Löwner's Operator and Spectral Functions
in Euclidean Jordan Algebras

Defeng Sun[*] and Jie Sun[†]

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Abstract

We study analyticity, differentiability, and semismoothness of Löwner's operator and spectral functions under the framework of Euclidean Jordan algebras. In particular, we show that many optimization-related classical results in the symmetric matrix space can be generalized within this framework. For example, the metric projection operator over any symmetric cone defined in a Euclidean Jordan algebra is shown to be strongly semismooth. The research also raises several open questions, whose answers would be of strong interest for optimization research.

1   Introduction

We are interested in functions (scalar valued or vector valued) associated with Euclidean Jordan algebras. Details on Euclidean Jordan algebras can be found in Koecher's 1962 lecture notes [24] and the monograph by Faraut and Korányi [15]. Here we briefly describe the properties of Euclidean Jordan algebras that are necessary for defining our functions. For research on interior point methods for optimization problems under the framework of Euclidean Jordan algebras, we refer to [16, 46] and references therein, and for research on P-properties of complementarity problems, see [20, 54].

Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{V}$ be a finite-dimensional vector space over $\mathbb{F}$ endowed with a bilinear mapping $(x, y) \rightarrow x \cdot y$ (product) from $\mathbb{V} \times \mathbb{V}$ into $\mathbb{V}$. The pair $\mathbb{A} := (\mathbb{V}, \cdot)$ is called an *algebra*. For a given $x \in \mathbb{V}$, let $\mathcal{L}(x)$ be the linear operator of $\mathbb{V}$ defined by

$$\mathcal{L}(x)y := x \cdot y \quad \text{for every } y \in \mathbb{V}.$$

An algebra $\mathbb{A}$ is said to be a *Jordan algebra* if, for all $x, y \in \mathbb{V}$:

(i) $x \cdot y = y \cdot x$;

(ii) $x \cdot (x^2 \cdot y) = x^2 \cdot (x \cdot y)$, where $x^2 := x \cdot x$.

For a Jordan algebra $\mathbb{A} = (\mathbb{V}, \cdot)$, we call $x \cdot y$ the Jordan product of x and y.

A Jordan algebra $\mathbb{A}$ is not necessarily associative. That is, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ may not hold in general. However, it is power associative, i.e., for any $x \in \mathbb{V}$, $x^r \cdot x^s = x^{r+s}$ for all integers $r, s \geq 1$ [15, Proposition II.1.2]. If for some element $e \in \mathbb{V}$, $x \cdot e = e \cdot x = x$ for all $x \in \mathbb{V}$, then e is called a *unit element* of $\mathbb{A}$. The unit element, if exists, is unique. A Jordan algebra $\mathbb{A}$ does not necessarily have a unit element. In this paper $\mathbb{A} = (\mathbb{V}, \cdot)$ is always assumed to have a unit element $e \in \mathbb{V}$. Let $\mathbb{F}[X]$ denote the algebra over $\mathbb{F}$ of polynomials in one variable with coefficients in $\mathbb{F}$. For $x \in \mathbb{V}$, define $\mathbb{F}(x) := \{p(x) \,:\, p \in \mathbb{F}[X]\}$ and $\mathbb{J}(x) := \{p \in \mathbb{F}[X] \,:\, p(x) = 0\}$. $(\mathbb{F}(x), \cdot)$ is a subalgebra generated by x and e and $\mathbb{J}(x)$ is an ideal. Since $\mathbb{F}[X]$ is a principal ring, the ideal $\mathbb{J}(x)$ is generated by a monic polynomial which is called the *minimal polynomial* of x [24, Page 28]. For an introduction on the concepts of rings, ideals and others in algebra, see [30, 56].

For $x \in \mathbb{V}$, let $\zeta(x)$ be the degree of the minimal polynomial of x, which can be equivalently defined as

$$\zeta(x) := \min\{\, k \,:\, \{e, x, x^2, \ldots, x^k\} \text{ are linearly dependent}\,\}.$$

This number is always bounded by $\dim(\mathbb{V})$, the dimension of $\mathbb{V}$. Then the *rank* of $\mathbb{A}$ is well defined by

$$r := \max\{\, \zeta(x) \,:\, x \in \mathbb{V}\,\}.$$

An element $x \in \mathbb{V}$ is said to be *regular* if $\zeta(x) = r$. The set of regular elements is open and dense in $\mathbb{V}$ and there exist polynomials $a_1, a_2, \ldots, a_r : \mathbb{V} \to \mathbb{F}$, which are polynomials in the coordinates of $x \in \mathbb{V}$ for some fixed basis, such that the minimal polynomial of every regular element x is given by

$$t^r - a_1(x)t^{r-1} + a_2(x)t^{r-2} + \cdots + (-1)^r a_r(x).$$

The polynomials $a_1, a_2, \ldots, a_r$ are uniquely determined and a_j is homogeneous of degree j, i.e., $a_j(ty) = t^j a_j(y)$ for every $t \in \mathbb{F}$ and $y \in \mathbb{V}$, $j = 1, 2, \ldots, r$ [15, Proposition II.2.1]. The polynomial $t^r - a_1(x)t^{r-1} + a_2(x)t^{r-2} + \cdots + (-1)^r a_r(x)$ is called the *characteristic polynomial* of x. For a regular x, the minimal polynomial and the characteristic polynomial are the same. We call $\mathrm{tr}(x) := a_1(x)$ and $\det(x) := a_r(x)$ the *trace* and the *determinant* of x, respectively.

A Jordan algebra $\mathbb{A} = (\mathbb{V}, \cdot)$, with a unit element $e \in \mathbb{V}$, defined over the real field $\mathbb{R}$ is called a *Euclidean Jordan algebra*, or *formally real Jordan algebra*, if there exists a positive definite symmetric bilinear form on $\mathbb{V}$ which is associative; in other words, there exists on $\mathbb{V}$ an inner product denoted by $\langle \cdot, \cdot \rangle_{\mathbb{V}}$ such that for all $x, y, z \in \mathbb{V}$:

(iii) $\langle x \cdot y, z \rangle_{\mathbb{V}} = \langle y, x \cdot z \rangle_{\mathbb{V}}$.

A Euclidean Jordan algebra is called *simple* if it is not the direct sum of two Euclidean Jordan algebras. Every Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras [15, Proposition III.4.4].

Here is an example of (simple) Euclidean Jordan algebras. Let $\mathbb{S}^m$ be the space of $m \times m$ real symmetric matrices. An inner product on this space is given by

$$\langle X, Y \rangle_{\mathbb{S}^m} := \mathrm{Tr}(XY),$$

where for $X, Y \in \mathbb{S}^m$, XY is the usual matrix multiplication of X and Y and $\mathrm{Tr}(XY)$ is the trace of matrix XY. Then, $(\mathbb{S}^m, \cdot)$ is a Euclidean Jordan algebra with the Jordan product given by

$$X \cdot Y = \frac{1}{2}(XY + YX), \quad X, Y \in \mathbb{S}^m.$$

In this case, the unit element is the identity matrix I in $\mathbb{S}^m$.

Recall that an element $c \in \mathbb{V}$ is said to be *idempotent* if $c^2 = c$. Two idempotents c and q are said to be *orthogonal* if $c \cdot q = 0$. One says that $\{c_1, c_2, \ldots, c_k\}$ is a *complete* system of orthogonal idempotents if

$$c_j^2 = c_j, \quad c_j \cdot c_i = 0 \text{ if } j \neq i, \ j, i = 1, 2, \ldots, k, \quad \text{and} \quad \sum_{j=1}^{k} c_j = e.$$

An idempotent is said to be *primitive* if it is nonzero and cannot be written as the sum of two other nonzero idempotents. We call a complete system of orthogonal primitive idempotents a *Jordan frame*. Then, we have the following important *spectral decomposition* theorem.

Theorem 1 ([15, Theorem III.1.2]) *Suppose that* $\mathbb{A} = (\mathbb{V}, \cdot)$ *is a Euclidean Jordan algebra and the rank of* $\mathbb{A}$ *is* r. *Then for any* $x \in \mathbb{V}$, *there exists a Jordan frame* $\{c_1, c_2, \ldots, c_r\}$ *and real numbers* $\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)$, *arranged in the decreasing order* $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$, *such that*

$$x = \sum_{j=1}^{r} \lambda_j(x) c_j = \lambda_1(x) c_1 + \lambda_2(x) c_2 + \cdots + \lambda_r(x) c_r.$$

The numbers $\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)$ *(counting multiplicities), which are uniquely determined by* x, *are called the eigenvalues and* $\sum_{j=1}^{r} \lambda_j(x) c_j$ *the spectral decomposition of* x. *Furthermore,*

$$\mathrm{tr}(x) = \sum_{j=1}^{r} \lambda_j(x) \quad \text{and} \quad \det(x) = \prod_{j=1}^{r} \lambda_j(x).$$

In fact, the above theorem is called the second version of the spectral decomposition, on which our analysis relies. It also follows readily that a Jordan frame has exactly r elements. The arrangement that $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$ allows us to consider the function $\lambda : \mathbb{V} \to \mathbb{R}^r$. Strictly speaking, the Jordan frame $\{c_1, c_2, \ldots, c_r\}$ in the spectral decomposition of x also depends on x. We do not write this dependence explicitly for the sake of simplicity in notation. Let $\sigma(x)$ be the set consisting of all distinct eigenvalues of x. Then $\sigma(x)$ contains at least one element and at most r. For each $\mu_i \in \sigma(x)$, denote $J_i(x) := \{j : \lambda_j(x) = \mu_i\}$ and

$$b_i(x) := \sum_{j \in J_i(x)} c_j.$$

Obviously, $\{b_i(x) : \mu_i \in \sigma(x)\}$ is a complete system of orthogonal idempotents. From Theorem 1, we obtain

$$x = \sum_{\mu_i \in \sigma(x)} \mu_i b_i(x),$$

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which is essentially the first version of the spectral decomposition stated in [15] as the uniqueness of $\{b_i(x) : \mu_i \in \sigma(x)\}$ is guaranteed by [15, Theorem III.1.1].

Since, by [15, Proposition III.1.5], a Jordan algebra $\mathbb{A} = (\mathbb{V}, \cdot)$ over $\mathbb{R}$ with a unit element $e \in \mathbb{V}$ is Euclidean if and only if the symmetric bilinear form $\mathrm{tr}(x \cdot y)$ is positive definite, we may define another inner product on $\mathbb{V}$ by $\langle x, y \rangle := \mathrm{tr}(x \cdot y), x, y \in \mathbb{V}$. By the associativity of $\mathrm{tr}(\cdot)$ [15, Proposition II.4.3], we know that the inner product $\langle \cdot, \cdot \rangle$ is also associative, i.e., for all $x, y, z \in \mathbb{V}$, it holds that $\langle x \cdot y, z \rangle = \langle y, x \cdot z \rangle$. Thus, for each $x \in \mathbb{V}$, $\mathcal{L}(x)$ is a symmetric operator with respect to this inner product in the sense that

$$\langle \mathcal{L}(x)y, z \rangle = \langle y, \mathcal{L}(x)z \rangle, \ \forall \ y, z \in \mathbb{V}.$$

Let $\| \cdot \|$ be the norm on $\mathbb{V}$ induced by this inner product

$$\|x\| := \sqrt{\langle x, x \rangle} = \Big(\sum_{j=1}^{r} \lambda_j^2(x) \Big)^{1/2}, \ \ x \in \mathbb{V}.$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a scalar valued function. Then, it is natural to define a vector valued function associated with the Euclidean Jordan algebra $\mathbb{A} = (\mathbb{V}, \cdot)$ [4, 25] by

$$\phi_{\mathbb{V}}(x) := \sum_{j=1}^{r} \phi(\lambda_j(x))c_j = \phi(\lambda_1(x))c_1 + \phi(\lambda_2(x))c_2 + \cdots + \phi(\lambda_r(x))c_r, \qquad (1)$$

where $x \in \mathbb{V}$ has the spectral decomposition $x = \sum_{j=1}^{r} \lambda_j(x)c_j$. In a seminal paper [35], Löwner initiated the study of $\phi_{\mathbb{V}}$ for the case $\mathbb{V} = \mathbb{S}^m$. Korányi [25] extended Löwner's result on the monotonicity of $\phi_{\mathbb{S}^m}$ to $\phi_{\mathbb{V}}$. For nonsmooth analysis of $\phi_{\mathbb{V}}$ over the Euclidean Jordan algebra associated with symmetric matrices, see [6, 7, 50] and over the Euclidean Jordan algebra associated with the second order cone (SOC), see [5, 17]. In recognition of Löwner's contribution, we call $\phi_{\mathbb{V}}$ Löwner's operator (function). When $\phi(t) = t_+ := \max(0, t), t \in \mathbb{R}$, Löwner's operator becomes the *metric projection operator*

$$x_+ = (\lambda_1(x))_+ c_1 + (\lambda_2(x))_+ c_2 + \cdots + (\lambda_r(x))_+ c_r$$

over the convex cone

$$\mathcal{K} := \{y^2 : y \in \mathbb{V}\}$$

under the inner product $\langle \cdot, \cdot \rangle$. Actually, $\mathcal{K}$ is a *symmetric cone* [15, Theorem III.2.1], i.e., $\mathcal{K}$ is a self-dual homogeneous closed convex cone.

Recall that a function $f : \mathbb{R}^r \to (-\infty, +\infty]$ is said to be *symmetric* if for any permutation matrix P in $\mathbb{R}^r$, $f(v) = f(Pv)$, i.e, the function value $f(v)$ does not change by permuting the coordinates of $v \in \mathbb{R}^r$. Then, the *spectral function* $f \circ \lambda : \mathbb{V} \to \mathbb{R}$ is defined as

$$(f \circ \lambda)(x) = f(\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)). \qquad (2)$$

See [33] for a survey and [41] for the latest development of the properties of $f \circ \lambda$ associated with $(\mathbb{S}^m, \cdot)$. In this paper, we shall study various differential properties of $f \circ \lambda$ and $\phi_{\mathbb{V}}$ associated with the Euclidean Jordan algebras in a unified way.

The organization of this paper is as follows. In Section 2, we present several basic results needed for further discussion. In section 3, we study important properties of the eigenvalues, Jordan frames and Löwner's operator over simple Euclidean Jordan algebras. We then investigate the differential properties of the spectral functions in Section 4 and conclude the paper in Section 5.

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2 The Building Blocks

Let $\mathbb{V}$ be the linear space $\mathbb{C}^n$ or $\mathbb{R}^n$. A function $g : \mathbb{V} \to \mathbb{F}$ is said to be *analytic* at $\bar{z} \in \mathbb{V}$ if there exists a neighborhood $\mathcal{N}(\bar{z})$ of $\bar{z}$ such that g in $\mathcal{N}(\bar{z})$ can be expanded into an absolutely convergent power series in $z - \bar{z}$:

$$\sum_{j_1, j_2, \ldots, j_n = 0}^{\infty} \bar{a}_{j_1 j_2 \cdots j_n} (z_1 - \bar{z}_1)^{j_1} (z_2 - \bar{z}_2)^{j_2} \cdots (z_n - \bar{z}_n)^{j_n},$$

where all $\bar{a}_{j_1 j_2 \cdots j_n} \in \mathbb{F}$. If g is analytic at $\bar{z} \in \mathbb{V} = \mathbb{R}^n$, then g is also called *real analytic* at $\bar{z}$.

