Isogeny graphs in cryptography

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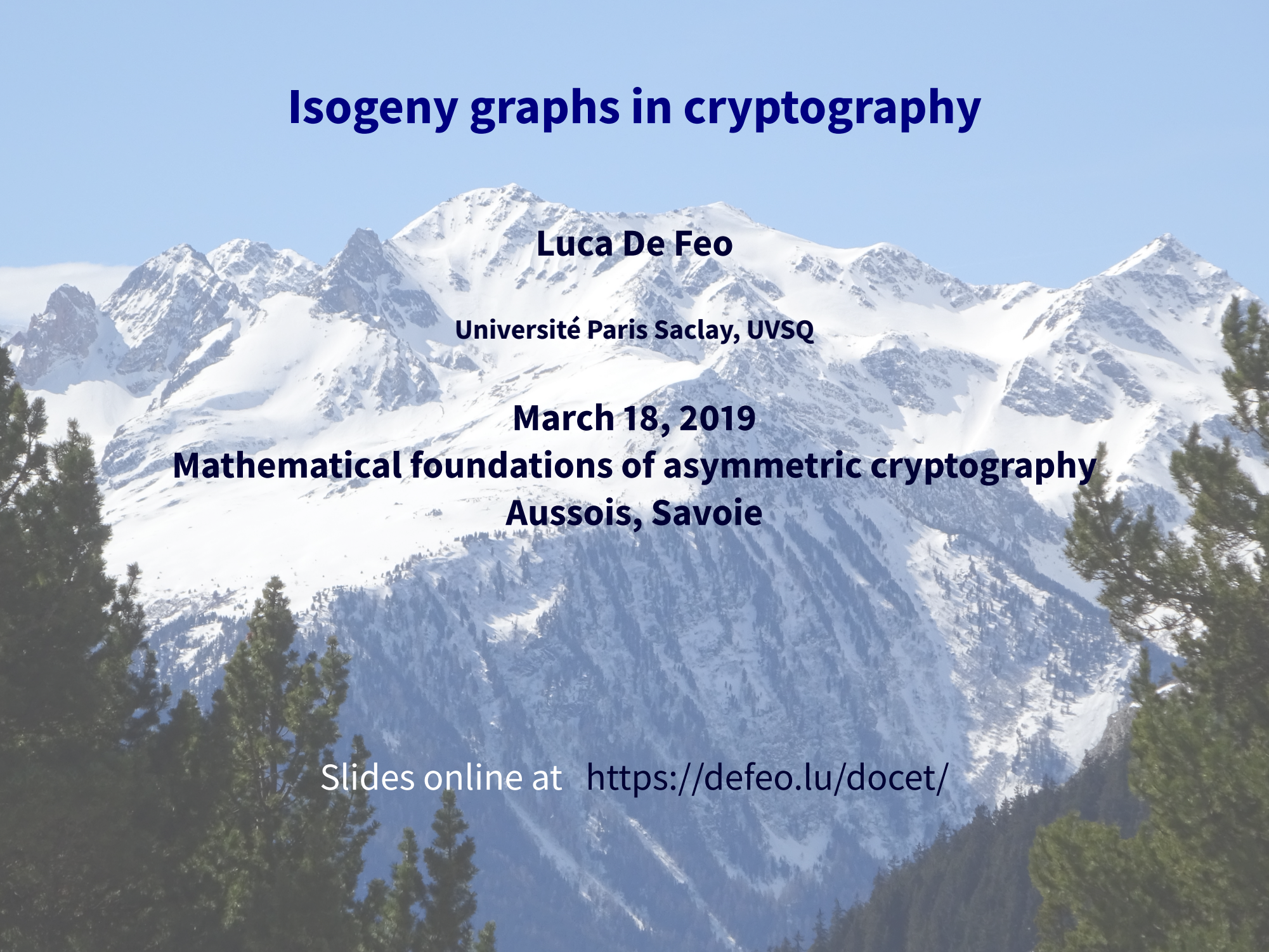
Mathematical foundations of asymmetric cryptography
Aussois, Savoie



