Isogeny graphs in cryptography

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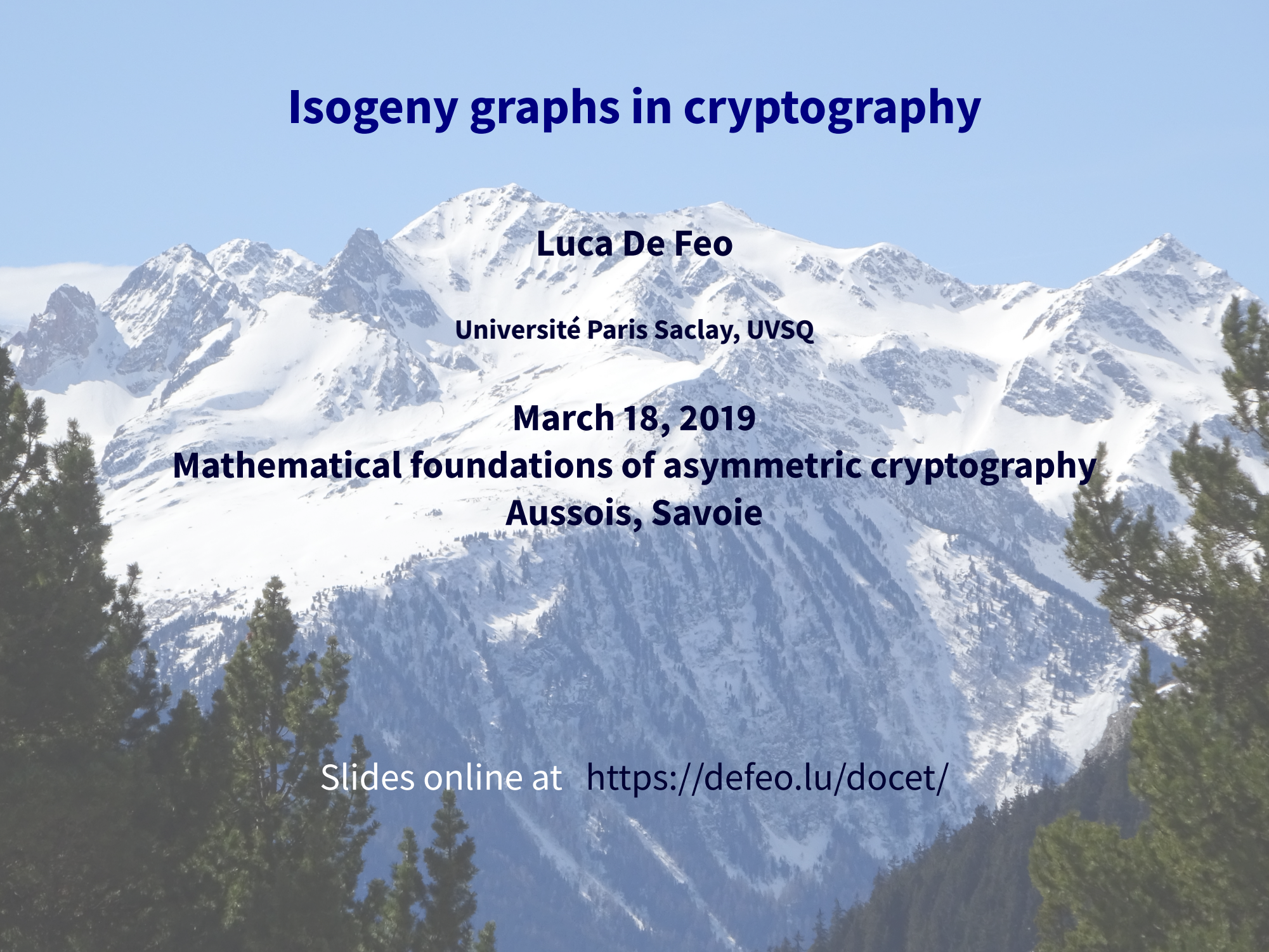
March 18, 2019
Mathematical foundations of asymmetric cryptography
Aussois, Savoie



Slides online at https://defeo.lu/docet/

Overview

1. Isogeny graphs
   - Elliptic Curves
   - Isogenies
   - Isogeny graphs
   - Endomorphism rings
   - Ordinary graphs
   - Supersingular graphs

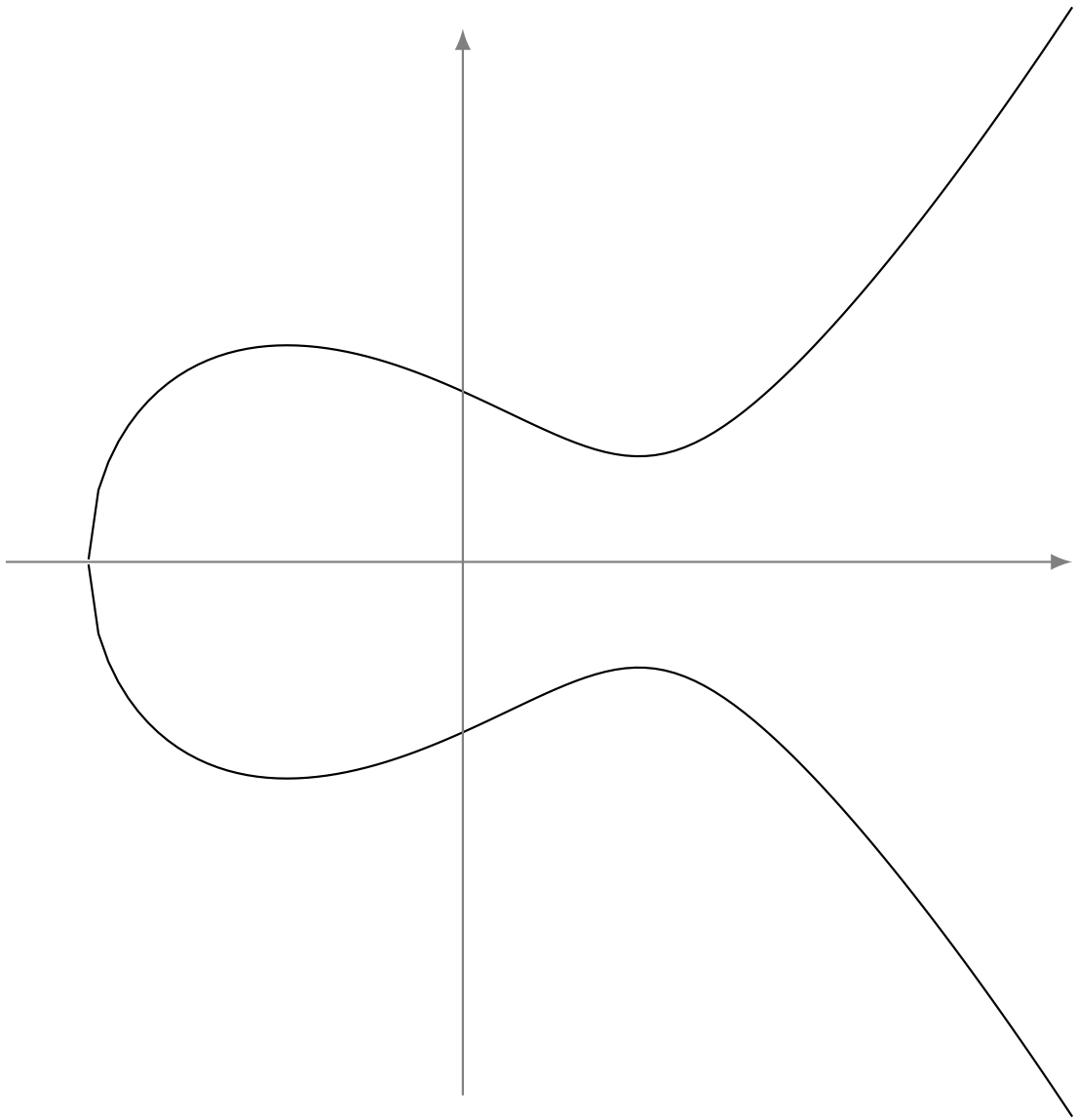
2. Cryptography
   - Isogeny walks and Hash functions
   - Pairing verification and Verifiable Delay Functions
   - Key exchange
   - Open Problems

Elliptic curves

Let k be a field of characteristic $\neq 2, 3$.
An elliptic curve *defined over* k is the locus in the projective space $\mathbb{P}^2(\bar{k})$ of an equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3,$$

where $a, b \in k$ and $4a^3 + 27b^2 \neq 0$.



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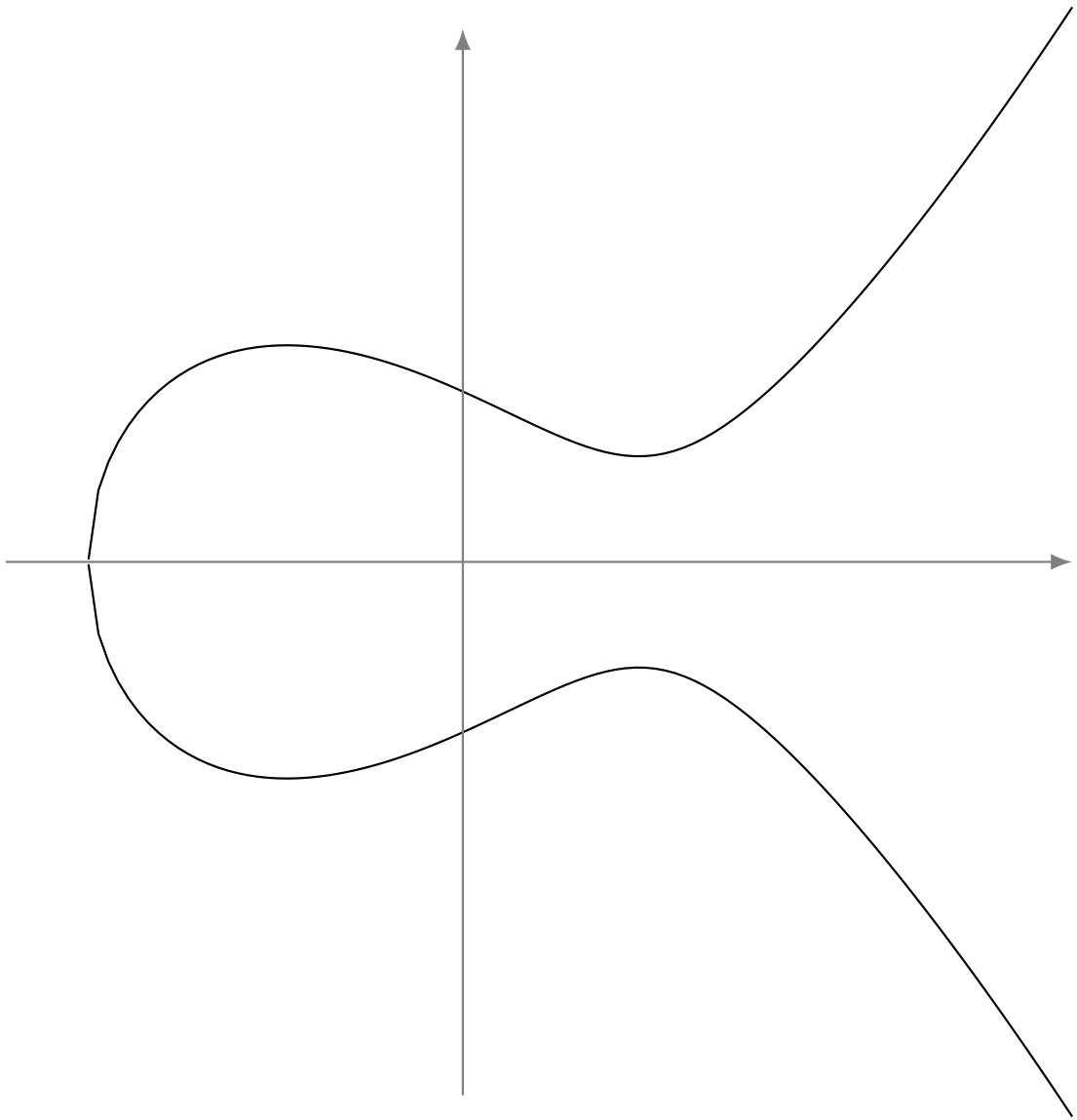

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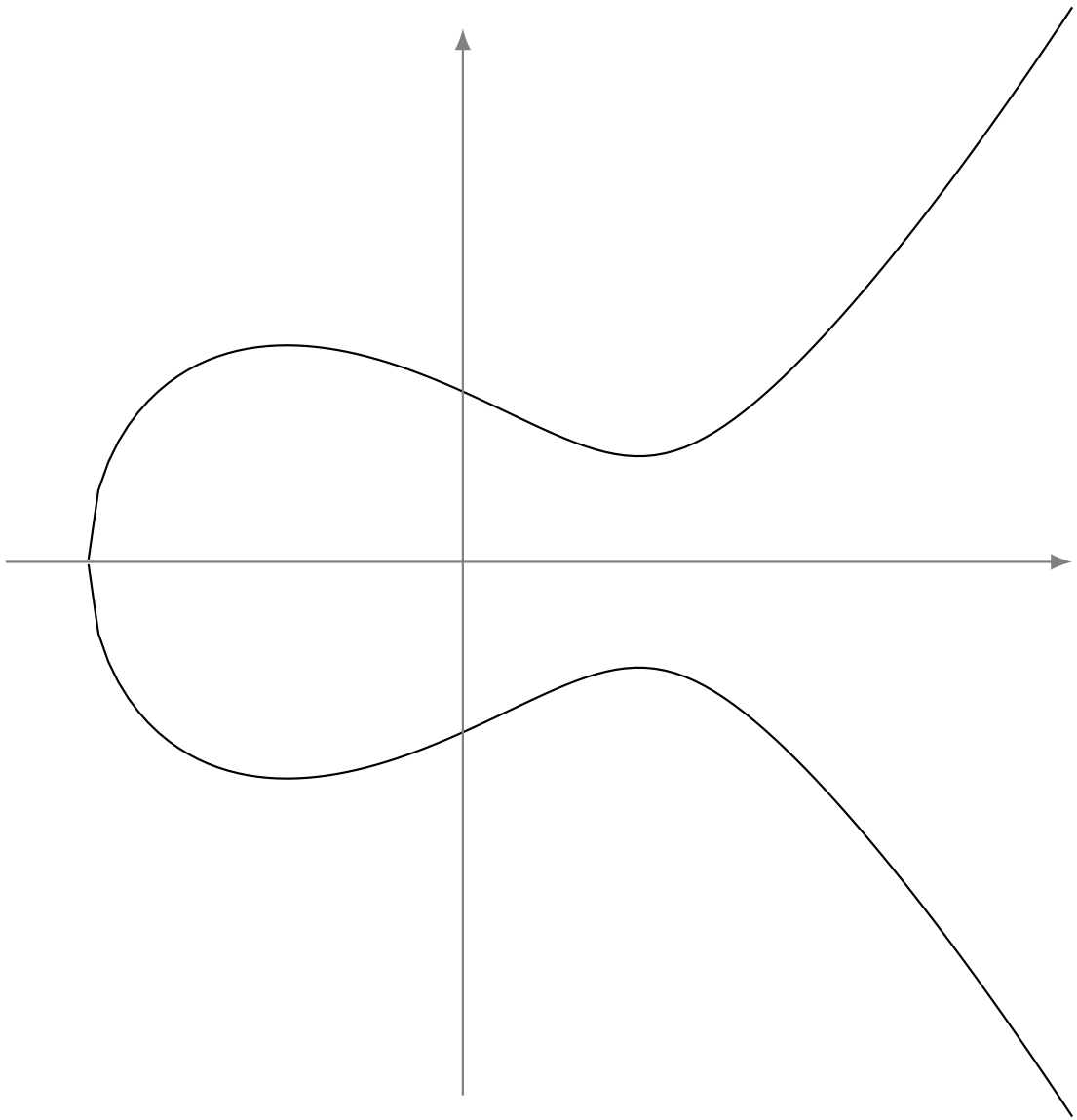
- $\mathcal{O} = (0 : 1 : 0)$ is the point at infinity;
- $y^2 = x^3 + ax + b$ is the affine Weierstrass equation.

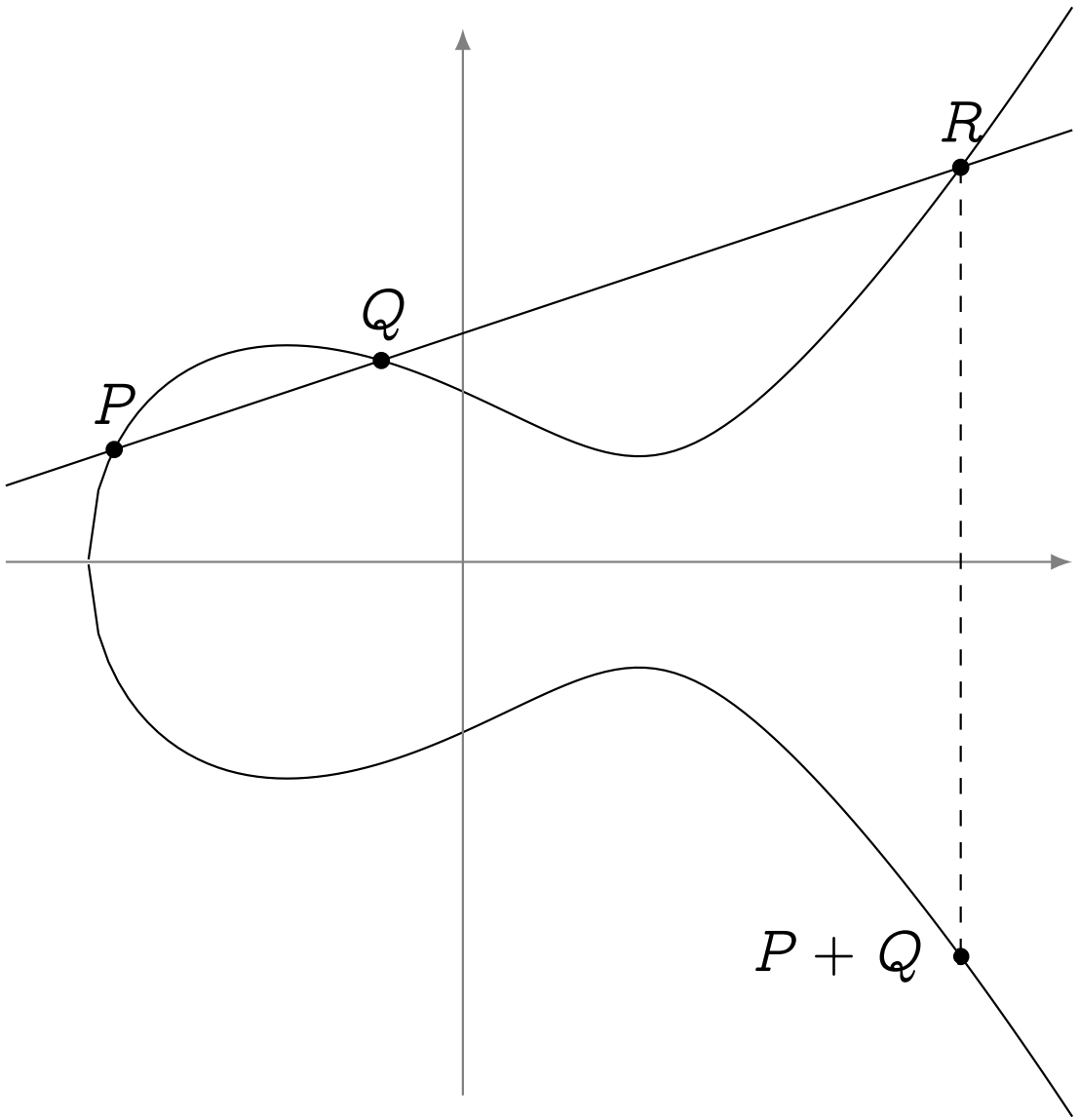


The group law

Bezout's theorem

Every line cuts E in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.



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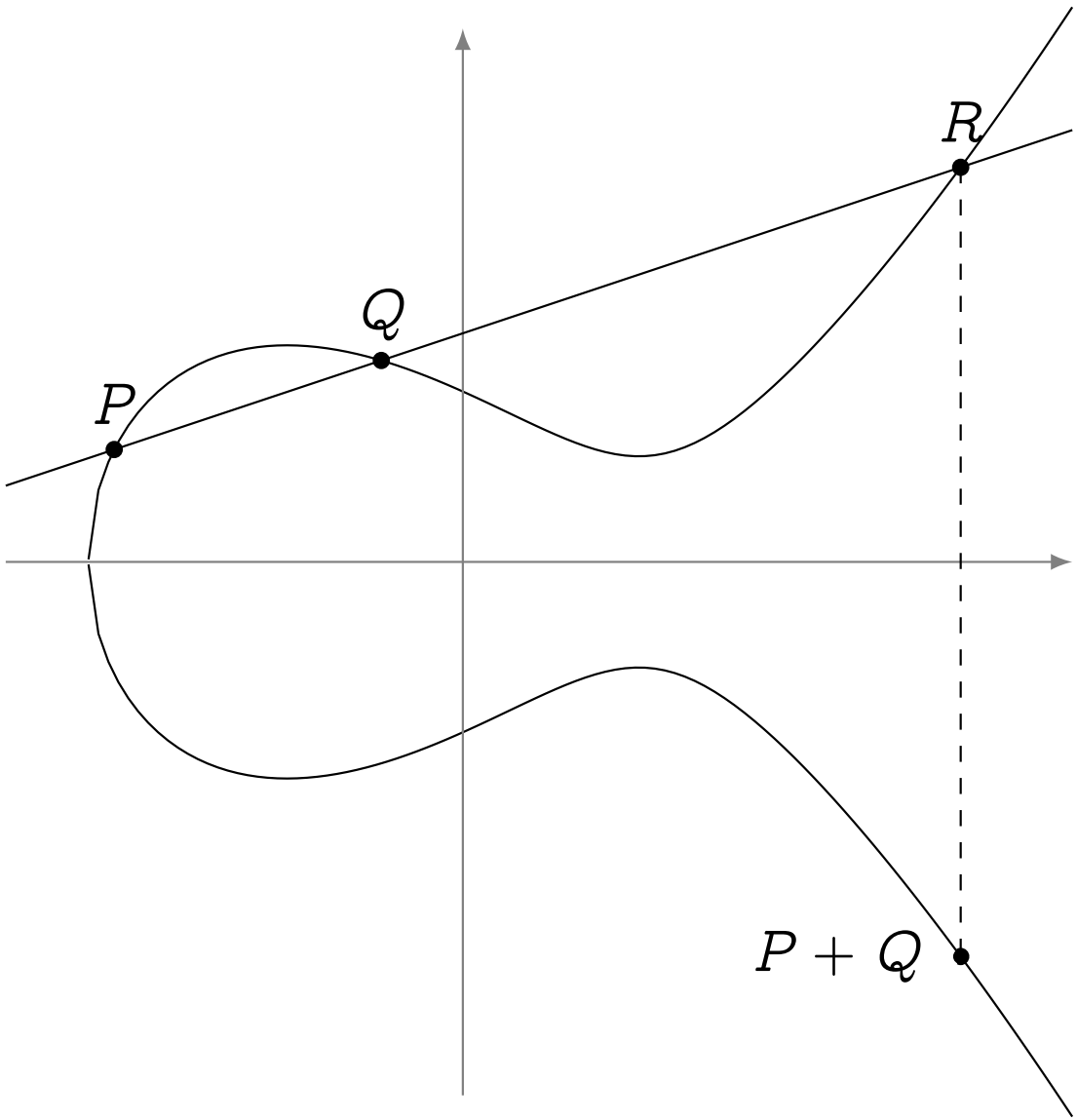

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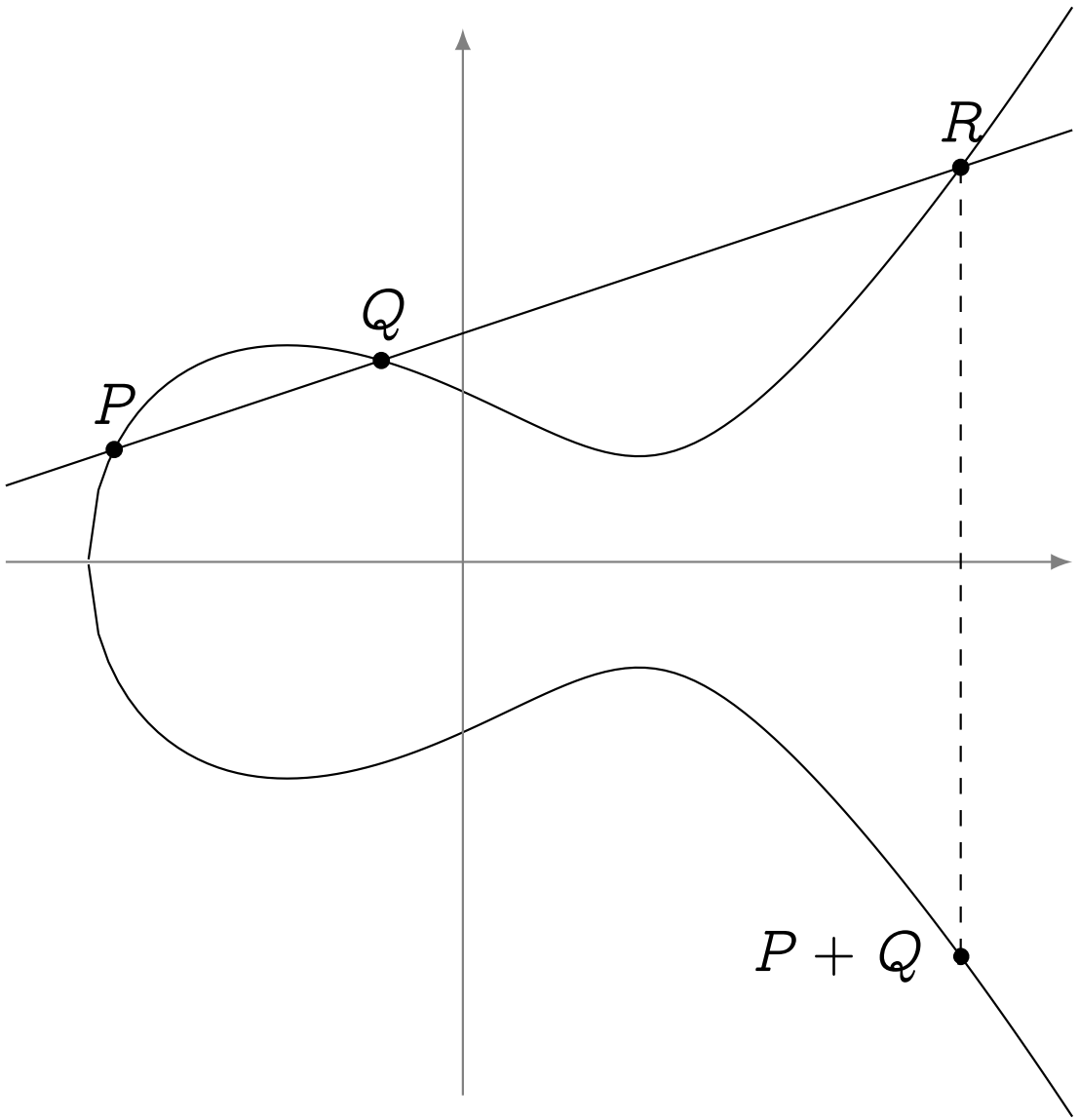
Every line cuts E in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.

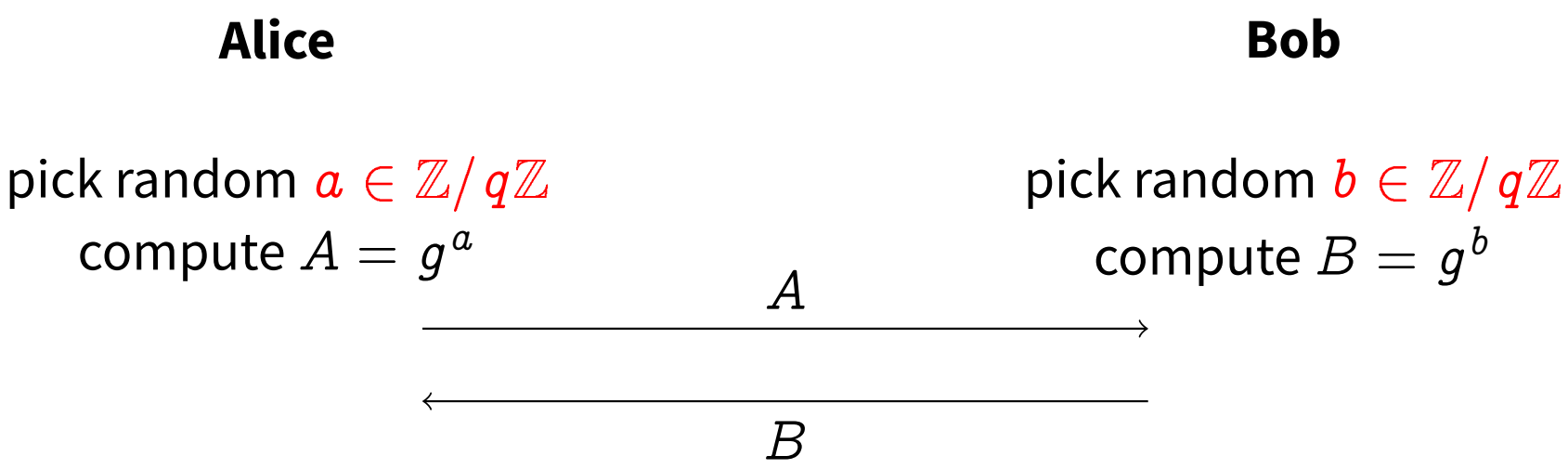
- The law is algebraic (it has *formulas*);
- The law is commutative;
- $\mathcal{O}$ is the group identity;
- Opposite points have the same x-value.



Why should I care? (Diffie–Hellman key exchange)

Goal: Alice and Bob have never met before. They are chatting over a public channel, and want to agree on a shared secret to start a private conversation.

Setup: They agree on a (large) cyclic group $G = \langle g \rangle$ of (prime) order q.



Shared secret is $B^a = g^{ab} = A^b$

Brief history of DH key exchange

1976 Diffie & Hellman publish New directions in cryptography, suggest using $G = \mathbb{F}_p^*$.

1978 Pollard publishes his discrete logarithm algorithm ($O(\sqrt{\#G})$ complexity).

1980 Miller and Koblitz independently suggest using elliptic curves $G = E(\mathbb{F}_p)$.

1994 Shor publishes his quantum polynomial time discrete logarithm / factoring algorithm.

2005 NSA standardizes elliptic curve key agreement (ECDH) and signatures ECDSA.

2017 $\sim 70\%$ of web traffic is secured by ECDH and/or ECDSA.

2017 NIST launches post-quantum competition, says “not to bother moving to elliptic curves, if you haven’t yet”.

Why should I care? (cont'd)

But, also:

- Elliptic Curve Factoring Method (Lenstra '85);
- Elliptic Curve Primality Proving (Atkin, Morain '86-'93);
- Efficient normal bases for finite fields (Couveignes, Lercier '10);
- ...

What are elliptic curves?

For mathematicians

- The smooth projective curves of genus 1;
- The simplest abelian varieties (dimension 1);
- Finitely generated abelian groups of mysterious free rank (aka BSD conjecture);
- What you use to make examples.

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For cryptographers

- Finite abelian groups (often cyclic);
- Easy to compute the order;
- "2-dimensional" generalizations of μ_k (the roots of unity of k)…
- …with bilinear maps (aka pairings)!

