6 The Linear Complementarity Problem

In this chapter, we briefly introduce the linear complementarity problem and present a complementary pivoting algorithm for solving it. Problems of this type arise frequently in engineering applications, game theory, economics and KKT conditions for linear and quadratic programming problems.

Definition 6.1 *Let M be a given a $p \times p$ matrix and let q be a given p vector. The linear complementarity problem (LCP for short) is to find vectors w and z such that*

$$w - Mz = q, \tag{6.1}$$

$$w_j \geq 0, \quad z_j \geq 0 \qquad \text{for } j = 1, \ldots, p, \tag{6.2}$$

$$w_j z_j = 0 \qquad \text{for } j = 1, \ldots, p \tag{6.3}$$

or to conclude that no such solution exists.

Here (w_j, z_j) is a pair of **complementary variables**. A solution (w, z) to the above system is called a **complementary feasible solution**. Moreover, such a solution is a **complementary basic feasible solution** if (w, z) is a basic feasible solution to (6.1) and (6.2) with one variable of the pair (w_j, z_j) basic for each $j = 1, \ldots, p$.

Consider the following quadratic programming problem

$$\begin{aligned} \min \quad & c^T x + \tfrac{1}{2} x^T H x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0, \end{aligned}$$

where c is an n vector, b is an m vector, A is an $m \times n$ matrix, and H is an $n \times n$ symmetric matrix. Denoting the Lagrangian multiplier vectors of the constraints $Ax \leq b$ and $-x \leq 0$ by u and v, respectively, and denoting the vector of slack variables by y, the KKT conditions can be written

$$\begin{aligned} Ax + y &= b \\ -Hx - A^T u + v &= c \\ x^T v = 0, \quad u^T y &= 0 \\ x, y, u, v &\geq 0. \end{aligned}$$

Now, letting

$$M = \begin{bmatrix} 0 & -A \\ A^T & H \end{bmatrix}, \; q = \begin{bmatrix} b \\ c \end{bmatrix}, \; w = \begin{pmatrix} y \\ v \end{pmatrix}, \; z = \begin{pmatrix} u \\ x \end{pmatrix},$$

we can write the KKT conditions as the LCP

$$w - Mz = q, \quad w^T z = 0, \quad (w, z) \geq 0.$$

Thus, by solving the LCP we can find a KKT point of the quadratic programming, in particular, a KKT point of the linear programming if $H \equiv 0$.

Let e_j denote a unit vector with a 1 in the jth position, and let m_j denote the jth column of M for $j = 1, \ldots, p$. A cone spanned by any p vectors obtained by selecting one vector from each pair e_j, and $-m_j$ for $j = 1, \ldots, p$, is called a **complementary cone** associated with matrix M that defines the system (6.1)-(6.3). Note that there are 2^p such complementary cones, and the above system has a solution if and only q belongs to at least one such cone. Also observe that if q belongs to a particular complementary cone and its generators constitute a basis, that is, they are linearly independent, then the corresponding solution is a complementary basic feasible solution, and vice versa. Furthermore, a square matrix M is called a $\mathbf{Q}-$ **matrix** if the corresponding system (6.1)-(6.3) has a solution for each $q \in \Re^p$.

Using the concept of complementary cones to characterize a solution to linear complementarity problems, we can cast (6.1)-(6.3) as an optimization problem in the following manner. Define a binary variable y_j to take on a value of zero or one accordingly as the variables w_j or z_j is permitted to be positive from complementary pair (w_j, z_j) for each $j = 1, \ldots, p$, and consider the following **zero-one bilinear programming problem (BLP)**:

$$\begin{aligned} \min \quad & \sum_{j=1}^{p} y_j w_j + (1 - y_j) z_j \\ \text{s.t.} \quad & w - Mz = q \\ & w \geq 0, z \geq 0, \text{ and } y \text{ binary.} \end{aligned} \tag{6.4}$$

Note that the objective function value for **BLP** is zero for any feasible solution if and only if

$$y_j w_j = (1 - y_j) z_j = 0 \text{ for each } j = 1, \ldots, p,$$

since all the objective terms are nonnegative. Moreover, this happens at optimality if and only if $w_j z_j = 0$ for each $j = 1, \ldots, p$ because of the binariness of y. Hence, a solution (w, z) is a solution to LCP if and only if it is part of an optimal solution to **BLP** with a zero objective function value.