Let $\mathfrak{L}(\mathbb{V})$ be the vector space of linear operators from $\mathbb{V}$ into itself. Denote by $\mathcal{I} \in \mathfrak{L}(\mathbb{V})$ the identity operator, i.e., for all $x \in \mathbb{V}$, $\mathcal{I}x = x$. For any $\mathcal{T} \in \mathfrak{L}(\mathbb{V})$ the *spectrum* $\sigma(\mathcal{T})$ of $\mathcal{T}$ is the set of complex numbers ζ such that $\zeta \mathcal{I} - \mathcal{T}$ is not one-to-one. By the definition of $\sigma(\mathcal{T})$, for any $\mu \in \sigma(\mathcal{T})$, there exists a vector $0 \neq v \in \mathbb{V}$ such that $(\mathcal{T} - \mu \mathcal{I})v = 0$. The number μ is called an *eigenvalue* of $\mathcal{T}$, and any corresponding v is called an eigenvector. Suppose that $\mathbb{M}_1, \mathbb{M}_2, \ldots, \mathbb{M}_s$ are s linear subspaces in $\mathbb{V}$ such that $\mathbb{V} = \mathbb{M}_1 + \mathbb{M}_2 + \cdots + \mathbb{M}_s$ and for all $u_j \in \mathbb{M}_j$ such that $\sum_{j=1}^{s} u_j = 0$ implies $u_j = 0$, $j = 1, 2, \ldots, s$. Then $\mathbb{V}$ is the *direct sum* of $\mathbb{M}_1, \mathbb{M}_2, \ldots, \mathbb{M}_s$ and is denoted by

$$\mathbb{V} = \mathbb{M}_1 \oplus \mathbb{M}_2 \oplus \cdots \oplus \mathbb{M}_s.$$

Each $x \in \mathbb{V}$ can be expressed in a unique way of the form $x = u_1 + u_2 + \cdots + u_s$, $u_j \in \mathrm{M}_j$, $j = 1, 2, \ldots, s$. Denote operators $\mathcal{P}_j \in \mathfrak{L}(\mathbb{V})$ by

$$\mathcal{P}_j x = u_j, \ \ j = 1, 2, \ldots, s.$$

The $\mathcal{P}_j$ is called the *projection operator* onto $\mathbb{M}_j$ along $\mathbb{M}_1 \oplus \cdots \oplus \mathbb{M}_{j-1} \oplus \mathbb{M}_{j+1} \oplus \cdots \oplus \mathbb{M}_s$, $j = 1, 2, \ldots, s$. According to [23, Page 21], we have

$$\mathcal{P}_j^2 = \mathcal{P}_j, \ \ \mathcal{P}_j \mathcal{P}_i = 0 \text{ if } i \neq j, \ i, j = 1, 2, \ldots, s, \ \ \sum_{j=1}^{s} \mathcal{P}_j = \mathcal{I}. \tag{3}$$

Conversely, let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_s \in \mathfrak{L}(\mathbb{V})$ be operators satisfying (3). If we write $\mathbb{M}_j := \mathcal{P}_j(\mathbb{V})$, then $\mathbb{V}$ is an direct sum of $\mathbb{M}_j$, $j = 1, 2, \ldots, s$. Here for any operator $\mathcal{T} \in \mathfrak{L}(\mathbb{V})$, $\mathcal{T}(\mathbb{V})$ is the range space of $\mathcal{T}$.

If $\mathbb{M}_1, \mathbb{M}_2, \ldots, \mathbb{M}_s$ are mutually orthogonal with respect to an inner product $\langle \cdot, \cdot \rangle$, then $\mathbb{V} = \mathbb{M}_1 \oplus \mathbb{M}_2 \oplus \cdots \oplus \mathbb{M}_s$ is called the *orthogonal direct sum* of $\mathbb{M}_1, \mathbb{M}_2, \ldots, \mathbb{M}_s$ and $\mathcal{P}_j$ is the *orthogonal projection operator* onto $\mathbb{M}_j$ with respect to the inner product $\langle \cdot, \cdot \rangle$, $j = 1, 2, \ldots, s$. The orthogonal projection operators $\{\mathcal{P}_j : j = 1, 2, \ldots, s\}$ satisfy

$$\mathcal{P}_j = \mathcal{P}_j^*, \mathcal{P}_j^2 = \mathcal{P}_j, \ \ \mathcal{P}_j \mathcal{P}_i = 0 \text{ if } i \neq j, \ i, j = 1, 2, \ldots, s, \ \ \sum_{j=1}^{s} \mathcal{P}_j = \mathcal{I}, \tag{4}$$

where $\mathcal{P}_j^*$ is the adjoint (operator) of $\mathcal{P}_j$, $j = 1, 2, \ldots, s$. For details, see [23, Chapter 1].

2.1 Functions of Symmetric Operators and Symmetric Matrices

Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{R}^n$ with an inner product $\langle \cdot, \cdot \rangle$. Let $\mathfrak{S}^n \subset \mathfrak{L}(\mathbb{R}^n)$ be the set consisting of all symmetric operators in $\mathfrak{L}(\mathbb{R}^n)$. Let $\mathcal{X}$ be a fixed but arbitrary symmetric operator in $\mathfrak{S}^n$. The *representation* of the symmetric operator $\mathcal{X}$ with respect to the basis $\{u_1, u_2, \ldots, u_n\}$ is the matrix $X \in \mathbb{S}^n$ defined by

$$[\mathcal{X}u_1 \ \mathcal{X}u_2 \ \cdots \ \mathcal{X}u_n] = [u_1 \ u_2 \ \cdots \ u_n]X \,, \tag{5}$$

where $[u_1 \ u_2 \ \cdots \ u_n]$ is the matrix of columns $u_1, u_2, \ldots,$ and u_n. Conversely, for any given $X \in \mathbb{S}^n$, the operator defined by (5) is a symmetric operator in $\mathfrak{S}^n$.

Let $\mathcal{O}^n$ be the set of $n \times n$ real orthogonal matrices. Then for any $X \in \mathbb{S}^n$, there exist an orthogonal matrix $V \in \mathcal{O}^n$ and n real values $\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X)$, arranged in the decreasing order $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$, such that X has the following spectral decomposition

$$X = V \operatorname{diag}(\lambda(X))V^T = \sum_{j=1}^n \lambda_j(X) v_j v_j^T \,, \tag{6}$$

where v_j is the jth column of V, $j = 1, 2, \ldots, n$. Denote $\tilde{v}_j = [u_1 \ u_2 \ \cdots \ u_n]v_j$, $j = 1, 2, \ldots, n$. Then $\{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n\}$ is another orthonormal basis of $\mathbb{R}^n$. Let $\mathcal{P}_j$ be the orthogonal projection operator onto the linear space spanned by $\tilde{v}_j$, i.e.,

$$\mathcal{P}_j x = \langle \tilde{v}_j, x \rangle \tilde{v}_j \,, \quad \forall \, x \in \mathbb{R}^n \,.$$

For each $j \in \{1, 2, \ldots, n\}$, $\mathcal{P}_j$ is a symmetric operator in $\mathfrak{S}(\mathbb{V})$ and its matrix with respect to the basis $\{u_1, u_2, \ldots, u_n\}$ is given by $P_j = v_j v_j^T$. Hence, the symmetric operator $\mathcal{X} \in \mathfrak{S}(\mathbb{V})$, with matrix X as its representation with respect to the basis $\{u_1, u_2, \ldots, u_n\}$, satisfies

$$\mathcal{X} = \sum_{j=1}^n \tilde{\lambda}_j(\mathcal{X}) \mathcal{P}_j \,, \tag{7}$$

where $\tilde{\lambda}_j(\mathcal{X}) := \lambda_j(X)$ is the jth largest eigenvalue of $\mathcal{X}$ (i.e., $\mathcal{X}$ and X share the same set of eigenvalues) with the corresponding eigenvector $\tilde{v}_j$, $j = 1, 2, \ldots, n$.

Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a symmetric function. Then one can define the scalar valued function $f \circ \lambda : \mathbb{S}^n \to \mathbb{R}$ by

$$(f \circ \lambda)(X) := f(\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X)) \,, \tag{8}$$

where $X \in \mathbb{S}^n$ has the spectral decomposition (6). The composite function $f \circ \lambda$ inherits many properties of f. See Lewis [33] for a survey. In [32], Lewis showed that f is (continuously) differentiable at $\lambda(X)$ if and only $f \circ \lambda$ is (continuously) differentiable at X and

$$\nabla(f \circ \lambda)(X) = V \operatorname{diag}(\nabla f(\lambda(X)))V^T \,, \tag{9}$$

which agrees with the formula given in Tsing, Fan, and Verriest [55, Theorem 3.1] when f is analytic at $\lambda(X)$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a scalar function. Then the matrix valued function $\phi_{\mathbb{S}^n}(X)$ at X is defined by

$$\phi_{\mathbb{S}^n}(X) := \sum_{j=1}^{n} \phi(\lambda_j(X)) v_j v_j^T = V \mathrm{diag}(\phi(\lambda_1(X)), \phi(\lambda_2(X)), \ldots, \phi(\lambda_n(X))) V^T. \quad (10)$$

Correspondingly, one may define $\phi_{\mathfrak{S}^n}(\mathcal{X})$ by

$$\phi_{\mathfrak{S}^n}(\mathcal{X}) := \sum_{j=1}^{n} \phi(\tilde{\lambda}_j(\mathcal{X})) \mathcal{P}_j, \quad (11)$$

where $\mathcal{X}$ is the symmetric operator with its representation given by the matrix X. By (6) and (7), we obtain

$$[\phi_{\mathfrak{S}^n}(\mathcal{X}) u_1 \ \phi_{\mathfrak{S}^n}(\mathcal{X}) u_2 \ \cdots \ \phi_{\mathfrak{S}^n}(\mathcal{X}) u_n] = [u_1 \ u_2 \ \cdots \ u_n] \phi_{\mathbb{S}^n}(X). \quad (12)$$

The functions $\phi_{\mathbb{S}^n}$ and $\phi_{\mathfrak{S}^n}$ have been well studied since Löwner [35]. See [3, 22].

Let ϕ be continuous in an open set containing $\sigma(X)$. Let φ_ϕ be any function such that φ_ϕ is differentiable at each $\lambda_j(X)$ and

$$(\varphi_\phi)'(\lambda_j(X)) = \phi(\lambda_j(X)), \ j = 1, 2, \ldots, n.$$

Define $f_\phi : \mathbb{R}^n \to \mathbb{R}$ by

$$f_\phi(x) := \sum_{i=1}^{n} \varphi_\phi(x_i), \ x \in \mathbb{R}^n. \quad (13)$$

Then, f_ϕ is symmetric and differentiable at $\lambda(X)$, and by (9),

$$\nabla(f_\phi \circ \lambda)(X) = \phi_{\mathbb{S}^n}(X) = \sum_{j=1}^{n} \phi(\lambda_j(X)) v_j v_j^T. \quad (14)$$

Let $\xi_1 > \xi_2 > \cdots > \xi_{\bar{n}}$ be all the $\bar{n}$ distinct values in $\sigma(X)$. For each $k = 1, 2, \ldots, \bar{n}$, let $J_k(X) := \{j : \lambda_j(X) = \xi_k\}$. Let $Y \in \mathbb{S}^n$ have the following spectral decomposition

$$Y = W \mathrm{diag}(\lambda(Y)) W^T = \sum_{j=1}^{n} \lambda_j(Y) w_j w_j^T,$$

with $\lambda_1(Y) \geq \lambda_2(Y) \geq \cdots \geq \lambda_n(Y)$ and $W \in \mathcal{O}^n$. Define

$$\tilde{P}_k(Y) = \sum_{j \in J_k(X)} w_j w_j^T. \quad (15)$$

Then, $X = \sum_{k=1}^{\bar{n}} \xi_k \tilde{P}_k(X)$ and $\phi_{\mathbb{S}^n}(X) = \sum_{k=1}^{\bar{n}} \phi(\xi_k) \tilde{P}_k(X)$.

For each $\xi_k \in \sigma(X)$, by taking $\phi_k(\zeta)$ be identically equal to one in an open neighborhood of ξ_k, and identically equal to zero in an open neighborhood of each ξ_j with $j \neq k$, we know that for all $Y \in \mathbb{S}^n$ sufficiently close to X,

$$\tilde{P}_k(Y) = (\phi_k)_{\mathbb{S}^n}(Y), \ k = 1, 2, \ldots, \bar{n}. \quad (16)$$

This equivalence and (14) allow us to state the analyticity of each operator $\tilde{P}_k$ at X, $k = 1, 2, \ldots, \bar{n}$; and the analyticity of $\phi_{\mathbb{S}^n}$ at X when ϕ is analytic in an open set containing $\sigma(X)$. First, we need the following theorem from [55, Theorem 3.1].

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Theorem 2 *Let $X \in \mathbb{S}^n$. Suppose that $f : \mathbb{R}^n \to (-\infty, \infty]$ is a symmetric function. If f is real analytic at the point $\lambda(X)$, then the composite function $f \circ \lambda$ is analytic at X.*

By (14), (15), (16), and Theorem 2, we have the following proposition, which does not require a proof.

Proposition 3 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be real analytic in an open set (may not be connected) containing $\sigma(X)$. Then, $\phi_{\mathbb{S}^n}(\cdot)$ is analytic at X and for all $Y \in \mathbb{S}^n$ sufficiently close to X,*

$$\phi_{\mathbb{S}^n}(Y) = \nabla(f_\phi \circ \lambda)(Y).$$

In particular, each $\tilde{P}_k(\cdot)$ is analytic at X, $k = 1, 2, \ldots, \bar{n}$.

2.2 Hyperbolic Polynomials

In order to study Löwner's operator $\phi_{\mathbb{V}}$ and the spectral function $f \circ \lambda$, we need some results under the framework of hyperbolic polynomials. Let $\mathbb{V}$ be a finite-dimensional inner product space over $\mathbb{R}$. Suppose that $p : \mathbb{V} \to \mathbb{R}$ is a homogeneous polynomial of degree r on $\mathbb{V}$ and $q \in \mathbb{V}$ with $p(q) \neq 0$. Then p is said to be *hyperbolic with respect to q*, if the univariate polynomial $t \mapsto p(x + tq)$ has only real zeros, for every $x \in \mathbb{V}$.

Let p be hyperbolic with respect to q of degree r. Then, for each $x \in \mathbb{V}$, $t \mapsto p(tq - x)$ has only real roots. Let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$ (counting multiplicities) be the r roots of $p(tq - x) = 0$. We say that $\lambda_j(x)$ is the jth *largest eigenvalue* of x (with respect to p and q). Then for $x \in \mathbb{V}$,

$$p(tq - x) = p(q) \prod_{j=1}^{r} (t - \lambda_j(x))$$

and

$$p(x + tq) = (-1)^r p(-tq - x) = p(q) \prod_{j=1}^{r} (t + \lambda_j(x)).$$

The univariate functional $t \mapsto p(tq - x)$ is the *characteristic polynomial* of x (with respect to p, in direction q). Let $\sigma_k(x) := \sum_{j=1}^{k} \lambda_j(x)$, $1 \leq k \leq r$, be the sum of the k largest eigenvalues of x.

A fundamental theorem of Gårding [18] shows that $\lambda_r(\cdot)$ is positively homogeneous and concave on $\mathbb{V}$. This implies that the (closed) hyperbolic cone

$$\mathcal{K}(p, q) := \{ x \ : \ \lambda_r(x) \geq 0 \},$$

associated with p in direction q, is convex. By exploring Gårding's theorem further, Bauschke et al. [2] showed that for each $1 \leq k \leq r$, $\sigma_k(\cdot)$ is positively homogeneous and convex on $\mathbb{V}$. This, by Rockafellar [44], implies that each $\lambda_j(\cdot)$ is locally Lipschitz continuous and directionally differentiable. Actually, by following Rellich's approach [43] for Hermitian matrices, we can further show that for any fixed $x \in \mathbb{V}$ and $h \in \mathbb{V}$, there exist r functions $\nu_1, \nu_2, \ldots, \nu_r : \mathbb{R} \to \mathbb{R}$, which are analytic at $\varepsilon = 0$, such that for all $\varepsilon \in \mathbb{R}$ sufficiently small,

$$\{\nu_1(\varepsilon), \nu_2(\varepsilon), \ldots, \nu_r(\varepsilon)\} = \{\lambda_1(x + \varepsilon h), \lambda_2(x + \varepsilon h), \ldots, \lambda_r(x + \varepsilon h)\}. \tag{17}$$

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The proof can be sketched as follows. For any $\varepsilon \in \mathbb{R}$,

$$p(tq - (x + \varepsilon h)) = p(q)(t^r + s_1(\varepsilon)t^{r-1} + \cdots + s_{r-1}(\varepsilon)t + s_r(\varepsilon)),$$

where $s_1, s_2, \ldots, s_r$ are polynomials of ε. Since p is hyperbolic with respect to q, all the roots of $t^r + s_1(\varepsilon)t^{r-1} + \cdots + s_{r-1}(\varepsilon)t + s_r(\varepsilon) = 0$ are reals when $\varepsilon \in \mathbb{R}$. Then, by a similar argument to the proof in Rellich [43, Page 31], we can conclude that there exist r functions $\nu_1, \nu_2, \ldots, \nu_r : \mathbb{R} \to \mathbb{R}$, which are analytic at $\varepsilon = 0$, such that (17) holds for all $\varepsilon \in \mathbb{R}$ sufficiently small.