Isomorphisms

Isomorphisms

The only invertible algebraic maps between elliptic curves are of the form

$$(x, y) \mapsto (u^2 x, u^3 y)$$

for some $u \in \bar{k}$.
They are group isomorphisms.

j-Invariant

Let $E : y^2 = x^3 + ax + b$, its j-invariant is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Two elliptic curves E, E' are isomorphic if and only if $j(E) = j(E')$.

Group structure

Torsion structure

Let E be defined over an algebraically closed field $\bar{k}$ of characteristic p.

$$E[m] \simeq \quad \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \qquad \text{if } p \nmid m,$$

$$E[p^e] \simeq \begin{cases} \mathbb{Z}/p^e\mathbb{Z} & \text{ordinary case,} \\ \{\mathcal{O}\} & \text{supersingular case.} \end{cases}$$

Finite fields (Hasse's theorem)

Let E be defined over a finite field $\mathbb{F}_q$, then

$$|\#E(\mathbb{F}_q) - q - 1| \leq 2\sqrt{q}.$$

In particular, there exist integers n_1 and $n_2 | \gcd(n_1, q-1)$ such that

$$E(\mathbb{F}_q) \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}.$$

What is scalar multiplication?

$$[n] \ : \ P \mapsto \underbrace{P + P + \cdots + P}_{n \text{ times}}$$

- A map $E \to E$,
- a group morphism,
- with finite kernel
  (the torsion group $E[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$),
- surjective (in the algebraic closure),
- given by rational maps of degree n^2.

What is ~~scalar multiplication~~ an isogeny?

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(Separable) isogenies $\Leftrightarrow$ finite subgroups:

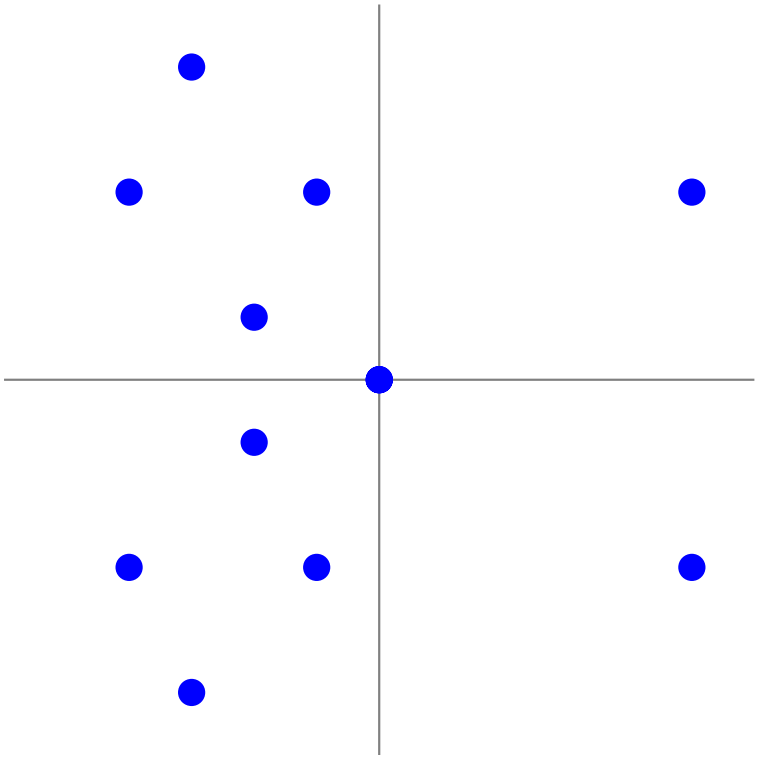
$$0 \longrightarrow H \longrightarrow E \xrightarrow{\phi} E' \to 0$$

The kernel H determines the image curve E' up to isomorphism

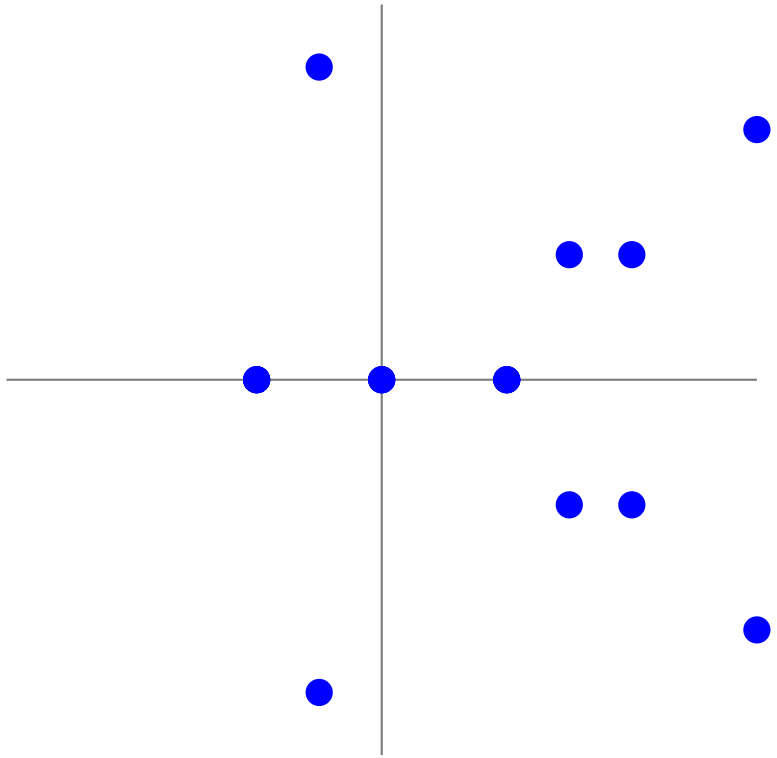
$$E/H \stackrel{\text{def}}{=} E'.$$

Isogenies: an example over $\mathbb{F}_{11}$

$$E \ : \ y^2 = x^3 + x$$



$$E' \ : \ y^2 = x^3 - 4x$$



$$\phi(x, y) = \left(\frac{x^2 + 1}{x}, \quad y \frac{x^2 - 1}{x^2} \right)$$

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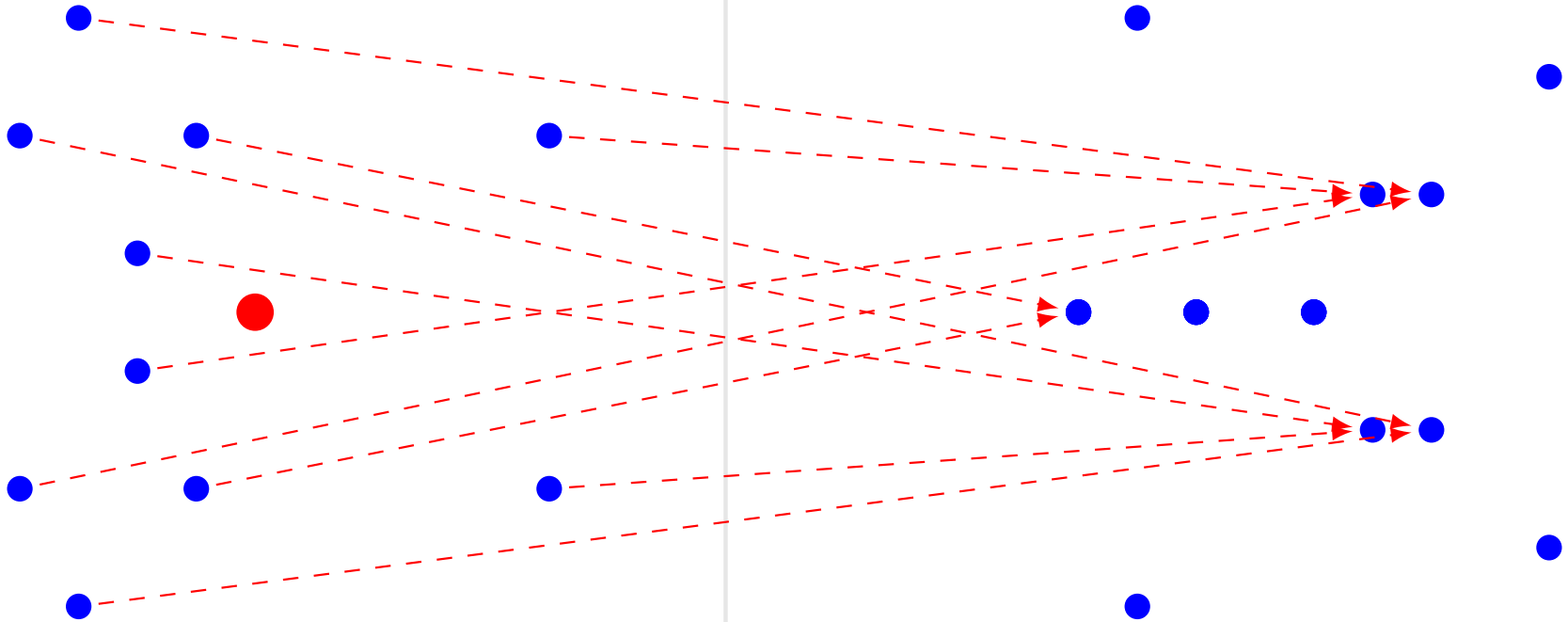

$$\phi(x, y) = \left(\frac{x^2 + 1}{x}, \quad y \frac{x^2 - 1}{x^2} \right)$$

- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

Isogeny properties

Let $\phi : E \to E'$ be an isogeny defined over a field k of characteristic p.

- $k(E)$ is the field of all rational functions from E to k;
- $\phi^* k(E')$ is the subfield of $k(E)$ defined as

$$\phi^* k(E') = \{f \circ \phi \mid f \in k(E')\}.$$

Degree, separability

1. The degree of ϕ is $\deg \phi = [k(E) : \phi^* k(E')]$. It is always finite.
2. ϕ is said to be separable, inseparable, or purely inseparable if the extension of function fields is.
3. If ϕ is separable, then $\deg \phi = \# \ker \phi$.
4. If ϕ is purely inseparable, then $\ker \phi = \{\mathcal{O}\}$ and $\deg \phi$ is a power of p.
5. Any isogeny can be decomposed as a product of a separable and a purely inseparable isogeny.

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The dual isogeny

Let $\phi : E \to E'$ be an isogeny of degree m. There is a unique isogeny $\hat{\phi} : E' \to E$ such that

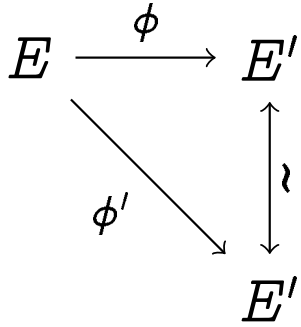
$$\hat{\phi} \circ \phi = [m]_E, \quad \phi \circ \hat{\phi} = [m]_{E'}.$$

$\hat{\phi}$ is called the dual isogeny of ϕ; it has the following properties:

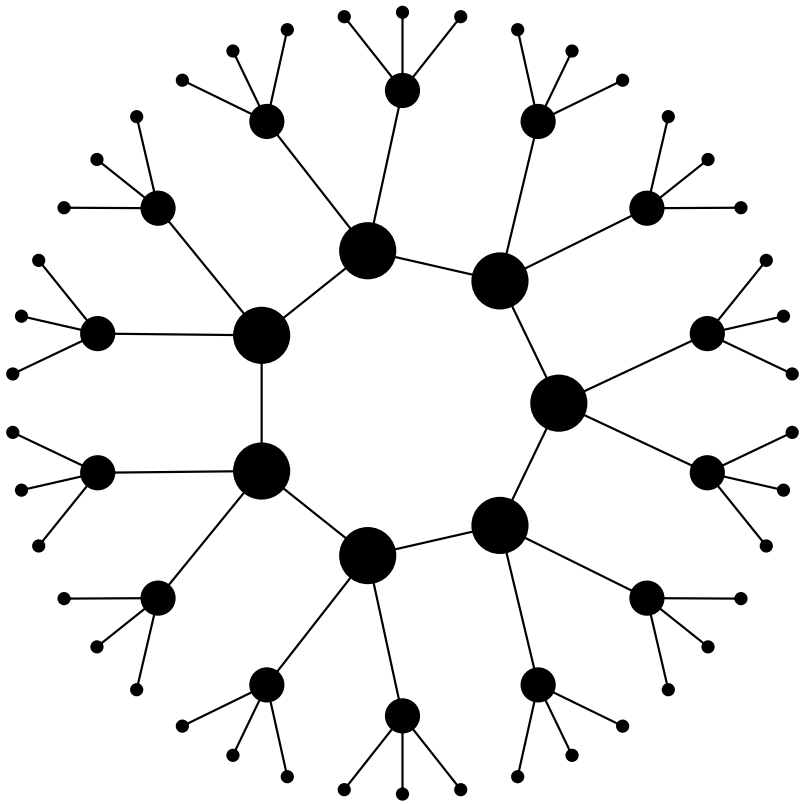
1. $\hat{\phi}$ is defined over k if and only if ϕ is;
2. $\widehat{\psi \circ \phi} = \hat{\phi} \circ \hat{\psi}$ for any isogeny $\psi : E' \to E''$;
3. $\widehat{\psi + \phi} = \hat{\psi} + \hat{\phi}$ for any isogeny $\psi : E \to E'$;
4. $\deg \phi = \deg \hat{\phi}$;
5. $\hat{\hat{\phi}} = \phi$.

Isogeny graphs

We look at the graph of elliptic curves with isogenies up to isomorphism. We say two isogenies ϕ, ϕ' are isomorphic if:



Example: Finite field, ordinary case, graph of isogenies of degree 3.



What do isogeny graphs look like?

Torsion subgroups (ℓ prime)

In an algebraically closed field:

$$E[\ell] = \langle P, Q \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

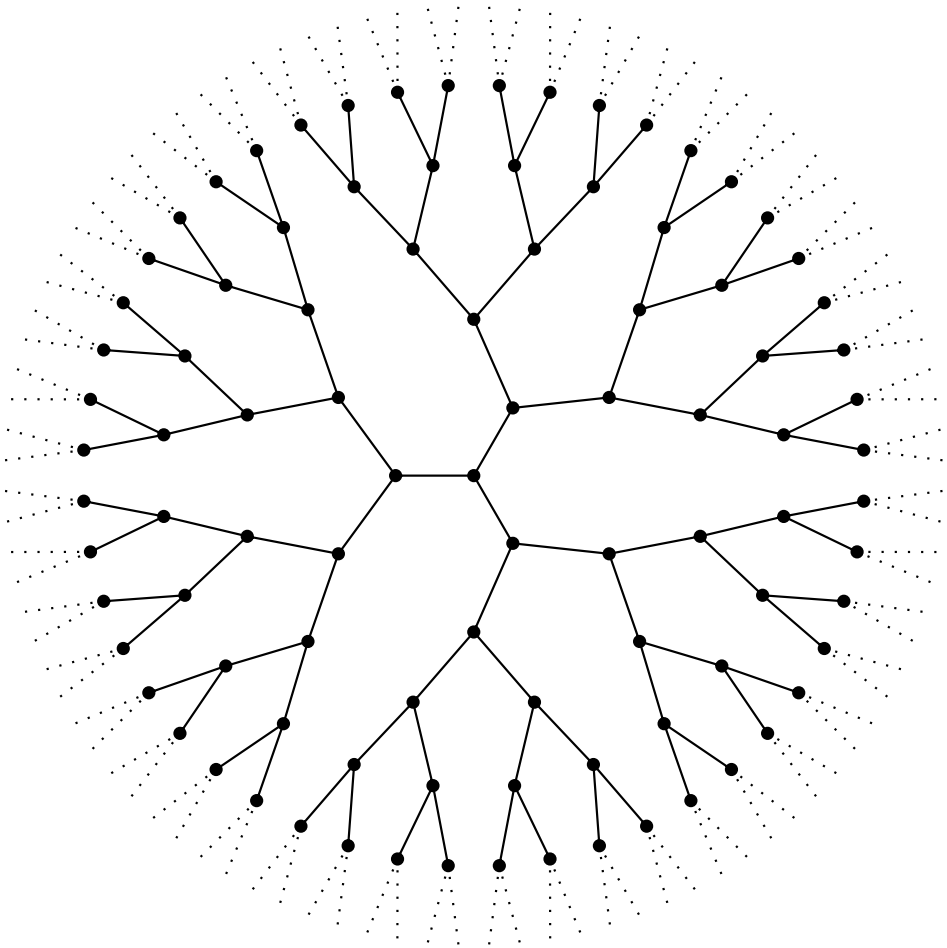
$$\Downarrow$$

There are exactly $\ell + 1$ cyclic subgroups $H \subset E$ of order ℓ:

$$\langle P + Q \rangle, \langle P + 2Q \rangle, \ldots, \langle P \rangle, \langle Q \rangle$$

$$\Downarrow$$

There are exactly $\ell + 1$ distinct isogenies of degree ℓ.



(non-CM) 2-isogeny graph over $\mathbb{C}$

What happens over a finite field $\mathbb{F}_p$?

Rational isogenies ($\ell \neq p$)

In the algebraic closure $\bar{\mathbb{F}}_p$

$$E[\ell] = \langle P, Q \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

However, an isogeny is defined over $\mathbb{F}_p$ only if its kernel is Galois invariant.

Enter the Frobenius map

$$\begin{aligned} \pi : E &\longrightarrow E \\ (x, y) &\longmapsto (x^p, y^p) \end{aligned}$$

E is seen here as a curve over $\bar{\mathbb{F}}_p$.

The Frobenius action on $E[\ell]$

$$\begin{aligned} \pi(P) &= aP + bQ \\ \pi(Q) &= cP + dQ \end{aligned}$$

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$$\pi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod \ell$$

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$$\pi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod \ell$$

We identify $\pi | E[\ell]$ to a conjugacy class in $\mathrm{GL}(\mathbb{Z}/\ell\mathbb{Z})$.

What happens over a finite field $\mathbb{F}_p$?

Galois invariant subgroups of $E[\ell]$

=

eigenspaces of $\pi \in \mathrm{GL}(\mathbb{Z}/\ell\mathbb{Z})$

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rational isogenies of degree ℓ

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How many Galois invariant subgroups?

- $\pi|E[\ell] \sim \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ $\rightarrow \ell + 1$ isogenies
- $\pi|E[\ell] \sim \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ with $\lambda \neq \mu$ $\rightarrow$ two isogenies
- $\pi|E[\ell] \sim \left(\begin{smallmatrix} \lambda & * \\ 0 & \lambda \end{smallmatrix}\right)$ $\rightarrow$ one isogeny
- $\pi|E[\ell]$ is not diagonalizable $\rightarrow$ no isogeny

Weil pairing

Let $(N, p) = 1$, fix any basis $E[N] = \langle R, S \rangle$. For any points $P, Q \in E[N]$

$$P = aR + bS$$
$$Q = cR + dS$$

the form $\det_N(P, Q) = \det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = ad - bc \in \mathbb{Z}/N\mathbb{Z}$
is bilinear, non-degenerate, and independent from the choice of basis.

Theorem

Let $E/\mathbb{F}_q$ be a curve, there exists a Galois invariant bilinear map

$$e_N : E[N] \times E[N] \longrightarrow \mu_N \subset \bar{\mathbb{F}}_q,$$

called the Weil pairing of order N, and a primitive N-th root of unity $\zeta \in \bar{\mathbb{F}}_q$ such that

$$e_N(P, Q) = \zeta^{\det_N(P,Q)}.$$

The degree k of the smallest extension such that $\zeta \in \mathbb{F}_{q^k}$ is called the embedding degree of the pairing.

Weil pairing and isogenies

Note

The Weil pairing is Galois invariant $\Leftrightarrow$ $\det(\pi | E[N]) = q$.

Theorem

Let $\phi : E \to E'$ be an isogeny and $\hat{\phi} : E' \to E$ its dual.
Let e_N be the Weil pairing of E and e'_N that of E'. Then, for

$$e_N(P, \hat{\phi}(Q)) = e'_N(\phi(P), Q),$$

for any $P \in E[N]$ and $Q \in E'[N]$.

Corollary

$$e'_N(\phi(P), \phi(Q)) = e_N(P, Q)^{\deg \phi}.$$

From local to global

Theorem (Hasse)

Let E be defined over a finite field $\mathbb{F}_q$. Its Frobenius map π satisfies a quadratic equation

$$\pi^2 - t\pi + q = 0$$

for some $|t| \leq 2\sqrt{q}$, called the trace of π. The trace t is coprime to q if and only if E is ordinary.

Endomorphisms

An isogeny $E \to E$ is also called an endomorphism. Examples:

- scalar multiplication $[n]$,
- Frobenius map π.

With addition and composition, the endomorphisms form a ring $\mathrm{End}(E)$.

The endomorphism ring

Theorem (Deuring)

Let E be an ordinary elliptic curve defined over a finite field $\mathbb{F}_q$.
Let π be its Frobenius endomorphism, and $D_\pi = t^2 - 4q < 0$ the discriminant of its minimal polynomial.
Then $\mathrm{End}(E)$ is isomorphic to an order $\mathcal{O}$ of the quadratic imaginary field $\mathbb{Q}(\sqrt{D_\pi})$.[a]

[a] An order is a subring that is a $\mathbb{Z}$-module of rank 2 (equiv., a 2-dimensional $\mathbb{R}$-lattice).

In this case, we say that E has complex multiplication (CM) by $\mathcal{O}$.

Theorem (Serre-Tate)

CM elliptic curves E, E' are isogenous iff $\mathrm{End}(E) \otimes \mathbb{Q} \simeq \mathrm{End}(E') \otimes \mathbb{Q}$.

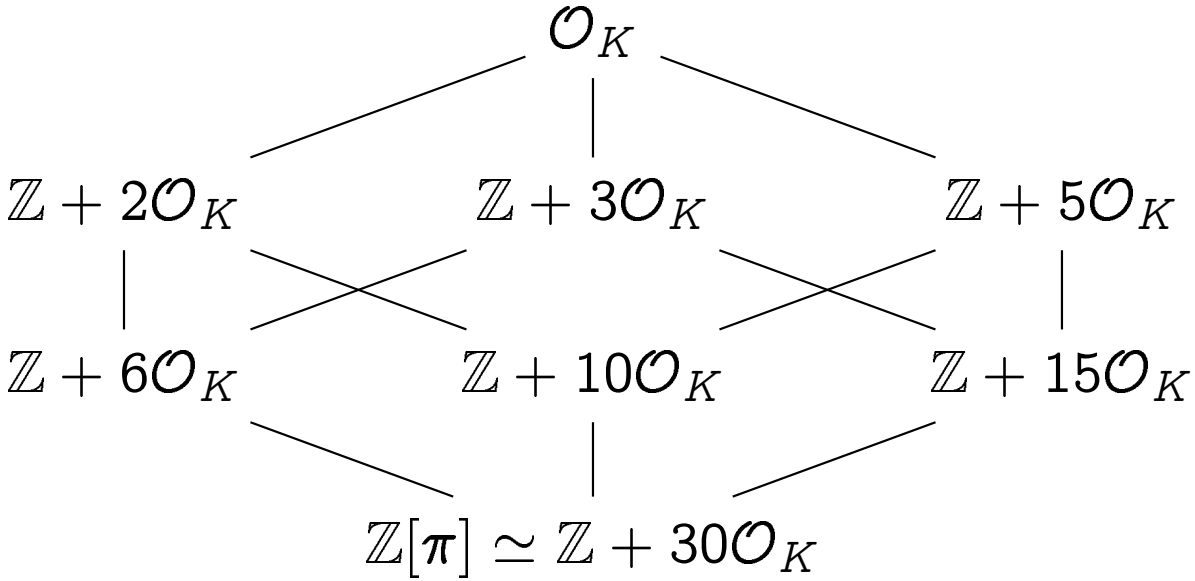
Corollary: $E/\mathbb{F}_p$ and $E'/\mathbb{F}_p$ are isogenous iff $\#E(\mathbb{F}_p) = \#E'(\mathbb{F}_p)$.

Endomorphism rings of ordinary curves

Classifying quadratic orders

Let K be a quadratic number field, and let $\mathcal{O}_K$ be its ring of integers.

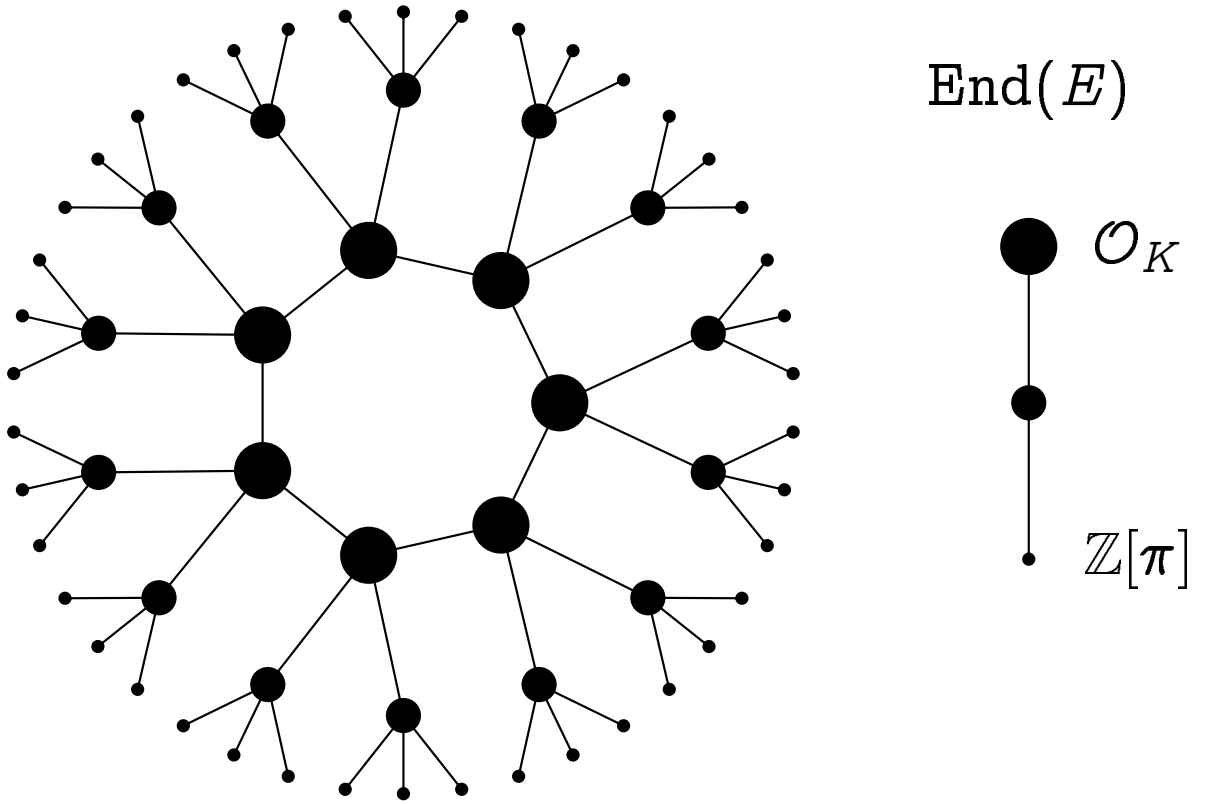
- Any order $\mathcal{O} \subset K$ can be written as $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ for an integer f, called the conductor of $\mathcal{O}$, denoted by $[\mathcal{O}_K : \mathcal{O}]$.
- If D_K is the discriminant of K, the discriminant of $\mathcal{O}$ is $f^2 D_K$.
- If $\mathcal{O}, \mathcal{O}'$ are two orders with discriminants D, D', then $\mathcal{O} \subset \mathcal{O}'$ iff $D' \mid D$.



Volcanology (Kohel 1996)

Let E, E' be curves with respective endomorphism rings $\mathcal{O}, \mathcal{O}' \subset K$.
Let $\phi : E \to E'$ be an isogeny of prime degree ℓ, then:

if $\mathcal{O} = \mathcal{O}'$, ϕ is horizontal;
if $[\mathcal{O}' : \mathcal{O}] = \ell$, ϕ is ascending;
if $[\mathcal{O} : \mathcal{O}'] = \ell$, ϕ is descending.

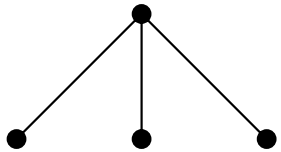


Ordinary isogeny volcano of degree $\ell = 3$.

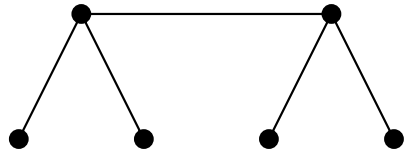
Volcanology (Kohel 1996)

Let E be ordinary,
$\mathrm{End}(E) \subset K$.

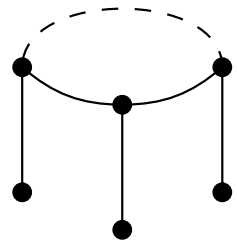
$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.



$\left(\frac{D_K}{\ell}\right) = -1$



$\left(\frac{D_K}{\ell}\right) = 0$



$\left(\frac{D_K}{\ell}\right) = +1$

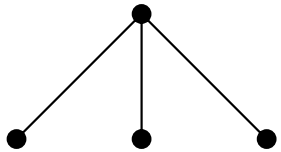
		Horizontal	**Ascending**	**Descending**
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		$\ell - \left(\frac{D_K}{\ell}\right)$
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	ℓ
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

Volcanology (Kohel 1996)

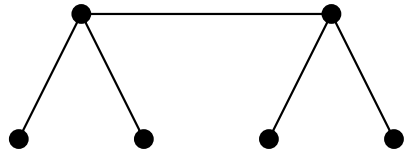
Let E be ordinary,
$\mathrm{End}(E) \subset K$.

$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.

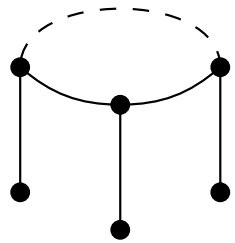
Height $= v_\ell([\mathcal{O}_K : \mathbb{Z}[\pi]])$.



$\left(\frac{D_K}{\ell}\right) = -1$



$\left(\frac{D_K}{\ell}\right) = 0$



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$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

Volcanology (Kohel 1996)

Let E be ordinary,
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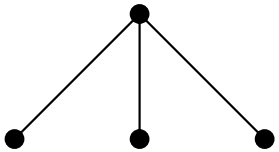
How large is the crater?



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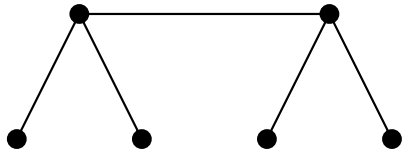
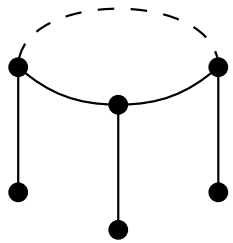


$\left(\frac{D_K}{\ell}\right) = 0$



$\left(\frac{D_K}{\ell}\right) = +1$

		Horizontal	Ascending	Descending
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$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

How large is the crater of a volcano?

Let $\mathrm{End}(E) = \mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$. Define

- $\mathcal{I}(\mathcal{O})$, the group of invertible fractional ideals,
- $\mathcal{P}(\mathcal{O})$, the group of principal ideals,

The class group

The class group of $\mathcal{O}$ is

$$\mathrm{Cl}(\mathcal{O}) = \mathcal{I}(\mathcal{O})/\mathcal{P}(O).$$

- It is a finite abelian group.
- Its order $h(\mathcal{O})$ is called the class number of $\mathcal{O}$.
- It arises as the Galois group of an abelian extension of $\mathbb{Q}(\sqrt{-D})$.

Complex multiplication

The $\mathfrak{a}$-torsion

- Let $\mathfrak{a} \subset \mathcal{O}$ be an (integral invertible) ideal of $\mathcal{O}$;
- Let $E[\mathfrak{a}]$ be the subgroup of E annihilated by $\mathfrak{a}$:

$$E[\mathfrak{a}] = \{P \in E \mid \alpha(P) = 0 \text{ for all } \alpha \in \mathfrak{a}\};$$

- Let $\phi : E \to E_{\mathfrak{a}}$, where $E_{\mathfrak{a}} = E/E[\mathfrak{a}]$.

