LARGE SCALE COMPOSITE OPTIMIZATION PROBLEMS WITH COUPLED OBJECTIVE FUNCTIONS: THEORY, ALGORITHMS AND APPLICATIONS

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To my parents

DECLARATION

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Cui Ying

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Contents

Summary

In this thesis, we conduct a thorough study on the theory and algorithms for large scale multi-block convex composite problems with coupled objective functions. These problems arise from a variety of application areas.

This thesis is mainly divided into two parts. The first part is focused on solving unconstrained multi-block large scale convex composite optimization problems with coupled objective functions. A two-block inexact majorized accelerated block coordinate descent method is proposed for problems without joint constraints, and the $O(1/k^2)$ iteration complexity is proved. We illustrate the implementation of this framework for solving the dual of an important class of composite least square problems that involves the nonsmooth regularized terms, the equations, the inequalities and the cone constraints. For solving subproblems, we adopt the inexact one cycle symmetric Gauss-Seidel technique proposed recently in [60] and a hybrid of the semismooth Newton-CG method and the accelerated proximal gradient method. The incorporation of the second order information plays a pivotal role in making our algorithms and the other existing ones more efficient. Numerical results demonstrate that our proposed methods outperform, by a large margin, existing different variants of the block coordinate descent methods.

The second part of this thesis is devoted to the study of the constrained convex composite optimization problems with joint linear constraints. A majorized alternating direction method of multipliers, which was discussed for problems with separable objective functions in the existing literature, is extended to deal with this class of problems. The global convergence and the ergodic and non-ergodic iteration complexities are presented. We also prove the linear convergence rate of the method for solving quadratically coupled problems under an error bound condition. For the purpose of deriving checkable conditions for the error bound, we present a characterization of the robust isolated calmness of the constrained problems penalized by the nuclear norm function via the second order sufficient optimality condition and the strict Robinson constraint qualification. The robust isolated clamness has its own interest beyond the implication of the error bound condition. For the convex composite nuclear norm problems, several equivalent conditions for the robust isolated calmness are obtained.

Chapter 1

Introduction

1.1 Motivations and related methods

This thesis is focused on designing and analyzing efficient algorithms for solving multi-block large scale convex optimization problems, with or without joint linear constraints. The first model under consideration is the following unconstrained optimization problems:

$$\boxed{\text{(P1)} \qquad \min \theta(u,v) := \sum_{i=1}^{s} p_i(u_i) + \sum_{j=1}^{t} q_j(v_j) + \phi(u,v), \qquad (1.1)}$$

where s and t are two given nonnegative integers, $u \equiv (u_1, u_2, \ldots, u_s) \in \mathcal{U}$ and $v \equiv (v_1, v_2, \ldots, v_t) \in \mathcal{V}$ are two groups of variables, $p_i : \mathcal{U}_i \to (-\infty, \infty]$, $i = 1, \ldots, s$ and $q_j : \mathcal{V}_j \to (-\infty, \infty]$, $j = 1, \ldots, t$ are simple closed proper convex functions (possibly nonsmooth), $\phi : \mathcal{U} \times \mathcal{V} \to (-\infty, \infty)$ is a smooth convex function whose gradient mapping is Lipschitz continuous, $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_s$ and $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2 \times \ldots \times \mathcal{V}_t$ are real finite dimensional Euclidean spaces each equipped with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Our aim is to solve large scale problems in the form of (1.1) of medium to high accuracy.

A natural extension of the unconstrained problem (1.1) in the two-block case is the following linearly constrained convex problems with coupled objective functions:

$$\begin{array}{|lll|}\text{(P2)} & \min & p(u)+q(v)+\phi(u,v), \\ & \text{s.t.} & \mathcal{A}^*u+\mathcal{B}^*v=c,\end{array} \tag{1.2}$$

where $p:\mathcal{U}\to(-\infty,+\infty]$ and $q:\mathcal{V}\to(-\infty,+\infty]$ are two nonsmooth closed proper convex functions, $\phi:\mathcal{U}\times\mathcal{V}\to\mathcal{R}$ is a smooth convex function with Lipschitz continuous gradient, $\mathcal{A}:\mathcal{X}\to\mathcal{U}$ and $\mathcal{B}:\mathcal{X}\to\mathcal{V}$ are two linear operators, $c\in\mathcal{X}$ is the given data, and $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{X}$ are finite dimensional Euclidean spaces. In this thesis, we also propose an algorithm to solve (1.2) and analyze the related properties.

Before going into details, we shall point out that throughout this thesis, we name the problems in the form of (1.1) unconstrained optimization problems. This terminology does not indicate that the problem (1.1) is free of any constraint. For example, we still allow the decision variables to stay in some convex sets by adding indicator functions over these sets in the objective functions. The expected way to interpret the word "unconstrained" is that no joint constraints across different blocks are allowed.

1.1.1 Unconstrained problems

Our first motivation to study the problems of the form (1.1) comes from the dual of the nonlinearly constrained strongly convex problems:

$$\begin{array}{ll}\min & f(x)+\theta(x) \\ \text{s.t.} & \mathcal{A}x=b, \quad g(x)\in\mathcal{C}, \quad x\in\mathcal{K},\end{array} \tag{1.3}$$

where $f:\mathcal{X}\to\mathcal{R}$ is a smooth and strongly convex function, $\theta:\mathcal{X}\to(-\infty,+\infty]$ is a closed proper convex function, $\mathcal{A}:\mathcal{X}\to\mathcal{Y}_E$ is a linear operator, $b\in\mathcal{Y}_E$ is the given data, $g:\mathcal{X}\to\mathcal{Y}_g$ is a smooth map, $\mathcal{C}\subseteq\mathcal{Y}_g$ and $\mathcal{K}\subseteq\mathcal{X}$ are two closed convex cones, and $\mathcal{X}$, $\mathcal{Y}_E$, $\mathcal{Y}_g$ are finite dimensional Euclidean spaces each equipped with an inner product $\langle\cdot,\cdot\rangle$ and its induced norm. In order to make (1.3) a convex problem, we further require g is $\mathcal{C}$-convexity [78, Example 4']:

$$g(\alpha x_1+(1-\alpha)x_2)-(\alpha g(x_1)+(1-\alpha)g(x_2))\in\mathcal{C},\ \forall\alpha\in(0,1).$$

In order to write down the dual of the problem (1.3), we introduce a slack variable $u = x \in \mathcal{X}$ and recast the problem (1.3) as

$$\begin{array}{ll} \min & f(x) + \theta(u) \\ \text{s.t.} & \mathcal{A}x = b, \quad g(x) \in \mathcal{C}, \quad x \in \mathcal{K}, \quad x = u. \end{array} \tag{1.4}$$

Given $(x, u, y, \lambda, s, z) \in \mathcal{X} \times \mathcal{X} \times \mathcal{Y}_E \times \mathcal{Y}_g \times \mathcal{X} \times \mathcal{X}$, the Lagrangian function of the problem (1.4) takes the form of

$$L(x, u; y, \lambda, s, z) := f(x) + \theta(u) - \langle y, \mathcal{A}x - b\rangle - \langle g(x), \lambda\rangle - \langle x, s\rangle - \langle z, x - u\rangle.$$

In this way, the dual of the problem (1.4) can be written as

$$\begin{array}{ll} \max & \psi(\mathcal{A}^*y + s + z, \lambda) + \langle b, y_E\rangle - \theta^*(-z) \\ \text{s.t.} & \lambda \in \mathcal{C}^*, \quad s \in \mathcal{K}^*, \end{array} \tag{1.5}$$

where the function $\psi : \mathcal{X} \times \mathcal{Y}_g \to \mathcal{R}$ is defined as

$$\psi(w, \lambda) := \inf_{x \in \mathcal{X}} \{f(x) - \langle w, x\rangle - \langle \lambda, g(x)\rangle\}, \quad \forall (w, \lambda) \in \mathcal{X} \times \mathcal{Y}_g. \tag{1.6}$$

By the assumption that g is $\mathcal{C}$-convexity, it is easy to see that $-\langle \lambda, g(x)\rangle$ is convex with respect to x for $\lambda \in \mathcal{C}^*$. Since f is assumed to be strongly convex, the optimal solution of the above problem is a singleton and thus ψ is continuously differentiable [30, Theorem 10.2.1]. In this way, the problem (1.5) falls into the framework of (1.1) with four blocks of variables (y, s, z, λ).

The model (1.3) includes many interesting applications. One particular example is the following regularized best approximation problem:

$$\begin{array}{ll} \min & \frac{1}{2}\|X - G\|^2 + \theta(X) \\ \text{s.t.} & \mathcal{A}X = b, \quad \mathcal{B}X \geq d, \quad X \in \mathcal{K}, \end{array} \tag{1.7}$$

where $\mathcal{A} : \mathcal{X} \to \mathcal{Y}_E$ and $\mathcal{B} : \mathcal{X} \to \mathcal{Y}_I$ are two linear operators, $G \in \mathcal{X}$, $b \in \mathcal{Y}_E$, $d \in \mathcal{Y}_I$ are given data, $\mathcal{K} \subseteq \mathcal{X}$ is a closed convex cone, $\theta : \mathcal{X} \to (-\infty, +\infty]$ is a simple proper closed convex function such as the indicator functions over the polyhedral or the norm functions. One can easily see that the problem (1.7) is a special case of (1.3) by taking $f(X) = \frac{1}{2}\|X - G\|^2$, $g(X) = \mathcal{B}X - d$ for any $X \in \mathcal{X}$ and letting $\mathcal{C} = \{y_I \in \mathcal{Y}_I : y_I \geq 0\}$. In this case, the function ψ defined in (1.6) can be computed explicitly as

$$\psi(w, \lambda) = -\frac{1}{2}\|G + w + \mathcal{B}^*\lambda\|^2 + \frac{1}{2}\|G\|^2 + \langle \lambda, d\rangle, \quad \forall (w, \lambda) \in \mathcal{X} \times \mathcal{Y}_I.$$

Hence, the dual of the problem (1.7) is given by

$$\begin{array}{ll} \max & -\frac{1}{2}\|G + \mathcal{A}^*y + \mathcal{B}^*\lambda + s + z\|^2 + \langle b, y\rangle + \langle \lambda, d\rangle - \theta^*(-z), \\ \text{s.t.} & \lambda \geq 0, \quad s \in \mathcal{K}^*. \end{array} \tag{1.8}$$

The study of the best approximation problems dates back to the 1980s, when the inequality constraint $\mathcal{B}X \in \mathcal{C}$ and the regularized term θ are not included. People are interested in its application to the interpolation with $\mathcal{K} = \{L_2[0, 1] \mid x \geq 0 \text{ a.e. on } [0, 1]\}$ at that time [48, 1, 23, 25, 26]. Around fifteen years ago, the positive semidefinite cone constraint is also under consideration with wide applications in calibrating the covariance (correlation) matrix in Finance [43]. For the related algorithms, see, for examples, [66, 75]. This model is further extended with the inequality constraints in [7, 36] with a dual approach. Recently, people also focus on the regularized least square problems with a nonsmooth regularized term θ in the objective function in order to impose different structures on the solutions, which are used in the under-sampling problems from the high dimensional data analysis. Two frequently used regularized terms are $\theta(\cdot) = \rho\|\cdot\|_1$ for the sparsity of a solution and $\rho\|\cdot\|_*$ for the low-rankness with given penalty parameter $\rho > 0$ [37, 13].

Another important application of the problem (1.3) is the projection onto the convex quadratically constrained positive semidefinite sets:

$$\begin{array}{ll} \min & \frac{1}{2}\|X - G\|^2 \\ \text{s.t.} & \frac{1}{2}\langle X, \mathcal{Q}_i X\rangle + \langle C_i, X\rangle + r_i \leq 0, \quad i = 1, \ldots, m, \\ & X \in \mathcal{S}^n_+, \end{array} \tag{1.9}$$

where $\mathcal{Q}_i \in \mathcal{S}^n_+, i = 1, 2, \ldots, m$ are given self-adjoint positive semidefinite linear operators, $C_i \in \mathcal{S}^n, i = 1, 2, \ldots, m$ are given matrices and $r_i \in \mathcal{R}, i = 1, 2, \ldots, m$

are given scalars. This is a reduced form of the convex quadratically constrained quadratic problems (CQCQP) considered in [92] by only considering the identity operator in the objective function.

Besides the dual of the problems with the form (1.3), many applications themselves belong to the unconstrained model (1.1). For example, in order to find and explore the structure in high dimensional data, people consider the following low rank and sparsity decomposition approach:

$$\min_{L,S\in\mathcal{R}^{m\times n}} \frac{1}{2}\|D-L-S\|^2+\lambda_1\|L\|_*+\lambda_2\|S\|_1,$$

where $D\in\mathcal{R}^{m\times n}$ is an observed matrix, and λ_1,λ_2 are two positive parameters. This model is named as the robust principle component analysis in [102]. A more complicated one is the robust matrix completion problem considered in [52]:

$$\begin{array}{ll} \displaystyle\min_{L,S\in\mathcal{R}^{m\times n}} & \displaystyle\frac{1}{N}\sum_{i=1}^{N}(Y_i-\langle X_i,L+S\rangle)^2+\lambda_1\|L\|_1+\lambda_2\|S\|_{2,1} \\ \text{s.t.} & \|L\|_\infty\le\alpha,\|S\|_\infty\le\alpha, \end{array} \tag{1.10}$$

where $\|L\|_\infty=\max\limits_{1\le i\le m,1\le j\le n}|L_{ij}|$, $\|S\|_{2,1}=\sum_{i=1}^n\|S_i\|_2$, N is the number of samples, $(X_i,Y_i)\in\mathcal{R}^{m\times n}\times\mathcal{R},\ i=1,2,\ldots,N$ are observations, and $\alpha>0$ is a given upper bound of $\|L\|_\infty$ and $\|S\|_\infty$. This model can be viewed as a two-block unconstrained problem by taking $p(L)=\lambda_1\|L\|_1+\delta_{\|L\|_\infty\le\alpha}(L)$ and $q(S)=\lambda_2\|S\|_{2,1}+\delta_{\|S\|_\infty\le\alpha}(S)$.

It is well known that the problem (1.1) can be solved by the block coordinate descent (BCD) method, where each block is updated sequentially based on the latest information [84, 95, 96]:

$$\left\{\begin{array}{l} u_1^{k+1}=\arg\min_{u_1}\theta(u_1,u_2^k,\ldots,u_s^k,v_1^k,v_2^k,\ldots,v_t^k),\\ \qquad\vdots\\ u_s^{k+1}=\arg\min_{u_s}\theta(u_1^{k+1},u_2^{k+1},\ldots,u_s,v_1^k,v_2^k,\ldots,v_t^k),\\ v_1^{k+1}=\arg\min_{v_1}\theta(u_1^{k+1},u_2^{k+1},\ldots,u_s^{k+1},v_1,v_2^k,\ldots,v_t^k),\\ \qquad\vdots\\ v_t^{k+1}=\arg\min_{v_t}\theta(u_1^{k+1},u_2^{k+1},\ldots,u_s^{k+1},v_1^{k+1},v_2^{k+1}\ldots,v_t). \end{array}\right.$$

To overcome the possible difficulty when solving the subproblems, people also study a variant framework called the block coordinate gradient descent method (BCGD) [98, 3]. A proximal gradient step is taken for each block of this method. Both the BCD and BCGD algorithms, or a hybrid of them, have the iteration complexity $O(1/k)$ for the generic model (1.1) [3, 47].

The Nesterov's acceleration technique [70, 71] is a powerful tool to solve one block convex optimization problems, with the attractive $O(1/k^2)$ convergence rate. However, in order to apply the accelerated proximal gradient (APG) method to the problem (1.1), a very large proximal term has to be added such that the whole problem can be treated as one block. This would no doubt cause the algorithm less efficient. People have already made several attempts to settle this issue. If there are no nonsmooth terms in (1.1), Beck and Tetruashvili [3] prove that the accelerated version of BCGD also enjoys the $O(1/k^2)$ complexity. Recently, Chambolle and Pock [9] show that if the optimization problem only involves two blocks and the coupled smooth term is quadratic, the $O(1/k^2)$ complexity can be achieved by a majorized accelerated BCD algorithm.

Another line of research focus on the randomized updating rule in order to accelerate the BCD-type method, which is initialized by Nesterov's innoative work in [73]. In his paper, Nesterov shows that without the nonsmooth terms, the accelerated BCD method could converge at $O(1/k^2)$ if different blocks are updated alternatively in a random order following the prescribed distribution. This idea is further extended by Fercoq and Richtárik [31] to solve the general problems of the form (1.1) with a large proximal term that proportional to the number of blocks. This is important progress in theory, but its numerical performance is far from satisfactory since a small proximal term is always preferred in practice.

The introduction of the inexactness is essential for efficiently solving the multi-block problems. Researchers have already incorporated this idea into different variants of the BCD and APG algorithms, see, for examples, [85, 49, 100, 93, 33]. There are several reasons to consider the inexactness. One is that many subproblems in the BCD-type algorithm do not have explicit solutions or their computational cost is very demanding, such as the problems involving the total variation regularizer (TV norm) in the image science, or the dual of the nonlinear constraint problems given in (1.5). Another reason, perhaps a more critical one, is that the inexactness allows us to tackle multi-block problems by combining several blocks together and solve them simultaneously by the Newton-type method. This idea has already been implemented in the inexact APG algorithm for solving a least square semidefinite programming in [91].

1.1.2 Linearly constrained problems

As mentioned in the begining of this chapter, we also consider the linearly constrained optimization problems with the coupled objective functions in the form of (1.2). Many interesting optimization problems belong to this class. One particular example is the following problem whose objective is the sum of a quadratic function and a squared distance function to a closed convex set:

$$
\begin{array}{ll}
\min & \frac{1}{2}\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \widetilde{\mathcal{Q}} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle + \frac{\rho}{2}\left\| \begin{pmatrix} u \\ v \end{pmatrix} - \Pi_{\mathcal{K}_1}\begin{pmatrix} u \\ v \end{pmatrix} \right\|^2, \\
\text{s.t.} & \mathcal{A}^* u + \mathcal{B}^* v = c, \\
& u \in \mathcal{K}_2, \ v \in \mathcal{K}_3,
\end{array}
\tag{1.11}
$$

where $\rho > 0$ is a penalty parameter, $\widetilde{\mathcal{Q}} : \mathcal{U} \times \mathcal{V} \to \mathcal{U} \times \mathcal{V}$ is a self-ajoint positive semidefinite linear operator, $\mathcal{K}_1 \subseteq \mathcal{U} \times \mathcal{V}$, $\mathcal{K}_2 \subseteq \mathcal{U}$ and $\mathcal{K}_3 \subseteq \mathcal{V}$ are closed convex sets and $\Pi_{\mathcal{K}_1}(\cdot,\cdot)$ denotes the metric projection onto $\mathcal{K}_1$. The reason behind this model is to treat different constraints separately, some of them need to be strictly satisfied, such as the equation constraints $\mathcal{A}^* u + \mathcal{B}^* v = c$ and $u \in \mathcal{K}_2$, $v \in \mathcal{K}_3$, and others are soft constraints like $(u, v) \in \mathcal{K}_1$, so that a penalized distance between (u, v) and $\mathcal{K}_1$ appears in the objective function.

One popular way to solve problem (1.2) is the augmented Lagrangian method (ALM). Given the Lagrangian multiplier $x \in \mathcal{X}$ of the linear constraint in (1.2), the augmented Lagrangian function associated with the parameter $\sigma > 0$ is defined as

$$\mathcal{L}_\sigma(u,v;x) = \theta(u,v) + \langle x, \mathcal{A}^*u + \mathcal{B}^*v - c\rangle + \frac{\sigma}{2}\|\mathcal{A}^*u + \mathcal{B}^*v - c\|^2, \quad (u,v) \in \mathcal{U}\times\mathcal{V}. \tag{1.12}$$

The ALM minimizes $\mathcal{L}_\sigma(u,v;x)$ with respect to (u,v) simultaneously regardless of whether the objective function is coupled or not before updating the Lagrangian multiplier x along the gradient ascent direction. Numerically, however, to minimize $\mathcal{L}_\sigma(u,v;x)$ with respect to (u,v) jointly may be a difficult task due to the non-separable structure of $\theta(\cdot,\cdot)$ combined with the nonsmoothness of $p(\cdot)$ and $q(\cdot)$.

When the objective function in (1.2) is separable for u and v, one can alleviate the numerical difficulty in the ALM by directly applying the alternating direction method of multipliers (ADMM). The iteration scheme of the ADMM works as follows:

$$\begin{cases} u^{k+1} = \arg\min\limits_u \mathcal{L}_\sigma(u, v^k; x^k), \\ v^{k+1} = \arg\min\limits_v \mathcal{L}_\sigma(u^{k+1}, v; x^k), \\ x^{k+1} = x^k + \tau\sigma(\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c), \end{cases} \tag{1.13}$$

where $\tau > 0$ is the step length. The global convergence of the ADMM with $\tau \in (0, \frac{1+\sqrt{5}}{2})$ and a separable objective function has been extensively studied in the literature, see, for examples, [34, 35, 38, 39, 28]. For a recent survey, see Eckstein and Yao [29]. Although it is possible to apply the ADMM directly to problem (1.2) even if $\phi(\cdot,\cdot)$ is not separable, its convergence analysis is largely non-existent. One way to deal with the non-separablity of $\phi(\cdot,\cdot)$ is to introduce a new variable $w \equiv (u,v) \in \mathcal{U}\times\mathcal{V}$. By letting

$$\widetilde{\mathcal{A}}^* = \begin{pmatrix} \mathcal{A} \\ \mathcal{I}_1 \\ 0 \end{pmatrix}, \quad \widetilde{\mathcal{B}}^* = \begin{pmatrix} \mathcal{B} \\ 0 \\ \mathcal{I}_2 \end{pmatrix}, \quad \widetilde{\mathcal{C}}^* = \begin{pmatrix} 0 & 0 \\ -\mathcal{I}_1 & 0 \\ 0 & -\mathcal{I}_2 \end{pmatrix}, \quad \tilde{c} = \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}$$

with identity maps $\mathcal{I}_1 : \mathcal{U} \to \mathcal{U}$ and $\mathcal{I}_2 : \mathcal{V} \to \mathcal{V}$, we can rewrite the optimization problem (1.11) equivalently as

$$\begin{aligned} \min_{u,v,w} \quad & \tilde{\theta}(u,v,w) := p(u) + q(v) + \phi(w), \\ \text{s.t.} \quad & \widetilde{\mathcal{A}}^*u + \widetilde{\mathcal{B}}^*v + \widetilde{\mathcal{C}}^*w = \tilde{c}. \end{aligned} \tag{1.14}$$

For given $\sigma > 0$, the corresponding augmented Lagrangian function for problem (1.14) is

$$\widetilde{L}_\sigma(u,v,w;x) = \tilde{\theta}(u,v,w) + \langle x, \widetilde{\mathcal{A}}^*u + \widetilde{\mathcal{B}}^*v + \widetilde{\mathcal{C}}^*w - \tilde{c}\rangle + \frac{\sigma}{2}\|\widetilde{\mathcal{A}}^*u + \widetilde{\mathcal{B}}^*v + \widetilde{\mathcal{C}}^*w - \tilde{c}\|^2,$$

where $(u,v,w) \in \mathcal{U}\times\mathcal{V}\times(\mathcal{U}\times\mathcal{V})$ and $x \in \mathcal{X}$. Directly applying the 3-Block ADMM yields the following framework:

$$\begin{cases} u^{k+1} = \arg\min\limits_u \widetilde{\mathcal{L}}_\sigma(u, v^k, w^k; x^k), \\ v^{k+1} = \arg\min\limits_v \widetilde{\mathcal{L}}_\sigma(u^{k+1}, v, w^k; x^k), \\ w^{k+1} = \arg\min\limits_w \widetilde{\mathcal{L}}_\sigma(u^{k+1}, v^{k+1}, w; x^k), \\ x^{k+1} = x^k + \tau\sigma(\widetilde{\mathcal{A}}^*u^{k+1} + \widetilde{\mathcal{B}}^*v^{k+1} + \widetilde{\mathcal{C}}^*w^{k+1} - \tilde{c}), \end{cases}$$

where $\tau > 0$ is the step length. Even though numerically the 3-block ADMM works well for many applications, generally it is not a convergent algorithm even if τ is as small as 10^{-8} as shown in the counterexamples given by Chen et al. [10].

Unlike the case with separable objective functions, there are very few papers on the ADMM targeting the problem (1.2) except for the work of Hong et al. [45], where the authors studied a majorized multi-block ADMM for linearly constrained optimization problems with non-separable objectives. When specialized to the 2-block case for problem (1.2), their algorithm works as follows:

$$\begin{cases} u^{k+1} = \arg\min\limits_u \{p(u) + \langle x^k, \mathcal{A}^*u\rangle + \hat{h}_1(u; u^k, v^k)\}, \\ v^{k+1} = \arg\min\limits_v \{q(v) + \langle x^k, \mathcal{B}^*v\rangle + \hat{h}_2(v; u^{k+1}, v^k)\}, \\ x^{k+1} = x^k + \alpha_k\sigma(\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c), \end{cases} \tag{1.15}$$

where $\hat{h}_1(u; u^k, v^k)$ and $\hat{h}_2(v; u^{k+1}, v^k)$ are majorization functions of $\phi(u,v) + \frac{\sigma}{2}\|\mathcal{A}^*u + \mathcal{B}^*v - c\|^2$ at (u^k, v^k) and (u^{k+1}, v^k), respectively and $\alpha_k > 0$ is the step length. Hong et al. [45] provided a very general convergence analysis of their majorized ADMM assuming that the step length α_k is a sufficiently small fixed number or converging to zero, among other conditions. Since a large step length is almost always desired in practice, one needs to develop a new convergence theorem beyond the one in [45].

The complexity of the ADMM has also been extensively studied in the literature for the problems with separable objective functions. Monteiro and Svaiter [68] show the ergodic complexity of the KKT system for the block-decomposition algorithms, which includes the classical ADMM with $\tau = 1$. When the proximal terms are only required to be positive semidefinite in the subproblems, Shefi and Teboulle [87] show the $O(1/k)$ ergodic complexity for the primal objective value and the feasiblity. Davis and Yin [16] further improve the above complexities to $o(1/k)$.

There are also some work focused on the linear convergence rate of the ADMM. When the problem under consideration only involves convex quadratic functions, the classical ADMM and its variant are shown to converge linearly with $\tau = 1$ [4, 41]. Deng and Yin [18] show that besides the convex quadratic programming, the linear convergence rate also holds if either p or q is strongly convex and smooth, among other conditions. Hong and Luo [46] further prove that if the step length τ allows to take sufficiently small value, the ADMM for solving multi-block problems achieves a linear convergence rate under an error bound condition. Also by assuming an error bound condition, Han, Sun and Zhang [40] establish the linear rate of the semi-proximal ADMM with $\tau \in (0, \frac{\sqrt{5}+1}{2})$.

In order to know the error bound and the linear convergence rate of ADMM can be achieved by which kind of problems, we also concern the sensitivity and stability to the composite constrained optimization problems:

$$\begin{aligned} \min \quad & f(x) + \theta(x), \\ \text{s.t.} \quad & h(x) \in \mathcal{P}, \end{aligned} \tag{1.16}$$

where $f : \mathcal{X} \to \mathcal{R}$ is a twice continuously differentiable function, $\theta : \mathcal{X} \to \mathcal{Y}$ is a closed proper convex function (not necessarily smooth), $h : \mathcal{X} \to \mathcal{Y}$ is a twice continuously differentiable mapping, $\mathcal{P} \subseteq \mathcal{Y}$ is a convex polyhedral, and $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional real Euclidean spaces.

The sensitivity and stability analysis, being the core of the theoretical study in the optimization community, has been dramatically pushed during the past several decades. There are several issues about the stability of optimization problems. For example, people care about whether the perturbed problems have non-empty solution sets under the assumption that the original problem has at least one KKT solution, and whether the distance between the two KKT solution sets can be bounded by the norm of the perturbation parameters. We say an optimization problem is stable if both of the above two questions have affirmative answers. A relative weaker condition is the semi-stability of an optimization problem, for which we only care about the Lipschitz continuity of those perturbed problems with non-empty KKT solution sets. An important application of the stability for optimization problems is the so-called Lipschitz error bound condition, which plays an important role in the convergence rate study of a bunch of algorithms. Many algorithms could archive the linear convergence rate instead of the genetic sublinear rate under the error bound conditions. For the examples of such algorithms, see [65, 63, 64, 97].

When $\theta = 0$, $h = (h_1, h_2)$ with $h_1 : \mathcal{X} \to \mathcal{R}^m$ and $h_2 : \mathcal{X} \to \mathcal{R}^q$, $\mathcal{P} = \{0\}^m \times \mathcal{R}^q_+$ the problem (1.16) reduces to the conventional nonlinear programming, which has quite complete theory about the stability subject to data perturbation. In particular, Dontchev and Rockafellar [24] show that in this case, the KKT system is robust isolated calm under canonical perturbations at a local optimal solution if and only if the strict Mangasarian-Fromovitz constraint qualification and the second order sufficient optimality condition hold. However, much less has been known if θ is a non-polyhedral function, such as the indicator function over a non-polyhedral set. Among them, we have some known results under a class of relatively "nice" set, which is called $\mathcal{C}^2$-cone reducible in the sense of Bonnans and Shapior [5, Definition 3.135]. It contains the polyhedral sets, the second order cone, the positive semidefinite cone, and their Cartesian product. In [106, 40], the authors characterize the isolated calmness of the nonlinear positive semidefinite programming by the second order sufficient condition and the strict Robinson constraint qualification. Similar type of isolated calmness characterization for the Ky Fan $k-$norm conic programming is provided by Liu and Pan [62]. A recent work of Zhou and So [108] shows that for a special class of unconstrained nuclear norm minimization (where the strict Robinson constraint qualification holds automatically), its error bound can be implied by the strict complementarity condition at a solution point.

1.2 Contributions

The main contributions of this thesis are two-folds. Firstly, we propose an inexact majorized accelerated block coordinate descent method (iABCD) in order to solve multi-block unconstrained convex problems. In the existing literature, problems of this nature are usually solved by (random) block coordinate descent type algorithms. However, it has been observed from extensive numerical experiments that the Nesterov's acceleration technique, which was originally designed for single-block problems, could dramatically improve the performance of the multi-block problems even when they are updated in an alternative fashion. We adopt a decomposition procedure in order to incorporate the acceleration technique to the multi-block problems. That is, even though (P1) consists of multiple blocks, we would first view u and v as two big blocks and focus on designing algorithms for the following problems:

$$\text{(P1-1)} \qquad \min \quad p(u) + q(v) + \phi(u, v), \tag{1.17}$$

where the functions p and q are given as $p(u) \equiv \sum_{i=1}^{s} p_i(u_i)$ and $q(v) \equiv \sum_{j=1}^{t} q_j(v_j)$. We establish the $O(1/k^2)$ complexity for the iABCD to solve the problem (1.17). However, each block, either u or v, may still be contributed by many separable non-smooth functions and a coupled smooth function. It is highly possible that no analytical solutions exist for the subproblems. To settle this issue, different methods are provided according to the structure of the subproblems, which includes the inexact one cycle symmetric Gauss-Seidel technique, the matrix Cauchy-Schwarz inequality and a hybrid of the APG and the semismooth Newton-CG method (APG-SNCG). We test the numerical performance of the iABCD framework on solving the dual of the projection onto the intersection of the equations, the inequalities, the positive semidefinite cone and the nonnegative cone, where four block variables appear in the dual problem. In particular, we solve the inequality and the nonnegative cone constraints together by the APG-SNCG method. It is very powerful for finding a solution of medium to high accuracy without adding a large proximal term by the incorporation of the second order information. The numerical results suggest that (i) the APG-SNCG method could universally improve the performance in the implementation among different frameworks; (ii) the iABCD is much more efficient than the BCD-type methods and the randomized ABCD-type methods for solving multi-block unconstrained problems.

Secondly, we consider the majorized alternating direction method of multipliers (mADMM) for solving linearly constrained convex problems with coupled objective functions, which is only discussed for problems with separable objective functions before. By making use of nonsmooth analysis, especially the generalized Mean-Value Theorem, we establish the global convergence, the ergodic and non-ergodic complexities of the mADMM with a large step length with the step length $\tau \in (0, \frac{1+\sqrt{5}}{2})$. We also explore the linear convergence rate for the quadratically coupled problems under an error bound assumption. In order to understand more about the error bound conditions, we also study the stability of the nonconvex constrained composite optimization problems involving the nuclear norm, which is also of its own interest. We fully characterize the robust isolated calmness property by the second order sufficient condition and the strict Robinson constraint qualification. We also explore several equivalent characterization by the dual information for convex constrained composite nuclear norm problems. In particular, the above mentioned isolated calmness results imply the error bound for the linearly constrained quadratic nuclear norm problems and thus the mADMM converges linearly when solving problems of this class.

1.3 Thesis organization

The rest of the thesis is organized as follows. In Chapter 2, we provide the preliminaries that will be used in the subsequent discussions. In Chapter 3, we first discuss the $O(1/k^2)$ iteration complexity of a two block majorized accelerated block coordinate descent algorithm. It follows by an extension to allow inexact solutions for each block and the same order of complexity is obtained. We demonstrate the implementation of this inexact framework to the dual of the regularized least square problems with equation, inequality and cone constraints. In Chapter 4, we analyze the convergence properties of a majorized alternating direction method of multipliers for solving the two-block linearly constrained convex problems with coupled objective functions. The linear convergence rate is also shown for the problems with quadratically coupled objective functions. Chapter 5 is devoted to the sensitivity and stability analysis for the constrained composite optimization problems. We show that for the constrained nuclear norm minimization problems, the robust isolated calmness holds for the KKT system if and only if both the second order sufficient condition and the strict Robinson constraint qualification hold at the reference point. Numerical examples and results are provided in Chapter 6, where we compare the performance of our algorithms to a class of least square problems several variants of the block coordinate descent method and the randomized accelerated block coordinate descent method. Finally we conclude the thesis in Chapter 7.

Chapter 2

Preliminaries

2.1 Notation

- Let n be a given integer. We use $\mathcal{S}^n$ to denote the space of all $n \times n$ symmetric matrices, $\mathcal{S}^n_+$ to denote the space of all $n \times n$ positive semidefinite matrices, $\mathcal{S}^n_{++}$ to denote the space of all $n \times n$ positive definite matrices, and $\mathcal{O}^n$ be the set of all $n \times n$ orthogonal matrices. For any given $X, Y \in \mathcal{S}^n$, we write $X \succeq Y$ if $X - Y \in \mathcal{S}^n_+$ and $X \succ Y$ if $X - Y \in \mathcal{S}^n_{++}$. In particular, we use the notation $X \succeq 0$ to indicate $X \in \mathcal{S}^n_+$ and $X \succ 0$ to indicate $X \in \mathcal{S}^n_{++}$.

- Denote $\mathcal{X}$ as a finite dimensional Euclidean space endowed with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$, and $\mathcal{M} : \mathcal{X} \to \mathcal{X}$ as a self-adjoint positive semidefinite linear operator. We write $\mathcal{M}^{\frac{1}{2}}$ as a self-adjoint positive semidefinite linear operator such that $\mathcal{M}^{\frac{1}{2}}\mathcal{M}^{\frac{1}{2}} = \mathcal{M}$, which always exists. For any $x, y \in \mathcal{X}$, we define $\langle x, y \rangle_{\mathcal{M}} := \langle x, \mathcal{M}y \rangle$ and $\|x\|_{\mathcal{M}} := \sqrt{\langle x, \mathcal{M}x \rangle}$.

- Given a set $\mathcal{S} \subseteq \mathcal{X}$, we denote $\text{conv}\{S\}$ as the convex hull of $\mathcal{S}$.

- Given a closed convex set $\mathcal{C} \subseteq \mathcal{X}$ and a point $x \in \mathcal{C}$, denote $\mathcal{T}_{\mathcal{C}}(x)$ as the tangent cone of C at x and $\mathcal{N}_{\mathcal{C}}(x)$ as the normal cone of C at x. We define $\text{dist}(x, \mathcal{C}) := \inf_{y \in \mathcal{C}} \|x - y\|$.

- Given a closed convex cone $\mathcal{K} \subseteq \mathcal{X}$, denote $\mathcal{K}^*$ as the dual cone of $\mathcal{K}$ and $\mathcal{K}^\circ$ as the polar cone of $\mathcal{K}$.

- Given a convex function $f : \mathcal{X} \to (-\infty, +\infty]$, we use dom$f$ to denote the effective domain of f, and epi$f$ to denote the epigraph of f. We also use the notation f^* to denote the Fenchel's conjugate function of f, and Prox_f as the proximal mapping of f. We use the notation $f'(x;d)$ to denote the directional derivative of f at $x \in \mathcal{X}$ along the direction $d \in \mathcal{X}$ if it exists, and it is given by
$$f'(x;d) := \lim_{t \downarrow 0} \frac{f(x+td) - f(x)}{t}.$$
Furthermore, we say f is a $L\mathcal{C}^1$ function if it is continuously differentiable and its gradient is Lipschitz continuous, and we say f is $\mathcal{C}^2$ if it is twice continuously differentiable.

- Given a matrix $X \in \mathcal{R}^{m\times n}$, we denote $\|X\|_*$ as the nuclear norm of X, i.e., the sum of all the singular values of X, and $\|X\|_2$ as the spectral norm of X, i.e., the largest singular value of X. We also use $\mathrm{tr}(X)$ to represent the trace of X, i.e., the sum of all the diagonal entries of X.

- Given a set of matrices $X := (X_1, X_2, \ldots X_s) \in \mathcal{R}^{n_1\times m_1} \times \mathcal{R}^{n_2\times m_2} \times \ldots \times \mathcal{R}^{n_s\times m_s}$ for some positive integers s, $n_1, n_2, \ldots, n_s$ and $m_1, m_2, \ldots, m_s$, we denote $\mathrm{Diag}(X)$ as a block diagonal matrix whose ith main block diagonal is given by X_i for $i \in \{1, 2, \ldots, s\}$.

2.2 Nonsmooth analysis

In this section, we list the useful results related to the generalized Mean-Value Theorem of smooth functions, the semismoothness, the Moreau-Yosida regularization and the spectral operators.

Assume that $\phi : \mathcal{W} \to (-\infty, +\infty)$ is a smooth convex function whose gradient mapping is Lipschitz continuous, where $\mathcal{W}$ is a real finite dimensional Euclidean space. Then $\nabla^2\phi(\cdot)$ exists almost everywhere and the following Clarke's generalized Hessian at given $w \in \mathcal{W}$ is well defined [12]:

$$\partial^2\phi(w) = \text{conv}\{\lim_{w^k \to w} \nabla^2\phi(w^k), \nabla^2\phi(w^k)\,\text{exists}\}, \tag{2.1}$$

where "conv$\{S\}$" denotes the convex hull of a given set S. Note that $\mathcal{W}$ is self-adjoint and positive semidefinite, i.e., $\mathcal{W} \succeq 0$, for any $\mathcal{W} \in \partial^2\phi(w)$, $w \in \mathcal{U} \times \mathcal{V}$. In [44], Hiriart-Urruty and Nguyen provide a second order Mean-Value Theorem for ϕ, which states that for any w' and w in $\mathcal{U} \times \mathcal{V}$, there exists $z \in [w', w]$ and $\mathcal{W} \in \partial^2\phi(z)$ such that

$$\phi(w) = \phi(w') + \langle \nabla\phi(w'), w - w' \rangle + \frac{1}{2}\langle w - w', \mathcal{W}(w - w')\rangle,$$

where $[w', w]$ denotes the line segment connecting w' and w.

Since $\nabla\phi$ is globally Lipschitz continuous, there exist two self-adjoint positive semidefinite linear operators $\mathcal{Q}$ and $\mathcal{H} : \mathcal{W} \to \mathcal{W}$ such that for any $w \in \mathcal{W}$,

$$\mathcal{Q} \preceq \mathcal{W} \preceq \mathcal{Q} + \mathcal{H}, \quad \forall\, \mathcal{W} \in \partial^2\phi(w). \tag{2.2}$$

Thus, for any $w, w' \in \mathcal{W}$, we have

$$\phi(w) \geq \phi(w') + \langle \nabla\phi(w'), w - w' \rangle + \frac{1}{2}\|w' - w\|^2_{\mathcal{Q}} \tag{2.3}$$

and

$$\phi(w) \leq \hat{\phi}(w; w') := \phi(w') + \langle \nabla\phi(w'), w - w' \rangle + \frac{1}{2}\|w' - w\|^2_{\mathcal{Q}+\mathcal{H}}. \tag{2.4}$$

In the following, we introduce the concept of the semismoothness. Denote $F : \mathcal{X} \to \mathcal{Y}$ as a Lipschitz continuous function. Then F is Fréchet differentiable almost everywhere (see, e.g., [82, Sesction 9.J]). Let D_F be the set of points in $\mathcal{X}$ such that F is differentiable and $F'(x^k)$ be the Jacobian of F at $x^k \in D_F$. The Bouligand subdifferential (B-subdifferential) of F at $x \in \mathcal{X}$ is defined as

$$\partial_B F(x) = \{\lim_{x^k \to x} F'(x_k),\ x^k \in D_F\}.$$

The Clarke's generalized Jacobian of F at $x \in \mathcal{X}$ is defined as

$$\partial F(x) = \text{conv}\{\partial_B F(x)\}.$$

The concept of semismoothness was first introduced by Mifflin [67] for functionals and lated on extended by Qi and Sun [76] for vector-valued functions.

Definition 2.1. **[G-Semismoothness and semismoothness]** *Let $F : \mathcal{O} \subseteq \mathcal{X} \to \mathcal{Y}$ be a locally Lipschitz continuous function on the open set $\mathcal{O}$. F is said to be G-semismooth at a point $x \in \mathcal{O}$ if for any $\Delta x \in \mathcal{X}$ and $V \in \partial F(x + \Delta x)$ with $\Delta x \to 0$,*

$$F(x + \Delta x) - F(x) - V\Delta x = o(\|\Delta x\|).$$

F is said to be strongly G-semismooth at $x \in \mathcal{X}$ if F is semismooth at x and for any $\Delta x \in \mathcal{X}$ and $V \in \partial F(x + \Delta x)$ with $\Delta x \to 0$,

$$F(x + \Delta x) - F(x) - V\Delta x = O(\|\Delta x\|^2).$$

If F is also directionally differentiable at x in the above definition, then F is said to be semismooth and strongly semismooth, respectively.

The Moreau-Yosida regularization is a useful tool in the nonsmooth optimization. Below we introduce this concept and list some frequently used properties related to it.

Definition 2.2. **[The Moreau-Yosida regularization]** *Let $f : \mathcal{X} :\to (-\infty, +\infty]$ be a closed proper convex function. The Moreau-Yosida regularization $\psi_f : \mathcal{X} \to \mathcal{R}$ associated with the function f is defined as*

$$\psi_f(x) = \min_{z \in \mathcal{X}} \left\{ f(z) + \frac{1}{2}\|z - x\|^2 \right\}. \tag{2.5}$$

The following proposition comes from Moreau [69] and Yosida [104].

Proposition 2.1. *For any given $x \in \mathcal{X}$, the above problem admits a unique solution.*

Thus, given any $x \in \mathcal{X}$, we call the unique solution of the problem (2.5) the proximal point of x associated with f and denote it as $\mathrm{Prox}_f(x)$, i.e. ,

$$\mathrm{Prox}_f(x) = \arg\min_{z \in \mathcal{X}} \left\{ f(z) + \frac{1}{2}\|z - x\|^2 \right\}.$$

Moreover, the single-valued mapping $\text{Prox}_f : \mathcal{X} \to \mathcal{X}$ is called the proximal mapping associated with the function f.

The following proposition shows the nice behaviours of the Moreau-Yosida regularization and the proximal mappings.

Proposition 2.2. [55, **Theorem XV. 4.1.4 and Theorem XV.4.1.7**] *Let $f : \mathcal{X} :\to (-\infty, +\infty]$ be a closed proper convex function. Then the following statements hold:*

(i) $\arg\min_{x\in\mathcal{X}} f(x) = \arg\min_{x\in\mathcal{X}} \psi_f(x)$.

(ii) Both Prox_f *and* $\mathcal{Q}_f := \mathcal{I} - \text{Prox}_f$ *are firmly non-expansive, i.e., for any $x, y \in \mathcal{X}$,*

$$\begin{aligned}\|\text{Prox}_f(x) - \text{Prox}_f(y)\|^2 &\leq \langle \text{Prox}_f(x) - \text{Prox}_f(y), x - y\rangle,\\ \|\mathcal{Q}_f(x) - \mathcal{Q}_f(y)\|^2 &\leq \langle \mathcal{Q}_f(x) - \mathcal{Q}_f(y), x - y\rangle.\end{aligned}$$

Therefore, Prox_f *and* $\mathcal{Q}_f$ *are globally Lipschitz continuous with modulus* 1.

Proposition 2.3. [**Moreau decomposition**] *Let $f : \mathcal{X} \to (-\infty, +\infty]$ be a closed proper convex function and f^* be its conjugate. Then for any $x \in \mathcal{X}$, it can be decomposed as*

$$x = \text{Prox}_f(x) + \text{Prox}_{f^*}(x).$$

In fact, if the nonsmooth function f equals to $\delta_{\mathcal{C}}(\cdot)$, the indicator function of a given closed convex set $C \subseteq \mathcal{X}$, the proximal mapping associated with f reduces to

$$\text{Prox}_{\delta_{\mathcal{C}}}(x) = \arg\min_{z\in\mathcal{X}} \left\{ \delta_{\mathcal{C}}(x) + \frac{1}{2}\|z - x\|^2 \right\} = \Pi_C(x),$$

i.e., the projection operator of the set C. In this way, one can take the proximal mapping of a nonsmooth function as a generalization of the projection operator.

The last concept in this section is the spectral operators. Given a matrix $X \in \mathcal{R}^{m\times n}$, denote its singular value decomposition (SVD) as

$$X = U[\text{Diag}(\sigma(X))\ 0]V^T = U[\text{Diag}(\sigma(X))\ 0][V_1\ V_2]^T = U\text{Diag}(\sigma(X))V_1^T, \quad (2.6)$$

where $U \in \mathcal{O}^m$, $V := [V_1\ V_2] \in \mathcal{O}^n$ with $V_1 \in \mathcal{R}^{n\times m}$ and $V_2 \in \mathcal{R}^{n\times(n-m)}$ are the singular vectors of X, and $\text{Diag}(\sigma(X)) := \text{Diag}(\sigma_1(X), \sigma_2(X), \ldots, \sigma_m(X))$ are the singular values of X with $\sigma_1(X) \geq \sigma_2(X) \geq \ldots \geq \sigma_m(X)$ being arranged in a non-increasing order. Denote

$$\mathcal{O}^{m,n} := \{(U,V) \in \mathcal{O}^m \times \mathcal{O}^n : X = U[\Sigma(X)\ 0]V^T\}.$$

Let $\mathcal{Q}_k \in \mathcal{R}^{k\times k}$ denote the set of all permutation matrices that have exactly one entry being 1 in each row and column and 0 elsewhere. Let $\mathcal{Q}_k^{\pm} \in \mathcal{R}^{k\times k}$ denote the set of all signed permutation matrices that have exactly one entry being ± 1 in each row and column and 0 elsewhere. In the following we introduce the concepts of symmetric vector-valued functions and the spectral operators associated with symmetric vector-valued functions.

Definition 2.3. *A function $f : \mathcal{R}^k \to \mathcal{R}^k$ is said to be symmetric if*

$$f(x) = Q^T f(Qx), \quad \forall Q \in \mathcal{Q}_k^{\pm}, \quad \forall x \in \mathcal{R}^k.$$

Definition 2.4. **[Spectral operator]** *Given the SVD of $X \in \mathcal{R}^{m\times n}$ as in (2.6), the spectral operator $F : \mathcal{R}^{m\times n} \to \mathcal{R}^{m\times n}$ associated with the function $f : \mathcal{R}^m \to \mathcal{R}^m$ is defined as*

$$F(X) := U[\mathrm{Diag}(f(\sigma(X))\ 0]V^T,$$

where $(U,V) \in \mathcal{O}^{m\times n}(X)$.

Definition 2.5. **[Hadamard directionally differentiable]** *A function $f : \mathcal{O} \subseteq \mathcal{X} \to \mathcal{Y}$ is said to be Hadamard directionally differentiable at $x \in \mathcal{O}$ if the limit*

$$\lim_{t\downarrow 0, h'\to h} \frac{f(x+th') - f(x)}{t} \quad \textit{exists for any } h \in \mathcal{X}.$$

In his Ph.D thesis, Ding shows that the above given spectral operator is well-defined [19, Theorem 3.1]. Moreover, the Hadamard directional differentiability of the spectral operator F, among other differential properties, depends on the the Hadamard directional differentiability of f and the directional derivative at a given point can be characterized explicitly. Before introducing the formula of the directional derivatives, we first give several notations.

Denote the following two index sets regrading the singular values of X:

$$a := \{1 \leq i \leq m : \sigma_i(X) > 0\}, \quad b := \{1 \leq i \leq m : \sigma_i(X) = 0\}.$$

We further denote the distinct nonzero singular values of X as $\mu_1(X) > \mu_2(X) > \ldots > \mu_r(X)$ for some nonnegative integer r and divide the set a into the following r subsets:

$$a = \bigcup_{1 \leq l \leq r} a_l, \quad a_l := \{i \in a : \sigma_i(X) = \mu_l(X)\}, \quad l = 1, 2, \ldots, r.$$

Assume that $f : \mathcal{R} \to \mathcal{R}$ is directional differentiable. Denote the directional derivative of f at $\sigma = \sigma(X)$ as $\phi(\cdot) = f'(\sigma; \cdot)$, which can be further decomposed according to the partition of the singular values as

$$\phi(h) = (\phi_1(h), \ldots, \phi_r(h), \phi_{r+1}(h)), \quad \forall h \in \mathcal{R}^m,$$

where $\phi_l(h) \in \mathcal{R}^{|a_l|}$ for $l = 1, 2, \ldots, r$ and$\phi_{r+1}(h) \in \mathcal{R}^{|b|}$.

Define a space $\mathcal{W}$ as $\mathcal{W} = \mathcal{S}^{|a_1|} \times \mathcal{S}^{|a_2|} \times \ldots \times \mathcal{S}^{|a_l|} \times \mathcal{R}^{|b| \times (n-|a|)}$ and a spectral operator $\Phi : \mathcal{W} \to \mathcal{W}$ with respect to the symmetric mapping ϕ as

$$\Phi(W) := (\Phi_1(W), \ldots, \Phi_r(W), \Phi_{r+1}(W)),$$

where

$$\Phi_l(W) := \begin{cases} P_l \mathrm{Diag}(\phi_l(\kappa(W)) P_l^T, & \text{if } 1 \leq l \leq r, \\ M \mathrm{Diag}(\phi_l(\kappa(W)) N_1^T, & \text{if } l = r + 1, \end{cases}$$

with $\kappa(W) = (\lambda(W_1), \ldots, \lambda(W_r), \sigma(W_{r+1})) \in \mathcal{R}^m$, $\mathcal{P}_l \in \mathcal{O}^{|a_l|}(W_l)$ and $(M, [N_1, N_2]) \in \mathcal{O}^{|b|, n-|a|}(W_{r+1})$ with $N_1 \in \mathcal{R}^{(n-|a|) \times |b|}$ and $N_2 \in \mathcal{R}^{(n-|a|) \times (n-m)}$.

We also define two linear matrix operators $S : \mathcal{R}^{k \times k} \to \mathcal{S}^k$ and $T : \mathcal{R}^{k \times k} \to \mathcal{R}^{k \times k}$ as

$$S(X) = \frac{1}{2}(X + X^T), \quad T(X) = \frac{1}{2}(X - X^T), \quad \forall X \in \mathcal{R}^{k \times k}.$$

Denote

$$D(H) := \big(S(U_{a_1}^T H V_{a_1}), \ldots, S(U_{a_r}^T H V_{a_r}), U_b^T H [V_b, V_2]\big) \in \mathcal{W},$$

and for any $W = (W_1, \ldots, W_r, W_{r+1}) \in \mathcal{W}$, $\widehat{\Phi}(W) \in \mathcal{R}^{m\times n}$ is defined by

$$\widehat{\Phi}(W) := \begin{pmatrix} \mathrm{Diag}(\Phi_1(W), \ldots, \Phi_r(W)) & \\ & \Phi_{r+1}(W) \end{pmatrix}.$$

Furthermore, denote three matrices $\mathcal{E}_1(\sigma), \mathcal{E}_2(\sigma) \in \mathcal{R}^{m\times m}$ and $F(\sigma) \in \mathcal{R}^{m\times(n-m)}$ (depending on X) as:

$$(\mathcal{E}_1(\sigma))_{ij} := \begin{cases} \dfrac{f_i(\sigma) - f_j(\sigma)}{\sigma_i - \sigma_j} & \text{if } \sigma_i \neq \sigma_j \\ 0 & \text{otherwise} \end{cases}, \quad i, j \in \{1, 2, \ldots, m\},$$

$$(\mathcal{E}_2(\sigma))_{ij} := \begin{cases} \dfrac{f_i(\sigma) + f_j(\sigma)}{\sigma_i + \sigma_j} & \text{if } \sigma_i + \sigma_j \neq 0 \\ 0 & \text{otherwise} \end{cases}, \quad i, j \in \{1, 2, \ldots, m\},$$

$$(\mathcal{F}(\sigma))_{ij} := \begin{cases} \dfrac{f_i(\sigma)}{\sigma_i} & \text{if } \sigma_i \neq 0 \\ 0 & \text{otherwise} \end{cases}, \quad i \in \{1, 2, \ldots, m\}, \quad j \in \{1, 2, \ldots, n-m\}.$$

Theorem 2.1. [**19**, **Theorem 3.4**] *Given the SVD of $X \in \mathcal{R}^{m\times n}$ as in (2.6), the spectral operator $F : \mathcal{R}^{m\times n} \to \mathcal{R}^{m\times n}$ associated with the function $f : \mathcal{R}^m \to \mathcal{R}^m$ is Hadamard directional differentiable if and only if f is Hadamard directional differentiable at $\sigma = \sigma(X)$. Moreover, F is directional differentiable and the directional derivative at $X \in \mathcal{R}^{m\times n}$ along any given direction $H \in \mathcal{R}^{m\times n}$ can be computed as*

$$F'(X; H) = U f^{[1]}(X; H) V^T,$$

where $(U, V) \in \mathcal{O}^{m\times n}(X)$ and $f^{[1]}(X; H)$ is given by

$$f^{[1]}(X; H) := [\mathcal{E}_1(\sigma) \circ S(\widetilde{H}_1) + \mathcal{E}_2(\sigma) \circ T(\widetilde{H}_1) \quad F(\sigma) \circ \widetilde{H}_2] + \widehat{\Phi}(D(H)),$$

with $\widetilde{H} := [\widetilde{H}_1 \ \widetilde{H}_2]$, $\widetilde{H}_1 := U^T H V_1$ and $\widetilde{H}_2 := U^T H V_2$.

We shall use these results in Chapter 5 to characterize the proximal mapping of the nuclear norm function and its directional derivative.

2.3 The one cycle symmetric Gauss-Seidel technique and the matrix Cauchy-Schwarz inequality

In this section, we review the one cycle symmetric Gauss-Seidel (sGS) technique proposed recently by Li, Sun and Toh [59]. It is a powerful tool to decompose quadratic coupled multi-block problems into separate ones by adding a particular designed semi-proximal term to the original problem, which plays an important role in our subsequent algorithm designs for solving large scale convex least square problems.

Mathematically speaking, the sGS technique targets at solving the following unconstrained nonsmooth convex optimization problem approximately:

$$\min_{x} f(x_1) + \frac{1}{2}\langle x,\, \mathcal{H}x\rangle - \langle r,\, x\rangle, \tag{2.7}$$

where $x \equiv (x_1, x_2, \ldots, x_s) \in \mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_s$ with $s \geq 2$ being a given integer and all $\mathcal{X}_i$ being assumed to be real finite dimensional Euclidean spaces, $f : \mathcal{X}_1 \to (-\infty, +\infty]$ is a given closed proper convex function (possibly nonsmooth), $\mathcal{H} : \mathcal{X} \to \mathcal{X}$ is a given self-adjoint positive semidefinite linear operator and $r \equiv (r_1, r_2, \ldots, r_s) \in \mathcal{X}$ is a given vector.

The difficulty of solving the problem (2.7) comes from the combination of the nonsmooth part f and the joint smooth quadratic function $\frac{1}{2}\langle x,\, \mathcal{H}x\rangle - \langle r,\, x\rangle$. For most of the applications with complicated operator $\mathcal{H}$, it is impossible to obtain the analytic expression of the optimal solution x^*.

For notational convenience, we denote the quadratic function in (2.7) as

$$h(x) = \frac{1}{2}\langle x, \mathcal{H}x\rangle - \langle r, x\rangle,$$

and the block decomposition of the operator $\mathcal{H}$ as:

$$\mathcal{H}x \equiv \begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} & \cdots & \mathcal{H}_{1s} \\ \mathcal{H}_{12}^* & \mathcal{H}_{22} & \cdots & \mathcal{H}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}_{1s}^* & \mathcal{H}_{2s}^* & \cdots & \mathcal{H}_{ss} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{pmatrix}, \tag{2.8}$$

where $\mathcal{H}_{ii} : \mathcal{X}_i \to \mathcal{X}_i,\ i = 1, \ldots, s$ are self-adjoint positive semidefinite linear operators and $\mathcal{H}_{ij} : \mathcal{X}_j \to \mathcal{X}_i,\ i = 1, \ldots, s-1,\ j > i$ are linear operators whose conjugate are given by $\mathcal{H}_{ij}^*$. We also write the upper triangular and diagonal parts of the operator $\mathcal{H}$ as

$$\mathcal{M} := \begin{pmatrix} \mathbf{0} & \mathcal{H}_{12} & \cdots & \mathcal{H}_{1s} \\ & \ddots & \cdots & \vdots \\ & & \mathbf{0} & \mathcal{H}_{(s-1)s} \\ & & & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathcal{D} := \mathrm{Diag}(\mathcal{H}_{11}, \ldots, \mathcal{H}_{ss}).$$

Note that $\mathcal{H} = \mathcal{D} + \mathcal{M} + \mathcal{M}^*$. Here, we further assume that

$$\mathcal{D} \succ 0. \tag{2.9}$$

In order to allow solving the problems inexactly, we write the following two error tolerance vectors:

$$\delta' \equiv (\delta'_1, \ldots, \delta'_s), \quad \delta^+ \equiv (\delta^+_1, \ldots, \delta^+_s),$$

where $\delta'_i, \delta^+_i \in \mathcal{X}_i$ for $i = 1, \ldots, m$ with $\delta'_1 = 0$.

The key ingredient of sGS decomposition technique is to construct an extra semi-proximal term based on the original problem that can decouple the joint quadratic function. It essentially relies on the following self-adjoint positive semidefinite linear operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ and the error term $\Delta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$:

$$\begin{aligned} \mathcal{T} &:= \mathcal{M}\mathcal{D}^{-1}\mathcal{M}^*, \\ \Delta(\delta', \delta^+) &:= \delta' + (\mathcal{D} + \mathcal{M})\mathcal{D}^{-1}(\delta^+ - \delta'). \end{aligned} \tag{2.10}$$

Denote $x_{\leq i} := (x_1, x_2, \ldots, x_i), \quad x_{\geq i} := (x_i, x_{i+1}, \ldots, x_s)$ for $i = 0, \ldots, s+1$, with the convention that $x_{\leq 0} = x_{\geq s+1} = \emptyset$. The following sGS Decomposition Theorem shows that one cycle of inexact symmetric Gauss-Seidel type sequential updating of the variables $x_1, \ldots, x_s$ is equivalent to solving a semi-proximal majorization of the original problem (2.7) inexactly with respect to all the components simultaneously.

Theorem 2.2. [**60, Theorem 2.1**] *Assume that the condition* (2.9) *holds, i.e., the self-adjoint linear operators* $\mathcal{H}_{ii}$, $i = 1, \ldots, s$ *are positive definite. Then, it holds that*

$$\widehat{\mathcal{H}} := \mathcal{H} + \mathcal{T} = (\mathcal{D} + \mathcal{M})\mathcal{D}^{-1}(\mathcal{D} + \mathcal{M}^*) \succ 0. \tag{2.11}$$

Furthermore, given $y \in \mathcal{X}$ *and for* $i = s, \ldots, 2$, *define* $x'_i \in \mathcal{X}_i$ *by*

$$\begin{aligned} x'_i &:= \arg\min_{x_i \in \mathcal{X}_i} \ \phi(y_1) + h(y_{\leq i-1}, x_i, x'_{\geq i+1}) - \langle \delta'_i, x_i \rangle \\ &= \mathcal{H}_{ii}^{-1}\big(r_i + \delta'_i - \sum_{j=1}^{i-1} \mathcal{H}^*_{ji} y_j - \sum_{j=i+1}^{s} \mathcal{H}_{ij} x'_j\big). \end{aligned} \tag{2.12}$$

Then the optimal solution x^+ *defined by*

$$x^+ := argmin_x \left\{ \phi(x_1) + h(x) + \frac{1}{2}\|x - y\|^2_{\mathcal{T}} - \langle x, \Delta(\delta', \delta^+) \rangle \right\} \tag{2.13}$$

can be obtained exactly via

$$\left\{ \begin{aligned} x_1^+ &= \arg\min_{x_1} \ \phi(x_1) + h(x_1, x'_{\geq 2}) - \langle \delta_1^+, x_1 \rangle, \\ x_i^+ &= \arg\min_{x_i} \ \phi(x_1^+) + h(x^+_{\leq i-1}, x_i, x'_{\geq i+1}) - \langle \delta_i^+, x_i \rangle \\ &= \mathcal{H}_{ii}^{-1}(r_i + \delta_i^+ - \textstyle\sum_{j=1}^{i-1} \mathcal{H}^*_{ji} x_j^+ - \sum_{j=i+1}^{s} \mathcal{H}_{ij} x'_j), \quad i = 2, \ldots, s. \end{aligned} \right. \tag{2.14}$$

It is easy to see that $\widehat{\mathcal{H}} \succeq \mathcal{H}$, such that we automatically majorize the original smooth function h by using the sGS technique.

Now we turn to discuss another majorization technique, the matrix type Cauchy-Schwarz inequality, that would also be useful for solving the convex optimization problems with quadratic coupled objective functions.

Recall the form of the self-adjoint positive semidefinite operator $\mathcal{H}$ given in (2.8). Denote

$$\widetilde{\mathcal{H}} := \mathrm{Diag}\,\big(\widetilde{\mathcal{H}}_{11}, \widetilde{\mathcal{H}}_{22}, \ldots, \widetilde{\mathcal{H}}_{ss}\big), \tag{2.15}$$

where for $i = 1, \ldots, s$, $\widetilde{\mathcal{H}}_{ii}$ are self-adjoint positive semidefinite operators defined as

$$\widetilde{\mathcal{H}}_{ii} := \mathcal{H}_{ii} + \sum_{j \neq i} (\mathcal{H}_{ij}\mathcal{H}_{ij}^*)^{1/2}.$$

Then by [58, Proposition 2.9], the following inequality always hold:

$$\mathcal{H} \preceq \widetilde{\mathcal{H}}. \tag{2.16}$$

We call (2.16) the matrix Cauchy-Schwarz inequality since it is in fact the matrix analogue of the classic Cauchy-Schwarz inequality.

From the above inequalities one can see that similarly as $\widehat{\mathcal{H}}$ given in (2.11), the operator $\widetilde{\mathcal{H}}$ is also a kind of upper bound of the original Hessian matrix $\mathcal{H}$. It can be easily observed that when the original operator $\mathcal{H}$ is nearly block-wise orthogonal, i.e., $\|\mathcal{H}_{ij}\mathcal{H}_{ij}^*\|$ is very small for $i \neq j$, the operator $\widetilde{H}$ would be quite tight estimation of $\mathcal{H}$. One can refer to Li's Ph.D thesis [58, Section 3.2] for a detailed comparison between the sGS-type majorization operator $\widehat{\mathcal{H}}$ defined in (2.11) and the matrix Cauchy-Schwarz-type majorization operator $\widetilde{\mathcal{H}}$ defined in (2.15).