Let $\mathbb{A} = (\mathbb{V}, \cdot)$ be a Euclidean Jordan algebra of rank r introduced in Section 1. By letting $p(x) := \det(x)$, $x \in \mathbb{V}$, we see from Theorem 1 that p is hyperbolic with respect to e of degree r since $p(e) = \det(e) = 1 \neq 0$. Furthermore, p is complete, i.e.,

$$\{x \in \mathbb{V} : \lambda(x)\} = \{0\}.$$

Moreover, p is *isometric* in the sense of [2, Definition 5.1], i.e., for all $x, y \in \mathbb{V}$, there exists $z \in \mathbb{V}$ such that

$$\lambda(z) = \lambda(y) \quad \text{and} \quad \lambda(x + z) = \lambda(x) + \lambda(z).$$

The latter fact can be shown as follows. Let x and y be arbitrary but fixed vectors in $\mathbb{V}$. Then, by Theorem 1, there exists a Jordan frame $\{c_1, c_2, \ldots, c_r\}$ such that

$$x = \sum_{j=1}^{r} \lambda_j(x)c_j = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r.$$

Let $z := \sum_{j=1}^{r} \lambda_j(y)c_j$. Then, $\lambda(z) = \lambda(y)$ and

$$x + z = \sum_{j=1}^{r} (\lambda_j(x) + \lambda_j(y))c_j,$$

which, together with $\lambda(z) = \lambda(y)$, implies that $\lambda(x + z) = \lambda(x) + \lambda(y) = \lambda(x) + \lambda(z)$. The isometric property of p thus holds. Therefore, by [2, Corollary 3.3 and Theorem 5.5] and (17), we have

Proposition 4 *Let $\mathbb{A} = (\mathbb{V}, \cdot)$ be a Euclidean Jordan algebra and $f : \mathbb{R}^r \to (-\infty, \infty]$ be a symmetric convex function. The following results hold.*

(i) *For each $1 \leq k \leq r$, $\sigma_k(\cdot)$ is positively homogeneous and convex on $\mathbb{V}$.*

(ii) *$f \circ \lambda$ is differentiable at x if and only if f is differentiable at $\lambda(x)$ and*

$$\{z \in \mathbb{V} : \lambda(z) = \nabla f(\lambda(x)), \ \langle x, z \rangle = \lambda(x)^T \lambda(z)\} = \{\nabla (f \circ \lambda)(x)\}.$$

(iii) *For any $x, h \in \mathbb{V}$, the eigenvalues of $x + \varepsilon h$, $\varepsilon \in \mathbb{R}$ can be arranged to be analytic at $\varepsilon = 0$.*

2.3 Semismoothness

Let $\mathbb{X}$ and $\mathbb{Y}$ be two finite dimensional inner product spaces over the field $\mathbb{R}$. Let $\mathcal{O}$ be an open set in $\mathbb{X}$ and $\Phi : \mathcal{O} \subseteq \mathbb{X} \to \mathbb{Y}$ be a locally Lipschitz continuous function on the open set $\mathcal{O}$. By Rademacher's theorem, Φ is almost everywhere (in the sense of Lebesgue measure) differentiable (in the sense of Fréchet) in $\mathcal{O}$. Let $\mathcal{D}_\Phi$ be the set of points in $\mathcal{O}$ where Φ is differentiable. Let $\Phi'(x)$, which is a linear mapping from $\mathbb{X}$ to $\mathbb{Y}$, denote the derivative of Φ at $x \in \mathcal{O}$ if Φ is differentiable at x. Then, the B(ouligand)-subdifferential of Φ at $x \in \mathcal{O}$, denoted by $\partial_B \Phi(x)$, is the set of V such that $V = \{\lim_{k \to \infty} \Phi'(x_k)\}$, where $\{x_k\} \in \mathcal{D}_\Phi$ is a sequence converging to x. Clarke's generalized Jacobian of Φ at x is the convex hull of $\partial_B \Phi(x)$ (see [10]), i.e., $\partial \Phi(x) = \text{conv}\{\partial_B \Phi(x)\}$. It follows from the work of Warga on derivative containers [57, Theorem 4] that the set $\partial \Phi(x)$ is actually "blind" to sets of Lebesgue measure zero (see [10, Theorem 2.5.1] for the case that $\mathbb{Y} = \mathbb{R}$), i.e., if S is any set of Lebesgue measure zero in $\mathbb{X}$, then

$$\partial \Phi(x) = \text{conv}\{\lim_{k \to \infty} \Phi'(x_k) \,:\, x_k \to x\,,\ x_k \in \mathcal{D}_\Phi\,,\ x_k \notin S\}\,. \tag{18}$$

Semismoothness was originally introduced by Mifflin [36] for functionals, and was used to analyze the convergence of bundle type methods [31, 37, 48] for nondifferentiable optimization problems. In particular, it plays a key role in establishing the convergence of the BT-trust region method for solving optimization problems with equilibrium constraints. For studying the superlinear convergence of Newton's method for solving nondifferentiable equations, Qi and Sun [42] extended the definition of semismoothness to vector valued functions. There are several equivalent ways for defining the semismoothness. We find the following definition of semismoothness convenient.

Definition 5 *Let $\Phi : \mathcal{O} \subseteq \mathbb{X} \to \mathbb{Y}$ be a locally Lipschitz continuous function on the open set $\mathcal{O}$. We say that Φ is semismooth at a point $x \in \mathcal{O}$ if*

(i) Φ is directionally differentiable at x; and

(ii) for any $y \to x$ and $V \in \partial \Phi(y)$,

$$\Phi(y) - \Phi(x) - V(y - x) = o(\|y - x\|)\,. \tag{19}$$

In Definition 5, part (i) and part (ii) do not imply each other. Condition (19) in part (ii), together with a nonsingularity assumption on $\partial \Phi$ at a solution point, was used by Kummer [26] before [42] to prove the superlinear convergence of Newton's method for locally Lipschitz equations. Φ is said to be G-semismooth at x if condition (19) holds. A stronger notion than semismoothness is γ-order semismoothness with $\gamma > 0$. For any $\gamma > 0$, Φ is said to be γ-order G-semismooth (respectively, γ-order semismooth) at x, if Φ is G-semismooth (respectively, semismooth) at x and for any $y \to x$ and $V \in \partial \Phi(y)$,

$$\Phi(y) - \Phi(x) - V(y - x) = O(\|y - x\|^{1+\gamma})\,. \tag{20}$$

In particular, Φ is said to be strongly G-semismooth (respectively, strongly semismooth) at x if Φ is 1-order G-semismooth (respectively, 1-order semismooth) at x, We say that Φ is

G-semismooth (respectively, semismooth, p-order G-semismooth, p-order semismooth) on a set $\mathcal{Z} \subseteq \mathcal{O}$ if Φ is G-semismooth (respectively, semismooth, γ-order G-semismooth, γ-order semismooth) at every point of $\mathcal{Z}$. G-semismoothness was used in [19] and [40] to obtain inverse and implicit function theorems and stability analysis for nonsmooth equations.

Lemma 6 *Let $\Phi : \mathcal{O} \subseteq \mathbb{X} \to \mathbb{Y}$ be locally Lipschitz near $x \in \mathcal{O}$. Let $\gamma > 0$ be a constant. If S is a set of Lebesgue measure zero in $\mathbb{X}$, then Φ is G-semismooth (γ-order G-semismooth) at x if and only if for any $y \to x$, $y \in \mathcal{D}_\Phi$, and $y \notin S$,*

$$\Phi(y) - \Phi(x) - \Phi'(y)(y - x) = o(||y - x||) \quad (= O(||y - x||^{1+\gamma})). \tag{21}$$

Proof. By examining the proof of [51, Theorem 3.7] and making use of (18), one can prove the conclusion without difficulty. We omit the details. □

Lemma 6 is useful in proving the semismoothness of Lipschitz functions. It first appeared in [51] for the case $S = \emptyset$ and has been used in [6, 8, 41]. Next, we shall use this lemma to show that a continuous selection of fintely many G-semismooth (respectively, γ-order G-semismooth) functions is still G-semismooth (respectively, γ-order G-semismooth). The latter will be used to prove the strong semismoothness of eigenvalue functions over the Euclidean Jordan algebras.

Let $\Phi_1, \Phi_2, \cdots, \Phi_m : \mathcal{O} \subseteq \mathbb{X} \to \mathbb{Y}$ be m continuous functions on the open set $\mathcal{O}$. A function $\Phi : \mathcal{O} \subseteq \mathbb{X} \to \mathbb{Y}$ is called a continuous selection of $\{\Phi_1, \Phi_2, \ldots, \Phi_m\}$ if Φ is a continuous function on $\mathcal{O}$ and for each $y \in \mathcal{O}$,

$$\Phi(y) \in \{\Phi_1(y), \Phi_2(y), \ldots, \Phi_m(y)\}.$$

For $x \in \mathcal{O}$, define the *active set* of Φ at x by

$$\mathcal{I}_\Phi(x) := \{j \,:\, \Phi_j(x) = \Phi(x)\,,\ j = 1, 2, \ldots, m\}$$

and the *essentially active set* of Φ at x by

$$\mathcal{I}_\Phi^e(x) := \{j \,:\, x \in \text{cl}(\text{int}\{y \in \mathcal{O} \,|\, \Phi_j(y) = \Phi(y)\})\,,\ j = 1, 2, \ldots, m\}\,,$$

where "cl" and "int" denote the closure and interior operations, respectively. The functions Φ_j, $j \in \mathcal{I}_\Phi(x)$ are called *active selection functions* at x. An active selection function Φ_j is called *essentially active* at x if $j \in \mathcal{I}_\Phi^e(x)$. In the proof of [47, Proposition 4.1.1], Scholtes actually showed that for every $x \in \mathcal{O}$, there exists an open neighborhood $\mathcal{N}(x)(\subseteq \mathcal{O})$ of x such that

$$\Phi(y) \in \{\Phi_j(y) \,:\, j \in \mathcal{I}_\Phi^e(x)\}\,,\ y \in \mathcal{N}(x)\,. \tag{22}$$

Proposition 7 *Let $\Phi_1, \Phi_2, \ldots, \Phi_m : \mathcal{O} \subseteq \mathbb{X} \to \mathbb{Y}$ be m continuous functions on an open set $\mathcal{O}$ and $\Phi : \mathcal{O} \subseteq \mathbb{X} \to \mathbb{Y}$ be a continuous selection of $\{\Phi_1, \Phi_2, \ldots, \Phi_m\}$. Let $x \in \mathcal{O}$ and $\gamma > 0$ be a constant. If all the essentially active selective functions Φ_j, $j \in \mathcal{I}_\Phi^e(x)$, at x are G-semismooth (respectively, semismooth, γ-order G-semismooth, γ-order semismooth) at x, then Φ is G-semismooth (respectively, semismooth, γ-order G-semismooth, γ-order semismooth) at x.*

Proof. Let $\mathcal{N}(x)(\subseteq \mathcal{O})$ be an open set of x such that (22) holds. Suppose that all Φ_j, $j \in \mathcal{I}_{\Phi}^e(x)$ are G-semismooth at x. By the definition of G-semismoothness, these functions Φ_j, $j \in \mathcal{I}_{\Phi}^e(x)$ are locally Lipschitz continuous functions on $\mathcal{N}(x)$. Then, by Hager [21] or [47, Proposition 4.12], Φ is locally Lipschitz continuous on the open set $\mathcal{N}(x)$.

Let $S_j := \mathcal{N}(x) \backslash \mathcal{D}_{\Phi_j}$, $j \in \mathcal{I}_{\Phi}^e(x)$ and

$$S := \bigcup_{j \in \mathcal{I}_{\Phi}^e(x)} S_j \,.$$

Since all $\{S_j \,:\, j \in \mathcal{I}_{\Phi}^e(x)\}$ are sets of Lebesgue measure zero, S is also a set of Lebesgue measure zero. By Lemma 6, in order to prove that Φ is also G-semismooth at x, we only need to show that for any $y \to x$, $y \in \mathcal{D}_{\Phi}$, and $y \notin S$,

$$\Phi(y) - \Phi(x) - \Phi'(y)(y - x) = o(||y - x||) \,. \tag{23}$$

For the sake of contradiction, assume that (23) does not hold. Then there exist a constant $\delta > 0$ and a sequence $\{y_k\}$ converging to x with $y_k \in \mathcal{D}_{\Phi} \cap \mathcal{N}(x)$ and $y_k \notin S$ such that

$$\|\Phi(y_k) - \Phi(x) - \Phi'(y_k)(y_k - x)\| \geq \delta \|y_k - x\|$$

for all k sufficiently large. Since $y_k \in \mathcal{D}_{\Phi} \cap \mathcal{N}(x)$ and $y_k \notin S$, we have for all k that

$$\Phi'(y_k)(y_k - x) \in \{(\Phi_j)'(y_k)(y_k - x) \,:\, j \in \mathcal{I}_{\Phi}^e(x)\} \,. \tag{24}$$

On the other hand, by the assumption that Φ_j, $j \in \mathcal{I}_{\Phi}^e(x)$ are G-semismooth at x, we have

$$\Phi_j(y_k) - \Phi_j(x) - (\Phi_j)'(y_k)(y_k - x) = o(||y_k - x||) \quad \text{as } k \to \infty \,, \ j \in \mathcal{I}_{\Phi}^e(x) \,,$$

which, together with (24) and the fact that $\Phi(x) = \Phi_j(x)$, $j \in \mathcal{I}_{\Phi}^e(x)$ implies

$$\begin{aligned} \Phi(y_k) - \Phi(x) - \Phi'(y_k)(y_k - x) \ &\in \{\Phi_j(y_k) - \Phi_j(x) - (\Phi_j)'(y_k)(y_k - x) \,:\, j \in \mathcal{I}_{\Phi}^e(x)\} \\ &= o(||y^k - x||) \quad \text{as } k \to \infty \,. \end{aligned}$$

So a contradiction is derived. This contradiction shows that (23) holds. Thus Φ is G-semismooth at x.

To prove that Φ is semismooth at x when all Φ_j, $j \in \mathcal{I}_{\Phi}^e(x)$ are semismooth at x, we only need to show that Φ is directionally differentiable at x if all Φ_j, $j \in \mathcal{I}_{\Phi}^e(x)$ are directionally differentiable at x. The latter can be derived from the proof of [27, Proposition 2.5]. In fact, Kuntz and Scholtes only proved that Φ is directionally differentiable at x under the assumption that all Φ_j, $j \in \mathcal{I}_{\Phi}^e(x)$ are continuously differentiable functions. A closer examination reveals that their proof is still valid if one replaces the derivatives of Φ_j, $j \in \mathcal{I}_{\Phi}^e(x)$ at x by their directional derivatives. Also see [39, Lemma 1] for this result.