Then $\mathrm{End}(E_{\mathfrak{a}}) = \mathcal{O}$ (i.e., ϕ is horizontal).

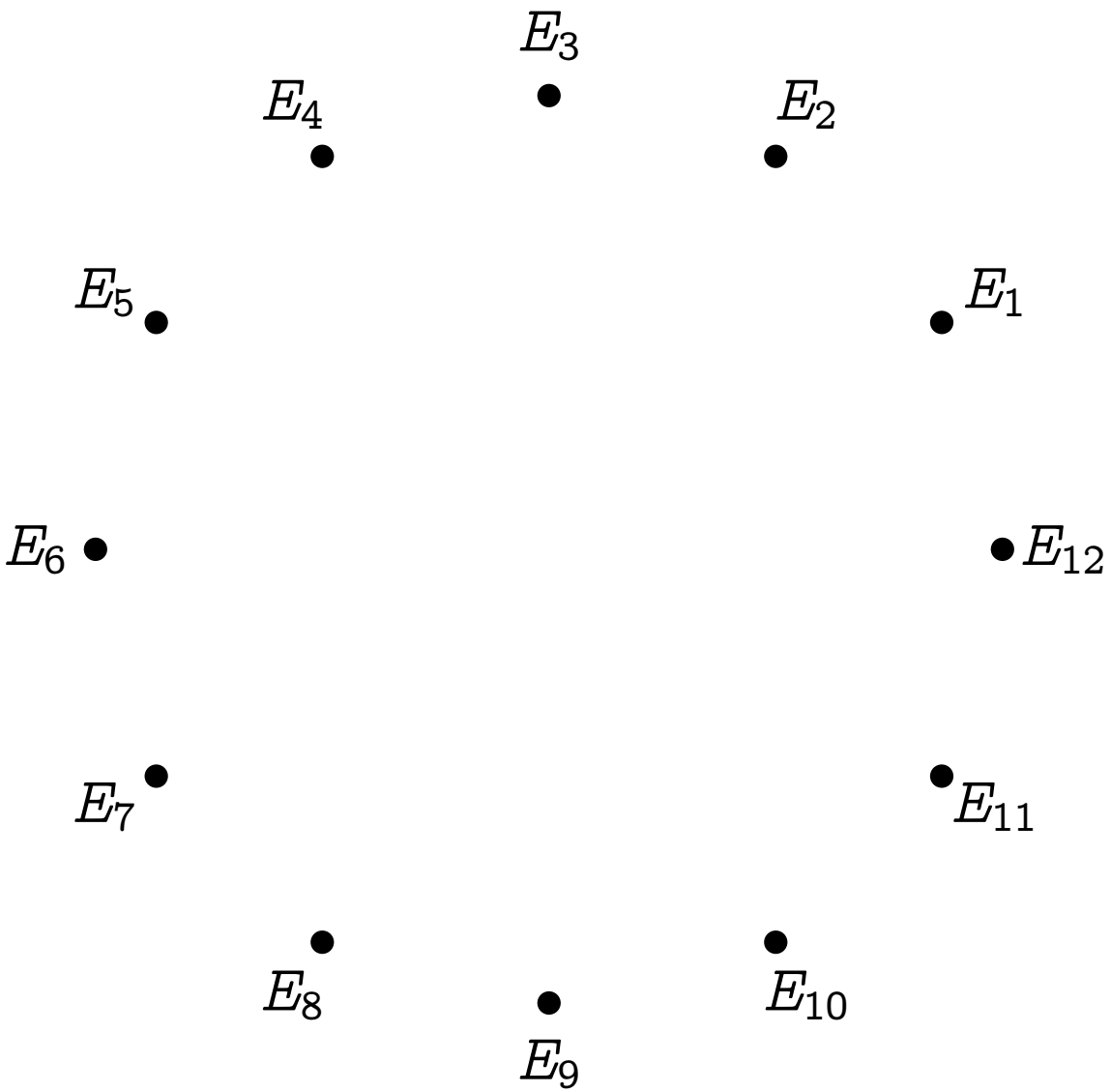
Theorem (Complex multiplication)

The action on the set of elliptic curves with complex multiplication by $\mathcal{O}$ defined by $\mathfrak{a} * j(E) = j(E_{\mathfrak{a}})$ factors through $\mathrm{Cl}(\mathcal{O})$, is faithful and transitive.

Corollary

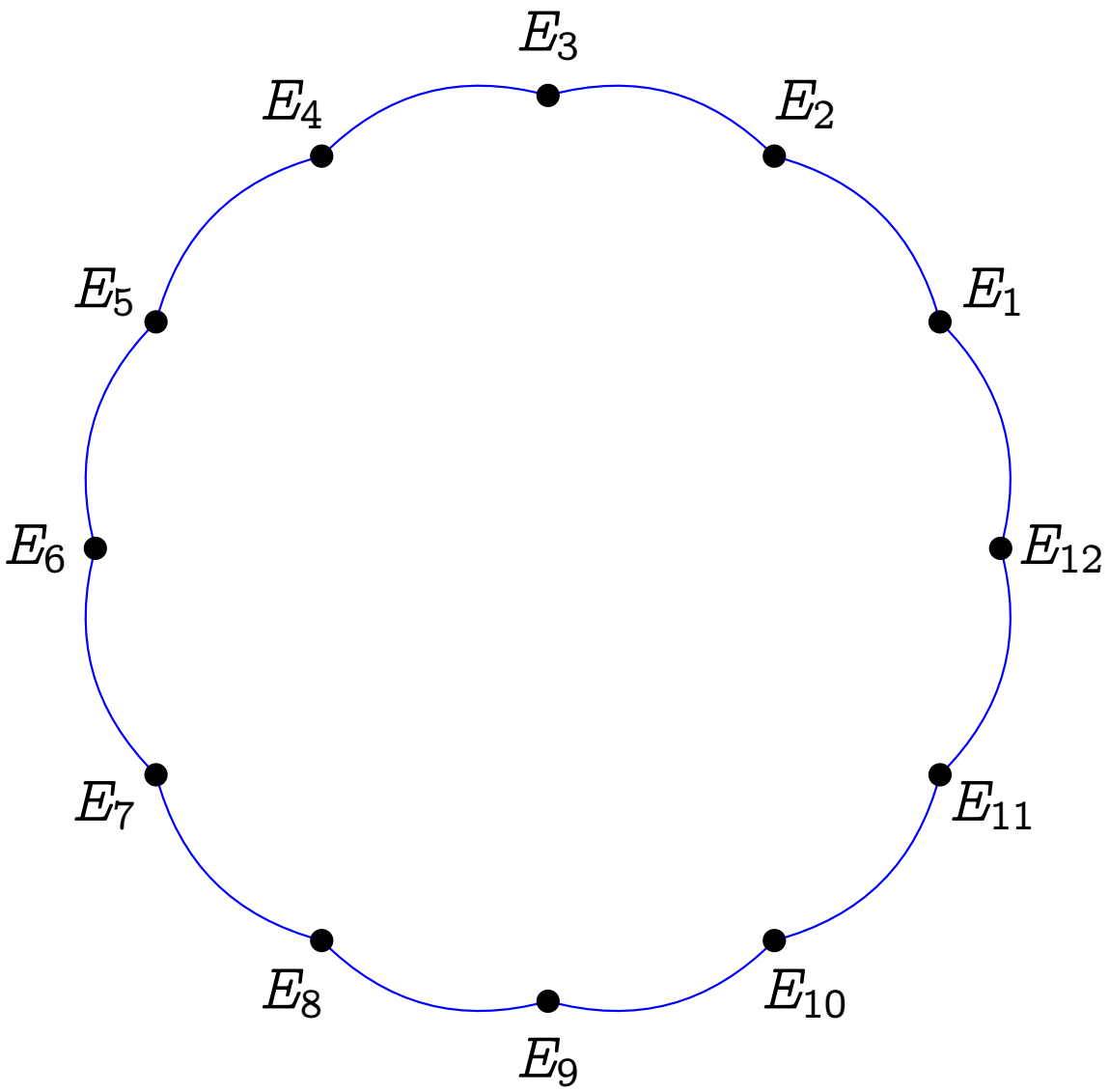
Let $\mathrm{End}(E)$ have discriminant D. Assume that $\left(\frac{D}{\ell}\right) = 1$, then E is on a crater of size N of an ℓ-volcano, and $N \mid h(\mathrm{End}(E))$

Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

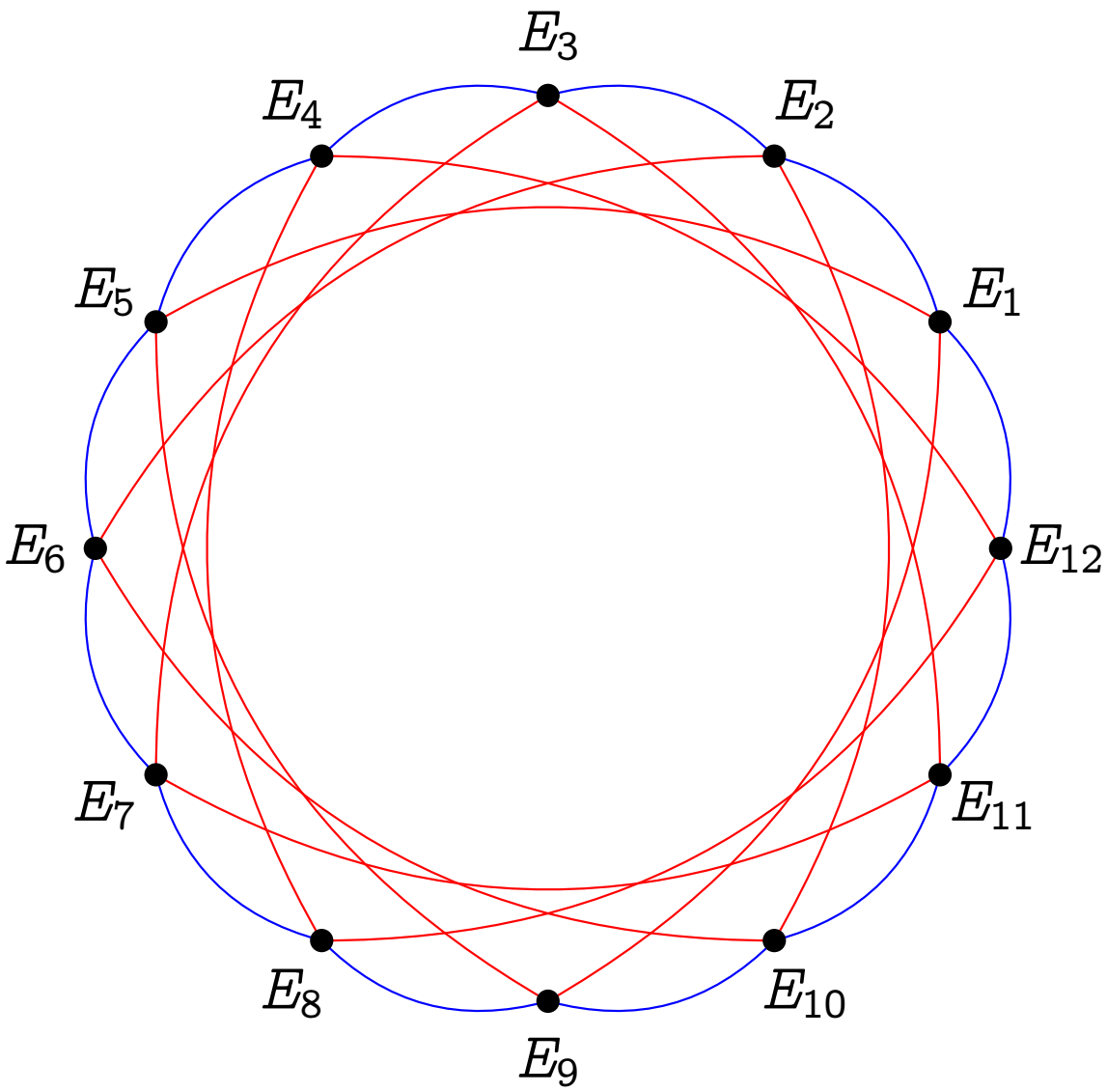
Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).
Edges are horizontal isogenies of bounded prime degree.

— degree 2

Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).
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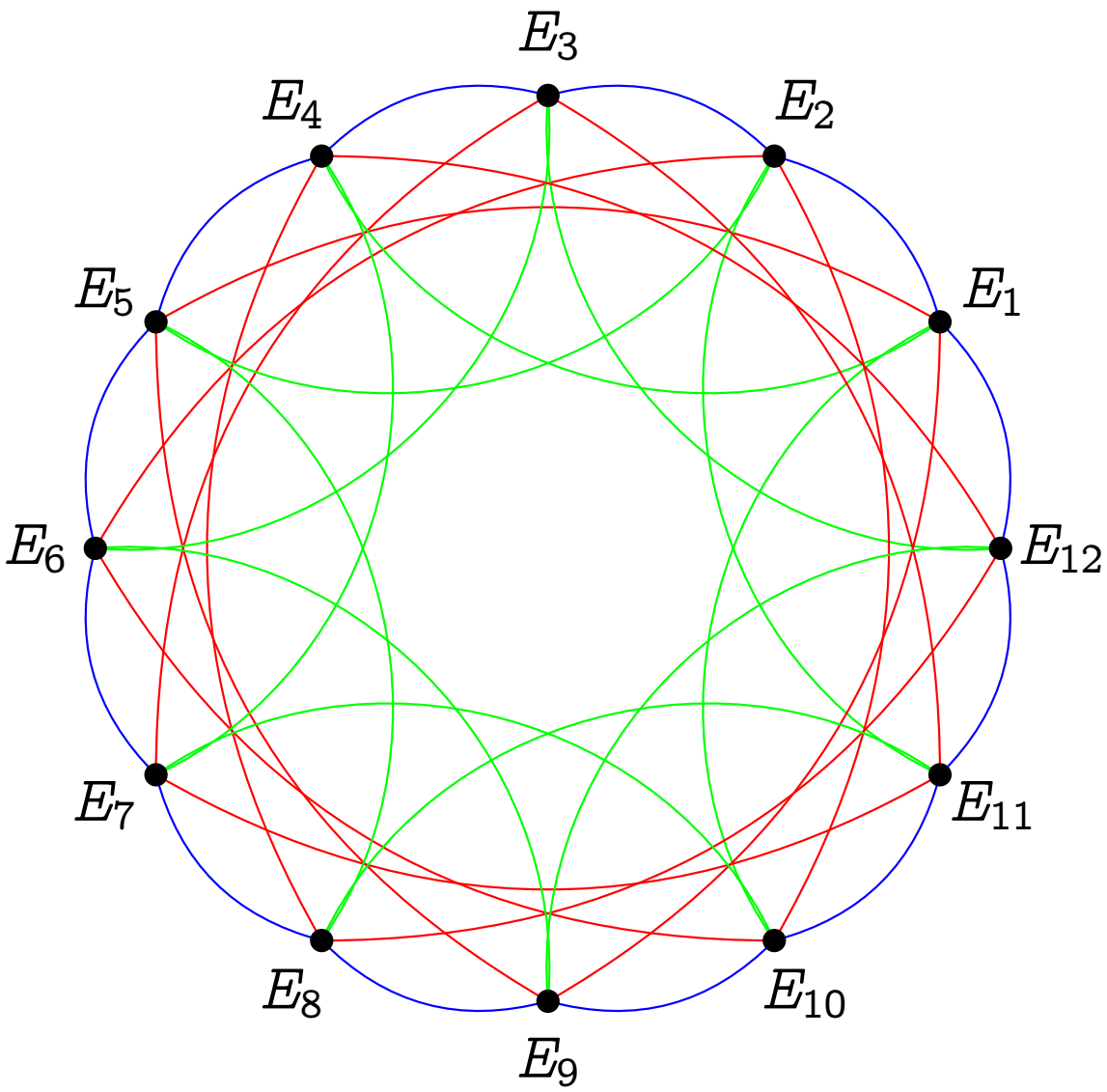
— degree 2

— degree 3

Complex multiplication graphs



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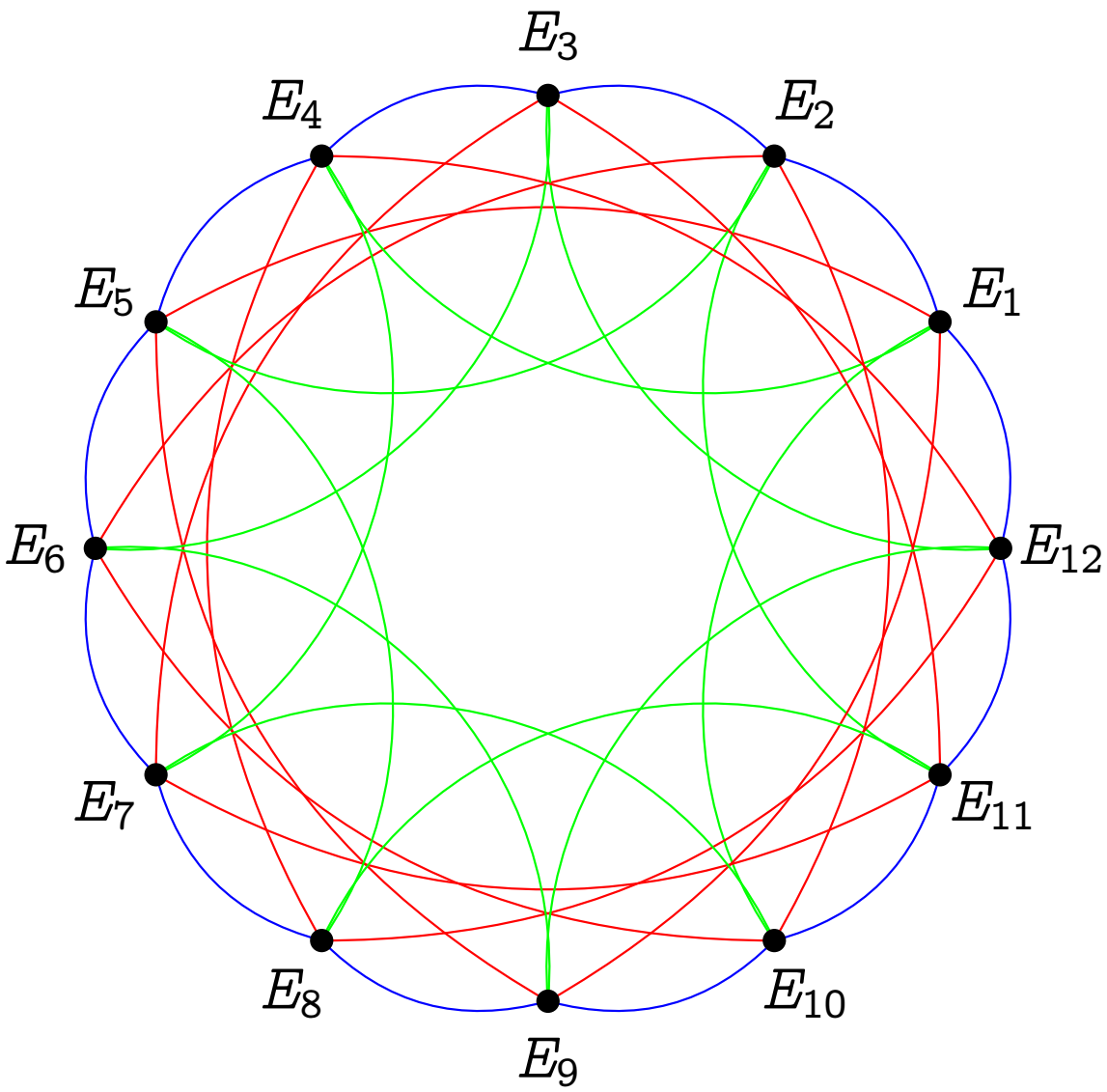
Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

— degree 5

Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).
Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

— degree 5

Isomorphic to a Cayley graph of $\mathrm{Cl}(\mathcal{O}_K)$.

Supersingular endomorphisms

Recall, a curve E over a field $\mathbb{F}_q$ of characteristic p is supersingular iff

$$\pi^2 - t\pi + q = 0$$

with $t = 0 \mod p$.

Case: $t = 0 \quad \Rightarrow \quad D_\pi = -4q$

- Only possibility for $E/\mathbb{F}_p$,
- $E/\mathbb{F}_p$ has CM by an order of $\mathbb{Q}(\sqrt{-p})$, similar to the ordinary case.

Case: $t = \pm 2\sqrt{q} \quad \Rightarrow \quad D_\pi = 0$

- General case for $E/\mathbb{F}_q$, when q is an even power.
- $\pi = \pm\sqrt{q}$, hence no complex multiplication.

We will ignore marginal cases: $t = \pm\sqrt{q}, \pm\sqrt{2q}, \pm\sqrt{3q}$.

Supersingular complex multiplication

Let $E/\mathbb{F}_p$ be a supersingular curve, then $\pi^2 = -p$, and

$$\pi = \begin{pmatrix} \sqrt{-p} & 0 \\ 0 & -\sqrt{-p} \end{pmatrix} \mod \ell$$

for any ℓ s.t. $\left(\frac{-p}{\ell}\right) = 1$.

Theorem (Delfs and Galbraith 2016)

Let $\mathrm{End}_{\mathbb{F}_p}(E)$ denote the ring of $\mathbb{F}_p$-rational endomorphisms of E. Then

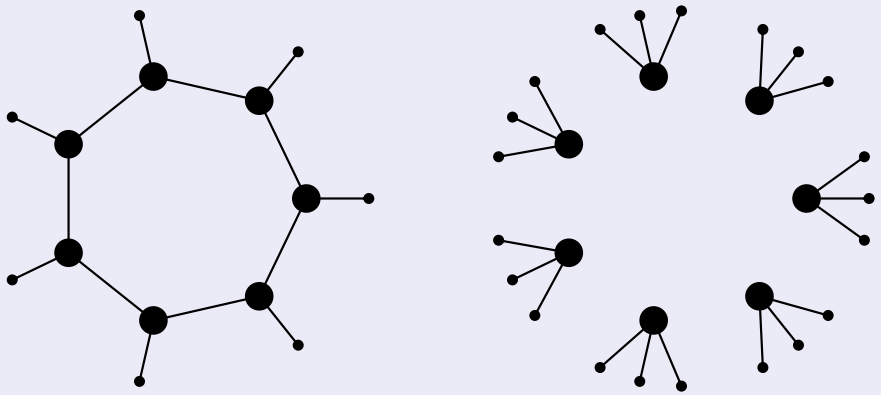
$$\mathbb{Z}[\pi] \subset \mathrm{End}_{\mathbb{F}_p}(E) \subset \mathbb{Q}(\sqrt{-p}).$$

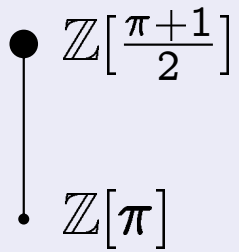
Orders of $\mathbb{Q}(\sqrt{-p})$

- If $p = 1 \mod 4$, then $\mathbb{Z}[\pi]$ is the maximal order.
- If $p = -1 \mod 4$, then $\mathbb{Z}[\frac{\pi+1}{2}]$ is the maximal order, and $[\mathbb{Z}[\frac{\pi+1}{2}] : \mathbb{Z}[\pi]] = 2$.

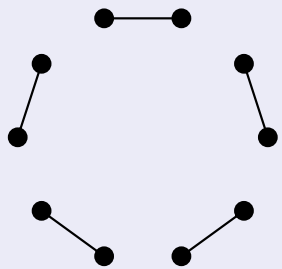
Supersingular CM graphs

2-volcanoes, $p = -1 \bmod 4$





2-graphs, $p = 1 \bmod 4$



- $\mathbb{Z}[\pi]$

All other ℓ-graphs are cycles of horizontal isogenies iff $\left(\frac{-p}{\ell}\right) = 1$.

The full endomorphism ring

Theorem (Deuring)

Let E be a supersingular elliptic curve, then

- E is isomorphic to a curve defined over $\mathbb{F}_{p^2}$;
- Every isogeny of E is defined over $\mathbb{F}_{p^2}$;
- Every endomorphism of E is defined over $\mathbb{F}_{p^2}$;
- $\mathrm{End}(E)$ is isomorphic to a maximal order in a quaternion algebra ramified at p and ∞.

In particular:

- If E is defined over $\mathbb{F}_p$, then $\mathrm{End}_{\mathbb{F}_p}(E)$ is strictly contained in $\mathrm{End}(E)$.
- Some endomorphisms do not commute!

An example

The curve of j-invariant 1728

$$E : y^2 = x^3 + x$$

is supersingular over $\mathbb{F}_p$ iff $p = -1 \mod 4$.

Endomorphisms

$\mathrm{End}(E) = \mathbb{Z}\langle \iota, \pi \rangle$, with:

- π the Frobenius endomorphism, s.t. $\pi^2 = -p$;
- ι the map

$$\iota(x, y) = (-x, iy),$$

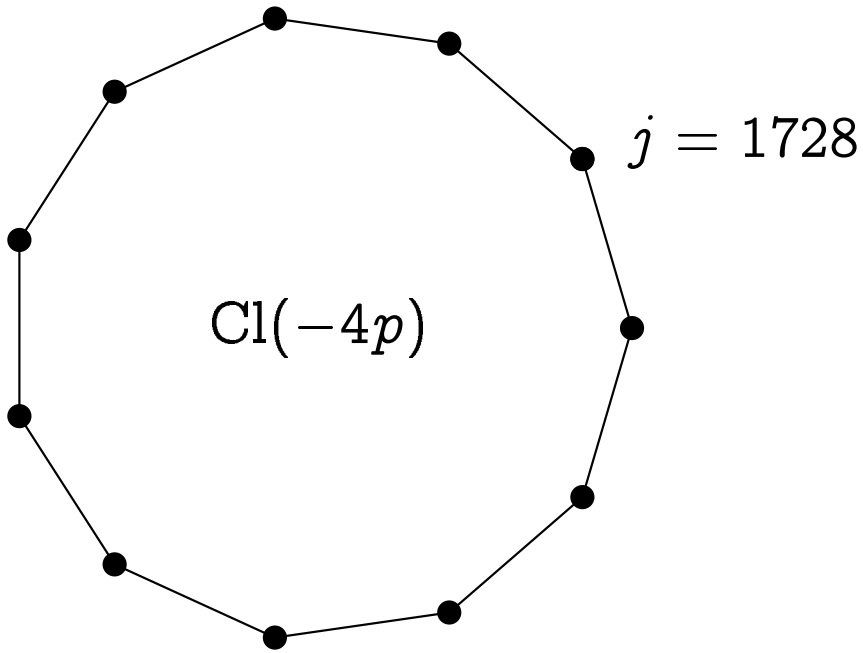
where $i \in \mathbb{F}_{p^2}$ is a 4-th root of unity. Clearly, $\iota^2 = -1$.

And $\iota\pi = -\pi\iota$.