Note that all the subproblems need to be solved sequentially in a symmetric Gauss-Seidel fashion using the sGS technique. However, by adopting the matrix Cauchy-Schwarz inequality we can solve different blocks simultaneously since the majorized problem is separable for all the s blocks.

2.4 A hybrid of the semismooth Newton-CG method and the accelerated proximal gradient method

In this section, we present a method for solving the following convex optimization problem:

$$\min \theta(x) := f(x) + g(x), \tag{2.17}$$

where $f : \mathcal{X} \to (-\infty, \infty)$ is a strongly convex smooth function whose gradient is Lipschitz continuous, $g : \mathcal{X} \to (-\infty, +\infty]$ is a simple convex function (possibly nonsmooth) in the sense that its proximal mapping is relatively easy to compute. Denote L_f as the Lipschitz constant of ∇f.

Define a function $F : \mathcal{X} \to \mathcal{X}$ as

$$F(x) := x - \mathrm{Prox}_{\alpha g}(x - \alpha \nabla f(x)), \tag{2.18}$$

where $\alpha > 0$ is a positive constant. It is well-known that $x^* \in \mathcal{X}$ is an optimal solution of the problem (2.17) if and only if $F(x^*) = 0$. Since f is assumed to be strongly convex, by solving $F(x) = 0$ we could obtain the unique optimal solution of the problem (2.17).

When the nonsmooth function g is vacant, the above problem reduces to solving a nonsmooth Lipschitz continuous equation

$$\nabla f(x) = 0.$$

Kummer [53] and Qi and Sun [76] show that if ∇f is semismooth and all $V \in \partial \nabla f(x^*)$ are nonsingular at the optimal solution x^*, the iteration with an initial guess $x^0 \in \mathcal{X}$ generated by a direct generalization of classical Newton's method

$$x^{k+1} = x^k - V_k^{-1} \nabla f(x^k), \quad V^k \in \partial \nabla f(x^k),\ k = 0, 1, 2, \ldots,$$

would converge superlinearly. This method can also be globalized by using the line search technique to the function f [75].

If a nonsmooth function g appears in the problem (2.17), it is not easy to directly globalize the semismooth Newton's method since the objective value may not decrease along the Newton's direction. However, in practice Newton's method usually outperforms the first order methods in the neighborhood of the optimal solution, as it incorporate the second order information into the searching direction. At the same time, the Nesterov's accelerated proximal gradient method (APG) [70] also achieves both the global convergence and the linear convergence rate for solving strongly convex problems [85]. Therefore, in order to get both the global convergence theoretically and a fast convergence rate numerically, we adopt a hybrid of the APG method and the semismooth Newton-CG method to solve (2.17).

In order to get a superlinear (quadratic) convergence of the semismooth Newton-CG method, we need F to be (strongly) semismooth and $\partial_B F$ to be nonsingular at the optimal solution. The following Proposition provides a convenient tool to compute and check the nonsingularity of the B-subdifferential for the composite function F. Its proof can be directly obtained by noting that the composite of semismooth functions is still semismooth [32] and the results in [36, Proposition 3.2].

Proposition 2.4. *Let F be defined by (2.18) and $x \in \mathcal{X}$. Suppose that ∇f is semismooth at x and $\mathrm{Prox}_g(\cdot)$ is semismooth at $x - \nabla f(x)$. Then the following statements are true:*
(i) F is semismooth at x;
(ii) For any $h \in \mathcal{X}$, one has

$$\partial_B F(x)h \subseteq \hat{\partial}_B F(x)h := h - \partial_B \mathrm{Prox}_g(x - \nabla f(x))(h - \partial_B \nabla f(x)(h)).$$

Moreover, if $\mathcal{I} - \mathcal{S}(\mathcal{I} - \mathcal{V})$ is nonsingular for any $\mathcal{S} \in \partial_B \mathrm{Prox}_g(x - \nabla f(x))$ and $\mathcal{V} \in \partial_B \nabla f(x)$, then any element in $\partial_B F(x)$ is nonsingular.

In fact, the proximal mapping $\mathrm{Prox}_g(\cdot)$ is (strongly) semismooth for many frequently used functions g, like the indicator function over polyhedral sets and the symmetric cones [89, 88], or the Ky Fan k-norm functions [19]. In the following, we present the framework of our hybrid methods consisting of the APG algorithm and the semismooth Newton-CG method.

SNCG-APG: A hybrid of the semismooth Newton-CG method and the accelerated proximal gradient to solve problem (2.17).

Choose an initial point $x^1 \in \mathcal{X}$, positive constants $\eta, \gamma \in (0,1)$, $\rho \in (0, 1/2)$, and a positive integer $m_0 > 0$. Set $k = 1$.

Step 1: Select $\mathcal{V}^k \in \hat{\partial}_B F(x^k)$ and apply the conjugate gradient (CG) method to find an approximate solution d^k to

$$\mathcal{V}^k d + F(x^k) = 0, \tag{2.19}$$

such that

$$R^k := \mathcal{V}^k d^k + F(x^k), \quad \text{and } \|R^k\| \leq \eta_k \|F(x^k)\|, \tag{2.20}$$

where $\eta_k = \min\{\eta, \|F(x^k)\|\}$. If (2.20) is achievable, go to Step 2. Otherwise, go to Step 3.

Step 2: Let $m_k \leq m_0$ be the smallest nonnegative integer m such that

$$\|F(x^k + \rho^m d^k)\| \leq \gamma \|F(x^k)\|. \tag{2.21}$$

If (2.21) is achievable, set $t_k := \rho^{m_k}$ and $x^{k+1} = x^k + t_k d^k$. Replace k by $k+1$ and go to Step 1. Otherwise if $m_k > m_0$ and (2.21) still fails, go to step 3.

Step 3: Set $x^{k_1} = \tilde{x}^{k_0} = x^k$, $\beta_{k_1} = 1$ and $i = 1$, compute

$$\begin{cases} \tilde{x}^{k_i} = \mathrm{Prox}_{g/L_f}(x^{k_i} - \nabla f(x^{k_i})/L_f), \\ \beta_{k_{i+1}} := \frac{1}{2}\left(1 + \sqrt{1 + 4\beta_{k_i}^2}\right) \\ x^{k_{i+1}} := \tilde{x}^{k_i} + \dfrac{\beta_{k_i} - 1}{\beta_{k_{i+1}}}(\tilde{x}^{k_i} - \tilde{x}^{k_{i-1}}). \end{cases}$$

If $\|F(x^{k_{i+1}})\| \leq \gamma \|F(x^k)\|$, set $x^{k+1} = x^{k_{i+1}}$. Replace k by $k+1$ and go to Step 1. Otherwise, set $i = i + 1$ and continue the above iteration.

Remark 2.1. *Since f is assumed to be strongly convex, the iteration sequence $\{x^{k_i}\}$ generated by the APG algorithm always converges to the unique optimal solution x^* of the problem (2.17), and this further indicates that $F(x^{k_i}) \to 0$ by the continuity of* *the proximal mapping. Therefore, the APG algorithm can be viewed as a safeguard of the global convergence in the above framework.*

Remark 2.2. *It is known from Rademacher's Theorem that the Lipschitz continuous function F is differentiable almost everywhere. Assume that (2.20) is achievable at a differentiable point x^k and $\|F(x^k)\| \neq 0$, then $\|F(x)\|^2$ is differentiable at x^k and*

$$\begin{aligned}\|F(x^k+td^k)\|^2 &= \|F(x^k)+t(R^k-F(x^k))+o(t)\|^2\\ &= \|F(x^k)\|^2+t\langle F(x^k),R^k-F(x^k)\rangle+o(t)\\ &\leq \|F(x^k)\|^2+t(\eta_k-1)\|F(x^k)\|^2+o(t).\end{aligned}$$

Since $\eta_k \leq \eta < 1$, we have $\|F(x^k+td^k)\| < \|F(x^k)\|$ for t sufficiently small such that d^k is a descent direction of $\|F(x)\|$ at x^k. Thus, the direction obtained by the (2.20) is a descent direction with probability 1.

Remark 2.3. *In her Ph.D thesis [27, Lemma 4.5], Du shows that for any sequence $\{y^k\} \subseteq \mathcal{X}$ converges to x^*, by letting*

$$\begin{cases}\tilde{y}^k = \mathrm{Prox}_g(y^k-\nabla f(y^k)),\\ r^k = y^k-\tilde{y}^k+\nabla f(\tilde{y}^k)-\nabla f(y^k),\end{cases}$$

we have $r^k \in \partial\theta(\tilde{y}^k)$ and $\lim_{k\to\infty}\|r^k\| = 0$. Note that in order to embedded the SNCG-APG method into the iABCD framework that will be discussed in the subsequent chapter, we shall terminate the whole algorithm if

$$x^{k+1} = \arg\min_x\{\theta(x)+\langle x,\delta_x\rangle\}, \tag{2.22}$$

where $\delta_x \in \mathcal{X}$ is an error vector that satisfies $\|\delta_x\| \leq \varepsilon$, with $\varepsilon > 0$ being the required tolerance. From the above discussions one can see that this stopping criterion is always checkable as (2.22) is equivalent to $-\delta_x \in \partial\theta(x^{k+1})$.

Remark 2.4. *The Newton's equation (2.19) may not be a symmetric linear system in general, so that the conjugate gradient method cannot directly applied to it. One can still use the BiCGStab iterative solver of van der Vorst [99] to fix this issue.*

2.5 The sensitivity analysis

2.5.1 The optimality conditions and constraint qualifications

In this section, we show the first and second order optimality conditions and constraint qualifications for the optimization problem with the form

$$\begin{aligned} \min \quad & f(x) + \theta(x) \\ \text{s.t.} \quad & h(x) \in \mathcal{P}, \end{aligned} \tag{2.23}$$

where $f : \mathcal{X} \to \mathcal{R}$ and $h : \mathcal{X} \to \mathcal{Y}$ are twice continuously differentiable functions, $\theta : \mathcal{X} \to (-\infty, +\infty]$ is a closed proper convex function (not necessarily smooth), $\mathcal{P} \subseteq \mathcal{Y}$ is a convex polyhedral set, and $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional real Euclidean spaces.

It is easy to see that the problem (2.23) can be equivalently written in a conic optimization form:

$$\begin{aligned} \min \quad & f(x) + t \\ \text{s.t.} \quad & h(x) \in \mathcal{P}, \quad (x, t) \in \mathcal{K}, \end{aligned} \tag{2.24}$$

where $\mathcal{K} := \text{epi}\theta$ is a closed convex set.

The constraint qualifications of the problems with the form (2.24) are extensively studied by Bonnans and Shapiro in the book [6]. For convenience of the later work to study the isolated calmness for a class of constrained nuclear norm regularized problems, we transform the conditions about (2.24), which are imposed in a higher dimensional space, to the ones directly on the original problem (2.23), without lifting the dimension. This enables us to focus on the properties of the nonsmooth function θ directly, instead of referring to its epigraph. Similar kinds of transformation have also been done in [6, Section 3.4.1], where the form of the composite optimization problems is a little bit different from ours.

First we provide some necessary knowledge about the first and second order tangent sets and the directional (epi)derivatives. These concepts are adopted by Bonnans and Shapiro in [6, Section 2.2, 3.2].

Given a subset $\mathcal{K} \subseteq \mathcal{X}$ and $x \in \mathcal{K}$, define the contingent cone as

$$\mathcal{T}_{\mathcal{K}}(x) = \limsup_{\rho \downarrow 0} \frac{\mathcal{K} - x}{\rho},$$

and the inner tangent cone as

$$\mathcal{T}_{\mathcal{K}}^{i}(x) = \liminf_{\rho \downarrow 0} \frac{\mathcal{K} - x}{\rho}.$$

If $\mathcal{K}$ is convex, we have $\dfrac{\mathcal{K} - x}{\rho}$ is a monotone decreasing function of ρ such that $\mathcal{T}_{\mathcal{K}}(x) = \mathcal{T}_{\mathcal{K}}^{i}(x)$ for any $x \in \mathcal{K}$ [6, Proposition 2.55]. And in this case we denote both $\mathcal{T}_{\mathcal{K}}(x)$ and $\mathcal{T}_{\mathcal{K}}^{i}(x)$ as $\mathcal{T}_{\mathcal{K}}(x)$ and call it the tangent cone of $\mathcal{K}$ at x.

We also need the second order tangent sets of $\mathcal{K}$. Given $x \in \mathcal{X}$ and the direction $d \in \mathcal{X}$, define the inner second order tangent set as

$$\mathcal{T}_{\mathcal{K}}^{i,2}(x; d) := \liminf_{\rho \downarrow 0} \frac{\mathcal{K} - x - \rho d}{\frac{1}{2}\rho^2},$$

and the outer second order tangent set as

$$\mathcal{T}_{\mathcal{K}}^{2}(x; d) := \limsup_{\rho \downarrow 0} \frac{\mathcal{K} - x - \rho d}{\frac{1}{2}\rho^2}.$$

However, different from the first order tangent cones, the inner and outer second order tangent sets are not necessarily identical in general, even if the set $\mathcal{K}$ is closed and convex.

One way to characterize the tangent cone of epiθ is via its generalized directional derivatives, which are called the directional epiderivatives in [6]. Define the lower and upper directional epiderivatives of $\theta : \mathcal{X} \to (-\infty, +\infty]$ at $x \in \mathcal{X}$ along the direction $h \in \mathcal{X}$ as

$$\theta_{-}^{\downarrow}(x; h) := \liminf_{\substack{\rho \downarrow 0 \\ h' \to h}} \frac{\theta(x + \rho h') - \theta(x)}{\rho},$$

and

$$\theta_{+}^{\downarrow}(x; h) := \sup_{\{\rho_n\} \in \Sigma} \left(\liminf_{\substack{n \to \infty \\ h' \to h}} \frac{\theta(x + \rho_n h') - \theta(x)}{\rho_n} \right),$$

where Σ denotes the set of positive real sequences $\{\rho_n\}$ converging to 0.

If $\theta_{-}^{\downarrow}(x; \cdot) = \theta_{+}^{\downarrow}(x; \cdot)$ for any $x \in \mathcal{X}$, we say θ is directional epidifferentiable with the directional epiderivative $\theta^{\downarrow}(x; \cdot)$. In particular, if θ is a closed convex function, we know from [6, Proposition 2.58] that θ is always directional epidifferentiable at $x \in \mathrm{dom}\theta$, and the following relationship holds:

$$\mathcal{T}_{\mathrm{epi}\theta}\big(x, \theta(x)\big) = \mathrm{epi}\theta^{\downarrow}(x; \cdot) = \{(d_1, d_2) \in \mathcal{X} \times \mathcal{R}, \theta^{\downarrow}(x; d_1) \leq d_2\}. \tag{2.25}$$

Here we take a remark that the directional epiderivative of a function θ may be different from the conventional directional derivative (denoted as $\theta'(x; \cdot)$). Even for a proper convex function θ such that both of them exist in $\mathrm{dom}\theta$, we can only obtain the following relationship [6, Theorem 2.58 and Theorem 2.60]:

$$\theta^{\downarrow}(x; \cdot) = \mathrm{cl}\theta'(x; \cdot), \quad \forall x \in \mathrm{dom}\theta.$$

Those functions that are both Lipschitz continuous and satisfying $\theta^{\downarrow}(x; \cdot) = \theta'(x; \cdot)$, $\forall x \in \mathrm{dom}\theta$ are named as "regular functions". In particular, a convex and Lipschitz continuous function is always regular [6, Theorem 2.126].

If both $\theta^{\downarrow}_{-}(x; d)$ and $\theta^{\downarrow}_{+}(x; d)$ are finite for $x \in \mathrm{dom}\theta$ and $d \in \mathcal{X}$, we can further define the lower and upper second order directional epiderivative of the function θ as

$$\theta^{\downarrow\downarrow}_{-}(x; d, h) := \liminf_{\substack{\rho \downarrow 0 \\ h' \to h}} \frac{\theta(x + \rho d + \frac{1}{2}\rho^2 h) - \theta(x) - \rho \theta^{\downarrow}_{-}(x; d)}{\frac{1}{2}\rho^2},$$

and

$$\theta^{\downarrow\downarrow}_{+}(x; d, h) := \sup_{\rho_n \in \Sigma} \left(\liminf_{\substack{n \to \infty \\ h' \to h}} \frac{\theta(x + \rho d + \frac{1}{2}\rho^2 h) - \theta(x) - \rho \theta^{\downarrow}_{+}(x; d)}{\frac{1}{2}\rho^2} \right).$$

Similarly with the first order variational analysis, the outer and inner second order tangent sets of the $\mathrm{epi}\theta$ are closely related to the lower and upper second order directional epiderivative of θ. In particular, for any $x \in \mathrm{dom}\theta$, if $\theta^{\downarrow}_{+}(x; d)$ and $\theta^{\downarrow}_{-}(x; d)$ are finite, we have [6, proposition 3.41]

$$\mathcal{T}^{i,2}_{\mathrm{epi}\theta}((x, \theta(x)); (d, \theta^{\downarrow}_{+}(x; d)) = \mathrm{epi}\theta^{\downarrow\downarrow}_{+}(x; d, \cdot), \tag{2.26}$$

and

$$\mathcal{T}^{2}_{\mathrm{epi}\theta}((x, \theta(x)); (d, \theta^{\downarrow}_{-}(x; d)) = \mathrm{epi}\theta^{\downarrow\downarrow}_{-}(x; d, \cdot). \tag{2.27}$$

Since $\mathcal{T}_{\mathcal{K}}^2(x;d)$ and $\mathcal{T}_{\mathcal{K}}^{i,2}(x;d)$ may be different for a closed convex set $\mathcal{K}$ at $x \in \mathcal{K}$ and $d \in \mathcal{X}$, by the relationship (2.27) we see the lower and upper second order directional epiderivatives could be unequal for a proper closed convex function.

After the above preparations, we now go back to the discussions about the optimality conditions of the original optimization problem (2.23) and its variant (2.24). For any $(x,t,y,z,\tau) \in \mathcal{X} \times \mathcal{R} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{R}$, the Lagrangian function of (2.24) can be written as

$$\mathcal{L}(x,t;y,z,\tau) := f(x) + t + \langle y, h(x)\rangle + \langle z, x\rangle + t\tau.$$

We call $(\bar{x},\bar{t}) \in \mathcal{X} \times \mathcal{R}$ a stationary point of the problem (2.24) and $(\bar{y},\bar{z},\bar{\tau})$ a Lagrangian multiplier if $(\bar{x},\bar{t},\bar{y},\bar{z},\bar{\tau})$ satisfies the following KKT system:

$$\begin{cases} \nabla f(\bar{x}) + \nabla h(\bar{x})\bar{y} + \bar{z} = 0, \\ \bar{\tau} = -1, \\ \bar{y} \in \mathcal{N}_{\mathcal{P}}(h(\bar{x})), \\ (\bar{z},\bar{\tau}) \in \mathcal{N}_{\mathcal{K}}\big((\bar{x},\bar{t})\big), \end{cases} \tag{2.28}$$

where $\mathcal{N}_C(s)$ denotes the normal cone of a given convex set C at the point $s \in C$.

By [12, Corollary 2.4.9], we have that

$$(z,-1) \in \mathcal{N}_{\mathcal{K}}\big(x, \theta(x)\big) \Longleftrightarrow z \in \partial\theta(x), \quad \forall x, z \in \mathcal{X}. \tag{2.29}$$

Thus, we call $\bar{x}$ a stationary point of the problem (2.23) and $\bar{y}$ a Lagrangian multiplier of (2.23) if $(\bar{x},\bar{y})$ satisfy that:

$$\begin{cases} 0 \in \nabla f(\bar{x}) + \partial\theta(\bar{x}) + \nabla h(\bar{x})\bar{y}, \\ \bar{y} \in \mathcal{N}_{\mathcal{P}}(h(\bar{x})), \end{cases} \tag{2.30}$$

which by [80, Theorem 23.2] is equivalent to

$$\begin{cases} f'(\bar{x})d + \langle \bar{y}, h'(\bar{x})d\rangle + \theta^{\downarrow}(\bar{x};d) \geq 0, \quad \forall d \in \mathcal{X}, \\ \bar{y} \in \mathcal{N}_{\mathcal{P}}(h(\bar{x})), \end{cases} \tag{2.31}$$

We denote $\overline{\mathcal{M}}(\bar{x},\bar{t})$ as the set of all the Lagrangian multipliers at $(\bar{x},\bar{t})$ with respect to the problem (2.24), and $\mathcal{M}(\bar{x})$ as the set of all the Lagrangian multipliers at $\bar{x}$ with respect to the original problem (2.23).

Robinson's constraint qualification (RCQ) at a feasible solution $(\bar{x},\theta(\bar{x}))$ of the problem (2.24) takes the form of

$$0 \in \operatorname{int}\left\{ \begin{pmatrix} (h(\bar{x}),0) \\ (\bar{x},\theta(\bar{x})) \end{pmatrix} + \begin{pmatrix} (h'(\bar{x}),0) \\ (\mathcal{I},1) \end{pmatrix} (\mathcal{X}\times\mathcal{R}) - \begin{pmatrix} \mathcal{P}\times\{0\} \\ \mathrm{epi}\theta \end{pmatrix} \right\}. \tag{2.32}$$

In order to transform the RCQ and others from the lifting problem (2.24) to the original one (2.23), It is known that the RCQ (2.32) holds at $(\bar{x},\theta(\bar{x}))$ if and only if $\overline{\mathcal{M}}(\bar{x},\theta(\bar{x}))$ associated with the problem (2.24) is nonempty, convex and compact [109]. Then by combining [6, Proposition 2.97], we can immediately get the following proposition, which shows the corresponding equivalence in a lower dimensional space.

Proposition 2.5. *Let $\bar{x}$ be a local optimal solution of the problem (2.23). Then the set of Lagrangian multipliers $\mathcal{M}(\bar{x})$ of (2.23) is nonempty, convex and compact if and only if*

$$\boxed{0 \in \operatorname{int}\left\{ \begin{pmatrix} h(\bar{x}) \\ \bar{x} \end{pmatrix} + \begin{pmatrix} h'(\bar{x}) \\ \mathcal{I} \end{pmatrix} \mathcal{X} - \begin{pmatrix} \mathcal{P} \\ \mathrm{dom}\theta \end{pmatrix} \right\}.} \tag{2.33}$$

Furthermore, (2.33) holds if and only if

$$\boxed{\begin{pmatrix} h'(\bar{x}) \\ \mathcal{I} \end{pmatrix} \mathcal{X} + \begin{pmatrix} \mathcal{T}_{\mathcal{P}}(h(\bar{x})) \\ \mathcal{T}_{\mathrm{dom}\theta}(\bar{x}) \end{pmatrix} = \begin{pmatrix} \mathcal{Y} \\ \mathcal{X} \end{pmatrix}.} \tag{2.34}$$

Remark 2.5. *The relationships (2.33) or (2.34) would be called the RCQ of the original problem (2.23) in the subsequent discussions.*

The RCQ condition only guarantees the existence and boundedness of the multipliers. More restrictive conditions are needed if we require the multiplier set to be a singleton for the problem (2.24). A frequently used one, which is called strict Robinson's constraint qualification (SRCQ) in the literature, serves this purpose. Specifically, it makes the following assumption at the stationary point $(\bar{x}, \theta(\bar{x}))$ with respect to $(\bar{y}, \bar{z}, -1) \in \mathcal{M}(\bar{x}, \theta(\bar{x}))$:

$$\begin{pmatrix} (h'(\bar{x}), 0) \\ (\mathcal{I}, 1) \end{pmatrix} (\mathcal{X} \times \mathcal{R}) + \begin{pmatrix} \mathcal{T}_{\mathcal{P}}(h(\bar{x})) \\ \mathcal{T}_{\mathcal{K}}(\bar{x}, \theta(\bar{x})) \end{pmatrix} \cap (\bar{y}, \bar{z}, -1)^{\perp} = \begin{pmatrix} \mathcal{Y} \\ \mathcal{X} \times \mathcal{R} \end{pmatrix}. \tag{2.35}$$

As before, we would also reduce the SRCQ imposed on the problem (2.24) to a lower dimensional space with respect to the original problem (2.23). Define a set-valued mapping $\mathcal{T}^{\theta} : \mathrm{dom}\theta \times \mathcal{X}$ associated with a closed proper convex function θ as

$$\mathcal{T}^{\theta}(x, z) := \{d \in \mathcal{X} : \theta^{\downarrow}(x; d) = \langle d, z \rangle\}, \quad \forall (x, z) \in \mathrm{dom}\theta \times \mathcal{X}. \tag{2.36}$$

Proposition 2.6. *Let $\bar{x}$ be a local optimal solution of the problem (2.23). Assume $\mathcal{M}(\bar{x})$ is nonempty. Suppose the following condition holds at $\bar{x}$ with respect to $\bar{y} \in \mathcal{M}(\bar{x})$:*

$$\boxed{\begin{pmatrix} h'(\bar{x}) \\ \mathcal{I} \end{pmatrix} \mathcal{X} + \begin{pmatrix} \mathcal{T}_{\mathcal{P}}(h(\bar{x})) \cap \bar{y}^{\perp} \\ \mathcal{T}^{\theta}(\bar{x}, -\nabla_x l(\bar{x}, \bar{y})) \end{pmatrix} = \begin{pmatrix} \mathcal{Y} \\ \mathcal{X} \end{pmatrix},} \tag{2.37}$$

where $\mathcal{T}^{\theta}(\cdot, \cdot)$ is define as in (2.36). Then $\mathcal{M}(\bar{x}) = \{\bar{y}\}$ is a singleton.

Let $(\bar{x}, \bar{t}) \in \mathcal{X} \times \mathcal{R}$ be a feasible point of the problem (2.24), then the critical cone of $\mathcal{C}(\bar{x}, \bar{t})$ of the problem (2.24) takes the form of

$$\begin{aligned} \mathcal{C}_{\theta}(\bar{x}, \bar{t}) := \{(d_1, d_2) \in \mathcal{X} \times \mathcal{R} : h'(\bar{x})d_1 \in \mathcal{T}_{\mathcal{P}}(h(\bar{x})), (d_1, d_2) \in \mathcal{T}_{\mathcal{K}}\big(\bar{x}, \bar{t}\big), \\ f'(\bar{x})d_1 + d_2 \leq 0\}. \end{aligned}$$

Furthermore, if $(\bar{x}, \theta(\bar{x}))$ is a local optimal solution of the problem (2.24) and

$\mathcal{M}(\bar{x}, \theta(\bar{x}))$ is nonempty, then for any $(\bar{y}, \bar{z}, \bar{\tau}) \in \mathcal{M}(\bar{x}, \bar{y})$,

$$\begin{aligned}\mathcal{C}_\theta(\bar{x}, \theta(\bar{x})) &= \{(d_1, d_2) \in \mathcal{X} \times \mathcal{R} : h'(\bar{x})d_1 \in \mathcal{T}_\mathcal{P}(h(\bar{x})), (d_1, d_2) \in \mathcal{T}_\mathcal{K}\big(\bar{x}, \theta(\bar{x})\big), \\ &\qquad f'(\bar{x})d_1 + d_2 = 0\} \\ &= \{(d_1, d_2) \in \mathcal{X} \times \mathcal{R} : h'(\bar{x})d_1 \in \mathcal{T}_\mathcal{P}(h(\bar{x})) \cap \bar{y}^\perp, \\ &\qquad (d_1, d_2) \in \mathcal{T}_\mathcal{K}\big(\bar{x}, \theta(\bar{x})\big) \cap (\bar{z}, \bar{\tau})^\perp\}.\end{aligned} \tag{2.38}$$

The following proposition provides the connection of the critical cones between the problems (2.23) and (2.24), for which the proof can be directly obtained by using the formula (2.25).

Proposition 2.7. *Let $\bar{x} \in \mathcal{X}$ be a feasible point of the problem (2.23). Then $(d_1, d_2) \in \mathcal{C}_\theta(\bar{x}, \theta(\bar{x}))$ if and only if*

$$(d_1, d_2) \in \widetilde{\mathcal{C}}(\bar{x}, \theta(\bar{x})) := \{(d_1, d_2) \in \mathcal{X} \times \mathcal{R} : d_1 \in \mathcal{C}(\bar{x}),\ \theta^\downarrow(\bar{x}; d_1) \le d_2 \le -f'(\bar{x})d_1\},$$

where $\mathcal{C}(\bar{x})$ is defined as

$$\boxed{\mathcal{C}(\bar{x}) := \left\{d \in \mathcal{X} : h'(\bar{x})d \in \mathcal{T}_\mathcal{P}(h(\bar{x})),\ f'(\bar{x})d + \theta^\downarrow(\bar{x}; d) \le 0\right\}. \qquad (2.39)}$$

Furthermore, if $\bar{x}$ is a a local optimal solution of the problem (2.23) and $\mathcal{M}(\bar{x})$ is non-empty, then $\mathcal{C}(\bar{x})$ defined in (2.39) can be written as

$$\boxed{\mathcal{C}(\bar{x}) = \left\{d \in \mathcal{X} : h'(\bar{x})d \in \mathcal{T}_\mathcal{P}(h(\bar{x})),\ d \in \mathcal{T}^\theta(\bar{x}, -\nabla f(\bar{x}))\right\}, \qquad (2.40)}$$

where $\mathcal{T}^\theta(\cdot, \cdot)$ is define as in (2.36).

2.5.2 Calmness, metric subregularity and error bounds

Denote $\mathcal{U}$ and $\mathcal{V}$ be two finite dimensional real Euclidean spaces and $F : \mathcal{U} \to \mathcal{V}$ a multi-valued mapping. The graph of the mapping F is defined as $\mathrm{gph}(F) := \{(u, v) \in \mathcal{U} \times \mathcal{V} : v \in F(u)\}$. In the following, we introduce the concept of the (isolated) calmness and the metric subregular of a multi-valued mapping from $\mathcal{U}$ to $\mathcal{V}$, which comes from, e.g., [82, 9(30)].

Definition 2.6. [calmness and isolated calmness] *Denote* $\mathcal{B}_v := \{v \in \mathcal{V} : \|v\| \leq 1\}$. *We say a multi-valued mapping* $F : \mathcal{U} \to \mathcal{V}$ *is calm at* $u_0 \in \mathcal{U}$ *if* $(\bar{u}, \bar{v}) \in gph(F)$ *and there exist a constant* $\eta_1 > 0$ *and a neighborhood* $\mathcal{N}(\bar{u})$ *of* $\bar{u}$ *such that*

$$F(u) \subseteq F(\bar{u}) + \eta_1 \|u - \bar{u}\| \mathcal{B}_v, \quad \forall u \in \mathcal{N}(\bar{u}).$$

Furthermore, we say F *is isolated calm*[1] *at* $\bar{u} \in \mathcal{U}$ *for* $\bar{v} \in \mathcal{V}$ *if* $(\bar{u}, \bar{v}) \in gph(F)$ *and there exist a constant* $\eta_2 > 0$ *and neighborhoods* $\mathcal{N}(\bar{u})$ *of* $\bar{u}$ *and* $\mathcal{N}(\bar{v})$ *of* $\bar{v}$ *such that*

$$F(u) \cap \mathcal{N}(\bar{v}) \subseteq \{\bar{v}\} + \eta_2 \|u - \bar{u}\| \mathcal{B}_v, \quad \forall u \in \mathcal{N}(\bar{u}).$$

Definition 2.7. [Metric subregularity] *We say a multi-valued mapping* $F : \mathcal{U} \to \mathcal{V}$ *is metric subregular at* $\bar{u} \in \mathcal{U}$ *for* $\bar{v} \in \mathcal{V}$ *if* $(\bar{u}, \bar{v}) \in gph(F)$ *and there exist a constant* $\kappa > 0$ *and a neighborhood* $\mathcal{N}(\bar{u})$ *of* $\bar{u}$ *such that*

$$\text{dist}(u, F^{-1}(\bar{v})) \leq \kappa \text{dist}(\bar{v}, F(u)), \quad \forall u \in \mathcal{N}(\bar{u}).$$

One can see that the calmness of a multi-valued mapping $F : \mathcal{U} \to \mathcal{V}$ at $\bar{u} \in \mathcal{U}$ for $\bar{v} \in \mathcal{V}$ is in fact equivalently to the metric sub-regularity of F^{-1} at $\bar{v}$ for $\bar{u}$ if $(\bar{u}, \bar{v}) \in \text{gph}(F)$. They are important tools in the study of perturbation analysis and error bounds of the optimization problems. For a nice survey about this topic, see [74].

It may be difficult to check the calmness or the metric subregular of a given multi-valued mapping by definition directly, since infinity many points on the graph of the reference points may be involved. Fortunately, the following criterion holds for a special class of multi-valued mappings: the sub-differential of convex functions.

Theorem 2.3. [2, Theorem 3.3] *Let* $\mathcal{H}$ *be a real Hilbert space endowed with the inner product* $\langle \cdot, \cdot \rangle$ *and* $f : \mathcal{H} \to (-\infty, +\infty]$ *be a proper lower semicontinuous convex function. Let* $\bar{v}, \bar{x} \in \mathcal{H}$ *satisfy* $\bar{v} \in \partial f(\bar{x})$. *Then* ∂f *is metric subregular at* $\bar{x}$ *for* $\bar{v}$ *if and only if there exists a neighborhood* $\mathcal{U}$ *of* $\bar{x}$ *and a positive constant* c *such that*

$$f(x) \geq f(\bar{x}) + \langle \bar{v}, x - \bar{x} \rangle + c\,dist^2(x, (\partial f)^{-1}(\bar{v})), \quad \forall x \in \mathcal{U}. \tag{2.41}$$

[1]The isolated calmness is also called the upper Lipschitz continuity in the literature, such as in [22, 56].

Based on the above description, we can easily show that the subgradient of the indicator function over the positive semidefinite cone is metric subregular.

Theorem 2.4. *Denote $\delta_{\mathcal{S}^n_+}(\cdot)$ as the indicator function over the positive semidefinite cone in $\mathcal{S}^n$. Then $\partial\delta_{\mathcal{S}^n_+}$ is metric subregular at any $\bar{x} \in \mathcal{S}^n_+$ for $\bar{v} \in \partial\delta_{\mathcal{S}^n_+}(\bar{x})$.*

Proof. In order to show the metric subregular of $\partial\delta_{\mathcal{S}^n_+}$ by using Theorem 2.3, it suffices to verify that for any $\bar{v} \in \partial\delta_{\mathcal{S}^n_+}(\bar{x})$, there exist a constant $c > 0$ and a neighborhood $\mathcal{N}(\bar{x})$ of $\bar{x}$ such that

$$0 \geq \langle \bar{v}, x - \bar{x}\rangle + c\,\mathrm{dist}^2(x, \partial\delta_{\mathcal{S}^n_+}(\bar{v})), \quad \forall x \in \mathcal{N}(\bar{x}) \cap \mathcal{S}^n_+. \tag{2.42}$$

Note that $\bar{v} \in \partial\delta_{\mathcal{S}^n_+}(\bar{x})$ is equivalent to $\bar{x} = \Pi_{\mathcal{S}^n_+}(\bar{x} + \bar{v})$. Suppose that $\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \ldots \geq \bar{\lambda}_n$ are the eigenvalues of $\bar{x} + \bar{v}$ arranged in the non-increasing order. Denote

$$\alpha := \{i : \bar{\lambda}_i > 0,\ 1 \leq i \leq n\},\ \beta := \{i : \bar{\lambda}_i = 0,\ 1 \leq i \leq n\},\ \gamma := \{i : \bar{\lambda}_i < 0,\ 1 \leq i \leq n\}.$$

Then there exists an orthogonal matrix $\overline{P} \in \mathcal{O}^n$ such that

$$\bar{x} = \overline{P} \begin{pmatrix} \Lambda_\alpha & & \\ & 0 & \\ & & 0_\gamma \end{pmatrix} \overline{P}^T, \quad \bar{v} = \overline{P} \begin{pmatrix} 0_\alpha & & \\ & 0 & \\ & & \Lambda_\gamma \end{pmatrix} \overline{P}^T,$$

where $\Lambda_\alpha \succ 0$ is a diagonal matrix whose diagonal entries are $\bar{\lambda}_i > 0$ for $i \in \alpha$ and $\Lambda_\gamma \prec 0$ is a diagonal matrix whose diagonal entries are $\bar{\lambda}_j < 0$ for $j \in \gamma$. The subgradint of $\delta_{\mathcal{S}^n_+}$ at $\bar{v}$ can be expressed explicitly as

$$\partial\delta_{\mathcal{S}^n_+}(\bar{v}) = \{h \in \mathcal{S}^n : [\overline{P}_\alpha\ \overline{P}_\beta]^T h [\overline{P}_\alpha\ \overline{P}_\beta] \succeq 0,\ \overline{P}_\gamma^T h \overline{P} = 0,\ \overline{P}^T h \overline{P}_\gamma = 0\}.$$

Let $\delta = \min\{1/2, \lambda_{|\alpha|}/2\} > 0$ and denote $\mathcal{N}_\delta(\bar{x}) = \{x \in \mathcal{S}^n : \|x - \bar{x}\| \leq \delta\}$. Consider an arbitrary $x \in \mathcal{S}^n_+ \cap \mathcal{N}_\delta(\bar{x})$. We write $\tilde{x} = \overline{P}^T x \overline{P}$ and decompose $\tilde{x}$ into the following nine blocks:

$$\tilde{x} \equiv \begin{pmatrix} \tilde{x}_{\alpha\alpha} & \tilde{x}_{\alpha\beta} & \tilde{x}_{\alpha\gamma} \\ \tilde{x}_{\alpha\beta}^T & \tilde{x}_{\beta\beta} & \tilde{x}_{\beta\gamma} \\ \tilde{x}_{\alpha\gamma}^T & \tilde{x}_{\beta\alpha}^T & \tilde{x}_{\gamma\gamma} \end{pmatrix}.$$

Then it is easy to see $\Pi_{\partial\delta_{\mathcal{S}^n_+}(\bar v)}(x) = \overline{P}\begin{pmatrix} \tilde x_{\alpha\alpha} & \tilde x_{\alpha\beta} & 0 \\ \tilde x_{\alpha\beta}^T & \tilde x_{\beta\beta} & 0 \\ 0 & 0 & 0 \end{pmatrix}\overline{P}^T$, so that

$$\mathrm{dist}(x, \partial\delta_{\mathcal{S}^n_+}(\bar v))^2 = 2\|\tilde x_{\alpha\gamma}\|^2 + 2\|\tilde x_{\beta\gamma}\|^2 + \|\tilde x_{\gamma\gamma}\|^2. \tag{2.43}$$

Denote $\Lambda(\tilde x_{\alpha\alpha}) = \mathrm{Diag}\,(\lambda_1(\tilde x_{\alpha\alpha}), \ldots, \lambda_{|\alpha|}(\tilde x_{\alpha\alpha}))$ as a diagonal matrix whose diagonal entries are the eigenvalues of $\tilde x_{\alpha\alpha}$ arranged in the non-increasing order. Since $\|\Lambda(\tilde x_{\alpha\alpha}) - \Lambda_\alpha\| \le \|\tilde x_{\alpha\alpha} - \Lambda_\alpha\| \le \delta$, we see that $\lambda_1(\tilde x_{\alpha\alpha}) \le \bar\lambda_1 + 1/2$ and $\lambda_{\min}(\tilde x_{\alpha\alpha}) \ge \delta > 0$. Then from $\tilde x_{\gamma\gamma} - \tilde x_{\alpha\gamma}^T \tilde x_{\alpha\alpha}^{-1} \tilde x_{\alpha\gamma} \succeq 0$ we get

$$\|\tilde x_{\alpha\gamma}\|^2 = \mathrm{tr}(\tilde x_{\alpha\gamma}^T \tilde x_{\alpha\gamma}) \le \lambda_1(\tilde x_{\alpha\alpha})\mathrm{tr}(\tilde x_{\gamma\gamma}) \le \frac{\bar\lambda_1 + 1/2}{-\bar\lambda_{|\alpha|+|\beta|+1}}\langle \tilde x_{\gamma\gamma}, -\Lambda_\gamma\rangle. \tag{2.44}$$

Moreover, we obtain from $\begin{pmatrix} \tilde x_{\beta\beta} & \tilde x_{\beta\gamma} \\ \tilde x_{\beta\gamma}^T & \tilde x_{\gamma\gamma} \end{pmatrix} \succeq 0$ that

$$\tilde x_{ij}^2 \le \tilde x_{ii}\tilde x_{jj} \le \frac{\delta}{-\bar\lambda_j}\tilde x_{jj}(-\bar\lambda_j), \quad \forall i \in \beta, j \in \gamma.$$

and therefore,

$$\|\tilde x_{\beta\gamma}\|^2 = \sum_{i\in\beta, j\in\gamma} \tilde x_{ij}^2 \le \frac{|\beta|}{-2\bar\lambda_{|\alpha|+|\beta|+1}}\langle \tilde x_{\gamma\gamma}, -\Lambda_\gamma\rangle. \tag{2.45}$$

In view of (2.43), (2.44), (2.45) and

$$\|\tilde x_{\gamma\gamma}\|^2 = \sum_{i,j\in\gamma} x_{ij}^2 \le \sum_{i\in\gamma} x_{ii}^2 + 2\sum_{i,j\in\gamma} x_{ii}x_{jj} = (\sum_{i\in\gamma} x_{ii})^2 \le \frac{1}{-2\bar\lambda_{|\alpha|+|\beta|+1}}\langle \tilde x_{\gamma\gamma}, -\Lambda_\gamma\rangle,$$

we obtain that for any $x \in \mathcal{S}^n_+ \cap \mathcal{N}_\delta(\bar x)$,

$$\begin{aligned} & \langle \bar v, x - \bar x\rangle + \frac{-\bar\lambda_{|\alpha|+|\beta|+1}}{2\bar\lambda_1 + 3/2 + |\beta|}\mathrm{dist}^2(x, \partial\delta_{\mathcal{S}^n_+}(\bar v)) \\ = \;& \langle \tilde x_{\gamma\gamma}, \Lambda_\gamma\rangle + \frac{-\bar\lambda_{|\alpha|+|\beta|+1}}{2\bar\lambda_1 + 3/2 + |\beta|}(2\|\tilde x_{\alpha\gamma}\|^2 + 2\|\tilde x_{\beta\gamma}\|^2 + \|\tilde x_{\gamma\gamma}\|^2) \le 0. \end{aligned}$$

Thus, the proof is completed by letting $c := \frac{-\bar\lambda_{|\alpha|+|\beta|+1}}{2\bar\lambda_1+3/2+|\beta|} > 0$ in (2.42). $\square$

The following Lemma provides a convenient tool for checking the isolated calmness via the directional derivative, which is modified from the classical results that are based on the non-singularity of the graphical derivative at the origin [50, 56].

Lemma 2.1. *Let $\mathcal{U}$ and $\mathcal{V}$ be two finite dimensional real Euclidean spaces and $F : \mathcal{U} \to \mathcal{V}$ is a continuous mapping. Let $(u_0, v_0) \in \mathcal{U} \times \mathcal{V}$ satisfing $F(u_0) = v_0$. Suppose that F is locally Lipschitz continuous around u_0 and directional differentiable at u_0. Then F^{-1} is isolated calm at v_0 for u_0 if and only if*

$$F'(u_0; d) = 0 \Longrightarrow d = 0, \quad \forall d \in \mathcal{V}.$$

The isolated calmness does not require the locally nonemptyness of the multi-valued mapping near the reference point. A stronger property is the following robust isolated calmness.

Definition 2.8. [Locally nonempty-valued] *We say a multi-valued mapping $F : \mathcal{U} \to \mathcal{V}$ is locally nonempty-valued at $\bar{u} \in \mathcal{U}$ for $\bar{v} \in \mathcal{V}$ if there exist neighborhoods $\mathcal{N}(\bar{u})$ of $\bar{u}$ and $\mathcal{N}(\bar{v})$ of $\bar{v}$ such that*

$$F(u) \cap \mathcal{N}(\bar{v}) \neq \emptyset, \quad \forall u \in \mathcal{N}(\bar{u}).$$

Definition 2.9. [Robust isolated calmness] *We say a multi-valued mapping $F : \mathcal{U} \to \mathcal{V}$ is robust isolated calm at $\bar{u} \in \mathcal{U}$ for $\bar{v} \in \mathcal{V}$ if F is both isolated calm and locally nonempty valued at $\bar{u} \in \mathcal{U}$ for $\bar{v} \in \mathcal{V}$.*

2.5.3 Equivalence of the isolated calmness in different forms

We show several different forms of the isolated calmness for the problem (2.23) are in fact equivalent to each other, and they can further imply an error bound condition.

The KKT system of the problem (2.23) takes the form of

$$\begin{cases} 0 \in \nabla_x l(\bar{x}, \bar{y}) + \partial\theta(\bar{x}), \\ \bar{y} \in \mathcal{N}_{\mathcal{P}}(h(\bar{x})). \end{cases} \tag{2.46}$$

Denote the natural map $G : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ associated with the inclusion (2.46) as

$$G(x, y) := \begin{pmatrix} x - \operatorname{Prox}_\theta\big(x - \nabla_x l(x, y)\big) \\ h(x) - \Pi_{\mathcal{P}}(h(x) + y) \end{pmatrix}, \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}, \tag{2.47}$$

and the normal map $G^{\text{nor}} : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ associated with the inclusion (2.46) as

$$G^{\text{nor}}(z, s) := \begin{pmatrix} \nabla_x l\big(\text{Prox}_\theta(z), \text{Prox}_{\delta^*_{\mathcal{P}}}(s)\big) + z - \text{Prox}_\theta(z) \\ h(\text{Prox}_\theta(z)) - \Pi_{\mathcal{P}}(s) \end{pmatrix}, \quad \forall (z, s) \in \mathcal{X} \times \mathcal{Y}. \tag{2.48}$$

It is easy to see that $(\bar{x}, \bar{y})$ satisfies the KKT condition of the problem (2.23) in the sense of (2.46) if and only if $G(\bar{x}, \bar{y}) = 0$, which is also equivalent to $G^{\text{nor}}(\bar{z}, \bar{s}) = 0$ with $(\bar{z}, \bar{s}) = (\bar{x} - \nabla_x l(\bar{x}, \bar{y}), h(\bar{x}) + \bar{y})$. In fact, there is a change of variable $(x, y) = \big(\text{Prox}_\theta(z), \text{Prox}_{\delta^*_{\mathcal{P}}}(s)\big)$ between the natural map and the normal map, and clearly $(\bar{x}, \bar{y}) = \big(\text{Prox}_\theta(\bar{z}), \text{Prox}_{\delta^*_{\mathcal{P}}}(\bar{s})\big)$. One can refer to the monograph [30, Section 1.5.2] of Facchinei and Pang for a detailed discussion about the relationships between the two maps.

We also consider the perturbation of the problem (2.23) with the form

$$\begin{aligned} \min \quad & f(x) + \theta(x) - \langle \delta_1, x \rangle, \\ \text{s.t.} \quad & h(x) - \delta_2 \in \mathcal{P}, \end{aligned} \tag{2.49}$$

where $\delta \equiv (\delta_1, \delta_2) \in \mathcal{X} \times \mathcal{Y}$ is the perturbation parameter. Similarly as (2.47), for a given $\delta \in \mathcal{X} \times \mathcal{Y}$, we could thus define a natural map $G^{\text{nat}}_\delta : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ associated with the perturbed problem (2.49) as

$$G^{\text{nat}}_\delta(x, y) := \begin{pmatrix} x - \text{Prox}_\theta\big(x - \nabla_x l(x, y) + \delta_1\big) \\ h(x) - \delta_2 - \Pi_{\mathcal{P}}(h(x) - \delta_2 + y) \end{pmatrix}, \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}. \tag{2.50}$$

For later discussions, we also write an extended natural map $\widetilde{G} : \mathcal{X} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y}$ as

$$\widetilde{G}(x, y, \delta) := \begin{pmatrix} G^{\text{nat}}_\delta(x, y) \\ \delta \end{pmatrix}, \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}, \quad \forall \delta \in \mathcal{X} \times \mathcal{Y}. \tag{2.51}$$

For a given $\delta \in \mathcal{X} \times \mathcal{Y}$, we denote $S_{\text{KKT}}(\delta)$ as the set of all the KKT points of the perturbed problem (2.49), i.e.,

$$S_{\text{KKT}}(\delta) := \left\{ (\bar{x}, \bar{y}) : G^{\text{nat}}_\delta(\bar{x}, \bar{y}) = 0 \right\}.$$

One convenience of the normal map is its translational property that if the KKT system (2.46) of the original problem (2.23) is translated by a constant $\delta \in \mathcal{X} \times \mathcal{Y}$, its associated normal map G^{nor} is also translated by the same constant δ, i.e.,

$$S_{\text{KKT}}(\delta) = \left\{ \left(\text{Prox}_\theta(\bar{z}), \text{Prox}_{\delta_{\mathcal{P}}^*}(\bar{s})\right) : \; G^{\text{nor}}(\bar{z}, \bar{s}) = \delta \right\}. \tag{2.52}$$

Note that when the perturbation parameter $\delta = 0$, the problem (2.49) reduces to the original problem and thus, $(\bar{x}, \bar{y}) \in \Omega$ if and only if $G^{\text{nat}}_{\bar{\delta}}(\bar{x}, \bar{y}) = 0$ for $\bar{\delta} = 0$, or equivalently $\widetilde{G}(\bar{x}, \bar{y}, \bar{\delta}) = 0$. In real applications, the data is usually inaccurate, which makes the study of the problem (2.49) necessary and important. In the following, we shall show that at the KKT point $(\bar{x}, \bar{y})$ of the problem (2.23), the isolated calmness of $\widetilde{G}^{-1}$ at the origin point with respect to $(\bar{x}, \bar{y}, 0)$ is equivalent to the isolated calmness of G^{-1} at the origin point with respect to $(\bar{x}, \bar{y})$, and is also the same with $(G^{\text{nor}})^{-1}$ at the origin point with respect to $(\bar{z}, \bar{s}) = (\bar{x} - \nabla_x l(\bar{x}, \bar{y}), h(\bar{x}) + \bar{y})$.

Theorem 2.5. *Suppose that $(\bar{x}, \bar{y}) \in \mathcal{X} \times \mathcal{Y}$ satisfies that $G(\bar{x}, \bar{y}) = 0$. Let $(\bar{z}, \bar{s}) = (\bar{x} - \nabla_x l(\bar{x}, \bar{y}), h(\bar{x}) + \bar{y})$. Then the following conditions are equivalent:*

(i) G^{-1} is isolated calm at the origin for $(\bar{x}, \bar{y})$.

(ii) $\widetilde{G}^{-1}$ is isolated calm at the origin for $(\bar{x}, \bar{y}, 0)$.

(iii) $(G^{nor})^{-1}$ is isolated calm at the origin for $(\bar{z}, \bar{s})$.

(iv) S_{KKT} is isolated calm at the origin for $(\bar{x}, \bar{y})$.

Proof. First we show (i) $\Longleftrightarrow$ (ii). Suppose that G^{-1} is isolated calm at the origin for $(\bar{x}, \bar{y})$. Let $(d_x, d_y, d_\delta) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y}$ satisfy that $\widetilde{G}'((\bar{x}, \bar{y}, 0); (d_x, d_y, d_\delta)) = 0$, where $d_\delta := (d_{\delta_1}, d_{\delta_2})$ with $d_{\delta_1} \in \mathcal{X}$, $d_{\delta_2} \in \mathcal{Y}$. Then we have

$$\begin{cases} d_x - \text{Prox}_\theta'(\bar{x} - \nabla_x l(\bar{x}, \bar{y}); d_x - \nabla^2_{xx} l(\bar{x}, \bar{y}) d_x - \nabla h(\bar{x}) d_y + d_{\delta_1}) = 0, \\ h'(\bar{x}) d_x - d_{\delta_2} - \Pi_{\mathcal{P}}'(h(\bar{x}) + \bar{y}; h'(\bar{x}) d_x - d_{\delta_2} + d_y) = 0, \\ (d_{\delta_1}, d_{\delta_2}) = 0, \end{cases} \tag{2.53}$$

which can be reduced to $d_\delta = (d_{\delta_1}, d_{\delta_2}) = 0$ and

$$\begin{cases} d_x - \text{Prox}_\theta'(\bar{x} - \nabla_x l(\bar{x}, \bar{y}); d_x - \nabla^2_{xx} l(\bar{x}, \bar{y}) d_x - \nabla h(\bar{x}) d_y) = 0, \\ h'(\bar{x}) d_x - \Pi_{\mathcal{P}}'(h(\bar{x}) + \bar{y}; h'(\bar{x}) d_x + d_y) = 0, \end{cases} \tag{2.54}$$

or equivalently, $G'((\bar{x},\bar{y});(d_x,d_y)) = 0$. Thus, we get $(d_x,d_y) = 0$ by Lemma 2.1 and the isolated calmness assumption of G^{-1} at the origin for $(\bar{x},\bar{y})$, which further indicates that $(d_x,d_y,d_\delta) = 0$. By applying Lemma 2.1 again we know $\widetilde{G}^{-1}$ is isolated calm at the origin for $(\bar{x},\bar{y},0)$.

Conversely, suppose $\widetilde{G}^{-1}$ is isolated calm at the origin for $(\bar{x},\bar{y},0)$. Let $(\tilde{d}_x,\tilde{d}_y) \in \mathcal{X}\times\mathcal{Y}$ be a direction such that $G'((\bar{x},\bar{y});(\tilde{d}_x,\tilde{d}_y)) = 0$, which is equivalent for (2.54) holds at $(\bar{x},\bar{y})$ along the direction $(\tilde{d}_x,\tilde{d}_y)$. Then by letting $\tilde{d}_\delta = 0_{\mathcal{X}\times\mathcal{Y}}$, we see (2.53) holds for $(\tilde{d}_x,\tilde{d}_y,\tilde{d}_\delta)$, and thus, $\widetilde{G}'((\bar{x},\bar{y},0);(\tilde{d}_x,\tilde{d}_y,\tilde{d}_\delta)) = 0$. Hence, $(\tilde{d}_x,\tilde{d}_y,\tilde{d}_\delta) = 0$ by the isolated calmness of $\widetilde{G}^{-1}$ at the origin for $(\bar{x},\bar{y},0)$. This non-singularity of $G'((\bar{x},\bar{y});(\cdot,\cdot))$ indicates that G^{-1} is isolated calm at the origin for $(\bar{x},\bar{y})$.

Now we focus on the equivalence between (i) and (iii). Note that under the condition $G(\bar{x},\bar{y}) = 0$ and $(\bar{z},\bar{s}) = (\bar{x} - \nabla_x l(\bar{x},\bar{y}), h(\bar{x}) + \bar{y})$, we have $(\bar{x},\bar{y}) = \big(\mathrm{Prox}_\theta(\bar{z}), \mathrm{Prox}_{\delta^*_{\mathcal{P}}}(\bar{s})\big)$ and $G^{\mathrm{nor}}(\bar{z},\bar{s}) = 0$. Assume that G^{-1} is isolated calm at the origin with respect to $(\bar{x},\bar{y})$. Again we let $(d_z,d_s) \in \mathcal{X}\times\mathcal{Y}$ satisfy that $(G^{\mathrm{nor}})'((\bar{z},\bar{s});(d_z,d_s)) = 0$, i.e.,

$$\begin{cases} \nabla^2_{xx} l\big(\mathrm{Prox}_\theta(\bar{z}), \mathrm{Prox}_{\delta^*_{\mathcal{P}}}(\bar{s})\big)\mathrm{Prox}'_\theta(\bar{z};d_z) \\ \qquad\qquad +\nabla h(\mathrm{Prox}_\theta(\bar{z}))\mathrm{Prox}'_{\delta^*_{\mathcal{P}}}(\bar{s};d_s) + d_z - \mathrm{Prox}'_\theta(\bar{z};d_z) = 0, \\ h'(\mathrm{Prox}_\theta(\bar{z}))\mathrm{Prox}'_\theta(\bar{z};d_z) - \Pi'_{\mathcal{P}}(\bar{s};d_s) = 0. \end{cases} \tag{2.55}$$

Let $(d_x,d_y) = (\mathrm{Prox}'_\theta(\bar{z};d_z), \mathrm{Prox}'_{\delta^*_{\mathcal{P}}}(\bar{s},d_s))$. Then the first equation of (2.55) indicates that

$$\begin{aligned} d_z &= \mathrm{Prox}'_\theta(\bar{z};d_z) - \nabla^2_{xx} l\big(\mathrm{Prox}_\theta(\bar{z}), \mathrm{Prox}_{\delta^*_{\mathcal{P}}}(\bar{s})\big)\mathrm{Prox}'_\theta(\bar{z};d_z) - \nabla h(\mathrm{Prox}_\theta(\bar{z}))\mathrm{Prox}'_{\delta^*_{\mathcal{P}}}(\bar{s};d_s) \\ &= d_x - \nabla^2_{xx} l(\bar{x},\bar{y})d_x - \nabla h(\bar{x})d_y, \end{aligned}$$

and the second equation of (2.55) implies that

$$d_s = \Pi'_{\mathcal{P}}(\bar{s};d_s) + \mathrm{Prox}'_{\delta^*_{\mathcal{P}}}(\bar{s},d_s) = h'(\bar{x})d_x + d_y.$$

Substituting the above equations back into (2.55), we see that the equation (2.54) is true, so that $G'((\bar{x},\bar{y});(d_x,d_y)) = 0$. Then $(d_x,d_y) = 0$ by the non-singularity of

$G'((\bar{x},\bar{y});(\cdot,\cdot))$ from Lemma 2.1, which further shows that $(d_z, d_s) = 0$ and thus, $(G^{\mathrm{nor}})^{-1}$ is isolated calm at the origin for $(\bar{z},\bar{s})$.

Conversely, assume that $(G^{\mathrm{nor}})^{-1}$ is isolated calm at the origin for $(\bar{z},\bar{s}) = (\bar{x} - \nabla_x l(\bar{x},\bar{y}), h(\bar{x}) + \bar{y})$. Let $(\tilde{d}_x, \tilde{d}_y) \in \mathcal{X}\times\mathcal{Y}$ satisfies that $G'((\bar{x},\bar{y});(\tilde{d}_x,\tilde{d}_y)) = 0$ such that (2.54) holds for $(\bar{x},\bar{y})$ along the direction $(\tilde{d}_x,\tilde{d}_y)$. Now we construct a direction in $\mathcal{X}\times\mathcal{Y}$ as

$$(\tilde{d}_z, \tilde{d}_s) = (\tilde{d}_x - \nabla^2_{xx} l(\bar{x},\bar{y})\tilde{d}_x - \nabla h(\bar{x})\tilde{d}_y, h'(\bar{x})\tilde{d}_x + \tilde{d}_y).$$

By the first equation of (2.54) we have $\tilde{d}_x = \mathrm{Prox}'_\theta(\bar{z};\tilde{d}_z)$, which, after substituted into the second equation of (2.54), indicates that

$$h'(\mathrm{Prox}_\theta(\bar{z}))\mathrm{Prox}'_\theta(\bar{z};\tilde{d}_z) - \Pi'_{\mathcal{P}}(\bar{s};\tilde{d}_s) = 0. \tag{2.56}$$

This also shows that

$$\tilde{d}_y = \tilde{d}_s - h'(\bar{x})\tilde{d}_x = \tilde{d}_s - h'(\mathrm{Prox}_\theta(\bar{z}))\mathrm{Prox}'_\theta(\bar{z};\tilde{d}_z) = \tilde{d}_s - \Pi'_{\mathcal{P}}(\bar{s};\tilde{d}_s) = \mathrm{Prox}'_{\delta^*_{\mathcal{P}}}(\bar{s},\tilde{d}_s).$$

Therefore, we can further obtain that

$$\begin{aligned}\tilde{d}_z &= \tilde{d}_x - \nabla^2_{xx} l(\bar{x},\bar{y})\tilde{d}_x - \nabla h(\bar{x})\tilde{d}_y \\ &= \mathrm{Prox}'_\theta(\bar{z};\tilde{d}_z) - \nabla^2_{xx} l(\mathrm{Prox}_\theta(\bar{z}), \mathrm{Prox}_{\delta^*_{\mathcal{P}}}(\bar{s}))\mathrm{Prox}'_\theta(\bar{z};\tilde{d}_z) - \nabla h(\mathrm{Prox}_\theta(\bar{z}))\mathrm{Prox}'_{\delta^*_{\mathcal{P}}}(\bar{s},\tilde{d}_s).\end{aligned}$$

Together with (2.56), we know $(G^{\mathrm{nor}})'((\bar{z},\bar{s});(\tilde{d}_z,\tilde{d}_s)) = 0$ such that $(\tilde{d}_z,\tilde{d}_s) = 0$ by Lemma 2.1 and the isolated calmness of $(G^{\mathrm{nor}})^{-1}$ at the origin for $(\bar{z},\bar{s})$. Then $(\tilde{d}_x,\tilde{d}_y) = (\mathrm{Prox}'_\theta(\bar{z};\tilde{d}_z), \mathrm{Prox}'_{\delta^*_{\mathcal{P}}}(\bar{s},\tilde{d}_s)) = 0$, which indicates the isolated calmness of G^{-1} at the origin for $(\bar{x},\bar{y})$.

Till now we have show that (i)$\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii). It is easy to obtain (ii)$\Longrightarrow$ (iv) by the definition of the isolated calmness as follows. Since $\widetilde{G}^{-1}$ is isolated calm at the origin for $(\bar{x},\bar{y},0)$, there exists a neighborhood $\mathcal{N}(\bar{x},\bar{y},0) \subseteq \mathcal{X}\times\mathcal{Y}\times(\mathcal{X}\times\mathcal{Y})$ of $(\bar{x},\bar{y},0)$, a neighborhood $\mathcal{N}(0) \subseteq \mathcal{X}\times\mathcal{Y}\times(\mathcal{X}\times\mathcal{Y})$ of 0 and a positive constant η such that

$$\widetilde{G}^{-1}(\Delta)\cap\mathcal{N}(\bar{x},\bar{y},0) \subseteq \{(\bar{x},\bar{y},0)\} + \eta\|\Delta\|B_U, \quad \forall\Delta := (\Delta_1,\Delta_2,\Delta_3,\Delta_4) \in \mathcal{N}(0),$$

where $B_U := \{(x, y, \delta) \in \mathcal{X} \times \mathcal{Y} \times (\mathcal{X} \times \mathcal{Y}) : \|x\|^2 + \|y\|^2 + \|\delta\|^2 \leq 1\}$. They by letting $(\Delta_3, \Delta_4) = 0 \in \mathcal{X} \times \mathcal{Y}$ in the above inclusion, we immediately have the isolated calmness of the $S_{\rm KKT}$ at the origin for $(\bar{x}, \bar{y})$.