Similarly, one can prove that Φ is γ-order G-semismooth (respectively, γ-order semismooth) x if all Φ_j, $j \in \mathcal{I}_{\Phi}^e(x)$, are γ-order G-semismooth (respectively, γ-order semismooth) at x. $\quad \square$

3 Eigenvalues, Jordan Frames and Löwner's Operator

Let $\mathbb{A} = (\mathbb{V}, \cdot)$ be a Jordan algebra (not necessarily Euclidean). An important part in the theory of Jordan algebras is the *Peirce decomposition*. Let $c \in \mathbb{V}$ be a nonzero idempotent. Then, by [15, Proposition III.1.3], we know that c satisfies $2\mathcal{L}^3(c) - 3\mathcal{L}^2(c) + \mathcal{L}(c) = 0$ and the distinct eigenvalues of the symmetric operator $\mathcal{L}(c)$ are $0, \frac{1}{2}$ and 1. Let $\mathbb{V}(c, 1), \mathbb{V}(c, \frac{1}{2})$, and $\mathbb{V}(c, 0)$ be the three corresponding eigenspaces, i.e.,

$$\mathbb{V}(c, i) := \{ x \in \mathbb{V} \, : \, \mathcal{L}(c)x = ix \}, \quad i = 1, \frac{1}{2}, 0 \,.$$

Then $\mathbb{V}$ is the orthogonal direct sum of $\mathbb{V}(c, 1), \mathbb{V}(c, \frac{1}{2})$, and $\mathbb{V}(c, 0)$. The decomposition

$$\mathbb{V} = \mathbb{V}(c, 1) \oplus \mathbb{V}(c, \frac{1}{2}) \oplus \mathbb{V}(c, 0)$$

is called the *Peirce decomposition* of $\mathbb{V}$ with respect to the nonzero idempotent c.

In the sequel we assume that $\mathbb{A} = (\mathbb{V}, \cdot)$ is a simple Euclidean Jordan algebra of rank r and $\dim(\mathbb{V}) = n$. Then, from the spectral decomposition theorem we know that an idempotent c is primitive if and only if $\dim(V(c, 1)) = 1$ [15, Page 65].

Let $\{c_1, c_2, \ldots, c_r\}$ be a Jordan frame of $\mathbb{A}$. From [15, Lemma IV.1.3], we know that the operators $\mathcal{L}(c_j), j = 1, 2, \ldots, r$ commute and admit a simultaneous diagonalization. For $i, j \in \{1, 2, \ldots, r\}$, define the following spaces

$$\mathbb{V}_{ii} := \mathbb{V}(c_i, 1) = \mathbb{R}c_i$$

and when $i \neq j$,

$$\mathbb{V}_{ij} := \mathbb{V}(c_i, \frac{1}{2}) \cap \mathbb{V}(c_j, \frac{1}{2}) \,.$$

Then, from [15, Theorem IV.2.1], we have the following proposition.

Proposition 8 *The space $\mathbb{V}$ is the orthogonal direct sum of subspaces $\mathbb{V}_{ij}$ $(1 \leq i \leq j \leq r)$, i.e., $\mathbb{V} = \bigoplus_{i \leq j} \mathbb{V}_{ij}$. Furthermore,*

$$\mathbb{V}_{ij} \cdot \mathbb{V}_{ij} \subset \mathbb{V}_{ii} + \mathbb{V}_{jj},$$

$$\mathbb{V}_{ij} \cdot \mathbb{V}_{jk} \subset \mathbb{V}_{ik}, \text{ if } i \neq k,$$

$$\mathbb{V}_{ij} \cdot \mathbb{V}_{kl} = \{0\}, \text{ if } \{i, j\} \cap \{k, l\} = \emptyset.$$

For any $i \neq j \in \{1, 2, \ldots, r\}$ and $s \neq t \in \{1, 2, \ldots, r\}$, by [15, Corollary IV.2.6], we have

$$\dim(\mathbb{V}_{ij}) = \dim(\mathbb{V}_{st}) \,.$$

Let d denote this dimension. Then

$$n = r + \frac{d}{2}r(r - 1) \,. \tag{25}$$

For $x \in \mathbb{V}$ we define

$$\mathcal{Q}(x) := 2\mathcal{L}^2(x) - \mathcal{L}(x^2) \,.$$

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The operator $\mathcal{Q}$ is called the *quadratic representation* of $\mathbb{V}$. Let $x \in \mathbb{V}$ have the spectral decomposition $x = \sum_{j=1}^{r} \lambda_j(x)c_j$, where $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$ are the eigenvalues of x and $\{c_1, c_2, \ldots, c_r\}$ (depending on x) the corresponding Jordan frame. Let $\mathfrak{C}(x)$ be the set consisting of all such Jordan frames at x. For $i, j \in \{1, 2, \ldots, r\}$, let $\mathcal{C}_{ij}(x)$ be the orthogonal projection operator onto $\mathbb{V}_{ij}$. Then, by [15, Theorem IV.2.1],

$$\mathcal{C}_{jj}(x) = \mathcal{Q}(c_j) \text{ and } \mathcal{C}_{ij}(x) = 4\mathcal{L}(c_i)\mathcal{L}(c_j) = 4\mathcal{L}(c_j)\mathcal{L}(c_i) = \mathcal{C}_{ji}(x), \ i, j = 1, 2, \ldots, r. \quad (26)$$

By Proposition 8 and (4), the orthogonal projection operators $\{\mathcal{C}_{ij}(x) : i, j = 1, 2, \ldots, r\}$ satisfy

$$\mathcal{C}_{ij}(x) = \mathcal{C}_{ij}^*(x), \ \mathcal{C}_{ij}^2(x) = \mathcal{C}_{ij}(x), \ \mathcal{C}_{ij}(x)\mathcal{C}_{kl}(x) = 0 \text{ if } \{i,j\} \neq \{k,l\}, i, j, k, l = 1, 2, \ldots, r$$

and

$$\sum_{1 \leq i \leq j \leq r} \mathcal{C}_{ij}(x) = \mathcal{I}.$$

From (26), one can obtain easily that

$$\mathcal{C}_{jj}(x)e = c_j \text{ and } \mathcal{C}_{ij}(x)e = 4c_i \cdot c_j = 0 \text{ if } i \neq j, \ i, j = 1, 2, \ldots, r. \quad (27)$$

From $\sum_{l=1}^{r} c_j = e$ and (26), we get for each $j \in \{1, 2, \ldots, r\}$ that

$$\mathcal{L}(c_j) = \mathcal{L}(c_j)\mathcal{I} = \mathcal{L}(c_j)\mathcal{L}(e) = \sum_{l=1}^{r} \mathcal{L}(c_j)\mathcal{L}(c_l) = \mathcal{L}^2(c_j) + \frac{1}{4}\sum_{\substack{l=1 \\ l \neq j}}^{r} \mathcal{C}_{jl}(x),$$

which, together with the facts that $\mathcal{Q}(c_j) = 2\mathcal{L}^2(c_j) - \mathcal{L}(c_j)$ and $\mathcal{C}_{jj}(x) = \mathcal{Q}(c_j)$, implies

$$\mathcal{L}(c_j) = \mathcal{C}_{jj}(x) + \frac{1}{2}\sum_{\substack{l=1 \\ l \neq j}}^{r} \mathcal{C}_{jl}(x).$$

Therefore, we have the following spectral decomposition theorem for $\mathcal{L}(x)$, $\mathcal{L}(x^2)$, and $\mathcal{Q}(x)$ (cf. [24, Chapter V, §5 and Chapter VI, §4].)

Theorem 9 *Let $x \in \mathbb{V}$ have the spectral decomposition $x = \sum_{j=1}^{r} \lambda_j(x)c_j$. Then the symmetric operator $\mathcal{L}(x)$ has the spectral decomposition*

$$\mathcal{L}(x) = \sum_{j=1}^{r} \lambda_j(x)\mathcal{C}_{jj}(x) + \sum_{1 \leq j < l \leq r} \frac{1}{2}(\lambda_j(x) + \lambda_l(x))\mathcal{C}_{jl}(x)$$

with the spectrum $\sigma(\mathcal{L}(x))$ consisting of all distinct numbers in $\{\frac{1}{2}(\lambda_j(x) + \lambda_l(x)) : j, l = 1, 2, \ldots, r\}$, $\mathcal{L}(x^2)$ has the spectral decomposition

$$\mathcal{L}(x^2) = \sum_{j=1}^{r} \lambda_j^2(x)\mathcal{C}_{jj}(x) + \sum_{1 \leq j < l \leq r} \frac{1}{2}(\lambda_j^2(x) + \lambda_l^2(x))\mathcal{C}_{jl}(x)$$

with the spectrum $\sigma(\mathcal{L}(x^2))$ consisting of all distinct numbers in $\{\frac{1}{2}(\lambda_j^2(x) + \lambda_l^2(x)) : j, l = 1, 2, \ldots, r\}$, and $\mathcal{Q}(x)$ has the spectral decomposition

$$\mathcal{Q}(x) = \sum_{j=1}^{r} \lambda_j^2(x) \mathcal{C}_{jj}(x) + \sum_{1 \leq j < l \leq r} \lambda_j(x) \lambda_l(x) \mathcal{C}_{jl}(x)$$

with the spectrum $\sigma(\mathcal{Q}(x))$ consisting of all distinct numbers in $\{\lambda_j(x)\lambda_l(x) : j, l = 1, 2, \ldots, r\}$.

Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{V}$. For any $y \in \mathbb{V}$, let $L(y)$, $Q(y)$, $C_{jl}(y)$, ... be the corresponding (matrix) representations of $\mathcal{L}(y)$, $\mathcal{Q}(y)$, $\mathcal{C}_{jl}(y)$, ... with respect to the basis $\{u_1, u_2, \ldots, u_n\}$. Let $\tilde{e}$ denote the coefficients of e with respect to the basis $\{u_1, u_2, \ldots, u_n\}$, i.e.,

$$e = \sum_{j=1}^{n} \langle e, u_j \rangle \, u_j = U\tilde{e}\,,$$

where $U = [u_1 \ u_2 \ \cdots \ u_n]$.

Let $\mu_1 > \mu_2 > \cdots > \mu_{\bar{r}}$ be all the $\bar{r}$ distinct values in $\sigma(x)$. Then there exist $0 = r_0 < r_1 < r_2 < \cdots < r_{\bar{r}} = r$ such that

$$\lambda_{r_{i-1}+1}(x) = \lambda_{r_{i-1}+2}(x) = \cdots = \lambda_{r_i}(x) = \mu_i\,, \quad i = 1, 2, \ldots, \bar{r}\,. \tag{28}$$

Let $\xi_1 > \xi_2 > \cdots > \xi_{\bar{n}}$ be all the $\bar{n}$ distinct values in $\sigma(\mathcal{L}(x))$ and

$$J_k(\mathcal{L}(x)) = \{(j, l) : \frac{1}{2}(\lambda_j(x) + \lambda_l(x)) = \xi_k\,, \ 1 \leq j \leq l \leq r\}\,, \quad k = 1, 2, \ldots, \bar{n}\,.$$

Then, by Theorem 9, there exist indices $n_1, n_2, \ldots, n_{\bar{r}} \in \{1, 2, \ldots, \bar{n}\}$ such that

$$\mu_i = \xi_{n_i}\,, \quad i = 1, 2, \ldots, \bar{r}\,.$$

For each $i \in \{1, 2, \ldots, \bar{r}\}$, denote $J_i(x) := \{j : \lambda_j(x) = \mu_i\}$. Let $y \in \mathbb{V}$ have the spectral decomposition $y = \sum_{j=1}^{r} \lambda_j(y)c_j(y)$ with $\lambda_1(y) \geq \lambda_2(y) \geq \cdots \geq \lambda_r(y)$ being its eigenvalues and $\{c_1(y), c_2(y), \ldots, c_r(y)\} \in \mathfrak{C}(y)$ the corresponding Jordan frame. Define

$$b_i(y) := \sum_{j \in J_i(x)} c_j(y)\,, \quad i = 1, 2, \ldots, \bar{r}\,.$$

Proposition 10 *Let $x \in \mathbb{V}$ have the spectral decomposition $x = \sum_{j=1}^{r} \lambda_j(x)c_j$. Then,*

(i) $\mathfrak{C}(x)$ is compact and $\mathfrak{C}(\cdot)$ is upper semi-continuous at x. Furthermore, for each $i \in \{1, 2, \ldots, \bar{r}\}$, $b_i(\cdot)$ is analytic at x.

(ii) For each $m \in \{1, 2, \ldots, r\}$, $\sigma_m(\cdot)$ is positively homogeneous and convex on $\mathbb{V}$.

(iii) For each $i \in \{1, 2, \ldots, \bar{r}\}$ and $r_{i-1} \leq m < r_i$,

$$\partial_B \sigma_m(x) = \sum_{j=1}^{i-1} b_j(x) + \left\{ \sum_{l=r_{i-1}+1}^{m} \bar{c}_l : \{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_r\} \in \mathfrak{C}(x) \right\} \tag{29}$$

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and the directional derivative of $\sigma_m(\cdot)$ at x, for any $0 \neq h \in \mathbb{V}$, is given by

$$(\sigma_m)'(x;h) = \sum_{j=1}^{i-1} \langle b_j(x), h \rangle + \max_{\{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_r\} \in \mathfrak{C}(x)} \sum_{l=r_{i-1}+1}^{m} \langle \bar{c}_l, h \rangle \,.$$

(iv) The function $\lambda(\cdot)$ is strongly semismooth on $\mathbb{V}$.

Proof. (i) The compactness of $\mathfrak{C}(x)$ follows directly from the fact that every primitive idempotent has norm one and the upper semi-continuity of $\mathfrak{C}(\cdot)$ follows from the continuity of $\lambda(\cdot)$ and Theorem 1.

Next, we consider the analyticity of $b_i(\cdot)$ at x, $i = 1, 2, \ldots, \bar{r}$. By the definitions of $J_i(x)$ and $J_{n_i}(L(x))$, one can see that

$$j \in J_i(x) \text{ if and only if } (j,j) \in J_{n_i}(L(x)), i = 1, 2, \ldots, \bar{r} \,.$$

Hence, by (27), for each $i \in \{1, 2, \ldots, \bar{r}\}$,

$$\begin{aligned}
b_i(y) &= \sum_{j \in J_i(x)} c_j(y) = \Big(\sum_{j \in J_i(x)} \mathcal{C}_{jj}(y) \Big) e = \Big(\sum_{(j,l) \in J_{n_i}(L(x))} \mathcal{C}_{jl}(y) \Big) e \\
&= U \Big(\sum_{(j,l) \in J_{n_i}(L(x))} C_{jl}(y) \Big) \tilde{e} \,,
\end{aligned} \tag{30}$$

which, together with (15), implies that for all y sufficiently close to x and for each $i \in \{1, 2, \ldots, \bar{r}\}$,

$$b_i(y) = U \Big(\sum_{(j,l) \in J_{n_i}(L(x))} C_{jl}(y) \Big) \tilde{e} = U \tilde{P}_{n_i}(L(y)) \tilde{e} \,.$$

Then from Proposition 3 and the linearity of $L(\cdot)$ we know that for each $i \in \{1, 2, \ldots, \bar{r}\}$, $b_i(\cdot)$ is analytic at x.

(ii) This is a special case of part (i) of Proposition 4.