Slides online at https://defeo.lu/docet/

Overview

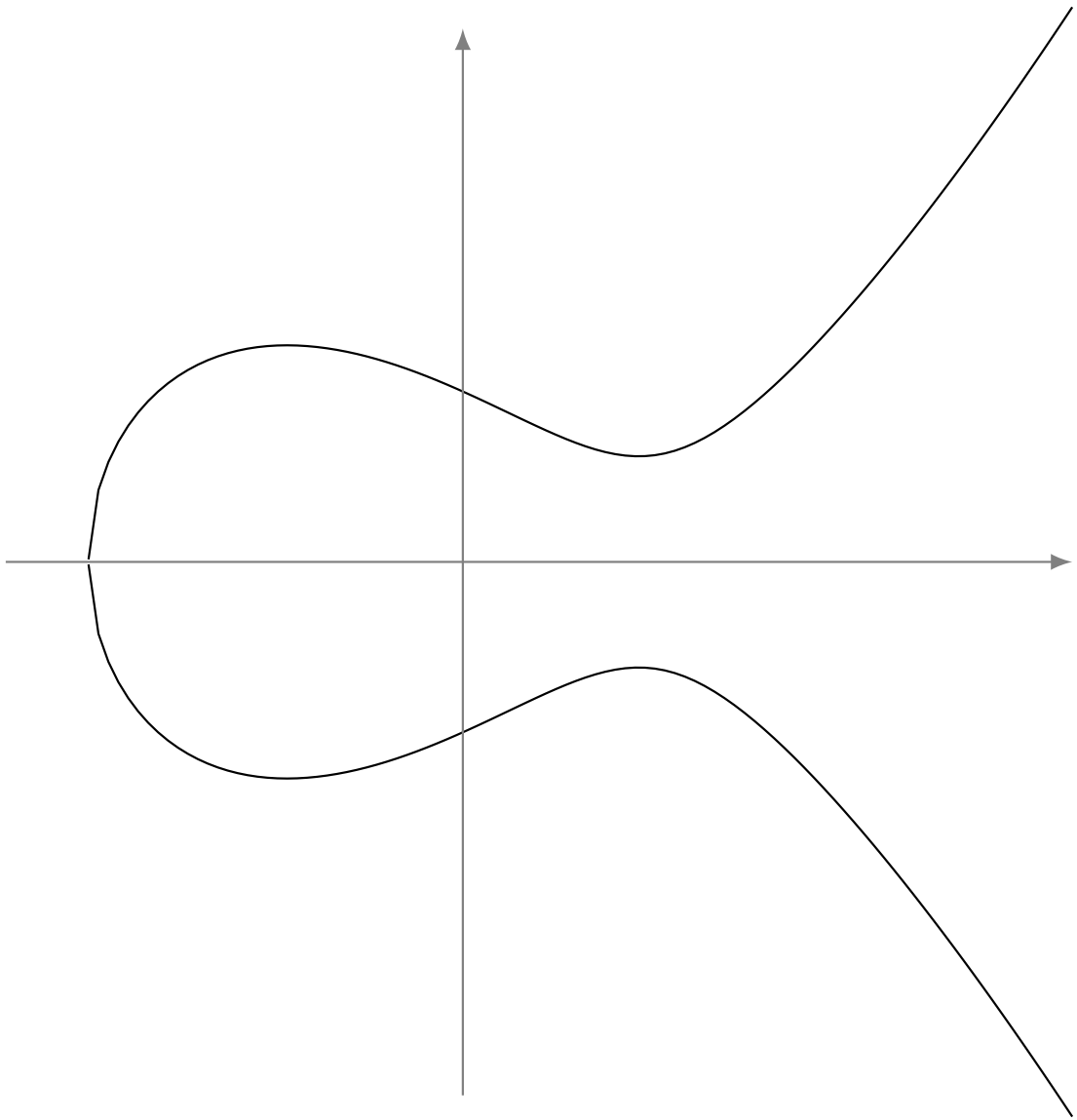
1. Isogeny graphs
   - Elliptic Curves
   - Isogenies
   - Isogeny graphs
   - Endomorphism rings
   - Ordinary graphs
   - Supersingular graphs
2. Cryptography
   - Isogeny walks and Hash functions
   - Pairing verification and Verifiable Delay Functions
   - Key exchange
   - Proofs of knowledge and Signatures
   - Open Problems

Elliptic curves

Let k be a field of characteristic $\neq 2, 3$.
An elliptic curve *defined over* k is the locus in the projective space $\mathbb{P}^2(\bar{k})$ of an equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3,$$

where $a, b \in k$ and $4a^3 + 27b^2 \neq 0$.



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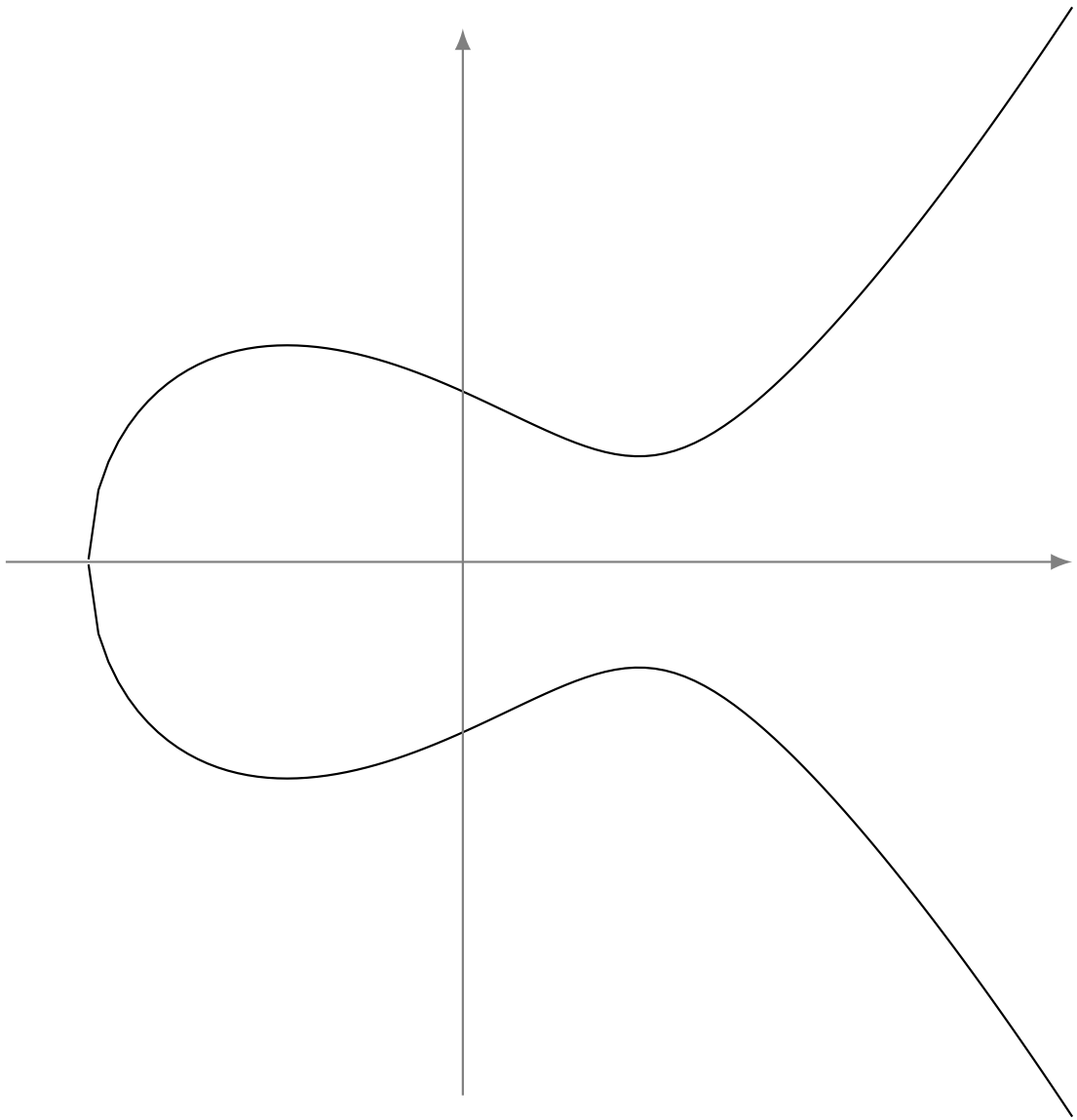
- $\mathcal{O} = (0 : 1 : 0)$ is the point at infinity;