To complete the statement of this Theorem, it suffices to show (iv) $\Longrightarrow$ (iii). By the definition of the isolated calm of $S_{\rm KKT}$ at the origin for $(\bar{x}, \bar{y})$, we know that there exist positive constants $\eta > 0$, $\epsilon > 0$ and a neighborhood $\mathcal{N}(0) \subseteq \mathcal{X} \times \mathcal{Y}$ of 0, such that for any $\delta \in \mathcal{N}(0)$ and any $(x, y) \in S_{\rm KKT}(\delta) \cap \{(x, y) \in \mathcal{X} \times \mathcal{Y} : \|x - \bar{x}\| + \|y - \bar{y}\| < \epsilon\}$, it holds

$$\|x - \bar{x}\| + \|y - \bar{y}\| \leq \eta\|\delta\|. \tag{2.57}$$

Note that for any $\delta := (\delta_1, \delta_2) \in \mathcal{N}(0)$ and any $(z_\delta, s_\delta) \in (G^{\rm nor})^{-1}(\delta) \cap \{(z, s) \in \mathcal{X} \times \mathcal{Y} : \|z - \bar{z}\| + \|s - \bar{s}\| < \epsilon\}$, we have $(x_\delta, y_\delta) := (\mathrm{Prox}_\theta(z_\delta), \mathrm{Prox}_{\delta^*_{\mathcal{P}}}(s_\delta)) \in S_{\rm KKT}(\delta) \cap \{(x, y) \in \mathcal{X} \times \mathcal{Y} : \|x - \bar{x}\| + \|y - \bar{y}\| < \epsilon\}$. Therefore, we get $\|x - \bar{x}\| + \|y - \bar{y}\| \leq \eta\|\delta\|$ by the inequality (2.57). Since $G^{\rm nor}(z_\delta, s_\delta) = \delta$, we have $(z_\delta, s_\delta) = (x_\delta + \delta_1 - \nabla_x l(x_\delta, y_\delta), h(x_\delta) + y_\delta - \delta_2)$ by the relationship (2.52). This further implies that

$$\begin{aligned} & \|z_\delta - \bar{z}\| + \|s_\delta - \bar{s}\| \\ \leq\ & \|x_\delta - \bar{x}\| + \|\nabla_x l(x_\delta, y_\delta) - \nabla_x l(\bar{x}, \bar{y})\| + \|h(x_\delta) - h(\bar{x})\| + \|y_\delta - \bar{y}\| + \|\delta\| \\ \leq\ & (1 + L)(\|x_\delta - \bar{x}\| + \|y_\delta - \bar{y}\|) + \|\delta\| \\ \leq\ & ((1 + L)\eta + 1)\|\delta\|, \end{aligned}$$

for some constant $L > 0$, where the second inequality comes from the assumption that f and h are $\mathcal{C}^2$ functions and (x_δ, y_δ) is restricted in a bounded neighborhood of $(\bar{x}, \bar{y})$. Thus, $(G^{\rm nor})^{-1}$ is isolated calm at the origin for $(\bar{z}, \bar{s})$ by definition. $\square$

Chapter 3

An inexact majorized accelerated block coordinate descent method for multi-block unconstrained problems

In this chapter, we focus on designing and analyzing efficient algorithms for solving the unconstrained convex optimization problems with coupled objective functions. Recall the compact two-block form of such kind of problems (1.17) given in the Chapter 1:

$$\min \quad p(u) + q(v) + \phi(u, v), \tag{3.1}$$

where $p : \mathcal{U} \to (-\infty, +\infty]$ and $q : \mathcal{V} \to (-\infty, +\infty]$ are two convex functions (possibly nonsmooth), $\phi : \mathcal{U} \times \mathcal{V} \to (-\infty, +\infty)$ is a smooth convex function with Lipschitz continuous gradient mapping, and $\mathcal{U}$ and $\mathcal{V}$ are real finite dimensional Euclidean spaces each equipped with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. This would be the core model that we focus on throughout this chapter.

Note that here we do not require p and q to admit explicit expressions of the proximal mappings, which is different from the conventional settings that frequently imposed in the existing literature. This change of settings allows us to handle the corresponding multi-block unconstrained problems with the form (1.1) given in the Chapter 1, by taking $u = (u_1, u_2, \ldots, u_s)$ as one block and $v = (v_1, v_2, \ldots, v_t)$

as the other block. In this way, the nonsmooth functions p and q are composite ones as $p(u) \equiv \sum_{i=1}^{s} p_i(u_i)$ and $q(v) \equiv \sum_{j=1}^{t} q_j(v_j)$ for any $u = (u_1, u_2, \ldots, u_s) \in \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_s$ and $v = (v_1, v_2, \ldots, v_t) \in \mathcal{V}_1 \times \mathcal{V}_2 \times \ldots \times \mathcal{V}_t$, which certainly fail to have explicit proximal mappings for most cases.

The first part of this chapter is devoted to the study of the two-block majorized accelerated block coordinate descent (ABCD) method. Following that, we extend our ABCD algorithm to an inexact version in the second part. This extension is critical and essential to our subsequent numerical implementation since our ultimate goal is to solve multi-block unconstrained optimization problems, instead of merely two blocks. By allowing inexact solutions of each block, we are able to adopt various well-studied iterative algorithms to solve the subproblems that involving two or more nonsmooth functions simultaneously.

Before presenting our proposed algorithms and their theoretical properties, we first introduce several notations and the majorization technique for the smooth function ϕ.

Throughout this chapter, we denote $w \equiv (u, v) \in \mathcal{U} \times \mathcal{V}$. Since $\nabla\phi$ is assumed to be globally Lipschitz continuous, we know from (2.2) that there exist two self-adjoint positive semidefinite linear operators $\mathcal{Q}$ and $\widehat{\mathcal{Q}} : \mathcal{U} \times \mathcal{V} \to \mathcal{U} \times \mathcal{V}$ such that for any $w, w' \in \mathcal{U} \times \mathcal{V}$, it holds

$$\phi(w) \geq \phi(w') + \langle \nabla\phi(w'), w - w' \rangle + \frac{1}{2}\|w' - w\|_{\mathcal{Q}}^2 \tag{3.2}$$

and

$$\phi(w) \leq \hat{\phi}(w; w') := \phi(w') + \langle \nabla\phi(w'), w - w' \rangle + \frac{1}{2}\|w' - w\|_{\widehat{\mathcal{Q}}}^2. \tag{3.3}$$

We further decompose the operators $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$ into the following block structures:

$$\mathcal{Q}w \equiv \begin{pmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^* & \mathcal{Q}_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad \widehat{\mathcal{Q}}w \equiv \begin{pmatrix} \widehat{\mathcal{Q}}_{11} & \widehat{\mathcal{Q}}_{12} \\ \widehat{\mathcal{Q}}_{12}^* & \widehat{\mathcal{Q}}_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad \forall w = (u, v) \in \mathcal{U} \times \mathcal{V},$$

where $\mathcal{Q}_{11}, \widehat{\mathcal{Q}}_{11} : \mathcal{U} \to \mathcal{U}$ and $\mathcal{Q}_{22}, \widehat{\mathcal{Q}}_{22} : \mathcal{V} \to \mathcal{V}$ are self-adjoint positive semidefinite linear operators, and $\mathcal{Q}_{12}, \widehat{\mathcal{Q}}_{12} : \mathcal{V} \to \mathcal{U}$ are two linear mappings whose adjoints are given by $\mathcal{Q}_{12}^*$ and $\widehat{\mathcal{Q}}_{12}^*$, respectively.

There are infinite many choices of $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$ that satisfy the inequality (3.2) and (3.3), but the operator $\widehat{\mathcal{Q}}$ is always expected to be as small as possible such that the majorized function $\hat{\phi}$ could tightly approximate the original counterpart ϕ. We need the following assumption on the selection of the operators $\mathcal{Q}$ and $\widehat{Q}$ for ensuring the $O(1/k^2)$ complexity of our proposed algorithms.

Assumption 3.1. *There exist two self-adjoint positive semidefinite linear operators $\mathcal{D}_1 : \mathcal{U} \to \mathcal{U}$ and $\mathcal{D}_2 : \mathcal{V} \to \mathcal{V}$ such that*

$$\widehat{\mathcal{Q}} = \mathcal{Q} + \mathrm{Diag}\,(\mathcal{D}_1, \mathcal{D}_2).$$

Furthermore, $\widehat{\mathcal{Q}}$ satisfies that $\widehat{\mathcal{Q}}_{11} \succ 0$ and $\widehat{\mathcal{Q}}_{22} \succ 0$.

In order to simplify the subsequent discussions, we denote a positive semidefinite operator $\mathcal{H} : \mathcal{U} \times \mathcal{V} \to \mathcal{U} \times \mathcal{V}$ as

$$\mathcal{H} := \mathrm{Diag}(\mathcal{D}_1, \mathcal{D}_2 + \mathcal{Q}_{22}). \tag{3.4}$$

Furthermore, we denote Ω as the optimal solution set of (3.1), which is assumed to be non-empty.

3.1 The $O(1/k^2)$ complexity for the exact method

In this section, we first state our ABCD method for solving the problem (3.1). Following that we provide the $O(1/k^2)$ complexity analysis of the proposed algorithm.

Generally speaking, our algorithms can be treated as an accelerated version of the alternating minimization type algorithms for the two-block problems. It is well-known that Nesterov's classical accelerated proximal gradient (APG) algorithm, which enjoys an impressive $O(1/k^2)$ complexity, only works for single-block problems. When the problems involve more than one separable nonsmooth terms and they are updated in an alternative fashion, people usually discard the acceleration technique and prefer the (proximal) block coordinate descent (BCD) type algorithms. However, theoretically the complexity of the BCD-type algorithms is

$O(1/k)$ in the best case [83, 3], and numerically, much experiment in the past shows that the acceleration technique may substantially improve the efficiency of the algorithms, see, for example, the numerical comparison in [91]. This makes the study of the accelerated BCD-type algorithms critical, both in the theoretical sense and for the numerical applications. Below is the framework of our ABCD algorithm:

ABCD: A majorized accelerated block coordinate descent algorithm

Choose an initial point $(u^1, v^1) = (\tilde{u}^0, \tilde{v}^0) \in \mathrm{dom}(p) \times \mathrm{dom}(q)$. Set $k := 1$ and $t_1 = 1$. Iterate until convergence:

Step 1. Compute $\tilde{u}^k = \arg\min_{u \in \mathcal{U}} \{p(u) + \hat{\phi}(u, v^k; w^k)\}$.

Step 2. Compute $\tilde{v}^k = \arg\min_{v \in \mathcal{V}} \{q(v) + \hat{\phi}(\tilde{u}^k, v; w^k)\}$.

Step 3. Compute

$$\left\{\begin{array}{l} t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}, \\ \begin{pmatrix} u^{k+1} \\ v^{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{u}^k \\ \tilde{v}^k \end{pmatrix} + \dfrac{t_k - 1}{t_{k+1}} \begin{pmatrix} \tilde{u}^k - \tilde{u}^{k-1} \\ \tilde{v}^k - \tilde{v}^{k-1} \end{pmatrix}. \end{array}\right.$$

The subsequent analysis is strongly motivated by a recent paper of Chambolle and Pock [9], where the joint objective function ϕ is taken to be quadratic. They show that the $O(1/k^2)$ complexity still holds for the above method, where the two blocks u and v are updated alternatively. We extend their nice results to a more general class of problems of the form (3.1), where the function ϕ is only required to be smooth, and prove the correpsonding $O(1/k^2)$ complexity of the objective values.

The next proposition shows an important property of the objective values at the current iteration point, which is essential to prove the main global complexity result.

Proposition 3.1. *Suppose that Assumption 3.1 holds. Let the sequences* $\{\tilde{w}^k\} := \{(\tilde{u}^k, \tilde{v}^k)\}$ *and* $\{w^k\} = \{(u^k, v^k)\}$ *be generated by the ABCD algorithm. Then for* *any $k \geq 1$, it holds*

$$\theta(\tilde{w}^k) - \theta(w) \leq \frac{1}{2}\|w - w^k\|_{\mathcal{H}}^2 - \frac{1}{2}\|w - \tilde{w}^k\|_{\mathcal{H}}^2, \quad w \in \mathcal{U} \times \mathcal{V}. \tag{3.5}$$

Proof. In the ABCD iteration scheme, the optimality condition for $(\tilde{u}^k, \tilde{v}^k)$ is

$$\begin{cases} 0 \in & \partial p(\tilde{u}^k) + \nabla_u \phi(w^k) + \widehat{\mathcal{Q}}_{11}(\tilde{u}^k - u^k), \\ 0 \in & \partial q(\tilde{v}^k) + \nabla_v \phi(w^k) + \mathcal{Q}_{12}^*(\tilde{u}^k - u^k) + \widehat{\mathcal{Q}}_{22}(\tilde{v}^k - v^k). \end{cases} \tag{3.6}$$

Therefore, by the convexity of the functions p and q, we have

$$\begin{cases} p(u) \geq p(\tilde{u}^k) + \langle u - \tilde{u}^k, -\nabla_u \phi(w^k) - \widehat{\mathcal{Q}}_{11}(\tilde{u}^k - u^k)\rangle, \quad \forall u \in \mathcal{U} \\ q(v) \geq q(\tilde{v}^k) + \langle v - \tilde{v}^k, -\nabla_v \phi(w^k) - \mathcal{Q}_{12}^*(\tilde{u}^k - u^k) - \widehat{\mathcal{Q}}_{22}(\tilde{v}^k - v^k)\rangle, \quad \forall v \in \mathcal{V}. \end{cases} \tag{3.7}$$

By the inequalities (3.2) and (3.3), we know that

$$\begin{cases} \phi(\tilde{w}^k) \leq \phi(w^k) + \langle \nabla\phi(w^k), \tilde{w}^k - w^k\rangle + \frac{1}{2}\|\tilde{w}^k - w^k\|_{\widehat{\mathcal{Q}}}^2, \\ \phi(w) \geq \phi(w^k) + \langle \nabla\phi(w^k), w - w^k\rangle + \frac{1}{2}\|w - w^k\|_{\mathcal{Q}}^2, \end{cases} \tag{3.8}$$

which, imply, that

$$\phi(w) - \phi(\tilde{w}^k) \geq \langle \nabla\phi(w^k), w - \tilde{w}^k\rangle + \frac{1}{2}\|w - w^k\|_{\mathcal{Q}}^2 - \frac{1}{2}\|\tilde{w}^k - w^k\|_{\widehat{\mathcal{Q}}}^2. \tag{3.9}$$

By the Cauchy-Schwarz inequality, we can also get that

$$\begin{aligned} \langle \tilde{u}^k - u, \, \mathcal{Q}_{12}(\tilde{v}^k - v^k)\rangle &= \left\langle \mathcal{Q}(\tilde{w}^k - w), \begin{pmatrix} 0 \\ \tilde{v}^k - v^k \end{pmatrix} \right\rangle - \langle \mathcal{Q}_{22}(\tilde{v}^k - v), \tilde{v}^k - v^k\rangle \\ &\leq \frac{1}{2}(\|\tilde{w}^k - w\|_{\mathcal{Q}}^2 + \|\tilde{v}^k - v^k\|_{\mathcal{Q}_{22}}^2) - \frac{1}{2}(\|\tilde{v}^k - v\|_{\mathcal{Q}_{22}}^2 \\ &\quad + \|\tilde{v}^k - v^k\|_{\mathcal{Q}_{22}}^2 - \|v^k - v\|_{\mathcal{Q}_{22}}^2) \\ &= \frac{1}{2}\|\tilde{w}^k - w\|_{\mathcal{Q}}^2 + \frac{1}{2}(\|v^k - v\|_{\mathcal{Q}_{22}}^2 - \|\tilde{v}^k - v\|_{\mathcal{Q}_{22}}^2). \end{aligned} \tag{3.10}$$

Summing up the inequalities (3.7) and (3.9) and substituting them into (3.10), we can obtain that

$$
\begin{aligned}
\theta(w)-\theta(\tilde{w}^k) \;&\geq \tfrac{1}{2}\|w-w^k\|_{\mathcal{Q}}^2-\tfrac{1}{2}\|\tilde{w}^k-w^k\|_{\widehat{\mathcal{Q}}}^2-\langle w-\tilde{w}^k,\widehat{\mathcal{Q}}(\tilde{w}^k-w^k)\rangle\\
&\quad -\langle \tilde{u}^k-u,\mathcal{Q}_{12}(\tilde{v}^k-v^k)\rangle\\
&\geq \tfrac{1}{2}\|w-w^k\|_{\mathcal{Q}}^2-\tfrac{1}{2}\|\tilde{w}^k-w^k\|_{\widehat{\mathcal{Q}}}^2-\tfrac{1}{2}(\|w-w^k\|_{\widehat{\mathcal{Q}}}^2-\|w-\tilde{w}^k\|_{\widehat{\mathcal{Q}}}^2\\
&\quad -\|\tilde{w}^k-w^k\|_{\widehat{\mathcal{Q}}}^2)-\tfrac{1}{2}(\|\tilde{w}^k-w\|_{\mathcal{Q}}^2+\|v^k-v\|_{\mathcal{Q}_{22}}^2-\|\tilde{v}^k-v\|_{\mathcal{Q}_{22}}^2)\\
&= \tfrac{1}{2}\|w-\tilde{w}^k\|_{\mathcal{H}}^2-\tfrac{1}{2}\|w-w^k\|_{\mathcal{H}}^2,
\end{aligned}
\tag{3.11}
$$

where the last equation is obtained from Assumption 3.1. □

Based on the previous proposition, we can show the following $O(1/k^2)$ complexity of the objective values for our proposed ABCD algorithm.

Theorem 3.1. *Suppose that Assumption 3.1 holds and the solution set Ω of the problem (3.1) is non-empty. Let $w^*=(u^*,v^*)\in\Omega$. Then the sequence $\{\tilde{w}^k\}:=\{(\tilde{u}^k,\tilde{v}^k)\}$ generated by the ABCD algorithm satisfies that*

$$
\theta(\tilde{w}^k)-\theta(w^*)\leq\frac{2\|\tilde{w}^0-w^*\|_{\mathcal{H}}^2}{(k+1)^2},\quad \forall k\geq 1. \tag{3.12}
$$

Proof. Taking $w=\dfrac{(t_k-1)\tilde{w}^{k-1}+w^*}{t_k}$ in (3.11) of the Proposition 3.1, we can see that for $k\geq 1$,

$$
\begin{aligned}
\theta\left(\frac{(t_k-1)\tilde{w}^{k-1}+w^*}{t_k}\right)-\theta(\tilde{w}^k)\geq\;& \frac{1}{2}\left\|\frac{(t_k-1)\tilde{w}^{k-1}+w^*}{t_k}-\tilde{w}^k\right\|_{\mathcal{H}}^2\\
&-\frac{1}{2}\left\|\frac{(t_k-1)\tilde{w}^{k-1}+w^*}{t_k}-w^k\right\|_{\mathcal{H}}^2.
\end{aligned}
\tag{3.13}
$$

From the convexity of the function θ and $t_k\geq 0$ we also know that

$$
\theta\left(\frac{(t_k-1)\tilde{w}^{k-1}+w^*}{t_k}\right)\leq(1-\frac{1}{t_k})\theta(\tilde{w}^{k-1})+\frac{1}{t_k}\theta(w^*). \tag{3.14}
$$

Therefore, from the inequalities (3.13) and (3.14) and the fact that $t_k^2-t_k=t_{k-1}^2$ for $k\geq 2$, we obtain that for $k=1$,

$$
\theta(\tilde{w}^1)-\theta(w^*)\;\leq\tfrac{1}{2}\|w^1-w^*\|_{\mathcal{H}}^2-\tfrac{1}{2}\|\tilde{w}^1-w^*\|_{\mathcal{H}}^2=\tfrac{1}{2}\|\tilde{w}^0-w^*\|_{\mathcal{H}}^2-\tfrac{1}{2}\|\tilde{w}^1-w^*\|_{\mathcal{H}}^2 \tag{3.15}
$$

and for $k \geq 2$,

$$
\begin{aligned}
& t_k^2[\theta(\tilde{w}^k) - \theta(w^*)] - (t_k^2 - t_k)[\theta(\tilde{w}^{k-1}) - \theta(w^*)] \\
= \;& t_k^2[\theta(\tilde{w}^k) - \theta(w^*)] - t_{k-1}^2[\theta(\tilde{w}^{k-1}) - \theta(w^*)] \\
\leq \;& \tfrac{1}{2}\|t_{k-1}\tilde{w}^{k-1} - w^* - (t_{k-1} - 1)\tilde{w}^{k-2}\|_{\mathcal{H}}^2 - \tfrac{1}{2}\|t_k\tilde{w}^k - w^* - (t_k - 1)\tilde{w}^{k-1}\|_{\mathcal{H}}^2.
\end{aligned}
\tag{3.16}
$$

Thus, we have that for $k \geq 1$,

$$
\begin{aligned}
& t_k^2[\theta(\tilde{w}^k) - \theta(w^*)] + \tfrac{1}{2}\|t_k\tilde{w}^k - w^* - (t_k - 1)\tilde{w}^{k-1}\|_{\mathcal{H}}^2 \\
\leq \;& t_{k-1}^2[\theta(\tilde{w}^{k-1}) - \theta(w^*)] + \tfrac{1}{2}\|t_{k-1}\tilde{w}^{k-1} - w^* - (t_{k-1} - 1)\tilde{w}^{k-2}\|_{\mathcal{H}}^2 \\
\leq \;& \cdots \\
\leq \;& t_1^2[\theta(\tilde{w}^1) - \theta(w^*)] + \tfrac{1}{2}\|t_1\tilde{w}^1 - w^* - (t_1 - 1)\tilde{w}^0\|_{\mathcal{H}}^2 \\
\leq \;& \tfrac{1}{2}\|\tilde{w}^0 - w^*\|_{\mathcal{H}}^2.
\end{aligned}
\tag{3.17}
$$

By noting that $t_k \geq \frac{k+1}{2}$, we can further get that for any $k \geq 1$,

$$
\theta(\tilde{w}^k) - \theta(w^*) \leq \frac{2\|\tilde{w}^0 - w^*\|_{\mathcal{H}}^2}{(k+1)^2}. \tag{3.18}
$$

This completes the proof. □

Remark 3.1. *Theorem 3.1 shows that the $O(1/k^2)$ complexity of the objective values is still true for the two-block accelerated BCD-type algorithms for problems of the form (3.1). However, the outline of the proof here cannot be easily extend to the problems with three or more blocks. We would thus use a different idea - introducing the inexactness, to settle this issue in the next section.*

3.2 An inexact accelerated block coordinate descent method

In this section, we extend the previously proposed ABCD algorithm to an inexact version. There are three main reasons to introduce the inexactness for the subproblems: one is that the proximal mapping of a nonsmooth function may not admit analytical solutions. Thus, only allowing the exact computation for the subproblem is unsuitable in practice. The second reason is that sometimes it is unnecessary to compute the solutions of each block exactly even though it is doable, especially at the early stage of the whole procedure. For example, if a subproblem of the ABCD is equivalent to solve a very large scale dense linear system, it is perhaps a good idea to use the iterative methods instead of running the direct solvers. The last reason, and in fact the most important one, is because our initial target is to solve multi-block convex composite optimization problems. Our principle to deal with the multi-block problems consists of two steps: the first step is to divide all the variables into two groups, following that we solve each group by either the Newton-type methods or others. Therefore, it is prominent to study the inexact ABCD algorithms (iABCD). There is one point worths to emphasize here: to divide the original multi-block problems into more than two blocks may not be efficient in practice for the convex composite conic optimization problems. Intuitively speaking, the more alternative updates in one cycle, the more away we are from the original problems.

Below is the framework of our proposed iABCD algorithm:

iABCD: An inexact majorized accelerated block coordinate descent algorithm

Choose an initial point $(u^1, v^1) = (\tilde{u}^0, \tilde{v}^0) \in \mathrm{dom}(p) \times \mathrm{dom}(q)$ and a nonnegative non-increasing sequence $\{\varepsilon_k\}$. Set $k := 1$ and $t_1 = 1$. Iterate until convergence:

Step 1. Choose error tolerance $\delta_u^k \in \mathcal{U}$, $\delta_v^k \in \mathcal{V}$ such that

$$\max\{\|\widehat{\mathcal{Q}}_{11}^{-1/2}\delta_u^k\|,\ \|\widehat{\mathcal{Q}}_{22}^{-1/2}\delta_v^k\|\} \leq \varepsilon_k.$$

Compute

$$\begin{cases} \tilde{u}^k = \arg\min\limits_{u\in\mathcal{U}}\{p(u) + \hat{\phi}(u, v^k; w^k) + \langle \delta_u^k, u\rangle\}, \\ \tilde{v}^k = \arg\min\limits_{v\in\mathcal{V}}\{q(v) + \hat{\phi}(\tilde{u}^k, v; w^k) + \langle \delta_v^k, v\rangle\}. \end{cases}$$

Step 2. Compute

$$\begin{cases} t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}, \\ \begin{pmatrix} u^{k+1} \\ v^{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{u}^k \\ \tilde{v}^k \end{pmatrix} + \dfrac{t_k - 1}{t_{k+1}} \begin{pmatrix} \tilde{u}^k - \tilde{u}^{k-1} \\ \tilde{v}^k - \tilde{v}^{k-1} \end{pmatrix}. \end{cases}$$

Lemma 3.1. *The sequence $\{t_k\}_{k\geq 1}$ generated by the iABCD satisfies the following properties:*

(a) $1 - \dfrac{1}{t_{k+1}} = \dfrac{t_k^2}{t_{k+1}^2}$. *(b)* $\dfrac{k+1}{2} \leq t_k \leq k$.

Proof. By noting that $t_{k+1}^2 - t_{k+1} = t_k^2$, the property (a) can be obtained directly. The property (b) holds via the following inequalities:

$$t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2} \leq \frac{1+1+2t_k}{2} = 1 + t_k \leq k + t_1 = k+1$$

and

$$t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2} \geq \frac{1+2t_k}{2} \geq \frac{k+2t_1}{2} = \frac{k+2}{2}.$$

□

Similarly as the exact case in the previous section, we shall also characterize the decrease of the objective values at the current iteration point for the iABCD algorithm in the following proposition. The proof of it can be derived with no difficulty based on the proof of Proposition 3.1 by slightly modify the optimality conditions at the iteration point $(\widetilde{u}^k, \widetilde{v}^k)$ for the iABCD algorithm. We omit the proof here for brevity.

Proposition 3.2. *Suppose that Assumption 3.1 holds. Let the sequences* $\{\tilde{w}^k\} := \{(\tilde{u}^k, \tilde{v}^k)\}$ *and* $\{w^k\} = \{(u^k, v^k)\}$ *be generated by the iABCD algorithm. Then for any* $k \geq 1$,

$$\theta(\tilde{w}^k) - \theta(w) \leq \frac{1}{2}\|w - w^k\|^2_{\mathcal{H}} - \frac{1}{2}\|w - \tilde{w}^k\|^2_{\mathcal{H}} + \varepsilon_k \|w - \tilde{w}^k\|_{\mathrm{Diag}(\widehat{\mathcal{Q}}_{11}, \widehat{\mathcal{Q}}_{22})}, \quad \forall w \in \mathcal{U} \times \mathcal{V}.$$

For $k \geq 1$, we denote

$$\bar{u}^k := \arg\min_{u \in \mathcal{U}} \{p(u) + \hat{\phi}(u, v^k; w^k)\}, \quad \bar{v}^k := \arg\min_{v \in \mathcal{V}} \{q(v) + \hat{\phi}(\bar{u}^k, v; w^k)\},$$

which are the exact solutions at the $(k+1)$th iteration, and $(\bar{u}^0, \bar{v}^0) = (u^1, v^1)$. Since $\widehat{\mathcal{Q}}_{11}$ and $\widehat{\mathcal{Q}}_{22}$ are assumed to be positive definite, the above two problems admit unique solutions and thus, $\bar{u}^k$ and $\bar{v}^k$ are well defined for $k \geq 0$. The following Lemma shows the gap between $(\bar{u}^k, \bar{v}^k)$ and $(\tilde{u}^k, \tilde{v}^k)$.

Lemma 3.2. *For any* $k \geq 1$, *we have the following inequalities:*

$$\|\bar{u}^k - u^*\|^2_{\widehat{\mathcal{Q}}_{11}} \leq \|u^k - u^*\|^2_{\mathcal{D}_{11}} + \|v^k - v^*\|^2_{\widehat{\mathcal{Q}}_{22}}. \tag{3.19}$$

$$\|\widehat{\mathcal{Q}}_{11}^{1/2}(\tilde{u}^k - \bar{u}^k)\| \leq \varepsilon_k, \quad \|\widehat{\mathcal{Q}}_{22}^{1/2}(\tilde{v}^k - \bar{v}^k)\| \leq (1 + \sqrt{2})\varepsilon_k. \tag{3.20}$$

Proof. By the optimality conditions at point $(\bar{u}^k, \bar{v}^k)$ and (u^*, v^*), we have that

$$\begin{cases} 0 \in & \partial p(\bar{u}^k) + \nabla_u \phi(w^k) + \widehat{\mathcal{Q}}_{11}(\bar{u}^k - u^k), \\ 0 \in & \partial p(u^*) + \nabla_u \phi(w^*). \end{cases} \tag{3.21}$$

By the monotone property of the subgradient operator ∂p, we get

$$\langle \bar{u}^k - u^*, \nabla_u \phi(w^k) - \nabla_u \phi(w^*) + \widehat{\mathcal{Q}}_{11}(\bar{u}^k - u^k) \rangle \leq 0. \tag{3.22}$$

Since $\nabla\phi$ is globally Lipschitz continuous, it is known from Clarke's Mean-Value Theorem [12, Proposition 2.6.5] that there exists a self-adjoint and positive semidefinite operator $\mathcal{W}^k \in \mathrm{conv}\{\partial^2\phi([w^{k-1}, w^k])\}$ such that

$$\nabla\phi(w^k) - \nabla\phi(w^{k-1}) = \mathcal{W}^k(w^k - w^{k-1}),$$

where the set $\mathrm{conv}\{\partial^2\phi[w^{k-1}, w^k]\}$ denotes the convex hull of all points $\mathcal{W} \in \partial^2\phi(z)$ for any $z \in [w^{k-1}, w^k]$. Denote $\mathcal{W}^k := \begin{pmatrix} \mathcal{W}_{11}^k & \mathcal{W}_{12}^k \\ (\mathcal{W}_{12}^k)^* & \mathcal{W}_{22}^k \end{pmatrix}$, where $\mathcal{W}_{11}^k : \mathcal{U} \to \mathcal{U}$, $\mathcal{W}_{22}^k : \mathcal{V} \to \mathcal{V}$ are self-adjoint positive semidefinite operators and $\mathcal{W}_{12}^k : \mathcal{U} \to \mathcal{V}$ is a linear operator. Then by Cauchy-Schwarz inequality, we have that

$$\begin{aligned}
& \langle \bar{u}^k - u^*, \nabla_u\phi(w^k) - \nabla_u\phi(w^*)\rangle \\
= \ & \langle \bar{u}^k - u^*, \mathcal{W}_{11}^k(u^k - u^*) + \mathcal{W}_{12}^k(v^k - v^*)\rangle \\
\geq \ & \tfrac{1}{2}(\|\bar{u}^k - u^*\|^2_{\mathcal{W}_{11}^k} + \|u^k - u^*\|^2_{\mathcal{W}_{11}^k} - \|\bar{u}^k - u^k\|^2_{\mathcal{W}_{11}^k}) - \tfrac{1}{2}(\|\bar{u}^k - u^*\|^2_{\mathcal{W}_{11}^k} + \|v^k - v^*\|^2_{\mathcal{W}_{22}^k}) \\
\geq \ & \tfrac{1}{2}(\|u^k - u^*\|^2_{\mathcal{Q}_{11}} - \|\bar{u}^k - u^k\|^2_{\widehat{\mathcal{Q}}_{11}}) - \tfrac{1}{2}\|v^k - v^*\|^2_{\widehat{\mathcal{Q}}_{22}} \\
= \ & \tfrac{1}{2}\|u^k - u^*\|^2_{\mathcal{Q}_{11}} - \tfrac{1}{2}\|\bar{u}^k - u^k\|^2_{\widehat{\mathcal{Q}}_{11}} - \tfrac{1}{2}\|v^k - v^*\|^2_{\widehat{\mathcal{Q}}_{22}}.
\end{aligned} \tag{3.23}$$

From (3.22) and (3.23) we can obtain that

$$\begin{aligned}
& \tfrac{1}{2}\|u^k - u^*\|^2_{\mathcal{Q}_{11}} - \tfrac{1}{2}\|\bar{u}^k - u^k\|^2_{\widehat{\mathcal{Q}}_{11}} - \tfrac{1}{2}\|v^k - v^*\|^2_{\widehat{\mathcal{Q}}_{22}} + \langle \bar{u}^k - u^*, \widehat{\mathcal{Q}}_{11}(\bar{u}^k - u^k)\rangle \\
= \ & \tfrac{1}{2}\|u^k - u^*\|^2_{\mathcal{Q}_{11}} - \tfrac{1}{2}\|\bar{u}^k - u^k\|^2_{\widehat{\mathcal{Q}}_{11}} - \tfrac{1}{2}\|v^k - v^*\|^2_{\widehat{\mathcal{Q}}_{22}} + \tfrac{1}{2}(\|\bar{u}^k - u^*\|^2_{\widehat{\mathcal{Q}}_{11}} \\
& +\|\bar{u}^k - u^k\|^2_{\widehat{\mathcal{Q}}_{11}} - \|u^k - u^*\|^2_{\widehat{\mathcal{Q}}_{11}}) \\
= \ & \tfrac{1}{2}\|\bar{u}^k - u^*\|^2_{\widehat{\mathcal{Q}}_{11}} - \tfrac{1}{2}\|u^k - u^*\|^2_{\mathcal{D}_1} - \tfrac{1}{2}\|v^k - v^*\|^2_{\widehat{\mathcal{Q}}_{22}} \\
\leq \ & 0,
\end{aligned} \tag{3.24}$$

which is equivalent to say $\|\bar{u}^k - u^*\|^2_{\widehat{\mathcal{Q}}_{11}} \leq \|u^k - u^*\|^2_{\mathcal{D}_{11}} + \|v^k - v^*\|^2_{\widehat{\mathcal{Q}}_{22}}$. This completes the proof of the first inequality.

In order to obtain bounds for $\|\widehat{\mathcal{Q}}_{11}^{1/2}(\tilde{u}^k - \bar{u}^k)\|$ and $\|\widehat{\mathcal{Q}}_{22}^{1/2}(\tilde{v}^k - \bar{v}^k)\|$, similarly by applying the optimality condition at point $(\bar{u}^k, \bar{v}^k)$ and $(\tilde{u}^k, \tilde{v}^k)$, we have that

$$\langle \widehat{\mathcal{Q}}_{11}(\tilde{u}^k - \bar{u}^k) + \delta_u^k, \tilde{u}^k - \bar{u}^k \rangle \leq 0,$$

$$\langle \mathcal{Q}_{12}^*(\tilde{u}^k - \bar{u}^k) + \widehat{\mathcal{Q}}_{22}(\tilde{v}^k - \bar{v}^k) + \delta_v^k, \tilde{v}^k - \bar{v}^k \rangle \leq 0.$$

From the first inequality we know that

$$\|\widehat{\mathcal{Q}}_{11}^{1/2}(\tilde{u}^k - \bar{u}^k)\| \leq \|\widehat{\mathcal{Q}}_{11}^{-1/2}\delta_u^k\| \leq \varepsilon_k. \tag{3.25}$$

From the second inequality we obtain that

$$\begin{aligned} \|\tilde{v}^k - \bar{v}^k\|_{\widehat{\mathcal{Q}}_{22}}^2 &\leq \|\widehat{\mathcal{Q}}_{22}^{-1/2}\delta_v^k\|\|\widehat{\mathcal{Q}}_{22}^{1/2}(\tilde{v}^k - \bar{v}^k)\| - \langle \mathcal{Q}_{12}^*(\tilde{u}^k - \bar{u}^k), \tilde{v}^k - \bar{v}^k \rangle \\ &\leq \|\widehat{\mathcal{Q}}_{22}^{-1/2}\delta_v^k\|\|\widehat{\mathcal{Q}}_{22}^{1/2}(\tilde{v}^k - \bar{v}^k)\| + \tfrac{1}{2}(\|\tilde{u}^k - \bar{u}^k\|_{\widehat{\mathcal{Q}}_{11}}^2 + \|\tilde{v}^k - \bar{v}^k\|_{\widehat{\mathcal{Q}}_{22}}^2), \end{aligned} \tag{3.26}$$

which further shows that

$$\begin{aligned} \|\tilde{v}^k - \bar{v}^k\|_{\widehat{\mathcal{Q}}_{22}}^2 &\leq 2\|\widehat{\mathcal{Q}}_{22}^{-1/2}\delta_v^k\|\|\widehat{\mathcal{Q}}_{22}^{1/2}(\tilde{v}^k - \bar{v}^k)\| + \|\tilde{u}^k - \bar{u}^k\|_{\widehat{\mathcal{Q}}_{11}}^2 \\ &\leq 2\varepsilon_k\|\widehat{\mathcal{Q}}_{22}^{1/2}(\tilde{v}^k - \bar{v}^k)\| + \varepsilon_k^2. \end{aligned} \tag{3.27}$$

By solving this inequality we obtain that

$$\|\widehat{\mathcal{Q}}_{22}^{1/2}(\tilde{v}^k - \bar{v}^k)\| \leq (1+\sqrt{2})\varepsilon_k. \tag{3.28}$$

This completes the proof of this Lemma. □

Based on the previous results, now we are ready to present the main theorem of this section, which shows that the iABCD algorithm also enjoys the nice $O(1/k^2)$ complexity.

Theorem 3.2. *Suppose that Assumption 3.1 holds and the solution set Ω of the problem (3.1) is non-empty. Let $w^* \in \Omega$. Assume that $\sum_{i=1}^{\infty} i\varepsilon_i < \infty$. Then the sequence $\{\tilde{w}^k\} := \{(\tilde{u}^k, \tilde{v}^k)\}$ generated by the iABCD algorithm satisfies that*

$$\theta(\tilde{w}^k) - \theta(w^*) \leq \frac{2\|\tilde{w}^0 - w^*\|_{\mathcal{H}}^2 + c_0}{(k+1)^2}, \quad \forall k \geq 1, \tag{3.29}$$

where c_0 is a constant number.

Proof. When $\tilde{w}^k = \bar{w}^k$ in Lemma 3.2, the corresponding $\varepsilon_k = 0$ since $\bar{w}^k = (\bar{u}^k, \bar{v}^k)$ exactly solves the subproblems in $(k+1)$th iteration, and thus we have

$$\theta(w) - \theta(\bar{w}^k) \geq \frac{1}{2}\|w - \bar{w}^k\|_{\mathcal{H}}^2 - \frac{1}{2}\|w - w^k\|_{\mathcal{H}}^2. \tag{3.30}$$

Taking $w = \dfrac{(t_k - 1)\bar{w}^{k-1} + w^*}{t_k}$ in (3.30), we can see that for $k \geq 1$,

$$\begin{aligned}\theta\left(\frac{(t_k - 1)\bar{w}^{k-1} + w^*}{t_k}\right) - \theta(\bar{w}^k) \geq\ & \frac{1}{2}\left\|\frac{(t_k - 1)\bar{w}^{k-1} + w^*}{t_k} - \bar{w}^k\right\|_{\mathcal{H}}^2 \\ & -\frac{1}{2}\left\|\frac{(t_k - 1)\bar{w}^{k-1} + w^*}{t_k} - w^k\right\|_{\mathcal{H}}^2.\end{aligned} \tag{3.31}$$

From the convexity of the function θ and $t_k \geq 0$, we know that

$$\theta\left(\frac{(t_k - 1)\bar{w}^{k-1} + w^*}{t_k}\right) \leq (1 - \frac{1}{t_k})\theta(\bar{w}^{k-1}) + \frac{1}{t_k}\theta(w^*). \tag{3.32}$$

Therefore, from the inequalities (3.31), (3.32) and the fact that $t_k^2 - t_k = t_{k-1}^2$ for $k \geq 2$, we obtain that for $k = 1$,

$$\begin{aligned}\theta(\bar{w}^1) - \theta(w^*) &\leq \tfrac{1}{2}\|w^1 - w^*\|_{\mathcal{H}}^2 - \tfrac{1}{2}\|\bar{w}^1 - w^*\|_{\mathcal{H}}^2 \\ &= \tfrac{1}{2}\|\tilde{w}^0 - w^*\|_{\mathcal{H}}^2 - \tfrac{1}{2}\|\bar{w}^1 - w^*\|_{\mathcal{H}}^2 \\ &= \tfrac{1}{2}\|\bar{w}^0 - w^*\|_{\mathcal{H}}^2 - \tfrac{1}{2}\|\bar{w}^1 - w^*\|_{\mathcal{H}}^2\end{aligned} \tag{3.33}$$

and for $k \geq 2$,

$$\begin{aligned}& t_k^2[\theta(\bar{w}^k) - \theta(w^*)] - (t_k^2 - t_k)[\theta(\bar{w}^{k-1}) - \theta(w^*)] \\ =\ & t_k^2[\theta(\bar{w}^k) - \theta(w^*)] - t_{k-1}^2[\theta(\bar{w}^{k-1}) - \theta(w^*)] \\ \leq\ & \tfrac{1}{2}\|t_{k-1}\tilde{w}^{k-1} - w^* - (t_{k-1} - 1)\tilde{w}^{k-2} + (t_k - 1)(\bar{w}^{k-1} - \tilde{w}^{k-1})\|_{\mathcal{H}}^2 \\ & -\tfrac{1}{2}\|t_k\bar{w}^k - w^* - (t_k - 1)\bar{w}^{k-1}\|_{\mathcal{H}}^2 \\ =\ & \tfrac{1}{2}\|\lambda^{k-1}\|_{\mathcal{H}}^2 - \langle \mathcal{H}\lambda^{k-1}, (t_{k-1} + t_k - 1)(\bar{w}^{k-1} - \tilde{w}^{k-1}) + (t_{k-1} - 1)(\bar{w}^{k-2} - \tilde{w}^{k-2})\rangle \\ & +\tfrac{1}{2}\|(t_{k-1} + t_k - 1)(\bar{w}^{k-1} - \tilde{w}^{k-1}) + (t_{k-1} - 1)(\bar{w}^{k-2} - \tilde{w}^{k-2})\|_{\mathcal{H}}^2 - \tfrac{1}{2}\|\lambda^k\|_{\mathcal{H}}^2,\end{aligned} \tag{3.34}$$

where $\lambda^k := t_k\bar{w}^k - w^* - (t_k - 1)\bar{w}^{k-1}$. By Lemma 3.1 (b), Lemma 3.2 and the nonincreasing property of $\{\varepsilon_k\}$, we have that

$$
\begin{aligned}
& \|\mathcal{H}^{1/2}(t_{k-1}+t_k-1)(\bar{w}^{k-1}-\tilde{w}^{k-1})+(t_{k-1}-1)(\bar{w}^{k-2}-\tilde{w}^{k-2}))\| \\
\leq\ & (t_{k-1}+t_k-1)\|\mathcal{H}^{1/2}\|\|\bar{w}^{k-1}-\tilde{w}^{k-1}\|+(t_{k-1}-1)\|\mathcal{H}^{1/2}\|\|\bar{w}^{k-2}-\tilde{w}^{k-2}\| \\
\leq\ & c_1(k-1)\varepsilon_{k-1},
\end{aligned}
$$

where $c_1 := 3\|\mathcal{H}^{1/2}\|(\|\widehat{\mathcal{Q}}_{11}^{-1/2}\|+(1+\sqrt{2})\|\widehat{\mathcal{Q}}_{22}^{-1/2}\|)$. Thus, we obtain for $k \geq 1$ that

$$
\begin{aligned}
& t_k^2[\theta(\bar{w}^k)-\theta(w^*)]+\tfrac{1}{2}\|\lambda^k\|_{\mathcal{H}}^2 \\
\leq\ & t_{k-1}^2[\theta(\bar{w}^{k-1})-\theta(w^*)]+\tfrac{1}{2}\|\lambda^{k-1}\|_{\mathcal{H}}^2+c_1(k-1)\varepsilon_{k-1}\|\mathcal{H}^{1/2}\lambda^{k-1}\|+\tfrac{1}{2}c_1^2(k-1)^2\varepsilon_{k-1}^2 \\
\leq\ & \cdots \\
\leq\ & t_1^2[\theta(\bar{w}^1)-\theta(w^*)]+\tfrac{1}{2}\|\lambda^1\|_{\mathcal{H}}^2+c_1\sum_{i=1}^{k-1}i\varepsilon_i\|\mathcal{H}^{1/2}\lambda^i\|+\frac{1}{2}c_1^2\sum_{i=1}^{k-1}i^2\varepsilon_i^2 \\
\leq\ & \tfrac{1}{2}\|\bar{w}^0-w^*\|_{\mathcal{H}}^2+c_1\sum_{i=1}^{k-1}i\varepsilon_i\|\mathcal{H}^{1/2}\lambda^i\|+\frac{1}{2}c_1^2\sum_{i=1}^{k-1}i^2\varepsilon_i^2.
\end{aligned}
\tag{3.35}
$$

Now we will show the above inequality in fact indicates the boundness of the sequence $\{\mathcal{H}^{1/2}\lambda^k\}$. There exists a subsequence $\{\lambda^{k_m}\}$ of $\{\lambda^k\}$ satisfies that $\lambda^{k_1} = \lambda^1$ and for $m \geq 2$, $\{\lambda^{k_m} : \ \|\mathcal{H}^{1/2}\lambda^{k_m}\| > \|\mathcal{H}^{1/2}\lambda^i\| \quad \forall i < k_m\}$. (This subsequence may only contain finite terms.) Then for any $k_m \geq 1$, we have that

$$
\begin{aligned}
\|\mathcal{H}^{1/2}\lambda^{k_m}\| &\leq \max\{1, \frac{\|\bar{w}^0-w^*\|_{\mathcal{H}}^2+2c_1\sum_{i=1}^{k_m-1}i\varepsilon_i\|\mathcal{H}^{1/2}\lambda^i\|+c_1^2\sum_{i=1}^{k_m-1}i^2\varepsilon_i^2}{\|\mathcal{H}^{1/2}\lambda^{k_m}\|}\} \\
&\leq \max\{1, \|\bar{w}^0-w^*\|_{\mathcal{H}}^2+2c_1\sum_{i=1}^{k_m-1}i\varepsilon_i\frac{\|\mathcal{H}^{1/2}\lambda^i\|}{\|\mathcal{H}^{1/2}\lambda^{k_m}\|}+c_1^2\sum_{i=1}^{k_m-1}i^2\varepsilon_i^2\} \\
&\leq \max\{1, \|\bar{w}^0-w^*\|_{\mathcal{H}}^2+2c_1\sum_{i=1}^{k_m-1}i\varepsilon_i+c_1^2\sum_{i=1}^{k_m-1}i^2\varepsilon_i^2\} \\
&\leq \max\{1, \|\bar{w}^0-w^*\|_{\mathcal{H}}^2+2c_1\sum_{i=1}^{\infty}i\varepsilon_i+c_1^2\sum_{i=1}^{\infty}i^2\varepsilon_i^2\},
\end{aligned}
\tag{3.36}
$$

where the third inequality is obtained by the definition of the subsequence λ^{k_m}. Since $\|\mathcal{H}^{1/2}\lambda^i\| \leq \|\mathcal{H}^{1/2}\lambda^{k_m}\|$ for $i \leq k_m$, we can further obtain that for any $k \geq 1$,

$$\|\mathcal{H}^{1/2}\lambda^k\| \leq c_2 := \max\{1, \|\bar{w}^0 - w^*\|_{\mathcal{H}}^2 + 2c_1 \sum_{i=1}^{\infty} i\varepsilon_i + c_1^2 \sum_{i=1}^{\infty} i^2\varepsilon_i^2\}. \tag{3.37}$$

Now we estimate the bound for $\|\bar{w}^{k+1} - w^*\|$. By letting $w = w^*$ in (3.30), we have

$$\begin{aligned}
\|\mathcal{H}^{1/2}(\bar{w}^{k+1} - w^*)\| \ &\leq \|\mathcal{H}^{1/2}(w^{k+1} - w^*)\| \\
&= \|\mathcal{H}^{1/2}((1 + \tfrac{t_k - 1}{t_{k+1}})\tilde{w}^k - \tfrac{t_k - 1}{t_{k+1}}\tilde{w}^{k-1} - w^*)\| \\
&\leq (1 - \tfrac{1}{t_{k+1}})\|\mathcal{H}^{1/2}(\bar{w}^k - w^*)\| + \tfrac{1}{t_{k+1}}\|\mathcal{H}^{1/2}(t_k\bar{w}^k - (t_k - 1)\bar{w}^{k-1} - w^*)\| \\
&\quad + (1 + \tfrac{t_k - 1}{t_{k+1}})\|\mathcal{H}^{1/2}(\tilde{w}^k - \bar{w}^k)\| + \tfrac{t_k - 1}{t_{k+1}}\|\mathcal{H}^{1/2}(\tilde{w}^{k-1} - \bar{w}^{k-1})\| \\
&\leq \frac{t_k^2}{t_{k+1}^2}\|\mathcal{H}^{1/2}(\bar{w}^k - w^*)\| + \frac{c_2}{t_{k+1}} + 3c_1\varepsilon_k,
\end{aligned} \tag{3.38}$$

where the last inequality is obtained by Lemma 3.1 (a). Thus similarly we can get that

$$\begin{aligned}
\frac{t_k^2}{t_{k+1}^2}\|\mathcal{H}^{1/2}(\bar{w}^k - w^*)\| \ &\leq \frac{t_k^2}{t_{k+1}^2}(\frac{t_{k-1}^2}{t_k^2}\|\mathcal{H}^{1/2}(\bar{w}^{k-1} - w^*)\| + \frac{c_2}{t_k} + 3c_1\varepsilon_{k-1}) \\
\frac{t_k^2}{t_{k+1}^2}\frac{t_{k-1}^2}{t_k^2}\|\mathcal{H}^{1/2}(\bar{w}^{k-1} - w^*)\| \ &\leq \frac{t_k^2}{t_{k+1}^2}\frac{t_{k-1}^2}{t_k^2}(\frac{t_{k-2}^2}{t_{k-1}^2}\|\mathcal{H}^{1/2}(\bar{w}^{k-2} - w^*)\| + \frac{c_2}{t_{k-1}} + 3c_1\varepsilon_{k-2}) \\
\vdots \qquad &\leq \qquad \vdots \\
\frac{t_k^2}{t_{k+1}^2}\frac{t_{k-1}^2}{t_k^2}\cdots\frac{t_2^2}{t_3^2}\|\mathcal{H}^{1/2}(\bar{w}^2 - w^*)\| \ &\leq \frac{t_k^2}{t_{k+1}^2}\frac{t_{k-1}^2}{t_k^2}\cdots\frac{t_2^2}{t_3^2}(\frac{t_1^2}{t_2^2}\|\mathcal{H}^{1/2}(\bar{w}^1 - w^*)\| + \frac{c_2}{t_2} + 3c_1\varepsilon_1).
\end{aligned} \tag{3.39}$$

Summing the above inequalities together, we can obtain that

$$\|\mathcal{H}^{1/2}(\bar{w}^{k+1} - w^*)\| \leq \frac{t_1^2}{t_{k+1}^2}\|\mathcal{H}^{1/2}(\bar{w}^1 - w^*)\| + c_2\sum_{i=1}^{k}\frac{t_{i+1}}{t_{k+1}^2} + 3c_1\sum_{i=1}^{k}\varepsilon_i. \tag{3.40}$$

By Lemma 3.1 (b), we have that for any $k \geq 1$,

$$\sum_{i=1}^{k}\frac{t_{i+1}}{t_{k+1}^2} \leq \frac{(3+k)k}{2(\frac{1}{2}k+1)^2} \leq 2.$$

Therefore, the inequality (3.40) implies that

$$\begin{aligned}\|\mathcal{H}^{1/2}(\bar{w}^{k+1}-w^*)\| &\leq \tfrac{4}{(k+2)^2}\|\mathcal{H}^{1/2}(\bar{w}^1-w^*)\|+2c_2+3c_1\textstyle\sum_{i=1}^{\infty}\varepsilon_i \\ &\leq c_3 := \tfrac{4}{9}\|\mathcal{H}^{1/2}(\bar{w}^1-w^*)\|+2c_2+3c_1\textstyle\sum_{i=1}^{\infty}\varepsilon_i.\end{aligned}$$

By the notation of the operator $\mathcal{H}$, we can further obtain that

$$\|\mathcal{D}_1^{1/2}(\bar{u}^{k+1}-u^*)\|\leq c_3, \quad \|\widehat{\mathcal{Q}}_{22}^{1/2}(\bar{v}^{k+1}-v^*)\|\leq c_3. \tag{3.41}$$

The next step is to prove the boundness of the term $\|t_k\bar{u}^k-u^*-(t_k-1)\bar{u}^{k-1}\|$. By noting that

$$\begin{aligned}\|\widehat{\mathcal{Q}}_{11}^{1/2}(\bar{u}^k-u^*)\| &\leq \|\mathcal{D}_1^{1/2}(u^k-u^*)\|+\|\widehat{\mathcal{Q}}_{22}^{1/2}(v^k-v^*)\| \\ &\leq (1+\tfrac{t_{k-1}-1}{t_k})(\|\mathcal{D}_1^{1/2}(\tilde{u}^{k-1}-u^*)\|+\|\widehat{\mathcal{Q}}_{22}^{1/2}(\tilde{v}^{k-1}-v^*)\| \\ &\quad +\tfrac{t_{k-1}-1}{t_k}(\|\mathcal{D}_1^{1/2}(\tilde{u}^{k-2}-u^*)\|+\|\widehat{\mathcal{Q}}_{22}^{1/2}(\tilde{v}^{k-2}-v^*)\|) \\ &\leq 2(\|\mathcal{D}_1^{1/2}(\tilde{u}^{k-1}-u^*)\|+\|\widehat{\mathcal{Q}}_{22}^{1/2}(\tilde{v}^{k-1}-v^*)\|)+\|\mathcal{D}_1^{1/2}(\tilde{u}^{k-2}-u^*)\| \\ &\quad +\|\widehat{\mathcal{Q}}_{22}^{1/2}(\tilde{v}^{k-2}-v^*)\| \\ &\leq 3(2c_3+(2+\sqrt{2})\varepsilon_{k-2}),\end{aligned} \tag{3.42}$$

we have

$$\begin{aligned}\|t_k\tilde{u}^k-u^*-(t_k-1)\tilde{u}^{k-1}\|_{\widehat{\mathcal{Q}}_{11}} &\leq t_k(\|\bar{u}^k-\tilde{u}^k\|+\|\bar{u}^k-u^*\|) \\ &\quad +(t_k-1)(\|\bar{u}^{k-1}-u^*\|+\|\bar{u}^{k-1}-\tilde{u}^{k-1}\|) \\ &\leq (2t_k-1)((7+3\sqrt{2})\varepsilon_{k-3}+6c_3).\end{aligned}$$

Finally, by Proposition 3.2 at $w = \dfrac{(t_k-1)\tilde{w}^{k-1}+w^*}{t_k}$, we see that

$$
\begin{aligned}
& t_k^2[\theta(\tilde{w}^k)-\theta(w^*)] + \tfrac{1}{2}\|t_k\tilde{w}^k - w^* - (t_k-1)\tilde{w}^{k-1}\|_{\mathcal{H}}^2 \\
\le\ & t_{k-1}^2[\theta(\tilde{w}^{k-1})-\theta(w^*)] + \tfrac{1}{2}\|t_{k-1}\tilde{w}^{k-1} - w^* - (t_{k-1}-1)\tilde{w}^{k-2}\|_{\mathcal{H}}^2 \\
& +\varepsilon_k\|t_k\tilde{w}^k - w^* - (t_k-1)\tilde{w}^{k-1}\|_{\mathrm{Diag}(\widehat{Q}_{11},\,\widehat{Q}_{22})} \\
\le\ & \cdots \\
\le\ & t_1^2[\theta(\tilde{w}^1)-\theta(w^*)] + \tfrac{1}{2}\|t_1\tilde{w}^1 - w^* - (t_1-1)\tilde{w}^0\|_{\mathcal{H}}^2 \\
& +\sum_{i=1}^{k}\varepsilon_i\|t_i\tilde{w}^i - w^* - (t_i-1)\tilde{w}^{i-1}\|_{\mathrm{Diag}(\widehat{Q}_{11},\,\widehat{Q}_{22})} \\
\le\ & \tfrac{1}{2}\|\tilde{w}^0 - w^*\|_{\mathcal{H}}^2 + \sum_{i=1}^{k}\varepsilon_i\|t_i\tilde{w}^i - w^* - (t_i-1)\tilde{w}^{i-1}\|_{\mathrm{Diag}(\widehat{Q}_{11},\,\widehat{Q}_{22})} \\
\le\ & \tfrac{1}{2}\|\tilde{w}^0 - w^*\|_{\mathcal{H}}^2 + \sum_{i=1}^{k}(2t_i-1)((7+3\sqrt{2})\varepsilon_i + 6c_3) + (1+\sqrt{2})\varepsilon_i + c_2)\varepsilon_i \\
\le\ & \tfrac{1}{2}\|\tilde{w}^0 - w^*\|_{\mathcal{H}}^2 + (8+4\sqrt{2})\sum_{i=1}^{\infty}(2i-1)\varepsilon_i^2 + (6c_3+c_2)\sum_{i=1}^{\infty}(2i-1)\varepsilon_i \\
\le\ & \tfrac{1}{2}\|\tilde{w}^0 - w^*\|_{\mathcal{H}}^2 + \tfrac{1}{4}c_0,
\end{aligned}
$$

where

$$
c_0 = \ 4(8+4\sqrt{2})\sum_{i=1}^{\infty}(2i-1)\varepsilon_i^2 + 4(6c_3+c_2)\sum_{i=1}^{\infty}(2i-1)\varepsilon_i
$$

is a finite value by the assumption that $\sum_{i=1}^{\infty} i\varepsilon_i < \infty$. By noting that $t_k \ge \frac{k+1}{2}$, we could obtain the inequality (3.29). This completes the proof. $\square$

3.3 An application: the regularized projection onto the intersection of equations, inequalities and convex sets

In this chapter, we discuss how to apply the iABCD framework provided in the previous section to an important class of least square problems: given an arbitrary point, find its nearest point that satisfies many equality and inequality constraints as well as stays in the intersection of some non-polyhedral sets.

The optimization model of the above mentioned least square problems takes the following form:

$$\begin{array}{ll} \min\limits_{X} & \frac{1}{2}\|X-G\|^2 + \sum\limits_{i=1}^{s} \theta_i(X) \\ \text{s.t.} & \mathcal{A}X = b, \quad \mathcal{B}X \in \mathcal{Q}, \quad X \in \mathcal{K}, \end{array} \tag{3.43}$$

where s, q are positive integers, $\theta_i(\cdot) : \mathcal{X} \to (-\infty, \infty]$ for $i = 1, \ldots, s$ are convex functions (possibly nonsmooth), $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{B} : \mathcal{X} \to \mathcal{Z}$ are linear operators, $\mathcal{Q} := \{z \in \mathcal{Z} : l \le z \le u\}$ with $l, z \in \mathcal{Z}$ being lower and upper bounds, $\mathcal{K} \subseteq \mathcal{X}$ is a convex cone, $G \in \mathcal{X}$ and $b \in \mathcal{Y}$ are given data, and $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are real finite dimensional Euclidean spaces each equipped with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\|\cdot\|$. In particular, one can let $\theta_i(\cdot) = \delta_{C_i}(\cdot)$ if the variable is required to stay within some convex set C_i. The nonsmooth function θ_i can also be chosen as regularization terms to impose different structures of the solutions, such as $\|\cdot\|_1$, $\|\cdot\|_{2,1}$ or $\|\cdot\|_*$ for the (column-wise) sparsity or low rank structure, and that is the reason we name the problem (3.43) as *regularized* projection.

By introducing variables $Y_i = X$ for $i = 1 \ldots s$, the problem (3.43) can be equivalently written as

$$\begin{array}{ll} \min\limits_{X,Y_1,\ldots Y_s} & \dfrac{1}{2}\|X-G\|^2 + \sum\limits_{i=1}^{s} \theta_i(Y_i) \\ \text{s.t.} & \mathcal{A}X = b, \quad \mathcal{B}X \in \mathcal{Q}, \quad X \in \mathcal{K}, \quad X = Y_i, \ i = 1, \ldots, s. \end{array} \tag{3.44}$$

The dual problem of (3.44) takes the following form:

$$\begin{aligned} \min \theta(y, S, z, Z_1, \ldots Z_s) := \ & \frac{1}{2}\|\mathcal{A}^* y + S + \mathcal{B}^* z + \sum_{i=1}^{s} Z_i + G\|^2 - \langle b, y \rangle \\ & + \delta_{\mathcal{K}^*}(S) + \delta_{\mathcal{Q}^*}(-z) + \sum_{i=1}^{s} \theta_i^*(-Z_i), \end{aligned} \tag{3.45}$$

where $\theta_i^*(\cdot)$ denotes the convex conjugate function of $\theta(\cdot)$ for $i = 1, \cdots, s$ and $\delta_C(\cdot) := \begin{cases} 0 & x \in C \\ +\infty & \text{otherwise} \end{cases}$ is the indicator function of a given convex set C.

For notational convenience, let $\mathcal{W} \equiv \mathcal{Y} \times \mathcal{X} \times \mathcal{Z} \times \mathcal{X} \times \ldots \times \mathcal{X}$ and $W \equiv (y, S, z, Z_1, \ldots, Z_s) \in \mathcal{W}$. We write the smooth part of the dual objective function (3.45) as

$$\phi(W) := \frac{1}{2}\|\mathcal{A}^* y + S + \mathcal{B}^* z + \sum_{i=1}^{s} Z_i + G\|^2 - \langle b, y\rangle. \tag{3.46}$$

Note that only two blocks of variables are allowed in the iABCD framework. Hence, we need to group the dual variables $(y, S, z, Z_1, \ldots, Z_s)$ into two parts. Motivated by the success of many previous large scale computation with similar type of constraints, for example, [103, 91], we prefer to put the "difficult" ones like y and S together as one block $U \equiv (y, S)$ and all others as the other block $V \equiv (z, Z_1, \ldots, Z_s)$. As discussed in Section 2.3, the block U can be solved via the inexact sGS technique.

In the following, we focus on the approaches for solving the block $V = (z, Z_1, \ldots, Z_s)$. For notational convenience, we denote $C^k := \mathcal{A}^* \tilde{y}^k + \widetilde{S}^k + G$ with $k \geq 1$ as the iteration number. Therefore, for the kth step, the subproblem that under consideration before any majoirzation is

$$\min_{z, Z_1, \ldots, Z_s} \frac{1}{2}\|\mathcal{B}^* z + \sum_{i=1}^{s} Z_i + C^k\|^2 + \delta_{\mathcal{Q}^*}(-z) + \sum_{i=1}^{s} \theta_i^*(-Z_i). \tag{3.47}$$

Since there are still $s + 1$ nonsmooth functions involved in this subproblem, one could apply the matrix Cauchy-Schwartz inequality to the block in order to the term $\frac{1}{2}\|\mathcal{B}^* z + \sum_{i=1}^{s} Z_i + C^k\|^2$ to obtain a relatively tight and convenience majorization function. In many real applications (as shown in the numerical examples sections), the linear inequality constraint $\mathcal{B}X \in \mathcal{Q}$ in the primal form (3.43) is challenging to solve because of the ultra large scale of the operator $\mathcal{B}$. A practical way to deal with it is to divide the operator $\mathcal{B}$ and the dual variable z into $q \geq 1$ parts as

$$\begin{cases} (\mathcal{B}X)^T \equiv (\mathcal{B}_1 X, \mathcal{B}_2 X, \ldots, \mathcal{B}_q X)^T, \quad \forall X \in \mathcal{X}, \\ \mathcal{B}^* z \equiv (\mathcal{B}_1^* z_1, \mathcal{B}_2^* z_2, \ldots, \mathcal{B}_q^* z_q), \quad \forall z \equiv (z_1, z_2, \ldots z_q) \in \mathcal{Z}_1 \times \mathcal{Z}_2 \times \ldots \times \mathcal{Z}_q, \end{cases}$$

where $\mathcal{B}_i : \mathcal{X} \to \mathcal{Z}_i$ and $\mathcal{Z}_1 \times \mathcal{Z}_2 \times \ldots \times \mathcal{Z}_q = \mathcal{Z}$. By the matrix Cauchy-Schwarz inequality (2.16), it is easy to see that

$$(\mathcal{B}^* \; \mathcal{I} \; \ldots \; \mathcal{I})^* (\mathcal{B}^* \; \mathcal{I} \; \ldots \; \mathcal{I}) \preceq (s+1)\mathrm{Diag}\,(\mathcal{M}_1, \ldots, \mathcal{M}_q, \mathcal{M}_{q+1}, \ldots, \mathcal{M}_{q+s}),$$

where

$$\mathcal{M}_i := \begin{cases} \mathcal{B}_i\mathcal{B}_i^* + \displaystyle\sum_{j=1,\ldots,q,\, j\neq i} (\mathcal{B}_i\mathcal{B}_j^*\mathcal{B}_j\mathcal{B}_i^*)^{1/2} & \text{for } i = 1,\ldots q, \\ \mathcal{I} & \text{for } i = q+1,\ldots,q+s. \end{cases}$$

The reasons behind the decomposition of the operator $\mathcal{B}$ and then majorize them with the above matrix Cauchy-Schwarz inequality are that in many real applications, the operator $\mathcal{B}$ usually are very sparse or contains some orthogonal structure between different rows. If this is the case, the quadratic terms involving $\sum_{j=1,\ldots,q,\, j\neq i} (\mathcal{B}_i\mathcal{B}_j^*\mathcal{B}_j\mathcal{B}_i^*)^{1/2}$ would be quite small compared to the block diagonal parts $\mathcal{B}_i\mathcal{B}_i^*$.