(iii) From part (ii) of Proposition 4, the definition of $\partial_B \sigma_m(x)$, and part (i) of this proposition, we obtain

$$\partial_B \sigma_m(x) \subseteq \sum_{j=1}^{i-1} b_j(x) + \left\{ \sum_{l=r_{i-1}+1}^{m} \bar{c}_l \; : \; \{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_r\} \in \mathfrak{C}(x) \right\} \,.$$

For any $\{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_r\} \in \mathfrak{C}(x)$, by considering

$$y_k := \sum_{\substack{j=1 \\ j \neq i}}^{\bar{r}} \mu_j b_j(x) + \sum_{l=r_{i-1}+1}^{r_i} (\mu_i - l/k) \bar{c}_l,$$

we can see that $y_k \to x$ and from (ii) of Proposition 4, for all k sufficiently large,

$$(\sigma_m)'(y_k) = \sum_{j=1}^{i-1} b_j(x) + \sum_{l=r_{i-1}+1}^{m} \bar{c}_l \,.$$

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Hence, (29) holds. The form of $(\sigma_m)'(x;h)$ can be obtained by

$$(\sigma_m)'(x;h) = \max_{v \in \partial \sigma_m(x)} \langle v, h \rangle \,.$$

(iv) Since convex functions are semismooth [36], from part (ii) we have already known that $\lambda(\cdot)$ is semismooth on $\mathbb{V}$. By Theorem 9, for each $x \in \mathbb{V}$ and $j \in \{1, 2, \ldots, r\}$,

$$\lambda_j(x) \in \{\lambda_1(L(x)), \lambda_2(L(x)), \ldots, \lambda_n(L(x))\} \,,$$

where $\lambda_k(L(x))$ is the k-th largest eigenvalue of the symmetric matrix $L(x)$ (note that $L(x)$ is the matrix representation of $\mathcal{L}(x)$), $k = 1, 2, \ldots, n$. It is known [52, Theorem 4.7] that for each $k \in \{1, 2, \ldots, n\}$, $\lambda_k(\cdot)$ is strongly semismooth on $\mathbb{S}^n$. Hence, by the linearity of $L(\cdot)$ and the continuity of $\lambda_j(\cdot)$, from Proposition 7 we derive the conclusion that $\lambda_j(\cdot)$ is strongly semismooth on $\mathbb{V}$, $j = 1, 2, \ldots, r$. Thus, $\lambda(\cdot)$ is also strongly semismooth on $\mathbb{V}$. $\square$

Remark 11 *From part (iii) of Proposition 4, we know that for any given $x, h \in \mathbb{V}$, the eigenvalues of $x + \varepsilon h$, $\varepsilon \in \mathbb{R}$ can be arranged to be analytic at $\varepsilon = 0$. If $\mathbb{A}$ is the Euclidean Jordan algebra of symmetric matrices, the eigenvectors of $x + \varepsilon h$ can also be chosen to be analytic at $\varepsilon = 0$ [43, Chapter 1]. It is not clear whether this is true for all Euclidean Jordan algebras.*

Part (i) of Proposition 10 says that for each $i \in \{1, 2, \ldots, \bar{r}\}$, $b_i(\cdot)$ is analytic at x. In the sequel, we establish an explicit formula of the derivative of $b_i'(x)$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a scalar valued function and $\phi_{\mathbb{V}}(\cdot)$ be Löwner's operator defined by (1). Let $\tau \in \mathbb{R}^r$. Suppose that ϕ is differentiable at τ_i, $i = 1, 2, \ldots, r$. Define the *first divided difference* $\phi^{[1]}(\tau)$ of ϕ at τ as the $r \times r$ symmetric matrix with its ijth entry $(\phi^{[1]}(\tau))_{ij}$ given by $[\tau_i, \tau_j]_\phi$, where

$$[\tau_i, \tau_j]_\phi := \begin{cases} \dfrac{\phi(\tau_i) - \phi(\tau_j)}{\tau_i - \tau_j} & \text{if } \tau_i \neq \tau_j \\ \phi'(\tau_i) & \text{if } \tau_i = \tau_j \end{cases} \,, \quad i, j = 1, 2, \ldots, r \,. \tag{31}$$

By Proposition 8, the fact that $\dim(V(c_j, 1)) = 1$, $\langle c_j, c_j \rangle = 1$, and the definition of the quadratic operator $\mathcal{Q}$, for any vector $h \in \mathbb{V}$ and each $j \in \{1, 2, \ldots, r\}$, there exists $\alpha_j(h) \in \mathbb{R}$ such that

$$\alpha_j(h)c_j = \mathcal{Q}(c_j)h = 2\mathcal{L}^2(c_j)h - \mathcal{L}(c_j^2)h = 2c_j \cdot (c_j \cdot h) - c_j \cdot h \,,$$

which implies

$$\alpha_j(h) = 2\langle c_j, c_j \cdot (c_j \cdot h) \rangle - \langle c_j, c_j \cdot h \rangle = \langle c_j, c_j \cdot h \rangle = \langle c_j, h \rangle$$

and

$$2c_j \cdot (c_j \cdot h) = c_j \cdot h + \langle c_j, h \rangle c_j \,. \tag{32}$$

Therefore, any vector $h \in \mathbb{V}$ can be written as

$$h = \sum_{j=1}^r \mathcal{C}_{jj}(x)h + \sum_{1 \leq j < l \leq r} \mathcal{C}_{jl}(x)h = \sum_{j=1}^r \langle c_j, h \rangle c_j + \sum_{1 \leq j < l \leq r} 4c_j \cdot (c_l \cdot h) \,. \tag{33}$$

Korányi [25, Page 74] proved the following result, which generalized Löwner's result [35] on symmetric matrices (see Donoghue [14, Chapter VIII] for a detailed proof on this) to Euclidean Jordan algebras.

Lemma 12 *Let $x = \sum_{j=1}^{r} \lambda_j(x)c_j$. Let (a,b) be an open interval in $\mathbb{R}$ that contains $\lambda_j(x)$, $j = 1, 2, \ldots, r$. If ϕ is continuously differentiable on (a,b), then $\phi_{\mathbb{V}}$ is differentiable at x and its derivative, for any $h \in \mathbb{V}$, is given by*

$$(\phi_{\mathbb{V}})'(x)h = \sum_{j=1}^{r} (\phi^{[1]}(\lambda(x)))_{jj}\langle c_j, h\rangle\, c_j + \sum_{1 \leq j < l \leq r} 4(\phi^{[1]}(\lambda(x)))_{jl}c_j \cdot (c_l \cdot h)\,. \qquad (34)$$

By (32), we can write (34) equivalently as

$$(\phi_{\mathbb{V}})'(x)h = 2\sum_{i=1}^{\bar{r}}\sum_{l=1}^{\bar{r}}[\mu_i, \mu_l]_\phi b_i(x) \cdot (b_l(x) \cdot h) - \sum_{i=1}^{\bar{r}} \phi'(\mu_i)b_i(x) \cdot h\,, \qquad (35)$$

where the fact $c_j \cdot (c_l \cdot h) = \mathcal{L}(c_j)\mathcal{L}(c_l)h = \mathcal{L}(c_l)\mathcal{L}(c_j)h = c_l \cdot (c_j \cdot h)$, $j \neq l = 1, 2, \ldots, r$ is used. Now, we can calculate $b_i'(x)$, $i \in \{1, 2, \ldots, \bar{r}\}$ is used. Pick an $\varepsilon > 0$ such that

$$(\mu_j - \varepsilon, \mu_j + \varepsilon) \cap (\mu_l - \varepsilon, \mu_l + \varepsilon) = \emptyset, \ 1 \leq j < l \leq \bar{r}\,. \qquad (36)$$

For each $i \in \{1, 2, \ldots, \bar{r}\}$, let ϕ_i be a continuously differentiable function on $(-\infty, \infty)$ such that ϕ_i is identically one on the interval $(\mu_i - \varepsilon, \mu_i + \varepsilon)$ and is identically zero on all other intervals $(\mu_j - \varepsilon, \mu_j + \varepsilon)$, $i \neq j = 1, 2, \ldots, \bar{r}$. Then for all y sufficiently close to x, $b_i(y) = (\phi_i)_{\mathbb{V}}(y)$. Hence, by Lemma 12 and (35), the derivative of $b_i(\cdot)$ at x, for any $h \in \mathbb{V}$, is given by

$$b_i'(x)h = \sum_{1 \leq j < l \leq r} 4(\phi_i^{[1]}(\lambda(x)))_{jl}c_j \cdot (c_l \cdot h) = \sum_{\substack{l=1 \\ l \neq i}}^{\bar{r}} \frac{4}{\mu_i - \mu_l}b_i(x) \cdot (b_l(x) \cdot h)\,. \qquad (37)$$

Based on the proof in [25, Page 74], we shall show in the next proposition that $\phi_{\mathbb{V}}$ is (continuously) differentiable at x if and only if $\phi(\cdot)$ is (continuously) differentiable at $\lambda_j(x)$, $j = 1, 2, \ldots, r$.

Theorem 13 *Let $x = \sum_{j=1}^{r} \lambda_j(x)c_j$. The function $\phi_{\mathbb{V}}$ is (continuously) differentiable at x if and only if for each $j \in \{1, 2, \ldots, r\}$, ϕ is (continuously) differentiable at $\lambda_j(x)$. In this case, the derivative of $\phi_{\mathbb{V}}(\cdot)$ at x, for any $h \in \mathbb{V}$, is given by (34), or equivalently by (35).*

Proof. " $\Longleftarrow$ " Suppose that for each $j \in \{1, 2, \ldots, r\}$, ϕ is differentiable at $\lambda_j(x)$. As in [25, Lemma], we first consider the special case that $\phi(\lambda_j(x)) = \phi'(\lambda_j(x)) = 0$, $j = 1, 2, \ldots, r$. Then, by the Lipschitz continuity of $\lambda(\cdot)$ and Proposition 10, for any $h \in \mathbb{V}$ with $h \to 0$ and $x + h = \sum_{j=1}^{r} \lambda_j(x+h)c_j(x+h)$ with $\lambda_1(x+h) \geq \lambda_2(x+h) \geq \cdots \geq \lambda_r(x+h)$ and $\{c_1(x+h), c_2(x+h), \ldots, c_r(x+h)\} \in \mathfrak{C}(x+h)$ we have

$$
\begin{aligned}
\phi_{\mathbb{V}}(x+h) &= \sum_{j=1}^{r} \phi(\lambda_j(x+h))c_j(x+h) \\
&= \sum_{j=1}^{r} \left(\phi(\lambda_j(x+h)) - \phi(\lambda_j(x))\right)c_j(x+h) \\
&= \sum_{j=1}^{r} \left(\phi'(\lambda_j(x))(\lambda_j(x+h) - \lambda_j(x)) + o(|\lambda_j(x+h) - \lambda_j(x)|)\right)c_j(x+h) \\
&= \sum_{j=1}^{r} o(|\lambda_j(x+h) - \lambda_j(x)|)c_j(x+h) = o(\|h\|)\,.
\end{aligned}
$$

Hence, $\phi_\mathbb{V}$ is differentiable at x and $(\phi_\mathbb{V})'(x)h = 0$ for all $h \in \mathbb{V}$, which satisfies (34).

Next, we consider the general case. Let $p(\cdot)$ be a polynomial function such that $p(\lambda_j(x)) = \phi(\lambda_j(x))$ and $p'(\lambda_j(x)) = \phi'(\lambda_j(x))$, $j = 1, 2, \ldots, r$. The existence of such a polynomial is guaranteed by the theory on Hermite interpolation (cf. [29, Section 5.2]). Hence, by the above proof it follows that the function $(\phi - p)_\mathbb{V}$ is differentiable at x. By noting from Lemma 12 that $p_\mathbb{V}$ is differentiable at x, we know that $\phi_\mathbb{V}$ is differentiable at x and the derivative of $\phi_\mathbb{V}(\cdot)$ at x, for any $h \in \mathbb{V}$, is given by (34), which is equivalent to (35).

Now, we show that $\phi_\mathbb{V}$ is continuously differentiable at x if for each $j \in \{1, 2, \ldots, r\}$, ϕ is continuously differentiable at $\lambda_j(x)$. It has already been proved that $\phi_\mathbb{V}$ is differentiable in an open neighborhood of x. By (34), for any y sufficiently close to x the derivative of $\phi_\mathbb{V}$ at y, for any $h \in \mathbb{V}$, can be written by

$$(\phi_\mathbb{V})'(y)h = \sum_{j=1}^{r} (\phi^{[1]}(\lambda(y)))_{jj} \langle c_j(y), h \rangle c_j(y) + \sum_{1 \le j < l \le r} 4(\phi^{[1]}(\lambda(y)))_{jl} c_j(y) \cdot (c_l(y) \cdot h),$$

where y has the spectral decomposition $y = \sum_{j=1}^{r} \lambda_j(y) c_j(y)$ with $\lambda_1(y) \ge \lambda_2(y) \ge \cdots \ge \lambda_r(y)$ and $\{c_1(y), c_2(y), \ldots, c_r(y)\} \in \mathfrak{C}(y)$. From the continuity of $\lambda(\cdot)$ and the assumption we know that for any $1 \le j \le l \le r$ and $y \to x$, if $\lambda_j(x) \ne \lambda_l(x)$, then

$$(\phi^{[1]}(\lambda(y)))_{jl} \to (\phi^{[1]}(\lambda(x)))_{jl};$$

and if $\lambda_j(x) = \lambda_l(x)$, then from the mean value theorem,

$$(\phi^{[1]}(\lambda(y)))_{jl} = \phi'(\tau_{jl}(y)) \to \phi'(\lambda_j(x)),$$

where $\tau_{jl}(y) \in [\lambda_l(y), \lambda_j(y)]$. Therefore, any accumulation point of $(\phi_\mathbb{V})'(y)h$ for $y \to x$ can be written as

$$\sum_{j=1}^{r} (\phi^{[1]}(\lambda(x)))_{jj} \langle \bar{c}_j, h \rangle \bar{c}_j + \sum_{1 \le j < l \le r} 4(\phi^{[1]}(\lambda(x)))_{jl} \bar{c}_j \cdot (\bar{c}_l \cdot h)$$

for some $\{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_r\} \in \mathfrak{C}(x)$. This, together with (34), implies that for any $h \in \mathbb{V}$,

$$(\phi_\mathbb{V})'(y)h \to (\phi_\mathbb{V})'(x)h.$$

The continuity of $(\phi_\mathbb{V})'$ at x is then proved.

" $\Longrightarrow$ " To prove that for each $i \in \{1, 2, \ldots, \bar{r}\}$, ϕ is (continuously) differentiable at μ_i, we consider the composite function of $\phi_\mathbb{V}$ and $u_i(t) := x + t b_i(x)$, $t \in \mathbb{R}$. For any $t \in \mathbb{R}$,

$$\phi_\mathbb{V}(u_i(t)) = \sum_{\substack{j=1 \\ j \ne i}}^{\bar{r}} \phi(\mu_j) b_j(x) + \phi(\mu_i + t) b_i(x),$$

which implies

$$\phi(\mu_i + t) \langle b_i(x), b_i(x) \rangle = \langle b_i(x), \phi_\mathbb{V}(u_i(t)) \rangle = \langle b_i(x), \phi_\mathbb{V}(x + t b_i(x)) \rangle.$$

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Since $\langle b_i(x), b_i(x)\rangle > 0$ and $\phi_{\mathbb{V}}$ is (continuously) differentiable at x, ϕ is (continuously) differentiable at μ_i with

$$\phi'(\mu_i) = \langle b_i(x), (\phi_{\mathbb{V}})'(x)b_i(x)\rangle/\|b_i(x)\|^2.$$

The proof is completed. $\square$

Remark 14 *Theorem 13 extends the results on the differentiability of Löwner's function over the symmetric matrices in [6, 34, 50] and over SOCs [5] to all Euclidean Jordan algebras. The approach adopted here follows the works of [35] and [25] and will be used to study the twice differentiability of the spectral function over Euclidean Jordan algebras.*

Next, we consider the (strong) semismoothness of $\phi_{\mathbb{V}}$ at $x \in \mathbb{V}$. We achieve this by establishing the connection between $\phi_{\mathbb{V}}(x)$ and $\phi_{\mathbb{S}^n}(L(x))$. According to Theorem 9 and the definition of $\phi_{\mathbb{S}^n}$,

$$\phi_{\mathbb{S}^n}(L(x)) = \sum_{j=1}^{r} \phi(\lambda_j(x))C_{jj}(x) + \sum_{1 \leq j < l \leq r} \phi\Big(\frac{1}{2}(\lambda_j(x) + \lambda_l(x))\Big)C_{jl}(x).$$

Thus, by (27), we obtain

$$\phi_{\mathbb{V}}(x) = U\phi_{\mathbb{S}^n}(L(x))\tilde{e}. \tag{38}$$

In particular, by taking $\phi(t) = t_+ = \max(0, t)$, $t \in \mathbb{R}$, we get

$$x_+ = U(L(x))_+\tilde{e}. \tag{39}$$

Hence, we have the following result.