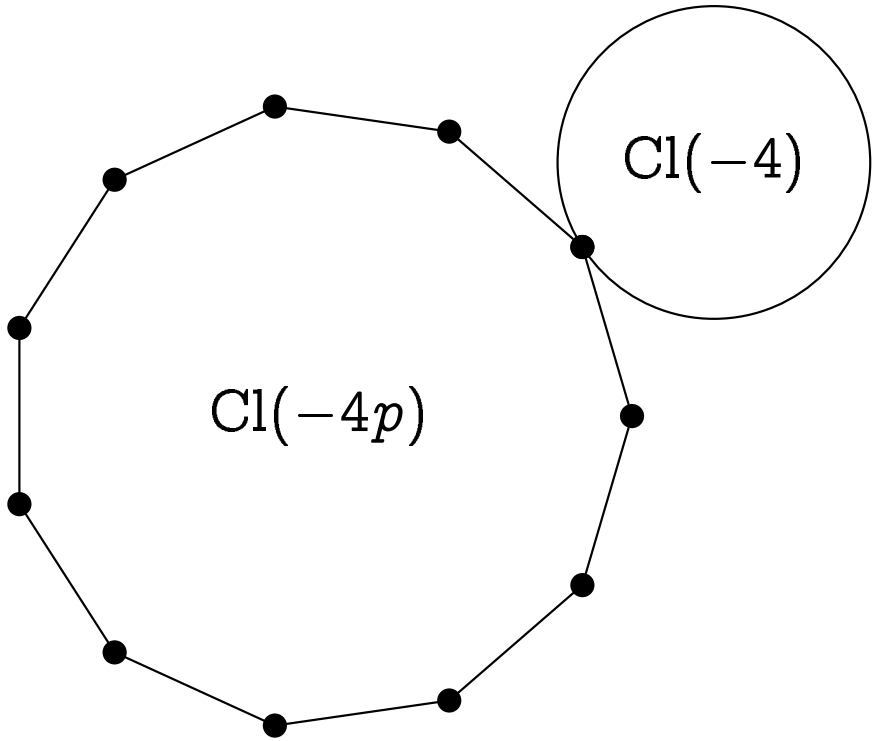
Class group action party

- $j = 1728$

Class group action party



Class group action party



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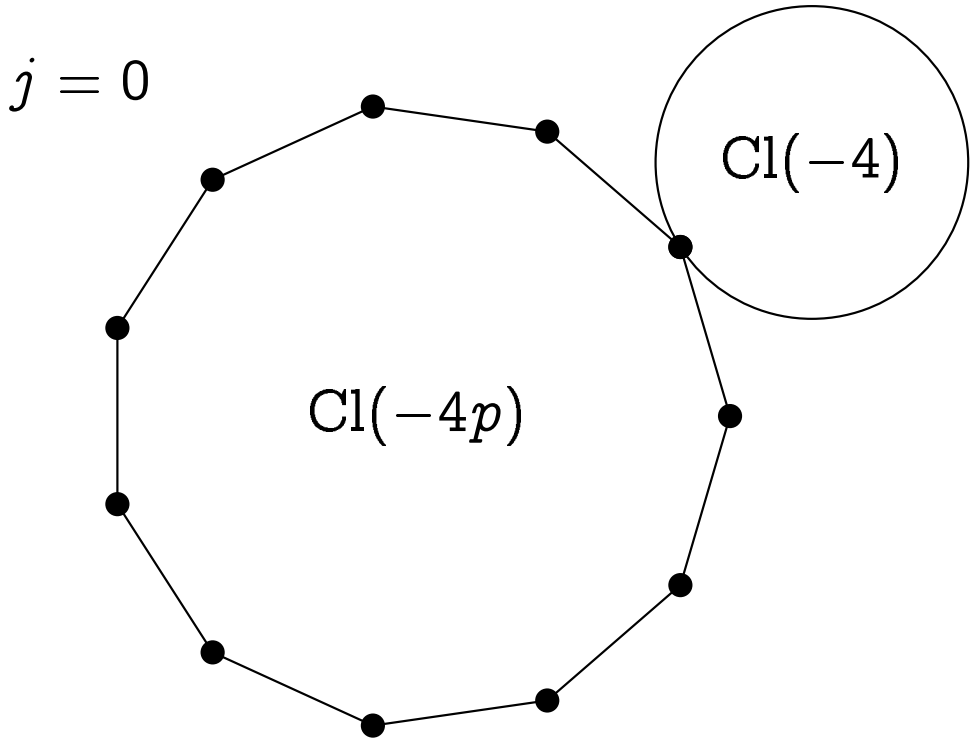

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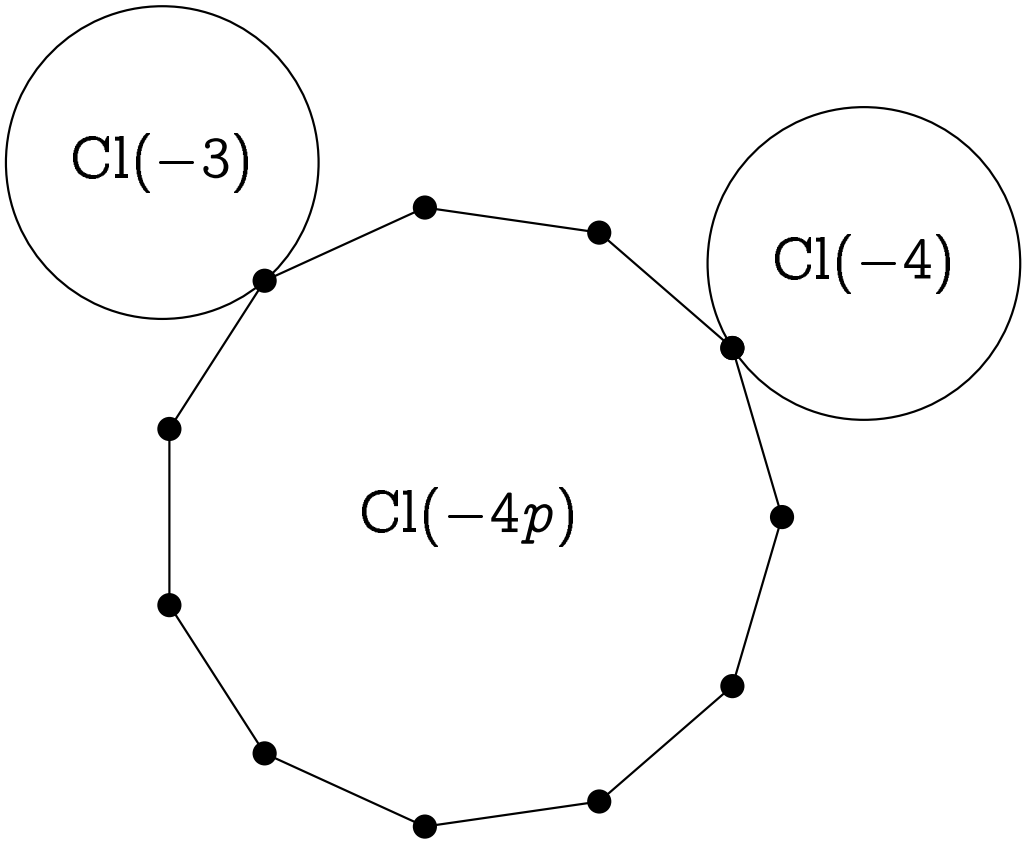
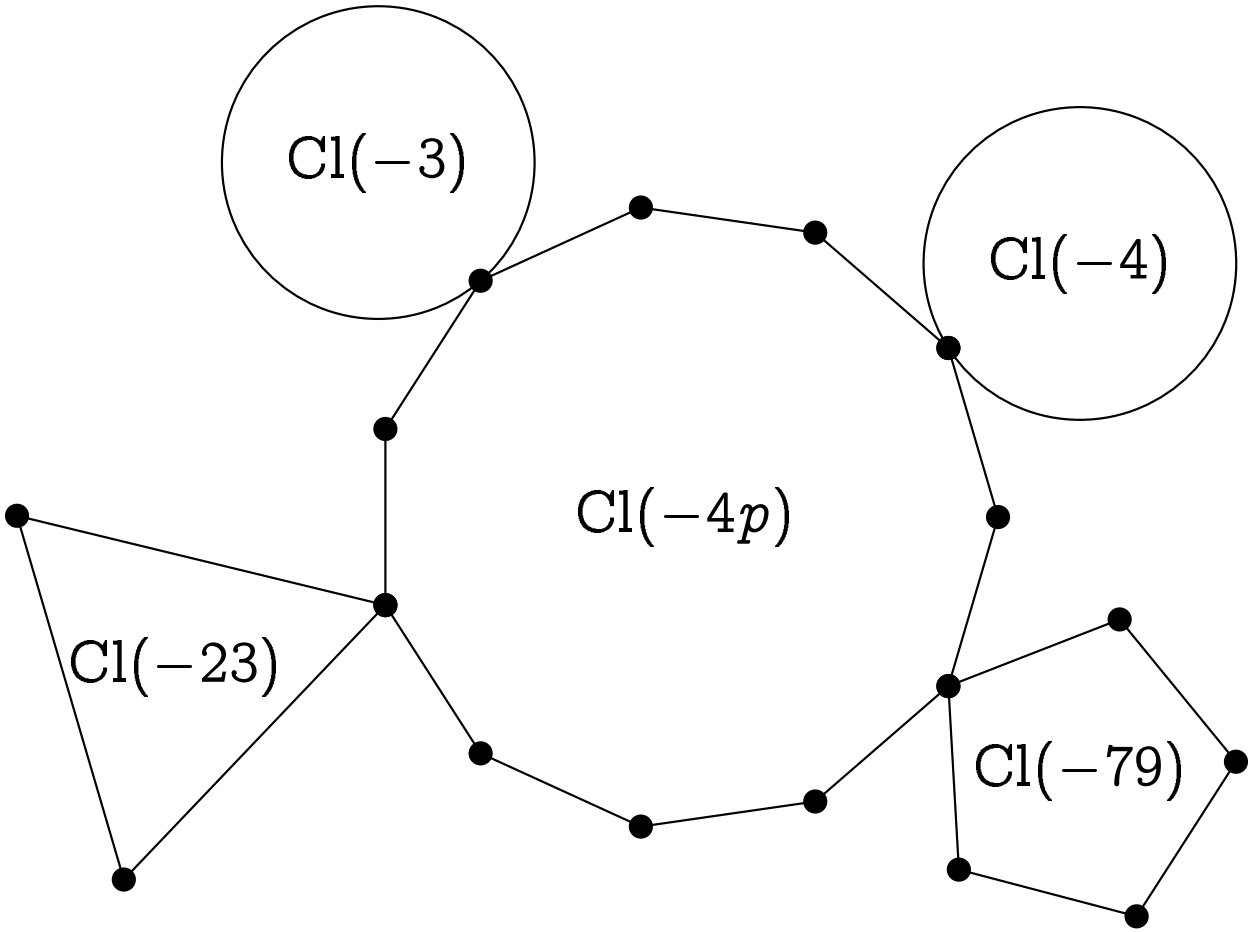


Class group action party



Quaternion algebra?! WTF?[2]

The quaternion algebra $B_{p,\infty}$ is:

- A 4-dimensional $\mathbb{Q}$-vector space with basis $(1, i, j, k)$;
- A non-commutative division algebra[1] $B_{p,\infty} = \mathbb{Q}\langle i, j\rangle$ with the relations:
$$i^2 = a, \quad j^2 = -p, \quad ij = -ji = k,$$
for some $a < 0$ (depending on p).
- All elements of $B_{p,\infty}$ are quadratic algebraic numbers;
- $B_{p,\infty} \otimes \mathbb{Q}_\ell \simeq \mathcal{M}_{2\times 2}(\mathbb{Q}_\ell)$ for all $\ell \neq p$.
I.e., endomorphisms restricted to $E[\ell^e]$ are just 2×2 matrices $\bmod \ell^e$.
- $B_{p,\infty} \otimes \mathbb{R} \simeq \mathcal{M}_{2\times 2}(\mathbb{R})$.

[1] All elements have inverses.

[2] What The Field?

Supersingular graphs

- Quaternion algebras have many maximal orders.
- For every maximal order type of $B_{p,\infty}$ there are 1 or 2 curves over $\mathbb{F}_{p^2}$ having endomorphism ring isomorphic to it.
- There is a unique isogeny class of supersingular curves over $\bar{\mathbb{F}}_p$ of size $\approx p/12$.
- Left ideals act on the set of maximal orders like isogenies.
- The graph of ℓ-isogenies is $(\ell+1)$-regular.

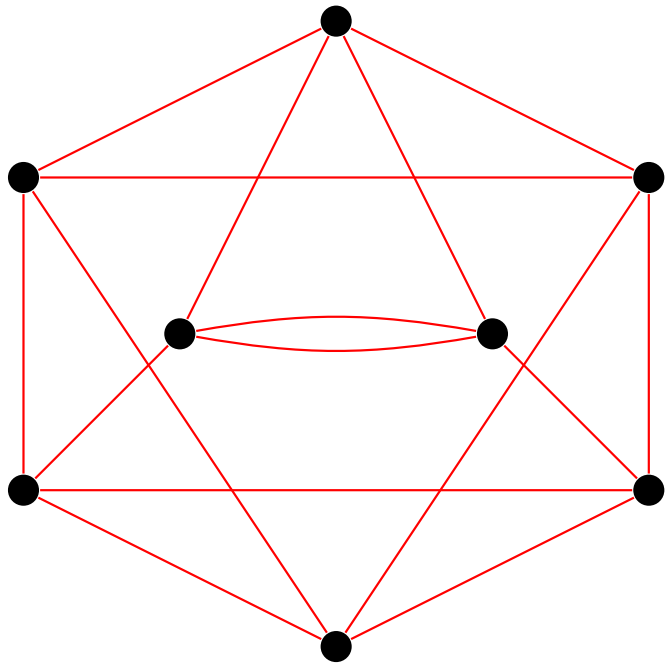


Figure: 3-isogeny graph on $\mathbb{F}_{97^2}$.

Graphs lexicon

Degree:	Number of (outgoing/ingoing) edges.
k-regular:	All vertices have degree k.
Connected:	There is a path between any two vertices.
Distance:	The length of the shortest path between two vertices.
Diamater:	The longest distance between two vertices.
$\lambda_1 \geq \cdots \geq \lambda_n$:	The (ordered) eigenvalues of the adjacency matrix.

Expander graphs

Proposition

If G is a k-regular graph, its largest and smallest eigenvalues satisfy

$$k = \lambda_1 \geq \lambda_n \geq -k.$$

Expander families

An infinite family of connected k-regular graphs on n vertices is an expander family if there exists an $\epsilon > 0$ such that all non-trivial eigenvalues satisfy $|\lambda| \leq (1 - \epsilon)k$ for n large enough.

- Expander graphs have short diameter ($O(\log n)$);
- Random walks mix rapidly (after $O(\log n)$ steps, the induced distribution on the vertices is close to uniform).

Expander graphs from isogenies

Theorem (Pizer 1990, 1998)

Let ℓ be fixed. The family of graphs of supersingular curves over $\mathbb{F}_{p^2}$ with ℓ-isogenies, as $p \to \infty$, is an expander family[a].

[a] Even better, it has the Ramanujan property.

Theorem (Jao, Miller, and Venkatesan 2009)

Let $\mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$ be an order in a quadratic imaginary field. The graphs of all curves over $\mathbb{F}_q$ with complex multiplication by $\mathcal{O}$, with isogenies of prime degree bounded[a] by $(\log q)^{2+\delta}$, are expanders.

[a] May contain traces of GRH.

Overview

1. Isogeny graphs
   - Elliptic Curves
   - Isogenies
   - Isogeny graphs
   - Endomorphism rings
   - Ordinary graphs
   - Supersingular graphs

2. Cryptography
   - Isogeny walks and Hash functions
   - Pairing verification and Verifiable Delay Functions
   - Key exchange
   - Open Problems

History of isogeny-based cryptography

1996 Couveignes introduces the Hard Homogeneous Spaces (HHS). His work stays unpublished for 10 years.

2006 Rostovtsev & Stolbunov independently rediscover Couveignes ideas, suggest isogeny-based Diffie–Hellman as a quantum-resistant primitive.

2007 Charles, Goren & Lauter propose supersingular 2-isogeny graphs as a foundation for a "provably secure" hash function.

2011-2012 D., Jao & Plût introduce SIDH, an efficient post-quantum key exchange inspired by Couveignes, Rostovtsev, Stolbunov, Charles, Goren, Lauter.

2017 SIDH is submitted to the NIST competition (with the name SIKE, only isogeny-based candidate).

2018 Castryck, Lange, Martindale, Panny & Renes publish an efficient variant of HHS named CSIDH.

2019 New isogeny protocols: Signatures, Verifiable Delay Functions, …

Computing Isogenies

Vélu's formulas

Input: A subgroup $H \subset E$,

Output: The isogeny $\phi : E \to E/H$.

Complexity: $O(\ell)$ — Vélu 1971, ...

Why?
- Evaluate isogeny on points $P \in E$;
- Walk in isogeny graphs.

Computing Isogenies

Vélu's formulas

Input: A subgroup $H \subset E$,

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Complexity: $O(\ell)$ — Vélu 1971, ...

Why?
- Evaluate isogeny on points $P \in E$;
- Walk in isogeny graphs.

Explicit Isogeny Problem

Input: Curve E, (prime) integer ℓ

Output: All subgroups $H \subset E$ of order ℓ.

Complexity: $\tilde{\mathcal{O}}(\ell^2)$ — Elkies 1992

Why?
- List all isogenies of given degree;
- Count points of elliptic curves;
- Compute endomorphism rings of elliptic curves;
- Walk in isogeny graphs.

Computing Isogenies

Explicit Isogeny Problem (2)

Input: Curves E, E', isogenous of degree ℓ.

Output: The isogeny $\phi : E \to E'$ of degree ℓ.

Complexity: $O(\ell^2)$ — Elkies 1992; Couveignes 1996; Lercier and Sirvent 2008; De Feo 2011; De Feo, Hugounenq, Plût, and Schost 2016; Lairez and Vaccon 2016, ...

Why?

- Count points of elliptic curves.

Computing Isogenies

Explicit Isogeny Problem (2)

Input: Curves E, E', isogenous of degree ℓ.

Output: The isogeny $\phi : E \to E'$ of degree ℓ.

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Why?
- Count points of elliptic curves.

Isogeny Walk Problem

Input: Isogenous curves E, E'.

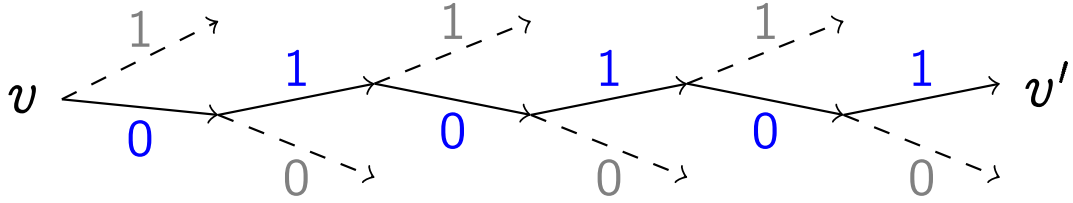
Output: An isogeny $\phi : E \to E'$ of smooth degree.

Complexity: Generically hard — Galbraith, Hess, and Nigel P. Smart 2002, ...

Why?
- Cryptanalysis (ECC);
- Foundational problem for isogeny-based cryptography.

Random walks and hash functions (circa 2006)

Any expander graph gives rise to a hash function.



$$H(010101) = v'$$

- Fix a starting vertex v;
- The value to be hashed determines a random path to v';
- v' is the hash.

(Denis X. Charles, Kristin E. Lauter, and Goren 2009) hash function (CGL)

- Use the expander graph of supersingular 2-isogenies;
- $\left.\begin{array}{r}\text{Collision resistance}\\ \text{2nd preimage resistance}\end{array}\right\} =$ hardness of finding cycles in the graph;
- Preimage resistance = hardness of finding a path from v to v'.

Hardness of CGL

Finding cycles

- Analogous to finding endomorphisms…
- …very bad idea to start from a curve with known endomorphism ring!
- Translation algortihm: elements of $B_{p,\infty} \leftrightarrow$ isogeny loops
  Doable in $\mathrm{polylog}(p)$.[a]

[a] Kohel, K. Lauter, Petit, and Tignol 2014; Eisenträger, Hallgren, K. Lauter, Morrison, and Petit 2018.

Finding paths $E \to E'$

- Analogous to finding connecting ideals between two maximal orders $\mathcal{O}, \mathcal{O}'$ (i.e. a left ideal $I \subset \mathcal{O}$ that is a right ideal of $\mathcal{O}'$).
- Poly-time equivalent to computing $\mathrm{End}(E)$ and $\mathrm{End}(E')$.[a]
- Best known algorithm to compute $\mathrm{End}(E)$ takes $\mathrm{poly}(p)$.[b]

[a] Eisenträger, Hallgren, K. Lauter, Morrison, and Petit 2018.
[b] Kohel 1996; Cerviño 2004.

Kohel, K. Lauter, Petit, and Tignol 2014 (KLPT)

Input:
- Maximal order $\mathcal{O} \subset B_{p,\infty}$ and associated curve E,
- Left ideal $I \subset \mathcal{O}$.

Output:
- Maximal order $\mathcal{O}' \subset B_{p,\infty}$ s.t. I connects $\mathcal{O}$ to $\mathcal{O}'$,
- Equivalent ideal J (i.e., also connecting $\mathcal{O}$ to $\mathcal{O}'$) of [smooth/power-smooth] norm.
- Isogeny walk associated to J.

- Complexity: $\mathrm{polylog}(p)$,
- Output size: $\mathrm{polylog}(p)$,
- Useful for:
  - "Shortening" isogeny walks (see VDFs),
  - "Reducing" isogeny walks (see Signatures),

  when these start from a curve with known endomorphism ring!
  (think $j = 0, 1728$ and other curves with small CM discriminant)

Sampling supersingular curves

How to sample:

- A supersingular curve $E/\mathbb{F}_p$?
- A supersingular curve $E/\mathbb{F}_{p^2}$?

Random walks

- Start from a supersingular curve E_0 with small CM discriminant (e.g.: $j = 1728$),
- Do a random walk $E_0 \to E$ until reaching the mixing bound ($O(\log(p))$ steps).

Problem: the random walk reveals $\mathtt{End}(E)$ via the KLPT algorithm.

Open problem

Give an algorithm to sample (uniformly) random supersingular curves in a way that does not reveal the endomorphism ring.

Boneh, Lynn, and Shacham 2004 signatures (BLS)

Setup:
- Elliptic curve $E/\mathbb{F}_p$, s.t $N|\#E(\mathbb{F}_p)$ for a large prime N,
- (Weil) pairing $e_N : E[N] \times E[N] \to \mathbb{F}_{p^k}$ for some small embedding degree k,
- A decomposition $E[N] = X_1 \times X_2$, with $X_1 = \langle P \rangle$.
- A hash function $H : \{0,1\}^* \to X_2$.

Private key: $s \in \mathbb{Z}/N\mathbb{Z}$.

Public key: sP.

Sign: $m \mapsto sH(m)$.

Verifiy: $e_N(P, sH(m)) = e_N(sP, H(m))$.

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{[s] \times 1} & X_1 \times X_2 \\ {\scriptstyle 1 \times [s]}\downarrow & & \downarrow{\scriptstyle e_N} \\ X_1 \times X_2 & \xrightarrow[e_N]{} & \mathbb{F}_{p^k} \end{array}$$

US patent 8,250,367[3]

Signatures from isogenies + pairings

- Replace the secret $[s] : E \to E$ with an isogeny $\phi : E \to E'$;
- Define decompositions

$$E[N] = X_1 \times X_2, \qquad E'[N] = Y_1 \times Y_2,$$

  s.t. $\phi(X_1) = Y_1$ and $\phi(X_2) = Y_2$;
- Define a hash function $H : \{0,1\}^* \to Y_2$.

$$\begin{array}{ccc} X_1 \times Y_2 & \xrightarrow{\phi \times 1} & Y_1 \times Y_2 \\ \Big\downarrow{\scriptstyle 1 \times \hat{\phi}} & & \Big\downarrow{\scriptstyle e'_N} \\ X_1 \times X_2 & \xrightarrow[e_N]{} & \mathbb{F}_{p^k} \end{array}$$

[3] Broker, Denis X Charles, and Kristin E Lauter 2012.

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Useless, but nice!

[3] Broker, Denis X Charles, and Kristin E Lauter 2012.

Verifiable Delay Functions

A Verifiable Delay Function (VDF) is a function $f : X \to Y$ s.t.:

- Evaluating f at random $x \in X$ is provably "slow" (e.g., $\mathrm{poly}(\#X)$),
- Given $x \in X$ and $y \in Y$,
  verifying that $f(x) = y$ can be done "fast" (e.g., $\mathrm{polylog}(\#X)$).

(non)-Example (time-lock puzzles)

- Take a trapdoor group G of (e.g., $G = \mathbb{Z}/N\mathbb{Z}$ with $N = pq$);
- Define $f : G \to G$ as $f(g) = g^{2^T}$:
  - Best algorithm if p, q known: compute $g^{2^T \bmod \varphi(pq)}$ — $\mathrm{polylog}(N)$
  - Best algorithm if p, q unknown: T squarings — $O(T)$

However, in VDFs we want to let anyone verify efficiently.

VDFs from groups of unknown order

Interactive verification protocol (Wesolowski 2019)

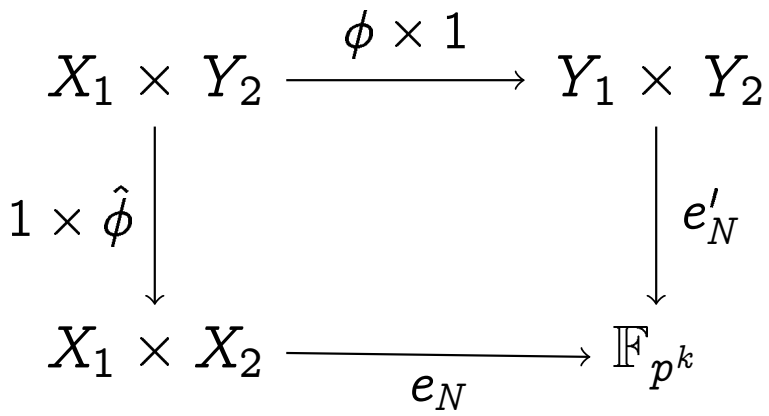
1. Verifier chooses a prime ℓ in a set of small primes $\mathcal{P}$;
2. Prover computes $2^T = a\ell + b$, sends g^{2^T}, g^a, b to verifier;
3. Verifier checks that
$$g^{2^T} = (g^a)^\ell g^b$$

Can be made non-interactive via Fiat-Shamir.

Candidate groups of unknown order:

- RSA groups $\mathbb{Z}/N\mathbb{Z}$, needs trusted third party to generate $N = pq$;
- Quadratic imaginary class groups $\mathrm{Cl}(-D)$ for large random discriminants $-D < 0$.

VDFs from isogenies and pairings



Setup:
- Supersingular curve $E/\mathbb{F}_p$ with (Weil) pairing e_N;
- Public isogeny $\phi : E \to E'$ of degree 2^T;
- The dual isogeny $\hat{\phi} : E' \to E$;
- A generator $\langle P \rangle = X_1 \subset E[N]$, compute $\phi(P)$.

Evaluate: On input a random $Q \in Y_2 \subset E'[N]$, compute $\hat{\phi}(Q)$.

Verify: Check that $e_N(P, \hat{\phi}(Q)) = e'_N(\phi(P), Q)$.

Security

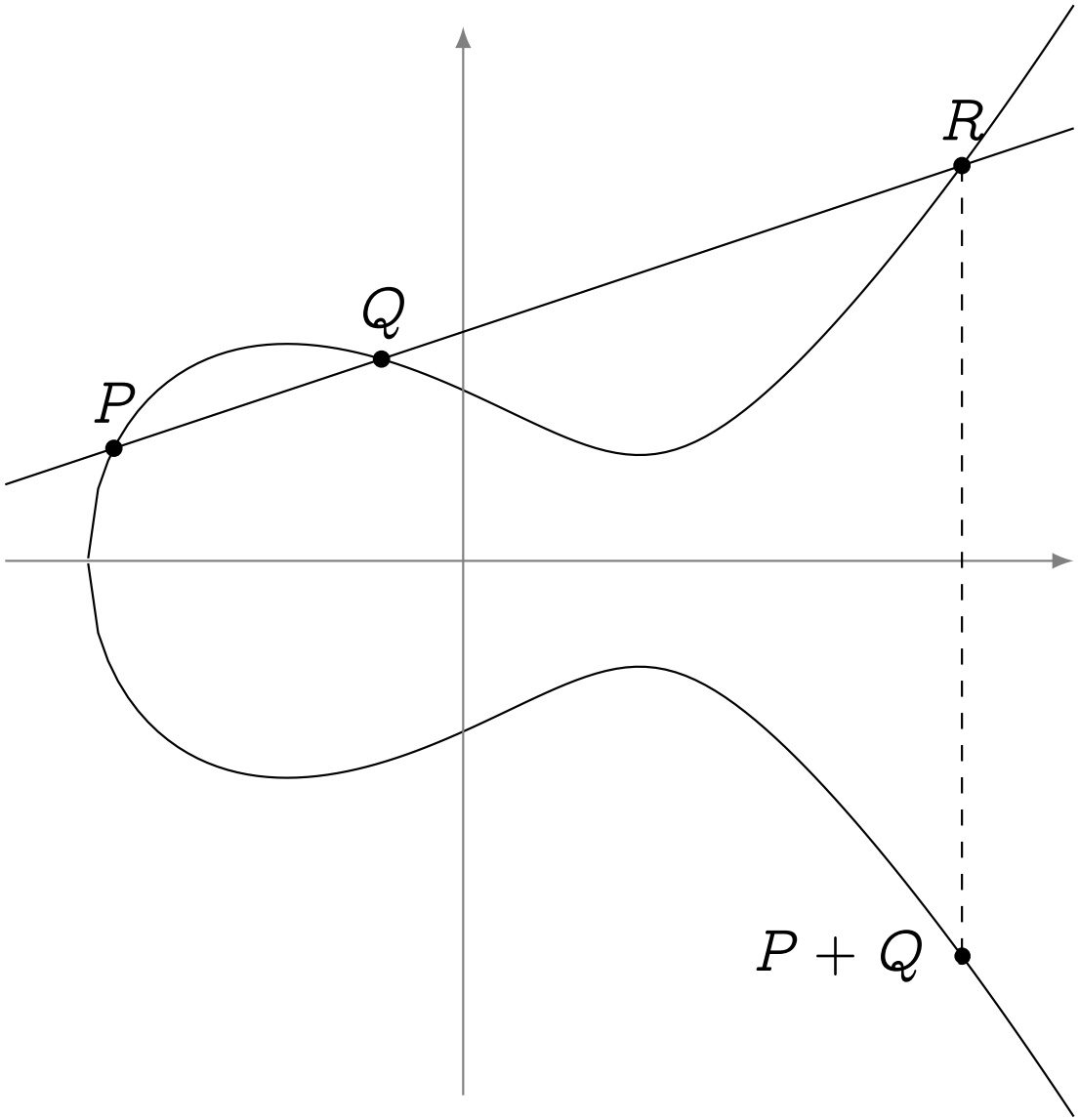
Obvious attack: Pairing inversion must be hard (non post-quantum).

Wanted: No better way to evaluate $\hat{\phi} : E' \to E$ than composing T degree 2 isogenies.

Shortcuts

- If we can find a shorter way from E to E', we can evaluate $\hat{\phi}$ faster.
- Shortcuts are easy to compute:
  - If the isogeny graph is small (excludes ordinary pairing friendly curves);
  - If $\mathrm{End}(E)$ or $\mathrm{End}(E')$ is known (via KLPT).
- Needed: choose $E/\mathbb{F}_p$ in a way that does not reveal $\mathrm{End}(E)$;
- Only known solution: let a trusted third party generate E.