Since $\mathcal{M}_1,\ldots,\mathcal{M}_q$ may still fails to be positive definite and difficult to compute, simple positive definite operators to majorize these operators are desirable. Here we adopt the following ideas: denote the eigenvalue decomposition of the operator $\mathcal{M}_i$ for $i = 1,\ldots,q$ as

$$\mathcal{M}_i = \sum_{j=1}^{r_i} \lambda_j^i P_j^i (P_j^i)^T, \quad i = 1,\ldots,q,$$

where $\lambda_1^i \geq \lambda_2^i \geq \ldots \geq \lambda_{r_i}^i$ are the eigenvalues of $\mathcal{M}_i$ with r_i being the rank of $\mathcal{M}_i$, and $P_1^i, P_2^i, \ldots, P_{r_i}^i$ are the orthogonal eigenvectors of $\mathcal{M}_i$. We choose the majorization operators for $i = 1,\ldots,q$ to be

$$\begin{aligned} \widehat{\mathcal{M}}_i := & \sum_{j=1}^{k_i} \lambda_j^i P_j^i (P_j^i)^T + \lambda_{k_i}^i \sum_{j=k_i+1}^{r_i} P_j^i (P_j^i)^T \\ = & \sum_{j=1}^{k_i} \lambda_j^i P_j^i (P_j^i)^T + \lambda_{k_i}^i (\mathcal{I} - \sum_{j=1}^{k_i} P_j^i (P_j^i)^T) \\ = & \sum_{j=1}^{k_i-1} (\lambda_j^i - \lambda_{k_i}) P_j^i (P_j^i)^T + \lambda_{k_i}^i \mathcal{I}, \end{aligned} \tag{3.48}$$

where $1 \leq k_i \leq l_i$ is a small integer that satisfies $\lambda_{k_i}^i > 0$. This majorization idea has also been adopted in [91] for efficiently solving the linear systems. After the positive definite proximal terms are obtained, each small block can be solved the the APG-SNCG algorithm provided in Section 2.4.

We end this section by a discussion on the generalization of the symmetric Gauss-Seidel technique to the problems involving two non-smooth terms. Consider the problems of the following form:

$$\min F(x) := f_1(x_1) + f_2(x_2) + \tfrac{1}{2}\langle x, \mathcal{H}x\rangle - \langle r, x\rangle, \tag{3.49}$$

where $x \equiv (x_1, x_2) \in \mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2$ with $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, $f_1 : \mathcal{X}_1 \to (-\infty, \infty]$, $f_2 : \mathcal{X}_2 \to (-\infty, \infty]$ are two closed proper convex functions (possibly nonsmooth), $\mathcal{H} : \mathcal{X} \to \mathcal{X}$ is a self-adjoint positive semidefinite linear operator, and $r \in \mathcal{X}$ is the given data. We decompose $\mathcal{H}$ and r according to the block structure of x such that for any $x \equiv (x_1, x_2) \in \mathcal{X}$,

$$\mathcal{H}x \equiv \begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{12}^* & \mathcal{H}_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \langle r, x\rangle \equiv \langle r_1, x_1\rangle + \langle r_2, x_2\rangle,$$

where $\mathcal{H}_{11} : \mathcal{X}_1 \to \mathcal{X}_1$ and $\mathcal{H}_{22} : \mathcal{X}_2 \to \mathcal{X}_2$ are self-adjoint positive semidefinite linear operators, and $\mathcal{H}_{12} : \mathcal{X}_2 \to \mathcal{X}_1$ is a linear mapping whose adjoint is given by $\mathcal{H}_{12}^*$. We further assume $\mathcal{H}_{11} \succ 0$ and $\mathcal{H}_{22} \succ 0$ as Theorem 2.2.

Define an operator $\mathcal{P}_{f_2} : \mathcal{X}_1 \to \mathcal{X}_2$ as

$$\mathcal{P}_{f_2}(x_1) := \arg\min_{x_2} \left\{ f_2(x_2) + \langle x, \mathcal{H}x\rangle - \langle r, x\rangle \right\}, \tag{3.50}$$

and two functions $g : \mathcal{X}_1 \to (-\infty, +\infty]$ and $\psi : \mathcal{X}_1 \to (-\infty, +\infty]$ as

$$\begin{cases} g(x_1) := f_2(\mathcal{P}_{f_2}(x_1)) + \frac{1}{2}\langle (x_1, \mathcal{P}_{f_2}(x_1)), \mathcal{H}(x_1, \mathcal{P}_{f_2}(x_1))\rangle - \langle r, (x_1, \mathcal{P}_{f_2}(x_1))\rangle, \\ \psi(x_1) := \frac{1}{2}\langle x_1, \mathcal{H}_{11}x_1\rangle - \langle r_1, x_1\rangle - g(x_1). \end{cases} \tag{3.51}$$

Since $\mathcal{H}_{22}$ is assumed to be positive definite, the above optimization problem has a unique solution and thus, $\mathcal{P}_{f_2}$ is well defined. The following Theorem can be obtained from [15] and [30, Theorem 10.2.1].

Proposition 3.3. *The function g is continuously differentiable, with the gradient given by*

$$\nabla g(x_1) = \mathcal{H}_{11}x_1 + \mathcal{H}_{12}\mathcal{P}_{f_2}(x_1) - r_1. \tag{3.52}$$

Furthermore, the function ψ is convex and continuously differentiable, with the gradient given by

$$\nabla\psi(x_1) = -\mathcal{H}_{12}\mathcal{P}_{f_2}(x_1). \tag{3.53}$$

The following Theorem states the property of the nonlinear one cycle symmetric Gauss-Seidel iteration for the problem of the form (3.49).

Theorem 3.3. *Given $\bar{x}_1 \in \mathcal{X}_1$. Define*

$$x_2' = \mathcal{P}_{f_2}(\bar{x}_1) := \arg\min_{x_2}\left\{f_2(x_2) + \frac{1}{2}\langle(\bar{x}_1, x_2), \mathcal{H}(\bar{x}_1, x_2)\rangle - \langle r_2, x_2\rangle\right\}.$$

Then

$$\begin{cases} {x_1}^+ = \arg\min_{x_1}\left\{f_1(x_1) + \frac{1}{2}\langle(x_1, x_2'), \mathcal{H}(x_1, x_2')\rangle - \langle r_1, x_1\rangle\right\} \\ {x_2}^+ = \mathcal{P}_{f_2}(x_1^+) := \arg\min_{x_2}\left\{f_2(x_2) + \frac{1}{2}\langle(x_1^+, x_2), \mathcal{H}(x_1^+, x_2)\rangle - \langle r_2, x_2\rangle\right\} \end{cases} \tag{3.54}$$

solves the problem

$$\min_x \left\{f_1(x_1) + f_2(x_2) + \frac{1}{2}\langle x, \mathcal{H}x\rangle - \langle r, x\rangle + \Delta_\psi(x_1, \bar{x}_1)\right\}, \tag{3.55}$$

where

$$\Delta_\psi(x_1, \bar{x}_1) := \psi(x_1) - \psi(\bar{x}_1) - \langle\nabla\psi(\bar{x}_1), x_1 - \bar{x}_1\rangle \geq 0.$$

Proof. Denote (x_1^*, x_2^*) as the optimal solution of the problem (3.55). Then by the optimality condition of the problem (3.55), we can obtain that

$$\begin{cases} 0 \in \partial f_1(x_1^*) + \mathcal{H}_{11}x_1^* + \mathcal{H}_{12}x_2^* - r_1 + \nabla\psi(x_1^*) - \nabla\psi(\bar{x}_1), \\ 0 \in \partial f_2(x_2^*) + \mathcal{H}_{12}^* x_1^* + \mathcal{H}_{22}x_2^* - r_2. \end{cases} \tag{3.56}$$

By the notation of the operator $\mathcal{P}_{f_2}(\cdot)$ and Proposition 3.3, we can equivalently write (3.56) as

$$\begin{cases} 0 \in \partial f_1(x_1^*) + \mathcal{H}_{11}x_1^* + \mathcal{H}_{12}\mathcal{P}_{f_2}(\bar{x}_1) - r_1, \\ 0 \in \partial f_2(x_2^*) + \mathcal{H}_{12}^* x_1^* + \mathcal{H}_{22}x_2^* - r_2. \end{cases} \tag{3.57}$$

Comparing (3.57) and the optimality conditions for the problem (3.54) at (x_1^+, x_2^+), we have that $(x_1^*, x_2^*) = (x_1^+, x_2^+)$. This completes the proof.

$\square$

Remark 3.2. *If $f_2 \equiv 0$, i.e. there is only one nonsmooth function $f_1(\cdot)$ in (3.49), then one can derive that*

$$\mathcal{P}_{f_2}(x_1) = \mathcal{H}_{22}^{-1}(r_2 - \mathcal{H}_{12}^* x_1).$$

This implies that $\Delta_\psi(x_1; \bar{x}_1) = \frac{1}{2}\|x_1 - \bar{x}_1\|^2_{\mathcal{H}_{12}\mathcal{H}_{22}^{-1}\mathcal{H}_{12}^*}$ for any $x_1, \bar{x}_1 \in \mathcal{X}_1$, which exactly covers the result of Theorem 2.2. In fact, Δ_ψ is the (semi-)Bregman distance function associated with the function ψ, which certainly includes the (semi-)Euclidean distance function.

Even though we obtain a nice extension of Theorem 2.2 from one nonsmooth term to two, there is a fatal disadvantage in the above procedure that the function ψ depends on the value of $\mathcal{P}_{f_2}(x_1)$. This causes a great difficulty for the algorithm to be embedded into the iABCD framework for solving the subproblems since the (semi-Bregman) distance proximal term Δ_ψ is neither fixed nor monotone decreasing. In order to obtain an efficient algorithm that can be used to solve the subproblems in the iABCD framework, perhaps we'd better give up the idea to solve x_1 and x_2 in an alternative fashion for the problem (3.49). In fact, we could substitute x_2 directly into the (3.49) and get a reduced optimization problem:

$$\min F(x_1, \mathcal{P}_{f_2}(x_1)) = f(x_1) + g(x_1), \tag{3.58}$$

where the function g is defined in (3.51). Note that g is a strongly convex function since $\mathcal{H}_{11} \succ 0$ and ∇g is continuously differentiable with Lipschitz continuous gradient by Proposition 3.3. Thus, the above problem can be efficiently solved inexactly by the APG-SNCG introduced in Section 2.4.

Chapter 4

A majorized alternating direction method of multipliers for linearly constrained problems with coupled objective functions

In this chapter, we focus on designing and analyzing the algorithm for solving the two block linearly constrained convex optimization problem (1.2). The (accelerated) block coordinate descent type algorithm that proposed in the previous chapter cannot apply to this kind of problems, since it is impossible to update a single block, say u, without violating the coupled constraint $\mathcal{A}^*u + \mathcal{B}^*v = c$.

As mentioned in the introduction chapter, a general and well-studied approach to solve the linearly constrained problem is the method of multipliers, which is also called the augmented Lagrangian method (ALM). Let the augmented Lagrangian function $\mathcal{L}_\sigma(\cdot,\cdot,\cdot)$ for the two-block problem (1.2) is defined as (1.12). Given an initial guess of the dual variable x^0 and parameters $\sigma, \tau > 0$, the framework of the ALM consists of the following iterations for the $(k+1)$th step:

$$\begin{cases} (u^{k+1}, v^{k+1}) \approx \arg\min\limits_{u,v} \mathcal{L}_\sigma(u, v; x^k), \\ x^{k+1} = x^k + \tau\sigma(\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c). \end{cases}$$

The global convergence and local linear convergence rate have been established by

Rockafellar in his influential papers [81, 79]. However, to obtain even an approximate solution (u^{k+1}, v^{k+1}) of the subproblem in the above framework is challenging and time-consuming, especially at the early stage of the algorithm such that the iteration sequences are far from the optimal solution set. Therefore, a popular and common practice is to solve u^{k+1} and v^{k+1} alternatively by the alternating direction method of multipliers (ADMM). However, when the objective function consists of a smooth coupled term $\phi(u, v)$, to solve u with fixed v^k may still be a difficult task, and the same situation occurs when solving the second block v. This motivates us to apply a proper majorization technique each step to the function $\phi(u, v)$ before solving the subproblems. In this way we named our modified algorithm the majorized ADMM (mADMM).

In this chapter, we analyze the global convergence, the ergodic and non-ergodic iteration complexity and the linear convergence rate of the mADMM applying to the linearly constrained convex optimzation problems (1.2), where the objective function consists of a smooth coupled term. Part of the results in this chapter have already been published in [14].

4.1 A majorized ADMM with coupled objective functions

The first part of this section is devoted to the framework of our mADMM for solving (1.2). Following that two important inequalities, which play essential roles for our convergence analysis, are presented. Throughout this chapter, we denote the primal variable as $w \equiv (u, v) \in \mathcal{U} \times \mathcal{V}$.

Assume that $\mathcal{Q}$ and $\mathcal{H}$ are defined as in (2.2) with respect to the function ϕ in (1.2), so that the inequalities (2.3) and (2.4) hold. In this section, we further assume that

$$\mathcal{H} = \operatorname{Diag}(\mathcal{D}_1, \mathcal{D}_2), \tag{4.1}$$

where $\mathcal{D}_1 : \mathcal{U} \to \mathcal{U}$ and $\mathcal{D}_2 : \mathcal{V} \to \mathcal{V}$ are two self-adjoint positive semidefinite linear operators. In fact, this kind of structure naturally appears in applications like (1.11), where the best possible lower bound of the generalized Hessian is $\widetilde{\mathcal{Q}}$ and the best possible upper bound of the generalized Hessian is $\widetilde{\mathcal{Q}}+\mathcal{I}$, where $\mathcal{I}:\mathcal{U}\times\mathcal{V}\to\mathcal{U}\times\mathcal{V}$ is the identity operator. For this case, the tightest estimation of $\mathcal{H}$ is $\mathcal{I}$, which is block diagonal. Since the coupled function $\phi(u,v)$ consists of two block variables u and v, the operators $\mathcal{Q}$ and $\mathcal{W}$ can be decomposed accordingly as $\mathcal{Q} = \begin{pmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^* & \mathcal{Q}_{22} \end{pmatrix}$ and $\mathcal{W} = \begin{pmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{12}^* & \mathcal{W}_{22} \end{pmatrix}$, where $\mathcal{W}_{11},\ \mathcal{Q}_{11}:\mathcal{U}\to\mathcal{U}$ and $\mathcal{W}_{22},\ \mathcal{Q}_{22}:\mathcal{V}\to\mathcal{V}$ are self-adjoint positive semidefinite linear operators, and $\mathcal{W}_{12},\ \mathcal{Q}_{12}:\mathcal{V}\to\mathcal{U}$ are two linear mappings whose adjoints are denoted by $\mathcal{W}_{12}^*$ and $\mathcal{Q}_{12}^*$, respectively. Denote $\eta\in[0,1]$ as a constant that satisfies

$$|\langle u, (\mathcal{W}_{12}-\mathcal{Q}_{12})v\rangle| \le \frac{\eta}{2}(\|u\|^2_{\mathcal{D}_1}+\|v\|^2_{\mathcal{D}_2}), \quad \forall\, \mathcal{W}\in\partial^2\phi(u,v),\ u\in\mathcal{U},\ v\in\mathcal{V}. \quad (4.2)$$

Note that (4.2) always holds true for $\eta = 1$ according to the Cauchy-Schwarz inequality.

Let $\sigma > 0$. For given $w' = (u',v') \in \mathcal{U}\times\mathcal{V}$, define the following majorized augmented Lagrangian function associated with (1.2):

$$\widehat{\mathcal{L}}_\sigma(w;(x,w')) := \ p(u)+q(v)+\hat{\phi}(w;w')+\langle x, \mathcal{A}^*u+\mathcal{B}^*v-c\rangle+\frac{\sigma}{2}\|\mathcal{A}^*u+\mathcal{B}^*v-c\|^2,$$

where $(w,x)=(u,v,x)\in\mathcal{U}\times\mathcal{V}\times\mathcal{X}$ and the majorized function $\hat{\phi}$ is given by (2.4). Then our proposed algorithm works as follows:

mADMM: A majorized alternating direction method of multipliers

Choose an initial point $(u^0,v^0,x^0)\in \mathrm{dom}(p)\times\mathrm{dom}(q)\times\mathcal{X}$ and parameters $\tau>0$. Let $\mathcal{S}$ and $\mathcal{T}$ be given self-adjoint positive semidefinite linear operators. Set $k:=0$. Iterate until convergence:

Step 1. Compute $u^{k+1} = \arg\min\limits_{u\in\mathcal{U}}\{\widehat{\mathcal{L}}_\sigma(u,v^k;(x^k,w^k))+\frac{1}{2}\|u-u^k\|^2_{\mathcal{S}}\}$.

Step 2. Compute $v^{k+1} = \arg\min\limits_{v\in\mathcal{V}}\{\widehat{\mathcal{L}}_\sigma(u^{k+1},v;(x^k,w^k))+\frac{1}{2}\|v-v^k\|^2_{\mathcal{T}}\}$.

Step 3. Compute $x^{k+1} = x^k+\tau\sigma(\mathcal{A}^*u^{k+1}+\mathcal{B}^*v^{k+1}-c)$.

The majorization idea of the ADMM has also been discussed in Hong et al. work [47], as shown in (1.15). There is one difference between our approach and theirs. We majorize $\phi(u,v)$ at (u^k, v^k) before the $(k+1)$th iteration while the majorization function in Hong et al.'s framework is based on (u^{k+1}, v^k) when updating v^{k+1}. Interestingly, if $\phi(\cdot,\cdot)$ merely consists of quadratically coupled functions and separable smooth functions, our mADMM is exactly the same as the one proposed by Hong et al. under a proper choice of the majorization functions. Moreover, for applications like (1.11), a potential advantage of our method is that we only need to compute the projection $\Pi_{\mathcal{K}_1}(\cdot,\cdot)$ once in order to compute $\nabla\phi(\cdot,\cdot)$ as a part of the majorization function within one iteration, while the procedure (1.15) needs to compute $\Pi_{\mathcal{K}_1}(\cdot,\cdot)$ at two different points (u^k, v^k) and (u^{k+1}, v^k).

Denote $\bar{w} \equiv (\bar{u}, \bar{v})$ as an optimal solution of the problem (1.2) and $\bar{x}$ as the corresponding multiplier. In order to prove the convergence of the proposed majorized ADMM, the following constraint qualification is needed:

Assumption 4.1. *There exists* $(\hat{u}, \hat{v}) \in \operatorname{ri}\,(\operatorname{dom}(p) \times \operatorname{dom}(q))$ *such that* $\mathcal{A}^*\hat{u} + \mathcal{B}^*\hat{v} = c$.

Let ∂p and ∂q be the sub-differential mappings of p and q, respectively. Define the set-valued mapping $\mathcal{F}$ by

$$F(u,v,x) := \nabla\phi(w) + \begin{pmatrix} \partial p(u) + \mathcal{A}x \\ \partial q(v) + \mathcal{B}x \end{pmatrix}, \quad (u,v,x) \in \mathcal{U} \times \mathcal{V} \times \mathcal{X}.$$

Under Assumption 4.1, $(\bar{u}, \bar{v})$ is optimal to (1.2) if and only if there exists $\bar{x} \in \mathcal{X}$ such that the following Karush-Kuhn-Tucker (KKT) condition holds:

$$\begin{cases} 0 \in F(\bar{u}, \bar{v}, \bar{x}), \\ \mathcal{A}^*\bar{u} + \mathcal{B}^*\bar{v} = c. \end{cases} \tag{4.3}$$

Furthermore, define the KKT mapping $\mathcal{R} : \mathcal{U} \times \mathcal{V} \times \mathcal{X} \to \mathcal{U} \times \mathcal{V} \times \mathcal{X}$ as follows:

$$R(u, v, x) = \begin{pmatrix} u - \mathrm{Prox}_p(u - (\nabla_u \phi(w) + \mathcal{A}x)) \\ v - \mathrm{Prox}_q(v - (\nabla_v \phi(w) + \mathcal{B}x)) \\ c - \mathcal{A}^* u - \mathcal{B}^* v \end{pmatrix}. \tag{4.4}$$

Denote Ω as the solution set to the above KKT optimality conditions. It is easy to see that the condition (4.3) holds at $(\bar{u}, \bar{v}, \bar{x}) \in \Omega$ if and only if $R(\bar{u}, \bar{v}, \bar{x}) = 0$ for $(\bar{u}, \bar{v}, \bar{x}) \in \mathcal{U} \times \mathcal{V} \times \mathcal{X}$.

Also by the assumptions that p and q are convex functions, we know $\partial p(\cdot)$ and $\partial q(\cdot)$ are maximal monotone operators. Then, for any $u, \hat{u} \in \mathrm{dom}(p)$, $\xi \in \partial p(u)$, and $\hat{\xi} \in \partial p(\hat{u})$, we have

$$\langle u - \hat{u}, \xi - \hat{\xi} \rangle \geq 0, \tag{4.5}$$

and similarly for any $v, \hat{v} \in \mathrm{dom}(q)$, $\zeta \in \partial q(v)$, and $\hat{\zeta} \in \partial q(\hat{v})$, we have

$$\langle v - \hat{v}, \zeta - \hat{\zeta} \rangle \geq 0. \tag{4.6}$$

On top of the above mentioned knowledge from the convex optimization, we need the following basic identity:

$$\langle \xi, \mathcal{G}\zeta \rangle = \frac{1}{2}(\|\xi\|_{\mathcal{G}}^2 + \|\zeta\|_{\mathcal{G}}^2 - \|\xi - \zeta\|_{\mathcal{G}}^2) = \frac{1}{2}(\|\xi + \zeta\|_{\mathcal{G}}^2 - \|\xi\|_{\mathcal{G}}^2 - \|\zeta\|_{\mathcal{G}}^2). \tag{4.7}$$

which holds for any ξ, ζ in the same space and a self-adjoint positive semidefinite operator $\mathcal{G}$. This identity would be frequently used in our convergence study of the mADMM.

Suppose that $\{(u^k, v^k, x^k)\}$ is the sequence generated by the mADMM algorithm. In order to simplify subsequent discussions, for any given parameter $\tau > 0$, we denote

$$\rho(\tau) := \min(\tau, 1 + \tau - \tau^2),$$

and for a given optimal solution $(\bar{u}, \bar{v}, \bar{x}) \in \Omega$ and $k \geq 0$, define

$$u_e^k := u^k - \bar{u}, \quad v_e^k := v^k - \bar{v}, \quad w_e^k := w^k - \bar{w}, \quad x_e^k := x^k - \bar{x},$$

$$\Delta u^{k+1} := u^{k+1} - u^k, \quad \Delta v^{k+1} := v^{k+1} - v^k, \quad \Delta w^{k+1} := w^{k+1} - w^k, \quad \Delta x^{k+1} := x^{k+1} - x^k.$$

For $k \geq 0$, we also use the following notations in the convergence study:

$$\begin{cases} \tilde{x}^{k+1} := x^k + \sigma(\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c), \\ \Xi_{k+1} := \|\Delta v^{k+1}\|^2_{\mathcal{D}_2+\mathcal{T}} + \eta\|\Delta u^{k+1}\|^2_{\mathcal{D}_1}, \\ \Theta_{k+1} := \|\Delta u^{k+1}\|^2_{\mathcal{S}} + \|\Delta v^{k+1}\|^2_{\mathcal{T}+\frac{1}{6}\mathcal{Q}_{22}} + \frac{1}{8}\|\Delta w^{k+1}\|^2_{\mathcal{Q}}, \\ \Gamma_{k+1} := \Theta_{k+1} + \frac{1}{3}(\tau^3\sigma)^{-1}\rho(\tau)\|\Delta x^{k+1}\|^2 + \rho(\tau)\|\Delta v^{k+1}\|^2_{\sigma\mathcal{B}\mathcal{B}^*} - \|\Delta u^{k+1}\|^2_{\eta\mathcal{D}_1} - \|\Delta v^{k+1}\|^2_{\eta\mathcal{D}_2} \end{cases} \tag{4.8}$$

and denote for $(u, v, x) \in \mathcal{U} \times \mathcal{V} \times \mathcal{X}$,

$$\begin{cases} \Phi_k(u, v, x) := (\tau\sigma)^{-1}\|x^k - x\|^2 + \|u^k - u\|^2_{\mathcal{D}_1+\mathcal{S}} + \|v^k - v\|^2_{\mathcal{Q}_{22}+\mathcal{D}_2+\mathcal{T}} + \frac{1}{4}\|w^k - w\|^2_{\mathcal{Q}} \\ \qquad\qquad\qquad +\sigma\|\mathcal{A}^* u + \mathcal{B}^* v^k - c\|^2, \\ \Psi_k(u, v, x) := \Phi_k(u, v, x) + \frac{3}{2}\|w^k - w\|^2_{\mathcal{Q}}. \end{cases} \tag{4.9}$$

Proposition 4.1. *Suppose that the solution set of problem (1.2) is nonempty and Assumption 4.1 holds. Assume that $\mathcal{S}$ and $\mathcal{T}$ are chosen such that the sequence $\{(u^k, v^k, x^k)\}$ is well defined. Then the following conclusions hold:*

(i) For $\tau \in (0, 1]$, we have that for any $k \geq 0$,

$$\begin{aligned} &\big(\Phi_{k+1}(\bar{u}, \bar{v}, \bar{x}) + \frac{1}{2}(1-\tau)\sigma\|\mathcal{A}^* u_e^{k+1} + \mathcal{B}^* v_e^{k+1}\|^2\big) \\ &\qquad\qquad -\big(\Phi_k(\bar{u}, \bar{v}, \bar{x}) + \frac{1}{2}(1-\tau)\sigma\|\mathcal{A}^* u_e^k + \mathcal{B}^* v_e^k\|^2\big) \\ \leq\;& -\big(\Theta_{k+1} + \sigma\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^k - c\|^2 + \frac{1}{2}(1-\tau)\sigma\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2\big). \end{aligned} \tag{4.10}$$

(ii) For $\tau \geq 0$, we have that for any $k \geq 1$,

$$\begin{aligned} &\big(\Psi_{k+1}(\bar{u}, \bar{v}, \bar{x}) + \Xi_{k+1} + \frac{1}{3}(4 - \tau - 2\min(\tau, \tau^{-1}))\sigma\|\mathcal{A}^* u_e^{k+1} + \mathcal{B}^* v_e^{k+1}\|^2\big) \\ &\qquad -\big(\Psi_k(\bar{u}, \bar{v}, \bar{x}) + \Xi_k + \frac{1}{3}(4 - \tau - 2\min(\tau, \tau^{-1}))\sigma\|\mathcal{A}^* u_e^k + \mathcal{B}^* v_e^k\|^2\big) \\ \leq\;& -\big(\Gamma_{k+1} + \frac{1}{6}\rho(\tau)\tau^{-1}\sigma\|\mathcal{A}^* \Delta u^{k+1} + \mathcal{B}^* \Delta v^{k+1}\|^2\big). \end{aligned} \tag{4.11}$$

Proof. In the mADMM iteration scheme, the optimality condition for (u^{k+1}, v^{k+1}) is

$$\left\{\begin{array}{ll} 0 \in & \partial p(u^{k+1}) + \nabla_u \phi(w^k) + \mathcal{A}\tilde{x}^{k+1} + (\mathcal{Q}_{11} + \mathcal{D}_1 + \mathcal{S})\Delta u^{k+1} - \sigma \mathcal{A}\mathcal{B}^* \Delta v^{k+1}, \\ 0 \in & \partial q(v^{k+1}) + \nabla_v \phi(w^k) + \mathcal{B}\tilde{x}^{k+1} + \mathcal{Q}_{12}^* \Delta u^{k+1} + (\mathcal{Q}_{22} + \mathcal{D}_2 + \mathcal{T})\Delta v^{k+1}, \end{array}\right. \tag{4.12}$$

which can be reformulated as

$$\left\{\begin{array}{l} -\mathcal{A}\tilde{x}^{k+1} - \nabla_u \phi(w^k) - (\mathcal{Q}_{11} + \mathcal{D}_1 + \mathcal{S})\Delta u^{k+1} + \sigma \mathcal{A}\mathcal{B}^* \Delta v^{k+1} \in \partial p(u^{k+1}), \\ -\mathcal{B}\tilde{x}^{k+1} - \nabla_v \phi(w^k) - (\mathcal{Q}_{22} + \mathcal{D}_2 + \mathcal{T})\Delta v^{k+1} - \mathcal{Q}_{12}^* \Delta u^{k+1} \in \partial q(v^{k+1}). \end{array}\right. \tag{4.13}$$

Since $(\bar{u}, \bar{v}, \bar{x})$ satisfies the KKT system (4.3), we also have that

$$\left\{\begin{array}{l} -\mathcal{A}\bar{x} - \nabla_u \phi(\bar{w}) \in \partial p(\bar{u}), \\ -\mathcal{B}\bar{x} - \nabla_v \phi(\bar{v}) \in \partial q(\bar{v}). \end{array}\right. \tag{4.14}$$

Therefore, by letting $u = u^{k+1}$, $\hat{u} = \bar{u}$, $v = v^{k+1}$ and $\hat{v} = \bar{v}$ in the inequalities (4.5) and (4.6), respectively, we are able to get that

$$\begin{array}{rl} 0 \leq & (\tau\sigma)^{-1}\langle x_e^{k+1}, x_e^k - x_e^{k+1}\rangle - (1-\tau)\sigma\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2 \\ & +\sigma\langle \mathcal{B}^*\Delta v^{k+1}, \mathcal{A}^* u_e^{k+1}\rangle - \langle \nabla\phi(w^k) - \nabla\phi(\bar{w}), w_e^{k+1}\rangle - \langle \mathcal{Q}_{12}^*\Delta u^{k+1}, v_e^{k+1}\rangle \\ & -\langle(\mathcal{Q}_{11} + \mathcal{D}_1 + \mathcal{S})\Delta u^{k+1}, u_e^{k+1}\rangle - \langle(\mathcal{Q}_{22} + \mathcal{D}_2 + \mathcal{T})\Delta v^{k+1}, v_e^{k+1}\rangle. \end{array} \tag{4.15}$$

By taking $(w, w') = (\bar{w}, w^k)$ and $(w^{k+1}, \bar{w})$ in (2.3), and $(w, w') = (w^{k+1}, w^k)$ in (2.4), we know that

$$\left\{\begin{array}{rl} \phi(\bar{w}) & \geq \phi(w^k) + \langle \nabla\phi(w^k), -w_e^k\rangle + \frac{1}{2}\|w_e^k\|_{\mathcal{Q}}^2, \\ \phi(w^{k+1}) & \geq \phi(\bar{w}) + \langle \nabla\phi(w), w_e^{k+1}\rangle + \frac{1}{2}\|w_e^{k+1}\|_{\mathcal{Q}}^2, \\ \phi(w^{k+1}) & \leq \phi(w^k) + \langle \nabla\phi(w^k), \Delta w^{k+1}\rangle + \frac{1}{2}\|\Delta w^{k+1}\|_{\mathcal{Q}+\mathcal{H}}^2. \end{array}\right. \tag{4.16}$$

Putting the above three inequalities together, we get

$$\langle \nabla\phi(w^k) - \nabla\phi(\bar{w}), w_e^{k+1}\rangle \geq \frac{1}{2}(\|w_e^k\|_{\mathcal{Q}}^2 + \|w_e^{k+1}\|_{\mathcal{Q}}^2) - \frac{1}{2}\|\Delta w^{k+1}\|_{\mathcal{Q}+\mathcal{H}}^2. \tag{4.17}$$

Substituting (4.17) into (4.15), we can further obtain that

$$\begin{aligned} 0 \leq\ & (\tau\sigma)^{-1}\langle x_e^{k+1}, x_e^k - x_e^{k+1}\rangle - (1-\tau)\sigma\|\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c\|^2 \\ & +\sigma\langle \mathcal{B}^*\Delta v^{k+1}, \mathcal{A}^*u_e^{k+1}\rangle - \|w_e^{k+1}\|_{\mathcal{Q}}^2 + \tfrac{1}{2}\|\Delta u^{k+1}\|_{\mathcal{D}_1}^2 + \tfrac{1}{2}\|\Delta v^{k+1}\|_{\mathcal{D}_2}^2 \\ & +\langle \Delta v^{k+1}, \mathcal{Q}_{12}^* u_e^{k+1}\rangle - \langle (\mathcal{D}_1 + \mathcal{S})\Delta u^{k+1}, u_e^{k+1}\rangle - \langle (\mathcal{D}_2 + \mathcal{T})\Delta v^{k+1}, v_e^{k+1}\rangle, \end{aligned} \tag{4.18}$$

where we take advantage of the following equality:

$$\begin{aligned} & \tfrac{1}{2}(\|\Delta w^{k+1}\|_{\mathcal{Q}}^2 - \|w_e^{k+1}\|_{\mathcal{Q}}^2 - \|w_e^k\|_{\mathcal{Q}}^2) - \langle \mathcal{Q}_{11}\Delta u^{k+1}, u_e^{k+1}\rangle - \langle \mathcal{Q}_{22}\Delta v^{k+1}, v_e^{k+1}\rangle \\ & -\langle \mathcal{Q}_{12}^*\Delta u^{k+1}, v_e^{k+1}\rangle \\ = \ & \tfrac{1}{2}(\|\Delta w^{k+1}\|_{\mathcal{Q}}^2 - \|w_e^{k+1}\|_{\mathcal{Q}}^2 - \|w_e^k\|_{\mathcal{Q}}^2) - \langle \Delta w^{k+1}, \mathcal{Q}w_e^{k+1}\rangle + \langle \mathcal{Q}_{12}\Delta v^{k+1}, u_e^{k+1}\rangle \\ = \ & -\|w_e^{k+1}\|_{\mathcal{Q}}^2 + \langle \mathcal{Q}_{12}\Delta v^{k+1}, u_e^{k+1}\rangle. \end{aligned} \tag{4.19}$$

From the identity (4.7) we can see that

$$\begin{aligned} \langle (\mathcal{D}_1 + \mathcal{S})\Delta u^{k+1}, u_e^{k+1}\rangle &= \frac{1}{2}(\|\Delta u^{k+1}\|_{\mathcal{D}_1+\mathcal{S}}^2 + \|u_e^{k+1}\|_{\mathcal{D}_1+\mathcal{S}}^2 - \|u_e^k\|_{\mathcal{D}_1+\mathcal{S}}^2), \\ \langle (\mathcal{D}_2 + \mathcal{T})\Delta v^{k+1}, v_e^{k+1}\rangle &= \frac{1}{2}(\|\Delta v^{k+1}\|_{\mathcal{D}_2+\mathcal{T}}^2 + \|v_e^{k+1}\|_{\mathcal{D}_2+\mathcal{T}}^2 - \|v_e^k\|_{\mathcal{D}_2+\mathcal{T}}^2), \\ \langle \Delta x^{k+1}, x_e^{k+1}\rangle &= \frac{1}{2}(\|\Delta x^{k+1}\|^2 + \|x_e^{k+1}\|^2 - \|x_e^k\|^2), \\ \langle \mathcal{Q}_{22}\Delta v^{k+1}, v_e^{k+1}\rangle &= \frac{1}{2}(\|\Delta v^{k+1}\|_{\mathcal{Q}_{22}}^2 + \|v_e^{k+1}\|_{\mathcal{Q}_{22}}^2 - \|v_e^k\|_{\mathcal{Q}_{22}}^2). \end{aligned} \tag{4.20}$$

These equalities enable us to reformulate (4.18) as

$$\begin{aligned} 0 \leq\ & \sigma\langle \mathcal{B}^*\Delta v^{k+1}, \mathcal{A}^*u_e^{k+1}\rangle + \langle \Delta v^{k+1}, \mathcal{Q}_{12}^*u_e^{k+1}\rangle - \tfrac{1}{2}(\|\Delta u^{k+1}\|_{\mathcal{S}}^2 + \|u_e^{k+1}\|_{\mathcal{D}_1+\mathcal{S}}^2 - \|u_e^k\|_{\mathcal{D}_1+\mathcal{S}}^2) \\ & -\frac{1}{2}(\|\Delta v^{k+1}\|_{\mathcal{T}}^2 + \|v_e^{k+1}\|_{\mathcal{D}_2+\mathcal{T}}^2 - \|v_e^k\|_{\mathcal{D}_2+\mathcal{T}}^2) - \frac{1}{2\tau\sigma}(\|\Delta x^{k+1}\|^2 + \|x_e^{k+1}\|^2 - \|x_e^k\|^2) \\ & -(1-\tau)\sigma\|\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c\|^2 - \|w_e^{k+1}\|_{\mathcal{Q}}^2. \end{aligned} \tag{4.21}$$

(i) Assume that $\tau \in (0,1]$. Then we get that

$$\begin{aligned}
\langle \Delta v^{k+1},\ \mathcal{Q}_{12}^* u_e^{k+1}\rangle &= \left\langle \begin{pmatrix} 0 \\ \Delta v^{k+1} \end{pmatrix},\ \mathcal{Q} w_e^{k+1} \right\rangle - \langle \mathcal{Q}_{22}\Delta v^{k+1}, v_e^{k+1}\rangle \\
&\leq (\frac{3}{4}\|w_e^{k+1}\|_{\mathcal{Q}}^2 + \frac{1}{3}\|\Delta v^{k+1}\|_{\mathcal{Q}_{22}}^2) - \frac{1}{2}(\|\Delta v^{k+1}\|_{\mathcal{Q}_{22}}^2 + \|v_e^{k+1}\|_{\mathcal{Q}_{22}}^2 \\
&\quad - \|v_e^k\|_{\mathcal{Q}_{22}}^2) \\
&= \frac{3}{4}\|w_4^{k+1}\|_{\mathcal{Q}}^2 + \frac{1}{2}(\|v_e^k\|_{\mathcal{Q}_{22}}^2 - \|v_e^{k+1}\|_{\mathcal{Q}_{22}}^2) - \frac{1}{6}\|\Delta v^{k+1}\|^2,
\end{aligned} \tag{4.22}$$

where the inequality is obtained by the Cauchy-Schwarz inequality[1]. By some simple manipulations we can also see that

$$\begin{aligned}
\sigma\langle \mathcal{B}^*\Delta v^{k+1}, \mathcal{A}^* u_e^{k+1}\rangle &= \frac{\sigma}{2}(\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2 - \|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^k - c\|^2) \\
&\quad + \frac{\sigma}{2}(\|\mathcal{B}^* v_e^k\|^2 - \|\mathcal{B}^* v_e^{k+1}\|^2).
\end{aligned} \tag{4.23}$$

Finally, by substituting (4.22) and (4.23) into (4.21) and recalling the definition of $\Phi_{k+1}(\cdot,\cdot,\cdot)$ and Θ_{k+1} in (4.8) and (4.9), we have that

$$\begin{aligned}
&[\Phi_{k+1}(\bar u, \bar v, \bar x) + \frac{1}{2}(1-\tau)\sigma\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2] \\
&\qquad - [\Phi_k(\bar u, \bar v, \bar x) + \frac{1}{2}(1-\tau)\sigma\|\mathcal{A}^* u^k + \mathcal{B}^* v^k - c\|^2] \\
&\leq -[\|\Delta u^{k+1}\|_{\mathcal{S}}^2 + \|\Delta v^{k+1}\|_{\mathcal{T}}^2 + \frac{1}{4}\|w_e^{k+1}\|_{\mathcal{Q}}^2 + \frac{1}{4}\|w_e^k\|_{\mathcal{Q}}^2 + \sigma\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^k - c\|^2 \\
&\qquad + \frac{1}{2}(1-\tau)\sigma\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2 + \frac{1}{2}(1-\tau)\sigma\|\mathcal{A}^* u^k + \mathcal{B}^* v^k - c\|^2] \\
&\leq -[\Theta_{k+1} + \sigma\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^k - c\|^2 + \tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2],
\end{aligned} \tag{4.24}$$

where the last inequality comes from the fact that

$$\|w_e^{k+1}\|_{\mathcal{Q}}^2 + \|w_e^{k+1}\|_{\mathcal{Q}}^2 \geq \tfrac{1}{2}\|\Delta w^{k+1}\|_{\mathcal{Q}}^2.$$

[1]The coefficient in this inequality is slightly different from the original one presented in the paper [14]. In recent work [11], the authors made a nice observation that the term $\left\langle \begin{pmatrix} 0 \\ \Delta v^{k+1} \end{pmatrix}, \mathcal{Q} w_e^{k+1} \right\rangle$ can be bounded by $\frac{3}{4}\|w_e^{k+1}\|_{\mathcal{Q}}^2 + \frac{1}{3}\|\Delta v^{k+1}\|_{\mathcal{Q}_{22}}^2$ instead of $\frac{1}{2}\|w_e^{k+1}\|_{\mathcal{Q}}^2 + \frac{1}{2}\|\Delta v^{k+1}\|_{\mathcal{Q}_{22}}^2$ (see the inequality (19) in Lemma 2.1 [11]). We adopt this modification to establish the convergence results under a weaker condition.

This completes the proof of part (i).

(ii) Assume that $\tau \geq 0$. In this part, first we shall estimate the following term

$$\sigma\langle \mathcal{B}^*\Delta v^{k+1}, \mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c\rangle + \langle \Delta v^{k+1}, \mathcal{Q}_{12}^*u_e^{k+1} + \mathcal{Q}_{22}v_e^{k+1}\rangle.$$

It follows from (4.13) that

$$\begin{cases} -\mathcal{B}\tilde{x}^{k+1} - \nabla_v\phi(w^k) - (\mathcal{Q}_{22} + \mathcal{D}_2 + \mathcal{T})\Delta v^{k+1} - \mathcal{Q}_{12}^*\Delta u^{k+1} \in \partial q(v^{k+1}), \\ -\mathcal{B}\tilde{x}^{k} - \nabla_v\phi(w^{k-1}) - (\mathcal{Q}_{22} + \mathcal{D}_2 + \mathcal{T})\Delta v^{k} - \mathcal{Q}_{12}^*\Delta u^{k} \in \partial q(v^{k}). \end{cases} \tag{4.25}$$

Since $\nabla\phi$ is globally Lipschitz continuous, it is known from Clarke's Mean-Value Theorem [12, Proposition 2.6.5] that there exists a self-adjoint and positive semidefinite operator $\mathcal{W}^k \in \operatorname{conv}\{\partial^2\phi([w^{k-1}, w^k])\}$ such that

$$\nabla\phi(w^k) - \nabla\phi(w^{k-1}) = \mathcal{W}^k\Delta w^k,$$

where the set $\operatorname{conv}\{\partial^2\phi[w^{k-1}, w^k]\}$ denotes the convex hull of all points $\mathcal{W} \in \partial^2\phi(z)$ for any $z \in [w^{k-1}, w^k]$. Denote $\mathcal{W}^k := \begin{pmatrix} \mathcal{W}_{11}^k & \mathcal{W}_{12}^k \\ (\mathcal{W}_{12}^k)^* & \mathcal{W}_{22}^k \end{pmatrix}$, where $\mathcal{W}_{11}^k : \mathcal{U} \to \mathcal{U}$, $\mathcal{W}_{22}^k : \mathcal{V} \to \mathcal{V}$ are self-adjoint positive semidefinite operators and $\mathcal{W}_{12}^k : \mathcal{U} \to \mathcal{V}$ is a linear operator. Combining (4.25) and the monotonicity of $\partial q(\cdot)$, we obtain that

$$\begin{aligned}
& -\langle \mathcal{B}(\tilde{x}^{k+1} - \tilde{x}^k), \Delta v^{k+1}\rangle - \langle \mathcal{Q}_{22}\Delta v^{k+1} + \mathcal{Q}_{12}^*\Delta u^{k+1}, \Delta v^{k+1}\rangle \\
\geq\ & \langle \nabla_v\phi(w^k) - \nabla_v\phi(w^{k-1}), \Delta v^{k+1}\rangle - \langle (\mathcal{Q}_{22} + \mathcal{D}_2 + \mathcal{T})\Delta v^k, \Delta v^{k+1}\rangle + \|\Delta v^{k+1}\|_{\mathcal{T}+\mathcal{D}_2}^2 \\
& -\langle \Delta u^k, \mathcal{Q}_{12}\Delta v^{k+1}\rangle \\
=\ & \langle \Delta u^k, (\mathcal{W}_{12}^k - \mathcal{Q}_{12})\Delta v^{k+1}\rangle - \langle (\mathcal{Q}_{22} + \mathcal{D}_2 + \mathcal{T} - \mathcal{W}_{22}^k)\Delta v^k, \Delta v^{k+1}\rangle + \|\Delta v^{k+1}\|_{\mathcal{T}+\mathcal{D}_2}^2 \\
\geq\ & -\frac{\eta}{2}(\|\Delta u^k\|_{\mathcal{D}_1}^2 + \|\Delta v^{k+1}\|_{\mathcal{D}_2}^2) - \frac{1}{2}(\|\Delta v^{k+1}\|_{\mathcal{T}+\mathcal{D}_2}^2 + \|\Delta v^k\|_{\mathcal{T}+\mathcal{D}_2}^2) + \|\Delta v^{k+1}\|_{\mathcal{T}+\mathcal{D}_2}^2 \\
=\ & \frac{1}{2}\|\Delta v^{k+1}\|_{\mathcal{T}+(1-\eta)\mathcal{D}_2}^2 - \frac{1}{2}\|\Delta v^k\|_{\mathcal{T}+\mathcal{D}_2}^2 - \frac{\eta}{2}\|\Delta u^k\|_{\mathcal{D}_1}^2,
\end{aligned}$$

where the second inequality is obtained from (4.2) and the fact that $\mathcal{W}_{22}^k \succeq \mathcal{Q}_{22}$. Therefore, with $\mu_{k+1} = (1-\tau)\sigma\langle \mathcal{B}^*\Delta v^{k+1}, \mathcal{A}^*u^k + \mathcal{B}^*v^k - c\rangle$, the cross term can be estimated as

$$\begin{aligned}
& \sigma\langle \mathcal{B}^*\Delta v^{k+1}, \mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c\rangle + \langle \mathcal{Q}_{12}^* u_e^{k+1} + \mathcal{Q}_{22}v_e^{k+1}, \Delta v^{k+1}\rangle \\
= \;& (1-\tau)\sigma\langle \mathcal{B}^*\Delta v^{k+1}, \mathcal{A}^*u^{k} + \mathcal{B}^*v^{k} - c\rangle + \langle \mathcal{B}^*\Delta v^{k+1}, \tilde{x}^{k+1} - \tilde{x}^{k}\rangle \\
& + \langle \mathcal{Q}_{12}^* u_e^{k} + \mathcal{Q}_{22}v_e^{k}, \Delta v^{k+1}\rangle + \langle \mathcal{Q}_{12}^* \Delta u^{k+1} + \mathcal{Q}_{22}\Delta v^{k+1}, \Delta v^{k+1}\rangle \\
\leq \;& \mu_{k+1} + (\frac{3}{4}\|w_e^k\|_{\mathcal{Q}}^2 + \frac{1}{3}\|\Delta v^{k+1}\|_{\mathcal{Q}_{22}}^2) - \frac{1}{2}\|\Delta v^{k+1}\|^2_{\mathcal{T}+(1-\eta)\mathcal{D}_2} + \frac{1}{2}\|\Delta v^{k}\|^2_{\mathcal{T}+\mathcal{D}_2} + \frac{\eta}{2}\|\Delta u^{k}\|^2_{\mathcal{D}_1}.
\end{aligned} \tag{4.26}$$

Finally, by the Cauchy-Schwarz inequality we know that

$$\mu_{k+1} \leq \begin{cases} \frac{1}{2}(1-\tau)\sigma(\|\mathcal{B}^*\Delta v^{k+1}\|^2 + \|\mathcal{A}^*u^k + \mathcal{B}^*v^k - c\|^2), & \tau \in]0,1], \\ \frac{1}{2}(\tau-1)\sigma(\tau\|\mathcal{B}^*\Delta v^{k+1}\|^2 + \tau^{-1}\|\mathcal{A}^*u^k + \mathcal{B}^*v^k - c\|^2), & \tau > 1. \end{cases} \tag{4.27}$$

Substituting (4.26) and (4.27) into (4.21) and by some manipulations, we can obtain that

$$\begin{aligned}
& (\Psi_{k+1}(\bar{u},\bar{v},\bar{x}) + \Xi_{k+1}) - (\Psi_k(\bar{u},\bar{v},\bar{x}) + \Xi_k) \\
\leq \;& -[\Gamma_{k+1} + \rho(\tau)\tau^{-1}\sigma\|\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c\|^2].
\end{aligned} \tag{4.28}$$

Note that

$$\begin{aligned}
& \|\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c\|^2 \\
= \;& \tfrac{1}{3}(\tau\sigma)^{-2}\|\Delta x^{k+1}\|^2 + \tfrac{1}{3}(\|\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c\|^2 - \|\mathcal{A}^*u^{k} + \mathcal{B}^*v^{k} - c\|^2) \\
& + \tfrac{1}{3}(\|\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c\|^2 + \|\mathcal{A}^*u^{k} + \mathcal{B}^*v^{k} - c\|^2) \\
\geq \;& \tfrac{1}{3}(\tau\sigma)^{-2}\|\Delta x^{k+1}\|^2 + \tfrac{1}{3}(\|\mathcal{A}^*u_e^{k+1} + \mathcal{B}^*v_e^{k+1}\|^2 - \|\mathcal{A}^*u_e^{k} + \mathcal{B}^*v_e^{k}\|^2) \\
& + \tfrac{1}{6}\|\mathcal{A}^*\Delta u^{k+1} + \mathcal{B}^*\Delta v^{k+1} - c\|^2.
\end{aligned} \tag{4.29}$$

Combining (4.28) and (4.29), we can obtain the inequality (4.11). This completes the proof of part (ii). □

4.2 The global convergence analysis

With all the preparations given in the previous sections, we can now discuss the main convergence results of this chapter.

4.2.1 The global convergence

First we prove that under mild conditions, the iteration sequence $\{(u^k, v^k)\}$ generated by the mADMM with $\tau \in (0, \frac{1+\sqrt{5}}{2})$ converges to an optimal solution of problem (1.2) and $\{x^k\}$ converges to an multiplier.

Theorem 4.1. *Suppose that the solution set of* (1.2) *is nonempty and Assumption* 4.1 *holds.*

(i) Assume that $\tau \in (0, 1]$. Suppose that

$$\mathcal{Q}_{11} + (1-\tau)\sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0, \quad \mathcal{Q}_{22} + (1-\tau)\sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0.$$

Then the generated sequence $\{(u^k, v^k)\}$ converges to an optimal solution of (1.2) *and $\{x^k\}$ converges to the corresponding multiplier.*

(ii) Assume that $\tau \in (0, \frac{1+\sqrt{5}}{2})$. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are chosen such that

$$\mathcal{Q}_{11} + \sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0, \quad \mathcal{Q}_{22} + \sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0$$

and for some $\alpha \in [0, 1)$,

$$\mathcal{M} := \frac{\alpha}{8}\mathcal{Q} + Diag\,(\alpha\mathcal{S} - \eta\mathcal{D}_1,\ \alpha\mathcal{T} - \eta\mathcal{D}_2) \succeq 0. \tag{4.30}$$

Then the generated sequence $\{(u^k, v^k)\}$ converges to an optimal solution of (1.2) *and $\{x^k\}$ converges to the corresponding multiplier.*

Proof. **(i)** Let $\tau \in (0, 1]$. The inequality (4.10) shows that $\{\Phi_{k+1}(\bar{u}, \bar{v}, \bar{x})\}$ is bounded, which implies that $\{\|x^{k+1}\|\}$, $\{\|w_e^{k+1}\|_{\mathcal{Q}}\}$, $\{\|u_e^{k+1}\|_{\mathcal{S}}\}$ and $\{\|v_e^{k+1}\|_{\mathcal{Q}_{22}+\sigma\mathcal{B}\mathcal{B}^*+\mathcal{T}}\}$ are all bounded. From the positive definiteness of $\mathcal{Q}_{22} + \sigma\mathcal{B}\mathcal{B}^* + \mathcal{T}$, we can see that $\{\|v_e^{k+1}\|\}$ is bounded. By using the inequalities

$$\begin{aligned} \|\mathcal{A}^*u_e^{k+1}\| &\leq \|\mathcal{A}^*u_e^{k+1} + \mathcal{B}^*v_e^{k+1}\| + \|\mathcal{B}^*v_e^{k+1}\| \leq \tau\sigma(\|x_e^{k+1}\| + \|x_e^k\|) + \|\mathcal{B}^*v_e^{k+1}\|, \\ \|u_e^{k+1}\|_{\mathcal{Q}_{11}} &\leq \|w_e^{k+1}\|_{\mathcal{Q}} + \|v_e^{k+1}\|_{\mathcal{Q}_{22}}, \end{aligned}$$

we know that the sequence $\{\|u_e^{k+1}\|_{\sigma\mathcal{A}\mathcal{A}^*+\mathcal{Q}_{11}}\}$ is also bounded. Therefore, the sequence $\{\|u_e^{k+1}\|_{\mathcal{Q}_{11}+\sigma\mathcal{A}\mathcal{A}^*+\mathcal{S}}\}$ is bounded. By the positive definiteness of $\mathcal{Q}_{11}+\sigma\mathcal{A}\mathcal{A}^*+$

$\mathcal{S}$, we know that $\{\|u_e^{k+1}\|\}$ is bounded. On the whole, the sequence $\{(u^k, v^k, x^k)\}$ is bounded. Thus, there exists a subsequence $\{(u^{k_i}, v^{k_i}, x^{k_i})\}$ converging to a cluster point, say $(u^\infty, v^\infty, x^\infty)$. Next we will prove that (u^∞, v^∞) is optimal to (1.2) and x^∞ is the corresponding multiplier.

The inequality (4.10) also implies that

$$\begin{aligned} \lim_{k\to\infty} \|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^k - c\| = 0, \quad \lim_{k\to\infty} (1-\tau)\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\| = 0, \\ \lim_{k\to\infty} \|\Delta w^{k+1}\|_{\mathcal{Q}+\mathrm{Diag}\,(\mathcal{S},\mathcal{T})} = 0, \quad \lim_{k\to\infty} \|\Delta v^{k+1}\|_{\mathcal{Q}_{22}} = 0. \end{aligned} \tag{4.31}$$

For $\tau \in (0,1)$, since $\lim_{k\to\infty} \|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\| = 0$, by using (4.31) we see that

$$\begin{aligned} &\lim_{k\to\infty} \|\mathcal{A}^* \Delta u^{k+1}\| \le \lim_{k\to\infty} (\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^k - c\| + \|\mathcal{A}^* u^k + \mathcal{B}^* v^k - c\|) = 0, \\ &\lim_{k\to\infty} \|\mathcal{B}^* \Delta v^{k+1}\| \le \lim_{k\to\infty} (\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\| + \|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^k - c\|) = 0, \\ &\lim_{k_i\to\infty} \|\Delta u^{k+1}\|_{\mathcal{Q}_{11}} \le \lim_{k_i\to\infty} (\|\Delta w^{k+1}\|_{\mathcal{Q}} + \|\Delta v^{k+1}\|_{\mathcal{Q}_{22}}) = 0, \end{aligned} \tag{4.32}$$

which implies $\lim_{k\to\infty} \|\Delta v^{k+1}\|_{\mathcal{T}+\sigma\mathcal{B}\mathcal{B}^*+\mathcal{Q}_{22}} = 0$ and $\lim_{k\to\infty} \|\Delta u^{k+1}\|_{\mathcal{S}+\sigma\mathcal{A}\mathcal{A}^*+\mathcal{Q}_{11}} = 0$. Therefore, for $\tau \in (0,1]$, we know that $\lim_{k\to\infty} \|\Delta v^{k+1}\|_{\mathcal{T}+(1-\tau)\sigma\mathcal{B}\mathcal{B}^*+\mathcal{Q}_{22}} = 0$ and $\lim_{k\to\infty} \|\Delta u^{k+1}\|_{\mathcal{S}+(1-\tau)\sigma\mathcal{A}\mathcal{A}^*+\mathcal{Q}_{11}} = 0$. By the positive definiteness of $\mathcal{S}+(1-\tau)\sigma\mathcal{A}\mathcal{A}^*+\mathcal{Q}_{11}$ and $\mathcal{T}+(1-\tau)\sigma\mathcal{B}\mathcal{B}^*+\mathcal{Q}_{22}$, we could get that $\lim_{k\to\infty} \|\Delta u^{k+1}\| = 0$ and $\lim_{k\to\infty} \|\Delta v^{k+1}\| = 0$.

Now taking limits on both sides of (4.12) along the subsequence $\{(u^{k_i}, v^{k_i}, x^{k_i})\}$, and by using the closedness of the graphs of ∂p, ∂q and the continuity of $\nabla\phi$, we obtain

$$0 \in F(u^\infty, v^\infty, x^\infty), \quad \mathcal{A}^* u^\infty + \mathcal{B}^* v^\infty = c.$$

This indicates that (u^∞, v^∞) is an optimal solution to (1.2) and x^∞ is the corresponding multiplier.

Since $(u^\infty, v^\infty, x^\infty)$ satisfies (4.3), all the above arguments involving $(\bar{u}, \bar{v}, \bar{x})$ can be replaced by $(u^\infty, v^\infty, x^\infty)$. Thus the subsequence $\{\Phi_{k_i}(u^\infty, v^\infty, x^\infty)\}$ converges to

0 as $k_i \to \infty$. Since $\{\Phi_{k_i}(u^\infty, v^\infty, x^\infty)\}$ is non-increasing, we obtain that

$$\begin{aligned}\lim_{k\to\infty} \Phi_{k+1}(u^\infty, v^\infty, x^\infty) &= \lim_{k\to\infty} \ (\tau\sigma)^{-1}\|x^{k+1} - x^\infty\|^2 + \|v^{k+1} - v^\infty\|^2_{\sigma\mathcal{B}\mathcal{B}^*+\mathcal{T}+\mathcal{Q}_{22}} \\ &\qquad +\|u^{k+1} - u^\infty\|^2_{\mathcal{S}} + \|w^{k+1} - w^\infty\|^2_{\mathcal{Q}} \\ &= 0.\end{aligned} \tag{4.33}$$

From this we can immediately get $\lim_{k\to\infty} x^{k+1} = x^\infty$ and $\lim_{k\to\infty} v^{k+1} = v^\infty$. Similar to inequality (4.32) we have that $\lim_{k\to\infty} \sigma\|\mathcal{A}^*(u^{k+1} - u^\infty)\| = 0$ and $\lim_{k\to\infty} \|u^{k+1} - u^\infty\|_{\mathcal{Q}_{11}} = 0$, which, together with (4.33), imply that $\lim_{k\to\infty} \|u^{k+1} - u^\infty\| = 0$ by the positive definiteness of $\mathcal{Q}_{11}+\mathcal{S}+\sigma\mathcal{A}\mathcal{A}^*$. Therefore, the whole sequence $\{(u^k, v^k, x^k)\}$ converges to $(u^\infty, v^\infty, x^\infty)$, the unique limit of the sequence. This completes the proof for the first case.

(ii) From the inequality (4.11) and the assumptions $\tau \in (0, \frac{1+\sqrt{5}}{2})$ and $\mathcal{M} \succeq 0$, we can obtain that $\Gamma_{k+1} \geq 0$ and $\min(1, 1 + \tau^{-1} - \tau) \geq 0$. Then both $\{\Psi_{k+1}(\bar{u}, \bar{v}, \bar{x})\}$ and $\{\Xi_{k+1}\}$ are bounded. Thus, by a similar approach to case (i), we see that the sequence $\{(u^k, v^k, x^k)\}$ is bounded. Therefore, there exists a subsequence $\{(u^{k_i}, v^{k_i}, x^{k_i})\}$ that converges to a cluster point, say $(u^\infty, v^\infty, x^\infty)$. Next we will prove that (u^∞, v^∞) is optimal to problem (1.2) and x^∞ is the corresponding multiplier. The inequality (4.11) also implies that

$$\begin{aligned}&\lim_{k\to\infty} \|\Delta x^{k+1}\| = \lim_{k\to\infty} (\tau\sigma)^{-1}\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\| = 0, \\ &\lim_{k\to\infty} \|\Delta w^{k+1}\|_{\mathcal{M}} = 0, \quad \lim_{k\to\infty} \|\Delta v^{k+1}\|_{\sigma\mathcal{B}\mathcal{B}^*+\mathcal{Q}_{22}+\mathcal{T}} = 0, \\ &\lim_{k\to\infty} \|\Delta w^{k+1}\|_{\mathcal{Q}} = 0, \quad \lim_{k\to\infty} \|\Delta u^{k+1}\|_{\mathcal{S}} = 0.\end{aligned}$$

By the relationship that

$$\begin{aligned}\lim_{k\to\infty} \|\mathcal{A}^*\Delta u^{k+1}\| &\leq \lim_{k\to\infty} (\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\| + \|\mathcal{A}^* u^k + \mathcal{B}^* v^k - c\| + \|\mathcal{B}^*\Delta v^{k+1}\|) \\ &= 0, \\ \lim_{k_i\to\infty} \|\Delta u^{k+1}\|_{\mathcal{Q}_{11}} &\leq \lim_{k_i\to\infty} (\|\Delta w^{k+1}\|_{\mathcal{Q}} + \|\Delta v^{k+1}\|_{\mathcal{Q}_{22}}) = 0,\end{aligned}$$

we can further get $\lim_{k\to\infty} \|\Delta u^{k+1}\|_{\mathcal{Q}_{11}+\sigma\mathcal{A}\mathcal{A}^*+\mathcal{S}} = 0$ and $\lim_{k\to\infty} \|\Delta v^{k+1}\|_{\mathcal{Q}_{22}+\sigma\mathcal{B}\mathcal{B}^*+\mathcal{T}} = 0$. Thus, by taking a subsequence of $\{(u^{k_i}, v^{k_i})\}$ if necessary, we can get $\lim_{k\to\infty} \|u^{k_i+1} -$

$u^{k_i}\| = 0$ and $\lim\limits_{k\to\infty} \|v^{k_i+1} - v^{k_i}\| = 0$. The remaining proof about the convergence of the whole sequence $\{(u^k, v^k, x^k)\}$ follows exactly the same as in case (i). This completes the proof for the second case. $\square$

Remark 4.1. *An interesting application of Theorem 4.1 is for the linearly constrained convex optimization problem with a quadratically coupled objective function of the form*

$$\phi(w) = \frac{1}{2}\langle w, \widetilde{\mathcal{Q}}w\rangle + f(u) + g(v),$$

where $\widetilde{\mathcal{Q}} : \mathcal{U} \times \mathcal{V} \to \mathcal{U} \times \mathcal{V}$ is a self-adjoint positive semidefinite linear operator, $f : \mathcal{U} \to (-\infty, \infty)$ and $g : \mathcal{V} \to (-\infty, \infty)$ are two convex smooth functions with Lipschitz continuous gradients. In this case, there exist four self-adjoint positive semidefinite operators $\Sigma_f, \widehat{\Sigma}_f : \mathcal{U} \to \mathcal{U}$ and $\Sigma_g, \widehat{\Sigma}_g : \mathcal{V} \to \mathcal{V}$ such that

$$\Sigma_f \preceq \xi \preceq \widehat{\Sigma}_f, \quad \forall \xi \in \partial^2 f(u),\ u \in \mathcal{U} \quad \text{and} \quad \Sigma_g \preceq \zeta \preceq \widehat{\Sigma}_g, \quad \forall \zeta \in \partial^2 g(v),\ v \in \mathcal{V},$$

where $\partial^2 f$ and $\partial^2 g$ are defined in (2.1). Then by letting $\mathcal{Q} = \widetilde{\mathcal{Q}} + Diag(\Sigma_f, \Sigma_g)$ in (2.3) and $\mathcal{Q} + \mathcal{H} = \widetilde{\mathcal{Q}} + Diag(\widehat{\Sigma}_f, \widehat{\Sigma}_g)$ in (2.4), we have $\eta = 0$ in (4.2). This implies that $\mathcal{M} \succeq 0$ always holds in (4.30). Therefore, for $\tau \in (0, \frac{1+\sqrt{5}}{2})$, the conditions for the convergence can be equivalently written as

$$\widetilde{\mathcal{Q}}_{11} + \Sigma_f + \mathcal{S} + \sigma\mathcal{A}\mathcal{A}^* \succ 0, \ \ \widetilde{\mathcal{Q}}_{22} + \Sigma_g + \mathcal{T} + \sigma\mathcal{B}\mathcal{B}^* \succ 0. \tag{4.34}$$

Note that (4.34) is necessary for the global convergence of the majorized ADMM even if $\widetilde{\mathcal{Q}} = 0$, i.e., the objective function of the original problem (1.2) is separable. Therefore, we recover the convergence conditions given in [57] for a majorized ADMM with semi-proximal terms.