Proposition 15 *The metric projection operator $(\cdot)_+$ is strongly semismooth on $\mathbb{V}$.*

Proof. It is proved in [51] that the metric projection operator $(\cdot)_+$ is strongly semismooth on $\mathbb{S}^n$. Since $L(\cdot)$ is a linear operator, from (39) we know that $(\cdot)_+$ is strongly semismooth on $\mathbb{V}$. $\square$

Let us consider another special, yet important, case. For any $\varepsilon \in \mathbb{R}$, define $\phi^\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$\phi^\varepsilon(t) := \sqrt{t^2 + \varepsilon^2}, \; t \in \Re.$$

Then the corresponding Löwner's operator $\phi_{\mathbb{V}}^\varepsilon$ takes the following form

$$\phi_{\mathbb{V}}^\varepsilon(x) = \sum_{j=1}^{r} \sqrt{\lambda_j^2(x) + \varepsilon^2}\, c_j = \sqrt{x^2 + \varepsilon^2 e},$$

which can be treated as the smoothed approximation to the "absolute value" function $|x| := \sqrt{x^2}$, $x \in \mathbb{V}$. On the other hand,

$$L(x^2) + \varepsilon^2 I = \sum_{j=1}^{r}(\lambda_j^2(x) + \varepsilon^2)C_{jj}(x) + \sum_{1 \leq j < l \leq r} \frac{1}{2}(\lambda_j^2(x) + \lambda_l^2(x) + 2\varepsilon^2)C_{jl}(x),$$

which implies

$$U\sqrt{L(x^2)+\varepsilon^2 I}\,\tilde{e}$$

$$= U\Big(\sum_{j=1}^r \sqrt{\lambda_j^2(x)+\varepsilon^2}\,C_{jj}(x) + \sum_{1\le j<l\le r}\frac{1}{\sqrt{2}}\sqrt{\lambda_j^2(x)+\lambda_l^2(x)+2\varepsilon^2}\,C_{jl}(x)\Big)\tilde{e}$$

$$= U\sum_{j=1}^r \sqrt{\lambda_j^2(x)+\varepsilon^2}\,C_{jj}(x)\tilde{e} = \sum_{j=1}^r \sqrt{\lambda_j^2(x)+\varepsilon^2}\,c_j = \sqrt{x^2+\varepsilon^2}e$$

$$= \phi_{\mathbb{V}}^\varepsilon(x)\,. \tag{40}$$

For $\varepsilon\in\mathbb{R}$ and $x\in\mathbb{V}$, let

$$\psi(\varepsilon,x) := \phi_{\mathbb{V}}^\varepsilon(x) = \sqrt{x^2+\varepsilon^2}e\,.$$

Then, by [53] and (40), we obtain the following result directly.

Proposition 16 *The function $\psi(\cdot,\cdot)$ is continuously differentiable at (ε,x) if $\varepsilon\neq 0$ and is strongly semismooth at $(0,x)$, $x\in\mathbb{V}$.*

Proposition 15 extends the strong semismoothness of $(\cdot)_+$ on symmetric matrices in [51] to Euclidean Jordan algebras. To study the strong semismoothness of Löwner's operator, we need to introduce another scalar valued function $\bar{\phi}:\mathbb{R}\to\mathbb{R}$. Let $\varepsilon>0$ be such that

$$(\xi_j-\varepsilon,\xi_j+\varepsilon)\cap(\xi_l-\varepsilon,\xi_l+\varepsilon)=\emptyset,\ 1\le j<l\le\bar{n}\,.$$

Then define $\bar{\phi}:\mathbb{R}\to\mathbb{R}$ by

$$\bar{\phi}(t) = \begin{cases} \phi(t) & \text{if } t\in\bigcup_{j=1}^{\bar{r}}(\mu_j-\varepsilon,\mu_j+\varepsilon) \\ 0 & \text{otherwise}\,. \end{cases}$$

Then, by using the fact that $\mathcal{C}_{jl}(y)e=0$ for $j\neq l$, for all y sufficiently close to x we have

$$\phi_{\mathbb{V}}(y) = U\phi_{\mathbb{S}^n}(L(y))\tilde{e}\,.$$

$$= U\Big(\sum_{j=1}^r \phi(\lambda_j(y))C_{jj}(y)\Big)\tilde{e}$$

$$= U\Big(\sum_{j=1}^r \bar{\phi}(\lambda_j(y))C_{jj}(y)\Big)\tilde{e}$$

$$= U\Big[\sum_{j=1}^r \bar{\phi}(\lambda_j(y))C_{jj}(y) + \sum_{1\le j<l\le r}\bar{\phi}\Big(\frac{1}{2}(\lambda_j(y)+\lambda_l(y))\Big)C_{jl}(y)\Big]\tilde{e}$$

$$= U\bar{\phi}_{\mathbb{S}^n}(L(y))\tilde{e}\,. \tag{41}$$

Theorem 17 *Let $\gamma\in(0,1]$ be a constant and $x=\sum_{j=1}^r \lambda_j(x)c_j$. $\phi_{\mathbb{V}}(\cdot)$ is (γ-order) semismooth at x if and only if for each $j\in\{1,2,\ldots,r\}$, $\phi(\cdot)$ is (γ-order) semismooth at $\lambda_j(x)$.*

Proof. We only need to consider the semismoothness as the proof for the γ-order semismoothness is similar.

" $\Longleftarrow$ " The definition of $\bar{\phi}(\cdot)$ and the assumption that for each $j \in \{1, 2, \ldots, r\}$, $\phi(\cdot)$ is semismooth at $\lambda_j(x)$ imply that $\bar{\phi}(\cdot)$ is semismooth at each $\frac{1}{2}(\lambda_j(x) + \lambda_l(x))$, $1 \le j \le l \le r$. Then by [6, Proposition 4.7] we know that $\bar{\phi}_{\mathbb{S}^n}(\cdot)$ is semismooth at $L(x)$. This, together with (41), shows that $\phi_{\mathbb{V}}(\cdot)$ is semismooth at x.

" $\Longrightarrow$ " To prove that for each $j \in \{1, 2, \ldots, r\}$, $\phi(\cdot)$ is semismooth at $\lambda_j(x)$ is equivalent to prove that for $i \in \{1, 2, \ldots, \bar{r}\}$, $\phi(\cdot)$ is semismooth at μ_i. For each $i \in \{1, 2, \ldots, \bar{r}\}$, let $u_i(t) := x + tb_i(x)$, $t \in \mathbb{R}$. For any $t \in \mathbb{R}$,

$$\phi_{\mathbb{V}}(u_i(t)) = \sum_{\substack{j=1 \\ j \ne i}}^{\bar{r}} \phi(\mu_j) b_j(x) + \phi(\mu_i + t) b_i(x) \,,$$

which implies

$$\phi(\mu_i + t) \langle b_i(x), b_i(x) \rangle = \langle b_i(x), \phi_{\mathbb{V}}(u_i(t)) \rangle = \langle b_i(x), \phi_{\mathbb{V}}(x + tb_i(x)) \rangle \,.$$

Since $\phi_{\mathbb{V}}$ is semismooth at x, $\langle b_i(x), \phi_{\mathbb{V}}(x + tb_i(x)) \rangle$ is semismooth at $t = 0$. Therefore, ϕ is semismooth at μ_i. $\qquad\square$

Remark 18 *The proof of Theorem 17 uses the semismoothness result of Löwner's function over symmetric matrices in [6]. It also provides a new proof on Löwner's function over SOCs considered in [5].*

4 Differential Properties of Spectral Functions

Let $\mathbb{A} = (\mathbb{V}, \cdot)$ be a simple Euclidean Jordan algebra of rank r and $\dim(\mathbb{V}) = n$. Let $x \in \mathbb{V}$ have the spectral decomposition $x = \sum_{j=1}^{r} \lambda_j(x) c_j$. Let $\mu_1 > \mu_2 > \cdots > \mu_{\bar{r}}$ be all the $\bar{r}$ distinct values in $\sigma(x)$ and $0 = r_0 < r_1 < r_2 < \cdots < r_{\bar{r}} = r$ be such that (28) holds.

For two vectors α and β in $\mathbb{R}^r$, we say that β *block-refines* α if $\alpha_j = \alpha_l$ whenever $\beta_j = \beta_l$ [32].

Lemma 19 *If $\lambda(x)$ block-refines α in $\mathbb{R}^r$, then the function $\alpha^T \lambda(\cdot)$ is differentiable at x with $\nabla(\alpha^T \lambda)(x) = \sum_{j=1}^{r} \alpha_j c_j$.*

Proof. Since $\lambda(x)$ block-refines α,

$$\alpha_{r_{i-1}+1} = \alpha_{r_{i-1}+2} = \cdots = \alpha_{r_i} \,, \quad i = 1, 2, \ldots, \bar{r} \,.$$

Let $y \in \mathbb{V}$ have the spectral decomposition $y = \sum_{j=1}^{r} \lambda_j(y) c_j(y)$ with $\lambda_1(y) \ge \lambda_2(y) \ge \cdots \ge \lambda_r(y)$. Let $\sigma_0 \equiv 0$. Then,

$$\alpha^T \lambda(y) = \sum_{j=1}^{r} \alpha_j \lambda_j(y) = \sum_{i=1}^{\bar{r}} \alpha_{r_i} \sum_{j=r_{i-1}+1}^{r_i} \lambda_j(y) = \sum_{i=1}^{\bar{r}} \alpha_{r_i} (\sigma_{r_i}(y) - \sigma_{r_{i-1}}(y)) \,.$$

By (ii) of Proposition 4, $\sigma_{r_i}(\cdot)$ is differentiable at x and

$$\nabla \sigma_{r_i}(x) = \sum_{j=1}^{r_i} c_j , \ \ i = 1, 2, \ldots, \bar{r} .$$

Hence, $\alpha^T \lambda(\cdot)$ is differentiable at x and

$$\nabla (\alpha^T \lambda)(x) = \sum_{i=1}^{\bar{r}} \alpha_{r_i} \sum_{j=r_{i-1}+1}^{r_i} c_j = \sum_{j=1}^r \alpha_j c_j .$$

This completes the proof. $\square$

Let $f : \mathbb{R}^r \to (-\infty, \infty]$ be a symmetric function. The properties on the symmetric function f in the following lemma are needed in our analysis. Parts (i) and (ii) can be checked directly (cf. [34, Lemma 2.1]). Parts (iii) and (iv) are implied by the proof of Case III in [34, Lemma 4.1].

Lemma 20 *Let $f : \mathbb{R}^r \to (-\infty, \infty]$ be a symmetric function and $\upsilon := \lambda(x)$. Let P be a permutation matrix such that $P\upsilon = \upsilon$.*

(i) *If f is differentiable at υ, then $\nabla f(\upsilon) = P^T \nabla f(\upsilon)$.*

(ii) *Let $l_i := r_{i+1} - r_i$, $i = 1, 2, \ldots, \bar{r}$. If f is twice differentiable at υ, then $\nabla^2 f(\upsilon) = P^T \nabla^2 f(\upsilon) P$. In particular,*

$$\nabla^2 f(\upsilon) = \begin{bmatrix} \eta_{11} E_{11} + \beta_{r_1} I_{l_1 \times l_1} & \eta_{12} E_{12} & \cdots & \eta_{1\bar{r}} E_{1\bar{r}} \\ \eta_{21} E_{21} & \eta_{22} E_{22} + \beta_{r_2} I_{l_2 \times l_2} & \cdots & \eta_{2\bar{r}} E_{2\bar{r}} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{\bar{r}1} E_{\bar{r}1} & \eta_{\bar{r}2} E_{\bar{r}2} & \cdots & \eta_{\bar{r}\bar{r}} E_{\bar{r}\bar{r}} + \beta_{r_{\bar{r}}} I_{l_{\bar{r}} \times l_{\bar{r}}} \end{bmatrix} ,$$

where for $i, j = 1, 2, \ldots, \bar{r}$, E_{ij} is the $l_i \times l_j$ matrix with all entries equal to one, $(\eta_{ij})_{i,j=1}^{\bar{r}}$ is a real symmetric matrix, $\beta := (\beta_1, \beta_2, \ldots, \beta_r)^T$ is a vector which is block refined by υ, and for each $i = 1, 2, \ldots, \bar{r}$, $I_{l_i \times l_i}$ is the $l_i \times l_i$ identity matrix. If $l_i = 1$ for some $i \in \{1, 2, \ldots, \bar{r}\}$, then we take $\eta_{ii} = 0$.

(iii) *Suppose that f is twice continuously differentiable at υ and $j \neq l \in \{1, 2, \ldots, r\}$ satisfy $\upsilon_j = \upsilon_l$. Then for any $\varsigma \in \mathbb{R}^r$ with $\varsigma_1 \geq \varsigma_2 \geq \cdots \geq \varsigma_r$, $\varsigma_j \neq \varsigma_l$, and $\varsigma \to \upsilon$,*

$$\frac{(\nabla f(\varsigma))_j - (\nabla f(\varsigma))_l}{\varsigma_j - \varsigma_l} \to (\nabla^2 f(\upsilon))_{jj} - (\nabla^2 f(\upsilon))_{jl} .$$

(iv) *Suppose that ∇f is locally Lipschitz continuous near υ with the Lipschitz constant $\kappa > 0$ and $j \neq l \in \{1, 2, \ldots, r\}$ satisfy $\upsilon_j = \upsilon_l$. Then for any $\varsigma \in \mathbb{R}^r$ with $\varsigma_1 \geq \varsigma_2 \geq \cdots \geq \varsigma_r$, $\varsigma_j \neq \varsigma_l$, and ς sufficiently close to υ,*

$$\left| \frac{(\nabla f(\varsigma))_j - (\nabla f(\varsigma))_l}{\varsigma_j - \varsigma_l} \right| \leq 3\kappa .$$

Theorem 21 [1] *Let $f : \mathbb{R}^r \to (-\infty, \infty]$ be a symmetric function. Then $f \circ \lambda$ is differentiable at $x = \sum_{j=1}^r \lambda_j(x) c_j$ if and only if f is differentiable at $\lambda(x)$, and in this case*

$$\nabla(f \circ \lambda)(x) = \sum_{j=1}^r (\nabla f(\lambda(x)))_j c_j \,.$$

Proof. " $\Longleftarrow$ " Let $\upsilon := \lambda(x)$. Since $\lambda(\cdot)$ is Lipschitz continuous, there exist constants $\tau > 0$ and $\delta_0 > 0$ such that

$$\|\lambda(y) - \lambda(x)\| \leq \tau \|y - x\|$$

for all $y \in \mathbb{V}$ satisfying $\|y - x\| \leq \delta_0$. For any given $\varepsilon > 0$, since f is differentiable at $\upsilon = \lambda(x)$, there exists a positive number $\delta(\leq \delta_0 \tau)$ such that for all $\varsigma \in \mathbb{R}^r$ satisfying $\|\varsigma - \upsilon\| \leq \delta$ it holds that

$$|f(\varsigma) - f(\upsilon) - (\nabla f(\upsilon))^T(\varsigma - \upsilon)| \leq \varepsilon \|\varsigma - \upsilon\| \,.$$

Hence, for all $y \in \mathbb{V}$ satisfying $\|y - x\| \leq \delta/\tau$,

$$|f(\lambda(y)) - f(\upsilon) - (\nabla f(\upsilon))^T(\lambda(y) - \upsilon)| \leq \varepsilon \|\lambda(y) - \upsilon\| \leq \tau \varepsilon \|y - x\| \,.$$

On the other hand, by part (i) of Lemma 20, υ block-refines $\nabla f(\upsilon)$. Then, by Lemma 19, we have

$$\left| (\nabla f(\upsilon))^T \lambda(y) - (\nabla f(\upsilon))^T \upsilon - \left\langle \sum_{j=1}^r (\nabla f(\upsilon))_j c_j, y - x \right\rangle \right| \leq \varepsilon \|y - x\|$$

for all y sufficiently close to x. By adding the two previous inequalities we obtain

$$\left| f(\lambda(y)) - f(\upsilon) - \left\langle \sum_{j=1}^r (\nabla f(\upsilon))_j c_j, y - x \right\rangle \right| \leq (\tau + 1)\varepsilon \|y - x\|$$

for all y sufficiently close to x. This shows that $f \circ \lambda$ is differentiable at x with

$$\nabla(f \circ \lambda)(x) = \sum_{j=1}^r (\nabla f(\lambda(x)))_j c_j.$$

" $\Longrightarrow$ " Suppose that $f \circ \lambda$ is differentiable at x. Then it is easy to see that f must be differentiable at $\lambda(x)$ because one may write

$$f(\varsigma) = (f \circ \lambda)\left(\sum_{j=1}^r \varsigma_j c_j \right)$$

for all $\varsigma \in \mathbb{R}^r$. $\square$

---

[1]When this paper was under review, Adrian S. Lewis kindly brought [1] to our attention, where Baes provided a similar proof to Theorem 21 without employing the hyperbolic polynomials as we did here.