Let's get back to Diffie-Hellman



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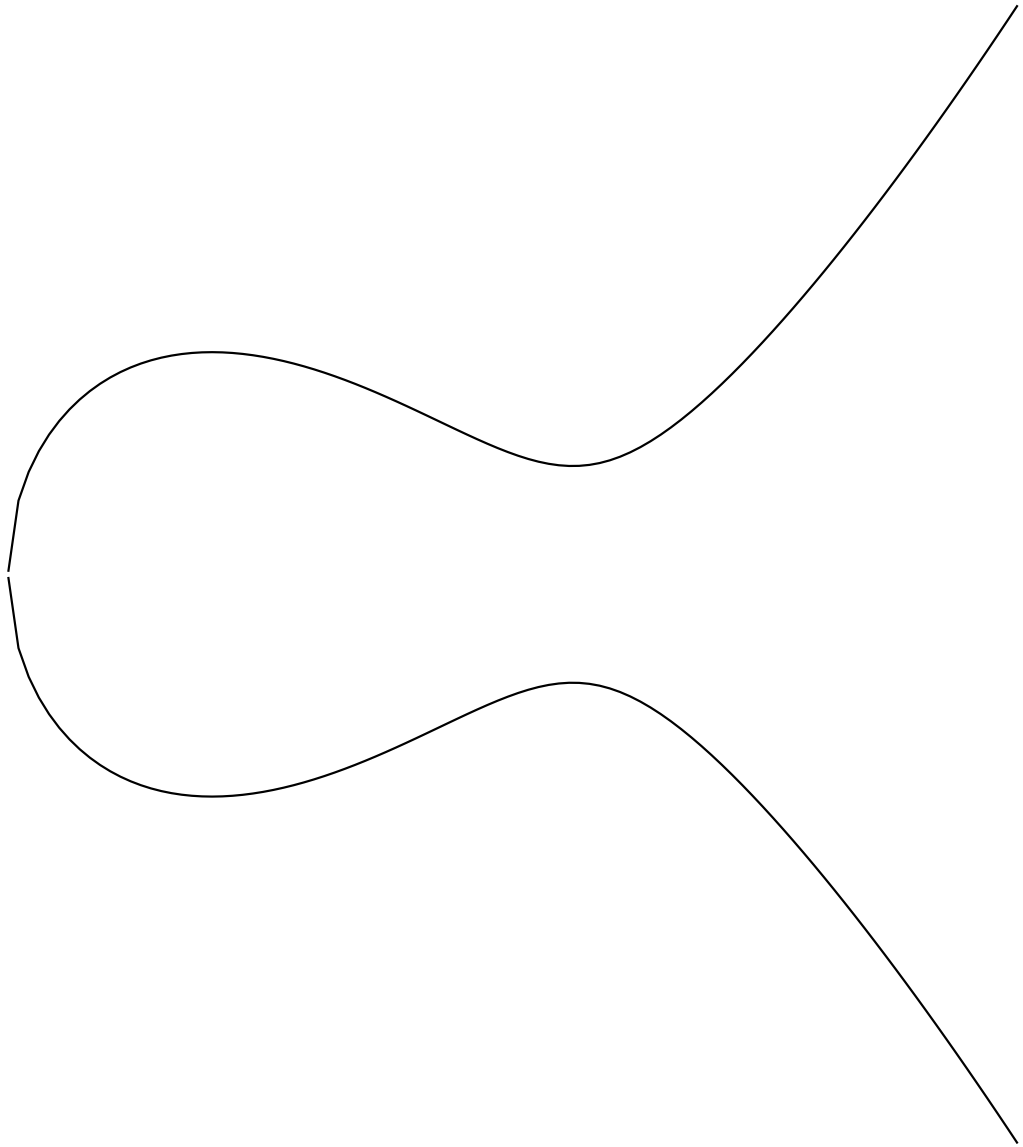

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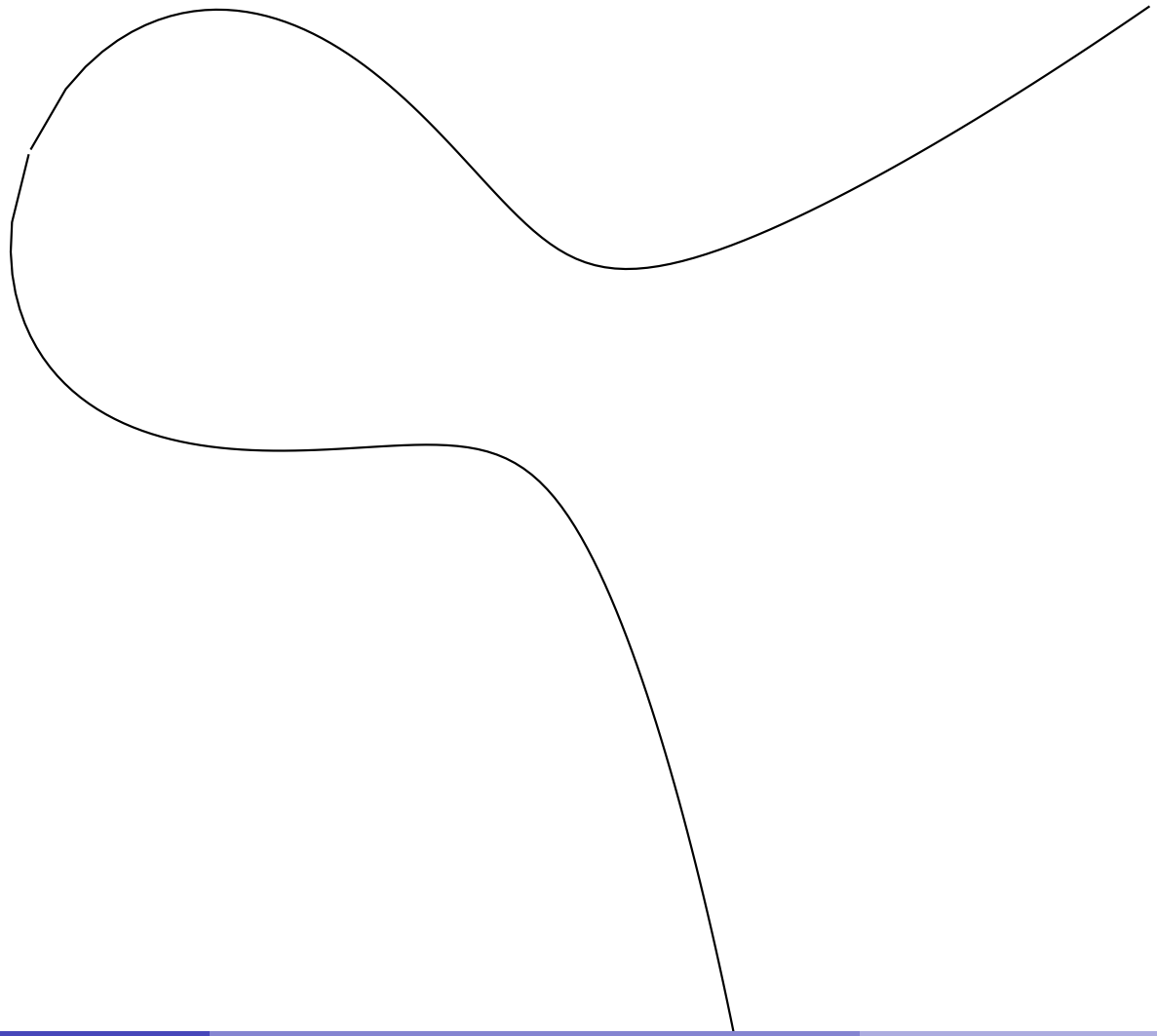

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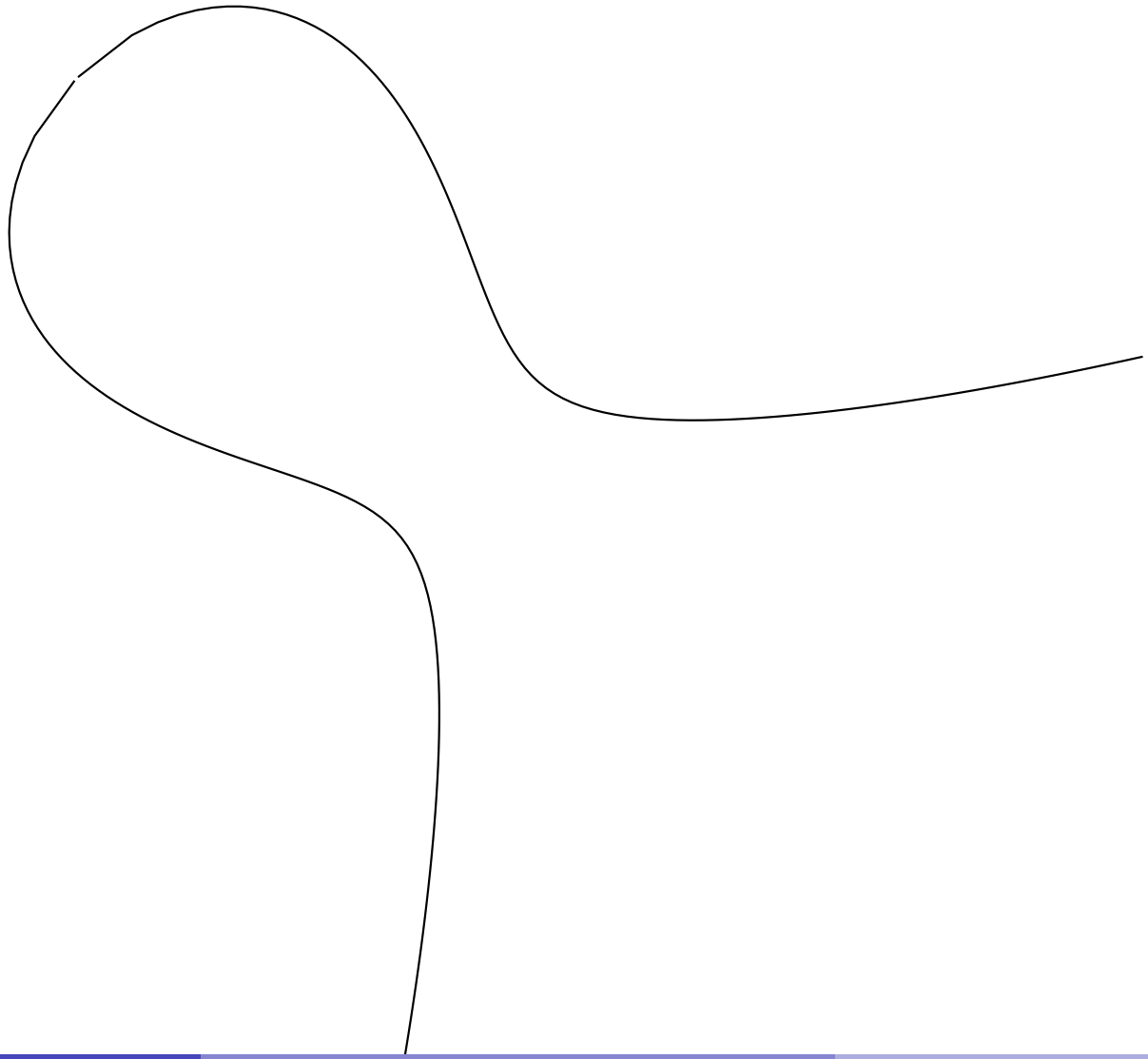

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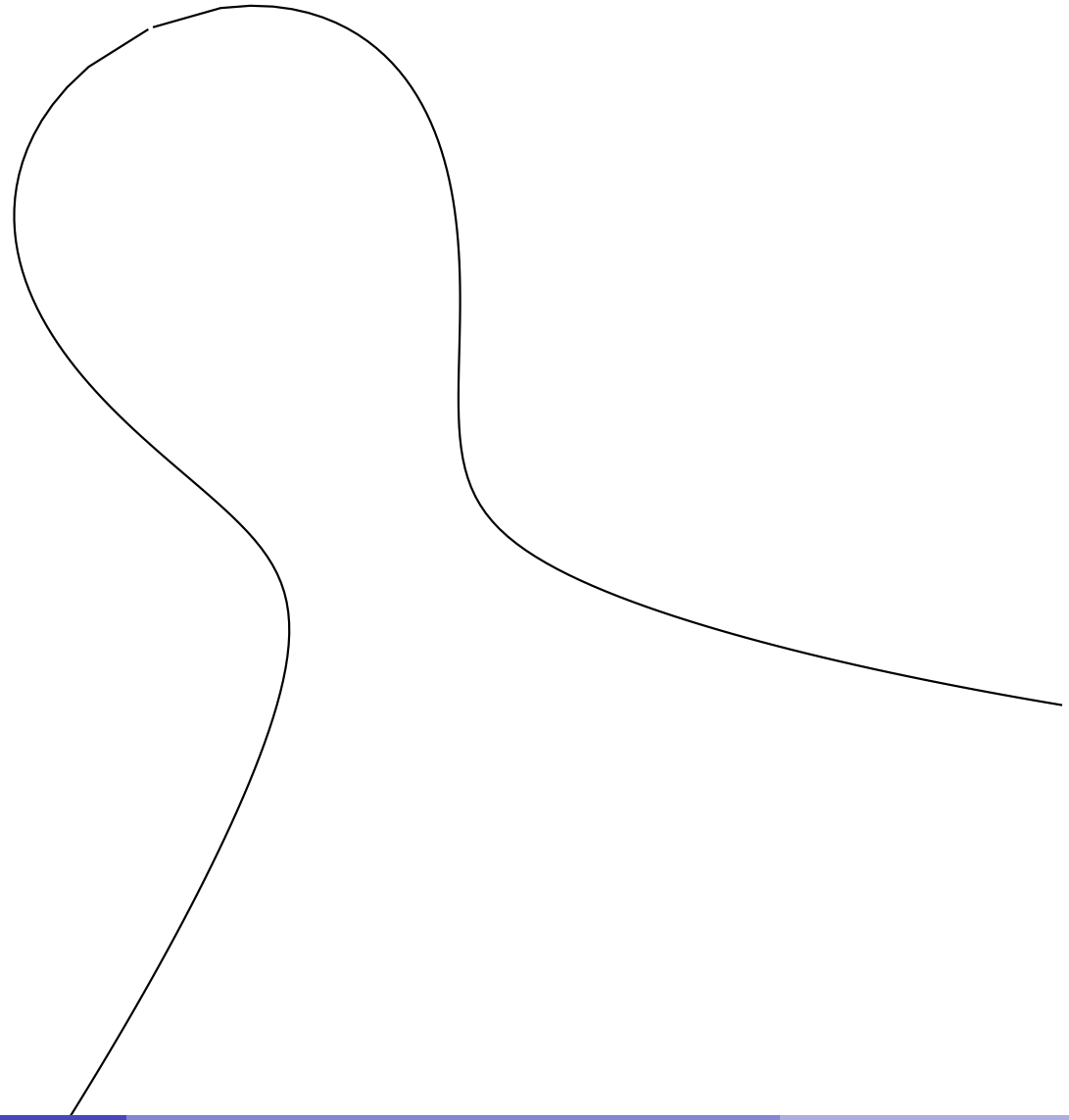

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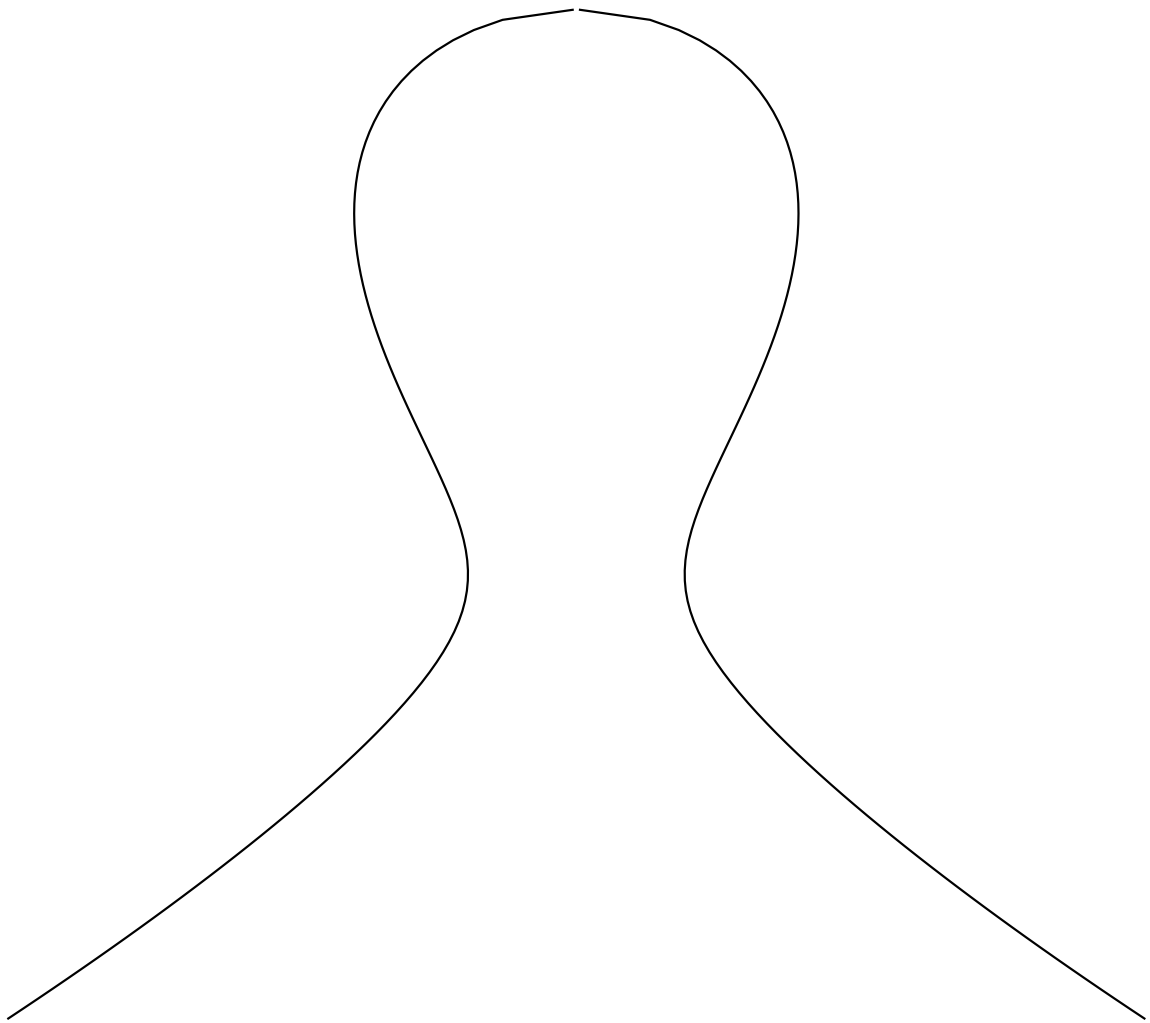
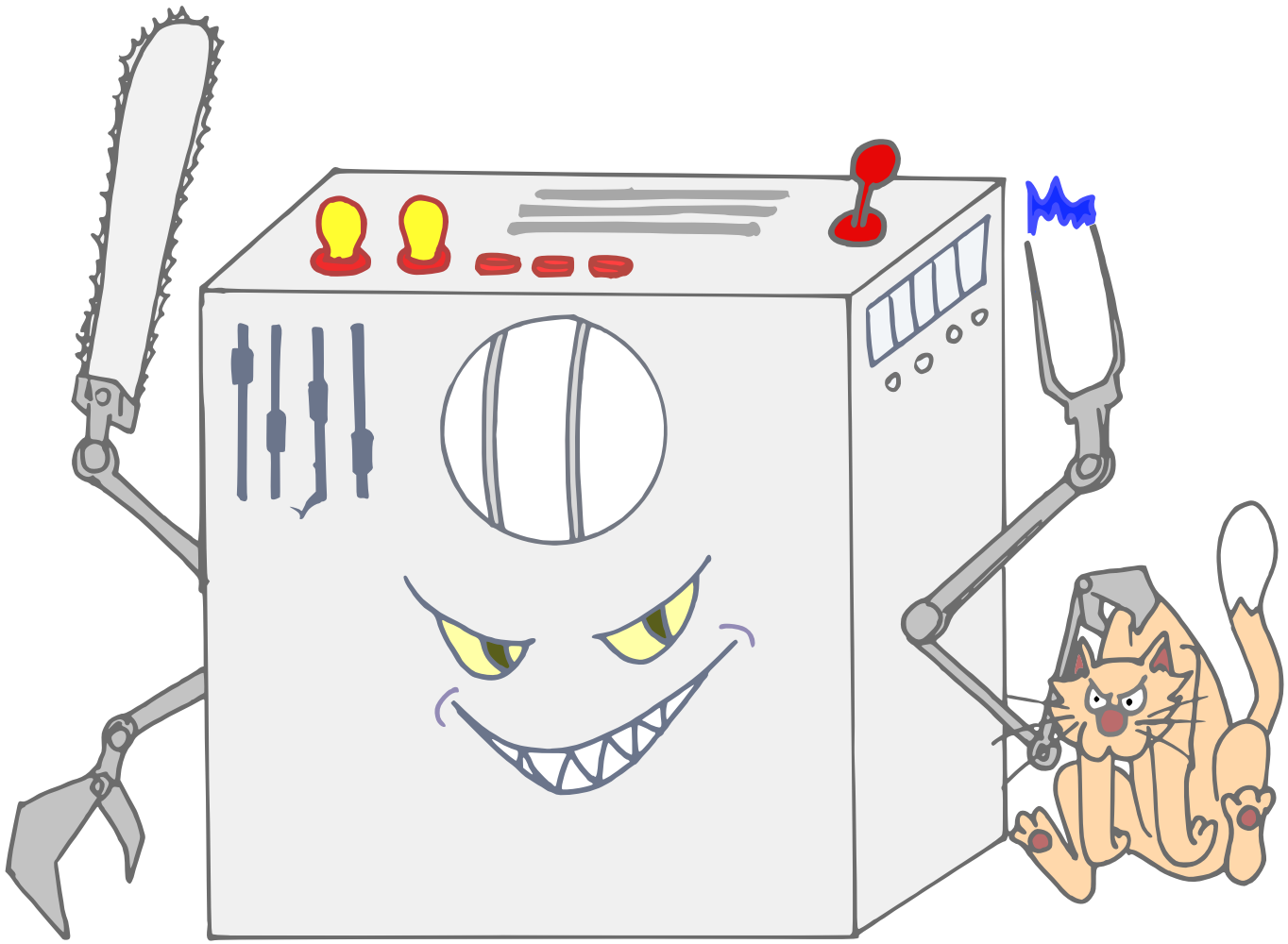


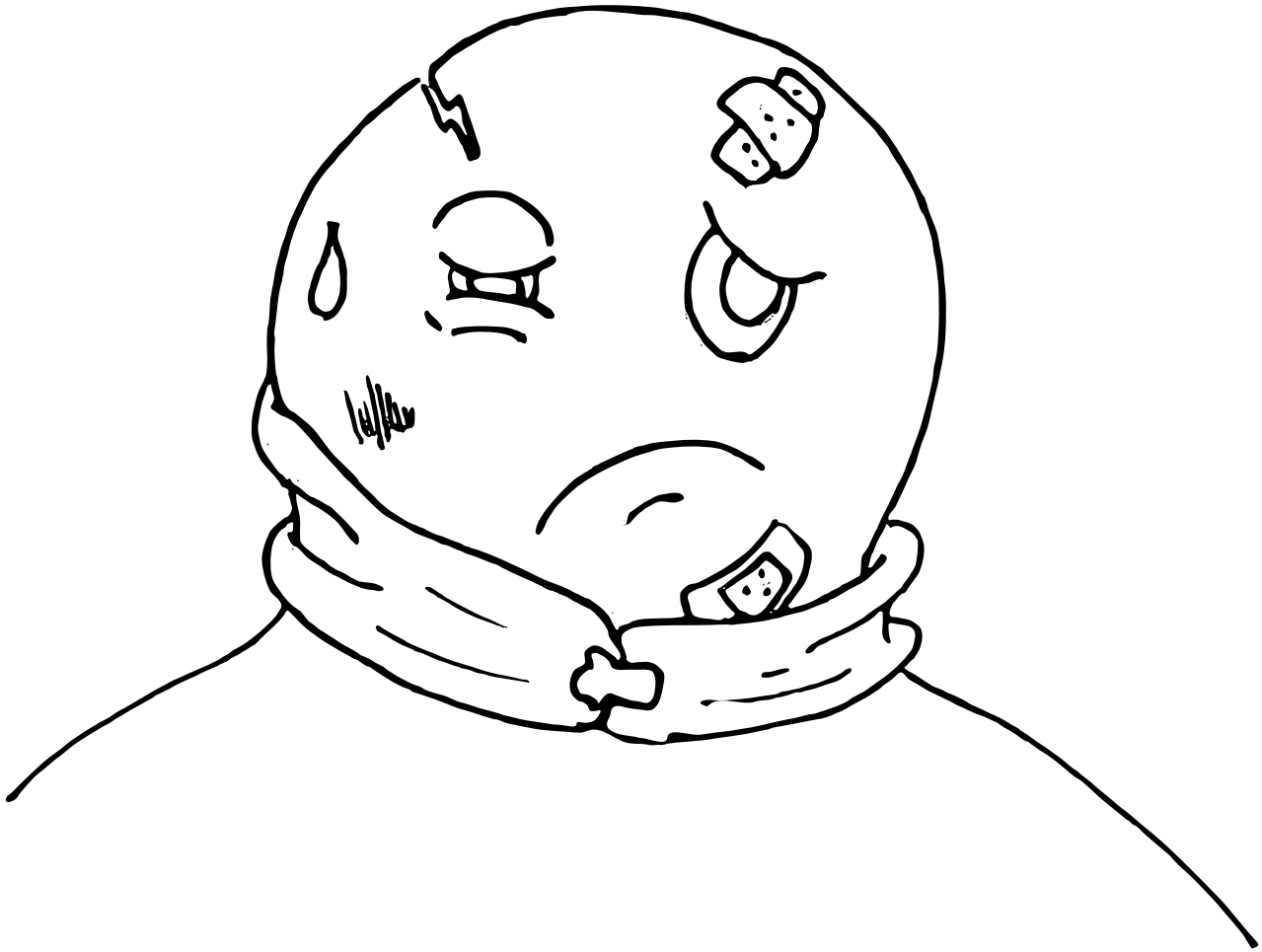
Elliptic curves

I power 70% of WWW traffic!

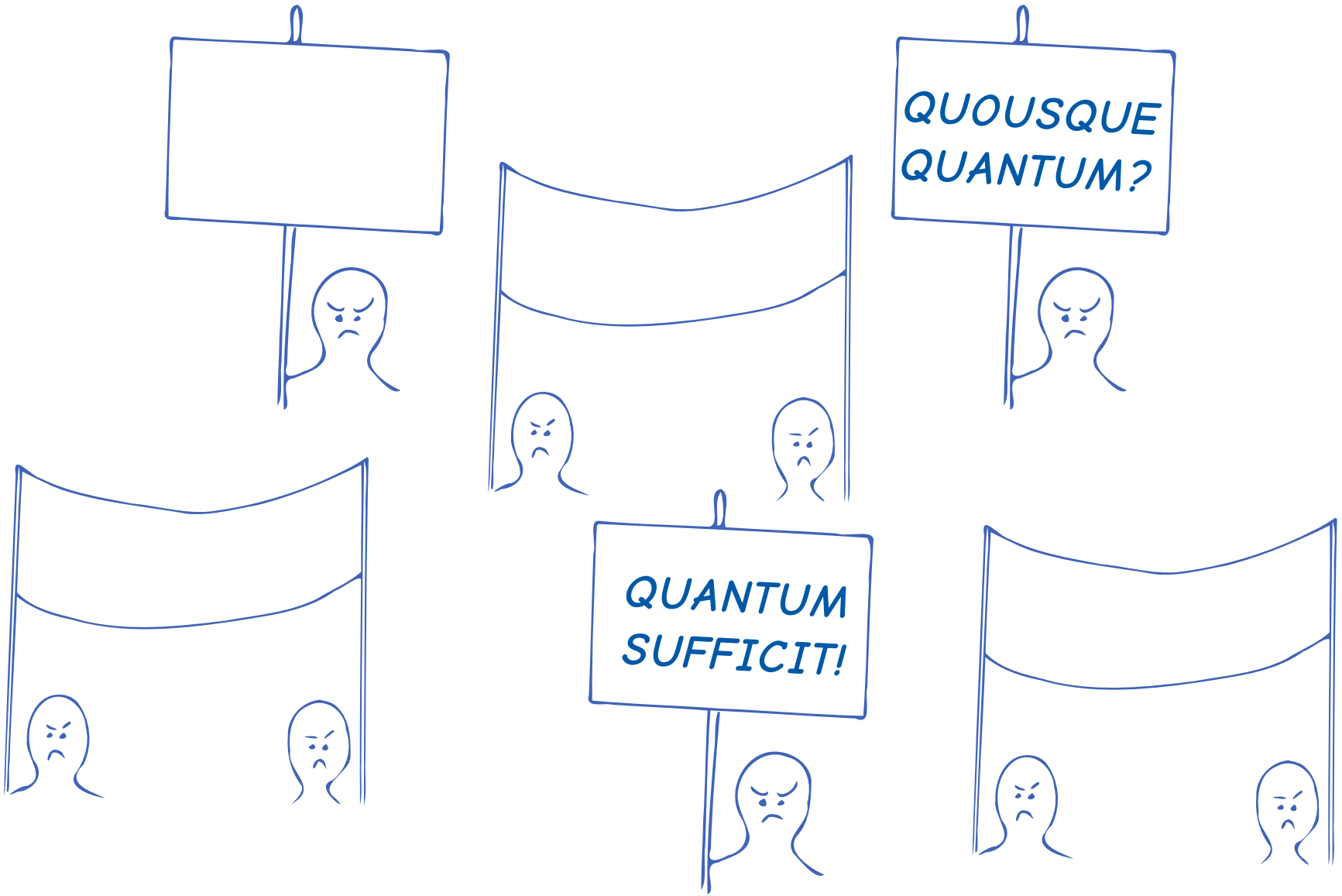
The Q Menace



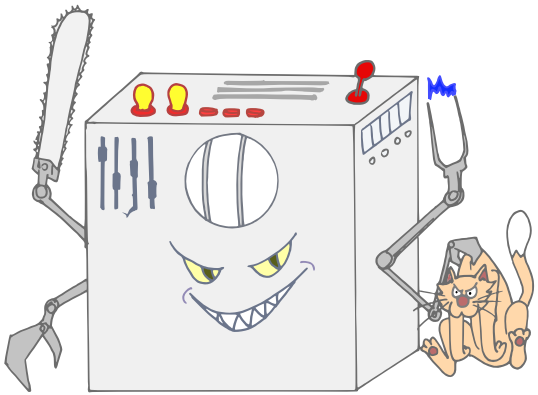
Post-quantum cryptographer?



Elliptic curves of the world, UNITE!



And so, they found a way around the Q...



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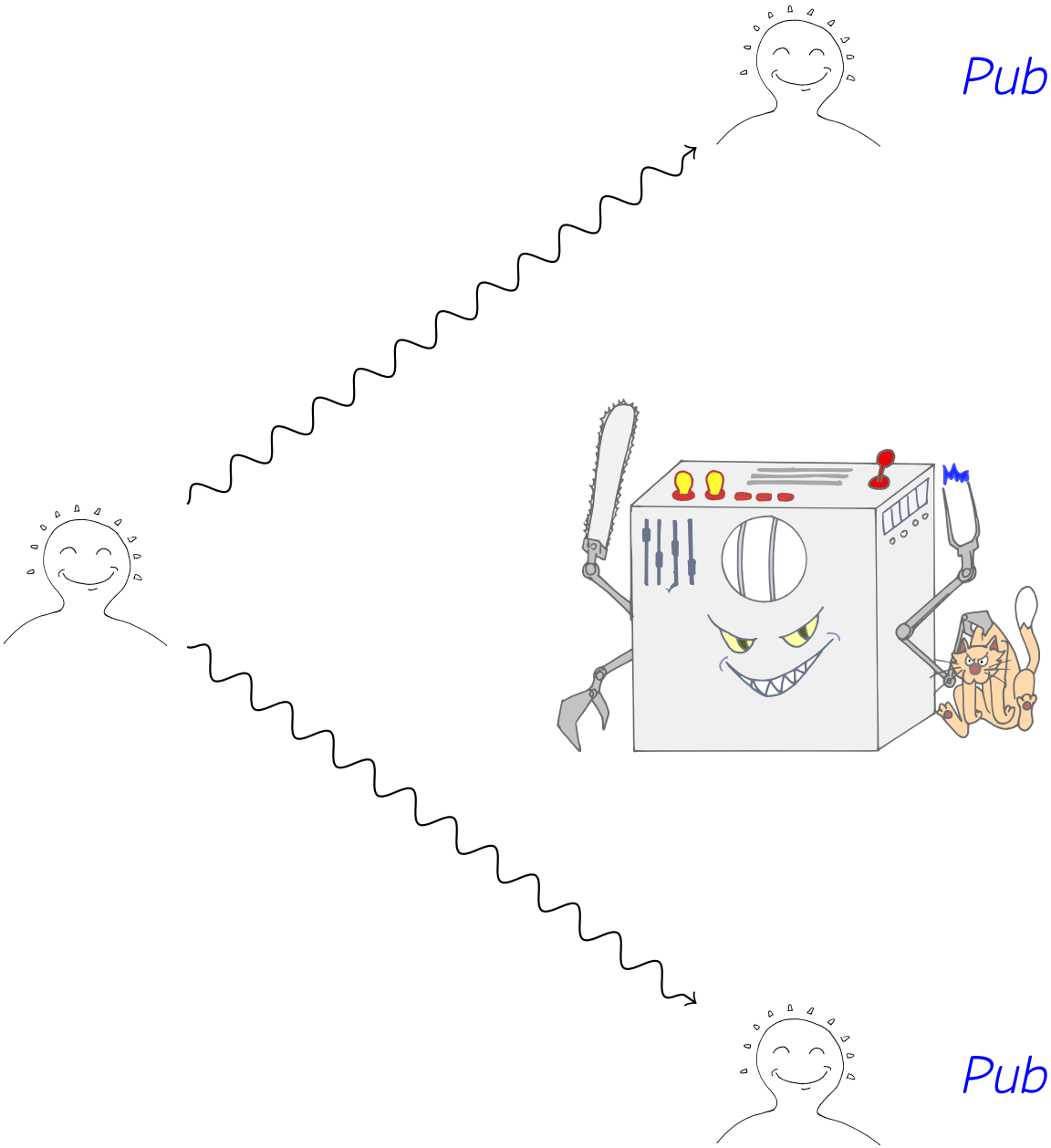