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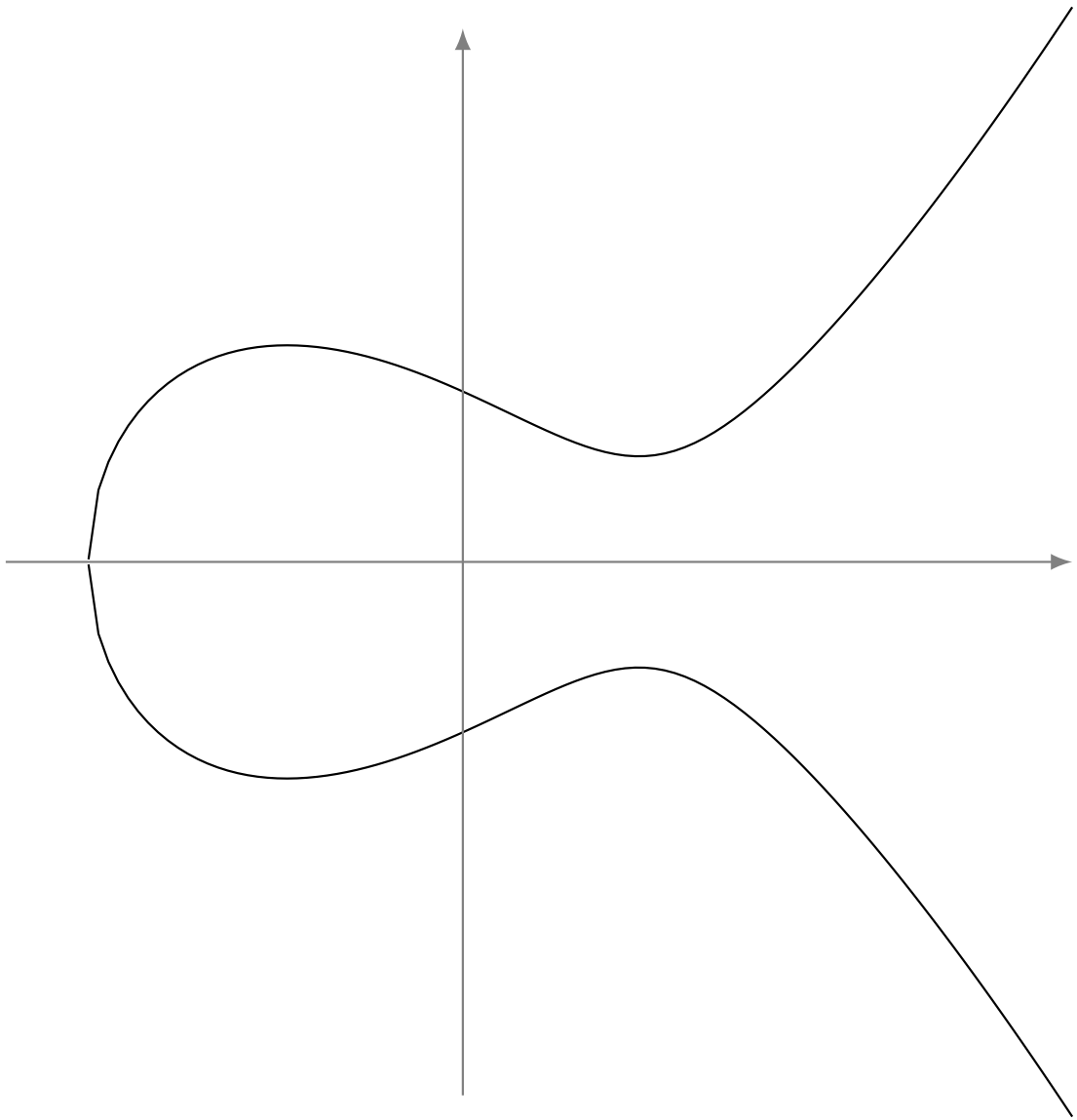
- $\mathcal{O} = (0 : 1 : 0)$ is the point at infinity;
- $y^2 = x^3 + ax + b$ is the affine Weierstrass equation.