4.2.2 The iteration complexity

In this section, we will present various non-ergodic and ergodic iteration complexity for the mADMM. The first part is devoted to the non-ergodic analysis, which shows the $O(1/k)$ (and $o(1/k)$) complexity in terms of the KKT optimality condition. The second part discuss the $O(1/k)$ ergodic complexity of the primal feasibility and the objective value.

Before showing the main result, we first present the following Lemma, which shows the decreasing property of the difference between two consecutive iteration points when the step length $\tau = 1$. This property has also been discussed by He and Yuan [42] for the classic ADMM with $\tau = 1$.

Lemma 4.1. *Assume that $\tau = 1$. Then for any $k \geq 0$, we have that*

$$\begin{aligned} &\|\Delta x^{k+1}\|^2_{\sigma^{-1}\mathcal{I}} + \|\Delta u^{k+1}\|^2_{\mathcal{S}} + \|\Delta v^{k+1}\|^2_{\mathcal{T}+\sigma\mathcal{B}\mathcal{B}^*+\mathcal{Q}_{22}} + \|\Delta w^{k+1}\|^2_{\widehat{\mathcal{Q}}} \\ &\qquad \leq \|\Delta x^{k}\|^2_{\sigma^{-1}\mathcal{I}} + \|\Delta u^{k}\|^2_{\mathcal{S}} + \|\Delta v^{k}\|^2_{\mathcal{T}+\sigma\mathcal{B}\mathcal{B}^*+\mathcal{Q}_{22}} + \|\Delta w^{k}\|^2_{\widehat{\mathcal{Q}}}. \end{aligned} \tag{4.35}$$

Proof. Since the step-length $\tau = 1$, the optimality conditions at the $(k+1)$th and kth iteration can be written as

$$\left\{ \begin{array}{l} -\mathcal{A}x^{k+1} - \nabla_u \phi(w^k) - (\mathcal{Q}_{11} + \mathcal{D}_1 + \mathcal{S})\Delta u^{k+1} + \sigma\mathcal{A}\mathcal{B}^*\Delta v^{k+1} \in \partial p(u^{k+1}), \\ -\mathcal{B}x^{k+1} - \nabla_v \phi(w^k) - (\mathcal{Q}_{22} + \mathcal{D}_2 + \mathcal{T})\Delta v^{k+1} - \mathcal{Q}_{12}^*\Delta u^{k+1} \in \partial q(v^{k+1}), \end{array} \right.$$

and

$$\left\{ \begin{array}{l} -\mathcal{A}x^{k} - \nabla_u \phi(w^{k-1}) - (\mathcal{Q}_{11} + \mathcal{D}_1 + \mathcal{S})\Delta u^{k} + \sigma\mathcal{A}\mathcal{B}^*\Delta v^{k} \in \partial p(u^{k}), \\ -\mathcal{B}x^{k} - \nabla_v \phi(w^{k-1}) - (\mathcal{Q}_{22} + \mathcal{D}_2 + \mathcal{T})\Delta v^{k} - \mathcal{Q}_{12}^*\Delta u^{k} \in \partial q(v^{k}), \end{array} \right.$$

By the monotonicity of the subdifferential of the convex functions p and q, we have the following inequality:

$$\begin{aligned} &\langle \Delta u^{k+1}, \mathcal{A}\Delta x^{k+1} + (\nabla_u \phi(w^k) - \nabla_u \phi(w^{k-1})) + (\mathcal{Q}_{11} + \mathcal{D}_1 + \mathcal{S})(\Delta u^{k+1} - \Delta u^k) \\ &\qquad -\sigma\mathcal{A}\mathcal{B}^*(\Delta v^{k+1} - \Delta v^k)\rangle \leq 0, \\ &\langle \Delta v^{k+1}, \mathcal{B}\Delta x^{k+1} + (\nabla_v \phi(w^k) - \nabla_v \phi(w^{k-1})) + (\mathcal{Q}_{22} + \mathcal{D}_2 + \mathcal{T})(\Delta v^{k+1} - \Delta v^k) \\ &\qquad +\mathcal{Q}_{12}^*(\Delta u^{k+1} - \Delta u^k)\rangle \leq 0. \end{aligned}$$

Adding the above two inequalities together, and by yielding the fact that

$$\Delta x^{k+1} - \Delta x^k = \sigma(\mathcal{A}^*\Delta u^{k+1} + \mathcal{B}^*\Delta v^{k+1}),$$

and the relationship that $\widehat{\mathcal{Q}} - \mathcal{Q} = \mathrm{Diag}(\mathcal{D}_1, \mathcal{D}_2)$, we get

$$\begin{aligned}\sigma^{-1}\langle \Delta x^{k+1}, \Delta x^{k+1} - \Delta x^k\rangle + \langle \Delta u^{k+1}, \mathcal{S}(\Delta u^{k+1} - \Delta u^k)\rangle + \langle \Delta v^{k+1}, \mathcal{T}(\Delta v^{k+1} - \Delta v^k)\rangle \\ +\langle \Delta w^{k+1}, \nabla\phi(w^k) - \nabla\phi(w^{k-1}) + \widehat{\mathcal{Q}}(\Delta w^{k+1} - \Delta w^k)\rangle \\ -\langle \Delta u^{k+1}, (\sigma\mathcal{A}\mathcal{B}^* + \mathcal{Q}_{12})(\Delta v^{k+1} - \Delta v^k)\rangle \le 0.\end{aligned} \tag{4.36}$$

By the globally Lipschitz continuous property of $\nabla\phi$, there exists a self-adjoint positive semidefinite linear operator $\mathcal{W}^k \in \mathrm{conv}\{\partial^2\phi([w^{k-1}, w^k])\}$ such that

$$\nabla\phi(w^k) - \phi(w^{k-1}) = \mathcal{W}^k \Delta w^k.$$

Substituting the above equation into (4.36) and by recalling the identity (4.7), we see that

$$\begin{aligned}&(\|\Delta x^{k+1}\|^2_{\sigma^{-1}\mathcal{I}} - \|\Delta x^k\|^2_{\sigma^{-1}\mathcal{I}} + \|\Delta x^{k+1} - \Delta x^k\|^2_{\sigma^{-1}\mathcal{I}}) + (\|\Delta u^{k+1}\|^2_{\mathcal{S}} - \|\Delta u^k\|^2_{\mathcal{S}} \\ &+\|\Delta u^{k+1} - \Delta u^k\|^2_{\mathcal{S}}) + (\|\Delta v^{k+1}\|^2_{\mathcal{T}} - \|\Delta v^k\|^2_{\mathcal{T}} + \|\Delta v^{k+1} - \Delta v^k\|^2_{\mathcal{T}}) + 2\|\Delta w^{k+1}\|^2_{\widehat{\mathcal{Q}}} \\ &-(\|\Delta w^{k+1}\|^2_{\widehat{\mathcal{Q}}-\mathcal{W}^k} - \|\Delta w^{k+1} - \Delta w^k\|^2_{\widehat{\mathcal{Q}}-\mathcal{W}^k} + \|\Delta w^k\|^2_{\widehat{\mathcal{Q}}-\mathcal{W}^k}) - (\|\Delta w^{k+1}\|^2_{\mathcal{Q}} \\ &+\|\Delta v^{k+1} - \Delta v^k\|^2_{\mathcal{Q}_{22}}) + (\|\Delta v^{k+1}\|^2_{\mathcal{Q}_{22}} + \|\Delta v^{k+1} - \Delta v^k\|^2_{\mathcal{Q}_{22}} - \|\Delta v^k\|^2_{\mathcal{Q}_{22}}) + (\|\Delta u^{k+1}\|^2_{\sigma\mathcal{A}\mathcal{A}^*} \\ &+\|\Delta v^{k+1}\|^2_{\sigma\mathcal{B}\mathcal{B}^*} - \|\Delta x^{k+1} - \Delta x^k\|^2_{\sigma^{-1}\mathcal{I}}) - (\|\Delta u^{k+1}\|^2_{\sigma\mathcal{A}\mathcal{A}^*} + \|\Delta v^k\|^2_{\sigma\mathcal{B}\mathcal{B}^*}) \le 0\end{aligned}$$

By rearranging the terms, the above inequality can be recast as

$$\begin{aligned}&\|\Delta x^{k+1}\|^2_{\sigma^{-1}\mathcal{I}} + \|\Delta u^{k+1}\|^2_{\mathcal{S}} + \|\Delta v^{k+1}\|^2_{\mathcal{T}+\sigma\mathcal{B}\mathcal{B}^*+\mathcal{Q}_{22}} + \|\Delta w^{k+1}\|^2_{\widehat{\mathcal{Q}}} \\ \le\ &\|\Delta x^k\|^2_{\sigma^{-1}\mathcal{I}} + \|\Delta u^k\|^2_{\mathcal{S}} + \|\Delta v^k\|^2_{\mathcal{T}+\sigma\mathcal{B}\mathcal{B}^*+\mathcal{Q}_{22}} + \|\Delta w^k\|^2_{\widehat{\mathcal{Q}}} - (\|\Delta u^{k+1} - \Delta u^k\|^2_{\mathcal{S}} \\ &+\|\Delta v^{k+1} - \Delta v^k\|^2_{\mathcal{T}} + \|\Delta w^{k+1} - \Delta w^k\|^2_{\widehat{\mathcal{Q}}-\mathcal{W}^k} + \|\Delta w^{k+1}\|^2_{\mathcal{W}^k-\mathcal{Q}} + \|\Delta w^k\|^2_{\mathcal{W}_k}) \\ \le\ &\|\Delta x^k\|^2_{\sigma^{-1}\mathcal{I}} + \|\Delta u^k\|^2_{\mathcal{S}} + \|\Delta v^k\|^2_{\mathcal{T}+\sigma\mathcal{B}\mathcal{B}^*+\mathcal{Q}_{22}} + \|\Delta w^k\|^2_{\widehat{\mathcal{Q}}},\end{aligned}$$

where the last inequality is obtained by (2.2). □

Denote two operators $\mathcal{O}_1 : \mathcal{U}\times\mathcal{V} \to \mathcal{U}\times\mathcal{V}$ and $\mathcal{O}_2 : \mathcal{U}\times\mathcal{V} \to \mathcal{U}\times\mathcal{V}$ as:

$$\mathcal{O}_1 := \frac{1}{8}\mathcal{Q} + \mathrm{Diag}(\mathcal{S} + (1-\tau)\sigma\mathcal{A}\mathcal{A}^*, \mathcal{T} + \frac{1}{6}\mathcal{Q}_{22} + (1-\tau)\sigma\mathcal{B}\mathcal{B}^*),$$

$$\mathcal{O}_2 := \frac{1}{8}\mathcal{Q} + \mathrm{Diag}\left(\mathcal{S} - \eta\mathcal{D}_1, \mathcal{T} + \frac{1}{6}\mathcal{Q}_{22} + \rho(\tau)\sigma\mathcal{B}\mathcal{B}^* - \eta\mathcal{D}_2\right) + \frac{1}{6}\rho(\tau)\tau^{-1}\sigma\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\end{pmatrix}(\mathcal{A}^*\ \mathcal{B}^*).$$

To prove the global nonergodic complexity of the mADMM, the following properties are also essential.

Lemma 4.2. *We have the following equivalent characterization of the positive definite properties of the operators:*

(i) For $\tau \in (0,1]$, *it holds that*

$$\mathcal{Q}_{11} + (1-\tau)\sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0 \quad \text{and} \quad \mathcal{Q}_{22} + (1-\tau)\sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0 \Longleftrightarrow \mathcal{O}_1 \succ 0.$$

(ii) For $\tau \in (0, \frac{\sqrt{5}+1}{2})$, *if there exists* $\alpha \in [0,1)$ *such that*

$$\frac{\alpha}{8}\mathcal{Q} + Diag\,(\alpha\mathcal{S} - \eta\mathcal{D}_1, \alpha\mathcal{T} - \eta\mathcal{D}_2) \succeq 0,$$

then we have

$$\mathcal{Q}_{11} + \sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0, \quad \mathcal{Q}_{22} + \sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0 \Longleftrightarrow \mathcal{O}_2 \succ 0.$$

Proof. (i) First, let us assume that $\mathcal{Q}_{11} + (1-\tau)\sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0$ and $\mathcal{Q}_{22} + (1-\tau)\sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0$. We prove $\mathcal{O}_1 \succ 0$ by contradiction. If there exists non-zero $w = (u,v) \in \mathcal{U} \times \mathcal{V}$ such that $\langle w, \mathcal{O}_1 w\rangle = 0$, by noting that $\tau \in (0,1]$, we have

$$\|w\|^2_{\mathcal{Q}} = 0, \quad \|u\|^2_{\mathcal{S}+(1-\tau)\sigma\mathcal{A}\mathcal{A}^*} = 0, \quad \|v\|^2_{\mathcal{T}+\mathcal{Q}_{22}+(1-\tau)\sigma\mathcal{B}\mathcal{B}^*} = 0.$$

This further implies $v = 0$ due to the assumption $\mathcal{Q}_{22} + (1-\tau)\sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0$. Then by substituting $v = 0$ into the above equations, we have $\|u\|^2_{\mathcal{Q}_{11}+(1-\tau)\mathcal{A}\mathcal{A}^*+\mathcal{S}} = 0$, so that $u = 0$ because of $\mathcal{Q}_{11} + (1-\tau)\sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0$. This contradicts the assumption that $w \neq 0$. Thus, we see $\mathcal{O}_1 \succ 0$.

Conversely, assume $\mathcal{O}_1 \succ 0$. Then for any $0 \neq \tilde{v} \in \mathcal{V}$, by letting $\tilde{w} := (0, \tilde{v}) \in \mathcal{U} \times \mathcal{V}$ we have $\langle \tilde{w}, \mathcal{O}_1 \tilde{w}\rangle = \langle \tilde{v}, (\frac{7}{24}\mathcal{Q}_{22} + (1-\tau)\sigma\mathcal{B}\mathcal{B}^* + \mathcal{T})\tilde{v}\rangle > 0$. Thus, we get $\frac{7}{24}\mathcal{Q}_{22} + (1-\tau)\sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0$, or equivalently, $\mathcal{Q}_{22} + (1-\tau)\sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0$. By the same approach we can also prove that $\mathcal{Q}_{11} + (1-\tau)\sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0$. This completes the proof.

(ii) The proof of this part is similar with (i) and hence, we only provide a brief one here. Suppose that $\mathcal{Q}_{11} + \sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0$ and $\mathcal{Q}_{22} + \sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0$, but $\mathcal{O}_2 \not\succ 0$. Then there exists nonzero $w = (u, v) \in \mathcal{U} \times \mathcal{V}$ such that $\|v\|^2_{\mathcal{T}+\mathcal{Q}_{22}+\sigma\mathcal{B}\mathcal{B}^*+(\frac{\alpha}{8}\mathcal{Q}_{22}+\alpha\mathcal{T}-\eta\mathcal{D}_2)} = 0$, so that $v = 0$. Moreover, we can obtain $\|u\|^2_{\mathcal{S}+\mathcal{Q}_{11}+\sigma\mathcal{A}\mathcal{A}^*+(\frac{\alpha}{8}\mathcal{Q}_{11}+\alpha\mathcal{S}-\eta\mathcal{D}_1)} = 0$, thus, $u = 0$. This contradicts with the assumption that $w = 0$. For the reverse direction, if $\mathcal{O}_2 \succ 0$, then by the same approach with (i) we could obtain $\mathcal{Q}_{11}+\mathcal{S}+\sigma\mathcal{A}\mathcal{A}^*-\eta\mathcal{D}_1 \succ 0$ and $\mathcal{Q}_{22}+\mathcal{T}+\sigma\mathcal{B}\mathcal{B}^*-\eta\mathcal{D}_2 \succ 0$, which further implies $\mathcal{Q}_{11}+\mathcal{S}+\sigma\mathcal{A}\mathcal{A}^* \succ 0$ and $\mathcal{Q}_{22}+\mathcal{T}+\sigma\mathcal{B}\mathcal{B}^* \succ 0$. This completes the proof. $\square$

Theorem 4.2. *Suppose that the solution set of (1.2) is nonempty and Assumption 4.1 holds. Let one of the conditions for the global convergence in Theorem 4.1 holds, i.e., either (i) or (ii) holds:*

(i) $\tau \in (0, 1]$, $\mathcal{Q}_{11} + (1-\tau)\sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0$ and $\mathcal{Q}_{22} + (1-\tau)\sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0$.

(ii) $\tau \in (0, \frac{1+\sqrt{5}}{2})$, $\mathcal{Q}_{11} + \sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0$, $\mathcal{Q}_{22} + \sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0$ *and there exists some* $\alpha \in [0, 1)$ *such that* $\frac{\alpha}{8}\mathcal{Q} + \mathit{Diag}\,(\alpha\mathcal{S} - \eta\mathcal{D}_1, \alpha\mathcal{T} - \eta\mathcal{D}_2) \succeq 0$.

Then there exists a constant C only depending on the initial point and the optimal solution set such that the sequence $\{(u^k, v^k, x^k)\}$ generated by the majorized ADMM satisfies that for $k \geq 1$,

$$\min_{1\leq i\leq k}\{\mathit{dist}^2(0, F(u^{i+1}, v^{i+1}, x^{i+1})) + \|\mathcal{A}^*u^{i+1} + \mathcal{B}^*v^{i+1} - c\|^2\} \leq C/k. \tag{4.37}$$

and for the limiting case we have that

$$\lim_{k\to\infty} k(\min_{1\leq i\leq k}\{\mathit{dist}^2(0, F(u^{i+1}, v^{i+1}, x^{i+1})) + \|\mathcal{A}^*u^{i+1} + \mathcal{B}^*v^{i+1} - c\|^2\}) = 0. \tag{4.38}$$

Furthermore, when the step length $\tau = 1$, it holds that

$$\lim_{k\to\infty} k(\mathit{dist}^2(0, F(u^{k+1}, v^{k+1}, x^{k+1})) + \|\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c\|^2) = 0. \tag{4.39}$$

i.e., the "$\min\limits_{1\leq i\leq k}$" can be removed from (4.38).

Proof. From the optimality condition for (u^{k+1}, v^{k+1}), we know that

$$\begin{pmatrix} -(1-\tau)\sigma\mathcal{A}(\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c) + (\sigma\mathcal{A}\mathcal{B}^* + \mathcal{Q}_{12})\Delta v^{k+1} - \mathcal{S}\Delta u^{k+1} \\ -(1-\tau)\sigma\mathcal{B}(\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c) - \mathcal{T}\Delta v^{k+1} \end{pmatrix}$$
$$-(\mathcal{Q} + \mathcal{H})\Delta w^{k+1} + \nabla\phi(w^{k+1}) - \nabla\phi(w^k) \in F(u^{k+1}, v^{k+1}, x^{k+1}).$$

Therefore, we can obtain that

$$
\begin{aligned}
& \mathrm{dist}^2(0, F(u^{k+1}, v^{k+1}, x^{k+1})) + \|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2 \\
\leq \; & 5\|\sigma \mathcal{A}\mathcal{B}^* \Delta v^{k+1}\|^2 + 5(1-\tau)^2\sigma^2(\|\mathcal{A}\|^2 + \|\mathcal{B}\|^2)\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2 \\
& +5\|(\mathcal{Q}+\mathcal{H})\Delta w^{k+1} - \nabla\phi(w^{k+1}) + \nabla\phi(w^k)\|^2 + 5\|\mathcal{Q}_{12}\Delta v^{k+1}\|^2 + 5\|\mathcal{T}\Delta v^{k+1}\|^2 \\
& +5\|\mathcal{S}\Delta u^{k+1}\|^2 + \|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2 \\
\leq \; & 5\sigma\|\mathcal{A}\|^2\|\Delta v^{k+1}\|^2_{\sigma\mathcal{B}\mathcal{B}^*} + (5(1-\tau)^2\sigma^2(\|\mathcal{A}\|^2 + \|\mathcal{B}\|^2) + 1)\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2 \\
& +5\|\sqrt{\mathcal{Q}_{12}^*\mathcal{Q}_{12}}\|\|\Delta v^{k+1}\|^2_{\sqrt{\mathcal{Q}_{12}^*\mathcal{Q}_{12}}} + 5\|\mathcal{H}\|\|\Delta w^{k+1}\|^2_{\mathcal{H}} + 5\|\mathcal{S}\|\|\Delta u^{k+1}\|^2_{\mathcal{S}} \\
& +5\|\mathcal{T}\|\|\Delta v^{k+1}\|^2_{\mathcal{T}} \\
\leq \; & C_1\|\Delta w^{k+1}\|^2_{\widehat{\mathcal{O}}} + C_2\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2,
\end{aligned}
\tag{4.40}
$$

where

$$
\begin{aligned}
& C_1 = 5\max(\sigma\|\mathcal{A}\|^2, \|\sqrt{\mathcal{Q}_{12}^*\mathcal{Q}_{12}}\|, \|\mathcal{H}\|, \|\mathcal{S}\|, \|\mathcal{T}\|), \\
& C_2 = 5(1-\tau)^2\sigma^2(\|\mathcal{A}\|^2 + \|\mathcal{B}\|^2) + 1, \\
& \widehat{\mathcal{O}} = \mathcal{H} + \mathrm{Diag}\,(\mathcal{S}, \mathcal{T} + \sigma\mathcal{B}\mathcal{B}^* + \sqrt{\mathcal{Q}_{12}^*\mathcal{Q}_{12}})
\end{aligned}
$$

and the second inequality comes from the fact that there exists some

$$
\mathcal{W}^k \in \mathrm{conv}\{\partial^2\phi([w^{k-1}, w^k])\}
$$

such that

$$
\begin{aligned}
& \|(\mathcal{Q}+\mathcal{H})\Delta w^{k+1} - \nabla\phi(w^{k+1}) + \nabla\phi(w^k)\|^2 \\
= \; & \|(\mathcal{Q}+\mathcal{H}-\mathcal{W}^k)\Delta w^{k+1}\|^2 \leq \|\mathcal{H}\|\|\Delta w^{k+1}\|^2_{\mathcal{H}}.
\end{aligned}
$$

Next we will estimate the upper bounds for $\|\Delta w^{k+1}\|^2_{\widehat{\mathcal{O}}}$ and $\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2$ by only involving the initial point and the optimal solution set under the two different conditions.

First, assume condition (i) holds. For $\tau \in (0, 1]$, by using (4.10) we have that for any $i \geq 1$,

$$\begin{aligned}
&\|\Delta w^{i+1}\|^2_{\frac{1}{8}\mathcal{Q}+\mathrm{Diag}\,(\mathcal{S},\mathcal{T}+\frac{1}{6}\mathcal{Q}_{22})} + \sigma\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^i-c\|^2 + \tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^{i+1}-c\|^2 \\
\leq\ & (\Phi_i(\bar{u},\bar{v},\bar{x}) + \tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^{k+1}+\mathcal{B}^*v_e^{k+1}\|^2) \\
& \qquad -(\Phi_{i+1}(\bar{u},\bar{v},\bar{x}) + \tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^{k}+\mathcal{B}^*v_e^{k}\|^2),
\end{aligned}$$

which, implies that,

$$\begin{aligned}
&\sum_{i=1}^{k}\big(\|\Delta w^{i+1}\|^2_{\frac{1}{8}\mathcal{Q}+\mathrm{Diag}\,(\mathcal{S},\mathcal{T}+\frac{1}{6}\mathcal{Q}_{22})} + \sigma\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^i-c\|^2 \\
&\qquad\qquad + \tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^{i+1}-c\|^2\big) \\
\leq\ & (\Phi_1(\bar{u},\bar{v},\bar{x}) + \tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^{1}+\mathcal{B}^*v_e^{1}\|^2) - (\Phi_{k+1}(\bar{u},\bar{v},\bar{x}) + \tfrac{1}{2}\|\mathcal{A}^*u_e^{k+1}+\mathcal{B}^*v_e^{k+1}\|^2) \\
\leq\ & \Phi_1(\bar{u},\bar{v},\bar{x}) + \tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^{1}+\mathcal{B}^*v_e^{1}\|^2.
\end{aligned}$$

This shows that

$$\begin{aligned}
\sum_{i=1}^{k}\|\Delta w^{i+1}\|^2_{\frac{1}{8}\mathcal{Q}+\mathrm{Diag}(\mathcal{S},\mathcal{T}+\frac{1}{6}\mathcal{Q}_{22})} &\leq \Phi_1(\bar{u},\bar{v},\bar{x}) + \frac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^{1}+\mathcal{B}^*v_e^{1}\|^2, \\
\sum_{i=1}^{k}\sigma\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^i-c\|^2 &\leq \Phi_1(\bar{u},\bar{v},\bar{x}) + \frac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^{1}+\mathcal{B}^*v_e^{1}\|^2, \\
\sum_{i=1}^{k}\frac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^{i+1}-c\|^2 &\leq \Phi_1(\bar{u},\bar{v},\bar{x}) + \frac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^{1}+\mathcal{B}^*v_e^{1}\|^2.
\end{aligned} \tag{4.41}$$

From the above three inequalities we can also get that

$$\begin{aligned}
&(1-\tau)\sum_{i=1}^{k}\|\Delta u^{i+1}\|^2_{\sigma\mathcal{A}\mathcal{A}^*} \\
\leq\ & (1-\tau)\sum_{i=1}^{k}(2\sigma\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^i-c\|^2 + 2\sigma\|\mathcal{A}^*u^{i}+\mathcal{B}^*v^i-c\|^2) \\
\leq\ & (6-2\tau)(\Phi_1(\bar{u},\bar{v},\bar{x}) + \tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^{1}+\mathcal{B}^*v_e^{1}\|^2),
\end{aligned}$$

and similarly,

$$\begin{aligned}
&(1-\tau)\sum_{i=1}^{k}\|\Delta v^{i+1}\|^2_{\sigma\mathcal{B}\mathcal{B}^*} \\
\leq\ & (1-\tau)\sum_{i=1}^{k}(2\sigma\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^i-c\|^2 + 2\sigma\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^{i+1}-c\|^2) \\
\leq\ & (6-2\tau)(\Phi_1(\bar{u},\bar{v},\bar{x}) + \tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^{1}+\mathcal{B}^*v_e^{1}\|^2).
\end{aligned}$$

With the notation of operator $\mathcal{O}_1$ we have that

$$\begin{aligned}\sum_{i=1}^{k}\|\Delta w^{i+1}\|^2_{\mathcal{O}_1} &= \sum_{i=1}^{k}\|\Delta w^{i+1}\|^2_{\frac{1}{8}\mathcal{Q}+\mathrm{Diag}(\mathcal{S},\mathcal{T}+\frac{1}{6}\mathcal{Q}_{22})}+\sum_{i=1}^{k}\|\Delta w^{i+1}\|^2_{(1-\tau)\sigma\mathrm{Diag}(\mathcal{A}\mathcal{A}^*,\mathcal{B}\mathcal{B}^*)}\\ &\leq (13-4\tau)(\Phi_1(\bar{u},\bar{v},\bar{x})+\tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^1+\mathcal{B}^*v_e^1\|^2).\end{aligned}\tag{4.42}$$

If $\tau\in(0,1)$, we further have that

$$\sum_{i=1}^{k}\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^{i+1}-c\|^2\leq 2(1-\tau)^{-1}\sigma^{-1}\Phi_1(\bar{u},\bar{v},\bar{x})+\|\mathcal{A}^*u_e^1+\mathcal{B}^*v_e^1\|^2.\tag{4.43}$$

If $\tau=1$, by the condition that $\mathcal{O}_1=\dfrac{1}{8}\mathcal{Q}+\mathrm{Diag}\,(\mathcal{S},\mathcal{T})\succ 0$, we have that

$$\begin{aligned}&\sum_{i=1}^{k}\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^{i+1}-c\|^2\\ \leq\ &\sum_{i=1}^{k}(2\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^{i}-c\|^2+2\|\Delta v^{i+1}\|^2_{\mathcal{B}\mathcal{B}^*})\\ \leq\ &\sum_{i=1}^{k}(2\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^{i}-c\|^2+2\|\mathcal{O}_1^{-\frac{1}{2}}\mathrm{Diag}\,(0,\mathcal{B}\mathcal{B}^*)\mathcal{O}_1^{-\frac{1}{2}}\|\|\Delta w^{i+1}\|^2_{\mathcal{O}_1})\\ \leq\ &(2\sigma^{-1}+(26-8\tau)\|\mathcal{O}_1^{-\frac{1}{2}}\mathrm{Diag}\,(0,\mathcal{B}\mathcal{B}^*)\mathcal{O}_1^{-\frac{1}{2}}\|)(\Phi_1(\bar{u},\bar{v},\bar{x})+\tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^1+\mathcal{B}^*v_e^1\|^2),\end{aligned}\tag{4.44}$$

where the second inequality is obtained by the fact that for any ξ, a self-adjoint positive definite operator $\mathcal{G}$ with square root $\mathcal{G}^{\frac{1}{2}}$ and a self-adjoint positive semidefinite operator $\widehat{G}$ defined in the same Hilbert space, it always holds that $\|\xi\|^2_{\widehat{\mathcal{G}}}=\langle\xi,\widehat{\mathcal{G}}\xi\rangle=\langle\xi,(\mathcal{G}^{\frac{1}{2}}\mathcal{G}^{-\frac{1}{2}})\widehat{\mathcal{G}}(\mathcal{G}^{-\frac{1}{2}}\mathcal{G}^{\frac{1}{2}})\xi\rangle=\langle\mathcal{G}^{\frac{1}{2}}\xi,(\mathcal{G}^{-\frac{1}{2}}\widehat{\mathcal{G}}\mathcal{G}^{-\frac{1}{2}})\mathcal{G}^{\frac{1}{2}}\xi\rangle\leq\|\mathcal{G}^{-\frac{1}{2}}\widehat{\mathcal{G}}\mathcal{G}^{-\frac{1}{2}}\|\|\xi\|^2_{\mathcal{G}}$.

Therefore, by using (4.40), (4.42) and the positive definiteness of operator $\mathcal{O}_1$ due to the assumptions that $\mathcal{Q}_{11}+\mathcal{S}+(1-\tau)\sigma\mathcal{A}\mathcal{A}^*\succ 0$ and $\mathcal{Q}_{22}+\mathcal{T}+(1-\tau)\sigma\mathcal{B}\mathcal{B}^*\succ 0$, we know that

$$\begin{aligned}&\min_{1\leq i\leq k}\{\mathrm{dist}^2(0,F(u^{i+1},v^{i+1},x^{i+1}))+\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^{i+1}-c\|^2\}\\ \leq\ &(\sum_{i=1}^{k}(\mathrm{dist}^2(0,F(u^{i+1},v^{i+1},x^{i+1}))+\|\mathcal{A}^*u^{i+1}+\mathcal{B}^*v^{i+1}-c\|^2))/k\\ \leq\ &C(\Phi_1(\bar{u},\bar{v},\bar{x})+\tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^1+\mathcal{B}^*v_e^1\|^2)/k,\end{aligned}$$

where

$$C = \begin{cases} C_1(13-4\tau)\|\mathcal{O}_1^{-\frac{1}{2}}\widehat{\mathcal{O}}\mathcal{O}_1^{-\frac{1}{2}}\| + C_2(1-\tau)^{-1}\sigma^{-1}, & \tau \in (0,1), \\ C_1(13-4\tau)\|\mathcal{O}_1^{-\frac{1}{2}}\widehat{\mathcal{O}}\mathcal{O}_1^{-\frac{1}{2}}\| + C_2(2\sigma^{-1} + (26-8\tau)\|\mathcal{O}_1^{-\frac{1}{2}}\mathrm{Diag}\,(0,\mathcal{B}\mathcal{B}^*)\mathcal{O}_1^{-\frac{1}{2}}\|), & \tau = 1. \end{cases}$$

To prove the limiting case (4.38), by using inequalities (4.42), (4.43), (4.44) and [57, Lemma 2.1], we have that

$$\min_{1\le i\le k}\|\Delta w^{i+1}\|^2_{\mathcal{O}_1} = o(1/k), \quad \min_{1\le i\le k}\|\mathcal{A}^*u^{i+1} + \mathcal{B}^*v^{i+1} - c\|^2 = o(1/k),$$

which together with (4.40), imply that

$$\begin{aligned} &\lim_{k\to\infty} k(\min_{1\le i\le k}\{\mathrm{dist}^2(0, F(u^{i+1}, v^{i+1}, x^{i+1})) + \|\mathcal{A}^*u^{i+1} + \mathcal{B}^*v^{i+1} - c\|^2\}) \\ \le\ &\lim_{k\to\infty} k(\min_{1\le i\le k}\{C_1\|\mathcal{O}_1^{-\frac{1}{2}}\widehat{\mathcal{O}}\mathcal{O}_1^{-\frac{1}{2}}\|\|\Delta w^{i+1}\|^2_{\mathcal{O}_1} + C_2\|\mathcal{A}^*u^{i+1} + \mathcal{B}^*v^{i+1} - c\|^2\}) = 0. \end{aligned}$$

Next, we shall prove (4.39) under the condition (i) and $\tau = 1$. By (4.41), (4.44) and the positive definiteness of the self-adjoint linear operator $\frac{1}{8}\mathcal{Q} + \mathrm{Diag}\,(\mathcal{S},\mathcal{T})$, we know that

$$\sum_{i=1}^{\infty}\|\Delta w^{i+1}\|^2 < \infty, \quad \sum_{i=1}^{\infty}\|\Delta x^{i+1}\|^2 < \infty. \tag{4.45}$$

Then, by using Lemma 4.1, (4.45) and [17, Lemma 1.2], we know that

$$\lim_{k\to\infty} k(\|\Delta w^{k+1}\|^2_{\mathcal{Q}+\mathcal{H}+\mathrm{Diag}\,(\mathcal{S},\mathcal{T}+\sigma\mathcal{B}\mathcal{B}^*+\mathcal{Q}_{22})} + \|\Delta x^{k+1}\|^2_{\sigma^{-1}\mathcal{I}}) = 0.$$

Since $\mathcal{Q} + \mathcal{H} + \mathrm{Diag}\,(\mathcal{S}, \mathcal{T} + \sigma\mathcal{B}\mathcal{B}^* + \mathcal{Q}_{22}) \succeq \mathcal{Q} + \mathrm{Diag}\,(\mathcal{S},\mathcal{T}) \succ 0$, we obtain that

$$\lim_{k\to\infty} k\|\Delta w^{k+1}\|^2 = 0, \quad \lim_{k\to\infty} k\|\Delta x^{k+1}\|^2_{\sigma^{-1}\mathcal{I}} = 0,$$

which, together with (4.40), imply

$$\lim_{k\to\infty} k(\mathrm{dist}^2(0, F(u^{k+1}, v^{k+1}, x^{k+1})) + \|\mathcal{A}^*u^{k+1} + \mathcal{B}^*v^{k+1} - c\|^2) = 0.$$

It completes the proof of the conclusions under condition (i).

For $\tau \in (0, \frac{1+\sqrt{5}}{2})$, We know from (4.11) that for $\tau \in (0, \frac{1+\sqrt{5}}{2})$ and any $k \ge 1$,

$$\begin{aligned} &\sum_{i=1}^{k}\|\Delta w^{i+1}\|^2_{\mathcal{O}_2} + \frac{1}{3}(\tau^3\sigma)^{-1}\rho(\tau)\|\Delta x^{i+1}\|^2 \\ \le\ &\Psi_1(\bar{u},\bar{v},\bar{x}) + \Xi_1 + \frac{1}{3}(4 - \tau - 2\min(\tau,\tau^{-1})\sigma\|\mathcal{A}^*u_e^1 + \mathcal{B}^*v_e^1\|^2. \end{aligned}$$

Thus, by the inequality (4.40) and the positive definiteness of $\mathcal{O}_2$ from Lemma 4.2, we have

$$
\begin{aligned}
& \min_{1\le i\le k}\{\mathrm{dist}^2(0, F(u^{i+1}, v^{i+1}, x^{i+1})) + \|\mathcal{A}^* u^{i+1} + \mathcal{B}^* v^{i+1} - c\|^2\} \\
\le \;& \Big(\sum_{i=1}^{k}(\mathrm{dist}^2(0, F(u^{i+1}, v^{i+1}, x^{i+1})) + \|\mathcal{A}^* u^{i+1} + \mathcal{B}^* v^{i+1} - c\|^2\Big)/k \\
\le \;& C'(\Psi_1(\bar{u}, \bar{v}, \bar{x}) + \Xi_1 + \tfrac{1}{3}(4-\tau-2\min(\tau,\tau^{-1})\sigma\|\mathcal{A}^* u_e^1 + \mathcal{B}^* v_e^1\|^2)/k,
\end{aligned}
$$

where $C' = C_1\|\mathcal{O}_2^{-\frac{1}{2}}\widehat{\mathcal{O}}\mathcal{O}_2^{-\frac{1}{2}}\| + 3C_2\tau\sigma^{-1}\rho(\tau)^{-1}$. The limiting property (4.38) can be derived in the same way as for the case under condition (i).

This completes the proof of Theorem 4.2. □

Remark 4.2. *Theorem 4.2 gives the non-ergodic complexity of the KKT optimality condition, which does not seem to be known even for the classic ADMM with separable objective functions. For the latter, related results about the non-ergodic iteration complexity for the primal feasibility and the objective functions of the special classic ADMM with $\tau = 1$ can be found in Davis and Yin [16]. When $\tau \neq 1$, instead of showing the behaviour of the current kth iteration point, our form of non-ergodic complexity states the property of the "best point among the first k iterations", indicating that the iteration sequence may satisfy the $O(1/k)$ tolerance of the KKT system before the kth step. Thus, it is interesting to see whether a slightly better result with the "$\min_{1\le i\le k}$" removed from (4.38) can be obtained.*

In the rest of this section, we discuss the ergodic iteration complexity of the mADMM for solving problem (1.2). For $k = 1, 2, \cdots$, denote

$$
\hat{x}^k = \frac{1}{k}\sum_{i=1}^{k}\tilde{x}^{i+1}, \quad \hat{u}^k = \frac{1}{k}\sum_{i=1}^{k} u^{i+1}, \quad \hat{v}^k = \frac{1}{k}\sum_{i=1}^{k} v^{i+1}, \quad \hat{w}^k = (\hat{u}^k, \hat{v}^k)
$$

and

$$
\begin{cases}
\Lambda_{k+1} = \|u_e^{k+1}\|^2_{\mathcal{D}_1+\mathcal{S}} + \|v_e^{k+1}\|^2_{\mathcal{D}_2+\mathcal{T}+\mathcal{Q}_{22}+\sigma\mathcal{B}\mathcal{B}^*} + (\tau\sigma)^{-1}\|x^{k+1}\|^2, \\
\overline{\Lambda}_{k+1} = \Lambda_{k+1} + \Xi_{k+1} + \|w_e^{k+1}\|^2_{\mathcal{Q}} + \max(1-\tau, 1-\tau^{-1})\sigma\|\mathcal{A}^* u^{k+1} + \mathcal{B}^* v^{k+1} - c\|^2.
\end{cases}
$$

Theorem 4.3. *Suppose that $\mathcal{S}$ and $\mathcal{T}$ are chosen such that*

$$\mathcal{Q}_{11} + \sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0, \quad \mathcal{Q}_{22} + \sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0.$$

Assume that either (a) $\tau \in (0, 1]$ or (b) $\tau \in (0, \frac{1+\sqrt{5}}{2})$ and (4.30) hold. Then there exist constants D_1 and D_2 that only depending on the initial point and the optimal solution set such that for $k \geq 1$, the following conclusions hold:
(i)

$$\|\mathcal{A}^*\hat{u}^k + \mathcal{B}^*\hat{v}^k - c\| \leq D_1/k. \tag{4.46}$$

(ii) For case (b), if we further assume that $\mathcal{S} - \eta\mathcal{D}_1 \succeq 0$ and $\mathcal{T} - \eta\mathcal{D}_2 \succeq 0$, then

$$|\theta(\hat{u}^k, \hat{v}^k) - \theta(\bar{u}, \bar{v})| \leq D_2/k. \tag{4.47}$$

The inequality (4.47) holds for case (a) without additional assumptions.

Proof. **(i)** Under the conditions for case (a), the inequality (4.10) indicates that $\{\Phi_{k+1}(\bar{u}, \bar{v}, \bar{x}) + \frac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^{k+1} + \mathcal{B}^*v_e^{k+1}\|^2\}$ is a non-increasing sequence, which implies that

$$\begin{aligned}(\tau\sigma)^{-1}\|x_e^{k+1}\|^2 &\leq \Phi_{k+1}(\bar{u}, \bar{v}, \bar{x}) + \tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^{k+1} + \mathcal{B}^*v_e^{k+1}\|^2 \\ &\leq \Phi_1(\bar{u}, \bar{v}, \bar{x}) + \tfrac{1}{2}(1-\tau)\sigma\|\mathcal{A}^*u_e^1 + \mathcal{B}^*v_e^1\|^2.\end{aligned}$$

Similarly, under the conditions for case (b), we can get from (4.11) that

$$\begin{aligned}(\tau\sigma)^{-1}\|x_e^{k+1}\|^2 &\leq \Psi_{k+1}(\bar{u}, \bar{v}, \bar{x}) + \Xi_{k+1} + \tfrac{1}{3}(4-\tau-2\min\{\tau, \tau^{-1}\})\sigma\|\mathcal{A}^*u_e^{k+1} + \mathcal{B}^*v_e^{k+1}\|^2 \\ &\leq \Psi_1(\bar{u}, \bar{v}, \bar{x}) + \Xi_1 + \tfrac{1}{3}(4-\tau-2\min\{\tau, \tau^{-1}\})\sigma\|\mathcal{A}^*u_e^1 + \mathcal{B}^*v_e^1\|^2.\end{aligned}$$

Therefore, in terms of the ergodic primal feasibility, we have that

$$\begin{aligned}\|\mathcal{A}^*\hat{u}^k + \mathcal{B}^*\hat{v}^k - c\|^2 &= \|\tfrac{1}{k}\sum_{i=1}^{k}(\mathcal{A}^*u^{i+1} + \mathcal{B}^*v^{i+1} - c)\|^2 \\ &= \|(\tau\sigma)^{-1}(x^{k+1} - x^1)\|^2/k^2 \\ &\leq 2\|(\tau\sigma)^{-1}x_e^{k+1}\|^2/k^2 + 2\|(\tau\sigma)^{-1}x_e^1\|^2/k^2 \leq C_3/k^2,\end{aligned} \tag{4.48}$$

where

$$C_3 := \begin{cases} 2(\tau\sigma)^{-1}\Phi_1(\bar{u}, \bar{v}, \bar{x}) + (\tau^{-1} - 1)\|\mathcal{A}^* u_e^1 + \mathcal{B}^* v_e^1\|^2 + 2\|(\tau\sigma)^{-1} x_e^1\|^2, & \text{case (a)}, \\ 2(\tau\sigma)^{-1}(\Psi_1(\bar{u}, \bar{v}, \bar{x}) + \Xi_1 + \frac{1}{3}(4 - \tau - 2\min\{\tau, \tau^{-1}\})\sigma\|\mathcal{A}^* u_e^1 + \mathcal{B}^* v_e^1\|^2) & \\ \qquad\qquad +2\|(\tau\sigma)^{-1} x_e^1\|^2, & \text{case (b)}. \end{cases}$$

Then by taking the square root on inequality (4.48), we can obtain (4.46).

(ii) For the complexity of primal objective values, first, we know from (4.3) that

$$p(u) \geq p(\bar{u}) + \langle -\mathcal{A}\bar{x} - \nabla_u \phi(\bar{w}), u - \bar{u}\rangle, \quad \forall u \in \mathcal{U},$$

$$q(v) \geq q(\bar{v}) + \langle -\mathcal{B}\bar{x} - \nabla_v \phi(\bar{w}), v - \bar{v}\rangle, \quad \forall v \in \mathcal{V}.$$

Therefore, summing them up and by noting $\mathcal{A}^*\bar{u} + \mathcal{B}^*\bar{v} = c$ and the convexity of function ϕ, we have that

$$\begin{aligned} \theta(u, v) - \theta(\bar{u}, \bar{v}) &\geq -\langle \bar{x}, \mathcal{A}^* u + \mathcal{B}^* v - c\rangle + \phi(w) - \phi(\bar{w}) - \langle \nabla\phi(\bar{w}), w - \bar{w}\rangle \\ &\geq -\langle \bar{x}, \mathcal{A}^* u + \mathcal{B}^* v - c\rangle, \quad \forall u \in \mathcal{U}, v \in \mathcal{V}. \end{aligned}$$

Thus, with $(u, v) = (\hat{u}^k, \hat{v}^k)$, it holds that

$$\begin{aligned} \theta(\hat{u}^k, \hat{v}^k) - \theta(\bar{u}, \bar{v}) &\geq -\langle \bar{x}, \mathcal{A}^*\hat{u}^k + \mathcal{B}^*\hat{v}^k - c\rangle \geq -\frac{1}{2}(\frac{1}{k}\|\bar{x}\|^2 + k\|\mathcal{A}^*\hat{u}^k + \mathcal{B}^*\hat{v}^k - c\|^2) \\ &\geq -\frac{1}{2}(\|\bar{x}\|^2 + C_3)/k, \end{aligned} \tag{4.49}$$

where C_3 is the same constant as in (4.48).

For the reverse part, by (2.3) and (2.4) we can obtain that for any $i \geq 1$,

$$\phi(w^{i+1}) \leq \phi(w^i) + \langle \nabla\phi(w^i), \Delta w^{i+1}\rangle + \frac{1}{2}\|\Delta w^{i+1}\|^2_{\mathcal{Q}+\mathcal{H}},$$

$$\phi(\bar{w}) \geq \phi(w^i) + \langle \nabla\phi(w^i), \bar{w} - w^i\rangle + \frac{1}{2}\|\bar{w} - w^i\|^2_{\mathcal{Q}},$$

which indicates that

$$\phi(w^{i+1}) - \phi(\bar{w}) \leq \langle \nabla\phi(w^i), w_e^{i+1}\rangle + \frac{1}{2}\|\Delta w^{i+1}\|^2_{\mathcal{Q}+\mathcal{H}} - \frac{1}{2}\|w_e^i\|^2_{\mathcal{Q}}. \tag{4.50}$$

By the inclusion (4.13) and the convexity of p and q, we have that

$$
\begin{aligned}
p(\bar{u}) \geq \ & p(u^{k+1}) + \langle u_e^{k+1}, \mathcal{A}\tilde{x}^{k+1} + \nabla_u \phi(w^k) + (\mathcal{Q}_{11} + \mathcal{D}_1 + \mathcal{S})\Delta u^{k+1} - \sigma \mathcal{A}\mathcal{B}^* \Delta v^{k+1} \rangle, \\
q(\bar{v}) \geq \ & q(v^{k+1}) + \langle v_e^{k+1}, \mathcal{B}\tilde{x}^{k+1} + \nabla_v \phi(w^k) + (\mathcal{Q}_{22} + \mathcal{D}_2 + \mathcal{T})\Delta v^{k+1} + \mathcal{Q}_{12}^* \Delta u^{k+1} \rangle.
\end{aligned} \tag{4.51}
$$

Thus, (4.51) and (4.50) imply that for $\tau \in (0, 1]$ and any $i \geq 1$,

$$
\begin{aligned}
& \theta(u^{i+1}, v^{i+1}) - \theta(\bar{u}, \bar{v}) \\
\leq \ & \frac{1}{2}\|\Delta w^{i+1}\|_{\mathcal{Q}+\mathcal{H}}^2 - \frac{1}{2}\|w_e^i\|_{\mathcal{Q}}^2 + \langle \bar{w} - w^{i+1}, \mathcal{Q}\Delta w^{i+1} \rangle + \langle \bar{u} - u^{i+1}, \mathcal{A}\tilde{x}^{i+1} \rangle \\
& + \langle \bar{v} - v^{i+1}, \mathcal{B}\tilde{x}^{i+1} \rangle - \langle \Delta v^{i+1}, \mathcal{Q}_{12}^*(\bar{u} - u^{i+1}) \rangle + \sigma \langle \mathcal{A}^* u_e^{i+1}, \mathcal{B}^* \Delta v^{i+1} \rangle \\
& + \langle \bar{u} - u^{i+1}, (\mathcal{D}_1 + \mathcal{S})\Delta u^{i+1} \rangle + \langle \bar{v} - v^{i+1}, (\mathcal{D}_2 + \mathcal{T})(v^{i+1} - v^i) \rangle \\
\leq \ & \frac{1}{2}(\Lambda_i - \Lambda_{i+1}) - \frac{1}{2}(\|\Delta u^{i+1}\|_{\mathcal{S}}^2 + \|\Delta v^{i+1}\|_{\mathcal{T}}^2 + \sigma\|\mathcal{A}^* u^{i+1} + \mathcal{B}^* v^i - c\|^2 \\
& + \sigma(1 - \tau)\|\mathcal{A}^* u^{i+1} + \mathcal{B}^* v^{i+1} - c\|^2) \leq \frac{1}{2}(\Lambda_i - \Lambda_{i+1}).
\end{aligned} \tag{4.52}
$$

Therefore, summing up the above inequalities over $i = 1, \cdots k$ and by using the convexity of function θ we can obtain that

$$
\theta(\hat{u}_k, \hat{v}_k) - \theta(\bar{u}, \bar{v}) \leq (\Lambda_1(u, v, x) - \Lambda_{k+1}(u, v, x))/2k \leq \Lambda_1/2k. \tag{4.53}
$$

The inequalities (4.49) and (4.53) indicate that (4.47) holds for case (a).

Next, assume that the conditions for case (b) hold. Similar to (4.52), we have that

$$
\begin{aligned}
& \theta(u^{i+1}, v^{i+1}) - \theta(\bar{u}, \bar{v}) \\
\leq \ & \frac{1}{2}(\overline{\Lambda}_i - \overline{\Lambda}_{i+1}) - \frac{1}{2}(\|\Delta u^{i+1}\|_{\mathcal{S}-\eta\mathcal{D}_1}^2 + \|\Delta v^{i+1}\|_{\mathcal{T}+\min(\tau, 1+\tau-\tau^2)\sigma\mathcal{B}\mathcal{B}^*-\eta\mathcal{D}_2}^2 \\
& + \min(1, 1 + \tau^{-1} - \tau)\sigma\|\mathcal{A}^* u^{i+1} + \mathcal{B}^* v^{i+1} - c\|^2).
\end{aligned}
$$

By the assumptions that $\mathcal{S} - \eta\mathcal{D}_1 \succeq 0$ and $\mathcal{T} - \eta\mathcal{D}_2 \succeq 0$, we can obtain that

$$
\theta(\hat{u}^k, \hat{v}^k) - \theta(\bar{u}, \bar{v}) \leq (\overline{\Lambda}_1 - \overline{\Lambda}_{k+1})/2k \leq \overline{\Lambda}_1/2k. \tag{4.54}
$$

Thus, by (4.53) and (4.54) we can obtain the inequality (4.47). $\square$

Remark 4.3. *The results in Theorem 4.3, which are on the ergodic complexity of the primal feasibility and the objective function, respectively, are extended from the work of Davis and Yin [16] on the classic ADMM with separable objective functions. However, there is no corresponding result available on the dual problem. Therefore, it will be very interesting to see if one can develop a more explicit ergodic complexity result containing all the three parts in the KKT condition.*

4.3 The convergence rate of the quadratically coupled problems

In this section, we focus on a special class of the general linearly constrained optimization problem (1.2), where the coupled smooth function ϕ is convex and quadratic. Specifically, the problems under consideration take the following form:

$$\begin{array}{ll} \min & p(u) + q(v) + \frac{1}{2}\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \mathcal{Q}\begin{pmatrix} u \\ v \end{pmatrix} \right\rangle \\ \text{s.t.} & \mathcal{A}^* u + \mathcal{B}^* v = c, \end{array} \tag{4.55}$$

where $p, q, \mathcal{A}^*, \mathcal{B}^*$ adopt the same setting as in the general form (1.2), and $\mathcal{Q} : \mathcal{U} \times \mathcal{V} \to \mathcal{U} \times \mathcal{V}$ is a self-adjoint positive semidefinite linear operator with the block structure

$$\mathcal{Q}\begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^* & \mathcal{Q}_{22} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \quad \forall u \in \mathcal{U}, v \in \mathcal{V}.$$

Similarly as in the previous sections, we denote $w := \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{U} \times \mathcal{V}$.

Note that for this special choice of the function ϕ, the Heissan defined in (2.1) is the constant operator $\mathcal{Q}$. Thus, both the upper and lower bound of $\partial^2\phi$ could be chosen as $\mathcal{Q}$, i.e., $\mathcal{D}_1 = 0$ and $\mathcal{D}_2 = 0$, and the corresponding parameter η defined in (4.2) would be 0.

The motivation of this section is to explore a better local convergence rate of the proposed algorithm, beyond the previously discussed global sublinear complexity. The major idea of this part is inspired by Han et al.'s recent paper [40], which shows that the (semi-proximal) ADMM for solving problems with separable objective functions actually converges linearly under an error bound assumption. We extend their nice results to our proposed mADMM algorithm for solving (4.55) with quadratic coupled objective functions.

Before showing the main results, we first provide our error bound assumption below.

Assumption 4.2. *For any $(\bar{w}, \bar{x}) \in \Omega$, there exist a positive constant η such that*

$$\operatorname{dist}((w, x), \Omega) \leq \eta \|R(w, x)\|, \quad \forall (w, x) \in \mathcal{N}(\bar{w}, \bar{x}), \tag{4.56}$$

where $R(\cdot, \cdot)$ is defined as in (4.4) for problem (4.55) and $\mathcal{N}(\bar{w}, \bar{x}) \subseteq \mathcal{U} \times \mathcal{V} \times \mathcal{X}$ is a neighborhood of $(\bar{w}, \bar{x})$.

Naturally one may ask that for which kinds of optimization problems would the error bound holds. In fact, this is a quite interesting and important question in optimization community. However, to the best of our knowledge, no complete answers have been obtained till now except the piecewise linear quadratic problems. In the next chapter, we would discuss the sufficient conditions to guarantee the inequality (4.56) for a particular kind of constrained optimization problems that involving the non-polyhedral nuclear norm function. Here we just leave it as a blanket assumption and establish the linear rate of convergence based on Assumption 4.2.

For simplicity of the subsequent discussions, given $\tau \in (0, \frac{\sqrt{5}+1}{2})$, we denote two positive parameters as

$$\alpha_1 := 4\|\mathcal{Q}\| + 4\max\{\|\mathcal{Q}_{11} + \mathcal{S}\|, \|\sqrt{\mathcal{Q}_{12}\mathcal{Q}_{12}^*}\|, \|\mathcal{Q}_{22} + \mathcal{T}\|, \sigma\|\mathcal{A}^*\mathcal{A}\|\},$$

$$\alpha_2(\tau) := 4(1 - \tau^{-1})^2(\|\mathcal{A}^*\mathcal{A}\| + \|\mathcal{B}^*\mathcal{B}\|) + (\tau\sigma)^{-2}, \quad \tau \in \mathcal{R},$$

and a self-adjoint positive semidefinite operator $\mathcal{E} : \mathcal{U} \times \mathcal{V} \to \mathcal{U} \times \mathcal{V}$ as

$$\mathcal{E} := \mathcal{Q} + \operatorname{Diag}(\mathcal{Q}_{11} + \mathcal{S} + \sqrt{\mathcal{Q}_{12}\mathcal{Q}_{12}^*},\ \mathcal{Q}_{22} + \mathcal{T} + \sigma\mathcal{B}\mathcal{B}^*).$$

The following proposition shows that the norm of the residue mapping R at the current iteration point can be bounded by the weighted norm of the difference between two latest consecutive iterations.

Proposition 4.2. *Suppose that the sequence $\{(w^k, x^k)\}$ is generated by the mADMM algorithm for problem (4.55). Then for any $k \geq 0$ and $\tau \in \mathcal{R}$, the following inequality always holds:*

$$\|R(w^{k+1}, x^{k+1})\|^2 \leq \alpha_1 \|\Delta w^{k+1}\|_{\mathcal{E}}^2 + \alpha_2(\tau)\|\Delta x^{k+1}\|^2. \tag{4.57}$$

Proof. For the problem (4.55), the optimality conditions at u^{k+1} and v^{k+1} can be reformulated in the form of proximal mapping as follows:

$$\begin{cases} u^{k+1} = \mathrm{Prox}_p\big(u^{k+1} - (\nabla_u \phi(w^k) + \mathcal{A}\tilde{x}^{k+1} + (\mathcal{Q}_{11} + \mathcal{S})\Delta u^{k+1} - \sigma \mathcal{A}\mathcal{B}^* \Delta v^{k+1})\big), \\ v^{k+1} = \mathrm{Prox}_q\big(v^{k+1} - (\nabla_v \phi(w^k) + \mathcal{B}\tilde{x}^{k+1} + \mathcal{Q}_{12}^* \Delta u^{k+1} + (\mathcal{Q}_{22} + \mathcal{T})\Delta v^{k+1})\big). \end{cases}$$

Thus, by recalling the KKT mapping defined in (4.4) and the non-expansive property of the proximal mappings, we have

$$\begin{array}{rl} & \|R(w^{k+1}, x^{k+1})\|^2 \\ \leq & 4\|\nabla\phi(w^{k+1}) - \nabla\phi(w^k)\|^2 + 4\|x^{k+1} - \tilde{x}^{k+1}\|_{\mathcal{A}^*\mathcal{A}+\mathcal{B}^*\mathcal{B}}^2 \\ & +4\max\{\|\mathcal{Q}_{11} + \mathcal{S}\|, \|\sqrt{\mathcal{Q}_{12}\mathcal{Q}_{12}^*})\|\}\|\Delta u^{k+1}\|_{\mathcal{Q}_{11}+\mathcal{S}+\sqrt{\mathcal{Q}_{12}\mathcal{Q}_{12}^*}}^2 \\ & +4\max\{\|\mathcal{Q}_{22} + \mathcal{T}\|, \sigma\|\mathcal{A}^*\mathcal{A}\|\}\|\Delta v^{k+1}\|_{\mathcal{Q}_{22}+\mathcal{T}+\sigma\mathcal{B}\mathcal{B}^*}^2 + (\tau\sigma)^{-2}\|\Delta x^{k+1}\|^2 \\ \leq & \alpha_1\|\Delta w^{k+1}\|_{\mathcal{E}}^2 + \alpha_2(\tau)\|\Delta x^{k+1}\|^2. \end{array}$$

□

We also denote the following two operators $\mathcal{M}_1 : \mathcal{U} \times \mathcal{V} \to \mathcal{U} \times \mathcal{V}$ and $\mathcal{M}_2 : \mathcal{U} \times \mathcal{V} \to \mathcal{U} \times \mathcal{V}$ as

$$\mathcal{M}_1 := \frac{7}{4}\mathcal{Q} + \kappa_1(\tau)\sigma \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} (\mathcal{A}^* \ \mathcal{B}^*) + \mathrm{Diag}(\mathcal{S},\, \mathcal{T} + \sigma\mathcal{B}\mathcal{B}^* + \mathcal{Q}_{22}),$$

$$\mathcal{M}_2 := \frac{1}{8}\mathcal{Q} + \kappa_2(\tau)\sigma \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} (\mathcal{A}^* \ \mathcal{B}^*) + \mathrm{Diag}(\mathcal{S},\, \mathcal{T} + \frac{1}{6}\mathcal{Q}_{22} + \rho(\tau)\sigma\mathcal{B}\mathcal{B}^*),$$

where $\kappa_1(\tau) = \frac{1}{3}(4 - \tau - 2\min\{\tau, \tau^{-1}\})$ and $\kappa_2(\tau) = \frac{1}{6}\rho(\tau)\tau^{-1}$ for given $\tau \in \mathcal{R}$.

The following Lemma characterize the relationship between the positive definiteness of several operators, which can be proved by the same way of Lemma 4.2 in the previous section. We omit the proof here for simplicity.

Lemma 4.3. *Let $\tau \in (0, \frac{1+\sqrt{5}}{2})$. Then we have the following equivalent characterization of the positive definite properties of the operators:*

$$\mathcal{Q}_{11} + \sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0 \quad \text{and} \quad \mathcal{Q}_{22} + \sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0 \Longleftrightarrow \mathcal{M}_2 \succ 0.$$

Now we are ready to present the main theorem of this section, which provides the linear convergence rate of the mADMM algorithm for solving (4.55) under Assumption 4.2 for the dual step length $\tau \in (0, \frac{1+\sqrt{5}}{2})$.