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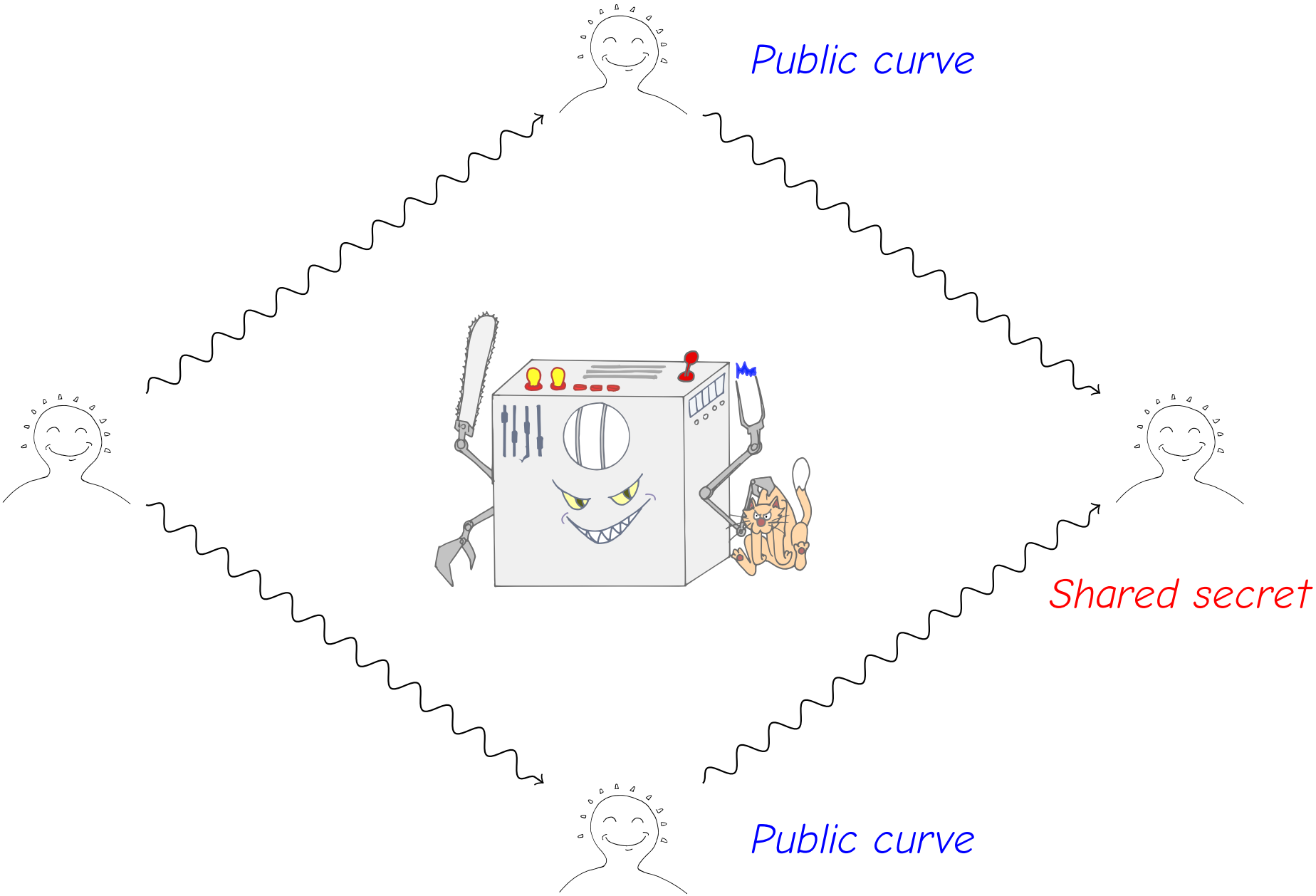
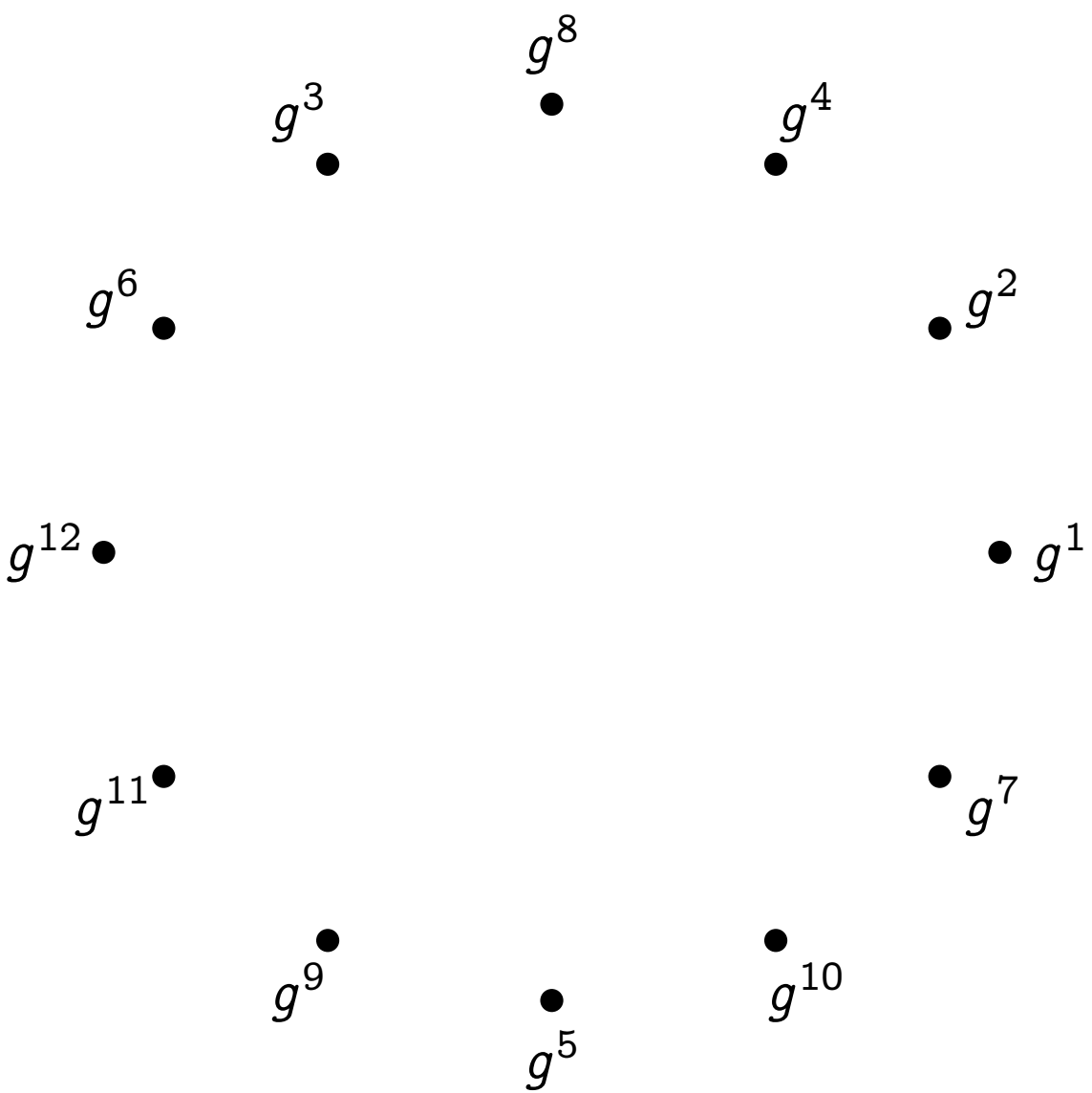
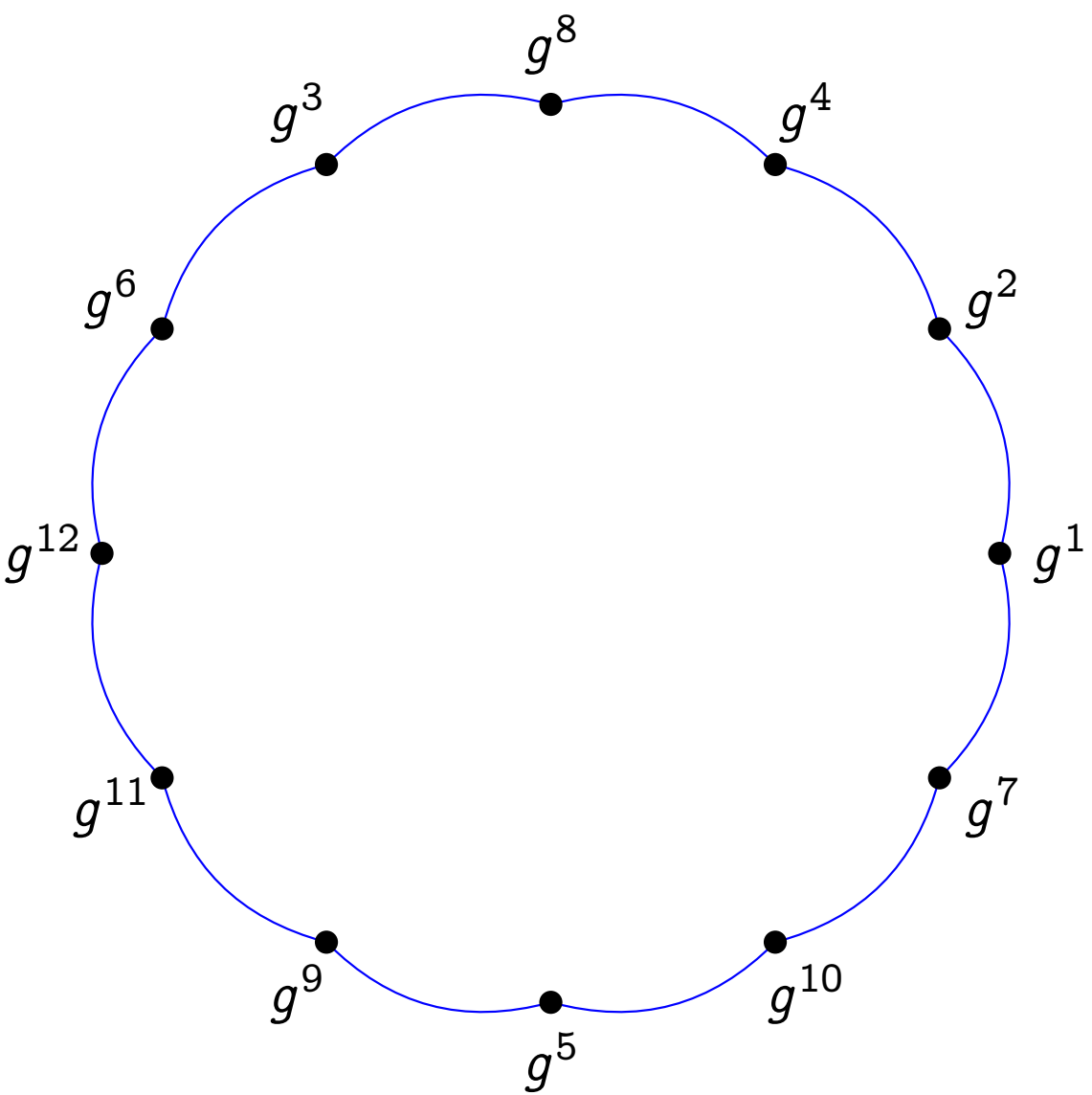


Expander graphs from groups



Let $G = \langle g \rangle$ be a cyclic group of order p.

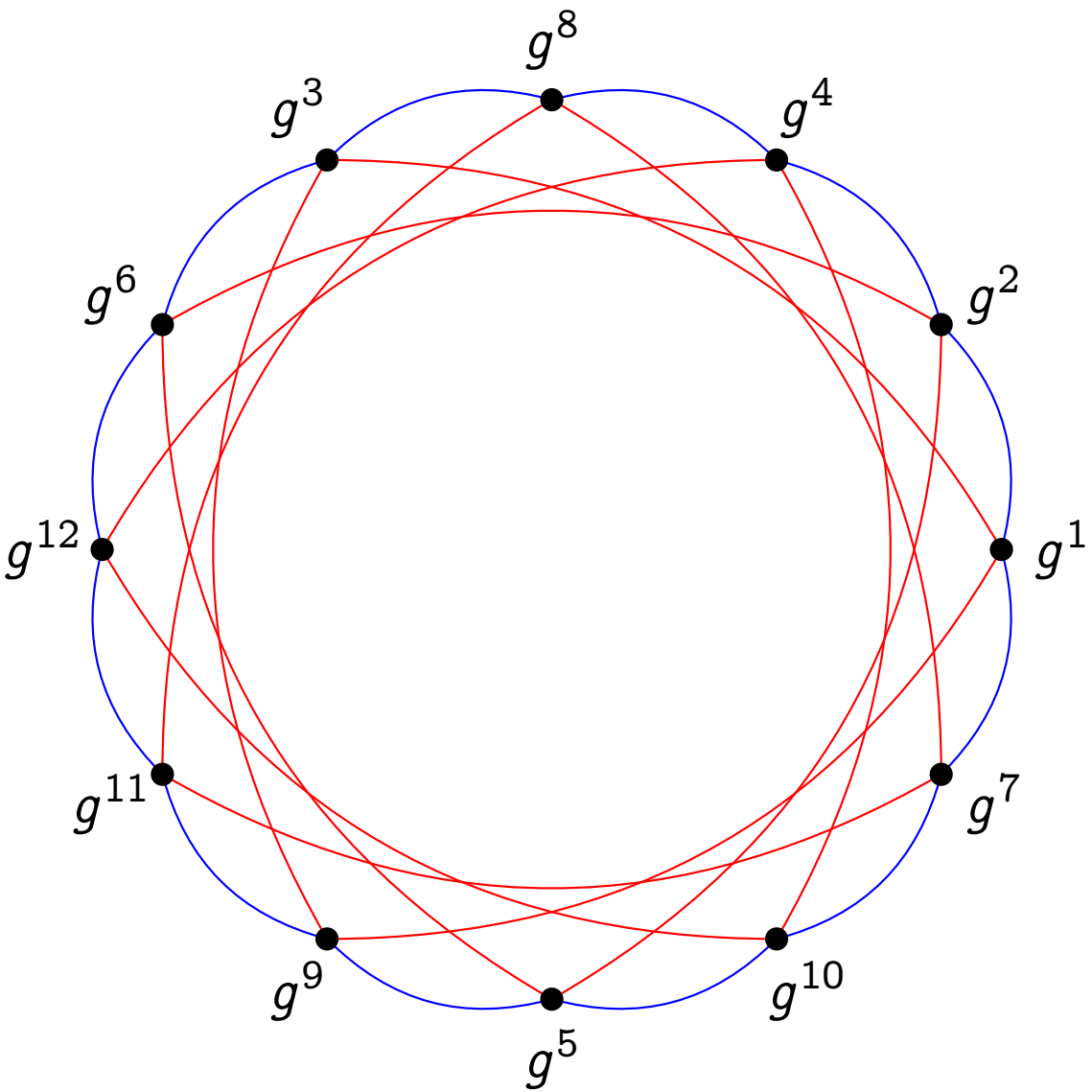
Expander graphs from groups



Let $G = \langle g \rangle$ be a cyclic group of order p.

$x \mapsto x^2$

Expander graphs from groups

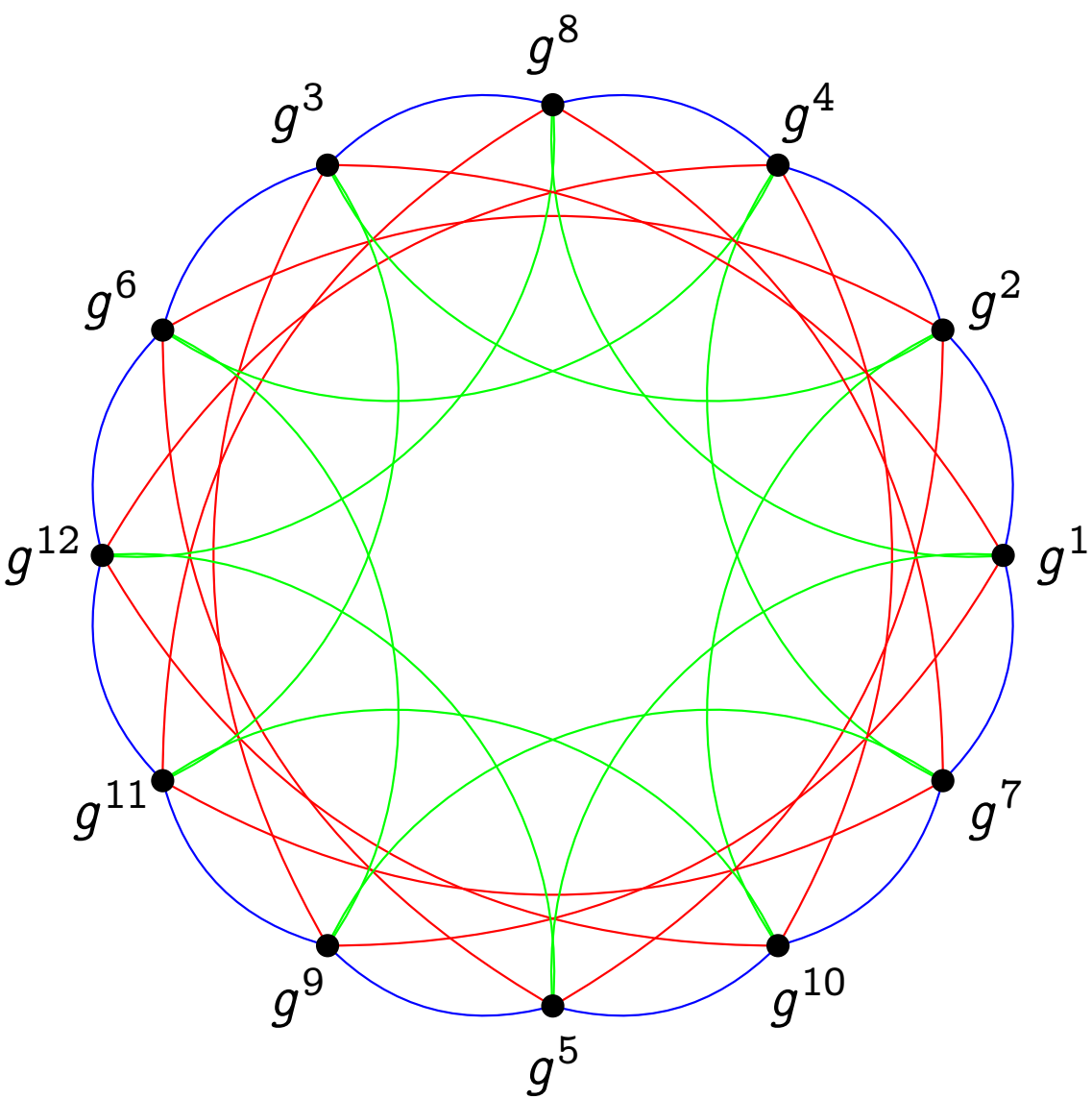


Let $G = \langle g \rangle$ be a cyclic group of order p.

— $x \mapsto x^2$

— $x \mapsto x^3$

Expander graphs from groups



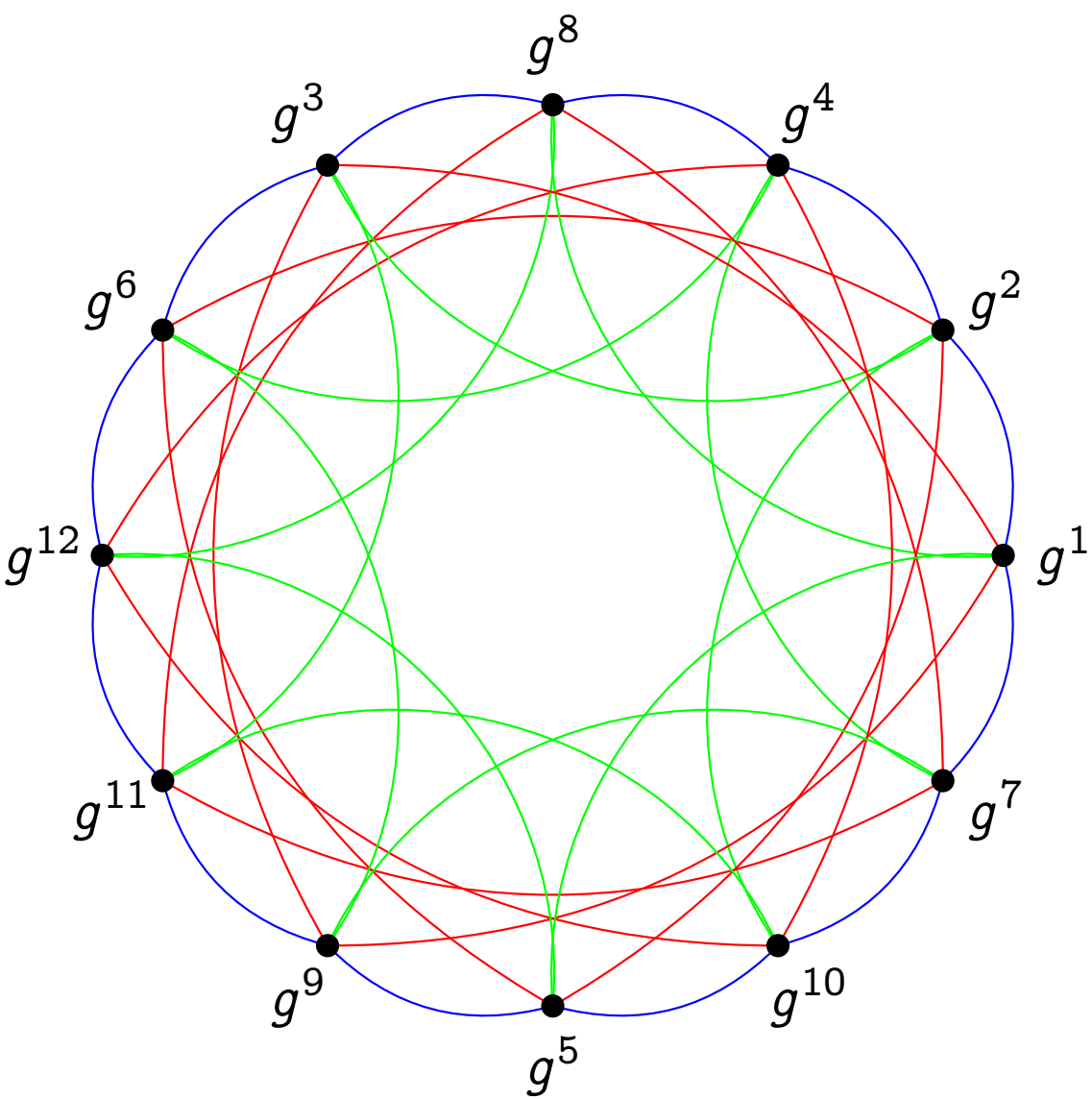
Let $G = \langle g \rangle$ be a cyclic group of order p.

— $x \mapsto x^2$

— $x \mapsto x^3$

— $x \mapsto x^5$

Expander graphs from groups



Let $G = \langle g \rangle$ be a cyclic group of order p. Let $S \subset (\mathbb{Z}/p\mathbb{Z})^\times$ s.t. $S^{-1} \subset S$.

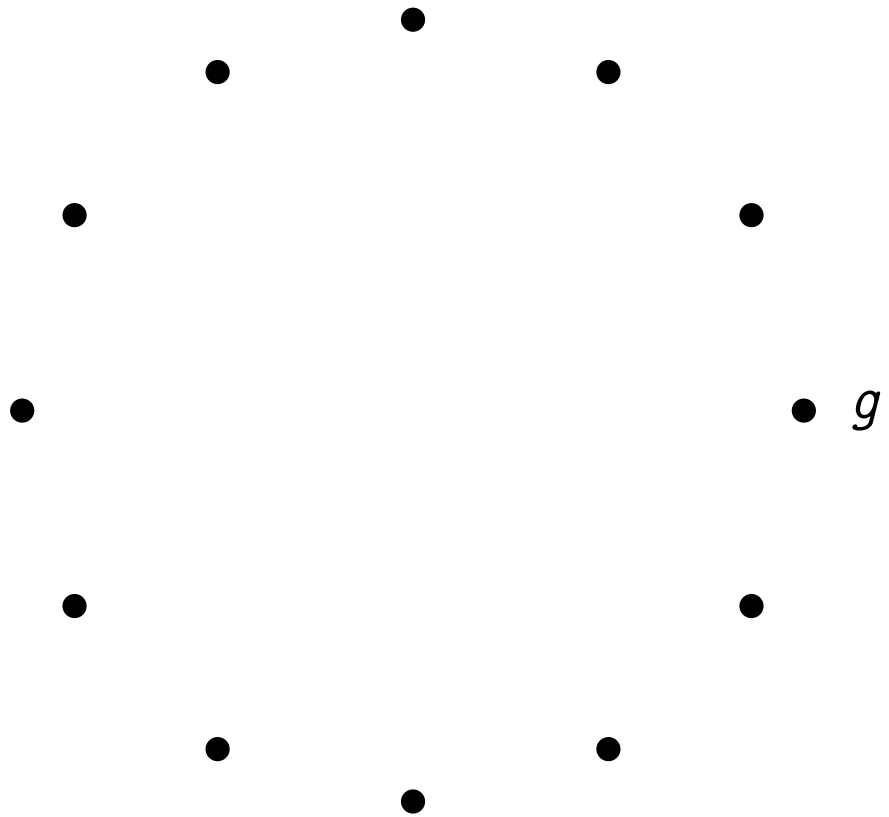
The Schreier graph of $(S, G \setminus \{1\})$ is (usually) an expander.

— $x \mapsto x^2$

— $x \mapsto x^3$

— $x \mapsto x^5$

Key exchange from Schreier graphs



Public parameters:

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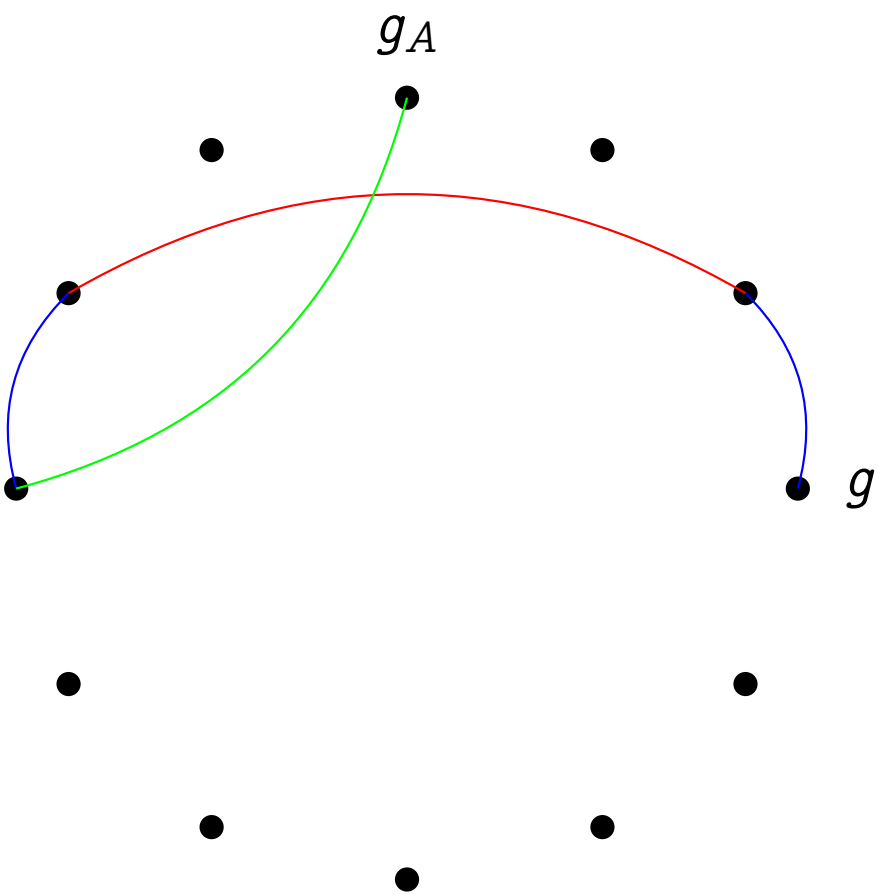
Key exchange from Schreier graphs



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1. **Alice** takes a secret random walk $s_A : g \to g_A$ of length $O(\log p)$;

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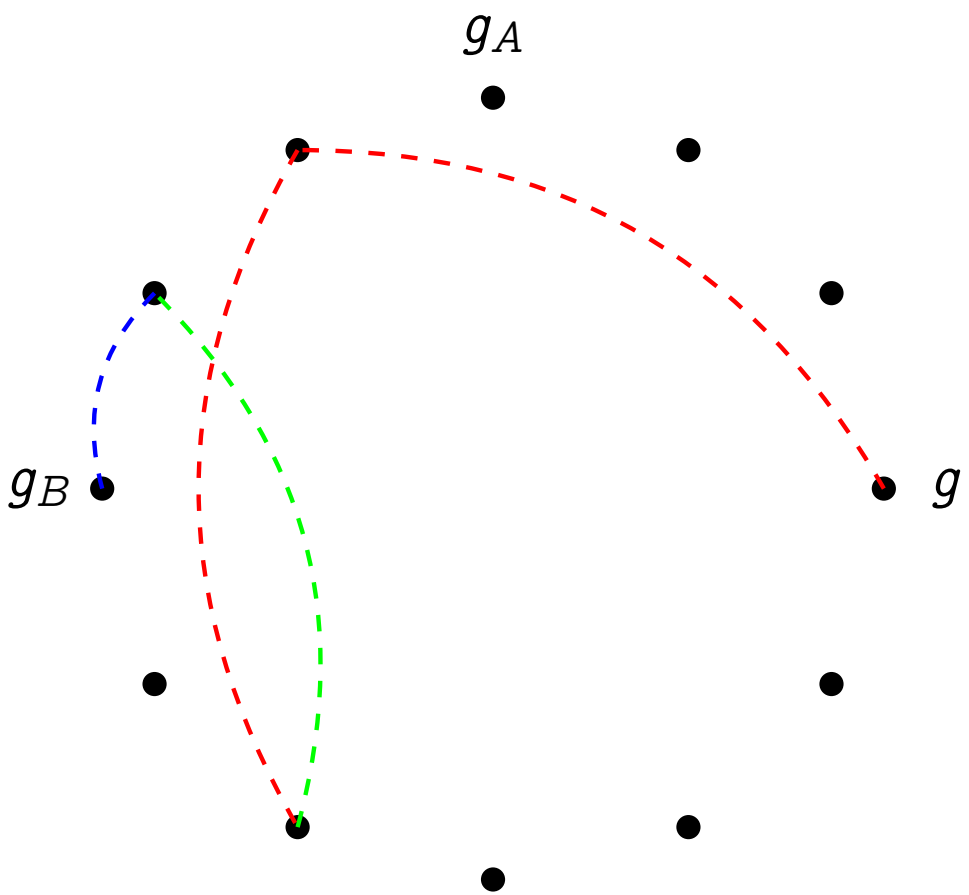

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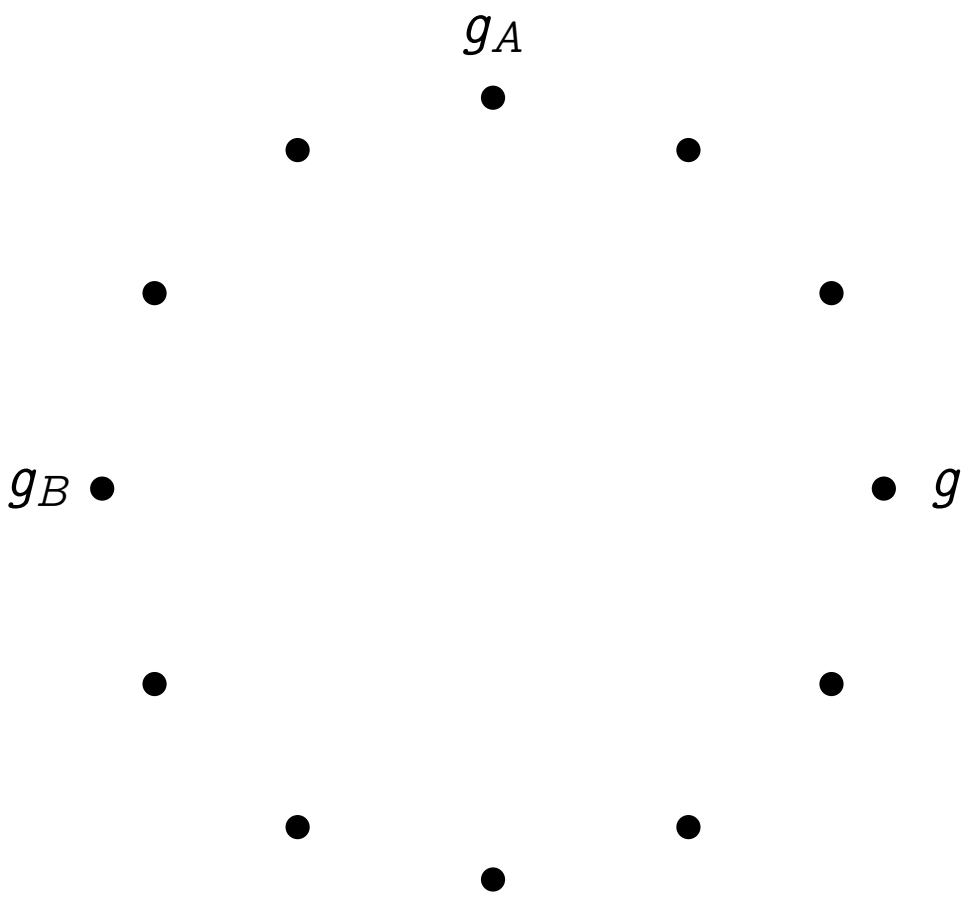
Key exchange from Schreier graphs