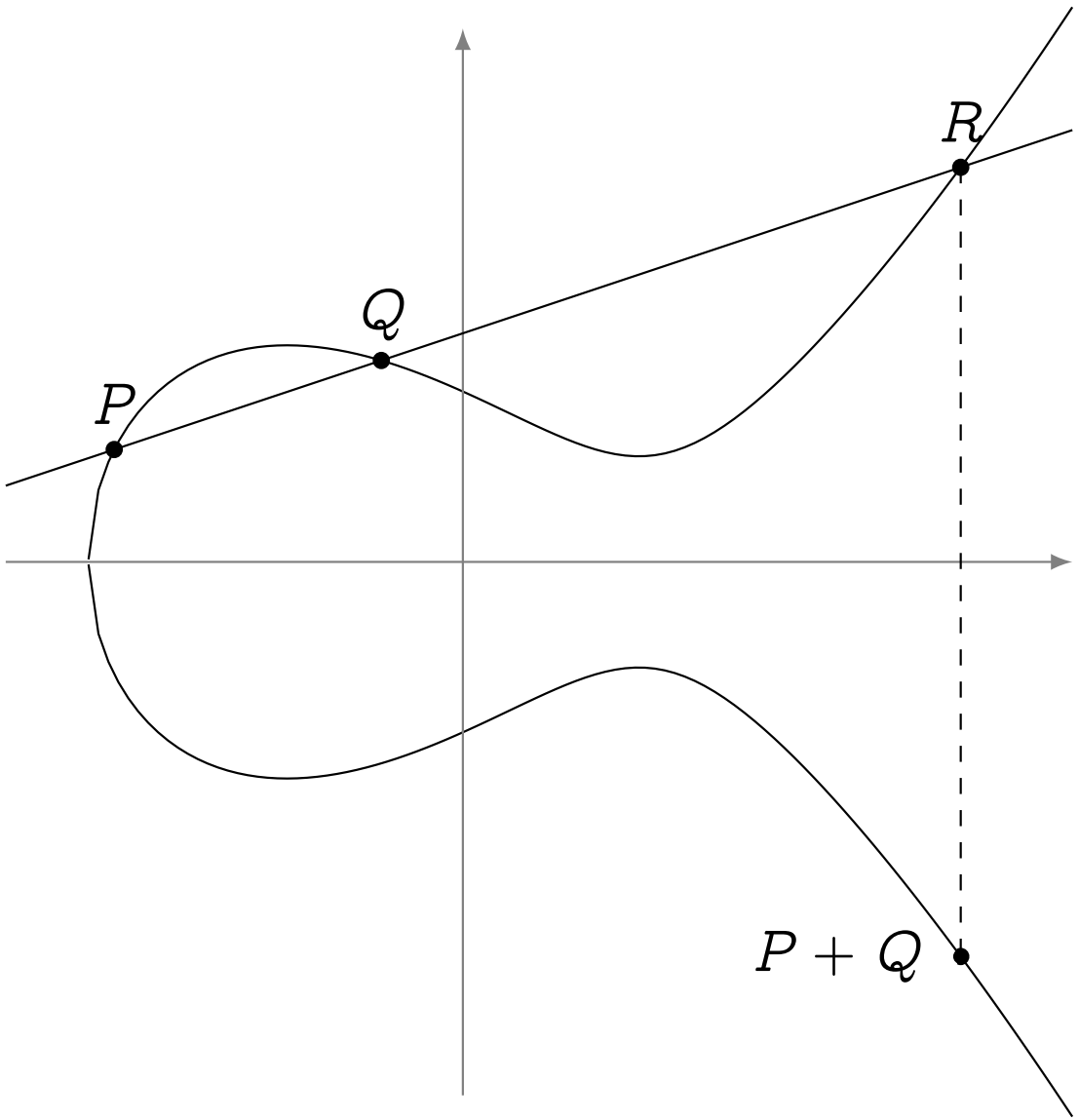


The group law

Bezout's theorem

Every line cuts E in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.