Remark 22 *Theorem 21 is a direct extension of the first derivative result in [32] on the spectral function over symmetric matrices.*

Let the symmetric function $f : \mathbb{R}^r \to (-\infty, \infty]$ be twice differentiable at $\upsilon := \lambda(x)$. Then by Lemma 20, $\nabla^2 f(\lambda(x))$ has the form as in part (ii) of Lemma 20. Let $\varepsilon > 0$ be such that (36) holds. Define $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{\phi}(t) = \begin{cases} \tilde{\beta}_j(x)t & \text{if } t \in (\upsilon_j - \varepsilon, \upsilon_j + \varepsilon), \; j = 1, 2, \dots, r \\ 0 & \text{otherwise}, \end{cases}$$

where $\tilde{\beta}(x)$ is the vector β defined in part (ii) of Lemma 20, i.e., for $r_{i-1} + 1 \le j \le r_i$,

$$\tilde{\beta}_j(x) = \begin{cases} (\nabla^2 f(\upsilon))_{jj} & \text{if } r_i - r_{i-1} = 1 \\ (\nabla^2 f(\upsilon))_{ll} - (\nabla^2 f(\upsilon))_{ls} & \text{if } r_{i-1} + 1 \le l \ne s \le r_i, \end{cases} \tag{42}$$

where $i = 1, 2, \dots, \bar{r}$. Then, by Theorem 13, $\tilde{\phi}_{\mathbb{V}}(\cdot)$ is continuously differentiable at x and its derivative, for any $h \in \mathbb{V}$, is given by

$$\begin{aligned} (\tilde{\phi}_{\mathbb{V}})'(x)h \;\; &= 2 \sum_{i=1}^{\bar{r}} \sum_{\substack{l=1 \\ l \ne i}}^{\bar{r}} \frac{\tilde{\beta}_{r_i}(x)\mu_i - \tilde{\beta}_{r_l}(x)\mu_l}{\mu_i - \mu_l} b_i(x) \cdot (b_l(x) \cdot h) \\ &\quad + \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x)[2b_i(x) \cdot (b_i(x) \cdot h) - b_i(x) \cdot h]. \end{aligned} \tag{43}$$

Define the symmetric matrix $\tilde{A}(x)$ as follows. Let $\tilde{a}_{jl}(x)$ be the jlth entry of $\tilde{A}(x)$. Then for $j, l = 1, 2, \dots, r$,

$$\tilde{a}_{jl}(x) := \begin{cases} 0 & \text{if } j = l \\ \tilde{\beta}_j(x) & \text{if } r_{i-1} + 1 \le j \ne l \le r_i \\ \dfrac{(\nabla f(\lambda(x)))_j - (\nabla f(\lambda(x)))_l}{\lambda_j(x) - \lambda_l(x)} & \text{otherwise}, \end{cases} \tag{44}$$

where $i = 1, 2, \dots, \bar{r}$.

Theorem 23 *Let $f : \mathbb{R}^r \to (-\infty, \infty]$ be a symmetric function and $x = \sum_{j=1}^{r} \lambda_j(x)c_j$. Then $f \circ \lambda$ is twice differentiable at x if and only if f is twice differentiable at $\lambda(x)$. In that case, the second derivative of $f \circ \lambda(\cdot)$ at x, for any $h \in \mathbb{V}$, is given by*

$$\nabla^2 (f \circ \lambda)(x)h = \sum_{j=1}^{r} \sum_{l=1}^{r} [2\tilde{a}_{jl}(x)c_j \cdot (c_l \cdot h) + (\nabla^2 f(\lambda(x)))_{jl} \langle c_l, h \rangle c_j]. \tag{45}$$

Proof. " $\Longleftarrow$ " By Theorem 21, for any $0 \ne h \in \mathbb{V}$ and h sufficiently small we have

$$\nabla (f \circ \lambda)(x + h) = \sum_{j=1}^{r} (\nabla f(\lambda(x + h)))_j c_j(x + h),$$

where $x + h = \sum_{j=1}^{r} \lambda_j(x+h)c_j(x+h)$ and $\{c_1(x+h), c_2(x+h), \dots, c_r(x+h)\} \in \mathfrak{C}(x+h)$. Hence, by [49] and the directional differentiability and the Lipschitz continuity of $\lambda(\cdot)$,

$$
\nabla(f \circ \lambda)(x+h) - \nabla(f \circ \lambda)(x)
$$

$$
= \sum_{j=1}^{r} (\nabla f(\lambda(x) + \lambda'(x;h) + o(\|h\|)))_j c_j(x+h) - \sum_{j=1}^{r} (\nabla f(\lambda(x)))_j c_j
$$

$$
= \sum_{j=1}^{r} (\nabla f(\lambda(x) + \lambda'(x;h)))_j c_j(x+h) - \sum_{j=1}^{r} (\nabla f(\lambda(x)))_j c_j + o(\|h\|)
$$

$$
= \sum_{j=1}^{r} (\nabla f(\lambda(x)))_j (c_j(x+h) - c_j) + \sum_{j=1}^{r} (\nabla^2 f(\lambda(x))\lambda'(x;h))_j c_j(x+h) + o(\|h\|)
$$

$$
= \sum_{i=1}^{\bar{r}} (\nabla f(\lambda(x)))_{r_i} (b_i(x+h) - b_i(x)) + \sum_{j=1}^{r} (\nabla^2 f(\lambda(x))\lambda'(x;h))_j c_j(x+h) + o(\|h\|),
$$

which, together with the analyticity of $b_i(\cdot)$, part (ii) of Lemma 20, and Proposition 10 gives

$$
\nabla(f \circ \lambda)(x+h) - \nabla(f \circ \lambda)(x) - \sum_{i=1}^{\bar{r}} (\nabla f(\lambda(x)))_{r_i} b_i'(x)h - (\tilde{\phi}_{\mathbb{V}})'(x)h
$$

$$
= \sum_{j=1}^{r} (\nabla^2 f(\lambda(x))\lambda'(x;h))_j c_j(x+h) - [\tilde{\phi}_{\mathbb{V}}(x+h) - \tilde{\phi}_{\mathbb{V}}(x)] + o(\|h\|)
$$

$$
= \sum_{i=1}^{\bar{r}} \sum_{l=1}^{\bar{r}} \eta_{il} \langle b_l(x), h \rangle \, b_i(x+h) + \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) \sum_{l=r_{i-1}+1}^{r_i} \lambda_l'(x;h) c_l(x+h)
$$
$$
- [\tilde{\phi}_{\mathbb{V}}(x+h) - \tilde{\phi}_{\mathbb{V}}(x)] + o(\|h\|)
$$

$$
= \sum_{i=1}^{\bar{r}} \sum_{l=1}^{\bar{r}} \eta_{il} \langle b_l(x), h \rangle \, (b_i(x) + o(\|h\|)) + \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) \sum_{l=r_{i-1}+1}^{r_i} [\lambda_l(x+h) - \lambda_l(x)] c_l(x+h)
$$
$$
- [\tilde{\phi}_{\mathbb{V}}(x+h) - \tilde{\phi}_{\mathbb{V}}(x)] + o(\|h\|)
$$

$$
= \sum_{i=1}^{\bar{r}} \sum_{l=1}^{\bar{r}} \eta_{il} \langle b_l(x), h \rangle \, b_i(x) - \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) \sum_{l=r_{i-1}+1}^{r_i} \lambda_l(x) c_l(x+h) + \tilde{\phi}_{\mathbb{V}}(x) + o(\|h\|)
$$

$$
= \sum_{i=1}^{\bar{r}} \sum_{l=1}^{\bar{r}} \eta_{il} \langle b_l(x), h \rangle \, b_i(x) - \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) \mu_i b_i(x+h) + \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) \mu_i b_i(x) + o(\|h\|)
$$

$$
= \sum_{i=1}^{\bar{r}} \sum_{l=1}^{\bar{r}} \eta_{il} \langle b_l(x), h \rangle \, b_i(x) - \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i} \mu_i b_i'(x)h + o(\|h\|),
$$

where $(\eta_{il})_{i,l=1}^{\bar{r}}$ is the symmetric matrix defined in part (ii) of Lemma 20. Therefore, $f \circ \lambda$ is twice differentiable at x and for any $h \in \mathbb{V}$,

$$
\nabla^2(f \circ \lambda)(x)h = \sum_{i=1}^{\bar{r}} (\nabla f(\lambda(x)))_{r_i} b_i'(x)h + (\tilde{\phi}_{\mathbb{V}})'(x)h
$$
$$
+ \sum_{i=1}^{\bar{r}} \sum_{l=1}^{\bar{r}} \eta_{il} \langle b_l(x), h \rangle \, b_i(x) - \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) \mu_i b_i'(x)h.
$$

By using (37) and (43) we obtain for any $h \in \mathbb{V}$,

$$\nabla^2 (f \circ \lambda)(x)h$$

$$= 2 \sum_{i=1}^{\bar{r}} \sum_{\substack{l=1 \\ l \neq i}}^{\bar{r}} \frac{(\nabla f(\lambda(x)))_{r_i} - (\nabla f(\lambda(x)))_{r_l}}{\mu_i - \mu_l} b_i(x) \cdot (b_l(x) \cdot h) + (\tilde{\phi}_{\mathbb{V}})'(x)(h)$$

$$+ \sum_{i=1}^{\bar{r}} \sum_{l=1}^{\bar{r}} \eta_{il} \langle b_l(x), h \rangle b_i(x) - 2 \sum_{i=1}^{\bar{r}} \sum_{\substack{l=1 \\ l \neq i}}^{\bar{r}} \frac{\tilde{\beta}_{r_i}(x)\mu_i - \tilde{\beta}_{r_l}(x)\mu_l}{\mu_i - \mu_l} b_i(x) \cdot (b_l(x) \cdot h)$$

$$= 2 \sum_{i=1}^{\bar{r}} \sum_{\substack{l=1 \\ l \neq i}}^{\bar{r}} \frac{(\nabla f(\lambda(x)))_{r_i} - (\nabla f(\lambda(x)))_{r_l}}{\mu_i - \mu_l} b_i(x) \cdot (b_l(x) \cdot h) + 2 \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) b_i(x) \cdot (b_i(x) \cdot h)$$

$$+ \sum_{i=1}^{\bar{r}} \sum_{l=1}^{\bar{r}} \eta_{il} \langle b_l(x), h \rangle b_i(x) - \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) b_i(x) \cdot h \, .$$

This, together with (32), (44), and part (ii) of Lemma 20, implies

$$\nabla^2 (f \circ \lambda)(x)h$$

$$= 2 \sum_{j=1}^{r} \sum_{l=1}^{r} \tilde{a}_{jl}(x) c_j \cdot (c_l \cdot h) + 2 \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) \sum_{j=r_{i-1}+1}^{r_i} c_j \cdot (c_j \cdot h)$$

$$+ \sum_{j=1}^{r} \sum_{l=1}^{r} (\nabla^2 f(\lambda(x)))_{jl} \langle c_l, h \rangle c_j - \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) \sum_{j=r_{i-1}+1}^{r_i} \langle c_j, h \rangle c_j - \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) b_i(x) \cdot h$$

$$= \sum_{j=1}^{r} \sum_{l=1}^{r} [2\tilde{a}_{jl}(x) c_j \cdot (c_l \cdot h) + (\nabla^2 f(\lambda(x)))_{jl} \langle c_l, h \rangle c_j]$$

$$+ \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) \sum_{j=r_{i-1}+1}^{r_i} c_j \cdot h - \sum_{i=1}^{\bar{r}} \tilde{\beta}_{r_i}(x) b_i(x) \cdot h \, .$$

Thus (45) holds.

" $\Longrightarrow$ " For any $\varsigma \in \mathbb{R}^r$, define

$$y = x + \sum_{j=1}^{r} \varsigma_j c_j = \sum_{j=1}^{r} (\lambda_j(x) + \varsigma_j) c_j \, .$$

Then, by Theorem 21, for all $\varsigma \in \mathbb{R}^r$ sufficiently small, f is differentiable at y and

$$\nabla(f \circ \lambda)(y) = \sum_{j=1}^{r} (\nabla f(\lambda(x) + \varsigma))_j c_j \, ,$$

which implies that

$$(\nabla f(\lambda(x) + \varsigma))_j = \langle \nabla(f \circ \lambda)(y), c_j \rangle \, , \ j = 1, 2, \ldots, r \, .$$

Thus f is twice differentiable at $\lambda(x)$. $\qquad \square$

The next theorem is about the continuity of $\nabla^2 (f \circ \lambda)(x)$. It is a direct consequence of Theorem 23 and parts (ii) and (iii) of Lemma 20.

Theorem 24 *Let $f : \mathbb{R}^r \to (-\infty, \infty]$ be a symmetric function and $x = \sum_{j=1}^r \lambda_j(x)c_j$. Then $f \circ \lambda$ is twice continuously differentiable at x if and only if f is twice continuously differentiable at $\lambda(x)$.*

Remark 25 *Theorems 23 and 24 extend the twice differentiability results in [34] on the spectral function over symmetric matrices to Euclidean Jordan algebras. This extension builds on known results of the symmetric function and the differentiability of Löwner's operator discussed in Section 3.*

Let $y \in \mathbb{V}$ have the spectral decomposition $y = \sum_{j=1}^r \lambda_j(y)c_j(y)$ with $\lambda_1(y) \geq \lambda_2(y) \geq \cdots \geq \lambda_r(y)$ and $\{c_1(y), c_2(y), \ldots, c_r(y)\} \in \mathfrak{C}(y)$. For any $1 \leq j < l \leq r$, there exist d mutually orthonormal vectors $\{v_{jl}^{(i)}(y)\}_{i=1}^d$ in $\mathbb{V}$ such that

$$\mathcal{C}_{jl}(y) = \sum_{i=1}^d \langle v_{jl}^{(i)}(y), \cdot \rangle \, v_{jl}^{(i)}(y) \,,$$

where d satisfies (25). Then

$$\left\{ c_1(y), c_2(y), \ldots, c_r(y), v_{jl}^{(1)}(y), v_{jl}^{(2)}(y), \ldots, v_{jl}^{(d)}(y), \; 1 \leq j < l \leq r \right\}$$

is an orthonormal basis of $\mathbb{V}$. Let $U(y)$ be the matrix formed by this basis, i.e., the first r columns of $U(y)$ are $c_1(y), c_2(y), \ldots, c_r(y)$ and the rest are $v_{jl}^{(i)}(y)$, $1 \leq j < l \leq r, 1 \leq i \leq d$. Let $\tilde{h}$ be the coefficients of $h := y - x$ with respect to the basis

$$\left\{ c_1(y), c_2(y), \ldots, c_r(y), v_{jl}^{(1)}(y), v_{jl}^{(2)}(y), \ldots, v_{jl}^{(d)}(y), \; 1 \leq j < l \leq r \right\} \,.$$

Then there exist numbers $\tilde{h}_j, \tilde{h}_{jl}^{(i)} \in \mathbb{R}$, $1 \leq j < l \leq r, 1 \leq i \leq d$ such that

$$h = U(y)\tilde{h} = \sum_{j=1}^r \tilde{h}_j c_j(y) + \sum_{1 \leq j < l \leq r} \sum_{i=1}^d \tilde{h}_{jl}^{(i)} v_{jl}^{(i)}(y) \,. \tag{46}$$

Let $\mathcal{D}_\lambda$ be the set of points in $\mathbb{V}$ where $\lambda(\cdot)$ is differentiable.

