex eigenvalues $1 \pm \sqrt{-1}$. However, if A is symmetric, i.e., $A = A^T$, all eigenvalues of A are *real* and for each eigenvalue $\mu \in \sigma(A)$, there exists a vector $0 \neq v \in \mathbb{R}^n$ such that $Av = \mu v$. More importantly, one can choose orthogonal eigenvectors $v_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$ such that

$$v_i^T v_i = 1, \quad v_i^T v_j = 0 \ (j \neq i), \ i, j = 1, 2, \ldots, n$$

and for each $\lambda_i \in \sigma(A)$, $Av_i = \lambda_i v_i$, $i = 1, 2, \ldots, n$. Let $Q := [v_1 \ v_2 \ \cdots v_n]$. Then $Q^T Q = QQ^T = I$ and

$$A = Q\mathrm{diag}(\lambda_1\,, \lambda_2\,, \cdots, \lambda_n)Q^T\,.$$

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called positive semidefinite, denoted by $A \succeq 0$, if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$. This is equivalent to say that each eigenvalue of A is nonnegative. If A is a symmetric matrix and $A \succeq 0$, then A has a unique "square root" given by

$$\sqrt{A} = Q\mathrm{diag}(\sqrt{\lambda_1}\,, \sqrt{\lambda_2}\,, \cdots, \sqrt{\lambda_n})Q^T\,.$$

One may simply check that $\sqrt{A}$ is symmetric, $\sqrt{A} \succeq 0$, and indeed $\sqrt{A}\sqrt{A} = A$.

When you use `MatLab` to calculate $\sqrt{A}$, you may first use the command `eig` as follows

$$[Q, D] = \mathtt{eig}(A)\,,$$

where Q is the above orthogonal matrix and the diagonal part of D contains all the eigenvalues of A.