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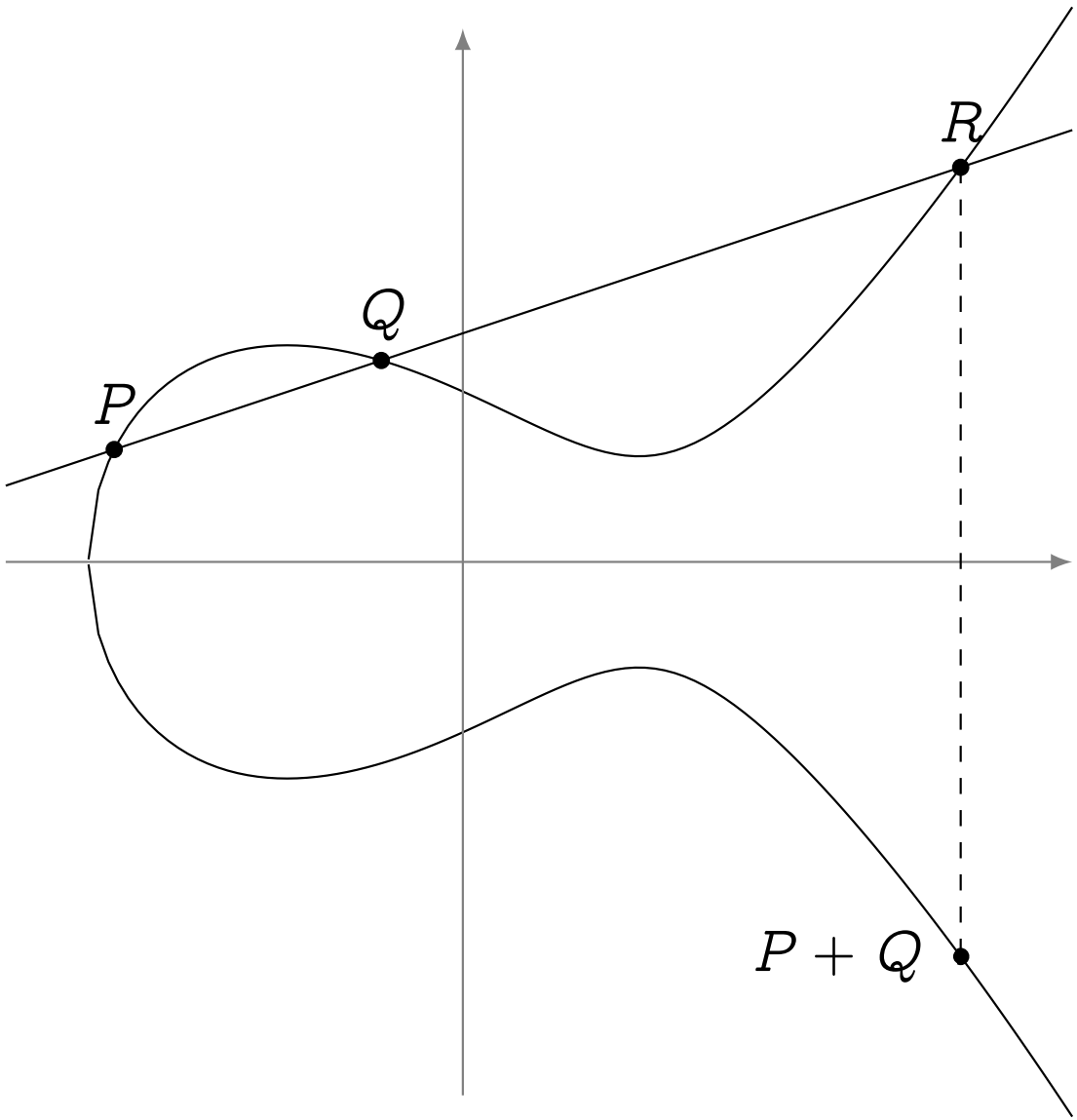

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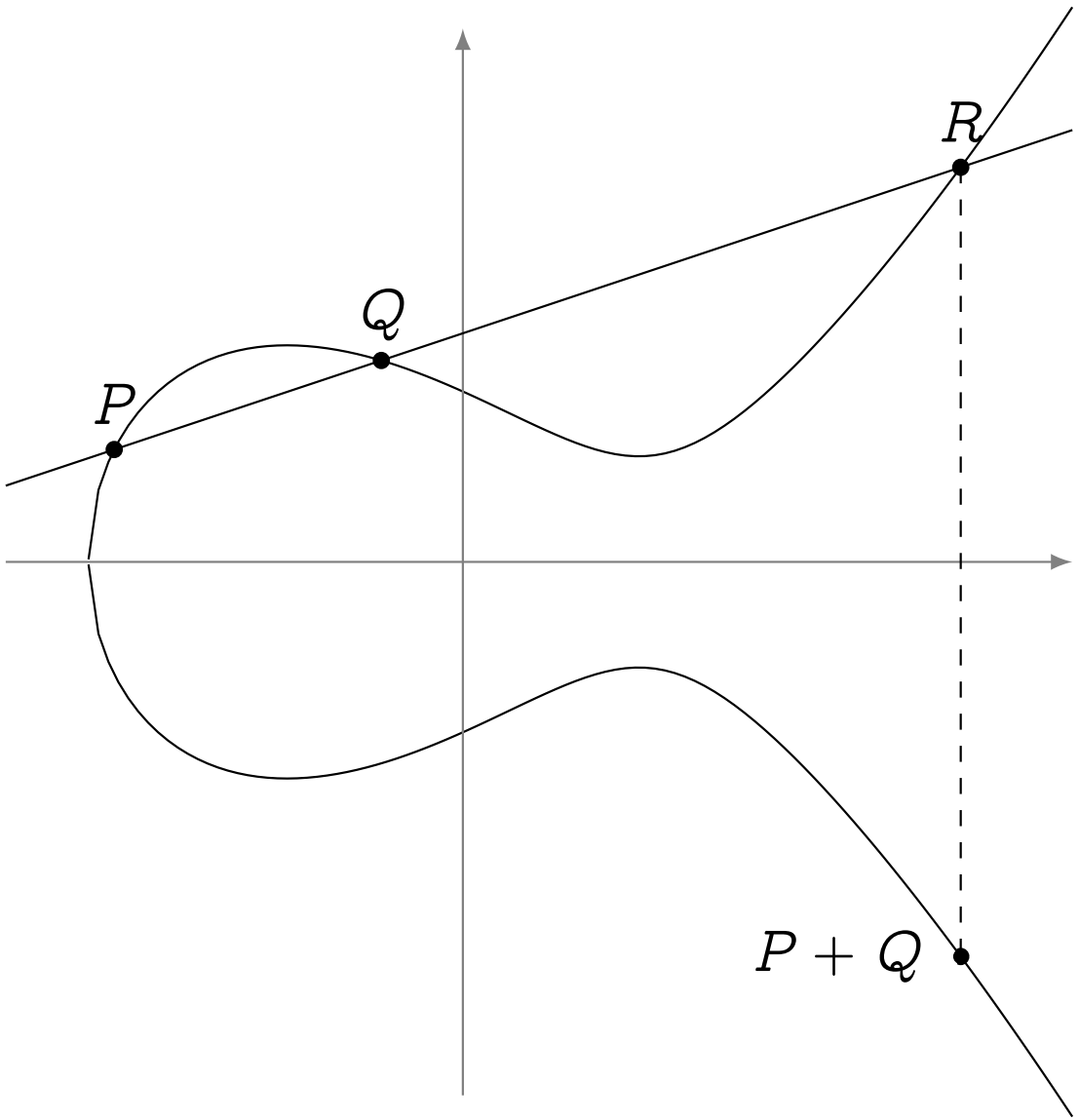
Every line cuts E in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.

