**NATIONAL UNIVERSITY OF SINGAPORE**
**Department of Mathematics**
**Semester 1 (2003/2004) MA4253 Mathematical Programming Tutorial 4**

Solution to selected questions

**Q1.** Similar as the one given in lecture notes.

**Q2.** Let

$$P = \{x \,|\, 4x_1 + 9x_2 \le 18, \; -2x_1 + 4x_2 \le 4\,, \; x_1\,, x_2\,, x_3 \ge 0\,\}\,.$$

Then any $x \in P$ can be written

$$x = \sum_{i=1}^{4} z_i x^i + z_5 x^5\,, \quad \sum_{i=1}^{4} z_i = 1\,, z_i \ge 0, i = 1, \dots, 5, \,,$$

where

$$x^1 = \begin{pmatrix} 4.5 \\ 0 \\ 0 \end{pmatrix}, x^2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, x^3 = \begin{pmatrix} \frac{18}{17} \\ \frac{26}{17} \\ 0 \end{pmatrix}, x^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, x^5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let

$$c = (-1 \;\; -1 \;\; 0)^T, \quad b = 2\,, \quad \text{and} \quad A = [1 \;\; -1 \;\; 1].$$

Then the Dantzig-Wolfe master program becomes

$$\begin{array}{rl} \min & \displaystyle\sum_{i=1}^{5} f_i z_i \\ \text{s.t.} & \displaystyle\sum_{i=1}^{4} p^i z_i + p^5 z_5 = b \\ & \displaystyle\sum_{i=1}^{4} z_i + 0 z_5 = 1 \\ & z_1, z_2, z_3, z_4, z_5 \ge 0\,, \end{array}$$

where

$$f_i = c^T x^i, \;\; p^i = A x^i, \; i = 1, \dots, 5\,.$$

Take $\Lambda = \{1, 2\}$. Then the restricted Dantzig-Wolfe master program becomes

$$\begin{array}{rl} \min & f_1 z_1 + f_2 z_2 \\ \text{s.t.} & p^1 z_1 + p^1 z_2 = b \\ & z_1 + z_2 = 1 \\ & z_1, z_2 \ge 0\,, \end{array}$$

i.e.,

$$\begin{array}{ll} \min & -4.5z_1 - 2z_2 \\ \text{s.t.} & 4.5z_1 + (-2)z_2 = 2 \\ & z_1 + z_2 = 1 \\ & z_1, z_2 \geq 0\,. \end{array}$$

The unique solution to the above problem is

$$z_1 = \frac{8}{13}\,, \quad z_2 = \frac{5}{13}\,.$$

The optimal dual multipliers are

$$\begin{bmatrix} u \\ u_0 \end{bmatrix} = \begin{bmatrix} -5/11 \\ -27/11 \end{bmatrix}.$$

The next step is to solve the subproblem

$$\begin{array}{ll} \min & (c - A^T u)^T x - u_0 \\ \text{s.t.} & x \in P\,. \end{array}$$

i.e.,

$$\begin{array}{ll} \min & (-6/11)x_1 + (-16/11)x_2 + (5/11)x_3 - (-27/11) \\ \text{s.t.} & 4x_1 + 9x_2 \leq 18 \\ & -2x_1 + 4x_2 \leq 4 \\ & x_1\,, x_2\,, x_3 \geq 0\,. \end{array}$$

The optimal solution to the subproblem is

$$x = x^3.$$

Since the optimal cost of the subproblem is

$$-\frac{524}{187} - (-27/11) < 0\,,$$

it follows that the reduced cost of the variable $z_3$ is negative, and this variable can enter the basis.

Let $\Lambda = \{1, 2, 3\}$ ... .

**Q3.** (a) The linear programming problem

$$\begin{array}{ll} \min & 10y_1 - 2y_2 + 4y_3 + 8y_4 + y_5 \\ \text{s.t.} & y_1 - 4y_2 - y_3 \geq 8 \\ & 2y_1 - y_2 + y_3 \geq 2 \\ & 3y_1 + y_4 + y_5 \geq 4 \\ & y_1 + 2y_4 - y_5 \geq 10 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0\,, \end{array}$$

can be written equivalently as

$$\begin{aligned} -\max \quad & b^T y \\ \text{s.t.} \quad & W^T y \le c \\ & -Iy \le 0\,. \end{aligned}$$

So its dual problem is

$$\begin{aligned} -\min \quad & c^T x + 0^T s \\ \text{s.t.} \quad & Wx - s = b \ \ (\text{i.e.}, -Wx + s = -b) \\ & x\,,\, s \ge 0\,, \end{aligned}$$

i.e.,

$$\begin{aligned} -\min \quad & (-8)x_1 + (-2)x_2 + (-4)x_3 + (-10)x_4 + 0^T s \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 + x_4 + s_1 = 10 \\ & -4x_1 - x_2 + s_2 = -2 \\ & -x_1 + x_2 + s_3 = 4 \\ & x_3 + 2x_4 + s_4 = 8 \\ & x_3 - x_4 + s_5 = 1 \\ & x\,,\, s \ge 0\,. \end{aligned} \tag{1}$$

This dual problem can be simplified as

$$\begin{aligned} -\min \quad & (-8)x_1 + (-2)x_2 + (-4)x_3 + (-10)x_4 + (0)x_5 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 10 \\ & -4x_1 - x_2 \le -2 \\ & -x_1 + x_2 \le 4 \\ & x_3 + 2x_4 \le 8 \\ & x_3 - x_4 \le 1 \\ & x \ge 0\,. \end{aligned} \tag{2}$$

Problem (2) is a block angular linear problem. Therefore, the Dantzig-Wolfe decomposition method can be applied to solve problem (2), and thus problem (1).

(b) Use the optimal basis matrix obtained in (a) for problem (1).