If q is nonnegative, then we immediately have a solution satisfying (6.1)-(6.3), by letting $w = q$ and $z = 0$. If $q \geq 0$ does not hold, a new column $\mathbf{1}$ and an artificial variable are introduced, leading to the following system, where $\mathbf{1}$ is a vector of p 1's:

$$w - Mz - \mathbf{1} z_0 = q \tag{6.5}$$

$$w_j \geq 0, z_j \geq 0, z_0 \geq 0 \text{ for } j = 1, \ldots, p \tag{6.6}$$

$$w_j z_j = 0 \text{ for } j = 1, \ldots, p \tag{6.7}$$

Letting

$$z_0 = \max\{-q_i : 1 \leq i \leq p\}, \; z = 0, \text{ and } w = q + \mathbf{1} z_0,$$

we obtain a starting solution to the above system. Through a sequence of pivots, to be specified later, we attempt to drive the artificial variable z_0 to level zero while satisfying (6.5)-(6.7), thus obtaining a solution to the linear complementarity problem.

Definition 6.2 *Consider the system defined by (6.5)-(6.7). A feasible solution* (w, z, z_0) *to this system is called* **an almost complementary basic feasible solution** *if*

1. (w, z, z_0) *is a basic feasible solution to (6.5) and (6.6).*
2. *Neither* w_s *nor* z_s *is basic, for some* $s \in \{1, \ldots, p\}$.
3. z_0 *is basic, and exactly one variable from each complementary pair* (w_j, z_j) *is basic, for* $j = 1, \ldots, p$ *and* $j \neq s$.

Given an almost complementary basic feasible solution (w, z, z_0), where w_s and z_s are both nonbasic, an **adjacent almost complementary basic feasible solution** $(\hat{w}, \hat{z}, \hat{z}_0)$ is obtained by introducing either w_s or z_s in the basis if pivoting drives a variable other than z_0 from the basis.

From the above definition, it is clear that each almost complementary basic feasible solution has, at most, two adjacent almost complementary solutions. If increasing w_s or z_s drives z_0 out of the basis or produces a ray of the set defined in (6.5) and (6.6), then we have less than two adjacent almost complementary basic feasible solutions.

Summary of Lemke's Complementary Pivoting Algorithm: Introducing the artificial variable z_0, Lemke's pivoting algorithm moves among adjacent almost complementary basic feasible solutions until either a complementary basic feasible solution is obtained or a direction indicating unboundedness of the region defined by (6.5) through (6.7) is found.

Step 0. If $q \geq 0$, stop; $(w, z) = (q, 0)$ is a complementary basic feasible solution. Otherwise, display the system defined by (6.5) and (6.6) in a tableau format. Let

$$-q_s = \max\{-q_i : 1 \leq i \leq p\},$$

and update the tableau by pivoting at row s and the z_0 column. Thus the basic variables z_0 and w_j for $j = 1, \ldots, p$ and $j \neq s$ are nonnegative. Let $y_s = z_s$.

Step 1. Let d_s be the updated column in the current tableau under the variable y_s. If $d_s \leq 0$, go the Step 4. Otherwise, determine the index r by the following minimum ratio test, where $\bar{q}$ is the updated right-hand side column denoting the values of the basic variables:

$$\frac{\bar{q}_r}{d_{rs}} = \min_{1 \leq i \leq p} \{ \frac{\bar{q}_i}{d_{is}} : d_{is} > 0 \}.$$

If the basic variable at row r is z_0, go to Step 3. Otherwise, go to Step 2.

Step 2. The basic variable at row r is either w_l or z_l for some $l \neq s$. The variable y_s enters the basis and the tableau is updated by pivoting at row r and the y_s column. If the variable that just left the basis is w_l, then let $y_s = z_l$; and if the variable that just left the basis is z_l, then let $y_s = w_l$. Return to Step 1.

Step 3. Here y_s enters the basis, and z_0 leaves the basis. Pivot at the y_s column and the z_0 row, producing a complementary basic feasible solution. Stop.

Step 4. Stop with ray termination. A ray

$$R = \{(w, z, z_0) + \lambda d : \lambda \geq 0\}$$

is found such that every point in R satisfies (6.5), (6.6), and (6.7). Here (w, z, z_0) is the almost complementary basic feasible solution associated with the last tableau, and d is an extreme direction of the set defined by (6.5) and (6.6), having a 1 in the row corresponding to y_s,$-d_s$ in the rows of the current basic variables and zero everywhere else.