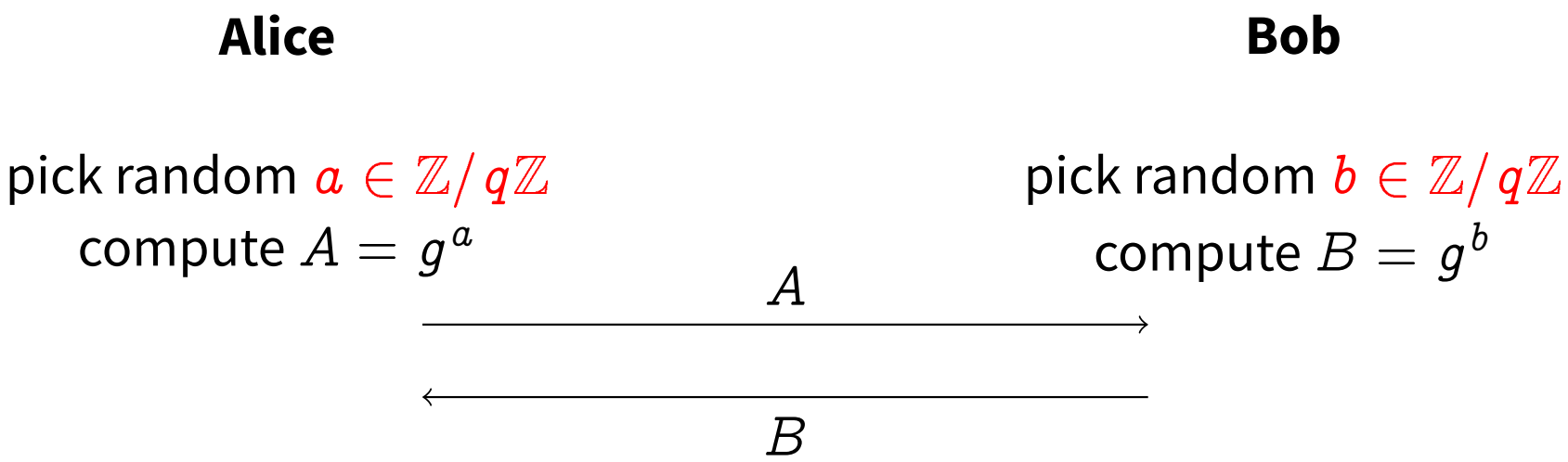
- The law is algebraic (it has *formulas*);
- The law is commutative;
- $\mathcal{O}$ is the group identity;
- Opposite points have the same x-value.



Why should I care? (Diffie–Hellman key exchange)

Goal: Alice and Bob have never met before. They are chatting over a public channel, and want to agree on a shared secret to start a private conversation.

Setup: They agree on a (large) cyclic group $G = \langle g \rangle$ of (prime) order q.



Shared secret is $B^a = g^{ab} = A^b$

Brief history of DH key exchange

1976 Diffie & Hellman publish New directions in cryptography, suggest using $G = \mathbb{F}_p^*$.

1978 Pollard publishes his discrete logarithm algorithm ($O(\sqrt{\#G})$ complexity).

1980 Miller and Koblitz independently suggest using elliptic curves $G = E(\mathbb{F}_p)$.

1994 Shor publishes his quantum polynomial time discrete logarithm / factoring algorithm.

2005 NSA standardizes elliptic curve key agreement (ECDH) and signatures ECDSA.

2017 $\sim 70\%$ of web traffic is secured by ECDH and/or ECDSA.

2017 NIST launches post-quantum competition, says “not to bother moving to elliptic curves, if you haven’t yet”.

Why should I care? (cont'd)

But, also:

- Elliptic Curve Factoring Method (Lenstra '85);
- Elliptic Curve Primality Proving (Atkin, Morain '86-'93);
- Efficient normal bases for finite fields (Couveignes, Lercier '10);
- ...

What are elliptic curves?

For mathematicians

- The smooth projective curves of genus 1;
- The simplest abelian varieties (dimension 1);
- Finitely generated abelian groups of mysterious free rank (aka BSD conjecture);
- What you use to make examples.

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For cryptographers

- Finite abelian groups (often cyclic);
- Easy to compute the order;
- "2-dimensional" generalizations of μ_k (the roots of unity of k)…
- …with bilinear maps (aka pairings)!