Theorem 4.4. *Suppose that the solution set of (4.55) is non-empty and Assumption 4.1 and 4.2 hold. Assume $\tau \in (0, \frac{1+\sqrt{5}}{2})$ and the following conditions hold:*

$$\mathcal{Q}_{11} + \sigma\mathcal{A}\mathcal{A}^* + \mathcal{S} \succ 0 \quad \text{and} \quad \mathcal{Q}_{22} + \sigma\mathcal{B}\mathcal{B}^* + \mathcal{T} \succ 0.$$

Then there exists a positive constant $\kappa < 1$ such that for any $k \geq 1$, it holds

$$\operatorname{dist}^2_{(\mathcal{M}_1,(\tau\sigma)^{-1}\mathcal{I})}\big((w^{k+1}, x^{k+1}), \Omega\big) + \|\Delta v^{k+1}\|^2_{\mathcal{T}} \leq \kappa\big(\operatorname{dist}^2_{(\mathcal{M}_1,(\tau\sigma)^{-1}\mathcal{I})}\big((w^k, x^k), \Omega\big) + \|\Delta v^k\|^2_{\mathcal{T}}\big). \tag{4.58}$$

Proof. Note that $\mathcal{D}_1 = 0$ and $\mathcal{D}_2 = 0$ in the mADMM for solving the quadratic coupled problem (4.55). Then for any $(\bar{w}, \bar{x}) \in \Omega$, we have from Proposition 4.1 (ii) that for any $k \geq 1$,

$$\begin{aligned} &(\|w^{k+1} - \bar{w}\|^2_{\mathcal{M}_1} + (\tau\sigma)^{-1}\|x^{k+1} - \bar{x}\|^2 + \|\Delta v^{k+1}\|^2_{\mathcal{T}}) \\ &\qquad - (\|w^k - \bar{w}\|^2_{\mathcal{M}_1} + (\tau\sigma)^{-1}\|x^k - \bar{x}\|^2 + \|\Delta v^k\|^2_{\mathcal{T}}) \\ \leq\ & -(\|\Delta w^{k+1}\|^2_{\mathcal{M}_2} + \tfrac{1}{3}(\sigma\tau^3)^{-1}\rho(\tau)\|\Delta x^{k+1}\|^2). \end{aligned} \tag{4.59}$$

Denote $(\bar{w}^k, \bar{x}^k)$ as the weighted projection of (w^k, x^k) to the solution set Ω, i.e.,

$$\begin{aligned} &\|w^k - \bar{w}^k\|^2_{\mathcal{M}_1} + (\tau\sigma)^{-1}\|x^k - \bar{x}^k\|^2 \\ =\ & \operatorname{dist}^2_{(\mathcal{M}_1,(\tau\sigma)^{-1}\mathcal{I})}\big((w^k, x^k), \Omega\big) := \inf_{(w,x)\in\Omega}\{\|w^k - w\|^2_{\mathcal{M}_1} + (\tau\sigma)^{-1}\|x^k - x\|^2\}. \end{aligned}$$

Thus, the inequality (4.59) can be recast as

$$\begin{aligned}
&(\|w^{k+1}-\bar{w}^k\|_{\mathcal{M}_1}^2+(\tau\sigma)^{-1}\|x^{k+1}-\bar{x}^k\|^2+\|\Delta v^{k+1}\|_{\mathcal{T}}^2)\\
&\qquad -(\mathrm{dist}^2_{(\mathcal{M}_1,(\tau\sigma)^{-1}\mathcal{I})}\big((w^k,x^k),\Omega\big)+\|\Delta v^k\|_{\mathcal{T}}^2)\\
\leq\ & -(\|\Delta w^{k+1}\|_{\mathcal{M}_2}^2+\tfrac{1}{3}(\sigma\tau^3)^{-1}\rho(\tau)\|\Delta x^{k+1}\|^2),
\end{aligned}$$

which further indicates that

$$\begin{aligned}
&(\mathrm{dist}^2_{(\mathcal{M}_1,(\tau\sigma)^{-1}\mathcal{I})}\big((w^{k+1},x^{k+1}),\Omega\big)+\|\Delta v^{k+1}\|_{\mathcal{T}}^2)\\
&\qquad -(\mathrm{dist}^2_{(\mathcal{M}_1,(\tau\sigma)^{-1}\mathcal{I})}\big((w^k,x^k),\Omega\big)+\|\Delta v^k\|_{\mathcal{T}}^2)\\
\leq\ & -(\|\Delta w^{k+1}\|_{\mathcal{M}_2}^2+\tfrac{1}{3}(\sigma\tau^3)^{-1}\rho(\tau)\|\Delta x^{k+1}\|^2).
\end{aligned} \tag{4.60}$$

By Assumption 4.2 and the global convergence of $\{(w^k,x^k)\}$ to some point in Ω, we know that there exists a positive integer k_0 such that for all $k\geq k_0$, the following inequality always holds:

$$\mathrm{dist}^2\big((w^{k+1},x^{k+1}),\Omega\big)\leq\eta^2\|R(w^{k+1},x^{k+1})\|^2.$$

Combining the above inequality with the inequality (4.57) in Proposition 4.2, we see that for $k\geq k_0$,

$$\mathrm{dist}^2\big((w^{k+1},x^{k+1}),\Omega\big)\leq\eta^2(\alpha_1\|\Delta w^{k+1}\|_{\mathcal{E}}^2+\alpha_2(\tau)\|\Delta x^{k+1}\|^2).$$

Hence, by recalling that $\mathcal{M}_2\succ 0$ from Lemma 4.3, we have for any $k\geq k_0$,

$$\begin{aligned}
&\mathrm{dist}^2_{(\mathcal{M}_1,(\tau\sigma)^{-1}\mathcal{I})}\big((w^{k+1},x^{k+1}),\Omega\big)+\|\Delta v^{k+1}\|_{\mathcal{T}}^2\\
\leq\ & \max\{\|\mathcal{M}_1\|,(\tau\sigma)^{-1}\}\mathrm{dist}^2\big((w^{k+1},x^{k+1}),\Omega\big)+\|\Delta v^{k+1}\|_{\mathcal{T}}^2\\
\leq\ & \eta^2\max\{\|\mathcal{M}_1\|,(\tau\sigma)^{-1}\}(\alpha_1\|\Delta w^{k+1}\|_{\mathcal{E}}^2+\alpha_2(\tau)\|\Delta x^{k+1}\|^2)+\|\Delta v^{k+1}\|_{\mathcal{T}}^2\\
\leq\ & \kappa_3(\tau)(\|\Delta w^{k+1}\|_{\mathcal{M}_2}^2+\tfrac{1}{3}(\sigma\tau^3)^{-1}\rho(\tau)\|\Delta x^{k+1}\|^2),
\end{aligned} \tag{4.61}$$

where $\kappa_3(\tau)=\eta^2\max\{\|\mathcal{M}_1\|,(\tau\sigma)^{-1}\}\max\{\|\mathcal{E}\|\|\mathcal{M}_2\|^{-1}\alpha_1,3\sigma\tau^3(\rho(\tau))^{-1}\alpha_2\}+1$ and the last inequality comes from the fact that $\|\Delta v^{k+1}\|_{\mathcal{T}}^2\leq\|\Delta w^{k+1}\|_{\mathcal{M}_2}^2$.

Substituting the inequality (4.61) into (4.60), we could obtain that for any $k\geq k_0$,

$$\begin{aligned}
&\mathrm{dist}^2_{(\mathcal{M}_1,(\tau\sigma)^{-1}\mathcal{I})}\big((w^{k+1},x^{k+1}),\Omega\big)+\|\Delta v^{k+1}\|_{\mathcal{T}}^2\\
\leq\ & \frac{\kappa_3(\tau)}{1+\kappa_3(\tau)}(\mathrm{dist}^2_{(\mathcal{M}_1,(\tau\sigma)^{-1}\mathcal{I})}\big((w^k,x^k),\Omega\big)+\|\Delta v^k\|_{\mathcal{T}}^2).
\end{aligned} \tag{4.62}$$

For $k < k_0$, define

$$\mu_k = 1 - \frac{\|\Delta w^{k+1}\|^2_{\mathcal{M}_2} + \frac{1}{3}(\sigma\tau^3)^{-1}\rho(\tau)\|\Delta x^{k+1}\|^2}{\mathrm{dist}^2_{(\mathcal{M}_1,(\tau\sigma)^{-1}\mathcal{I})}\big((w^k, x^k), \Omega\big) + \|\Delta v^k\|^2_{\mathcal{T}}} \in (0,1).$$

Then the inequality (4.60) can be rewritten as

$$\begin{aligned} & \mathrm{dist}^2_{(\mathcal{M}_1,(\tau\sigma)^{-1}\mathcal{I})}\big((w^{k+1}, x^{k+1}), \Omega\big) + \|\Delta v^{k+1}\|^2_{\mathcal{T}} \\ \le\ & \mu_k(\mathrm{dist}^2_{(\mathcal{M}_1,(\tau\sigma)^{-1}\mathcal{I})}\big((w^k, x^k), \Omega\big) + \|\Delta v^k\|^2_{\mathcal{T}}). \end{aligned} \tag{4.63}$$

Therefore, by letting $\kappa = \min\left\{\mu_1, \mu_2, \ldots, \mu_{k_0}, \dfrac{\kappa_3(\tau)}{1+\kappa_3(\tau)}\right\}$, we can see from (4.62) and (4.63) that the inequality (4.58) holds for any $k \ge 1$.

$\square$

Chapter 5

Characterization of the robust isolated calmness

One can see from the discussions in Section 4.3 that the error bound condition plays an important role in guaranteeing the linear convergence rate of the ADMM. In fact, the convergence rate of a rich class of first order algorithms can be established under the error bound conditions. For a nice survey about this topic, see [64]. This motivates us to explore more on the error bound conditions for the constrained composite programming. Recently, Han, Sun and Zhang [40] establish a certain error bound for the composite semidefinite optimization problems by the isolated calmness of the KKT system, and characterize the later property by the second order sufficient condition and the strict Robinson constraint qualification. In this chapter, we shall extend their nice work to a class of composite constrained nuclear norm problems with the form:

$$
\begin{aligned}
\min \quad & f(x) + \rho\theta(x) \\
\text{s.t.} \quad & h(x) \in \mathcal{P},
\end{aligned}
\tag{5.1}
$$

where $f : \mathcal{R}^{m\times n} \to \mathcal{R}$ is a twice continuously differentiable function, and $h : \mathcal{R}^{m\times n} \to \mathcal{Y}$ is a twice continuously differentiable mapping, $\rho > 0$ is a given positive penalty parameter, $\mathcal{P} \subseteq \mathcal{Y}$ is a closed convex polyhedral, $\theta : \mathcal{R}^{m\times n} \to \mathcal{R}$ denotes the nuclear norm function, i.e., $\theta(x) = \|x\|_*$ for all $x \in \mathcal{R}^{m\times n}$, and $\mathcal{Y}$ is a finite dimensional Euclidean space. Without lose of generality, we assume $\rho = 1$ in the objective function for simplicity.

5.1 The robust isolated calmness for the nuclear norm problems

In this section, we explore the robust isolated calmness (Definition 2.9) of the KKT system for the problem (5.1). Different from the positive semidefinite programming, the problem (5.1) is no longer a conic optimization problem. Fortunately, with the results prepared in Section 2.5.1, we are able to focus on the nuclear norm directly without referring to its epigraph. In the following, we would first analyze the variational properties related to the nuclear norm and its proximal mapping. Based on them, we provide a full picture about the robust isolated calmness of the KKT system for the problem (5.1).

Since the nuclear norm function θ is Lipschitz continuous and convex, we always have $\mathrm{dom}\theta = \mathcal{R}^{m\times n}$, and $\theta^{\downarrow}(x,\cdot) = \theta'(x;\cdot)$ for any $x \in \mathcal{R}^{m\times n}$ [6, Theorem 2.126]. Thus, all the directional epiderivatives of θ appeared in Section 2.5.1 would be replaced by its conventional directional derivative in this section. Furthermore, we have $\theta_{+}^{\downarrow\downarrow}(x;d,\cdot) = \theta_{+}''(x;d,\cdot)$ and $\theta_{-}^{\downarrow\downarrow}(x;d,\cdot) = \theta_{-}''(x;d,\cdot)$ for any $x, d \in \mathcal{R}^{m\times n}$ by the Lipschitz continuity of θ and its directional differentiability. Moreover, in his Ph.D thesis, Ding [19, Proposition 4.3] proves that the epigraph of the nuclear norm is $\mathcal{C}^2$-cone reducible[1] at every point $(x,t) \in \mathrm{epi}\theta$, and thus, second order regular[2] by [6, Proposition 3.136]. In this way, we have θ is second order directional differentiable by combining the equation (2.26) and (2.27).

We call $\bar{x} \in \mathcal{R}^{m\times n}$ a stationary point of the problem (5.1) and $\bar{y} \in \mathcal{Y}$ a multiplier

[1]For the definition of $\mathcal{C}^2$-cone reducible, see [6, Definition 3.135].

[2]For the definition of second order regular, see [6, Definition 3.85].

of $\bar{x}$ if $(\bar{x}, \bar{y})$ satisfies the following optimality condition in the sense of (2.31):

$$\begin{cases} f'(\bar{x})d + \langle \bar{y}, h'(\bar{x})d\rangle + \theta'(\bar{x}; d) \geq 0, \quad \forall d \in \mathcal{X}, \\ \bar{y} \in \mathcal{N}_{\mathcal{P}}(h(\bar{x})), \end{cases} \tag{5.2}$$

Denote $\mathcal{M}(\bar{x})$ as the set of all the multipliers at $\bar{x}$.

From Proposition 2.5 and by noting that $\mathrm{dom}\theta = \mathcal{R}^{m\times n}$, the multiplier set $\mathcal{M}(\bar{x})$ is nonempty, convex and compact at a local optimal solution $\bar{x} \in \mathcal{R}^{m\times n}$ for the problem (5.1) if and only if the following RCQ holds at $\bar{x}$:

$$0 \in \mathrm{int}\{h(\bar{x}) + h'(\bar{x})\mathcal{X} - \mathcal{P}\}. \tag{5.3}$$

Recall from Proposition 2.6 that the SRCQ of the problem (5.1) at a local optimal solution $\bar{x}$ and its multiplier $\bar{y} \in \mathcal{M}(\bar{x})$ is given by

$$\begin{pmatrix} h'(\bar{x}) \\ \mathcal{I} \end{pmatrix} \mathcal{X} + \begin{pmatrix} \mathcal{T}_{\mathcal{P}}(h(\bar{x})) \cap \bar{y}^{\perp} \\ \mathcal{T}^{\theta}(\bar{x}, -\nabla_x l(\bar{x}, \bar{y})) \end{pmatrix} = \begin{pmatrix} \mathcal{Y} \\ \mathcal{X} \end{pmatrix}. \tag{5.4}$$

Denote $\psi_{(x,d)}(\cdot) = \theta''(x; d, \cdot)$ for any $(x, d) \in \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n}$. Then the following no gap second order necessary and sufficient optimality conditions can be easily obtained by the Lipschitz continuity of the function θ and the results in [86, Theorem 3.45, Proposition 3.136 and Theorem 3.137].

Proposition 5.1. *Suppose that $\bar{x}$ is a local optimal solution of the problem (5.1) and the RCQ (5.3) holds at $\bar{x}$. Then the following second order necessary condition holds:*

$$\sup_{\bar{y} \in \mathcal{M}(\bar{x})} \left\{ \langle d, \nabla^2_{xx} l(\bar{x}, \bar{y}) d\rangle - \psi^*_{(\bar{x},d)}(-\nabla_x l(\bar{x}, \bar{y})) \right\} \geq 0, \quad \forall d \in \mathcal{C}(\bar{x}) \setminus \{0\}. \tag{5.5}$$

Conversely, let $\bar{x}$ be a feasible solution of the problem (5.1) and the RCQ (5.3) holds at $\bar{x}$. Then the following condition

$$\sup_{\bar{y} \in \mathcal{M}(\bar{x})} \left\{ \langle d, \nabla^2_{xx} l(\bar{x}, \bar{y}) d\rangle - \psi^*_{(\bar{x},d)}(-\nabla_x l(\bar{x}, \bar{y})) \right\} > 0, \quad \forall d \in \mathcal{C}(\bar{x}) \setminus \{0\}. \tag{5.6}$$

is necessary and sufficient for the quadratic growth condition at $\bar{x}$, i.e., there exist a constant $c > 0$ and a neighborhood $\mathcal{N}(\bar{x})$ of $\bar{x}$ such that for any feasible point $x \in \mathcal{N}(\bar{x})$, it holds

$$f(x) + \theta(x) \geq f(\bar{x}) + \theta(\bar{x}) + c\|x - \bar{x}\|^2. \tag{5.7}$$

5.1.1 The variational analysis of the nuclear norm

Let $A, B \in \mathcal{R}^{m\times n}$ satisfying $B \in \partial\theta(A)$ and denote $C := A + B$. By the optimality conditions of the proximal mapping, one can see that this is equivalent as

$$A = \mathrm{Prox}_{\theta}(C), \quad B = \mathrm{Prox}_{\theta^*}(C). \tag{5.8}$$

Suppose that C admits the following singular-value decomposition (SVD):

$$C = U[\Sigma(C)\ 0]V^T = U[\Sigma(C)\ 0][V_1\ V_2]^T = U\Sigma(C)V_1^T, \tag{5.9}$$

where $U \in \mathcal{O}^m$, $V := [V_1\ V_2] \in \mathcal{O}^n$ with $V_1 \in \mathcal{R}^{n\times m}$ and $V_2 \in \mathcal{R}^{n\times(n-m)}$ are the singular vectors of C, and $\Sigma(C) := \mathrm{Diag}(\sigma_1(C), \sigma_2(C), \ldots, \sigma_m(C))$ are the singular value of C with $\sigma_1(C) \geq \sigma_2(C) \geq \ldots \geq \sigma_m(C)$ being arranged in a non-increasing order.

It is known by [20] that given the SVD of C as (5.9), the SVD of A and B can be written as:

$$\begin{aligned} A &= U[\Sigma(A)\ 0]V^T = U\Sigma(A)V_1^T, \\ B &= U[\Sigma(B)\ 0]V^T = U\Sigma(B)V_1^T, \end{aligned} \tag{5.10}$$

where $\Sigma(A) := \mathrm{Diag}(\sigma_1(A), \sigma_2(A), \ldots, \sigma_m(A))$,$\Sigma(B) := \mathrm{Diag}(\sigma_1(B), \sigma_2(B), \ldots, \sigma_m(B))$ and

$$\sigma_i(A) = (\sigma_i(C) - 1)_+, \quad \sigma_i(B) = \sigma_i(C) - \sigma_i(A), \quad i = 1, 2, \ldots, m. \tag{5.11}$$

Obviously $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_m(A)$ and $\sigma_1(B) \geq \sigma_2(B) \geq \ldots \geq \sigma_m(B)$.

For simplicity of the subsequent discussions, we denote the following two index sets:

$$a := \{1 \leq i \leq m : \sigma_i(A) > 0\}, \quad b := \{1 \leq i \leq m : \sigma_i(A) = 0\}. \tag{5.12}$$

We further denote the distinct nonzero singular values of A as $\mu_1(A) > \mu_2(A) > \ldots > \mu_r(A)$ for some nonnegative integer r and divide the set a into the following r subsets:

$$a = \bigcup_{1\leq l\leq r} a_l, \quad a_l := \{i \in a : \sigma_i(A) = \mu_l(A)\}, \quad l = 1, 2, \ldots, r. \tag{5.13}$$

From the relationship (5.11), we can see that $0 \leq \sigma_i(B) \leq 1$ for $i \in b$. Based on it, we also divide the set b into the following three subsets:

$$b_1 := \{i \in b : \sigma_i(B) = 1\}, \quad b_2 := \{i \in b : 0 < \sigma_i(B) < 1\}, \quad b_3 := \{i \in b : \sigma_i(B) = 0\}. \tag{5.14}$$

In fact, the above denoted indices in (5.12), (5.13) and (5.14) can also be regarded as a classification about the singular values of C by noting the relationship (5.11) again. That is,

$$\begin{aligned}
& a = \{1 \leq i \leq m : \sigma_i(C) > 1\}, \quad b = \{1 \leq i \leq m : 0 \leq \sigma_i(C) \leq 1\}, \\
& a_l = \{i \in a : \sigma_i(C) = \mu_l(C)\}, \quad l = 1, 2, \ldots, r, \\
& b_1 := \{i \in b : \sigma_i(C) = 1\}, \quad b_2 := \{i \in b : 0 < \sigma_i(C) < 1\}, \quad b_3 := \{i \in b : \sigma_i(C) = 0\},
\end{aligned} \tag{5.15}$$

where $\mu_1(C) > \mu_2(C) > \ldots > \mu_r(C) > 1$ denotes the distinct singular values of C that are larger than 1.

It is known from Watson [101] that given the SVD of $A \in \mathcal{R}^{m\times n}$ in the form of (5.10) with the indices a and b defined in (5.12), the subgradient at A takes the following form:

$$\partial\theta(A) = \left\{U_a V_a^T + U_b W [V_b\ V_2]^T : W \in \mathcal{R}^{|b|\times(n-|a|)}, \|W\|_2 \leq 1\right\}. \tag{5.16}$$

Therefore, for any $H \in \mathcal{R}^{m\times n}$, the directional derivative at A along H can be computed as

$$\begin{aligned}
\theta'(A; H) &= \sup_{S\in\partial\theta(A)} \langle H, S\rangle \\
&= \mathrm{tr}(U_a^T H V_a) + \|U_b^T H [V_b\ V_2]\|_*.
\end{aligned} \tag{5.17}$$

Now we shall discuss the directional derivative of $\mathrm{Prox}_\theta(\cdot)$. Define a mapping $\phi : \mathcal{R}^m \to \mathcal{R}^m$ as

$$\phi(x) := \big((x_1-1)_+, (x_2-1)_+, \ldots, (x_m-1)_+\big), \quad \forall x = (x_1, x_2, \ldots, x_m) \in \mathcal{R}^m.$$

Suppose $C \in \mathcal{R}^{m\times n}$ admits the SVD in the form of (5.9) with

$$\sigma(C) := (\sigma_1(C), \sigma_2(C), \ldots, \sigma_m(C)).$$

By the relationship (5.10), we can rewrite the proximal mapping of the nuclear norm as

$$\mathrm{Prox}_\theta(C) = U\mathrm{Diag}\big(\phi(\sigma(C))\ 0\big)V^T. \tag{5.18}$$

Thus, Prox_θ is the spectral operator associated with the symmetric function ϕ in the sense of Definition 2.4. It is easy to see that ϕ is directional differentiable with the directional derivative at $x = (x_1, x_2, \ldots, x_m) \in \mathcal{R}^m$ along the direction $d := (d_1, d_2, \ldots, d_m) \in \mathcal{R}^m$ given by

$$\phi'(x; d) = (\phi_1'(x_1; d_1), \phi_2'(x_2; d_2), \ldots, \phi_m'(x_m; d_m)),$$

$$\text{where } \phi_i'(x_i; d_i) = \begin{cases} d_i & x_i > 1 \\ (d_i)_+ & x_i = 1 \\ 0 & x_i < 1 \end{cases} \quad \text{for } i = 1, 2, \ldots, m.$$

Given a positive integer p, define two linear matrix operators $S : \mathcal{R}^{p\times p} \to \mathcal{S}^p$ and $T : \mathcal{R}^{p\times p} \to \mathcal{R}^{p\times p}$ as

$$S(Y) = \frac{1}{2}(Y + Y^T), \quad T(Y) = \frac{1}{2}(Y - Y^T), \quad \forall Y \in \mathcal{R}^{p\times p}. \tag{5.19}$$

From Theorem 2.1, the directional derivative of $\mathrm{Prox}_\theta'(C; \cdot)$ takes the form of

$$\mathrm{Prox}_\theta'(C; H) = U \left(\begin{array}{c:c:c:c:c}
\Gamma_1(C, \widetilde{H}_1) & \multicolumn{3}{c:}{\Gamma_2(C, \widetilde{H}_1)} & \Gamma_4(C, \widetilde{H}_2) \\ \hdashline
 & P_{b_1}(\Lambda(S(\widetilde{H}_{b_1 b_1})))_+ P_{b_1}^T & 0 & 0 & \\ \cdashline{2-4}
\Gamma_3(C, \widetilde{H}_1) & 0 & 0_{b_2\times b_2} & 0 & 0 \\ \cdashline{2-4}
 & 0 & 0 & 0_{b_3\times b_3} &
\end{array}\right) V^T, \tag{5.20}$$

where $\widetilde{H} = [\widetilde{H}_1 \ \widetilde{H}_2] = [U^T HV_1 \ U^T HV_2]$, $S(\widetilde{H}_{b_1b_1})$ has the eigenvalue decomposition $S(\widetilde{H}_{b_1b_1}) = P_{b_1}\Lambda(S(\widetilde{H}_{b_1b_1}))P_{b_1}^T$ with $P_{b_1} \in \mathcal{O}^{|b_1|}(S(\widetilde{H}_{b_1b_1}))$, and the four blocks $\Gamma_1(C, \widetilde{H}_1) \in \mathcal{R}^{|a|\times|a|}$, $\Gamma_2(C, \widetilde{H}_1) \in \mathcal{R}^{|a|\times|b|}$, $\Gamma_3(C, \widetilde{H}_1) \in \mathcal{R}^{|b|\times|a|}$ and $\Gamma_4(C, \widetilde{H}_2) \in \mathcal{R}^{|a|\times(n-m)}$ admit the forms

$$\begin{cases} (\Gamma_1(C, \widetilde{H}_1))_{a_la_t} & = & (S(\widetilde{H}_1))_{a_la_t} + \dfrac{\mu_l(C) + \mu_t(C) - 2}{\mu_l(C) + \mu_t(C)}(T(\widetilde{H}_1))_{a_la_t}, \quad 1 \le l, t \le r, \\ \Gamma_2(C, \widetilde{H}_1) & = & \Xi_S \circ (S(\widetilde{H}_1))_{ab} + \Xi_T \circ (T(\widetilde{H}_1))_{ab}, \\ \Gamma_3(C, \widetilde{H}_1) & = & \Xi_S^T \circ (S(\widetilde{H}_1))_{ba} + \Xi_T^T \circ (T(\widetilde{H}_1))_{ba}, \\ \Gamma_4(C, \widetilde{H}_2) & = & \Xi_2 \circ (\widetilde{H}_2)_{a2}, \end{cases}$$

and $\Xi_S \in \mathcal{R}^{|a|\times|b|}$, $\Xi_T \in \mathcal{R}^{|a|\times|b|}$, $\Xi_2 \in \mathcal{R}^{|a|\times(n-m)}$ are given by

$$\begin{cases} (\Xi_S)_{ij} = \dfrac{\sigma_i(C) - 1}{\sigma_i(C) - \sigma_{j+|a|}(C)}, \ i = 1, 2, \ldots, |a|, \ j = 1, 2, \ldots, |b|, \\ (\Xi_T)_{ij} = \dfrac{\sigma_i(C) - 1}{\sigma_i(C) + \sigma_{j+|a|}(C)}, \ i = 1, 2, \ldots, |a|, \ j = 1, 2, \ldots, |b|, \\ (\Xi_2)_{ij} = \dfrac{\sigma_i(C) - 1}{\sigma_i(C)} \ i = 1, 2, \ldots, |a|, \ j = 1, 2, \ldots, n - m. \end{cases}$$

In [20], Ding computes $\psi^*_{(A,H)}(B)$ for $H \in \mathcal{R}^{m\times n}$ explicitly as follows, which would be the sigma term in the second order sufficient optimality condition of the problem (2.23):

$$\psi^*_{(A,H)}(B) = 2\sum_{l=1}^{r} \operatorname{tr}\big(\Omega_{a_l}(A, H)\big) + 2\langle \operatorname{Diag}(\sigma_b(B)), U_b^T HA^\dagger HV_b\rangle, \tag{5.21}$$

where $\sigma_b(B) = (\sigma_i(B))_{i\in b}$ and

$$\begin{aligned} \Omega_{a_l}(A, H) := \ & (S(\widetilde{H}_1))_{a_l}^T(\Sigma(A) - \mu_l(A)\mathcal{I}_m)^\dagger(S(\widetilde{H}_1))_{a_l} + (2\mu_l(A))^{-1}\widetilde{H}_{a_l2}\widetilde{H}_{a_l2}^T \\ & +(T(\widetilde{H}_1))_{a_l}^T(-\Sigma(A) - \mu_l(A)\mathcal{I}_m)^\dagger(T(\widetilde{H}_1))_{a_l}, \quad l = 1, 2, \ldots r, \end{aligned}$$

with $\widetilde{H} = [\widetilde{H}_1 \ \widetilde{H}_2] = [U^T HV_1 \ U^T HV_2]$.

Define a set-valued mapping $\mathcal{T}^\theta$ on $\mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n}$ as

$$\mathcal{T}^\theta(X, S) := \{H \in \mathcal{R}^{m\times n} : \theta'(X; H) = \langle H, S\rangle\}, \quad \forall (X, S) \in \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n}. \tag{5.22}$$

In the following, we present several properties related to the directional derivatives of θ, $\operatorname{Prox}_\theta$ and the sigma term generated by θ.

Lemma 5.1. *Suppose $A, B, C \in \mathcal{R}^{m\times n}$ satisfy the relationship (5.8) and the index sets a, b, b_1, b_2, b_3 are defined as in (5.12) and (5.14). Given any $H \in \mathcal{R}^{m\times n}$, denote $\widetilde{H} = U^T HV$ for U, V satisfying (5.9). Then the following conclusions hold:*

(i) $H \in \mathcal{T}^\theta(A, B)$ if and only if $\widetilde{H}$ has the following block structure:

$$\widetilde{H} = \left(\begin{array}{c:c:c:c:c} \widetilde{H}_{aa} & \multicolumn{3}{c:}{\widetilde{H}_{ab}} & \widetilde{H}_{a2} \\ \hdashline & \Pi_{\mathcal{S}_+^{|b_1|}}(\widetilde{H}_{b_1b_1}) & 0 & 0 & \\ \cdashline{2-4} \widetilde{H}_{ba} & 0 & 0_{b_2\times b_2} & 0 & 0 \\ \cdashline{2-4} & 0 & 0 & 0_{b_3\times b_3} & \end{array}\right). \tag{5.23}$$

where $\Pi_{\mathcal{S}_+^p}(\cdot)$ denotes the projection onto the $p \times p$ dimensional positive semidefinite cone.

(ii) $H \in \left(\mathcal{T}^\theta(A, B)\right)^\circ \Longleftrightarrow \mathrm{Prox}'_\theta(C; H) = 0$.

Proof. The result of part (i) comes from [20, proposition 10]. For part (ii), by the expression of $\mathrm{Prox}'_\theta(C; \cdot)$ in (5.20), we have $\mathrm{Prox}'_\theta(C; H) = 0$ for some $H \in \mathcal{R}^{m\times n}$ if and only if

$$\widetilde{H}_{aa} = 0, \quad \widetilde{H}_{ab} = 0, \quad \widetilde{H}_{ba} = 0, \quad \widetilde{H}_{a2} = 0, \quad \widetilde{H}_{b_1b_1} \preceq 0,$$

where $\widetilde{H} = U^T HV$. And from part (i) we can see that the above conditions are equivalent for $H \in \left(\mathcal{T}^\theta(A, B)\right)^\circ$. $\square$

Lemma 5.2. *Suppose $A, B, C \in \mathcal{R}^{m\times n}$ satisfy the relationship (5.8). Then for any $H, D \in \mathcal{R}^{m\times n}$, we have*

$$H = \mathrm{Prox}'_\theta(C; H + D) \Longleftrightarrow H \in \mathcal{T}^\theta(A, B) \quad \textit{and} \quad \langle H, D\rangle = -\psi^*_{(A,H)}(B). \tag{5.24}$$

Proof. First, by using the expression of the second order directional derivative for the eigenvalues and singular values [94, 105], we can compute the sigma-term

$\psi^*_{(A,H)}(B)$ explicitly [20, Proposition 14] as

$$
\begin{aligned}
\psi^*_{(A,H)}(B) = & \sum_{1\le l,t\le r} \frac{2}{-\mu_t(A)-\mu_l(A)}\|(T(\widetilde{H}_1))_{a_l a_t}\|^2 + \sum_{1\le l\le r} \frac{4}{-\mu_l(A)}\|(T(\widetilde{H}_1))_{a_l b_1}\|^2 \\
& + \sum_{\substack{1\le l\le r \\ 1\le i-|a|-|b_1|\le |b_2|}} \left(\frac{2(1-\sigma_i(B))}{-\mu_l(A)}\|(S(\widetilde{H}_1))_{a_l i}\|^2 + \frac{2(\sigma_i(B)+1)}{-\mu_l(A)}\|(T(\widetilde{H}_1))_{a_l i}\|^2 \right) \\
& + \sum_{\substack{1\le l\le r \\ 1\le i-|a|-|b_1|-|b_2|\le |b_3|}} \left(\frac{2}{-\mu_l(A)}\|(S(\widetilde{H}_1))_{a_l i}\|^2 + \frac{2}{-\mu_l(A)}\|(T(\widetilde{H}_1))_{a_l i}\|^2 \right) \\
& + \sum_{1\le l\le r} \frac{1}{-\mu_l(A)}\|(\widetilde{H}_2)_{a_l 2}\|^2.
\end{aligned}
\tag{5.25}
$$

Suppose $H = \text{Prox}'_\theta(C; H+D)$. Recall that $\text{Prox}'_\theta(C;\cdot)$ has the form (5.20). Thus, by letting $\widetilde{H} = U^T H V$ and $\tilde{d} = U^T D V$, we could directly obtain that $\widetilde{H}$ has a special block structure as

$$
\widetilde{H} = \left(\begin{array}{c:ccc:c}
\widetilde{H}_{aa} & & \widetilde{H}_{ab} & & \widetilde{H}_{a2} \\ \hdashline
 & \widetilde{H}_{b_1 b_1} & 0 & 0 & \\
\widetilde{H}_{ba} & 0 & 0_{b_2\times b_2} & 0 & 0 \\
 & 0 & 0 & 0_{b_3\times b_3} &
\end{array} \right),
$$

and $\widetilde{H}$ and $\tilde{d}$ further satisfy that

$$
\begin{aligned}
& \tilde{d}_{a_l a_t} = \frac{2}{\mu_l(C)+\mu_t(C)-2}(T(\widetilde{H}_1))_{a_l a_t},\ 1\le l,t\le r, \\
& (\tilde{d}_{ab})_{ij} = \frac{1}{\sigma_i(C)-1}(\widetilde{H}_{ab})_{ij} - \frac{\sigma_{j+|a|}(C)}{\sigma_i(C)-1}(\widetilde{H}_{ab})_{ji},\ i=1,2,\ldots|a|,\ j=1,2,\ldots,|b|, \\
& (\tilde{d}_{ba})_{ji} = \frac{1}{\sigma_i(C)-1}(\widetilde{H}_{ab})_{ji} - \frac{\sigma_{j+|a|}(C)}{\sigma_i(C)-1}(\widetilde{H}_{ab})_{ij},\ i=1,2,\ldots|a|,\ j=1,2,\ldots,|b|, \\
& (\tilde{d}_{a2})_{ij} = \frac{1}{\sigma_i(C)-1}(\widetilde{H}_{a2})_{ij}\ i=1,2,\ldots,|a|,\ j=1,2,\ldots,n-m, \\
& 0 \preceq \widetilde{H}_{b_1 b_1} = S(\widetilde{H}_{b_1 b_1}) \perp S(\tilde{d}_{b_1 b_1}) \preceq 0.
\end{aligned}
$$

This directly shows that $\theta'(A;H) = \langle B, H\rangle$ by Lemma 5.1 (i). Furthermore, we can compute the inner product between D and H as

$$\begin{aligned}
\langle D, H\rangle &= \langle \tilde{d}_{aa}, \widetilde{H}_{aa}\rangle + \langle \tilde{d}_{ab}, \widetilde{H}_{ab}\rangle + \langle \tilde{d}_{ba}, \widetilde{H}_{ba}\rangle + \langle \tilde{d}_{a2}, \widetilde{H}_{a2}\rangle \\
&= \sum_{1\le t,l\le r} \frac{2}{\mu_l(C)+\mu_t(C)-2}\|(T(\widetilde{H}_1))_{a_l a_t}\|^2 + \sum_{1\le l\le r} \frac{4}{\mu_l(C)-1}\|(T(\widetilde{H}_1))_{a_l b_1}\|^2 \\
&+ \sum_{\substack{1\le l\le r \\ 1\le i-|a|-|b_1|\le|b_2|}} \left(\frac{2(1-\sigma_i(C))}{\mu_l(C)-1}\|(S(\widetilde{H}_1))_{a_l i}\|^2 + \frac{2(\sigma_i(C)+1)}{\mu_l(C)-1}\|(T(\widetilde{H}_1))_{a_l i}\|^2\right) \\
&+ \sum_{\substack{1\le l\le r \\ 1\le i-|a|-|b_1|-|b_2|\le|b_3|}} \left(\frac{2}{\mu_l(C)-1}\|(S(\widetilde{H}_1))_{a_l i}\|^2 + \frac{2}{\mu_l(C)-1}\|(T(\widetilde{H}_1))_{a_l i}\|^2\right) \\
&\quad + \sum_{1\le l\le r} \frac{1}{\mu_l(C)-1}\|(\widetilde{H}_2)_{a_l 2}\|^2.
\end{aligned}$$

Comparing the above formula with (5.25), and by noting the relationship between the singular values of A and C in (5.11) and (5.15), we could obtain $\langle D, H\rangle = -\psi^*_{(A,H)}(B)$.

The reverse direction of this proposition can be shown by the above formula without any difficulty, and hence we omit it here.

□

5.1.2 The robust isolated calmness

After the preparation in the previous part, now we are ready to establish the robust isolated calmness of the KKT system by the second order sufficient condition and the SRCQ for the problem (5.1).

Recall the definition of the natural map for the problem (5.1) given in Section 2.5.3:

$$G(x,y) := \begin{pmatrix} x - \mathrm{Prox}_\theta\big(x - \nabla_x l(x,y)\big) \\ h(x) - \Pi_{\mathcal{P}}(h(x)+y) \end{pmatrix}, \quad \forall (x,y) \in \mathcal{R}^{m\times n} \times \mathcal{Y}. \tag{5.26}$$

We also consider the canonically perturbed problem with a given perturbation parameter $\delta := (\delta_1, \delta_2) \in \mathcal{R}^{m\times n} \times \mathcal{Y}$:

$$\begin{aligned}
\min \quad & f(x) + \theta(x) - \langle \delta_1, x\rangle, \\
\text{s.t.} \quad & h(x) - \delta_2 \in \mathcal{P},
\end{aligned} \tag{5.27}$$

where the set of all the optimal solutions are given by

$$S_{\mathrm{KKT}}(\delta) = \{(x,y) \in \mathcal{R}^{m\times n} \times \mathcal{Y} : \ \begin{array}{l} x - \mathrm{Prox}_\theta\big(x - \nabla_x l(x,y) + \delta_1\big) = 0, \\ h(x) - \delta_2 - \Pi_{\mathcal{P}}(h(x) - \delta_2 + y) = 0\}. \end{array} \tag{5.28}$$

The following Lemma is motivative by Klatte's work [51] on the isolated calmness property for the nonlinear programming.

Lemma 5.3. *Suppose that $\bar{x} \in \mathcal{R}^{m\times n}$ is a local optimal solution of the problem (5.1) and the RCQ (5.3) holds at $\bar{x}$. Let $\bar{y} \in \mathcal{M}(\bar{x})$. If G^{-1} is isolated calm at the origin for $(\bar{x},\bar{y})$ and there exists $\Delta x \in \mathcal{C}(\bar{x}) \setminus \{0\}$ such that*

$$\langle \Delta x, \nabla^2_{xx} l(\bar{x},\bar{y})\Delta x\rangle - \psi^*_{(\bar{x},\Delta x)}(-\nabla_x l(\bar{x},\bar{y})) = 0, \tag{5.29}$$

then there exists $\bar{d} \in \mathcal{C}(\bar{x})$ such that

$$\langle \bar{d}, \nabla^2_{xx} l(\bar{x},\bar{y})\Delta x\rangle - 2\Gamma_{(\bar{x},-\nabla_x l(\bar{x},\bar{y}))}(\bar{d},\Delta x) < 0, \tag{5.30}$$

where

$$\begin{aligned}\Gamma_{(\bar{x},-\nabla_x l(\bar{x},\bar{y}))}(d,h) := \textstyle\sum_{l=1}^r \mathrm{tr}\Big(&(S(\tilde{d}_1))^T_{a_l}(\Sigma(\bar{x}) - \mu_l(\bar{x})\mathcal{I}_m)^\dagger (S(\tilde{h}_1))_{a_l} + (2\mu_l(\bar{x}))^{-1}\tilde{d}_{a_l 2}\tilde{h}^T_{a_l 2} \\ &+(T(\tilde{d}_1))^T_{a_l}(-\Sigma(\bar{x}) - \mu_l(\bar{x})\mathcal{I}_m)^\dagger (T(\tilde{h}_1))_{a_l}\Big) + \langle \mathrm{Diag}(\sigma_b(-\nabla_x l(\bar{x},\bar{y}))), U_b^T d\bar{x}^\dagger h V_b\rangle.\end{aligned}$$

Proof. We prove the conclusion by contradiction. Suppose that there does not exist $\bar{d} \in \mathcal{C}(\bar{x})$ such that the inequality (5.30) holds. Since $\Gamma_{(\bar{x},-\nabla_x l(\bar{x},\bar{y}))}(d,h) = \Gamma_{(\bar{x},-\nabla_x l(\bar{x},\bar{y}))}(h,d)$ for any $d,h \in \mathcal{R}^{m\times n}$, we have by the assumption (5.29) that Δx is an optimal solution of the following linear positive semidefinite problem

$$\begin{array}{ll} \min\limits_{d} & \langle d, \nabla^2_{xx} l(\bar{x},\bar{y})\Delta x\rangle - \Gamma_{(\bar{x},-\nabla_x l(\bar{x},\bar{y}))}(d,\Delta x) - \Gamma_{(\bar{x},-\nabla_x l(\bar{x},\bar{y}))}(\Delta x,d) \\ \text{s.t.} & h'(\bar{x})d \in \mathcal{T}_{\mathcal{P}}(h(\bar{x})) \cap \bar{y}^\perp \\ & U_{b_1}^T dV_{b_1} \succeq 0, \quad U_{b_1}^T dV_{[b_2\ b_3]} = 0, \quad U_{[b_2\ b_3]}^T dV_b = 0, \quad U_b^T dV_2 = 0, \end{array} \tag{5.31}$$

where the constraint sets are in fact the equivalent conditions for $d \in \mathcal{C}(\bar{x})$ combined with $h'(\bar{x})d \in \bar{y}^\perp$ under the RCQ condition at $\bar{x}$. Obviously the RCQ condition for the problem (5.31) holds at Δx, then there exists $(\Delta y, \bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4) \in \mathcal{M}(\Delta x)$ such

that
$$\begin{cases} \nabla^2_{xx} l(\bar{x},\bar{y})\Delta x + \nabla h(\bar{x})\Delta y + \Xi(\Delta x, \bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4) = 0, \\ h'(\bar{x})\Delta x - \Pi_{\mathcal{T}_{\mathcal{P}}(h(\bar{x}))\cap \bar{y}^{\perp}}(h'(\bar{x})\Delta x + \Delta y) = 0, \\ U_{b_1}^T \Delta x V_{b_1} - \Pi_{\mathcal{S}_+^{|b_1|}}(U_{b_1}^T \Delta x V_{b_1} + \bar{\xi}_1) = 0, \end{cases} \tag{5.32}$$
where the term $\Xi(\Delta x, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4, \bar{\xi}_5)$ is given by
$$\begin{aligned}\Xi(\Delta x, \bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4) := \nabla_d \Big(\Gamma_{(\bar{x}, -\nabla_x l(\bar{x},\bar{y}))}(d, \Delta x) + \Gamma_{(\bar{x}, -\nabla_x l(\bar{x},\bar{y}))}(d, \Delta x) \Big)\Big|_{d=\Delta x} \\ + U_{b_1}\bar{\xi}_1 V_{b_1}^T + U_{b_1}\bar{\xi}_2 V_{[b_2\ b_3]}^T + U_{[b_2\ b_3]}\bar{\xi}_3 V_b^T + U_b \bar{\xi}_4 V_2^T.\end{aligned}$$

By Lemma 5.1 (i) and Δx is a feasible point of the problem (5.31), we have $\Delta x \in \mathcal{T}^\theta(\bar{x}, -\nabla_x l(\bar{x},\bar{y}))$. Furthermore, we see that
$$\begin{aligned} & \langle \Delta x, \Xi(\Delta x, \bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4)\rangle \\ = & -2\Gamma_{(\bar{x}, -\nabla_x l(\bar{x},\bar{y}))}(\Delta x, \Delta x) + \langle U_{b_1}^T \Delta x V_{b_1}, \bar{\xi}_1\rangle + \langle U_{b_1}^T \Delta x V_{[b_2\ b_3]}, \bar{\xi}_2\rangle + \langle U_{[b_2\ b_3]}^T \Delta x V_b, \bar{\xi}_3\rangle \\ & + \langle U_b^T \Delta x V_2, \bar{\xi}_4\rangle \\ = & -2\Gamma_{(\bar{x}, -\nabla_x l(\bar{x},\bar{y}))}(\Delta x, \Delta x) \\ = & -\psi^*_{(\bar{x},\Delta x)}(-\nabla_x l(\bar{x},\bar{y})), \end{aligned}$$
where the first equation holds by noting that $\Gamma_{(\bar{x}, -\nabla_x l(\bar{x},\bar{y}))}(d, \Delta x)$ and $\Gamma_{(\bar{x}, -\nabla_x l(\bar{x},\bar{y}))}(d, \Delta x)$ are linear functions of d, and the second equation comes from the feasibility of Δx for the problem (5.31). Therefore, we obtain from Lemma 5.2 that
$$\Delta x = \mathrm{Prox}'_\theta(\bar{x} - \nabla_x l(\bar{x},\bar{y}); \Delta x + \Xi(\Delta x, \bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4)).$$
Moreover, since $\Pi'_{\mathcal{P}}(h(\bar{x}) + \bar{y}; h'(\bar{x})\Delta x + \Delta y) = \Pi_{\mathcal{T}_{\mathcal{P}}(h(\bar{x}))\cap \bar{y}^{\perp}}(h'(\bar{x})\Delta x + \Delta y)$ by [30, Theorem 4.1.1], we conclude by combining (5.32) that
$$\begin{aligned} G'((\bar{x},\bar{y}); (\Delta x, \Delta y)) = & \begin{pmatrix} \Delta x - \mathrm{Prox}'_\theta(\bar{x} - \nabla_x l(\bar{x},\bar{y}); \Delta x - \nabla^2_{xx} l(\bar{x},\bar{y})\Delta x - \nabla h(\bar{x})\Delta y) \\ h'(\bar{x})\Delta x - \Pi'_{\mathcal{P}}(h(\bar{x}) + \bar{y}; h'(\bar{x})\Delta x + \Delta y) \end{pmatrix} \\ = & \ 0, \end{aligned}$$
for $\Delta x \neq 0$. This is a contradiction to the isolated calmness of G^{-1} at the origin for $(\bar{x},\bar{y})$ by Lemma 2.1. $\square$

The following proposition, which characterize the non-empty property of the $S_{\rm KKT}$ map, can be obtained directly from [21] by noting the second order regularity of the nuclear norm function.

Proposition 5.2. *Let $\bar{x}$ be an isolated local optimal solution of the problem (5.1) and the RCQ (5.3) holds at $\bar{x}$. Let $y \in \mathcal{M}(\bar{x})$ and assume that the SRCQ (5.4) holds at $\bar{x}$ with respect to $\bar{y}$. Then there exists a neighborhood $\mathcal{N}(\bar{x},\bar{y})$ of $(\bar{x},\bar{y})$ and $\mathcal{N}(0)$ of $0 \in \mathcal{X} \times \mathcal{Y}$ such that for any $\delta \in \mathcal{N}(0)$, we have $S_{KKT}(\delta) \cap \mathcal{N}(\bar{x},\bar{y}) \neq \emptyset$.*

Our first main result of this Chapter is given in the next theorem, which fully characterizes the robust isolated calmness of the problem (5.1) by the second order sufficient condition and the strict Robinson constraint qualification.

Theorem 5.1. *Suppose that $\bar{x} \in \mathcal{R}^{m\times n}$ is a feasible solution of the problem (5.1) and the RCQ (5.3) holds at $\bar{x}$. Let $\bar{y} \in \mathcal{M}(\bar{x})$. Then the following two statements are equivalent to each other:*
(i) The second order sufficient condition (5.6) holds at $\bar{x}$ and the SRCQ (5.4) holds at $\bar{x}$ with respect to $\bar{y} \in \mathcal{M}(\bar{x})$.
(ii) $\bar{x}$ is a locally optimal solution of the problem (5.1) and the multi-valued mapping S_{KKT} defined in (5.28) is robust isolated calm at the origin with respect to $(\bar{x},\bar{y})$.

Proof. (i)$\Longrightarrow$ (ii): Since the second order sufficient condition (5.6) holds at $\bar{x}$, $\bar{x}$ must be a local optimal solution. From Proposition 5.2 and the SRCQ assumption at $\bar{x}$ we see that the mapping $S_{\rm KKT}$ is nonempty valued. Thus, it suffices to show that $S_{\rm KKT}$ is isolated calm at $(\bar{x},\bar{y})$ for the origin, which by Theorem 2.5 is equivalent to the isolated calmness of G^{-1} at the origin with respect to $(\bar{x},\bar{y})$.

Under the assumption that the SRCQ holds at $\bar{x}$, we have $\bar{y}$ is the unique Lagrangian multiplier with respect to $\bar{x}$ of the problem (5.1) by Proposition 2.6. Let $(d_x, d_y) \in \mathcal{R}^{m\times n} \times \mathcal{Y}$ satisfies that $G'((\bar{x},\bar{y});(d_x,d_y)) = 0$, which is equivalent to say

$$\begin{cases} d_x - {\rm Prox}'_\theta(\bar{x} - \nabla_x l(\bar{x},\bar{y}); d_x - \nabla^2_{xx} l(\bar{x},\bar{y})d_x - \nabla h(\bar{x})d_y) = 0, \\ h'(\bar{x})d_x - \Pi'_{\mathcal{P}}(h(\bar{x}) + \bar{y}; h'(\bar{x})d_x + d_y) = 0. \end{cases} \tag{5.33}$$

By the optimality condition that $G(\bar{x},\bar{y})=0$, we have $\mathrm{Prox}_\theta(\bar{x}-\nabla_x l(\bar{x},\bar{y}))=\bar{x}$. Then from Lemma 5.2, we can obtain the first equation in (5.33) holds if and only if

$$\theta'(\bar{x};d_x)+\langle\nabla_x l(\bar{x},\bar{y}),d_x\rangle=0,$$

and

$$\langle d_x,\nabla^2_{xx}l(\bar{x},\bar{y})d_x+\nabla h(\bar{x})d_y\rangle=\psi^*_{(\bar{x},d_x)}(-\nabla_x l(\bar{x},\bar{y})). \tag{5.34}$$

Also by [40, Lemma 4.2] and the fact that $h(\bar{x})=\Pi_{\mathcal{P}}(h(\bar{x})+\bar{y})$, we have that the second equation in (5.33) holds if and only if

$$h'(\bar{x})d_x\in\mathcal{T}_{\mathcal{P}}(h(\bar{x}))\cap\bar{y}^\perp\quad\text{and}\quad\langle h'(\bar{x})d_x,d_y\rangle=0. \tag{5.35}$$

Thus, we have that

$$f'(\bar{x})d_x+\theta'(\bar{x};d_x)=f'(\bar{x})d_x-\langle\nabla_x l(\bar{x},\bar{y}),d_x\rangle=-\langle\nabla h(\bar{x})\bar{y},d_x\rangle=0,$$

so that $d_x\in\mathcal{C}(\bar{x})$. Moreover, from (5.34) and (5.35) we have

$$\langle d_x,\nabla^2_{xx}l(\bar{x},\bar{y})d_x\rangle-\psi^*_{(\bar{x},d_x)}(-\nabla_x l(\bar{x},\bar{y}))=0.$$

Since the second order sufficient condition holds at $\bar{x}$, we must have $d_x=0$. Now (5.33) can be simplified as

$$\begin{cases}\mathrm{Prox}'_\theta(\bar{x}-\nabla_x l(\bar{x},\bar{y});-\nabla h(\bar{x})d_y)=0,\\[2mm] \Pi'_{\mathcal{P}}(h(\bar{x})+\bar{y};d_y)=0.\end{cases}$$

By Lemma 5.1 (ii), we have from the first equation that

$$-\nabla h(\bar{x})d_y\in\big(\mathcal{T}^\theta(\bar{x},-\nabla_x l(\bar{x},\bar{y}))\big)^\circ,$$

where $\mathcal{T}^\theta(\bar{x},-\nabla_x l(\bar{x},\bar{y}))$ is defined by (5.22). Moreover, from [40, Lemma 4.2], we have that the second equation is equivalent to say

$$d_y\in(\mathcal{T}_{\mathcal{P}}(h(\bar{x}))\cap\bar{y}^\perp)^\circ.$$

Thus, we obtain that

$$\begin{pmatrix} -d_y \\ \nabla h(\bar{x}) d_y \end{pmatrix} \in \left(\begin{pmatrix} h'(\bar{x}) \\ \mathcal{I} \end{pmatrix} \mathcal{R}^{m\times n} + \begin{pmatrix} \mathcal{T}_{\mathcal{P}}(h(\bar{x})) \cap \bar{y}^{\perp} \\ \mathcal{T}^{\theta}(\bar{x}, -\nabla_x l(\bar{x}, \bar{y})) \end{pmatrix} \right)^{\circ},$$

so that $d_y = 0$ by the assumption that SRCQ (5.4) holds at $\bar{x}$ with respect to $\bar{y} \in \mathcal{M}(\bar{x})$.

Therefore, G^{-1} is isolated calm at the origin for $(\bar{x}, \bar{y})$ by Lemma 2.1.

(ii)$\Longrightarrow$ (i). This part is proved by contradiction. By using Theorem 5.2 again and the isolated calmness of G^{-1} at the origin for $(\bar{x}, \bar{y})$, we have $S_{\rm KKT}$ is isolated calm at $(\bar{x}, \bar{y})$ for the origin. First let us suppose the SRCQ (5.4) does not hold at $\bar{x}$ with respect to $\bar{y} \in \mathcal{M}(\bar{x})$. Then there exists nonzero $(d_x, d_y) \in \mathcal{R}^{m\times n} \times \mathcal{Y}$ such that

$$\begin{pmatrix} d_y \\ d_x \end{pmatrix} \in \left(\begin{pmatrix} h'(\bar{x}) \\ \mathcal{I} \end{pmatrix} \mathcal{R}^{m\times n} + \begin{pmatrix} \mathcal{T}_{\mathcal{P}}(h(\bar{x})) \cap \bar{y}^{\perp} \\ \mathcal{T}^{\theta}(\bar{x}, -\nabla_x l(\bar{x}, \bar{y})) \end{pmatrix} \right)^{\circ},$$

which is equivalent to

$$\begin{cases} \nabla h(\bar{x}) d_y + d_x = 0, \\ d_x \in \left(\mathcal{T}^{\theta}(\bar{x}, -\nabla_x l(\bar{x}, \bar{y}))\right)^{\circ} \\ d_y \in (\mathcal{T}_{\mathcal{P}}(h(\bar{x})) \cap \bar{y}^{\perp})^{\circ}. \end{cases} \tag{5.36}$$

From Lemma 5.1 (ii) and [40, Lemma 4.2], we get

$$\begin{cases} \mathrm{Prox}'_{\theta}(\bar{x} - \nabla_x l(\bar{x}, \bar{y}); -\nabla h(\bar{x}) d_y) = \mathrm{Prox}'_{\theta}(\bar{x} - \nabla_x l(\bar{x}, \bar{y}); d_x) = 0 \\ \Pi'_{\mathcal{P}}(h(\bar{x}) + \bar{y}; d_y) = 0. \end{cases}$$

Since (d_x, d_y) is assumed to be nonzero, we have that $d_y \neq 0$ by the first equation in (5.36). This shows that $G'((\bar{x}, \bar{y}); (0, d_y)) = 0$ along a nonzero direction $(0, d_y)$, which is a contradiction of the isolated calm of G^{-1} at the origin for $(\bar{x}, \bar{y})$ by Lemma 2.1.

In the following, we show the second order sufficient condition (5.6) holds at $\bar{x}$. We also prove this statement by contradiction. From the previous proof we know that the SRCQ (5.4) holds at $\bar{x}$ with respect to $\bar{y}$ under the isolated calmness assumption of G^{-1} at the origin for $(\bar{x}, \bar{y})$, and thus, the multiplier $\bar{y} \in \mathcal{M}(\bar{x})$ is unique.

By the assumption that the SRCQ holds at the local optimal solution $\bar{x}$, we have the second order necessary condition holds at $\bar{x}$, i.e.,

$$\langle d, \nabla^2_{xx} l(\bar{x}, \bar{y}) d \rangle - \psi^*_{(\bar{x},d)}(-\nabla_x l(\bar{x}, \bar{y})) \geq 0, \quad \forall d \in \mathcal{C}(\bar{x}) \setminus \{0\}.$$

Suppose the second order sufficient condition does not hold at $\bar{x}$. Since $\bar{x}$ is a local optimal solution of the problem (5.1), the second order necessary condition (5.5) holds at $\bar{x}$. Then there exists $d_x \in \mathcal{C}(\bar{x}) \setminus \{0\}$ such that

$$\langle d_x, \nabla^2_{xx} l(\bar{x}, \bar{y}) d_x \rangle - \psi^*_{(\bar{x},d_x)}(-\nabla_x l(\bar{x}, \bar{y})) = 0.$$

By Lemma 5.3 this further indicates that there exists $\bar{d}_x \in \mathcal{C}(\bar{x})$ such that (5.30) holds. Therefore, for any $t > 0$ sufficiently small, we have

$$\begin{aligned}
& \langle (d_x + t\bar{d}_x), \nabla^2_{xx} l(\bar{x}, \bar{y})(d_x + t\bar{d}_x) \rangle - \psi^*_{(\bar{x}, d_x + t\bar{d}_x)}(-\nabla_x l(\bar{x}, \bar{y})) \\
= \; & \langle (d_x + t\bar{d}_x), \nabla^2_{xx} l(\bar{x}, \bar{y})(d_x + t\bar{d}_x) \rangle - 2\Gamma_{(\bar{x}, -\nabla_x l(\bar{x},\bar{y}))}((d_x + t\bar{d}_x), (d_x + t\bar{d}_x)) \\
= \; & \langle d_x, \nabla^2_{xx} l(\bar{x}, \bar{y}) d_x \rangle - 2\Gamma_{(\bar{x}, -\nabla_x l(\bar{x},\bar{y}))}(d_x, d_x) \\
& + 2t\big(\langle d_x, \nabla^2_{xx} l(\bar{x}, \bar{y}) \bar{d}_x \rangle - 2\Gamma_{(\bar{x}, -\nabla_x l(\bar{x},\bar{y}))}(d_x, \bar{d}_x)\big) \\
& + t^2\big(\langle \bar{d}_x, \nabla^2_{xx} l(\bar{x}, \bar{y}) \bar{d}_x \rangle - 2\Gamma_{(\bar{x}, -\nabla_x l(\bar{x},\bar{y}))}(\bar{d}_x, \bar{d}_x)\big) < 0.
\end{aligned}$$

By noting that $d_x + t\bar{d}_x \in \mathcal{C}(\bar{x})$ since $\mathcal{C}(\bar{x})$ is a convex cone, we see the second order necessary condition (5.5) fails at $\bar{x}$ and thus, $\bar{x}$ cannot be a local optimal solution. This contradiction implies that the second order sufficient condition (5.6) holds at $\bar{x}$.

$\square$

5.2 The robust isolated calmness for the convex composite nuclear norm minimization problems

A specific and perhaps the most popular application of the nuclear norm minimization model (5.1) is the following convex composite nuclear norm minimization problem:

$$
\begin{aligned}
\min \quad & f(\mathcal{L}x) + \langle c, x\rangle + \|x\|_* \\
\text{s.t.} \quad & \mathcal{A}x = b, \quad x \in \mathcal{P},
\end{aligned} \tag{5.37}
$$

where $f : \mathcal{R}^l \to \mathcal{R}$ is a twice continuously differentiable and strongly convex function, $\mathcal{L} : \mathcal{R}^{m\times n} \to \mathcal{R}^l$ and $\mathcal{A} : \mathcal{R}^{m\times n} \to \mathcal{R}^e$ are linear operators, $c \in \mathcal{R}^{m\times n}$, $b \in \mathcal{R}^e$ are the given data and $\mathcal{P} \subseteq \mathcal{R}^{m\times n}$ is a convex polyhedral. Following the notation in the previous section, we denote $\theta : \mathcal{R}^{m\times n} \to \mathcal{R}$ as $\theta(\cdot) \equiv \|\cdot\|_*$.

The aim of this section is to provide more complete characterization of the robust isolated calmness for the KKT system, by also combining the information provided from the dual problem of (5.37). We show that the second order sufficient condition for the problem (5.37), which is required in Theorem 5.1, is in fact equivalent to the extended SRCQ of its dual problem, and vice versa. In this way, the error bound conditions can be checked by several equivalent conditions.

In order to write down the dual problem explicitly, we first reformulate the problem (5.37) by introducing auxiliary variables $w \in \mathcal{R}^l$, $u, v \in \mathcal{R}^{m\times n}$ and write (5.37) as

$$
\begin{aligned}
\min \quad & f(w) + \langle c, x\rangle + \theta(x) \\
\text{s.t.} \quad & \mathcal{A}x = b, \quad \mathcal{L}x = w, \quad x = v, \quad v \in \mathcal{P},
\end{aligned} \tag{5.38}
$$

where $\delta_{\mathcal{P}}(\cdot)$ is the indicator function of $\mathcal{P}$. The Lagrangian dual problem with respect to the problem (5.38) is given by

$$
\begin{aligned}
\max \quad & \langle b, y\rangle - f^*(-\xi) - \theta^*(-s) - \delta^*_{\mathcal{P}}(-z) \\
\text{s.t.} \quad & \mathcal{A}^*y + \mathcal{L}^*\xi + s + z = c.
\end{aligned} \tag{5.39}
$$

Define the KKT mapping $G_P : \mathcal{R}^{m\times n}\times\mathcal{R}^l\times\mathcal{R}^{m\times n}\times\mathcal{R}^e\times\mathcal{R}^l\times\mathcal{R}^{m\times n}$ associated with the primal problem (5.37) as

$$G_P(x,w,v,y,\xi,z) := \begin{pmatrix} -x+\mathrm{Prox}_\theta(x-c+\mathcal{A}^*y+\mathcal{L}^*\xi+z) \\ -\nabla f(w)+\xi \\ -v+\Pi_{\mathcal{P}}(v-z) \\ \mathcal{A}x-b \\ \mathcal{L}x-w \\ x-v \end{pmatrix}. \tag{5.40}$$

Suppose that $(\bar{x},\bar{w},\bar{v})\in\mathcal{R}^{m\times n}\times\mathcal{R}^l\times\mathcal{R}^{m\times n}$ is an optimal solution of the problem (5.38). We use $\mathcal{M}_P(\bar{x},\bar{w},\bar{v})$ to denote the set of multipliers $(\bar{y},\bar{\xi},\bar{z})\in\mathcal{R}^{m\times n}\times\mathcal{R}^l\times\mathcal{R}^{m\times n}$ such that $G_P(\bar{x},\bar{w},\bar{v},\bar{y},\bar{\xi},\bar{z})=0$, and call $(\bar{x},\bar{w},\bar{v},\bar{y},\bar{\xi},\bar{z})$ a KKT point of the problem (5.38).

The canonically perturbed problem of (5.38) takes the form of

$$\begin{aligned} \min\quad & f(w)+\langle c,x\rangle+\theta(x)-\langle x,\delta_1\rangle-\langle w,\delta_2\rangle \\ \text{s.t.}\quad & \mathcal{A}x=b-\delta_3,\quad \mathcal{L}x=w-\delta_4,\quad x=v-\delta_5,\quad v-\delta_6\in\mathcal{P}, \end{aligned} \tag{5.41}$$

where $\delta\equiv(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6)\in\mathcal{R}^{m\times n}\times\mathcal{R}^l\times\mathcal{R}^e\times\mathcal{R}^l\times\mathcal{R}^{m\times n}\times\mathcal{R}^{m\times n}$ is the perturbation parameter.

Similarly as in Section 2.5.3, for an given $\delta\equiv(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6)\in\mathcal{R}^{m\times n}\times\mathcal{R}^l\times\mathcal{R}^e\times\mathcal{R}^l\times\mathcal{R}^{m\times n}\times\mathcal{R}^{m\times n}$, we define the multi-valued mapping S_{KKT} for the problem (5.41) as

$$\begin{aligned} S_{\mathrm{KKT}}(\delta) := \; & \{(x,w,v,y,\xi,z)\in\mathcal{R}^{m\times n}\times\mathcal{R}^l\times\mathcal{R}^{m\times n}\times\mathcal{R}^e\times\mathcal{R}^l\times\mathcal{R}^{m\times n} : \\ & \quad x=\mathrm{Prox}_\theta(x-c+\mathcal{A}^*y+\mathcal{L}^*\xi+z+\delta_1),\ \nabla f(w)-\delta_2+\xi=0, \\ & \quad \mathcal{A}x=b-\delta_3,\ \mathcal{L}x=w-\delta_4,\ x=v-\delta_5,\ v-\delta_6=\Pi_{\mathcal{P}}(v-\delta_6-z)\}. \end{aligned}$$

Let $(\bar{x},\bar{w},\bar{v},\bar{y},\bar{\xi},\bar{z})\in\mathcal{R}^{m\times n}\times\mathcal{R}^l\times\mathcal{R}^{m\times n}\times\mathcal{R}^e\times\mathcal{R}^l\times\mathcal{R}^{m\times n}$ be a KKT point of the problem (5.38). It is known from Theorem 2.5 that the isolated calmness of G_P^{-1} at the origin with respect to $(\bar{x},\bar{w},\bar{v},\bar{y},\bar{\xi},\bar{z})$ is equivalent to the isolated calmness of S_{KKT} at $(\bar{x},\bar{w},\bar{v},\bar{y},\bar{\xi},\bar{z})$ with respect to the origin.