Lemma 6.1 *Suppose that each almost complementary basic feasible solution of the system (6.5) to (6.7) is nondegenerate; that is, each basic variable is positive. Then, none of the points generated by the complementary pivoting algorithm is repeated; and furthermore, the algorithm must stop in a finite number of steps.*

Lemma 6.2 *Suppose that each almost complementary basic feasible solution of the system (6.5) to (6.7) is nondegenerate. Suppose that the complementary pivoting algorithm is used to solve this system, and, further, suppose that ray termination occurs. In particular, assume that at termination we have the almost complementary basic feasible solution*

$$(\bar{w}, \bar{z}, \bar{z}_0)$$

and the extreme direction

$$(\hat{w}, \hat{z}, \hat{z}_0),$$

giving the ray

$$R = \{(\bar{w}, \bar{z}, \bar{z}_0) + \lambda(\hat{w}, \hat{z}, \hat{z}_0) : \lambda \geq 0\}.$$

Then,

1. $(\hat{w}, \hat{z}, \hat{z}_0) \neq (0, 0, 0)$, $(\hat{w}, \hat{z}) \geq 0$, $\hat{z}_0 \geq 0$.

2. $\hat{w} - M\hat{z} - \mathbf{1}\hat{z}_0 = 0$.

3. $\bar{w}^T\bar{z} = \bar{w}^T\hat{z} = \hat{w}^T\bar{z} = \hat{w}^T\hat{z} = 0$.

4. $\hat{z} \neq 0$.

5. $\hat{z}^T M\hat{z} = -\mathbf{1}^T\hat{z}\hat{z}_0 \leq 0$.

Definition 6.3 *Let M be a $p \times p$ matrix. Then M is said to be copositive*

$$\text{if } z^T M z \geq 0 \text{ foreach } z \geq 0 \,.$$

Furthermore, M is said to be copositive-plus if it is copositive and

$$\text{if } z \geq 0 \text{ and } z^T M Z = 0 \text{ implies that } (M + M^T)z = 0 \,.$$

Theorem 6.1 *Suppose that each almost complementary basic feasible solution of the system (6.5)-(6.7) is nondegenerate, and suppose that M is copositive-plus. The complementary pivoting algorithm stops in a finite number of steps. In particular, if the system defined by (6.1) and (6.2) is consistent, then the algorithm stops with a complementary basic feasible solution to the system defined by (6.1)-(6.3). On the other hand, if the system defined in (6.1) and (6.2) is inconsistent, then the algorithm stops with ray termination.*

Corollary 6.1 *If M has nonnegative entries, with positive diagonal elements, then the complementary pivoting algorithm stops in a finite number of steps with a complementary basic feasible solution.*

Proof. First, note that by the stated assumption on M, the system

$$w - Mz = q \,, \; (w, z) \geq 0$$

has a solution, say, by choosing z sufficiently large so that

$$w = Mz + q \geq 0 \,.$$

The result follows from the theorem by noting that M is copositive-plus. B.E.D.

Example 6.1 [Termination with a Complementary BFS] *We wish to find a solution to the LCP defined by*

$$M = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 1 & -1 & 2 & -2 \\ 1 & 2 & -2 & 4 \end{bmatrix} , \; q = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -6 \end{bmatrix} .$$

Initialization. Introduce the artificial z_0 and form the following tableau:

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	RHS
w_1	1	0	0	0	0	0	1	1	-1	2
w_2	0	1	0	0	0	0	-1	2	-1	2
w_3	0	0	1	0	-1	1	-2	2	-1	-2
w_4	0	0	0	1	-1	-2	2	-4	-1^*	-6

Note that

$$\min\{q_i : 1 \leq i \leq 4\} = q_4 ,$$

so that the pivot at row 4 and the z_0 column. Go to iteration 1 with $y_s = z_4$.

Iteration 1

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	RHS
w_1	1	0	0	-1	1	2	-1	5	0	8
w_2	0	1	0	-1	1	2	-3	6	0	8
w_3	0	0	1	-1	0	3	-4	6^*	0	4
z_0	0	0	0	-1	1	2	-2	4	1	6

Here, $y_s = z_4$ enters the basis. By the minimus ratio test, w_3 leaves the basis; so, for the purpose of the next iteration, $y_s = z_3$. We pivot at the w_3 row and the z_4 column, and we go to iteration 2.

Iteration 2

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	RHS
w_1	1	0	$-\frac{5}{6}$	$-\frac{1}{6}$	1	$-\frac{1}{2}$	$\frac{7}{3}^*$	0	0	$\frac{14}{3}$
w_2	0	1	-1	0	1	-1	1	0	0	4
z_4	0	0	$\frac{1}{6}$	$-\frac{1}{6}$	0	$\frac{1}{2}$	$-\frac{2}{3}$	1	0	$\frac{2}{3}$
z_0	0	0	$-\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{2}{3}$	0	1	$\frac{10}{3}$

Here, $y_s = z_3$ enters the basis. By the minimum ratio test, w_1 leaves the basis; so, for the purpose of the next iteration, $y_s = z_1$. We pivot at the w_1 row and the z_3 column, and we go to iteration 3.