Isomorphisms

Isomorphisms

The only invertible algebraic maps between elliptic curves are of the form

$$(x, y) \mapsto (u^2 x, u^3 y)$$

for some $u \in \bar{k}$.
They are group isomorphisms.

j-Invariant

Let $E : y^2 = x^3 + ax + b$, its j-invariant is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Two elliptic curves E, E' are isomorphic if and only if $j(E) = j(E')$.

Group structure

Torsion structure

Let E be defined over an algebraically closed field $\bar{k}$ of characteristic p.

$$E[m] \simeq \quad \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \qquad \text{if } p \nmid m,$$

$$E[p^e] \simeq \begin{cases} \mathbb{Z}/p^e\mathbb{Z} & \text{ordinary case,} \\ \{\mathcal{O}\} & \text{supersingular case.} \end{cases}$$

Finite fields (Hasse's theorem)

Let E be defined over a finite field $\mathbb{F}_q$, then

$$|\#E(\mathbb{F}_q) - q - 1| \leq 2\sqrt{q}.$$

In particular, there exist integers n_1 and $n_2 | (q-1)$ such that

$$E(\mathbb{F}_q) \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}.$$

What is scalar multiplication?

$$[n] \ : \ P \mapsto \underbrace{P + P + \cdots + P}_{n \text{ times}}$$

- A map $E \to E$,
- a group morphism,
- with finite kernel
  (the torsion group $E[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$),
- surjective (in the algebraic closure),
- given by rational maps of degree n^2.

What is scalar multiplication an isogeny?

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(Separable) isogenies $\Leftrightarrow$ finite subgroups:

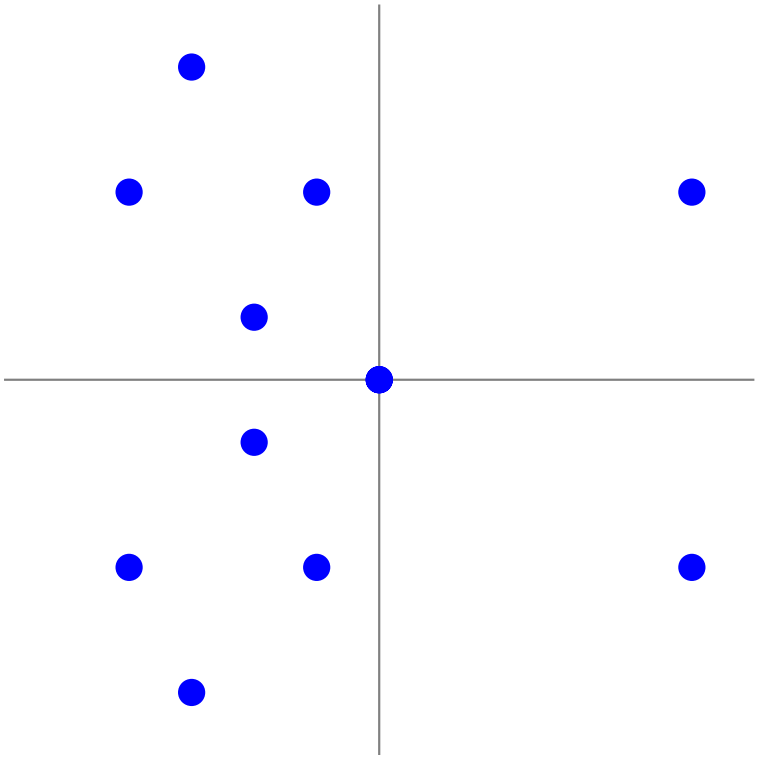
$$0 \longrightarrow H \longrightarrow E \xrightarrow{\phi} E' \to 0$$