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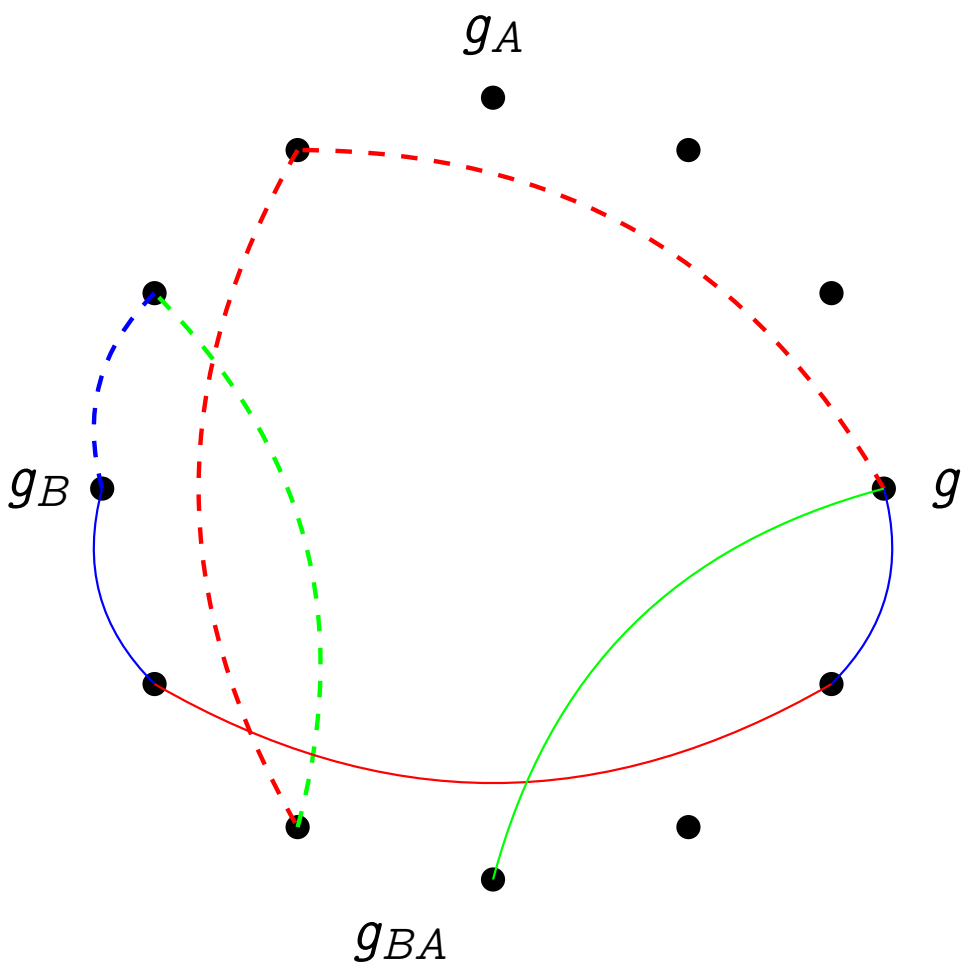
1. **Alice** takes a secret random walk $s_A : g \to g_A$ of length $O(\log p)$;
2. **Bob** does the same;
3. They publish g_A and g_B;

Key exchange from Schreier graphs



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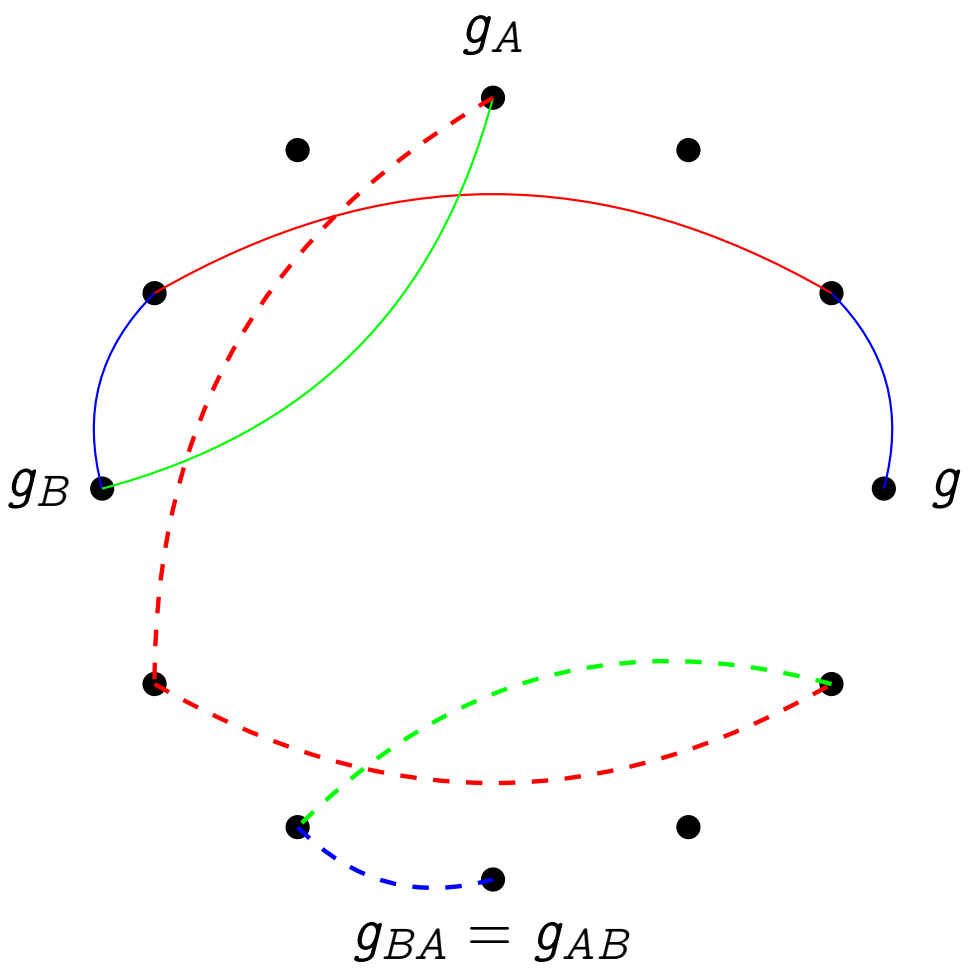
2. **Bob** does the same;
3. They publish g_A and g_B;
4. **Alice** repeats her secret walk s_A starting from g_B.

Key exchange from Schreier graphs



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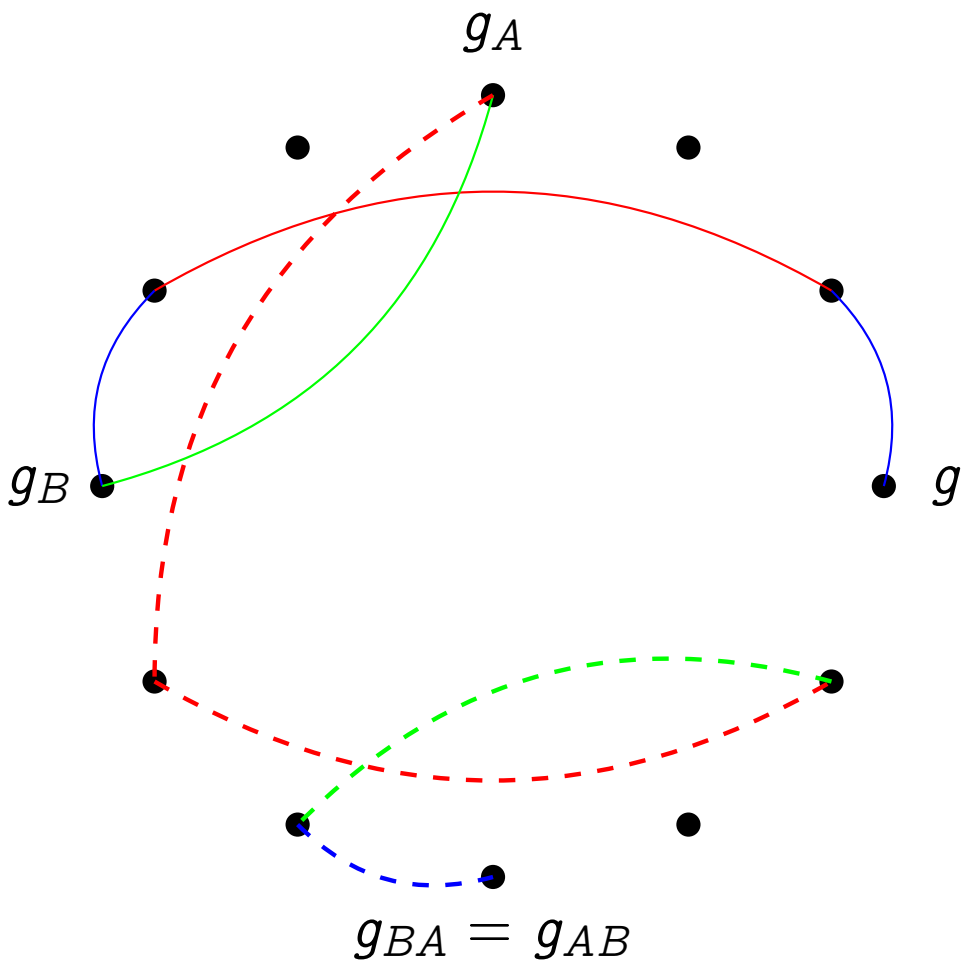
- A group $G = \langle g \rangle$ of order p;
- A subset $S \subset (\mathbb{Z}/p\mathbb{Z})^\times$.

1. **Alice** takes a secret random walk $s_A : g \to g_A$ of length $O(\log p)$;
2. **Bob** does the same;
3. They publish g_A and g_B;
4. **Alice** repeats her secret walk s_A starting from g_B.
5. **Bob** repeats his secret walk s_B starting from g_A.

Key exchange from Schreier graphs



Why does this work?

$$g_A = g^{2\cdot 3\cdot 2\cdot 5},$$
$$g_B = g^{3^2\cdot 5\cdot 2},$$
$$g_{BA} = g_{AB} = g^{2^3\cdot 3^3\cdot 5^2};$$

and g_A, g_B, g_{AB} are uniformly distributed in G…

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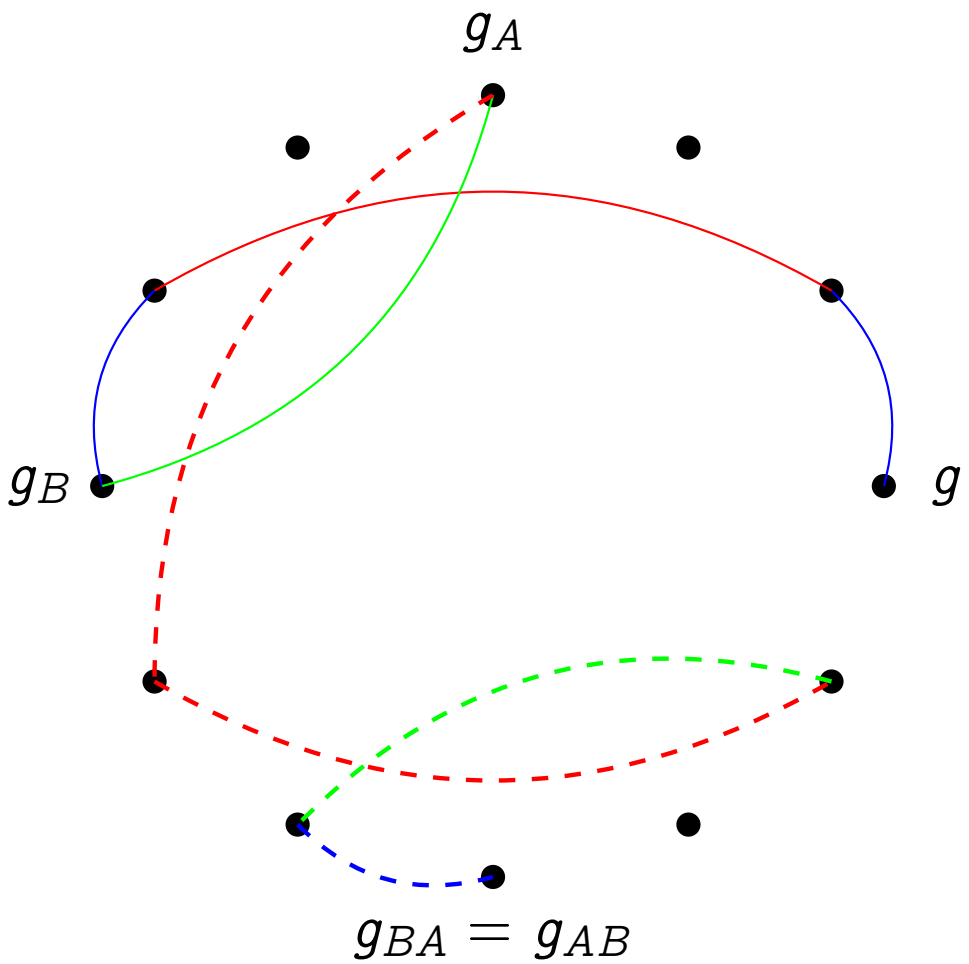

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and g_A, g_B, g_{AB} are uniformly distributed in G…

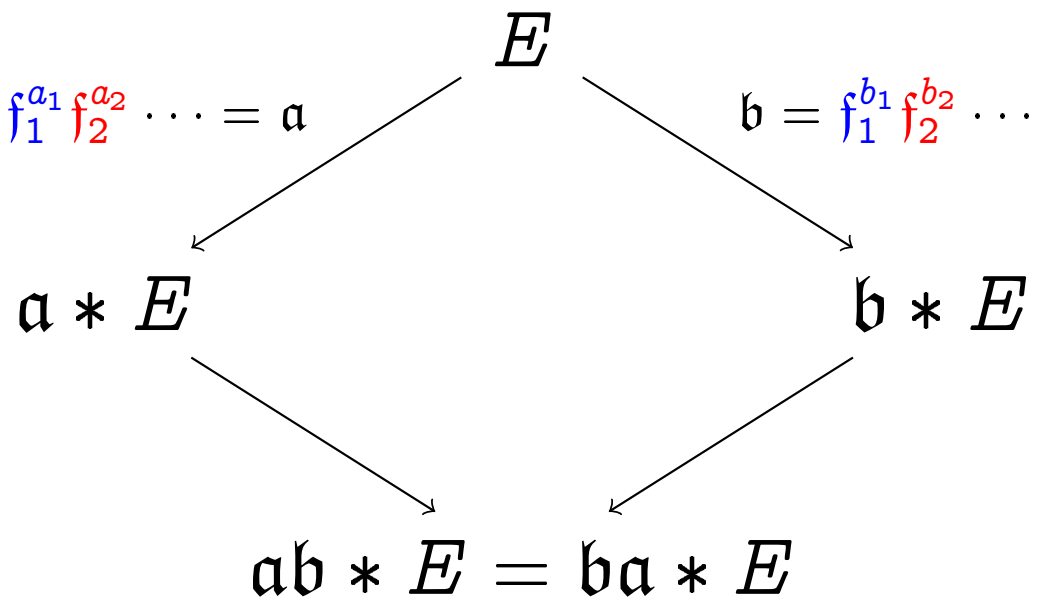
…Indeed, this is just a twisted presentation of the classical Diffie-Hellman protocol!

Key exchange in graphs of ordinary isogenies[4] (CRS)

Parameters:

- $E/\mathbb{F}_p$ ordinary elliptic curve, with Frobenius endomorphism $\pi \in \mathcal{O}$.
- (small) primes $\ell_1, \ell_2, \ldots$ such that $\left(\frac{D_\pi}{\ell_i}\right) = 1$.
- elements $\mathfrak{f}_1 = (\ell_1, \pi - \lambda_1)$, $\mathfrak{f}_2 = (\ell_2, \pi - \lambda_2), \ldots$ in $\mathrm{Cl}(\mathcal{O})$.

Secret data: Random walks $\mathfrak{a}, \mathfrak{b} \in \mathrm{Cl}(\mathcal{O})$ in the isogeny graph.



[4] Couveignes 2006; Rostovtsev and Stolbunov 2006.

Computing the action of $\mathrm{Cl}(\mathcal{O})$

Input: An ideal class $\mathfrak{a} = \mathfrak{l}_1^{a_1}\mathfrak{l}_2^{a_2}\cdots$.

Output: The elliptic curve $\mathfrak{a} * E$.

Algorithm: Let $\mathfrak{l}^n = (\ell, \pi - \lambda)^n$, repeat n times:

- Use **Elkies' algorithm** to find all (two) curves isogenous to E of degree ℓ,
- Choose the one such that $\ker\phi \subset \ker(\pi - \lambda)$.

Parameters size / performance

Adversary goal: Given $E, \mathfrak{a} * E$, find $\mathfrak{a}$;

Graph size: $\#\mathrm{Cl}(\mathcal{O}) \approx \sqrt{p}$;

Best (classical) attack: Meet-in-the-middle / Random-walk in $\sqrt{\#\mathrm{Cl}(\mathcal{O})}$;

For 2^{128} security: choose $\log p \sim 512$;

Time to evaluate the isogeny action[a]: Dozens of minutes!

[a] De Feo, Kieffer, and Smith 2018.

Vélu to the rescue?

Input: An ideal class $\mathfrak{a} = \mathfrak{f}_1^{a_1} \mathfrak{f}_2^{a_2} \cdots$.

Output: The elliptic curve $\mathfrak{a} * E$.

Algorithm: Let $\mathfrak{f}^n = (\ell, \pi - \lambda)^n$. Why not:

- *Presciently* find $H = E[\ell] \cap \ker(\pi - \lambda)$,
- Apply Vélu's formulas to H.

Speeding up the class group action

Problem: H must be in $E(\mathbb{F}_p)$ for Vélu's formulas to be efficient.

Idea[a]: Force $\begin{cases} p = -1 & \mod \ell, \\ \lambda = 1 & \mod \ell, \end{cases}$
so that $E[\ell] = H \subset E(\mathbb{F}_p)$.

[a] De Feo, Kieffer, and Smith 2018.

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How to waste an internship: Forcing $\lambda =$ Forcing $\#E =$ Very hard!

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How to waste an internship: Forcing λ = Forcing $\#E$ = Very hard!

Time to evaluate the isogeny action: Still 5 minutes!

[a] De Feo, Kieffer, and Smith 2018.

Supersingular to the rescue!

For all supersingular curves defined over $\mathbb{F}_p$,

$$\pi = \begin{pmatrix} \sqrt{-p} & 0 \\ 0 & -\sqrt{-p} \end{pmatrix} \mod \ell$$

CSIDH *(pron.: Seaside)*

Choose $p = -1 \mod \ell$ for many primes ℓ;

Hence, $\lambda = 1 \mod \ell$. Win!

Performance: Same security as CRS in less than 50ms![a]

[a] Castryck, Lange, Martindale, Panny, and Renes 2018.

Quantum security

Fact: Shor's algorithm does not apply to Diffie-Hellman protocols from group actions.

Subexponential attack $\exp(\sqrt{\log p \log\log p})$

- Reduction to the hidden shift problem by evaluating the class group action in quantum superposition[a] (subexpoential cost);
- Well known reduction from the hidden shift to the dihedral (non-abelian) hidden subgroup problem;
- Kuperberg's algorithm[b] solves the dHSP with a subexponential number of class group evaluations.
- Recent work[c] suggests that 2^{64}-qbit security is achieved somewhere in $512 < \log p < 1024$.

---

[a] Childs, Jao, and Soukharev 2014.

[b] Kuperberg 2005; Regev 2004; Kuperberg 2013.

[c] Bonnetain and Naya-Plasencia 2018; Bonnetain and Schrottenloher 2018; Biasse, Jacobson Jr, and Iezzi 2018.

Key exchange with supersingular curves (2011)

Good news: there is no action of a commutative class group.

Bad news: there is no action of a commutative class group.

Idea: Let Alice and Bob walk in two different isogeny graphs on the same vertex set.

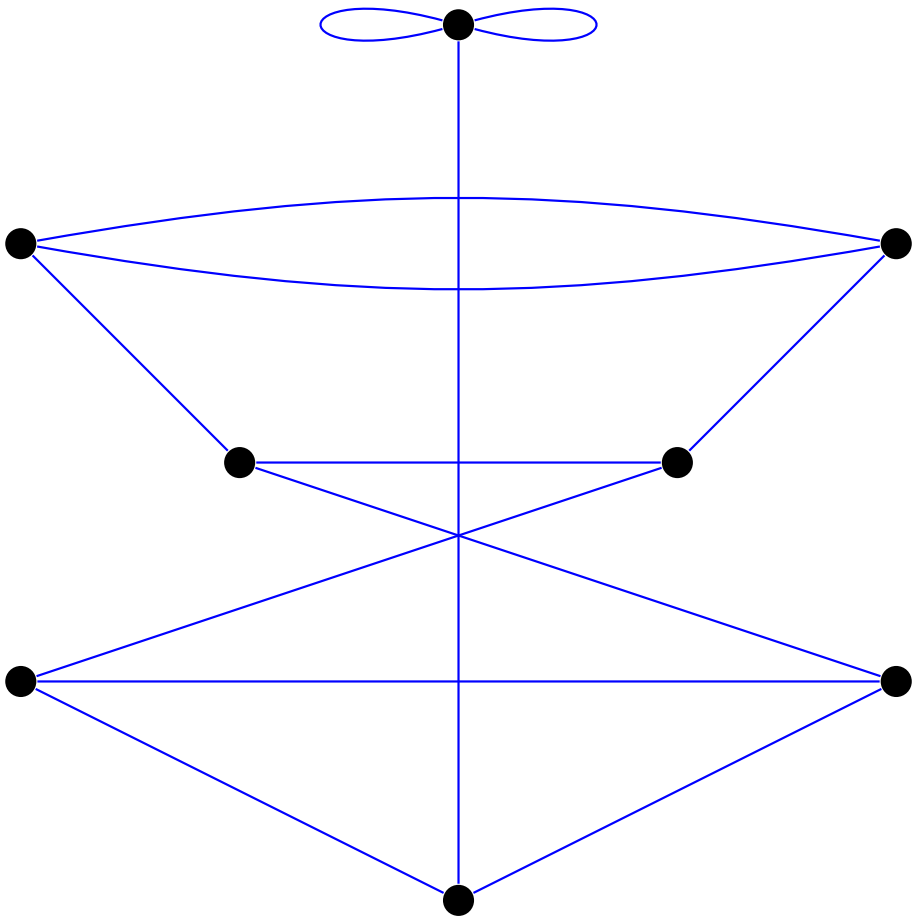

Figure: 2- and 3-isogeny graphs on $\mathbb{F}_{97^2}$.

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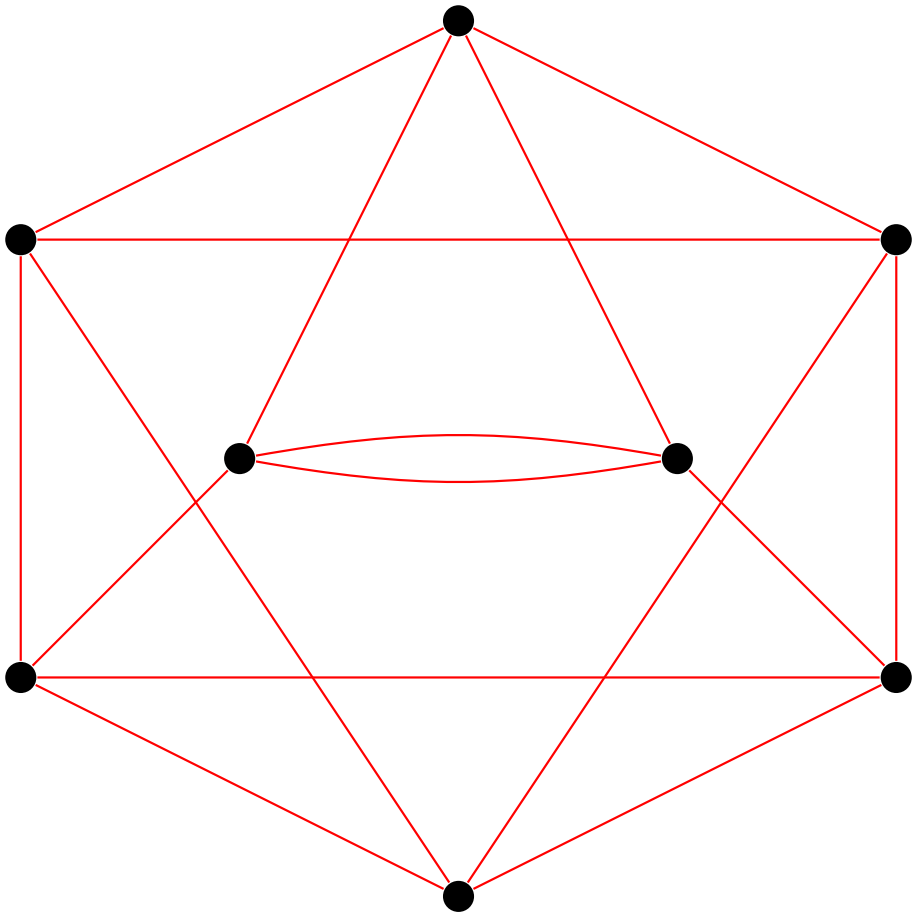

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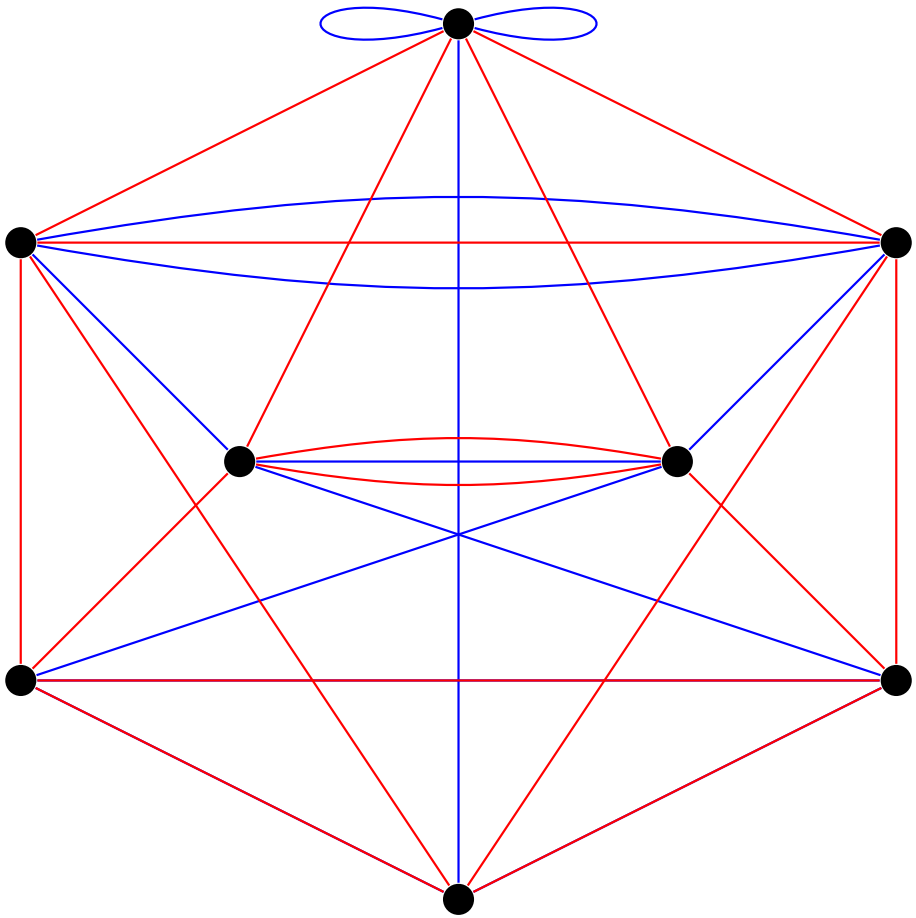


Figure: 2- and 3-isogeny graphs on $\mathbb{F}_{97^2}$.

Key exchange with supersingular curves (2011)

- Fix small primes ℓ_A, ℓ_B;
- No canonical labeling of the ℓ_A- and ℓ_B-isogeny graphs; however…

Walk of length e_A

$=$

Isogeny of degree $\ell_A^{e_A}$

$=$

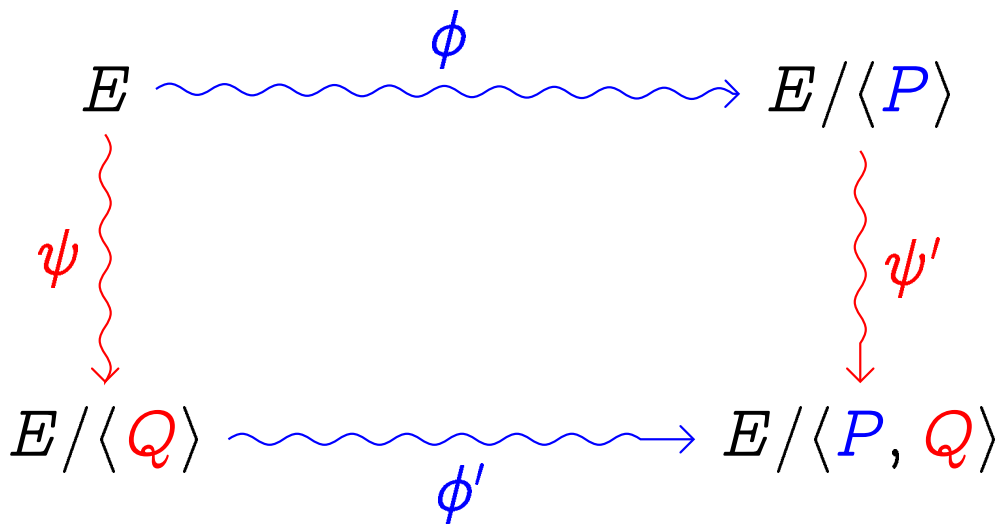
Kernel $\langle P \rangle \subset E[\ell_A^{e_A}]$

$$\ker \phi = \langle P \rangle \subset E[\ell_A^{e_A}]$$

$$\ker \psi = \langle Q \rangle \subset E[\ell_B^{e_B}]$$

$$\ker \phi' = \langle \psi(P) \rangle$$

$$\ker \psi' = \langle \phi(Q) \rangle$$



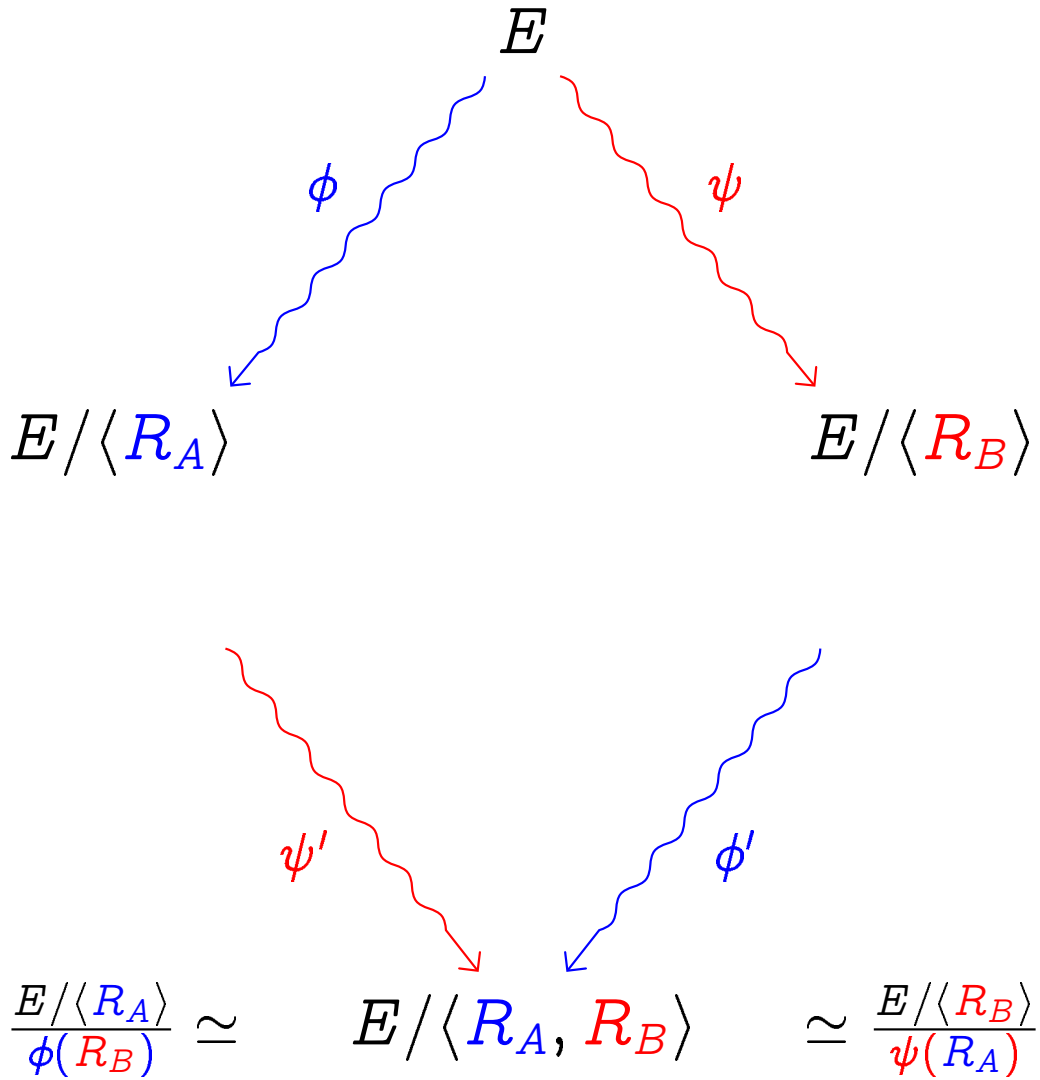
Supersingular Isogeny Diffie-Hellman[5]

Parameters:

- Prime p such that $p + 1 = \ell_A^a \ell_B^b$;
- Supersingular curve $E \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2$;
- $E[\ell_A^a] = \langle P_A, Q_A \rangle$;
- $E[\ell_B^b] = \langle P_B, Q_B \rangle$.