Lemma 26 *Let $x = \sum_{j=1}^r \lambda_j(x)c_j$. Then for any $1 \leq j < l \leq r$ such that $\lambda_j(x) = \lambda_l(x)$ and $y \to x$ with $y \in \mathcal{D}_\lambda$,*

$$\tilde{h}_{jl}^{(i)} = O(\|h\|^2) \,, \; i = 1, 2, \ldots, d \,.$$

Proof. By (i) and (iv) in Proposition 10 and Lemma 6, for any $y \to x$ with $y \in \mathcal{D}_\lambda$ we have

$$
\begin{aligned}
0 \;&= y - x - h = \sum_{j=1}^r \lambda_j(y)c_j(y) - \sum_{j=1}^r \lambda_j(x)c_j - h \\
&= \sum_{j=1}^r \left(\lambda_j(x) + \lambda_j'(y)h + O(\|h\|^2)\right)c_j(y) - \sum_{j=1}^r \lambda_j(x)c_j - h \\
&= \sum_{j=1}^r (\lambda_j(x) + \lambda_j'(y)h)c_j(y) - \sum_{j=1}^r \lambda_j(x)c_j - h + O(\|h\|^2) \\
&= \sum_{i=1}^{\bar{r}} \lambda_{r_i}(x)(b_i(y) - b_i(x)) + \sum_{j=1}^r \langle c_j(y), h \rangle \, c_j(y) - h + O(\|h\|^2) \,,
\end{aligned}
$$

which, together with the analyticity of b_i, (37), and (33), implies

$$
\begin{aligned}
0 &= \sum_{i=1}^{\bar{r}} \lambda_{r_i}(x) b_i'(y) h - \sum_{1 \leq j < l \leq r} \mathcal{C}_{jl}(y) h + O(\|h\|^2) \\
&= 4 \sum_{i=1}^{\bar{r}} \lambda_{r_i}(x) \sum_{\substack{s=1 \\ s \neq i}}^{\bar{r}} \frac{b_i(y) \cdot (b_s(y) \cdot h)}{\lambda_{r_i}(x) - \lambda_{r_s}(x)} - \sum_{1 \leq j < l \leq r} \mathcal{C}_{jl}(y) h + O(\|h\|^2) \\
&= 2 \sum_{i=1}^{\bar{r}} \sum_{\substack{s=1 \\ s \neq i}}^{\bar{r}} b_i(y) \cdot (b_s(y) \cdot h) - \sum_{1 \leq j < l \leq r} \mathcal{C}_{jl}(y) h + O(\|h\|^2) \\
&= 2 \sum_{j=1}^{r} \sum_{l=1}^{r} \omega_{jl} c_j(y) \cdot (c_l(y) \cdot h) - \sum_{1 \leq j < l \leq r} \mathcal{C}_{jl}(y) h + O(\|h\|^2),
\end{aligned}
$$

where for $j, l = 1, 2, \ldots, r$,

$$
\omega_{jl} = \begin{cases} 0 & \text{if } r_{t-1} + 1 \leq j, l \leq r_t \\ 1 & \text{otherwise}, \end{cases}
$$

$t = 1, 2, \ldots, \bar{r}$. Therefore, for $y \to x$ with $y \in \mathcal{D}_\lambda$,

$$
\begin{aligned}
0 &= \sum_{1 \leq j < l \leq r} \omega_{jl} \mathcal{C}_{jl}(y) h - \sum_{1 \leq j < l \leq r} \mathcal{C}_{jl}(y) h + O(\|h\|^2) \\
&= \sum_{1 \leq j < l \leq r} \sum_{i=1}^{d} (\omega_{jl} - 1) \tilde{h}_{jl}^{(i)} v_{jl}^{(i)}(y) + O(\|h\|^2),
\end{aligned}
$$

which implies that for $1 \leq j < l \leq r$ and $1 \leq i \leq d$,

$$
0 = (\omega_{jl} - 1) \tilde{h}_{jl}^{(i)} \langle v_{jl}^{(i)}(y), v_{jl}^{(i)}(y) \rangle + O(\|h\|^2) = (\omega_{jl} - 1) \tilde{h}_{jl}^{(i)} + O(\|h\|^2).
$$

By observing that for any $1 \leq j < l \leq r$, $\omega_{jl} = 0$ if $\lambda_j(x) = \lambda_l(x)$, we then complete the proof. $\square$

Theorem 27 *Let $f : \mathbb{R}^r \to (-\infty, \infty]$ be a symmetric function. Let $x = \sum_{j=1}^{r} \lambda_j(x) c_j$ and $\gamma \in (0, 1]$. Then $\nabla(f \circ \lambda)$ is (γ-order) G-semismooth at x if and only if ∇f is (γ-order) G-semismooth at $\lambda(x)$.*

Proof. " $\Longleftarrow$ " We only prove the case for the γ-order G-semismoothness. The case for the G-semismoothness can be obtained similarly. Suppose that ∇f is γ-order G-semismooth at $\upsilon := \lambda(x)$. By considering the convolution regularization of f (cf. [45, Section 9.K]), we can adapt the proof of [41, Proposition 4.3] for the case of symmetric matrices and use Lemma 20 and Theorems 23 and 24 to show that there exists an open set $\mathcal{O}(x)$ containing x such that $\nabla(f \circ \lambda)$ is Lipschitz continuous on $\mathcal{O}$. For brevity, we omit the details here.

By Theorem 23, $y \in \mathcal{D}_{\nabla(f \circ \lambda)}$, the set of differentiable points of $\nabla(f \circ \lambda)$ in $\mathcal{O}$, if and only if ∇f is differentiable at $\lambda(y)$. Since $\lambda(\cdot)$ is Lipschitz continuous on $\mathcal{O}$, the set $S := \bigcup_{j=1}^{r} S_j$

is a set of Lebesgue measure zero, where $S_j := \mathcal{O}\backslash\mathcal{D}_{\lambda_j}$, $j = 1, 2, \ldots, r$. Then for any $y \in \mathcal{D}_{\nabla(f \circ \lambda)}$ and $y \notin S$, $\nabla^2(f \circ \lambda)(y)$ exists and for any $h \in \mathbb{V}$,

$$\nabla^2(f \circ \lambda)(y)h = \sum_{j=1}^{r}\sum_{l=1}^{r}\left[2\tilde{a}_{jl}(y)c_j(y) \cdot (c_l(y) \cdot h) + (\nabla^2 f(\lambda(y)))_{jl}\langle c_l(y), h\rangle\, c_j(y)\right],$$

where for $j, l = 1, 2, \ldots, r$,

$$\tilde{a}_{jl}(y) = \begin{cases} 0 & \text{if } j = l \\ \dfrac{(\nabla f(\lambda(y)))_j - (\nabla f(\lambda(y)))_l}{\lambda_j(y) - \lambda_l(y)} & \text{otherwise}. \end{cases} \tag{47}$$

Let $h := y - x$. By Theorem 21 and Lemma 6, for any $y \to x$ with $y \in \mathcal{D}_{\nabla(f\circ\lambda)}$ and $y \notin S$,

$$\nabla(f \circ \lambda)(y) - \nabla(f \circ \lambda)(x)$$
$$= \sum_{j=1}^{r}(\nabla f(\lambda(y)))_j c_j(y) - \sum_{j=1}^{r}(\nabla f(\lambda(x)))_j c_j$$
$$= \sum_{j=1}^{r}(\nabla f(\lambda(x)) + \nabla^2 f(\lambda(y))(\lambda(y) - \lambda(x)))_j c_j(y) - \sum_{j=1}^{r}(\nabla f(\lambda(x)))_j c_j + O(\|h\|^{1+\gamma})$$
$$= \sum_{i=1}^{\bar{r}}(\nabla f(\lambda(x)))_{r_i}(b_i(x+h) - b_i(x)) + \sum_{j=1}^{r}(\nabla^2 f(\lambda(y))\lambda'(y)h)_j c_j(y) + O(\|h\|^{1+\gamma}),$$

which, together with (37) and part (iii) of Proposition 10, implies

$$\nabla(f \circ \lambda)(y) - \nabla(f \circ \lambda)(x)$$
$$= \sum_{i=1}^{\bar{r}}(\nabla f(\lambda(x)))_{r_i} b_i'(y)h + \sum_{j=1}^{r}\sum_{l=1}^{r}(\nabla^2 f(\lambda(y)))_{jl}\langle c_l(y), h\rangle\, c_j(y) + O(\|h\|^{1+\gamma}).$$

Therefore, for any $y \to x$ with $y \in \mathcal{D}_{\nabla(f\circ\lambda)}$ and $y \notin S$,

$$\nabla(f \circ \lambda)(y) - \nabla(f \circ \lambda)(x) - \nabla^2(f \circ \lambda)(y)h$$
$$= 2\sum_{i=1}^{\bar{r}}\sum_{\substack{s=1 \\ s \neq i}}^{\bar{r}}\frac{(\nabla f(\lambda(x)))_{r_i} - (\nabla f(\lambda(x)))_{r_s}}{\mu_i - \mu_s}b_i(y) \cdot (b_s(y) \cdot h)$$
$$\quad - 2\sum_{j=1}^{r}\sum_{l=1}^{r}\tilde{a}_{jl}(y)c_j(y) \cdot (c_l(y) \cdot h) + O(\|h\|^{1+\gamma})$$
$$= 2\sum_{j=1}^{r}\sum_{l=1}^{r}[\tilde{\omega}_{jl}(x) - \tilde{a}_{jl}(y)]c_j(y) \cdot (c_l(y) \cdot h) + O(\|h\|^{1+\gamma}),$$

where for $1 \leq j, l \leq r$,

$$\tilde{\omega}_{jl}(x) := \begin{cases} 0 & \text{if } r_{i-1} + 1 \leq j, l \leq r_i \\ \dfrac{(\nabla f(\lambda(x)))_j - (\nabla f(\lambda(x)))_l}{\lambda_j(x) - \lambda_l(x)} & \text{otherwise}, \end{cases} \tag{48}$$

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$i = 1, 2, \ldots, \bar{r}$. Let $\delta(h) := 2 \sum_{j=1}^{r} \sum_{l=1}^{r} [\tilde{\omega}_{jl}(x) - \tilde{a}_{jl}(y)] c_j(y) \cdot (c_l(y) \cdot h)$. Then, by the definition of $\mathcal{C}_{jl}(y)$ and (46), for $y \in \mathcal{D}_{\nabla(f \circ \lambda)}$ with $y \notin S$,

$$
\begin{aligned}
\delta(h) &= \sum_{1 \le j < l \le r} [\tilde{\omega}_{jl}(x) - \tilde{a}_{jl}(y)] \mathcal{C}_{jl}(y) h \\
&= \sum_{1 \le j < l \le r} [\tilde{\omega}_{jl}(x) - \tilde{a}_{jl}(y)] \sum_{i=1}^{d} \tilde{h}_{jl}^{(i)} v_{jl}^{(i)}(y) .
\end{aligned}
$$

We consider the following cases about $\tilde{\delta}_{jl}^{(i)}(h) := [\tilde{\omega}_{jl}(x) - \tilde{a}_{jl}(y)] \tilde{h}_{jl}^{(i)}$, $1 \le j < l \le r$, $1 \le i \le d$:

Case 1): $\lambda_j(x) = \lambda_l(x)$. In this case, by (48), Lemma 26, and part (iv) of Lemma 20,

$$
\tilde{\delta}_{jl}^{(i)}(h) = [\tilde{\omega}_{jl}(x) - \tilde{a}_{jl}(y)] \tilde{h}_{jl}^{(i)} = -\tilde{a}_{jl}(y) \tilde{h}_{jl}^{(i)} = O(\|h\|^2) .
$$

Case 2): $\lambda_j(x) \ne \lambda_l(x)$. In this case, by the Lipschitz continuity of $\nabla f(\cdot)$ and $\lambda(\cdot)$,

$$
\begin{aligned}
\tilde{\delta}_{jl}^{(i)}(h) &= \left[\frac{(\nabla f(\lambda(x)))_j - (\nabla f(\lambda(x)))_l}{\lambda_j(x) - \lambda_l(x)} - \frac{(\nabla f(\lambda(y)))_j - (\nabla f(\lambda(y)))_l}{\lambda_j(y) - \lambda_l(y)} \right] \tilde{h}_{jl}^{(i)} \\
&= \frac{O(\|h\|)}{(\lambda_j(x) - \lambda_l(x))(\lambda_j(y) - \lambda_l(y))} \tilde{h}_{jl}^{(i)} \\
&= O(\|h\| \|\tilde{h}\|) .
\end{aligned}
$$

Therefore, for any $y \to x$ with $y \in \mathcal{D}_{\nabla(f \circ \lambda)}$ and $y \notin S$,

$$
\nabla(f \circ \lambda)(y) - \nabla(f \circ \lambda)(x) - \nabla^2(f \circ \lambda)(y) h = O(\|h\|^{1+\gamma}) .
$$

This, by Lemma 6, shows that $\nabla(f \circ \lambda)$ is γ-order G-semismooth at x.

" $\Longrightarrow$ " This direction can be done easily by following the proof in the second part of Theorem 23. $\quad \square$

Remark 28 *Theorem 27 is about the G-semismoothness of $\nabla(f \circ \lambda)$ rather than the semismoothness of $\nabla(f \circ \lambda)$ as the directional derivative of $\nabla(f \circ \lambda)$ is not involved. For the spectral function over symmetric matrices, the latter has been done in [41]. It is not clear to us whether the result in [41] holds in general for Euclidean Jordan algebras.*

5 Conclusions

We have studied differential properties of Löwner's operator and spectral functions in Euclidean Jordan algebras. The approach consists of adaptations of known arguments for symmetric matrices and developments of new technical results. Compared to our knowledge of symmetric matrices, more research is needed for functions in Euclidean Jordan algebras. We conclude the discussion of this paper by listing below a few interesting questions, which we would like to know the answers in the near future.

Question 1. The eigenvalue function $\lambda(\cdot)$ defined over Euclidean Jordan algebras is directionally differentiable. Can we derive formulas on the directional derivative of $\lambda(\cdot)$ as was done in [28, Theorem 7] for the symmetric matrix case?

Question 2. Is that true as for the symmetric matrix case that for any given $x, h \in \mathbb{V}$, the eigenvectors of $x + \varepsilon h$ can be chosen to be analytic at $\varepsilon = 0$ (cf. Remark 11)?

Question 3. For the symmetric matrix case, it is proved that $\mathfrak{C}(\cdot)$ is upper Lipschitz continuous at x [7, 51, 52]. Can we extend this to Euclidean Jordan algebras? Lemma 26 presents a partial solution.

Question 4. Can we use the results in [11, 12, 13] for the symmetric matrix case to get explicit formulas for the higher-order derivatives of Löwner's operator over Euclidean Jordan algebras under sufficient differentiability of ϕ?

Question 5. The first- and second-order derivatives of the spectral function are established. What can we say about the higher-order derivatives?

Question 6. Under what conditions about ∇f, $\nabla(f \circ \lambda)$ is directionally differentiable? This question is related to Question 2.

Question 7. The metric projection operator over symmetric cones are proved to be strongly semismooth. What kind of differential properties can we say about the metric projection operator over the closed hyperbolic cone (cf. Section 2.2)? Or less ambitiously, over the closed homogeneous cone (cf. [9])?

Acknowledgments

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