Since f is assumed to be strongly convex, f^* is a smooth function. We define the KKT mapping $G_D : \mathcal{R}^e \times \mathcal{R}^l \times \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n}$ associated with the dual problem (5.39) as:

$$G_D(y, \xi, s, z, x) := \begin{pmatrix} -\mathcal{A}^* y - \mathcal{L}^* \xi - s - z + c \\ \mathcal{A}x - b \\ -\nabla f^*(-\xi) + \mathcal{L}x \\ s + \mathrm{Prox}_{\theta^*}(-s + x) \\ z + \mathrm{Prox}_{\delta_P^*}(-z + x) \end{pmatrix}. \tag{5.42}$$

Suppose that $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z}) \in \mathcal{R}^e \times \mathcal{R}^l \times \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n}$ is an optimal solution of the dual problem (5.39). Denote $\mathcal{M}_D(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$ as the set of multipliers such that $G_D(\bar{y}, \bar{\xi}, \bar{s}, \bar{z}, \bar{x}) = 0$ for $\bar{x} \in \mathcal{M}_D(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$.

Define multi-valued mappings $\mathcal{T}^{\theta^*} : \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n} \to \mathcal{R}^{m\times n}$ and $\mathcal{T}^{\delta_{\mathcal{P}}^*} : \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n} \to \mathcal{R}^{m\times n}$ as

$$\begin{aligned} \mathcal{T}^{\theta^*}(s, x) &:= \{d \in \mathcal{R}^{m\times n} : (\theta^*)'(s; d) = \langle d, x\rangle\}, \quad \forall (s, x) \in \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n}, \\ \mathcal{T}^{\delta_{\mathcal{P}}^*}(z, x) &:= \{d \in \mathcal{R}^{m\times n} : (\delta_{\mathcal{P}}^*)'(z; d) = \langle d, x\rangle\}, \quad \forall (z, x) \in \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n}. \end{aligned}$$

Definition 5.1. *Let $(\bar{x}, \bar{w}, \bar{v})$ be an optimal solution of the problem (5.38). Suppose that $\mathcal{M}_P(\bar{x}, \bar{w}, \bar{v}) \neq \emptyset$. We say the extended SRCQ for the problem (5.39) holds at $\mathcal{M}_P(\bar{x}, \bar{w}, \bar{v})$ with respect to $(\bar{x}, \bar{w}, \bar{v})$ if*

$$conv\left\{ \bigcup_{(\bar{y},\bar{\xi},\bar{z})\in\mathcal{M}_P(\bar{x},\bar{w},\bar{v})} \left(\mathcal{T}^{\theta^*}(-c + \mathcal{A}^*\bar{y} + \mathcal{L}^*\bar{\xi} + \bar{z}, \bar{x}) + \mathcal{T}^{\delta_{\mathcal{P}}^*}(\bar{z}, \bar{x}) \right) \right\} - \mathcal{A}^*\mathcal{R}^e - \mathcal{L}^*\mathcal{R}^l = \mathcal{R}^{m\times n}. \tag{5.43}$$

This is a generalization of the previous discussed SRCQ by considering all the points $(\bar{y}, \bar{\xi}, \bar{z})$ in the set $\mathcal{M}_P(\bar{x}, \bar{w}, \bar{v})$.

Given $A, B, C \in \mathcal{R}^{m\times n}$ satisfy the relationship (5.8), Ding [20, Proposition 12] characterize the set $\mathcal{T}^{\theta^*}(B, A)$, which shows the for any $H \in \mathcal{T}^{\theta^*}(B, A)$, it must admits the form

$$\widetilde{H} = \left(\begin{array}{c:c:c:c:c} T(\widetilde{H}_{aa}) & T(\widetilde{H}_{ab_1}) & \widetilde{H}_{ab_2} & \widetilde{H}_{ab_3} & \\ \hdashline T(\widetilde{H}_{b_1a}) & \Pi_{\mathcal{S}_-^{|b_1|}}(\widetilde{H}_{b_1b_1}) & \widetilde{H}_{b_1b_2} & \widetilde{H}_{b_1b_3} & \widetilde{H}_2 \\ \hdashline \widetilde{H}_{b_2a} & \widetilde{H}_{b_2b_1} & \widetilde{H}_{b_2b_2} & \widetilde{H}_{b_2b_3} & \\ \hdashline \widetilde{H}_{b_3a} & \widetilde{H}_{b_3b_1} & \widetilde{H}_{b_3b_2} & \widetilde{H}_{b_3b_3} & \end{array} \right), \tag{5.44}$$

where $\widetilde{H} = U^T HV$ and the operator T is defined as in (5.19).

Similarly as in Section 5.1, we define $\phi_{(s,h)} = (\theta^*)''(s; h, \cdot)$ for $(s, h) \in \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n}$, the parabolic second order directional derivative of θ^*. The conjugate function of $\phi_{(\bar{s},h)}(\cdot)$ is the sigma term of the second order sufficient condition for the dual problem. Before showing the main result, we first show an important observation about the sets $\mathcal{T}^{\theta}$, $\mathcal{T}^{\theta^*}$ and the two sigma term generated by θ and θ^* at the stationary point.

Lemma 5.4. *Suppose $A, B, C \in \mathcal{R}^{m\times n}$ satisfy the relationship (5.8). Then for any $H, D \in \mathcal{R}^{m\times n}$, we have*

(i)

$$H \in (\mathcal{T}^{\theta}(A, B))^{\circ} \Longleftrightarrow \phi^*_{(B,H)}(A) = 0 \quad \text{and} \quad H \in \mathcal{T}^{\theta^*}(B, A).$$

(ii)

$$H \in (\mathcal{T}^{\theta^*}(B, A))^{\circ} \Longleftrightarrow \psi^*_{(A,H)}(B) = 0 \quad \text{and} \quad H \in \mathcal{T}^{\theta}(A, B).$$

Proof. The conclusions of the two parts could be obtained by comparing the characterization of H between the two sides. For part (i), we have from [20, Proposition 16] that $\phi^*_{(B,H)}(A) = 0$ if and only if $\psi^*_{(A,H)}(B) = 0$ for $H \in \mathcal{R}^{m\times n}$. Then by combining Lemma 5.3 (i), Lemma 5.4 and (5.44), one can see that either $H \in (\mathcal{T}^{\theta^*}(B, A))^*$ or $\phi^*_{(B,H)}(A) = 0$ & $H \in \mathcal{T}^{\theta^*}(B, A)$ are equivalent to say

$$\widetilde{H}_{aa} = 0, \quad \widetilde{H}_{ab} = 0, \quad \widetilde{H}_{a2} = 0, \quad \widetilde{H}_{ba} = 0, \quad \widetilde{H}_{b_1b_1} \preceq 0.$$

Similarly for part (ii), either $H \in (\mathcal{T}^{\theta^*}(B, A))^*$ or $\psi^*_{(A,H)}(B) = 0$ & $H \in \mathcal{T}^{\theta}(A, B)$

can be characterized by

$$T(\widetilde{H}_{aa}) = 0; \quad T(\widetilde{H}_{ab_1}) = 0; \quad T(\widetilde{H}_{b_1a}) = 0; \quad \widetilde{H}_{b_1b_1} \succeq 0,$$
$$\widetilde{H}_{ab_2} = \widetilde{H}^T_{b_2a} = 0; \quad \widetilde{H}_{ab_3} = \widetilde{H}^T_{b_3a} = 0; \quad \widetilde{H}_2 = 0.$$

This completes the proof. □

The following proposition shows the equivalence between the primal second order sufficient condition and the dual extended SRCQ.

Proposition 5.3. *Let $(\bar{x}, \bar{w}, \bar{v}) \in \mathcal{R}^{m\times n} \times \mathcal{R}^l \times \mathcal{R}^{m\times n}$ be an optimal solution of the problem (5.38) and assume that $\mathcal{M}_P(\bar{x}, \bar{w}, \bar{v})$ is nonempty. Then the following two statements are equivalent to each other:*

(i) The second order sufficient condition holds at $(\bar{x}, \bar{w}, \bar{v})$ with respect to the problem (5.38):

$$\sup_{(\bar{y},\bar{\xi},\bar{z})\in\mathcal{M}_P(\bar{x},\bar{w},\bar{v})} \left\{\langle d_w, \nabla^2 f(\bar{w})d_w\rangle - \psi^*_{(\bar{x},d_x)}(-c + \mathcal{A}^*(\bar{y} + \mathcal{L}^*\bar{\xi} + \bar{z})\right\} > 0, \qquad \forall (d_x, d_w, d_v) \in \mathcal{C}(\bar{x}, \bar{w}, \bar{v}) \setminus \{0\}, \tag{5.45}$$

where the critical cone, in the sense of (2.40), is defined by

$$\begin{aligned}\mathcal{C}(\bar{x}, \bar{w}, \bar{v}) = \ & \{(d_x, d_w, d_v) \in \mathcal{R}^{m\times n} \times \mathcal{R}^l \times \mathcal{R}^{m\times n} : \mathcal{A}d_x = 0,\ \mathcal{L}d_x - d_w = 0,\\ & d_x - d_v = 0;\ f'(\bar{w})d_w = 0, d_x \in \mathcal{T}^\theta(\bar{x}, -c), d_v \in \mathcal{T}_\mathcal{P}(\bar{v})\}.\end{aligned} \tag{5.46}$$

(ii) The extended SRCQ (5.43) holds at $\mathcal{M}_P(\bar{x}, \bar{w}, \bar{v})$ with respect to $(\bar{x}, \bar{w}, \bar{v})$ for the dual problem (5.39).

Proof. First we prove (i) implies (ii) by contradiction. Suppose the condition (5.43) fails to hold at $\mathcal{M}_P(\bar{x}, \bar{w}, \bar{v})$. Denote

$$\mathcal{E} := \operatorname{conv}\left\{\bigcup_{(\bar{y},\bar{\xi},\bar{z})\in\mathcal{M}_P(\bar{x},\bar{w},\bar{v})} \left(\mathcal{T}^{\theta^*}(-c + \mathcal{A}^*\bar{y} + \mathcal{L}^*\bar{\xi} + \bar{z}, \bar{x}) + \mathcal{T}^{\delta^*_\mathcal{P}}(\bar{z}, \bar{x})\right)\right\} - \mathcal{A}^*\mathcal{R}^e - \mathcal{L}^*\mathcal{R}^l.$$

Then there exists $D \in \mathcal{R}^{m\times n}$ such that $D \notin \operatorname{cl}(\mathcal{E})$ since $\operatorname{cl}(\mathcal{E}) \neq \mathcal{R}^{m\times n}$ by [80, Theorem 6.3]. Note that $\operatorname{cl}(\mathcal{E})$ is a closed convex cone. We have by letting $\bar{d} := D - \Pi_{\operatorname{cl}(\mathcal{E})}(D) = \Pi_{(\operatorname{cl}(\mathcal{E}))^\circ}(D) \neq 0$ that

$$\langle H, \bar{d}\rangle \leq 0, \quad \forall H \in \operatorname{cl}(\mathcal{E}).$$

By recalling the notation of $\mathcal{E}$, the above inequality implies that $\mathcal{A}\bar{d} = 0$, $\mathcal{L}\bar{d} = 0$ and

$$\langle H, \bar{d}\rangle \leq 0, \quad \forall H \in \operatorname{conv}\left\{\bigcup_{(\bar{y},\bar{\xi},\bar{z})\in\mathcal{M}_P(\bar{x},\bar{w},\bar{v})} \left(\mathcal{T}^{\theta^*}(-c+\mathcal{A}^*\bar{y}+\mathcal{L}^*\bar{\xi}+\bar{z},\bar{x}) + \mathcal{T}^{\delta^*_{\mathcal{P}}}(\bar{z},\bar{x})\right)\right\}.$$

Let $(\bar{y},\bar{\xi},\bar{z}) \in \mathcal{M}_P(\bar{x},\bar{w},\bar{v})$. By the optimality condition that $\bar{v} = \Pi_{\mathcal{P}}(\bar{v}-\bar{z})$ and $\bar{x}-\bar{v} = 0$, we have by [40, Lemma 4.2] that

$$\mathcal{T}_{\mathcal{P}}(\bar{v}) \cap \bar{z}^{\perp} = (\mathcal{T}_{\delta_{\mathcal{P}^*}}(\bar{z},\bar{x}))^{\circ},$$

so that $\bar{d} \in \mathcal{T}_{\mathcal{P}}(\bar{v}) \cap \bar{z}^{\perp}$.

Similarly, we could also obtain from the optimality condition $\bar{x} = \operatorname{Prox}_{\theta}(\bar{x} - c + \mathcal{A}^*\bar{y} + \mathcal{L}^*)$ and Lemma 5.4 that

$$\psi^*_{(\bar{x},\bar{d})}(-c+\mathcal{A}^*\bar{y}+\mathcal{L}^*\bar{\xi}+\bar{z}) = 0 \quad \text{and} \quad \theta'(\bar{x};\bar{d}) = \langle -c+\mathcal{A}^*\bar{y}+\mathcal{L}^*\bar{\xi}+\bar{z}, \bar{d}\rangle.$$

Therefore, we have $\bar{d} \neq 0$ satisfies that

$$\mathcal{A}\bar{d} = 0, \quad \mathcal{L}\bar{d} = 0, \quad \langle \nabla f(\bar{w}), \mathcal{L}\bar{d}\rangle = 0, \quad \bar{d} \in \mathcal{T}_{\mathcal{P}}(\bar{v}),$$

and

$$0 = \theta'(\bar{x};\bar{d}) + \langle c - \mathcal{A}^*\bar{y} - \mathcal{L}^*\bar{\xi} - \bar{z}, \bar{d}\rangle = \theta'(\bar{x};\bar{d}) + \langle c, \bar{d}\rangle,$$

which implies that $(\bar{d}, \mathcal{L}\bar{d}, \bar{d}) \in \mathcal{C}(\bar{x},\bar{w},\bar{v}) \setminus \{0\}$.

Altogether, the above arguments show that for any $(\bar{y},\bar{\xi},\bar{z}) \in \mathcal{M}_P(\bar{x},\bar{w},\bar{v})$, there exists $(\bar{d}, \mathcal{L}\bar{d}, \bar{d}) \in \mathcal{C}(\bar{x},\bar{w},\bar{v}) \setminus \{0\}$ such that

$$\langle \mathcal{L}\bar{d}, \nabla^2 f(\bar{w})\mathcal{L}\bar{d}\rangle - \psi^*_{(\bar{x},\bar{d})}(-c+\mathcal{A}^*\bar{y}+\mathcal{L}^*\bar{\xi}+\bar{z}) = 0,$$

which contradicts the assumption that the second order sufficient condition (5.45) holds at $(\bar{x},\bar{w},\bar{v})$. This completes the proof of the first part.

The reverse direction can also be proven by contradiction. Suppose the second order sufficient condition fails to hold at an optimal point $(\bar{x},\bar{w},\bar{v})$. Then there exists $(\tilde{d}, \mathcal{L}\tilde{d}, \tilde{d}) \in \mathcal{C}(\bar{x},\bar{u},\bar{v}) \setminus \{0\}$ such that

$$\sup_{(\bar{y},\bar{\xi},\bar{z})\in\mathcal{M}_P(\bar{x},\bar{w},\bar{v})} \left\{\langle \mathcal{L}\tilde{d}, \nabla^2 f(\bar{w})\mathcal{L}\tilde{d}\rangle - \psi^*_{(\bar{x},\tilde{d})}(-c+\mathcal{A}^*\bar{y}+\mathcal{L}^*\bar{\xi}+\bar{z})\right\} = 0.$$

By noting that $\mathcal{Q} \succeq 0$ and the sigma-term is always non-positive, we have that

$$\mathcal{L}\tilde{d} = 0 \quad \text{and} \quad \psi^*_{(\bar{x},\tilde{d})}(-c + \mathcal{A}^*\bar{y} + \mathcal{L}^*\bar{\xi} + \bar{z}) = 0, \quad \forall (\bar{y}, \bar{\xi}, \bar{z}) \in \mathcal{M}_P(\bar{x}, \bar{w}, \bar{v}).$$

By the optimality conditions that $-c+\mathcal{A}^*\bar{y}+\mathcal{L}^*\bar{\xi}+\bar{z} \in \partial\theta(\bar{x})$ and $\bar{s} = c-\mathcal{A}^*\bar{y} - \mathcal{L}^*\bar{w} - \bar{z}$, we see

$$\langle -c, \tilde{d}\rangle = \theta'(\bar{x};\tilde{d}) \geq \langle -c + \mathcal{A}^*\bar{y} + \mathcal{L}^*\bar{w} + \bar{z}, \tilde{d}\rangle.$$

This implies that $\langle \bar{z}, \tilde{d}\rangle \geq 0$ since $(\mathcal{A} + \mathcal{L})\tilde{d} = 0$. From $\bar{z} \in \mathcal{N}_{\mathcal{P}}(\bar{v})$ and $\tilde{d} \in \mathcal{T}_{\mathcal{P}}(\bar{v})$, we also have $\langle \bar{z}, \tilde{d}\rangle \leq 0$. Thus, $\langle \bar{z}, \tilde{d}\rangle = 0$ and altogether,

$$\tilde{d} \in \mathcal{T}_{\mathcal{P}}(\bar{v}) \cap \bar{z}^{\perp}, \quad \psi^*_{(\bar{x},\tilde{d})}(-c+\mathcal{A}^*\bar{y}+\mathcal{L}^*\bar{\xi}+\bar{z}) = 0, \quad \theta'(\bar{x};\tilde{d}) = \langle -c+\mathcal{A}^*\bar{y}+\mathcal{L}^*\bar{\xi}+\bar{z}, \tilde{d}\rangle.$$

By [40, Lemma 4.2] and Lemma 5.4, the above conditions are the same as

$$\tilde{d} \in \big(\mathcal{T}_{\delta^*_{\mathcal{P}}}(\bar{z},\bar{x})\big)^{\circ} \cap \big(\mathcal{T}^{\theta^*}(-c+\mathcal{A}^*\bar{y}+\mathcal{L}^*\bar{\xi}+\bar{z},\bar{x})\big)^{\circ} = \Big(\mathcal{T}^{\theta^*}(-c+\mathcal{A}^*\bar{y}+\mathcal{L}^*\bar{\xi}+\bar{z},\bar{x}) + \mathcal{T}^{\delta^*_{\mathcal{P}}}(\bar{z},\bar{x})\Big)^{\circ}.$$

By the assumption that the extended SRCQ (5.43) holds at $(\bar{x}, \bar{w}, \bar{v})$ with respect to $\mathcal{M}_P(\bar{x}, \bar{w}, \bar{v})$, there exist $\tilde{y} \in \mathcal{R}^m$, $\tilde{\xi} \in \mathcal{R}^l$ and

$$\tilde{h} \in \operatorname{conv}\left\{ \bigcup_{(\bar{y},\bar{\xi},\bar{s},\bar{z}) \in \mathcal{M}_P(\bar{x},\bar{w},\bar{v})} \Big(\mathcal{T}^{\theta^*}(-c + \mathcal{A}^*\bar{y} + \mathcal{L}^*\bar{\xi} + \bar{z}, \bar{x}) + \mathcal{T}^{\delta^*_{\mathcal{P}}}(\bar{z},\bar{x})\Big)\right\}$$

such that $\tilde{d}$ can be decomposed as:

$$\tilde{d} = \tilde{h} - \mathcal{A}^*\tilde{y} - \mathcal{L}^*\tilde{\xi}.$$

By Carathéodory's Theorem, there exist $k \leq (mn+1)$, $u_i \in [0,1]$, $\sum_{i=1}^k \mu_i = 1$, $\tilde{h}_1^i \in \mathcal{T}^{\theta^*}(-c + \mathcal{A}^*\bar{y}_i + \mathcal{L}^*\bar{\xi}_i + \bar{z}_i, \bar{x})$ and $\tilde{h}_2^i \in \mathcal{T}^{\delta^*_{\mathcal{P}}}(\bar{z}_i, \bar{x})$ for some $(\bar{y}_i, \bar{\xi}_i, \bar{z}_i) \in \mathcal{M}_P(\bar{x}, \bar{w}, \bar{v})$ such that $\tilde{h} = \sum_{i=1}^k \mu_i(\tilde{h}_1^i + \tilde{h}_2^i)$. Then

$$\langle \tilde{d}, \tilde{d}\rangle = \langle \tilde{d}, \tilde{h} - \mathcal{A}^*\tilde{y} - \mathcal{L}^*\tilde{\xi}\rangle = \sum_{i=1}^k \mu_i \langle \tilde{d}, \tilde{h}_1^i + \tilde{h}_2^i\rangle \leq 0,$$

which contradicts the assumption that $\tilde{d} \neq 0$ and thus, the proof is finished.

$\square$

Remark 5.1. *Under our assumption that f is strongly convex and twice continuously differentiable, the second order sufficient condition (5.45) holds automatically. A more interesting result is to concern a continuously differentiable function f with Lipschitz continuous gradient. Under this setting, one could use the Clarke's generalized Hessian given in (2.1) to replace $\nabla^2 f$. All the previous and subsequent discussions should thus be directly extended to this case.*

Since the isolated calmness of the inverse of KKT mapping essentially needs the uniqueness of the KKT solution, we would like to further explore the equivalence between the primal second order sufficient condition and the dual SRCQ when the multiplier set is a singleton.

Corollary 5.1. *Suppose $(\bar{x}, \bar{w}, \bar{v}) \in \mathcal{R}^{m\times n} \times \mathcal{R}^l \times \mathcal{R}^{m\times n}$ is an optimal solution of the problem (5.38) and $\mathcal{M}_P(\bar{x}, \bar{w}, \bar{u}) = \{(\bar{y}, \bar{\xi}, \bar{z})\}$ is a singleton. Then the following two statements are equivalent:*

(i) The second order sufficient condition holds at $(\bar{x}, \bar{w}, \bar{u})$ with respect to the primal problem (5.38):

$$\langle d_w, \nabla^2 f(\bar{w}) d_w\rangle - \psi^*_{(\bar{x}, d_x)}(-c + \mathcal{A}^*\bar{y} + \mathcal{L}^*\bar{\xi} + \bar{z}) > 0, \quad \forall (d_x, d_w, d_v) \in \mathcal{C}(\bar{x}, \bar{w}, \bar{v}) \setminus \{0\}, \tag{5.47}$$

(ii) The SRCQ holds for the dual problem (5.39) at $(\bar{y}, \bar{\xi}, \bar{z})$ with respect to $(\bar{x}, \bar{w}, \bar{v})$:

$$\mathcal{T}^{\theta^*}(-c + \mathcal{A}^*\bar{y} + \mathcal{L}^*\bar{\xi} + \bar{z}, \bar{x}) + \mathcal{T}^{\delta^*_{\mathcal{P}}}(\bar{z}, \bar{x}) - \mathcal{A}^*\mathcal{R}^e - \mathcal{L}^*\mathcal{R}^l = \mathcal{R}^{m\times n}. \tag{5.48}$$

We also have parallel conclusions by switching the roles of the primal and dual problems in Proposition 5.4 and Corollary 5.1. The proof of the following proposition is quite similar to the proof of Proposition 5.4.

Proposition 5.4. *Suppose $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z}) \in \mathcal{R}^e \times \mathcal{R}^l \times \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n}$ is an optimal solution of the problem (5.39) and $\mathcal{M}_D(\bar{y}, \bar{\xi}, \bar{s}, \bar{z}) \neq \emptyset$. Then the following two statements are equivalent to each other:*

(i) The second order sufficient condition for the dual problem (5.39) holds at $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$:

$$\sup_{\bar{x}\in\mathcal{M}_D(\bar{y},\bar{\xi},\bar{s},\bar{z})} \left\{\langle d_\xi, \nabla^2 f^*(-\bar{\xi})d_\xi\rangle - \phi^*_{(-\bar{s},d_s)}(\bar{x})\right\} > 0, \quad \forall (d_y, d_\xi, d_s, d_z) \in \mathcal{C}(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})\backslash\{0\}, \tag{5.49}$$

where the critical cone is defined as

$$\begin{aligned}\mathcal{C}(\bar{y}, \bar{\xi}, \bar{s}, \bar{z}) := \ & \{(d_y, d_\xi, d_s, d_z) \in \mathcal{R}^e \times \mathcal{R}^l \times \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n} : \\ & \mathcal{A}^* d_y + \mathcal{L}^* d_\xi + d_s + d_z = 0, \\ & \langle b, d_y\rangle + \langle \nabla f^*(-\xi), -d_\xi\rangle + (\theta^*)'(-\bar{s}; -d_s) + (\delta^*_{\mathcal{P}})'(-\bar{z}; -d_z) = 0\}.\end{aligned} \tag{5.50}$$

(ii) The extended SRCQ holds at $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$ with respect to $\mathcal{M}_D(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$ for the primal problem (5.38):

$$\operatorname{conv}\left\{\bigcup_{\bar{x}\in\mathcal{M}_D(\bar{y},\bar{\xi},-\bar{s},\bar{z})}\left(\begin{pmatrix}\mathcal{A}\\ \mathcal{I}\end{pmatrix}\mathcal{T}^\theta(\bar{x}, -\bar{s}) - \begin{pmatrix}\{0\}\\ \mathcal{T}_{\mathcal{P}}(\bar{x}) \cap \bar{z}^\perp\end{pmatrix}\right)\right\} = \begin{pmatrix}\mathcal{R}^e\\ \mathcal{R}^{m\times n}\end{pmatrix}. \tag{5.51}$$

Proof. First we prove that (i) implies (ii). Suppose (ii) does not hold. Then by the same approach of the proof for Proposition 5.4, there exists nonzero $(\bar{d}_y, -\bar{d}_z) \in \mathcal{R}^e \times \mathcal{R}^{m\times n}$ such that

$$\begin{aligned}&\langle \bar{d}_y, h_y\rangle + \langle -\bar{d}_z, h_z\rangle \le 0, \\ &\quad \forall (h_y, h_z) \in \operatorname{conv}\left\{\bigcup_{\bar{x}\in\mathcal{M}_D(\bar{y},\bar{\xi},\bar{s},\bar{z})}\left(\begin{pmatrix}\mathcal{A}\\ \mathcal{I}\end{pmatrix}\mathcal{T}^\theta(\bar{x}, -\bar{s}) - \begin{pmatrix}\{0\}\\ \mathcal{T}_{\mathcal{P}}(\bar{x}) \cap \bar{z}^\perp\end{pmatrix}\right)\right\}.\end{aligned}$$

This implies that

$$\mathcal{A}^*\bar{d}_y - \bar{d}_z \in \big(\mathcal{T}^\theta(\bar{x}, -\bar{s})\big)^\circ \quad \text{and} \quad \bar{d}_z \in (\mathcal{T}_{\mathcal{P}}(\bar{x}) \cap \bar{z}^\perp)^\circ. \tag{5.52}$$

By Lemma 5.4, the first inclusion indicates that

$$\phi^*_{(-\bar{s},\mathcal{A}^*\bar{d}_y - \bar{d}_z)}(\bar{x}) = 0, \quad \text{and} \quad (\theta^*)'(-\bar{s}; \mathcal{A}^*\bar{d}_y - \bar{d}_z) = \langle \bar{x}, \mathcal{A}^*\bar{d}_y - \bar{d}_z\rangle.$$

Also by [40, Lemma 4.2], the second inclusion in (5.52) shows that $(\delta^*_{\mathcal{P}})'(-\bar{z}; -\bar{d}_z) = \langle \bar{d}_z, \bar{x}\rangle$.

Therefore, if we choose $h_s = \mathcal{A}^*\bar{d}_y - \bar{d}_z$, $h_\xi = 0 \in \mathcal{R}^l$, $h_y = -\bar{d}_y$ and $h_z = \bar{d}_z$, it is easy to see that

$$(h_y, h_\xi, h_s, h_z) \in \mathcal{C}(\bar{y}, \bar{\xi}, \bar{s}, \bar{z}) \setminus \{0\}, \quad \langle h_\xi, \nabla^2 f^*(-\bar{\xi})h_\xi\rangle = 0, \quad \psi^*_{(-\bar{s}, h_s)}(\bar{x}) = 0,$$

which is a contradiction of the second order sufficient condition (5.49) at $\bar{x}$.

Now we prove the reverse direction. Suppose the second order sufficient condition (5.49) does not hold at $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$. Then there exists $(\tilde{d}_y, \tilde{d}_\xi, \tilde{d}_s, \tilde{d}_z) \in \mathcal{C}(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})\setminus\{0\}$ such that

$$\langle \tilde{d}_\xi, \nabla^2 f^*(-\bar{\xi})d_\xi\rangle = 0, \quad \phi^*_{(-\bar{s}, \tilde{d}_s)}(\bar{x}) = 0.$$

Again by [20, Proposition 16], the second equation above holds if and only if $\psi^*_{(\bar{x}, \tilde{d}_s)}(-\bar{s}) = 0$. Then by the similar approach of the second part in Proposition 5.4, we could obtain $(\tilde{d}_y, \tilde{d}_\xi, \tilde{d}_s, \tilde{d}_z) = 0$. Thus, the proof is completed. □

Similarly as Corollary 5.1, we could obtain stronger results with the assumption that the multiplier set with respect to the dual problem is a singleton.

Corollary 5.2. *Suppose* $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z}) \in \mathcal{R}^e \times \mathcal{R}^l \times \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n}$ *is an optimal solution to the problem (5.39) with* $\mathcal{M}_D(\bar{y}, \bar{\xi}, \bar{s}, \bar{z}) = \{\bar{x}\}$. *Then the following two conditions are equivalent:*

(i) The second order sufficient condition for the dual problem (5.39) holds at $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$:

$$\langle d_\xi, \nabla^2 f^*(-\bar{\xi})d_\xi\rangle - \phi^*_{(-\bar{s}, d_s)}(\bar{x}) > 0, \quad \forall (d_y, d_\xi, d_s, d_z) \in \mathcal{C}(\bar{y}, \bar{\xi}, \bar{s}, \bar{z}) \setminus \{0\}. \tag{5.53}$$

(ii) The SRCQ for the primal problem (5.38) at $\bar{x}$ *with respect to* $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$:

$$\begin{pmatrix} \mathcal{A} \\ \mathcal{I} \end{pmatrix} \mathcal{T}^\theta(\bar{x}, -\bar{s}) - \begin{pmatrix} \{0\} \\ \mathcal{T}_{\mathcal{P}}(\bar{x}) \cap \bar{z}^\perp \end{pmatrix} = \begin{pmatrix} \mathcal{R}^e \\ \mathcal{R}^{m\times n} \end{pmatrix}. \tag{5.54}$$

Finally, as a combination of Theorem 5.1, Corollary 5.1 and Corollary 5.2, and noting that the second order sufficient condition at $(\bar{x}, \bar{w}, \bar{u})$ (or $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$) implies that $(\bar{x}, \bar{w}, \bar{v})$ (or $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$) is the unique optimal solution of (5.38) (or (5.39)), we are able to provide several equivalence characterization of the isolated calmness for the inverse of the KKT mapping at the origin point. They are summarized in the following theorem:

Theorem 5.2. *Let* $(\bar{x}, \bar{w}, \bar{u}, \bar{y}, \bar{\xi}, \bar{s}, \bar{z}) \in \mathcal{R}^{m\times n} \times \mathcal{R}^l \times \mathcal{R}^{m\times n} \times \mathcal{R}^e \times \mathcal{R}^l \times \mathcal{R}^{m\times n} \times \mathcal{R}^{m\times n}$ *satisfies* $G_P(\bar{x}, \bar{w}, \bar{u}, \bar{y}, \bar{\xi}, \bar{z}) = 0$ *and* $G_D(\bar{y}, \bar{\xi}, \bar{s}, \bar{z}, \bar{x}) = 0$. *Then the following conditions are equivalent to each other:*

(i) S_{KKT} *is robust isolated calm at the origin for* $(\bar{x}, \bar{w}, \bar{u}, \bar{y}, \bar{\xi}, \bar{z})$.

(ii) G_P^{-1} *is isolated calm at the origin with respect to* $(\bar{x}, \bar{w}, \bar{u}, \bar{y}, \bar{\xi}, \bar{z})$.

(iii) G_D^{-1} *is isolated calm at the origin with respect to* $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z}, \bar{x})$.

(iv) The second order sufficient condition (5.45) holds at $(\bar{x}, \bar{w}, \bar{u})$ *with respect to the primal problem (5.38) and the second order sufficient condition (5.49) holds at* $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$ *with respect to the dual problem (5.39).*

(v) The second order sufficient condition (5.45) holds at $(\bar{x}, \bar{w}, \bar{u})$ *for the primal problem (5.38) and the SRCQ (5.54) holds at* $\bar{x}$ *with respect to* $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$ *for the primal problem (5.38) .*

(vi) The second order sufficient condition (5.49) holds at $(\bar{y}, \bar{\xi}, \bar{s}, \bar{z})$ *for the dual problem (5.39) and the SRCQ (5.48) holds at* $(\bar{y}, \bar{\xi}, \bar{z})$ *with respect to* $(\bar{x}, \bar{w}, \bar{v})$ *for the dual problem (5.39) .*

5.3 Discussions on the calmness of the composite optimization problems

All the discussions in the Section 5.1 and 5.2 focused on the problems with isolated KKT solutions. Naturally one may ask that does there exist similar characterization about the calmness of the KKT system for the problem (5.1)? Unfortunately, to the best of our knowledge, there are no complete answers to this question till now if non-polyhedral functions θ are involved, including both the indicator function over the positive semidefinite cone and the nuclear norm function.

Let us first look at an example to have some ideas about the difficulty for the calmness property involving the non-polyhedral function. This example is modified from the one given by Zhou and So [108], with the initial purpose to demonstrate the failure of an error bound condition without the strict complementarity assumption for the unconstrained nuclear norm problem:

Consider the following problem

$$\begin{aligned} \min \quad & \tfrac{1}{2}\|\mathcal{A}x - b\|^2 + \|x\|_*, \\ \text{s.t.} \quad & \langle E, x\rangle \leq 1, \end{aligned} \tag{5.55}$$

and its dual

$$\begin{aligned} \min \quad & \tfrac{1}{2}\|z + b\|^2 + y \\ \text{s.t.} \quad & \|\mathcal{A}^* z + yE\|_2 \leq 1, \quad y \geq 0, \end{aligned} \tag{5.56}$$

where $E \in \mathcal{R}^{2\times 2}$ is a matrix with all entries equal to 1, $\|\cdot\|_2$ denotes the spectral norm, i.e., the largest singular value of a given matrix, the operator $\mathcal{A} : \mathcal{R}^{2\times 2} \to \mathcal{R}^2$ and the vector $b \in \mathcal{R}^2$ are defined as

$$\mathcal{A}x = B^{1/2}\mathrm{diag}(x), \quad \forall x \in \mathcal{R}^{2\times 2}, \quad \text{and} \quad b = B^{-1/2}\begin{pmatrix} 5/2 \\ -1 \end{pmatrix},$$

and the matrix B is given by $B = \begin{pmatrix} 3/2 & -2 \\ -2 & 3 \end{pmatrix} \succ 0.$

Given a parameter $\delta \in \mathcal{R}^{2\times 2}$, we consider the following canonically perturbation of the dual problem (5.56):

$$\begin{aligned} \min \quad & \tfrac{1}{2}\|z + b\|^2 + y \\ \text{s.t.} \quad & \|\mathcal{A}^* z + yE + \delta\|_2 \leq 1, \quad y \geq 0. \end{aligned} \tag{5.57}$$

Denote the multi-valued mapping $S_{\mathrm{KKT}} : \mathcal{R}^{2\times 2} \to \mathcal{R}^2 \times \mathcal{R} \times \mathcal{R}^{2\times 2}$ associated with the problem (5.57) by

$$\begin{aligned} S_{\mathrm{KKT}}(\delta) = \{(z, y, x) \in \mathcal{R}^2 \times \mathcal{R} \times \mathcal{R}^{2\times 2} : \quad & x = \mathrm{Prox}_{\|\cdot\|_*}(x - \mathcal{A}^* z - \delta - yE), \\ & z = \mathcal{A}x - b, \ y = \Pi_{\mathcal{R}_+}(y + \langle E, x\rangle - 1)\}. \end{aligned}$$

It is easy to check that the original problem (5.56) admits an unique solution $(\bar{z}, \bar{y}) =$

$\left(\mathcal{B}^{-1/2}\begin{pmatrix}-1\\-1\end{pmatrix},0\right)$ with an unique multiplier $\bar{x}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$. Moreover, the second order sufficient condition of the problem (5.56) holds at $(\bar{z},\bar{y})$. Now let us consider a sequence of perturbation $\delta_k=\begin{pmatrix}-\varepsilon_k&0\\0&\varepsilon_k\end{pmatrix}$ with $\varepsilon_k>0$. We can construct a sequence $(z_{\delta_k},y_{\delta_k},x_{\delta_k})\in S_{\rm KKT}(\delta_k)$ by letting

$$z_{\delta_k}=\mathcal{B}^{-1/2}\begin{pmatrix}-1+\varepsilon_k\\-1-\varepsilon_k\end{pmatrix},\quad y_{\delta_k}=0,\quad x_{\delta_k}=\begin{pmatrix}1+2\varepsilon_k&\tau_k\\\tau_k&\varepsilon_k\end{pmatrix},$$

where $|\tau_k|\leq\sqrt{\varepsilon_k+2\varepsilon_k^2}$. This indicates that the $S_{\rm KKT}$ cannot be calm at $(\bar{z},\bar{y},\bar{x})$ with respect to the origin since $\|(z_{\delta_k},y_{\delta_k},x_{\delta_k})-(\bar{z},\bar{y},\bar{x})\|=O(\|\sqrt{\varepsilon_k}\|)$ and $\|\delta_k\|=O(\|\varepsilon_k\|)$.

However, for the nonlinear programming problem, Dontchev and Rockafellar [24] show that the multi-valued mapping $S_{\rm KKT}$ is isolated calm under canonical perturbations for a locally optimal solution if and only if the strict Mangasarian-Fromovitz constraint qualification (which is equivalent to the uniqueness of the Lagrange multiplier in this case [54]) and the second order sufficient optimality condition hold. As mentioned above, both the uniqueness of the Lagrange multiplier and the second order sufficient condition hold for the problem (5.56), while the (isolated) calm still fails. This indicates that there exists a gap of the calmness property between the polyhedral and non-polyhedral problems. One can check that the SRCQ for the problem (5.57) does not hold. Therefore, a possible reason of this gap comes from the mismatch between the SRCQ and the uniqueness of the multiplier in the general conic problems.

Recently, there is nice work on the calmness of the solution mappings done by Zhou and So [108]. They consider a class of unconstrained convex problems:

$$\min\quad h(\mathcal{A}x)+\langle c,x\rangle+p(x)\tag{5.58}$$

where $\mathcal{A}:\mathcal{R}^{m\times n}\to\mathcal{R}^l$ is a linear operator, $h:\mathcal{R}^l\to(-\infty,+\infty]$ is smooth and strongly convex function on any compact convex set $\mathcal{V}\subseteq{\rm dom}(h)$, and $p:\mathcal{R}^{m\times n}\to$

$(-\infty, +\infty]$ is a closed convex proper function. Define the multi-valued mappings $\Gamma_f : \mathcal{R}^l \to \mathcal{R}^{m\times n}$, $\Gamma_p : \mathcal{R}^{m\times n} \to \mathcal{R}^{m\times n}$ and $\Gamma(y, g) : \mathcal{R}^l \times \mathcal{R}^{m\times n} \to \mathcal{R}^{m\times n}$ as

$$\begin{cases} \Gamma_f(y) := \{x \in \mathcal{R}^{m\times n} : \mathcal{A}x = y\}, \quad \forall y \in \mathcal{R}^l, \\ \Gamma_p(g) := \{x \in \mathcal{R}^{m\times n} : -g \in \partial p(x)\}, \quad \forall g \in \mathcal{R}^{m\times n}, \\ \Gamma(y, g) := \Gamma_f(y) \cap \Gamma_P(g), \quad \forall (y, g) \in \mathcal{R}^l \times \mathcal{R}^{m\times n}. \end{cases}$$

Zhou and So show that if the optimal solution set Ω of (5.58) is nonempty compact and contains $\bar{x}$, then $\Omega = \Gamma(\bar{y}, \bar{g})$, where $\bar{y} = \mathcal{A}\bar{x}$ and $\bar{g} = \mathcal{A}^*\nabla h(\bar{y}) + c$. Moreover, under the assumptions that $\{\Gamma_f(\bar{y}), \Gamma_p(\bar{g})\}$ is bounded linearly regular and ∂p is metrically subregular at $\bar{x}$ for $-\bar{g}$, the solution map Γ is calm at $(\bar{y}, \bar{g})$ for $\bar{x}$. In particular, the bounded linearly regular of $\{\Gamma_f(\bar{y}), \Gamma_p(\bar{g})\}$ holds if the strict complementarity condition is satisfied at some $\tilde{x} \in \Omega$:

$$0 \in \mathcal{A}^*\nabla h(\mathcal{A}\tilde{x}) + c + \mathrm{ri}(\partial p(\tilde{x})).$$

Besides, the metric subregularity of the sub-differential holds for the polyhedral functions [77], the vector ℓ_p norm for $p \in [1, 2]$ or $p = \infty$ [107], the nuclear norm [108], and the indicator function over the SDP cone given by Theorem 2.4. It is easy to see that the Cartesian product of the metric subregular mappings are also metric subregular. Therefore, we could get from Zhou and So's result that under the strict complementarity condition, the solution mapping is calm for the problem

$$\min_{x_1,x_2,\ldots,x_k} h(\sum_{i=1}^{k} \mathcal{A}_i x_i) + \sum_{i=1}^{k} \langle c_i, x_i\rangle + \sum_{i=1}^{k} p_i(x_i),$$

where $h : \mathcal{R}^l \to \mathcal{R}$ is smooth and strongly convex on any compact convex set $\mathcal{V} \subseteq \mathrm{dom}(h)$, $\mathcal{A}_i : \mathcal{X}_i \to \mathcal{R}^l$ are linear operators, $c_i \in \mathcal{X}_i$ are given data, and $p_i : \mathcal{X}_i \to (-\infty, +\infty]$ can be chosen from the polyhedral functions, the vector ℓ_p norm for $p \in [1, 2]$ or $p = \infty$, the nuclear norm function, and the indicator function over the SDP cone.

The above nice results provide a partial answer to the calmness of the solution mapping. From the discussions in the previous sections, we also know the calmness of the solution mappings should hold under the second order sufficient condition and the strict Robinson constraint qualification. It is thus very interesting to know whether a unified condition can be proposed to characterize the calmness of the optimization problems involving the non-polyhedral functions. We leave it as a future research topic.

Chapter 6

Numerical experiments

In this chapter, we test our iABCD algorithm discussed in Chapter 3 on solving the projection onto the intersection of the equations, inequalities and the doubly nonnegative cone constraints:

$$\begin{array}{ll} \min & \frac{1}{2}\|X-G\|^2 \\ \text{s.t.} & \mathcal{A}X = b, \quad \mathcal{B}X \geq d, \quad X \in \mathcal{S}^n_+ \cap \mathcal{N}, \end{array} \tag{6.1}$$

where $\mathcal{A}: \mathcal{S}^n \to \mathcal{R}^{m_E}$ and $\mathcal{B}: \mathcal{S}^n \to \mathcal{R}^{m_I}$ are linear operators and $\mathcal{A}$ is onto, $G \in \mathcal{S}^n$, $b \in \mathcal{R}^{m_E}$, $d \in \mathcal{R}^{m_I}$ are given data, and $\mathcal{N} := \{X \in \mathcal{S}^n : X \geq 0\}$ denotes the nonnegative cone.

The dual of (6.1) can be written as

$$\begin{array}{ll} \min & F(y,z,S,Z) := \frac{1}{2}\|\mathcal{A}^*y + \mathcal{B}^*z + S + Z + G\|^2 - \langle b, y\rangle - \langle d, z\rangle - \frac{1}{2}\|G\|^2 \\ \text{s.t.} & z \geq 0, \quad S \in \mathcal{S}^n_+, \quad Z \in \mathcal{N}. \end{array} \tag{6.2}$$

We implement our iABCD framework to solve the above dual form, where the variables $(y, S) \in \mathcal{R}^{m_E} \times \mathcal{S}^n$ are taken as one block, and the variables $(z, Z) \in \mathcal{R}^{m_I} \times \mathcal{S}^n$ are taken as the other one. As discussed in Section 3.3, the subproblem of the block (y, S) would be solved by the one cycle inexact sGS technique, and the subproblem of the block (z, Z) would be solved by the APG-SNCG method.

Denote $W \equiv (y, z, S, Z) \in \mathcal{W}$ with $\mathcal{W} := \mathcal{R}^{m_E} \times \mathcal{R}^{m_I} \times \mathcal{S}^n \times \mathcal{S}^n$. In order to implement the above idea, we majorize the smooth function F at $W' \in \mathcal{W}$ as

$$F(W) \leq \widehat{F}_1(W;W') := F(W) + \frac{1}{2}\|S - S'\|^2_{\mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\mathcal{A}} + \frac{\|\mathcal{B}\|}{2}\|z - z'\|^2, \quad \forall W \in \mathcal{W}. \tag{6.3}$$

Since F itself is a quadratic function, the linearization at any point must be itself and thus, the function $\widehat{F}_1$ satisfies the inequality (3.3) and Assumption 3.1. The first proximal term $\frac{1}{2}\|S - S'\|^2_{\mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\mathcal{A}}$ comes from the sGS technique to solve (y, S), and the second proximal term $\frac{\|\mathcal{B}\|}{2}\|z - z'\|^2$ aims to make the block (z, Z) strongly convex and the Newton's equation well-conditioned. The detailed framework of our algorithm is given below.

iABCD: An inexact majorized accelerated block coordinate descent algorithm to solve the problem (6.2) with APG-SNCG

Choose an initial point $W^1 = \widetilde{W}^0 \in \mathcal{W}$. Set $k = 1$ and $t_1 = 1$. Let the nonnegative error tolerance $\{\varepsilon_k\}$ satisfies $\sum_{i=1}^{\infty} i\varepsilon_i < \infty$. Iterate until convergence:

Step 1. Suppose $(\delta_y^k, \hat{\delta}_y^k, \delta_z^k, \delta_Z^k) \in \mathcal{R}^{m_E} \times \mathcal{R}^{m_E} \times \mathcal{R}^{m_I} \times \mathcal{S}^n$ are error vectors such that

$$\max\{\|\delta_y^k\|, \|\hat{\delta}_y^k\|, \|\delta_z^k\|, \|\delta_Z^k\|\} \leq \varepsilon_k.$$

Compute

$$\begin{cases} \tilde{y}^{k+1/2} = \arg\min_y\{\widehat{F}_1(y, S^k, z^k, Z^k; W^k) + \langle y, \delta_y^k\rangle\}, \\ \widetilde{S}^k = \Pi_{\mathcal{S}^n_+}(-\mathcal{A}^*\tilde{y}^{k+1/2} - \mathcal{B}^* z^k - Z^k - G), \\ \tilde{y}^k = \arg\min_y\{\widehat{F}_1(y, \widetilde{S}^k, z^k, Z^k; W^k) + \langle y, \hat{\delta}_y^k\rangle\}. \end{cases}$$

Then compute

$$(\tilde{z}^k, \widetilde{Z}^k) = \arg\min_{z,Z}\{\widehat{F}_1(\tilde{y}^k, \widetilde{S}^k, z, Z; W^k) + \langle z, \delta_z^k\rangle + \langle Z, \delta_Z^k\rangle : \ z \geq 0, \ Z \in \mathcal{N}\}.$$

Step 2. Compute $\begin{cases} t_{k+1} = \frac{1}{2}(1 + \sqrt{1 + 4t_k^2}), \\ W^{k+1} = \widetilde{W}^k + \dfrac{t_k - 1}{t_{k+1}}(\widetilde{W}^k - \widetilde{W}^{k-1}). \end{cases}$

The equation, inequality and cone constraints of our test examples (6.1) are generated from the doubly nonnegative relaxation of a binary integer nonconvex quadratic (ex-BIQ) programming that considered in [90] :

$$
\begin{array}{ll}
\min & \frac{1}{2}\langle Q, Y\rangle + \langle c, x\rangle \\
\text{s.t.} & \operatorname{Diag}(Y) = x, \quad \alpha = 1, \quad X = \begin{pmatrix} Y & x \\ x^T & \alpha \end{pmatrix} \in \mathcal{S}_+^n \cap \mathcal{N}, \\
& -Y_{ij} + x_i \geq 0, -Y_{ij} + x_j \geq 0, Y_{ij} - x_i - x_j \geq -1, \ \forall i < j, \ j = 2, \dots, n-1.
\end{array} \tag{6.4}
$$

The matrix G in the objective function of the problem (6.1) is taken to be $G = -\frac{1}{2}\begin{pmatrix} Q & c \\ c & 0 \end{pmatrix}$. The test data for Q and c in our numerical experiments are taken from Biq Mac Library maintained by Wiegele, which is available at http://biqmac.uni-klu.ac.at/biqmaclib.html.

Under a Slater's condition, the KKT optimality conditions of the problem (6.1) are given as follows:

$$
X = G + \mathcal{A}^* y + \mathcal{B}^* z + S + Z,
$$
$$
\mathcal{A}X = b, \quad \mathcal{B}X - d = \Pi_{R_+^{m_I}}(\mathcal{B}X - d - z), \quad X = \Pi_{\mathcal{S}_+^n}(X - S), \quad X = \Pi_{\mathcal{N}}(X - Z).
$$

We measure the accuracy of an approximate solution (y, z, S, Z) for (6.2) by the relative residue of the KKT system:

$$
\eta := \max\{\eta_1, \eta_2, \eta_3, \eta_4\},
$$

where

$$
\eta_1 := \frac{\|\mathcal{A}X - b\|}{1 + \|b\|}, \quad \eta_2 := \frac{\|\mathcal{B}X - d - \Pi_{\mathcal{R}^{m_I}}(\mathcal{B}X - d - z)\|}{1 + \|d\|}, \quad \eta_3 := \frac{\|X - \Pi_{\mathcal{S}_+^n}(X - S)\|}{1 + \|X\| + \|S\|},
$$
$$
\eta_4 := \frac{\|X - \Pi_{\mathcal{N}}(X - Z)\|}{1 + \|X\| + \|Z\|}, \quad X = G + \mathcal{A}^* y + \mathcal{B}^* z + S + Z.
$$

In addition, we compute the relative gap between the primal and dual objective functions:

$$
\eta_{\mathrm{g}} := \frac{\mathrm{obj_p} - \mathrm{obj_d}}{1 + |\mathrm{obj_p}| + |\mathrm{obj_d}|},
$$

where $\mathrm{obj_p} := \frac{1}{2}\|X - G\|^2$ and $\mathrm{obj_d} := -\frac{1}{2}\|\mathcal{A}^* y + \mathcal{B}^* z + S + Z + G\|^2 + \langle b, y\rangle + \langle d, z\rangle + \frac{1}{2}\|G\|^2$.

We stop the algorithms imABCD, BCD, mABCD, eRARBCG and iAPG if $\eta < \varepsilon$, where ε is the prescribed accuracy.

In order to demonstrate the importance for the incorporation of the second order information, we compare our iABCD method with the two-block accelerated block coordinate gradient descent algorithm proposed by Chambolle and Pock [9]. The two blocks are still taken as (y, S) and (z, Z). The iteration steps are given as follows:

ABCGD: An accelerated block coordinate gradient descent algorithm to solve the problem (6.2)

Choose an initial point $W^1 = \widetilde{W}^0 \in \mathcal{W}$. Set $k = 1$ and $t_1 = 1$. Iterate until convergence:

Step 1. Compute $R^{k+\frac{1}{2}} = \mathcal{A}^*y^k + \mathcal{B}^*z^k + S^k + Z^k + G$ and

$$\begin{cases} \tilde{y}^k = y^k - (\mathcal{A}\mathcal{A}^*)^{-1}(\mathcal{A}R^{k+\frac{1}{2}} - b)/2, \\ \widetilde{S}^k = \Pi_{\mathcal{S}^n_+}(S^k - R^{k+\frac{1}{2}}/2). \end{cases}$$

Step 2. Compute $R^k = \mathcal{A}^*\tilde{y}^k + \mathcal{B}^*z^k + \widetilde{S}^k + Z^k + G$ and

$$\begin{cases} \tilde{z}^k = \Pi_{\mathcal{R}^{m_I}_+}(z^k - (\mathcal{B}R^k - d)/(2\lambda_{\max}(\mathcal{B}\mathcal{B}^*))), \\ \widetilde{Z}^k = \Pi_{\mathcal{N}}(Z^k - R^k/2). \end{cases}$$

Step 3. Compute $\begin{cases} t_{k+1} = \frac{1}{2}(1 + \sqrt{1 + 4t_k^2}), \\ W^{k+1} = \widetilde{W}^k + \dfrac{t_k - 1}{t_{k+1}}(\widetilde{W}^k - \widetilde{W}^{k-1}). \end{cases}$

Note that the step length is $\frac{1}{2}$ when updating the variables for each proximal gradient step in order to solve the variables within each block simultaneously.

Figure 6.1 and Figure 6.2 show the performance profile of the iABCD and ABCGD algorithms for the large scale ex-BIQ problems with $\varepsilon = 10^{-6}$, where the detailed numerical results are provided in Table 6.1. A point (x, y) is in the performance profile curve of a method if and only if it can solve exactly $(100y)\%$ of all the tested problems at most x times slower than any other methods. The

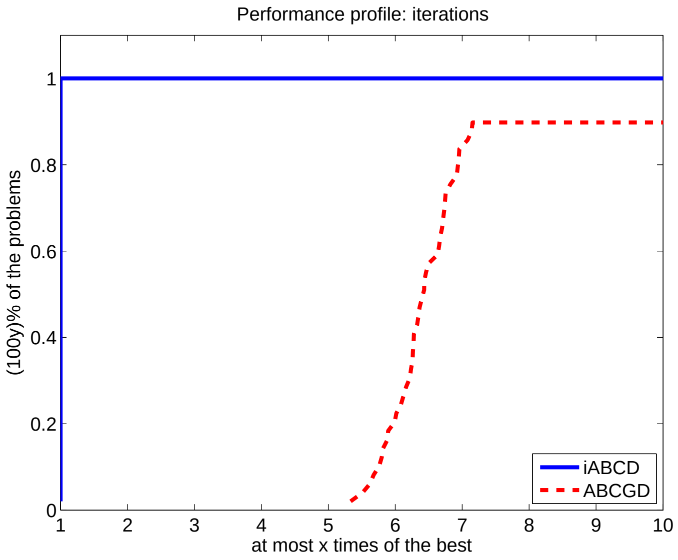

Figure 6.1: Performance profile of iABCD and ABCGD with $\varepsilon = 10^{-6}$

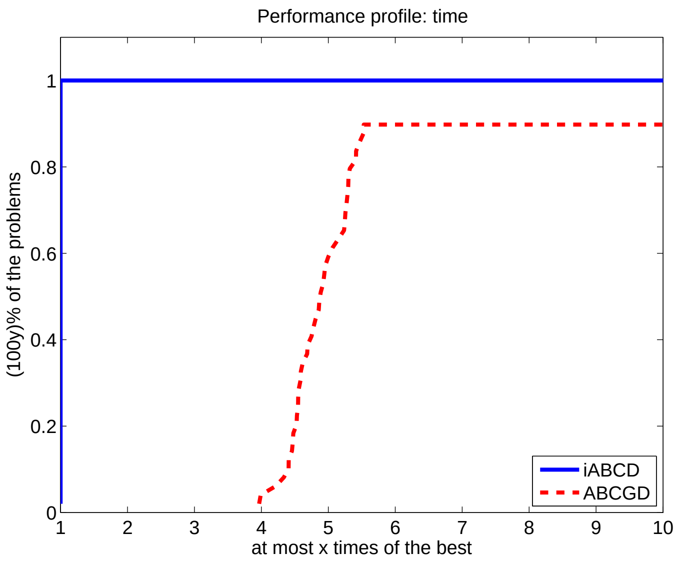

Figure 6.2: Performance profile of iABCD and ABCGD with $\varepsilon = 10^{-6}$

first four columns list the problem names, the dimension of the variable y (m_E), z (m_I) and the size of the matrix G (n_s), respectively. The last several columns provide the number of iterations, the relative residual η, the relative gap between the primal and dual objective values η_{gap}, and the computation times in the format of "hours:minutes:seconds". One can see from the performance profile that the ABCGD algorithm takes about 5 times iteration steps compared with the iABCD algorithm, and is around 4 times faster than the ABCGD in terms of computing time. In particular, the ABCGD method cannot solve all the large scale bdq500 problems within 50000 iterations, while our iABCD could obtain satisfied solutions by 6000 iterations. This indicates that even though the computational cost for each cycle of the iABCD method is larger than the ABCGD method, its overall performance is extremly good. In fact, the Newton system is well-conditioned in this case such that it only takes one or two CG iterations to obtain a satisfied Newton direction.

We also compare the iABCD with some other existing methods. The first one is the block coordinate descent algorithm (BCD), where we view (6.2) as a four-block problem. The block z is solved by the APG-SNCG algorithm, while other blocks have analytical solutions. The steps are given as follows:

BCD: An inexact block coordinate descent method to solve the problem (6.2)

Choose an initial point $W^1 \in \mathcal{W}$. Let $\{\varepsilon_k\}$ be a series of given summable error tolerance such that the error vector $\delta_z^k \in \mathcal{R}^{m_I}$ satisfies $\|\delta_z^k\| \leq \varepsilon_k$. Set $k = 1$. Iterate until convergence:

$$
\begin{cases}
y^{k+1} = (\mathcal{A}\mathcal{A}^*)^{-1}(b - \mathcal{A}(\mathcal{B}^* z^k + S^k + Z^k + G)), \\
S^{k+1} = \Pi_{\mathcal{S}_+^n}(-\mathcal{A}^* y^{k+1} - \mathcal{B}^* z^k - Z^k - G), \\
z^{k+1} = \arg\min\limits_{z \geq 0}\{F(y^{k+1}, S^{k+1}, z, Z^k) + \dfrac{\|\mathcal{B}\|}{2}\|z - z^k\|^2 + \langle \delta_z^k, z \rangle\}, \\
Z^{k+1} = \Pi_{\mathcal{N}}(-\mathcal{A}^* y^{k+1} - \mathcal{B}^* z^{k+1} - S^{k+1} - G).
\end{cases}
$$

The second framework is an enhanced version of the inexact accelerated randomized block coordinate descent method (eRABCG) that modified from [61], where we use the proximal terms $\frac{1}{2}\|y - y^k\|_{\mathcal{A}\mathcal{A}^*}$ instead of $\frac{1}{2}\|y - y^k\|^2_{\lambda_{\max}(\mathcal{A}\mathcal{A}^*)}$ when updating the block y^{k+1}, and $\frac{1}{2}\|z - z^k\|^2_{\mathcal{B}\mathcal{B}^*} + \frac{\|\mathcal{B}\|}{2}\|z - z^k\|^2$ when updating the block z^{k+1}. Similar idea has also been used in [91] as a comparison for solving a class of positive semidefinite feasibility problems. The detailed steps of the eRABCG are presented below.

eRABCG: An inexact enhanced randomized accelerated block coordinate descent algorithm with four blocks to solve the problem (6.2)

Choose an initial point $W^1 = \widetilde{W}^0 \in \mathcal{W}$. Set $k = 1$ and $\alpha_0 = \frac{1}{4}$. Let $\{\varepsilon_k\}$ be a series of given summable error tolerance such that the error vector $\delta_z^k \in \mathcal{R}^{m_I}$ satisfies $\|\delta_z^k\| \le \varepsilon_k$. Iterate until convergence:

Step 1. Compute $\alpha^k = \frac{1}{2}(\sqrt{\alpha_{k-1}^4 + 4\alpha_{k-1}^2} - \alpha_{k-1}^2)$.
Step 2. Compute $\widehat{W}^{k+1} = (1 - \alpha_k)\widehat{W}^k + \alpha_k \widetilde{W}^k$.
Step 3. Denote $\widehat{R}^k = \mathcal{A}^*\hat{y}^k + \mathcal{B}^*\hat{z}^k + \widehat{S}^k + \widehat{Z}^k + G$. Choose $i_k \in \{1, 2, 3, 4\}$ uniformly at random and update $\widetilde{W}_{i_k}^{k+1}$ according to the following rule if the k-th block is selected:

$$
\begin{cases}
i_k = 1: & \tilde{y}^{k+1} = (\mathcal{A}\mathcal{A}^*)^{-1}((b - \mathcal{A}\widehat{R}^k)/(4\alpha_k) + \mathcal{A}\mathcal{A}^*\tilde{y}_E^k), \\[2mm]
i_k = 2: & \tilde{z}^{k+1} = \arg\min\limits_{z \ge 0}\{\langle \nabla_z F(\widehat{W}^{k+1}), z\rangle + \dfrac{4\alpha_k}{2}\|z - \tilde{z}^k\|^2_{\mathcal{B}\mathcal{B}^* + \|\mathcal{B}\|\mathcal{I}} + \langle z, \delta_z^k \rangle\}, \\[2mm]
i_k = 3: & \widetilde{Z}^{k+1} = \Pi_{\mathcal{N}}(\tilde{Z}^k - \widehat{R}^k/(4\alpha_k)), \\[2mm]
i_k = 4: & \widetilde{S}^{k+1} = \Pi_{\mathcal{S}_+^n}(\tilde{S}^k - \widehat{R}^k/(4\alpha_k)).
\end{cases}
$$

Set $\widetilde{W}_i^{k+1} = \widetilde{W}_i^k$ for all $i \ne i_k$, $k = 1, 2, 3, 4$.

Step 4. Set $W_i^{k+1} = \begin{cases} \widehat{W}_i^k + 4\alpha_k(\widetilde{W}_i^{k+1} - \widetilde{W}_i^k), & i = i_k, \\ \widehat{W}_i^k, & i \ne i_k, \end{cases} \quad i = 1, 2, 3, 4.$

In order to know whether our proposed APG-SNCG method could universally improve the efficiency for different algorithms, we also test two variants of the BCD and eRABCG, where the block z are updated by the proximal gradient step. They are named as mBCD and eRABCG2. The numerical performance of two selected test examples are shown in Table 6.2. One can see that the mBCD and eRABCG2 perform much worse than their inexact counterpart, even though the same outer framework is adopted. This observation confirms the point that if there exists a computing intensive block (the block S in our test examples as the complexity of the eigenvalue decomposition is $O(n^3)$), a small proximal term is always preferred for other blocks in order to reduce the iteration numbers for the difficult block.

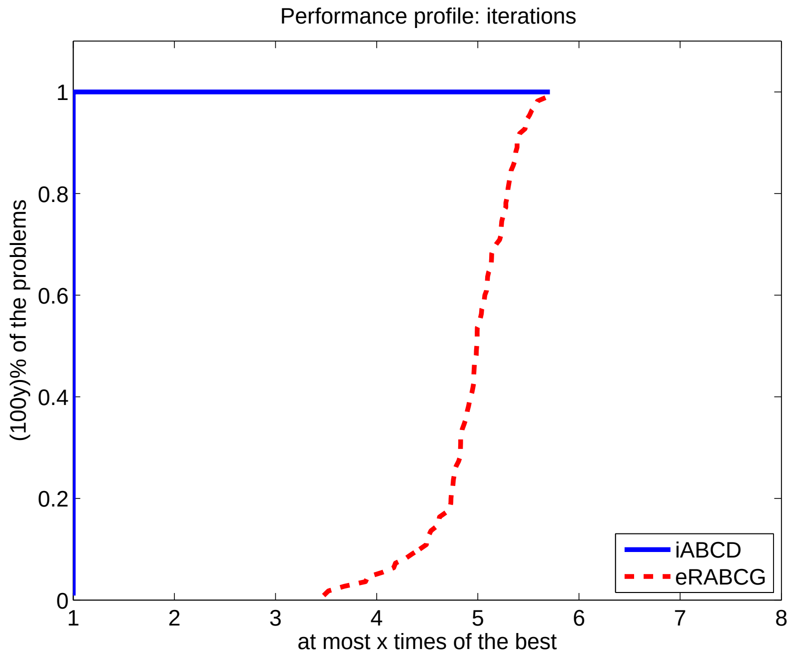

Figure 6.3: Performance profile of iABCD and eRABCG with accuracy $\varepsilon = 10^{-6}$.