Secret data:

- $R_A = m_A P_A + n_A Q_A$,
- $R_B = m_B P_B + n_B Q_B$,



[5] Jao and De Feo 2011; De Feo, Jao, and Plût 2014.

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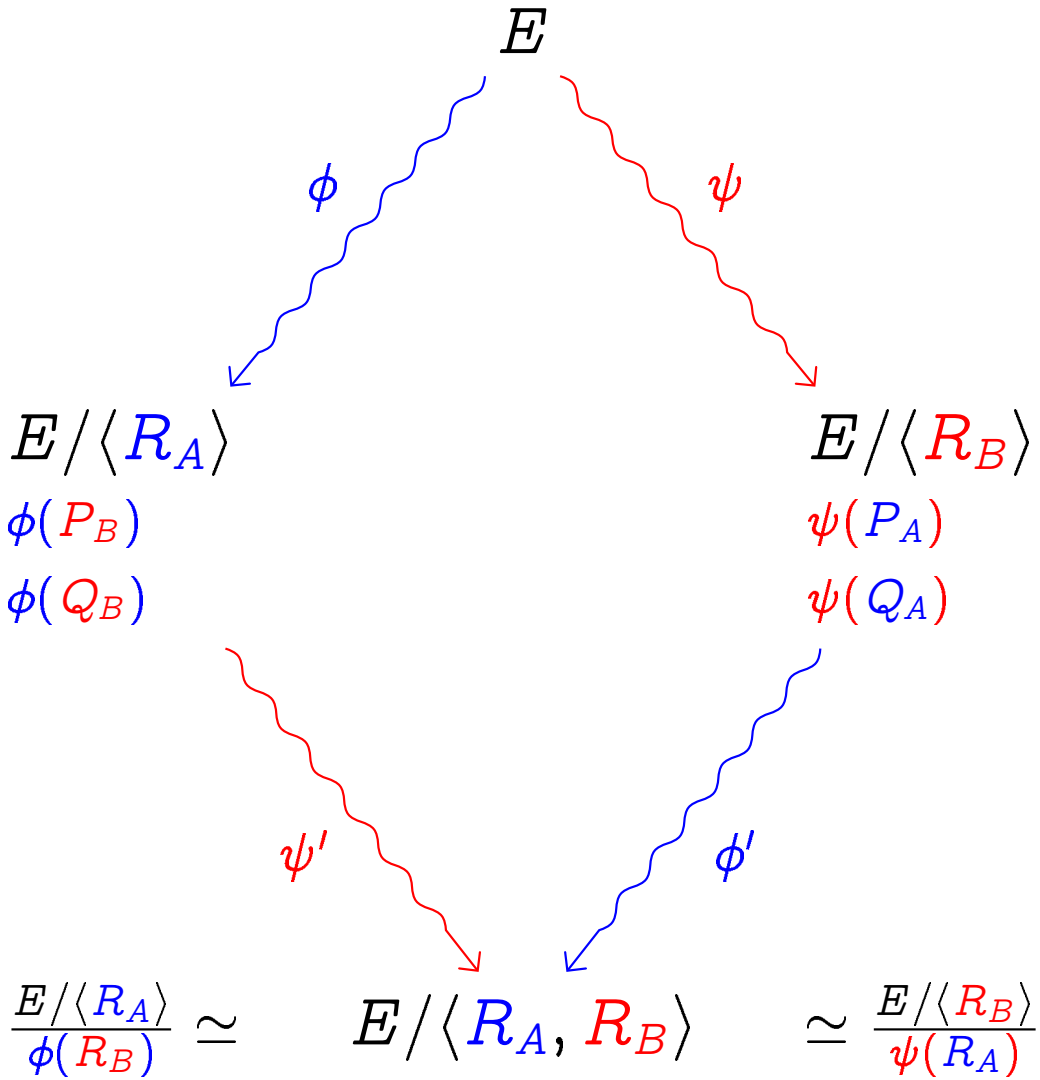

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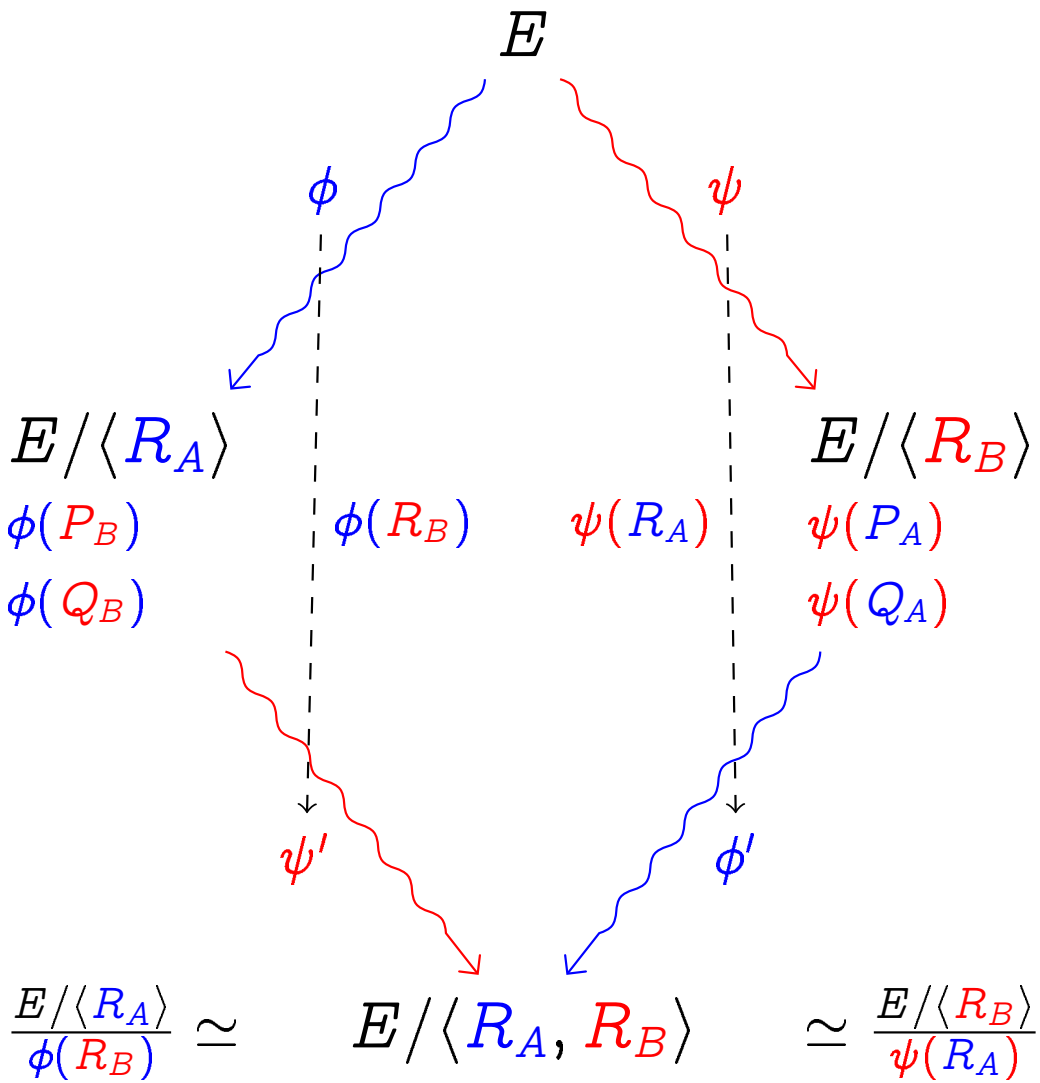
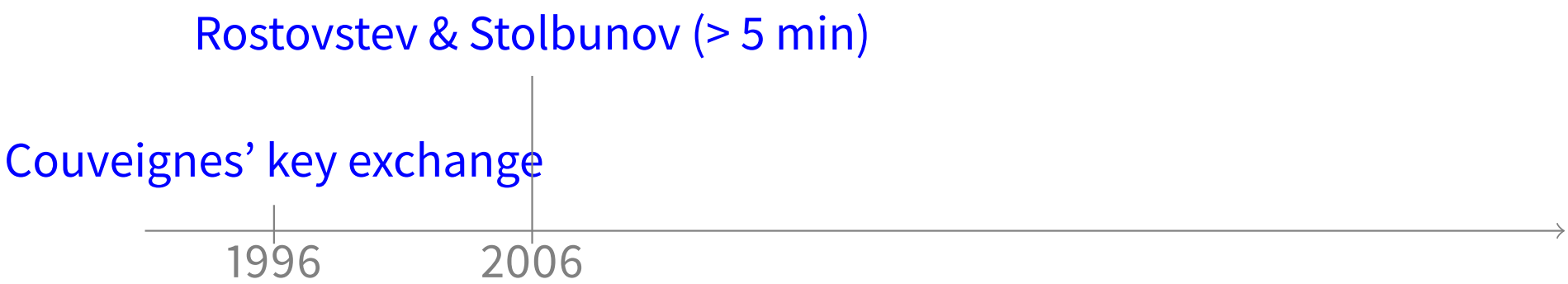


[5] Jao and De Feo 2011; De Feo, Jao, and Plût 2014.

From 10 minutes to 10ms in 20 years

Couveignes' key exchange

From 10 minutes to 10ms in 20 years



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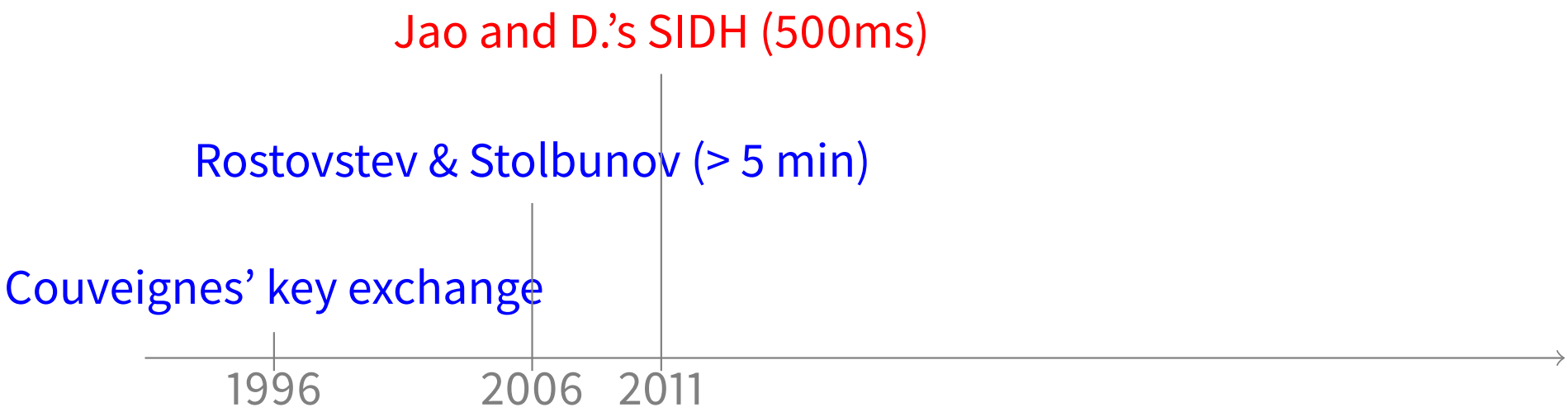

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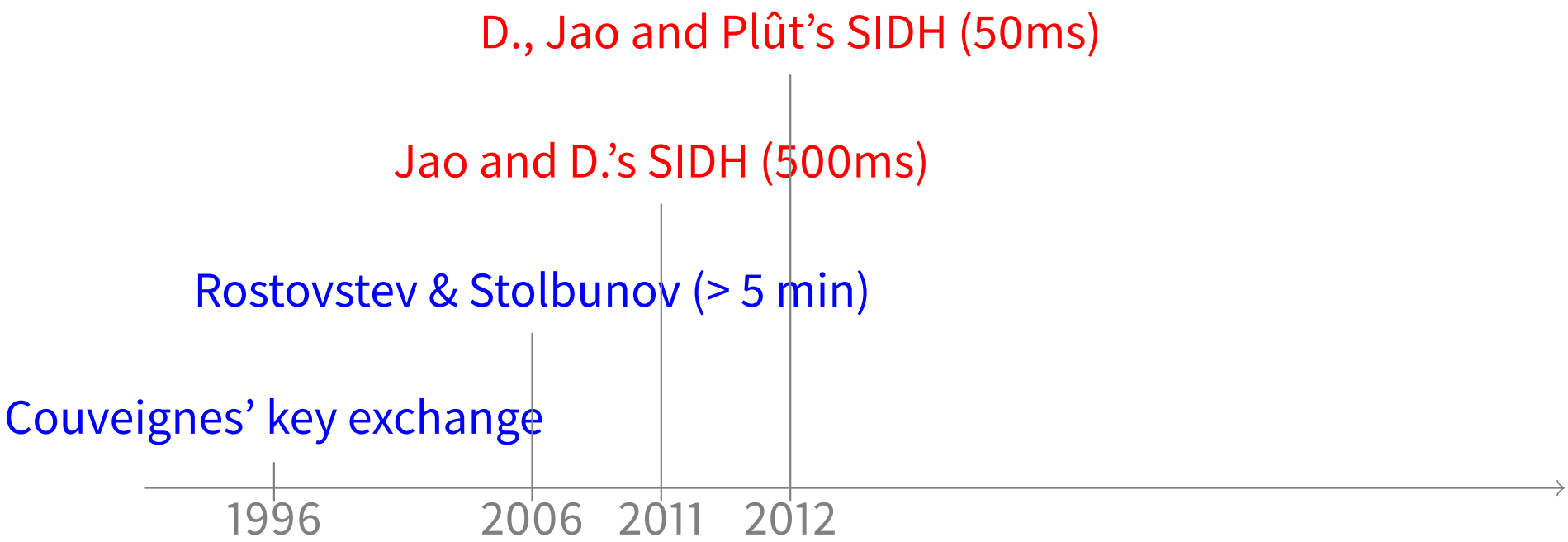

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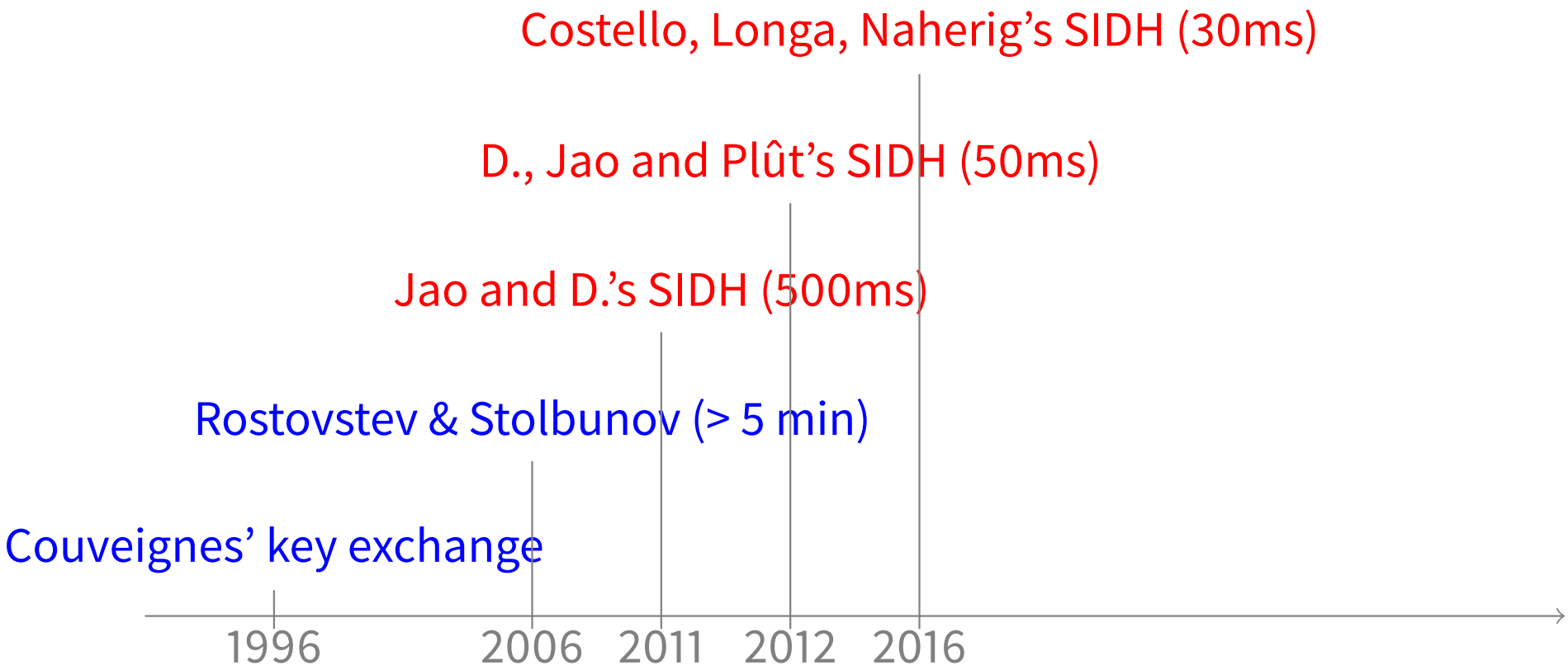

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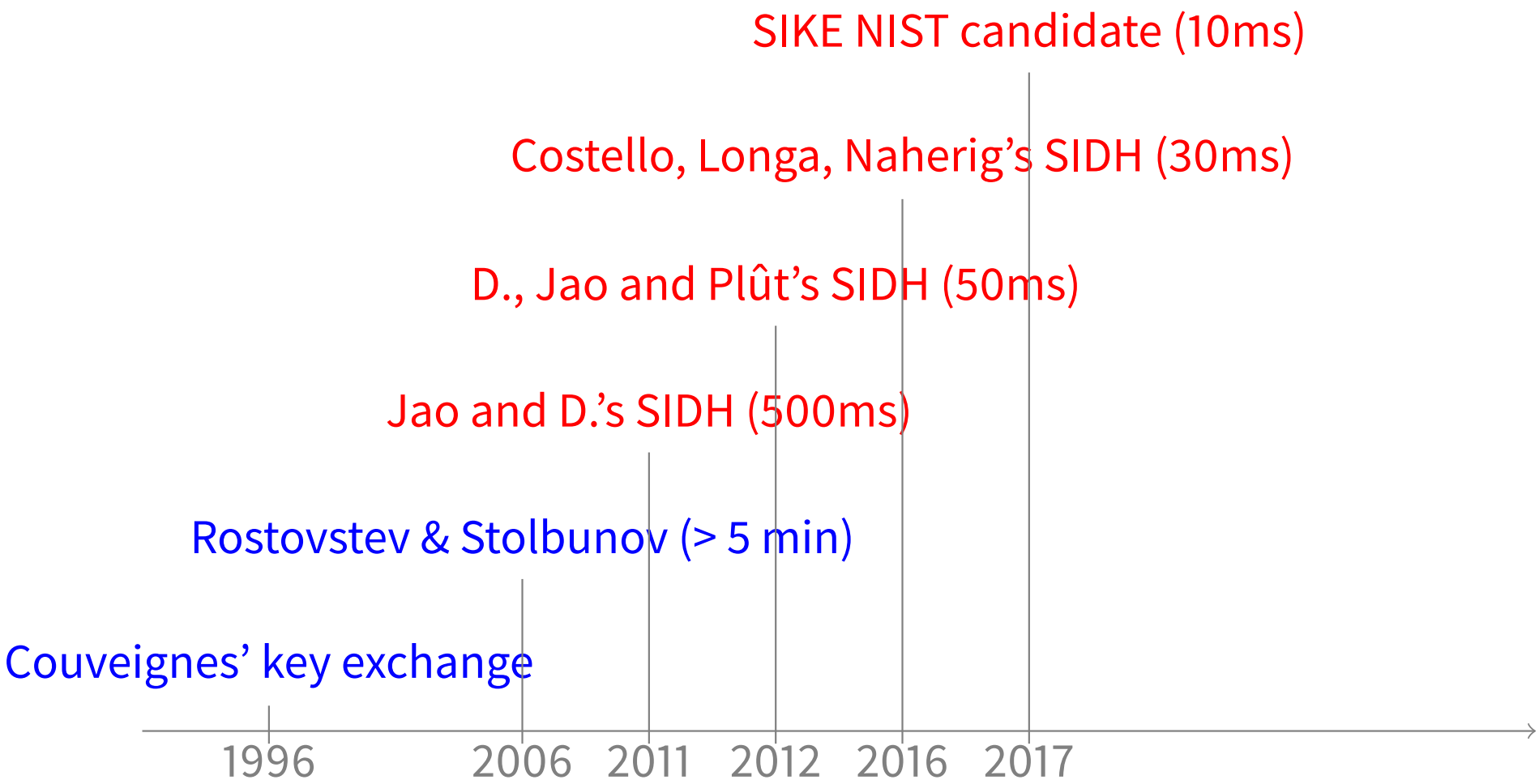

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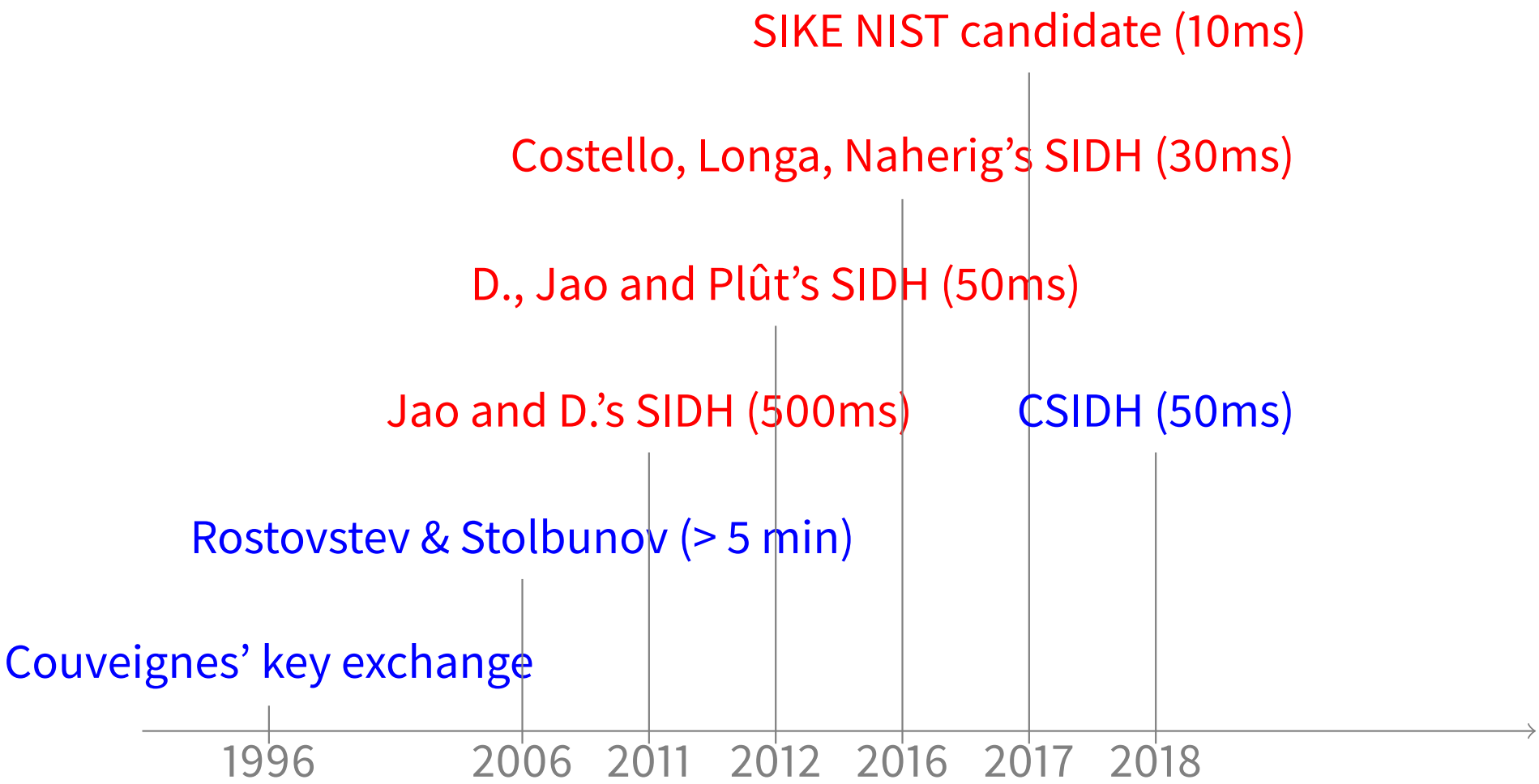
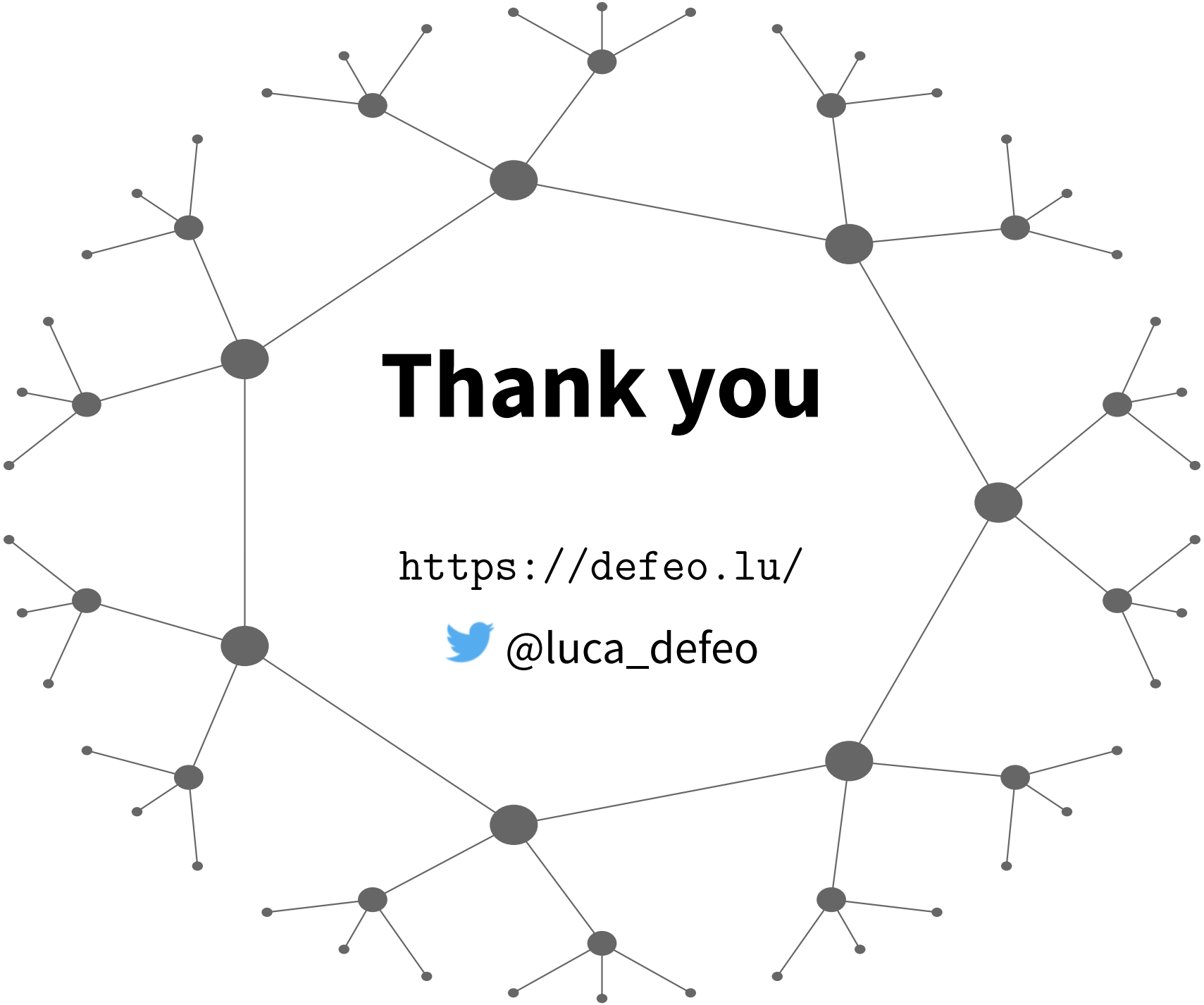


Open problems

- Precisely asses the quantum security of CRS/CSIDH.
- Give a convincing constant-time implementation of CSIDH.
- Find an efficient post-quantum isogeny-based signature scheme.
- Sample supersingular curves without revealing endomorphism rings.
- Compute endomorphism rings of supersingular curves.


Thank you
https://defeo.lu/
@luca_defeo

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