Iteration 3

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	RHS
z_3	$\frac{3}{7}$	0	$-\frac{5}{14}$	$-\frac{1}{14}$	$\frac{3}{7}$	$-\frac{3}{14}$	1	0	0	2
w_2	$-\frac{3}{7}$	1	$-\frac{9}{14}$	$\frac{1}{14}$	$\frac{4}{7}$	$-\frac{11}{14}$	0	0	0	2
z_4	$\frac{2}{7}$	0	$-\frac{1}{14}$	$-\frac{3}{14}$	$\frac{2}{7}$	$\frac{5}{14}$	0	1	0	2
z_0	$-\frac{2}{7}$	0	$-\frac{3}{7}$	$-\frac{2}{7}$	$\frac{5}{7}^*$	$\frac{1}{7}$	0	0	1	2

Here, $y_s = z_1$ enters the basis and z_0 leaves the basis, pivoting at the z_0 row and the z_1 column gives the complementary BFS represented by the following tableau:

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	RHS
z_3	$\frac{3}{5}$	0	$-\frac{1}{10}$	$\frac{1}{10}$	0	$-\frac{3}{10}$	1	0	$-\frac{3}{5}$	$\frac{4}{5}$
w_2	$-\frac{1}{5}$	1	$-\frac{3}{10}$	$\frac{3}{10}$	0	$-\frac{9}{10}$	0	0	$-\frac{4}{5}$	$\frac{2}{5}$
z_4	$\frac{2}{5}$	0	$\frac{1}{10}$	$-\frac{1}{10}$	0	$\frac{3}{10}$	0	1	$-\frac{2}{5}$	$\frac{6}{5}$
z_1	$-\frac{2}{5}$	0	$-\frac{3}{5}$	$-\frac{2}{5}$	1	$\frac{1}{5}$	0	0	$\frac{7}{5}$	$\frac{14}{5}$

The solution is

$$(w_1, w_2, w_3, w_4, z_1, z_2, z_3, z_4) = (0, \frac{2}{5}, 0, 0, \frac{14}{5}, 0, \frac{4}{5}, \frac{6}{5}).$$

Example 6.2 [Ray Termination] *We wish to find a solution to the LCP defined by*

$$M = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ -1 & 1 & 2 & -2 \\ 1 & -2 & -2 & 2 \end{bmatrix}, \ q = \begin{bmatrix} 1 \\ 4 \\ -2 \\ -4 \end{bmatrix}.$$

Initialization.

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	RHS
w_1	1	0	0	0	0	0	-1	1	-1	1
w_2	0	1	0	0	0	0	1	-2	-1	4
w_3	0	0	1	0	1	-1	-2	2	-1	-2
w_4	0	0	0	1	-1	2	2	-2	-1^*	-4

Note that

$$\min\{q_i \,:\, 1 \le i \le 4\} \;=\; q_4\,,$$

so that we pivot at row 4 and the z_0 column. Go to iteration 1 with $y_s = z_4$.

Iteration 1

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	RHS
w_1	1	0	0	-1	1	-2	-3	3	0	5
w_2	0	1	0	-1	1	-2	-1	0	0	8
w_3	0	0	1	-1	2	-3	-4	4^*	0	2
z_0	0	0	0	-1	1	-2	-2	2	1	4

Here, $y_s = z_4$ enters the basis. By the minimus ratio test, w_3 leaves the basis. The tableau is updated by pivoting at the w_3 row and the z_4 column, and we go to iteration 2 with $y_s = z_3$.

Iteration 2

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	RHS
w_1	1	0	$-\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	0	0	0	$\frac{7}{2}$
w_2	0	1	0	-1	1	-2	-1	0	0	8
z_4	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{3}{4}$	-1	1	0	$\frac{1}{2}$
z_0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	1	3

Here, $y_s = z_3$ should enter the basis. However, all the entries under the z_3 column are nonpositive, so we stop with ray termination. We have thus found the ray

$$R \;=\; \{\,(w,z,z_0) \;=\; (\frac{7}{2},8,0,0,0,0,0,\frac{1}{2},3) \;+\; \lambda\,(0,1,0,0,0,0,1,1,0) \,:\, \lambda \ge 0\,\}$$

where every point on the ray satisfies (6.5)-(6.7).