Table 6.3 list the results of the numerical performance of the iABCD, BCD and eRABCG methods, with the performance profile given in Figure 6.3 and Figure 6.4. One can see that the BCD algorithm is much less efficient compared with others, since all the test examples cannot be solved to the accuracy $\varepsilon = 10^{-6}$ within 50000 iteration steps (Therefore, we do not include the performance of the BCD in the performance profile). This phenomenon emphasizes the power of the acceleration

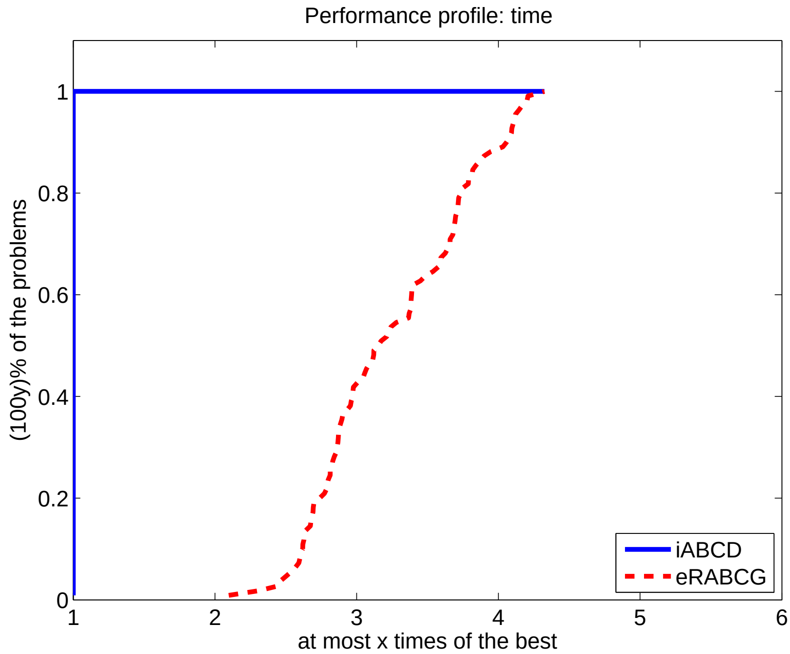

Figure 6.4: Performance profile of iABCD and eRABCG for with accuracy $\varepsilon = 10^{-6}$.

technique in solving unconstrained problems. One can find that the iABCD framework is 3.5 times faster than the eRABCG method, which is caused by the 4 times enlargement (the number of the blocks is 4 in the eRABCG method) of the proximal terms for the randomized-type accelerated block coordinate descent method.

Based on all the above observations, we shall draw a conclusion that the impressing numerical performance of the iABCD algorithm is mainly due to two reasons: one is the outer acceleration of the two-block coordinate descent framework, and the other is the inner acceleration by the proper incorporation of the second order information.

Table 6.1: The performance of iABCD and ABCGD with accuracy $\varepsilon = 10^{-6}$.

		iteration	η	$\eta_{\rm gap}$	time
problem	$m_E; m_I \mid n_s$	iabcd|abcgd	iabcd|abcgd	iabcd|abcgd	iabcd|abcgd
be100.1	101 ; 14850 | 101	5276 | 31048	9.9-7 | 9.9-7	-2.3-7 | -7.7-8	45 | 3:08
be100.2	101 ; 14850 | 101	5747 | 30645	9.9-7 | 9.9-7	-2.0-7 | -9.4-8	47 | 3:08
be100.3	101 ; 14850 | 101	5950 | 41172	9.9-7 | 9.9-7	-3.7-7 | -8.7-8	53 | 4:02

Table 6.1: The performance of iABCD and ABCGD with accuracy $\varepsilon = 10^{-6}$.

		iteration	η	η_{gap}	time
problem	$m_E; m_I \mid n_s$	iabcd|abcgd	iabcd|abcgd	iabcd|abcgd	iabcd|abcgd
be100.4	101 ; 14850 | 101	5704 | 36684	9.9-7 | 9.9-7	-2.7-7 | -7.2-8	49 | 3:48
be100.5	101 ; 14850 | 101	5762 | 39956	9.9-7 | 9.7-7	-3.1-7 | -7.6-8	51 | 4:05
be100.6	101 ; 14850 | 101	5769 | 36134	9.9-7 | 9.9-7	-1.8-7 | -7.9-8	48 | 3:40
be100.7	101 ; 14850 | 101	4994 | 28087	9.9-7 | 9.9-7	-3.3-7 | -8.4-8	40 | 2:53
be100.8	101 ; 14850 | 101	5613 | 33772	9.9-7 | 9.9-7	-1.5-7 | -5.3-8	45 | 3:32
be100.9	101 ; 14850 | 101	5763 | 40048	9.9-7 | 9.9-7	-2.7-7 | -8.1-8	51 | 4:10
be100.10	101 ; 14850 | 101	5260 | 32010	9.9-7 | 9.9-7	-3.7-7 | -8.8-8	43 | 3:46
be120.3.1	121 ; 21420 | 121	4120 | 27781	9.9-7 | 9.9-7	-2.2-7 | -5.6-8	41 | 3:42
be120.3.2	121 ; 21420 | 121	4106 | 23809	9.9-7 | 9.9-7	-2.3-7 | -6.3-8	40 | 3:00
be120.3.3	121 ; 21420 | 121	3548 | 21867	9.9-7 | 9.9-7	-9.8-8 | -6.6-8	35 | 2:53
be120.3.4	121 ; 21420 | 121	4745 | 31783	9.9-7 | 9.9-7	-3.1-7 | -6.7-8	47 | 4:08
be120.3.5	121 ; 21420 | 121	5637 | 31076	9.9-7 | 9.9-7	-4.9-8 | -7.1-8	58 | 3:51
be120.3.6	121 ; 21420 | 121	3946 | 26558	9.9-7 | 9.9-7	-1.8-7 | -4.8-8	39 | 3:26
be120.3.7	121 ; 21420 | 121	4169 | 26176	9.9-7 | 9.9-7	-2.7-7 | -6.5-8	41 | 3:37
be120.3.8	121 ; 21420 | 121	3793 | 23796	9.9-7 | 9.9-7	-1.6-7 | -3.8-8	35 | 3:11
be120.3.9	121 ; 21420 | 121	4951 | 28518	9.9-7 | 9.9-7	-2.0-7 | -5.2-8	52 | 3:58
be120.3.10	121 ; 21420 | 121	4264 | 24803	9.9-7 | 9.9-7	-3.4-7 | -5.3-8	42 | 3:06
be120.8.1	121 ; 21420 | 121	5671 | 32200	9.9-7 | 9.9-7	-3.5-7 | -8.1-8	58 | 4:26
be120.8.2	121 ; 21420 | 121	5897 | 35336	9.9-7 | 9.9-7	-3.1-7 | -7.1-8	1:02 | 4:39
be120.8.3	121 ; 21420 | 121	5199 | 33259	9.9-7 | 9.9-7	-4.9-7 | -9.2-8	52 | 4:35
be120.8.4	121 ; 21420 | 121	6688 | 40964	9.9-7 | 9.9-7	-4.0-7 | -8.0-8	1:12 | 5:26
be120.8.5	121 ; 21420 | 121	5828 | 41263	9.9-7 | 9.9-7	-3.6-7 | -6.3-8	1:03 | 5:40
be120.8.6	121 ; 21420 | 121	4735 | 29524	9.9-7 | 9.9-7	-4.8-7 | -7.8-8	47 | 3:52
be120.8.7	121 ; 21420 | 121	4456 | 29722	9.9-7 | 9.9-7	-3.8-7 | -6.5-8	44 | 3:36
be120.8.8	121 ; 21420 | 121	5979 | 35240	9.9-7 | 9.9-7	-2.8-7 | -6.4-8	1:01 | 4:52
be120.8.9	121 ; 21420 | 121	5788 | 37397	9.9-7 | 9.9-7	-3.3-7 | -8.5-8	1:02 | 4:42
be120.8.10	121 ; 21420 | 121	5636 | 35274	9.9-7 | 9.9-7	-3.5-7 | -8.0-8	58 | 4:53
be250.1	251 ; 93375 | 251	3958 | 25038	9.9-7 | 9.9-7	1.4-7 | -4.9-8	2:10 | 9:40
be250.2	251 ; 93375 | 251	4213 | 29313	9.9-7 | 9.9-7	-3.7-7 | -6.8-8	2:22 | 11:36
be250.3	251 ; 93375 | 251	4230 | 27211	9.9-7 | 9.8-7	-3.7-7 | -4.4-8	2:29 | 10:56
be250.4	251 ; 93375 | 251	4059 | 28985	9.9-7 | 9.9-7	-3.6-7 | -5.8-8	2:24 | 11:15
be250.5	251 ; 93375 | 251	4361 | 29277	9.9-7 | 9.9-7	-3.9-7 | -5.2-8	2:35 | 11:45
bqp100-1	101 ; 14850 | 101	7344 | 50000	9.9-7 | 1.1-6	-9.8-8 | -1.1-7	1:08 | 5:25

Table 6.1: The performance of iABCD and ABCGD with accuracy $\varepsilon = 10^{-6}$.

		iteration	η	η_{gap}	time
problem	$m_E; m_I \mid n_s$	iabcd|abcgd	iabcd|abcgd	iabcd|abcgd	iabcd|abcgd
bqp100-2	101 ; 14850 | 101	3799 | 24170	9.9-7 | 9.9-7	-1.5-7 | -6.7-8	30 | 2:48
bqp100-3	101 ; 14850 | 101	3630 | 22570	9.9-7 | 9.9-7	8.6-8 | -5.1-8	29 | 2:35
bqp100-4	101 ; 14850 | 101	4293 | 27893	9.9-7 | 9.9-7	-2.2-7 | -5.9-8	35 | 3:13
bqp100-5	101 ; 14850 | 101	5145 | 34243	9.9-7 | 9.9-7	-1.0-7 | -4.8-8	43 | 3:29
bqp500-1	501 ; 374250 | 501	6385 | 50000	9.9-7 | 1.3-6	-1.2-6 | -1.2-7	23:40 | 1:43:49
bqp500-2	501 ; 374250 | 501	6622 | 50000	9.9-7 | 1.7-6	-1.1-6 | -1.6-7	23:21 | 1:43:49
bqp500-3	501 ; 374250 | 501	6042 | 50000	9.9-7 | 1.1-6	-1.1-6 | -9.1-8	22:10 | 1:45:49
bqp500-4	501 ; 374250 | 501	5537 | 50000	9.9-7 | 1.2-6	-1.1-6 | -8.0-8	20:05 | 1:46:16
gka1e	201 ; 59700 | 201	5292 | 37861	9.9-7 | 9.9-7	-2.6-7 | -4.8-8	2:05 | 10:59
gka2e	201 ; 59700 | 201	4623 | 29338	9.9-7 | 9.9-7	-6.8-7 | -7.1-8	1:47 | 8:29
gka3e	201 ; 59700 | 201	6033 | 40016	9.9-7 | 9.9-7	-3.7-7 | -6.0-8	2:12 | 11:44
gka4e	201 ; 59700 | 201	7001 | 47779	9.9-7 | 9.9-7	-5.9-7 | -6.9-8	2:45 | 14:09
gka5e	201 ; 59700 | 201	6245 | 42175	9.9-7 | 9.9-7	-5.3-7 | -7.8-8	2:30 | 12:30

Table 6.2: The performance of eRABCG, eRABCG2, BCD and mBCD on ex-BIQ problems with accuracy $\varepsilon = 10^{-5}$.

<table>
<tr><td></td><td></td><td>iteration</td><td>η_{qsdp}</td><td>η_{gap}</td><td>time</td></tr>
<tr><td>problem</td><td>$m_E; m_I \mid n_s$</td><td>erabcg|erabcg2|bcd|mbcd</td><td>erabcg|erabcg2|bcd|mbcd</td><td>erabcg|erabcg2|bcd|mbcd</td><td>erabcg|erabcg2|bcd|mbcd</td></tr>
<tr><td>bqp50-1</td><td>51 ; 3675 | 51</td><td>5168 | 20172 | 110848 | 500000</td><td>9.9-6 | 9.9-6 | 9.9-6 | 7.6-3</td><td>-8.5-7 | -6.1-7 | -9.9-7 | 2.9-3</td><td>19 | 40 | 8:02 | 18:02</td></tr>
<tr><td>bqp100-1</td><td>101 ; 14850 | 101</td><td>7789 | 39167 | 203733 | 500000</td><td>9.3-6 | 9.0-6 | 9.9-6 | 1.1-2</td><td>-1.8-6 | -1.4-6 | -1.6-6 | 5.2-3</td><td>48 | 2:19 | 33:24 | 52:57</td></tr>
</table>

Table 6.3: The performance of iABCD, BCD and eRABCG with accuracy $\varepsilon = 10^{-6}$.

<table>
<tr><th></th><th></th><th>iteration</th><th>η</th><th>η_{gap}</th><th>time</th></tr>
<tr><th>problem</th><th>$m_E; m_I \mid n_s$</th><th>iabcd|bcd|erabcg</th><th>iabcd|bcd|erabcg</th><th>iabcd|bcd|erabcg</th><th>iabcd|bcd|erabcg</th></tr>
<tr><td>be100.1</td><td>101 ; 14850 | 101</td><td>5276 | 50000 | 27844</td><td>9.9-7 | 3.3-5 | 9.9-7</td><td>-2.3-7 | -6.1-6 | -4.8-7</td><td>45 |7:52 |2:39</td></tr>
<tr><td>be100.2</td><td>101 ; 14850 | 101</td><td>5747 | 50000 | 28405</td><td>9.9-7 | 3.6-5 | 9.9-7</td><td>-2.0-7 | -7.6-6 | -7.1-7</td><td>47 |7:50 |2:40</td></tr>
<tr><td>be100.3</td><td>101 ; 14850 | 101</td><td>5950 | 50000 | 30325</td><td>9.9-7 | 3.2-5 | 9.9-7</td><td>-3.7-7 | -5.1-6 | -7.1-7</td><td>53 |8:00 |3:00</td></tr>
<tr><td>be100.4</td><td>101 ; 14850 | 101</td><td>5704 | 50000 | 28722</td><td>9.9-7 | 3.7-5 | 9.9-7</td><td>-2.7-7 | -6.7-6 | -5.9-7</td><td>49 |7:49 |2:47</td></tr>
<tr><td>be100.5</td><td>101 ; 14850 | 101</td><td>5762 | 50000 | 29125</td><td>9.9-7 | 3.1-5 | 9.9-7</td><td>-3.1-7 | -4.5-6 | -7.0-7</td><td>51 |7:55 |2:52</td></tr>
<tr><td>be100.6</td><td>101 ; 14850 | 101</td><td>5769 | 50000 | 27866</td><td>9.9-7 | 3.3-5 | 9.9-7</td><td>-1.8-7 | -5.9-6 | -5.0-7</td><td>48 |7:49 |2:44</td></tr>
<tr><td>be100.7</td><td>101 ; 14850 | 101</td><td>4994 | 50000 | 27913</td><td>9.9-7 | 3.4-5 | 9.7-7</td><td>-3.3-7 | -8.9-6 | -5.5-7</td><td>40 |7:40 |2:46</td></tr>
<tr><td>be100.8</td><td>101 ; 14850 | 101</td><td>5613 | 50000 | 28035</td><td>9.9-7 | 3.5-5 | 9.9-7</td><td>-1.5-7 | -6.7-6 | -5.4-7</td><td>45 |7:46 |2:44</td></tr>
<tr><td>be100.9</td><td>101 ; 14850 | 101</td><td>5763 | 50000 | 28728</td><td>9.9-7 | 3.5-5 | 9.9-7</td><td>-2.7-7 | -4.0-6 | -4.5-7</td><td>51 |8:08 |2:52</td></tr>
<tr><td>be100.10</td><td>101 ; 14850 | 101</td><td>5260 | 50000 | 27868</td><td>9.9-7 | 3.3-5 | 9.9-7</td><td>-3.7-7 | -6.7-6 | -6.9-7</td><td>43 |7:49 |2:41</td></tr>
<tr><td>be120.3.1</td><td>121 ; 21420 | 121</td><td>4120 | 50000 | 22564</td><td>9.9-7 | 3.6-5 | 7.7-7</td><td>-2.2-7 | -1.0-5 | -6.5-7</td><td>41 |10:03 |2:45</td></tr>
<tr><td>be120.3.2</td><td>121 ; 21420 | 121</td><td>4106 | 50000 | 20388</td><td>9.9-7 | 3.7-5 | 9.9-7</td><td>-2.3-7 | -1.1-5 | -9.9-7</td><td>40 |10:02 |2:29</td></tr>
<tr><td>be120.3.3</td><td>121 ; 21420 | 121</td><td>3548 | 50000 | 18503</td><td>9.9-7 | 3.5-5 | 9.9-7</td><td>-9.8-8 | -1.2-5 | -9.9-7</td><td>35 |9:59 |2:15</td></tr>
<tr><td>be120.3.4</td><td>121 ; 21420 | 121</td><td>4745 | 50000 | 24812</td><td>9.9-7 | 3.7-5 | 9.9-7</td><td>-3.1-7 | -1.2-5 | -7.3-7</td><td>47 |10:04 |3:01</td></tr>
<tr><td>be120.3.5</td><td>121 ; 21420 | 121</td><td>5637 | 50000 | 27721</td><td>9.9-7 | 3.6-5 | 9.9-7</td><td>-4.9-8 | -8.9-6 | -6.2-7</td><td>58 |10:19 |3:28</td></tr>
<tr><td>be120.3.6</td><td>121 ; 21420 | 121</td><td>3946 | 50000 | 18775</td><td>9.9-7 | 3.3-5 | 9.8-7</td><td>-1.8-7 | -9.0-6 | -9.4-7</td><td>39 |10:05 |2:14</td></tr>
<tr><td>be120.3.7</td><td>121 ; 21420 | 121</td><td>4169 | 50000 | 22564</td><td>9.9-7 | 3.5-5 | 7.2-7</td><td>-2.7-7 | -1.2-5 | -6.2-7</td><td>41 |10:05 |2:48</td></tr>
<tr><td>be120.3.8</td><td>121 ; 21420 | 121</td><td>3793 | 50000 | 20388</td><td>9.9-7 | 3.3-5 | 9.9-7</td><td>-1.6-7 | -1.2-5 | -7.2-7</td><td>35 |9:56 |2:27</td></tr>
<tr><td>be120.3.9</td><td>121 ; 21420 | 121</td><td>4951 | 50000 | 22879</td><td>9.9-7 | 3.7-5 | 9.9-7</td><td>-2.0-7 | -1.0-5 | -5.0-7</td><td>52 |10:17 |2:32</td></tr>
<tr><td>be120.3.10</td><td>121 ; 21420 | 121</td><td>4264 | 50000 | 22877</td><td>9.9-7 | 3.3-5 | 9.9-7</td><td>-3.4-7 | -9.8-6 | -5.6-7</td><td>42 |10:33 |2:57</td></tr>
<tr><td>be120.8.1</td><td>121 ; 21420 | 121</td><td>5671 | 50000 | 28844</td><td>9.9-7 | 3.6-5 | 9.9-7</td><td>-3.5-7 | -9.8-6 | -1.0-6</td><td>58 |11:02 |3:43</td></tr>
</table>

Table 6.3: The performance of iABCD, BCD and eRABCG with accuracy $\varepsilon = 10^{-6}$.

			iteration			η			η_{gap}			time		
problem	$m_E; m_I$	n_s	iabcd	bcd	erabcg	iabcd	bcd	erabcg	iabcd	bcd	erabcg	iabcd	bcd	erabcg
be120.8.2	121 ; 21420	121	5897	50000	28010	9.9-7	3.8-5	9.9-7	-3.1-7	-7.1-6	-8.1-7	1:02	11:09	3:37
be120.8.3	121 ; 21420	121	5199	50000	28010	9.9-7	3.7-5	9.9-7	-4.9-7	-8.5-6	-9.0-7	52	11:02	3:35
be120.8.4	121 ; 21420	121	6688	50000	33698	9.9-7	3.6-5	9.9-7	-4.0-7	-6.3-6	-7.0-7	1:12	11:00	4:27
be120.8.5	121 ; 21420	121	5828	50000	28138	9.9-7	3.6-5	9.9-7	-3.6-7	-5.8-6	-7.5-7	1:03	11:05	3:42
be120.8.6	121 ; 21420	121	4735	50000	25034	9.9-7	3.6-5	9.9-7	-4.8-7	-9.8-6	-8.7-7	47	10:54	3:12
be120.8.7	121 ; 21420	121	4456	50000	23393	9.9-7	3.4-5	9.6-7	-3.8-7	-9.3-6	-7.8-7	44	10:44	3:01
be120.8.8	121 ; 21420	121	5979	50000	28880	9.9-7	3.7-5	9.9-7	-2.8-7	-8.9-6	-9.7-7	1:01	10:57	3:44
be120.8.9	121 ; 21420	121	5788	50000	28851	9.9-7	3.8-5	9.9-7	-3.3-7	-6.1-6	-8.2-7	1:02	11:16	3:48
be120.8.10	121 ; 21420	121	5636	50000	28138	9.9-7	3.6-5	9.6-7	-3.5-7	-7.7-6	-9.2-7	58	11:15	3:45
be150.3.1	151 ; 33525	151	5917	50000	28133	9.9-7	3.9-5	9.8-7	-1.8-7	-1.2-5	-9.3-7	1:23	15:44	5:08
be150.3.2	151 ; 33525	151	4565	50000	23958	9.9-7	3.9-5	9.7-7	-5.8-7	-1.4-5	-1.2-6	1:02	15:32	4:23
be150.3.3	151 ; 33525	151	4524	50000	23103	9.9-7	4.0-5	9.5-7	-4.9-7	-1.3-5	-8.4-7	1:02	15:34	4:09
be150.3.4	151 ; 33525	151	4506	50000	22609	9.9-7	3.7-5	9.3-7	-4.7-7	-1.3-5	-1.1-6	1:01	15:34	4:02
be150.3.5	151 ; 33525	151	5975	50000	28854	9.9-7	3.8-5	9.7-7	-3.0-7	-9.8-6	-9.9-7	1:24	15:53	5:12
be150.3.6	151 ; 33525	151	4131	50000	23008	9.9-7	3.9-5	9.9-7	-2.8-7	-1.7-5	-6.6-7	56	16:56	4:03
be150.3.7	151 ; 33525	151	4914	50000	23971	9.9-7	4.0-5	9.9-7	-4.9-7	-1.1-5	-1.2-6	1:08	15:41	4:20
be150.3.8	151 ; 33525	151	4354	50000	22976	9.9-7	3.6-5	9.9-7	-4.0-7	-1.4-5	-8.0-7	59	15:30	4:03
be150.3.9	151 ; 33525	151	5559	50000	25534	9.9-7	3.9-5	9.9-7	-2.6-7	-1.1-5	-1.0-6	1:22	15:46	4:38
be150.3.10	151 ; 33525	151	5647	50000	28752	9.9-7	3.9-5	9.9-7	-2.8-7	-1.2-5	-7.0-7	1:24	15:35	5:10
be150.8.1	151 ; 33525	151	5999	50000	29705	9.9-7	3.7-5	9.9-7	-4.0-7	-9.0-6	-1.0-6	1:26	15:48	5:19
be150.8.2	151 ; 33525	151	6516	50000	33468	9.9-7	3.7-5	9.8-7	-5.8-7	-8.3-6	-9.7-7	1:36	16:18	6:03

Table 6.3: The performance of iABCD, BCD and eRABCG with accuracy $\varepsilon = 10^{-6}$.

			iteration			η			η_{gap}			time		
problem	$m_E; m_I$	n_s	iabcd	bcd	erabcg	iabcd	bcd	erabcg	iabcd	bcd	erabcg	iabcd	bcd	erabcg
be150.8.3	151 ; 33525	151	6733	50000	32219	9.9-7	4.0-5	9.8-7	-5.5-7	-8.3-6	-9.9-7	1:40	16:10	6:00
be150.8.4	151 ; 33525	151	6063	50000	29344	9.9-7	3.8-5	9.8-7	-4.6-7	-8.0-6	-1.1-6	1:27	16:05	5:17
be150.8.5	151 ; 33525	151	6717	50000	33518	9.9-7	4.2-5	9.9-7	-4.9-7	-7.6-6	-8.2-7	1:44	16:20	6:18
be150.8.6	151 ; 33525	151	5953	50000	29344	9.9-7	3.6-5	9.8-7	-3.4-7	-9.7-6	-1.2-6	1:25	15:50	5:13
be150.8.7	151 ; 33525	151	5965	50000	31697	9.9-7	3.7-5	9.9-7	-4.7-7	-1.0-5	-1.2-6	1:25	15:44	5:47
be150.8.8	151 ; 33525	151	6396	50000	33130	9.9-7	3.9-5	8.0-7	-5.3-7	-7.9-6	-9.0-7	1:38	16:21	6:09
be150.8.9	151 ; 33525	151	5892	50000	28854	9.9-7	4.0-5	9.5-7	-4.0-7	-7.4-6	-8.1-7	1:25	16:15	5:19
be150.8.10	151 ; 33525	151	6262	50000	29344	9.9-7	4.0-5	9.7-7	-3.7-7	-9.4-6	-1.1-6	1:30	16:08	4:24
be200.3.1	201 ; 59700	201	6020	50000	28718	9.9-7	4.1-5	9.7-7	-4.1-7	-1.5-5	-1.0-6	2:27	23:30	7:18
be200.3.2	201 ; 59700	201	5440	50000	28026	9.9-7	4.3-5	9.8-7	-6.4-7	-1.6-5	-1.2-6	2:13	25:08	6:23
be200.3.3	201 ; 59700	201	6150	50000	30498	9.9-7	4.3-5	9.8-7	-4.6-7	-1.3-5	-1.4-6	2:31	22:09	6:46
be200.3.4	201 ; 59700	201	6079	50000	28715	9.9-7	4.4-5	9.8-7	-4.1-7	-1.3-5	-1.3-6	2:30	22:10	6:16
be200.3.5	201 ; 59700	201	6536	50000	33573	9.9-7	4.1-5	9.9-7	-5.8-7	-1.2-5	-1.3-6	2:42	22:04	7:43
be200.3.6	201 ; 59700	201	5193	50000	28028	9.9-7	4.3-5	9.9-7	-5.0-7	-1.6-5	-1.1-6	2:08	24:24	6:07
be200.3.7	201 ; 59700	201	5750	50000	28715	9.9-7	4.2-5	9.8-7	-1.7-7	-1.6-5	-1.1-6	2:22	23:04	6:19
be200.3.8	201 ; 59700	201	6087	50000	28820	9.9-7	4.1-5	9.9-7	-5.2-7	-1.4-5	-1.5-6	2:30	22:00	6:35
be200.3.9	201 ; 59700	201	6088	50000	28820	9.9-7	4.2-5	9.9-7	-3.9-7	-1.4-5	-1.4-6	2:32	22:16	6:29
be200.3.10	201 ; 59700	201	5464	50000	28026	9.9-7	4.0-5	9.6-7	-6.8-7	-1.6-5	-1.3-6	2:14	22:14	6:16
be200.8.1	201 ; 59700	201	6882	50000	34136	9.9-7	4.0-5	9.9-7	-7.2-7	-7.9-6	-1.5-6	2:53	23:08	8:00
be200.8.2	201 ; 59700	201	6345	50000	33798	9.9-7	4.0-5	9.8-7	-7.0-7	-1.0-5	-1.4-6	2:37	22:44	7:48
be200.8.3	201 ; 59700	201	7829	50000	39097	9.9-7	4.1-5	9.3-7	-6.4-7	-8.3-6	-1.0-6	3:17	23:29	9:09

Table 6.3: The performance of iABCD, BCD and eRABCG with accuracy $\varepsilon = 10^{-6}$.

<table>
<tr><th></th><th></th><th>iteration</th><th>η</th><th>η_{gap}</th><th>time</th></tr>
<tr><td>problem</td><td>$m_E; m_I \mid n_s$</td><td>iabcd|bcd|erabcg</td><td>iabcd|bcd|erabcg</td><td>iabcd|bcd|erabcg</td><td>iabcd|bcd|erabcg</td></tr>
<tr><td>be200.8.4</td><td>201 ; 59700 | 201</td><td>7475 | 50000 | 34483</td><td>9.9-7 | 4.2-5 | 9.8-7</td><td>-6.3-7 | -6.7-6 | -1.2-6</td><td>3:21 |24:03 |8:15</td></tr>
<tr><td>be200.8.5</td><td>201 ; 59700 | 201</td><td>7600 | 50000 | 39771</td><td>9.8-7 | 4.1-5 | 9.9-7</td><td>-7.1-7 | -6.8-6 | -9.9-7</td><td>3:24 |23:58 |9:40</td></tr>
<tr><td>be200.8.6</td><td>201 ; 59700 | 201</td><td>7164 | 50000 | 34143</td><td>9.9-7 | 3.9-5 | 9.9-7</td><td>-7.6-7 | -7.6-6 | -1.6-6</td><td>3:00 |23:11 |7:47</td></tr>
<tr><td>be200.8.7</td><td>201 ; 59700 | 201</td><td>5677 | 50000 | 29693</td><td>9.9-7 | 3.9-5 | 9.7-7</td><td>-7.4-7 | -1.5-5 | -1.9-6</td><td>2:20 |22:18 |5:42</td></tr>
<tr><td>be200.8.8</td><td>201 ; 59700 | 201</td><td>5620 | 50000 | 29427</td><td>9.9-7 | 3.9-5 | 9.9-7</td><td>-7.7-7 | -1.1-5 | -1.7-6</td><td>2:19 |21:33 |6:32</td></tr>
<tr><td>be200.8.9</td><td>201 ; 59700 | 201</td><td>7100 | 50000 | 34143</td><td>9.9-7 | 4.0-5 | 9.9-7</td><td>-7.1-7 | -7.4-6 | -1.5-6</td><td>3:01 |22:11 |7:38</td></tr>
<tr><td>be200.8.10</td><td>201 ; 59700 | 201</td><td>6981 | 50000 | 34136</td><td>9.9-7 | 3.9-5 | 9.7-7</td><td>-7.5-7 | -8.3-6 | -1.5-6</td><td>2:55 |21:58 |7:35</td></tr>
<tr><td>be250.1</td><td>251 ; 93375 | 251</td><td>3958 | 50000 | 19178</td><td>9.9-7 | 5.2-5 | 9.8-7</td><td>1.4-7 | -3.0-5 | -1.9-6</td><td>2:10 |35:43 |6:15</td></tr>
<tr><td>be250.2</td><td>251 ; 93375 | 251</td><td>4213 | 50000 | 23028</td><td>9.9-7 | 5.1-5 | 9.9-7</td><td>-3.7-7 | -2.8-5 | -1.1-6</td><td>2:22 |37:47 |7:25</td></tr>
<tr><td>be250.3</td><td>251 ; 93375 | 251</td><td>4230 | 50000 | 23195</td><td>9.9-7 | 5.1-5 | 9.8-7</td><td>-3.7-7 | -3.0-5 | -1.2-6</td><td>2:29 |37:37 |7:52</td></tr>
<tr><td>be250.4</td><td>251 ; 93375 | 251</td><td>4059 | 50000 | 23109</td><td>9.9-7 | 4.7-5 | 9.9-7</td><td>-3.6-7 | -2.4-5 | -1.1-6</td><td>2:24 |32:53 |7:45</td></tr>
<tr><td>be250.5</td><td>251 ; 93375 | 251</td><td>4361 | 50000 | 23141</td><td>9.9-7 | 5.2-5 | 9.9-7</td><td>-3.9-7 | -2.7-5 | -1.1-6</td><td>2:35 |38:43 |7:39</td></tr>
<tr><td>be250.6</td><td>251 ; 93375 | 251</td><td>4131 | 50000 | 22773</td><td>9.9-7 | 5.2-5 | 7.7-7</td><td>-2.4-7 | -2.9-5 | -8.7-7</td><td>2:25 |37:40 |7:46</td></tr>
<tr><td>be250.7</td><td>251 ; 93375 | 251</td><td>6019 | 50000 | 28518</td><td>9.9-7 | 4.9-5 | 9.9-7</td><td>-1.8-7 | -2.1-5 | -1.1-6</td><td>3:37 |33:21 |9:46</td></tr>
<tr><td>be250.8</td><td>251 ; 93375 | 251</td><td>3398 | 50000 | 18307</td><td>9.9-7 | 5.5-5 | 8.6-7</td><td>-5.0-7 | -3.3-5 | -1.4-6</td><td>2:00 |38:14 |5:56</td></tr>
<tr><td>be250.9</td><td>251 ; 93375 | 251</td><td>5299 | 50000 | 28464</td><td>9.9-7 | 4.9-5 | 9.9-7</td><td>-5.8-7 | -2.4-5 | -1.0-6</td><td>3:09 |33:24 |9:38</td></tr>
<tr><td>be250.10</td><td>251 ; 93375 | 251</td><td>4601 | 50000 | 24611</td><td>9.9-7 | 5.0-5 | 9.9-7</td><td>-3.7-7 | -2.6-5 | -1.1-6</td><td>2:39 |37:58 |8:09</td></tr>
<tr><td>bqp50-1</td><td>51 ; 3675 | 51</td><td>3219 | 50000 | 13417</td><td>9.9-7 | 2.3-5 | 9.8-7</td><td>-2.1-8 | -2.8-6 | -5.3-8</td><td>16 |4:03 |49</td></tr>
<tr><td>bqp50-2</td><td>51 ; 3675 | 51</td><td>4717 | 50000 | 19129</td><td>9.9-7 | 3.2-5 | 9.9-7</td><td>-3.7-8 | -3.5-6 | -1.3-7</td><td>21 |3:52 |1:00</td></tr>
<tr><td>bqp50-3</td><td>51 ; 3675 | 51</td><td>3252 | 50000 | 14389</td><td>9.9-7 | 2.9-5 | 9.9-7</td><td>-6.4-8 | -3.8-6 | -1.7-7</td><td>14 |3:49 |44</td></tr>
<tr><td>bqp50-4</td><td>51 ; 3675 | 51</td><td>3817 | 50000 | 14850</td><td>9.9-7 | 1.9-5 | 9.9-7</td><td>-3.6-8 | -1.3-6 | -5.7-8</td><td>18 |3:54 |51</td></tr>
</table>

Table 6.3: The performance of iABCD, BCD and eRABCG with accuracy $\varepsilon = 10^{-6}$.

<table>
<tr><th></th><th></th><th>iteration</th><th>η</th><th>η_{gap}</th><th>time</th></tr>
<tr><th>problem</th><th>$m_E; m_I \mid n_s$</th><th>iabcd|bcd|erabcg</th><th>iabcd|bcd|erabcg</th><th>iabcd|bcd|erabcg</th><th>iabcd|bcd|erabcg</th></tr>
<tr><td>bqp50-5</td><td>51 ; 3675 | 51</td><td>1887 | 50000 | 6945</td><td>9.8-7 | 1.2-5 | 9.9-7</td><td>3.4-8 | -1.1-6 | -7.6-8</td><td>09 |4:00 |24</td></tr>
<tr><td>bqp50-6</td><td>51 ; 3675 | 51</td><td>3234 | 50000 | 14601</td><td>9.9-7 | 3.2-5 | 9.9-7</td><td>-6.0-8 | -4.8-6 | -1.4-7</td><td>14 |3:50 |46</td></tr>
<tr><td>bqp50-7</td><td>51 ; 3675 | 51</td><td>2209 | 50000 | 11197</td><td>9.9-7 | 1.9-5 | 9.9-7</td><td>-1.6-8 | -1.9-6 | -5.9-8</td><td>10 |3:51 |36</td></tr>
<tr><td>bqp50-8</td><td>51 ; 3675 | 51</td><td>4293 | 50000 | 18393</td><td>9.9-7 | 3.1-5 | 9.9-7</td><td>-8.2-8 | -5.5-6 | -1.1-7</td><td>19 |3:52 |56</td></tr>
<tr><td>bqp50-9</td><td>51 ; 3675 | 51</td><td>4751 | 50000 | 18573</td><td>9.9-7 | 3.1-5 | 9.9-7</td><td>-1.0-8 | -4.0-6 | -8.5-8</td><td>22 |3:55 |1:04</td></tr>
<tr><td>bqp50-10</td><td>51 ; 3675 | 51</td><td>4145 | 50000 | 14601</td><td>9.9-7 | 3.2-5 | 9.9-7</td><td>-3.6-8 | -3.1-6 | -8.0-8</td><td>20 |3:57 |52</td></tr>
<tr><td>bqp100-1</td><td>101 ; 14850 | 101</td><td>7344 | 50000 | 25527</td><td>9.9-7 | 4.8-5 | 9.9-7</td><td>-9.8-8 | -6.1-6 | -2.7-7</td><td>1:08 |7:54 |2:22</td></tr>
<tr><td>bqp100-2</td><td>101 ; 14850 | 101</td><td>3799 | 50000 | 18857</td><td>9.9-7 | 4.3-5 | 9.8-7</td><td>-1.5-7 | -1.4-5 | -6.4-7</td><td>30 |7:22 |1:39</td></tr>
<tr><td>bqp100-3</td><td>101 ; 14850 | 101</td><td>3630 | 50000 | 18066</td><td>9.9-7 | 4.4-5 | 6.7-7</td><td>8.6-8 | -1.1-5 | -2.5-7</td><td>29 |7:22 |1:32</td></tr>
<tr><td>bqp100-4</td><td>101 ; 14850 | 101</td><td>4293 | 50000 | 22744</td><td>9.9-7 | 4.4-5 | 9.6-7</td><td>-2.2-7 | -1.0-5 | -3.6-7</td><td>35 |7:26 |1:58</td></tr>
<tr><td>bqp100-5</td><td>101 ; 14850 | 101</td><td>5145 | 50000 | 23136</td><td>9.9-7 | 4.5-5 | 9.9-7</td><td>-1.0-7 | -1.0-5 | -3.7-7</td><td>43 |7:25 |1:58</td></tr>
<tr><td>bqp100-6</td><td>101 ; 14850 | 101</td><td>4170 | 50000 | 20467</td><td>9.9-7 | 4.6-5 | 9.9-7</td><td>-2.3-7 | -9.6-6 | -6.3-7</td><td>34 |7:34 |1:46</td></tr>
<tr><td>bqp100-7</td><td>101 ; 14850 | 101</td><td>5697 | 50000 | 27700</td><td>9.9-7 | 4.7-5 | 9.9-7</td><td>-1.4-7 | -9.3-6 | -3.7-7</td><td>48 |7:28 |2:27</td></tr>
<tr><td>bqp100-8</td><td>101 ; 14850 | 101</td><td>5491 | 50000 | 22986</td><td>9.9-7 | 4.4-5 | 9.9-7</td><td>-6.4-8 | -8.3-6 | -4.7-7</td><td>46 |7:29 |2:00</td></tr>
<tr><td>bqp100-9</td><td>101 ; 14850 | 101</td><td>5481 | 50000 | 24863</td><td>9.9-7 | 4.1-5 | 9.9-7</td><td>-1.3-7 | -6.7-6 | -4.5-7</td><td>47 |7:28 |2:13</td></tr>
<tr><td>bqp100-10</td><td>101 ; 14850 | 101</td><td>5715 | 50000 | 24868</td><td>9.9-7 | 4.4-5 | 9.9-7</td><td>-6.9-8 | -1.0-5 | -5.7-7</td><td>45 |7:25 |2:10</td></tr>
<tr><td>bqp250-1</td><td>251 ; 93375 | 251</td><td>7230 | 50000 | 34885</td><td>9.9-7 | 4.7-5 | 9.9-7</td><td>-7.1-7 | -1.4-5 | -1.6-6</td><td>4:06 |32:10 |10:38</td></tr>
<tr><td>bqp250-2</td><td>251 ; 93375 | 251</td><td>6848 | 50000 | 34885</td><td>9.9-7 | 4.8-5 | 9.9-7</td><td>-7.7-7 | -1.5-5 | -1.7-6</td><td>3:51 |31:20 |10:17</td></tr>
<tr><td>bqp250-3</td><td>251 ; 93375 | 251</td><td>5352 | 50000 | 28522</td><td>9.9-7 | 4.8-5 | 9.7-7</td><td>-8.0-7 | -2.1-5 | -1.4-6</td><td>2:56 |35:00 |8:13</td></tr>
<tr><td>bqp250-4</td><td>251 ; 93375 | 251</td><td>6624 | 50000 | 33999</td><td>9.9-7 | 5.0-5 | 9.9-7</td><td>-6.2-7 | -1.3-5 | -1.4-6</td><td>3:46 |31:52 |10:08</td></tr>
<tr><td>bqp250-5</td><td>251 ; 93375 | 251</td><td>6933 | 50000 | 34834</td><td>9.9-7 | 4.7-5 | 9.9-7</td><td>-7.5-7 | -1.5-5 | -1.7-6</td><td>3:55 |31:36 |10:15</td></tr>
</table>

Table 6.3: The performance of iABCD, BCD and eRABCG with accuracy $\varepsilon = 10^{-6}$.

<table>
<tr><th></th><th></th><th>iteration</th><th>η</th><th>η_{gap}</th><th>time</th></tr>
<tr><td>problem</td><td>$m_E; m_I \mid n_s$</td><td>iabcd|bcd|erabcg</td><td>iabcd|bcd|erabcg</td><td>iabcd|bcd|erabcg</td><td>iabcd|bcd|erabcg</td></tr>
<tr><td>bqp250-6</td><td>251 ; 93375 | 251</td><td>6789 | 50000 | 34379</td><td>9.9-7 | 4.9-5 | 9.8-7</td><td>-6.1-7 | -1.4-5 | -1.5-6</td><td>3:49 |33:00 |10:16</td></tr>
<tr><td>bqp250-7</td><td>251 ; 93375 | 251</td><td>5894 | 50000 | 33670</td><td>9.9-7 | 5.0-5 | 9.9-7</td><td>-6.2-7 | -2.1-5 | -9.6-7</td><td>3:17 |34:30 |9:47</td></tr>
<tr><td>bqp250-8</td><td>251 ; 93375 | 251</td><td>7654 | 50000 | 34379</td><td>9.9-7 | 5.0-5 | 9.9-7</td><td>-6.7-7 | -1.1-5 | -1.4-6</td><td>4:32 |32:13 |10:27</td></tr>
<tr><td>bqp250-9</td><td>251 ; 93375 | 251</td><td>6089 | 50000 | 33694</td><td>9.9-7 | 4.8-5 | 9.9-7</td><td>-8.8-7 | -1.9-5 | -1.3-6</td><td>3:23 |32:00 |9:43</td></tr>
<tr><td>bqp250-10</td><td>251 ; 93375 | 251</td><td>7023 | 50000 | 34834</td><td>9.9-7 | 4.6-5 | 9.8-7</td><td>-6.1-7 | -1.7-5 | -1.6-6</td><td>3:55 |34:32 |10:20</td></tr>
</table>

Chapter 7

Conclusions

In this thesis, we study the algorithms for solving multi-block large scale convex composite optimization problems with coupled objective functions. In the first part of this thesis, we propose an inexact majorized accelerated block coordinate descent method for the two-block problems and prove the $O(1/k^2)$ iteration complexity. The introduction of the inexactness enables us to solve multi-block unconstrained problems by dividing all the variables into two groups. For an illustration purpose, we implement the iABCD framework to a class of composite least square problems with equations, inequalities and the convex set constraints. The subproblems are suggested to be solved by the inexact one-cycle symmetric Gauss-Seidel technique and the APG-SNCG method, where the later one incorporates the second order information in order to obtain an accurate solution of the subprolems within several steps. The convincing numerical results are presented to demonstrate the superior performance of our proposed iABCD framework.

In the second part of this thesis, we establish the various convergence properties of the mADMM for solving two block linearly constrained optimization problems with coupled objective functions, which greatly extend the previous theoretical results for the ADMM to solve problems with separable objective functions. We also prove the linear convergence rate for the quadratically coupled problems under an error bound condition. In addition, we study the robust isolated calmness for a class of constrained nuclear norm minimization problems that not necessarily to be convex. Our purpose here is to have a deeper understanding of the stability, as well as the error bound conditions, for problems involving non-polyhedral functions.

Many interesting problems related to the content of this thesis are still far from being settled. Below we list some research directions that deserve more explorations.

- Recently, Chambolle and Dossal [8] prove the convergence of the iteration sequence for a class of accelerated proximal gradient algorithms. We leave it as a future work to study the convergence property of the iteration sequence generated by the iABCD method. If this can be done, we shall further consider the convergence rate under the error bound conditions provided in this thesis.

- For the mADMM to solve linearly constrained problems with coupled objective functions, we only show the ergodic complexity for the primal objective values and the primal feasibility. It is still unknown whether a KKT-type ergodic complexity could be obtained, perhaps in the same spirit of Monteiro and Svaiter's work [68] on BD-HPE that include the classical ADMM with the dual step-length equals to 1. To the best of our knowledge, this has not been done if the semi-proximal terms are allowed in the subproblems even for solving the problems with separable objective functions.

- We show the linear convergence rate of the mADMM for problems with quadratically coupled objective functions. It is interesting to know whether the same type of result holds when the coupled objective function is only assumed to be smooth.

- In this thesis, we only consider the problems with smooth coupled objective functions. When the coupled objective function is nonsmooth, is it possible to incorporate the smoothing technique proposed by Nesterov [72] in our iABCD and mADMM frameworks and show the corresponding complexity?

- There are also unsolved questions about the stability and sensitivity analysis for the optimization problems involving nuclear norm or other non-polyhedral functions, such as the characterization of the calmness and the Aubin property by the constraint qualifications or others.

- The numerical results show that the hybrid of the APG and the semismooth Newton-CG algorithm is very efficient in solving the strongly convex quadratic problems with the nonnegative constraints. It is interesting to explore whether this algorithm can be applied to solve other convex SC^1 problems.

Bibliography

[1] L.-E. Andersson and T. Elfving, *An algorithm for constrained interpolation*, SIAM Journal on Scientific and Statistical Computing, 8 (1987), pp. 1012–1025.

[2] F. J. Aragón Artacho and M. H. Geoffroy, *Characterization of metric regularity of subdifferentials*, Journal of Convex Analysis, 15 (2008), pp. 365–380.

[3] A. Beck and L. Tetruashvili, *On the convergence of block coordinate descent type methods*, SIAM Journal on Optimization, 23 (2013), pp. 2037–2060.

[4] D. Boley, *Local linear convergence of the alternating direction method of multipliers on quadratic or linear programs*, SIAM Journal on Optimization, 23 (2013), pp. 2183–2207.

[5] J. F. Bonnans, R. Cominetti, and A. Shapiro, *Second order optimality conditions based on parabolic second order tangent sets*, SIAM Journal on Optimization, 9 (1999), pp. 466–492.

[6] J. F. Bonnans and A. Shapiro, *Perturbation analysis of optimization problems*, Springer Science & Business Media, 2000.

[7] S. Boyd and L. Xiao, *Least-squares covariance matrix adjustment*, SIAM Journal on Matrix Analysis and Applications, 27 (2005), pp. 532–546.

[8] A. Chambolle and C. Dossal, *On the convergence of the iterates of the "fast iterative shrinkage/thresholding algorithm"*, Journal of Optimization Theory and Applications, 166 (2015), pp. 1–15.

[9] A. Chambolle and T. Pock, *A remark on accelerated block coordinate descent for computing the proximity operators of a sum of convex functions.*, Optimization Online, (2015).

[10] C. Chen, B. He, Y. Ye, and X. Yuan, *The direct extension of ADMM for multi-block convex minimization problems is not necessarily convergent*, Mathematical Programming, 155 (2016), pp. 1–23.

[11] C. Chen, M. Li, X. Liu, and Y. Ye, *On the convergence of multi-block alternating direction method of multipliers and block coordinate descent method*, arXiv preprint arXiv:1508.00193, (2015).

[12] F. H. Clarke, *Optimization and nonsmooth analysis*, vol. 5, John Wiley and Sons, New York, 1983.

[13] Y. Cui, C. Leng, and D. Sun, *Sparse estimation of high-dimensional correlation matrices*, Computational Statistics & Data Analysis, (2014).

[14] Y. Cui, X. Li, D. Sun, and K.-C. Toh, *On the convergence properties of a majorized ADMM for linearly constrained convex optimization problems with coupled objective functions*, Journal of Optimization Theory and Applications, to appear.

[15] J. M. Danskin, *The theory of max-min and its application to weapons allocation problems*, vol. 5, Springer Science & Business Media, 2012.

[16] D. Davis and W. Yin, *Convergence rate analysis of several splitting schemes*, arXiv preprint arXiv:1406.4834, (2014).

[17] W. Deng, M.-J. Lai, Z. Peng, and W. Yin, *Parallel multi-block ADMM with O(1/k) convergence*, arXiv preprint arXiv:1312.3040, (2013).

[18] W. Deng and W. Yin, *On the global and linear convergence of the generalized alternating direction method of multipliers*, Journal of Scientific Computing, (2012), pp. 1–28.

[19] C. Ding, *An introduction to a class of matrix optimization problems*, PhD thesis, National University of Singapore, 2012.

[20] ——, *Variational analysis of the Ky Fan k-norm*, Preprint, (2015).

[21] C. Ding, D. Sun, and L. Zhang, *Characterization of the robust isolated calmness for a class of conic programming problems*, Preprint, (2016).

[22] A. Dontchev, *Characterizations of Lipschitz stability in optimization*, in Recent developments in well-posed variational problems, Springer, 1995, pp. 95–115.

[23] A. Dontchev and B. Kalchev, *Duality and well-posedness in convex interpolation*, Numerical functional analysis and optimization, 10 (1989), pp. 673–689.

[24] A. Dontchev and R. Rockafellar, *Characterizations of Lipschitzian stability in nonlinear programming*, Mathematical Reviews, 1001 (1998), p. 48107.

[25] A. L. Dontchev, H. Qi, and L. Qi, *Convergence of Newton's method for convex best interpolation*, Numerische Mathematik, 87 (2001), pp. 435–456.

[26] ——, *Quadratic convergence of Newton's method for convex interpolation and smoothing*, Constructive approximation, 19 (2003), p. 123.

[27] M. Du, *An inexact alternating direction method of multipliers for convex composite conic programming with nonlinear constraints*, PhD thesis, National University of Singapore, 2015.

[28] J. Eckstein and D. P. Bertsekas, *On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators*, Mathematical Programming, 55 (1992), pp. 293–318.

[29] J. Eckstein and W. Yao, *Understanding the convergence of the alternating direction method of multipliers: Theoretical and computational perspectives*, Pacific Journal on Optimization, to appear, (2014).

[30] F. Facchinei and J.-S. Pang, *Finite-dimensional variational inequalities and complementarity problems*, Springer Science & Business Media, 2007.

[31] O. Fercoq and P. Richtárik, *Accelerated, parallel and proximal coordinate descent*, arXiv preprint arXiv:1312.5799, (2013).

[32] A. Fischer, *Solution of monotone complementarity problems with locally Lipschitzian functions*, Mathematical Programming, 76 (1997), pp. 513–532.

[33] K. Fountoulakis and R. Tappenden, *Robust block coordinate descent*, arXiv preprint arXiv:1407.7573, (2014).

[34] D. Gabay, *Applications of the method of multipliers to variational inequalities*, vol. 15, Elsevier, 1983, pp. 299–331.

[35] D. Gabay and B. Mercier, *A dual algorithm for the solution of nonlinear variational problems via finite element approximation*, Computers & Mathematics with Applications, 2 (1976), pp. 17–40.

[36] Y. Gao and D. Sun, *Calibrating least squares covariance matrix problems with equality and inequality constraints*, SIAM Journal of Matrix Analsis and Applications, 31 (2009), pp. 1432–1457.

[37] ——, *A majorized penalty approach for calibrating rank constrained correlation matrix problems*, Preprint available at http://www. math. nus. edu. sg/˜ matsundf/MajorPen. pdf, (2010).

[38] R. Glowinski, *Lectures on numerical methods for non-linear variational problems, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, Notes by Vijayasundaram, G and Adimurthi, M*, vol. 65, Springer Berlin, 1980.

[39] R. Glowinski and A. Marroco, *Approximation by finite elements of order one and solution by penalization-duality of a class of nonlinear dirichlet problems*, Revue française d'automatique, informatique, recherche opérationnelle. Analyse numérique, 9 (1975), pp. 41–76.

[40] D. Han, D. Sun, and L. Zhang, *Linear rate convergence of the alternating direction method of multipliers for convex composite quadratic and semi-definite programming*, arXiv preprint arXiv:1508.02134, (2015).

[41] D. Han and X. Yuan, *Local linear convergence of the alternating direction method of multipliers for quadratic programs*, SIAM Journal on numerical analysis, 51 (2013), pp. 3446–3457.

[42] B. He and X. Yuan, *On non-ergodic convergence rate of Douglas-Rachford alternating direction method of multipliers*, Numerische Mathematik, 130 (2012), pp. 567–577.

[43] N. J. Higham, *Computing the nearest correlation matrix—a problem from finance*, IMA journal of Numerical Analysis, 22 (2002), pp. 329–343.

[44] J.-B. Hiriart-Urruty, J.-J. Strodiot, and V. H. Nguyen, *Generalized Hessian matrix and second-order optimality conditions for problems with $C^{1,1}$ data*, Applied mathematics and optimization, 11 (1984), pp. 43–56.

[45] M. Hong, T.-H. Chang, X. Wang, M. Razaviyayn, S. Ma, and Z.-Q. Luo, *A block successive upper bound minimization method of multipliers for linearly constrained convex optimization*, arXiv preprint arXiv:1401.7079, (2014).

[46] M. Hong and Z.-Q. Luo, *On the linear convergence of the alternating direction method of multipliers*, arXiv preprint arXiv:1208.3922, (2012).

[47] M. Hong, X. Wang, M. Razaviyayn, and Z.-Q. Luo, *Iteration complexity analysis of block coordinate descent methods*, arXiv preprint arXiv:1310.6957, (2013).

[48] L. D. Irvine, S. P. Marin, and P. W. Smith, *Constrained interpolation and smoothing*, Constructive approximation, 2 (1986), pp. 129–151.

[49] K. Jiang, D. Sun, and K.-C. Toh, *An inexact accelerated proximal gradient method for large scale linearly constrained convex SDP*, SIAM Journal on Optimization, 22 (2012), pp. 1042–1064.

[50] A. J. King and R. T. Rockafellar, *Sensitivity analysis for nonsmooth generalized equations*, Mathematical Programming, 55 (1992), pp. 193–212.

[51] D. Klatte, *Upper Lipschitz behavior of solutions to perturbed $C^{1,1}$ programs*, Mathematical Programming, 88 (2000), pp. 285–311.

[52] O. Klopp, K. Lounici, and A. B. Tsybakov, *Robust matrix completion*, arXiv preprint arXiv:1412.8132, (2014).

[53] B. Kummer et al., *Newton's method for non-differentiable functions*, Advances in mathematical optimization, 45 (1988), pp. 114–125.

[54] J. Kyparisis, *On uniqueness of Kuhn-Tucker multipliers in nonlinear programming*, Mathematical Programming, 32 (1985), pp. 242–246.

[55] C. Lemarechal and J.-B. Hiriart-Urruty, *Convex analysis and minimization algorithms II, vol. 306 of Grundlehren der mathematischen Wissenschaften*, Springer-Verlag Berlin Heidelberg, 1993.

[56] A. B. Levy, *Implicit multifunction theorems for the sensitivity analysis of variational conditions*, Mathematical programming, 74 (1996), pp. 333–350.

[57] M. Li, D. Sun, and K.-C. Toh, *A majorized ADMM with indefinite proximal terms for linearly constrained convex composite optimization*, arXiv preprint arXiv:1412.1911, (2014).

[58] X. Li, *A two-phase augmented Lagrangian method for convex composite quadratic programming*, PhD thesis, Department of Mathematics, National University of Singapore, 2015.

[59] X. Li, D. Sun, and K.-C. Toh, *A Schur complement based semi-proximal ADMM for convex quadratic conic programming and extensions*, Mathematical Programming, 155 (2014), pp. 1–41.

[60] ——, *QSDPNAL: A two-phase proximal augmented Lagrangian method for convex quadratic semidefinite programming*, arXiv preprint arXiv:1512.08872, (2015).

[61] Q. Lin, Z. Lu, and L. Xiao, *An accelerated proximal coordinate gradient method and its application to regularized empirical risk minimization*, arXiv preprint arXiv:1407.1296, (2014).

[62] Y. Liu and S. Pan, *Locally upper Lipschitz of the perturbed KKT system of Ky Fan k-norm matrix conic optimization problems*, arXiv preprint arXiv:1509.00681, (2015).

[63] Z.-Q. Luo and P. Tseng, *On the linear convergence of descent methods for convex essentially smooth minimization*, SIAM Journal on Control and Optimization, 30 (1992), pp. 408–425.

[64] ——, *Error bounds and convergence analysis of feasible descent methods: a general approach*, Annals of Operations Research, 46 (1993), pp. 157–178.

[65] F. J. Luque, *Asymptotic convergence analysis of the proximal point algorithm*, SIAM Journal on Control and Optimization, 22 (1984), pp. 277–293.

[66] J. Malick, *A dual approach to semidefinite least-squares problems*, SIAM Journal on Matrix Analysis and Applications, 26 (2004), pp. 272–284.

[67] R. Mifflin, *Semismooth and semiconvex functions in constrained optimization*, SIAM Journal on Control and Optimization, 15 (1977), pp. 959–972.

[68] R. D. Monteiro and B. F. Svaiter, *Iteration-complexity of block-decomposition algorithms and the alternating direction method of multipliers*, SIAM Journal on Optimization, 23 (2013), pp. 475–507.

[69] J.-J. Moreau, *Proximité et dualité dans un espace hilbertien*, Bulletin de la Société mathématique de France, 93 (1965), pp. 273–299.

[70] Y. Nesterov, *A method of solving a convex programming problem with convergence rate $O(1/k^2)$*, in Soviet Mathematics Doklady, vol. 27, 1983, pp. 372–376.

[71] ——, *Introductory lectures on convex optimization*, vol. 87, Springer Science & Business Media, 2004.

[72] Y. Nesterov, *Smooth minimization of non-smooth functions*, Mathematical programming, 103 (2005), pp. 127–152.

[73] ——, *Efficiency of coordinate descent methods on huge-scale optimization problems*, SIAM Journal on Optimization, 22 (2012), pp. 341–362.

[74] J.-S. Pang, *Error bounds in mathematical programming*, Mathematical Programming, 79 (1997), pp. 299–332.

[75] H. Qi and D. Sun, *A quadratically convergent Newton method for computing the nearest correlation matrix*, SIAM Journal on matrix analysis and applications, 28 (2006), pp. 360–385.

[76] L. Qi and J. Sun, *A nonsmooth version of Newton's method*, Mathematical programming, 58 (1993), pp. 353–367.

[77] S. M. Robinson, *Some continuity properties of polyhedral multifunctions*, Springer, 1981.

[78] R. T. Rockafellar, *Conjugate duality and optimization*, vol. 14, Philadelphia: Society for Industrial and Applied Mathematics, 1974.

[79] ——, *Augmented Lagrangians and applications of the proximal point algorithm in convex programming*, Mathematics of operations research, 1 (1976), pp. 97–116.

[80] ——, *Convex analysis*, Princeton university press, 1976.

[81] ——, *Monotone operators and the proximal point algorithm*, SIAM journal on control and optimization, 14 (1976), pp. 877–898.

[82] R. T. Rockafellar and R. J.-B. Wets, *Variational analysis*, vol. 317, Springer Science & Business Media, 2009.

[83] A. Saha and A. Tewari, *On the nonasymptotic convergence of cyclic coordinate descent methods*, SIAM Journal on Optimization, 23 (2013), pp. 576–601.

[84] S. Sardy, A. G. Bruce, and P. Tseng, *Block coordinate relaxation methods for nonparametric wavelet denoising*, Journal of computational and graphical statistics, 9 (2000), pp. 361–379.

[85] M. Schmidt, N. L. Roux, and F. R. Bach, *Convergence rates of inexact proximal-gradient methods for convex optimization*, in Advances in neural information processing systems, 2011, pp. 1458–1466.

[86] A. Shapiro, *Sensitivity analysis of nonlinear programs and differentiability properties of metric projections*, SIAM Journal on Control and Optimization, 26 (1988), pp. 628–645.

[87] R. Shefi and M. Teboulle, *Rate of convergence analysis of decomposition methods based on the proximal method of multipliers for convex minimization*, SIAM Journal on Optimization, 24 (2014), pp. 269–297.

[88] D. Sun and J. Sun, *Semismooth matrix-valued functions*, Mathematics of Operations Research, 27 (2002), pp. 150–169.

[89] ——, *Löwner's operator and spectral functions in Euclidean Jordan algebras*, Mathematics of Operations Research, 33 (2008), pp. 421–445.

[90] D. Sun, K.-C. Toh, and L. Yang, *A convergent proximal alternating direction method of multipliers for conic programming with 4-block constraints, arxiv preprint*, arXiv preprint arXiv:1404.5378, (2014).

[91] D. Sun, K.-C. Toh, and L. Yang, *An efficient inexact ABCD method for least squares semidefinite programming*, arXiv preprint arXiv:1505.04278, (2015).

[92] J. Sun and S. Zhang, *A modified alternating direction method for convex quadratically constrained quadratic semidefinite programs*, European Journal of Operational Research, 207 (2010), pp. 1210–1220.

[93] R. Tappenden, P. Richtárik, and J. Gondzio, *Inexact coordinate descent: complexity and preconditioning*, arXiv preprint arXiv:1304.5530, (2013).

[94] M. Torki, *Second-order directional derivatives of all eigenvalues of a symmetric matrix*, Nonlinear Analysis: Theory, Methods & Applications, 46 (2001), pp. 1133–1150.

[95] P. Tseng, *Dual coordinate ascent methods for non-strictly convex minimization*, Mathematical programming, 59 (1993), pp. 231–247.

[96] ——, *Convergence of a block coordinate descent method for nondifferentiable minimization*, Journal of optimization theory and applications, 109 (2001), pp. 475–494.

[97] ——, *Approximation accuracy, gradient methods, and error bound for structured convex optimization*, Mathematical Programming, 125 (2010), pp. 263–295.

[98] P. Tseng and S. Yun, *A coordinate gradient descent method for nonsmooth separable minimization*, Mathematical Programming, 117 (2009), pp. 387–423.

[99] H. A. Van der Vorst, *Bi-CGSTAB: A fast and smoothly converging variant of Bi-CG for the solution of nonsymmetric linear systems*, SIAM Journal on scientific and Statistical Computing, 13 (1992), pp. 631–644.

[100] S. Villa, S. Salzo, L. Baldassarre, and A. Verri, *Accelerated and inexact forward-backward algorithms*, SIAM Journal on Optimization, 23 (2013), pp. 1607–1633.

[101] G. A. Watson, *Characterization of the subdifferential of some matrix norms*, Linear Algebra and its Applications, 170 (1992), pp. 33–45.

[102] J. Wright, A. Ganesh, S. Rao, Y. Peng, and Y. Ma, *Robust principal component analysis: Exact recovery of corrupted low-rank matrices via convex optimization*, in Advances in neural information processing systems, 2009, pp. 2080–2088.

[103] L. Yang, D. Sun, and K.-C. Toh, *SDPNAL+: A majorized semismooth Newton-CG augmented Lagrangian method for semidefinite programming with nonnegative constraints*, Mathematical Programming Computation, 7 (2015), pp. 1–36.

[104] K. Yosida, *Functional analysis*, Springer, 1974.

[105] L. Zhang, N. Zhang, and X. Xiao, *On the second-order directional derivatives of singular values of matrices and symmetric matrix-valued functions*, Set-Valued and Variational Analysis, 21 (2013), pp. 557–586.

[106] Y. Zhang and L. Zhang, *On the upper Lipschitz property of the KKT mapping for nonlinear semidefinite optimization*, 2015.

[107] Z. Zhou, *Error bounds for structured convex programming: theory and applications*, PhD thesis, The Chinese University of Hong Kong, 2015.

[108] Z. Zhou and M.-C. S. Anthony, *A unified approach to error bounds for structured convex optimization problems*, Preprint.

[109] J. Zowe and S. Kurcyusz, *Regularity and stability for the mathematical programming problem in Banach spaces*, Applied mathematics and Optimization, 5 (1979), pp. 49–62.

LARGE SCALE COMPOSITE OPTIMIZATION PROBLEMS WITH COUPLED OBJECTIVE FUNCTIONS: THEORY, ALGORITHMS AND APPLICATIONS

CUI YING

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2016

Large scale composite optimization problems with coupled objective functions: theory, algorithms and applications **Cui Ying** **2016**