The kernel H determines the image curve E' up to isomorphism

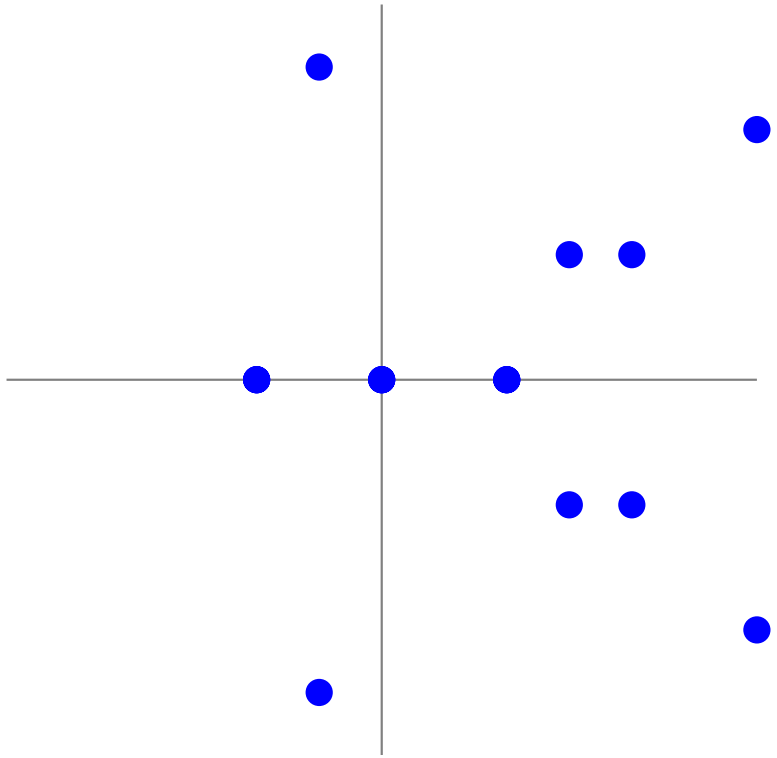
$$E/H \stackrel{\text{def}}{=} E'.$$

Isogenies: an example over $\mathbb{F}_{11}$

$E : y^2 = x^3 + x$



$E' : y^2 = x^3 - 4x$



$$\phi(x, y) = \left(\frac{x^2 + 1}{x}, \quad y \frac{x^2 - 1}{x^2} \right)$$

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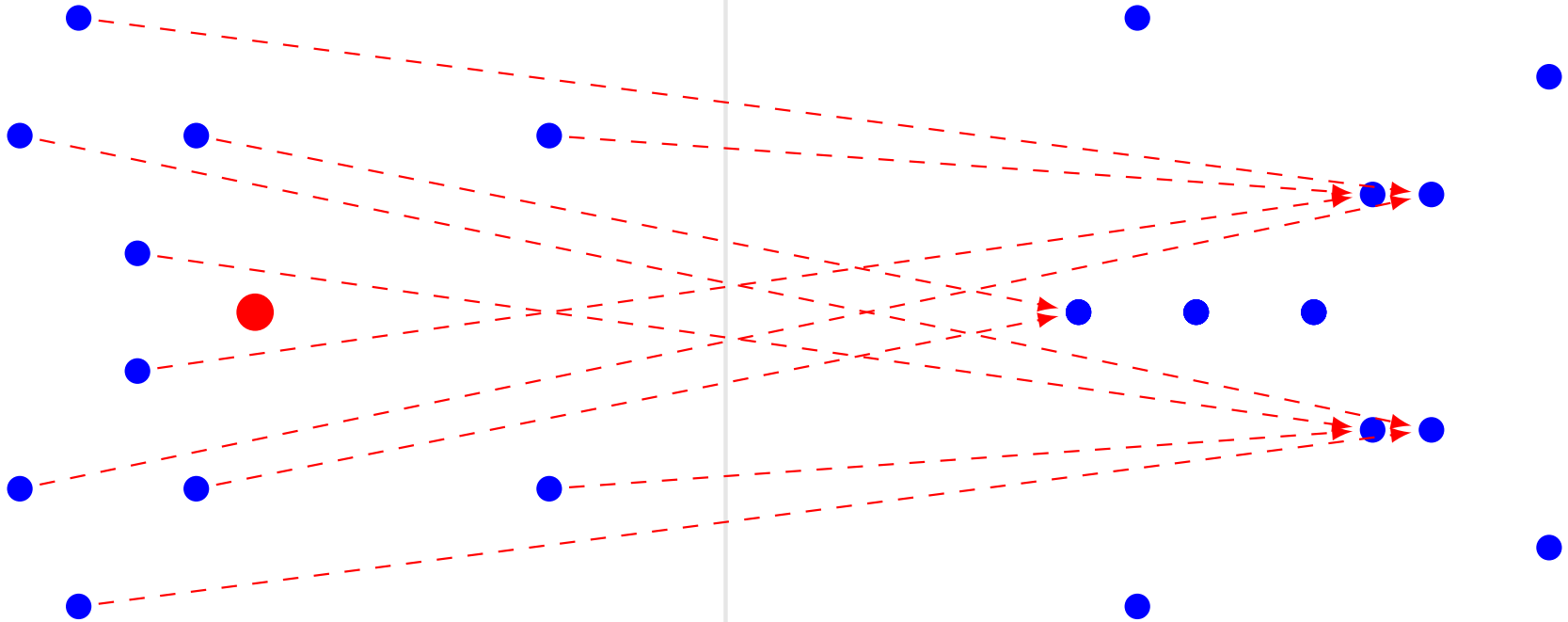

$$\phi(x, y) = \left(\frac{x^2 + 1}{x}, \quad y \frac{x^2 - 1}{x^2} \right)$$

- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

Isogeny properties

Let $\phi : E \to E'$ be an isogeny defined over a field k of characteristic p.

- $k(E)$ is the field of all rational functions from E to k;
- $\phi^* k(E')$ is the subfield of $k(E)$ defined as

$$\phi^* k(E') = \{f \circ \phi \mid f \in k(E')\}.$$

Degree, separability

1. The degree of ϕ is $\deg \phi = [k(E) : \phi^* k(E')]$. It is always finite.
2. ϕ is said to be separable, inseparable, or purely inseparable if the extension of function fields is.
3. If ϕ is separable, then $\deg \phi = \# \ker \phi$.
4. If ϕ is purely inseparable, then $\ker \phi = \{\mathcal{O}\}$ and $\deg \phi$ is a power of p.
5. Any isogeny can be decomposed as a product of a separable and a purely inseparable isogeny.

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The dual isogeny

Let $\phi : E \to E'$ be an isogeny of degree m. There is a unique isogeny $\hat{\phi} : E' \to E$ such that

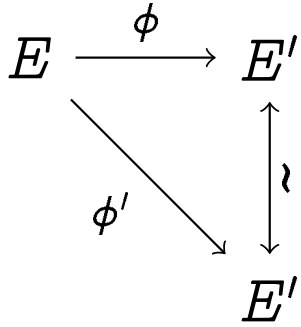
$$\hat{\phi} \circ \phi = [m]_E, \quad \phi \circ \hat{\phi} = [m]_{E'}.$$

$\hat{\phi}$ is called the dual isogeny of ϕ; it has the following properties:

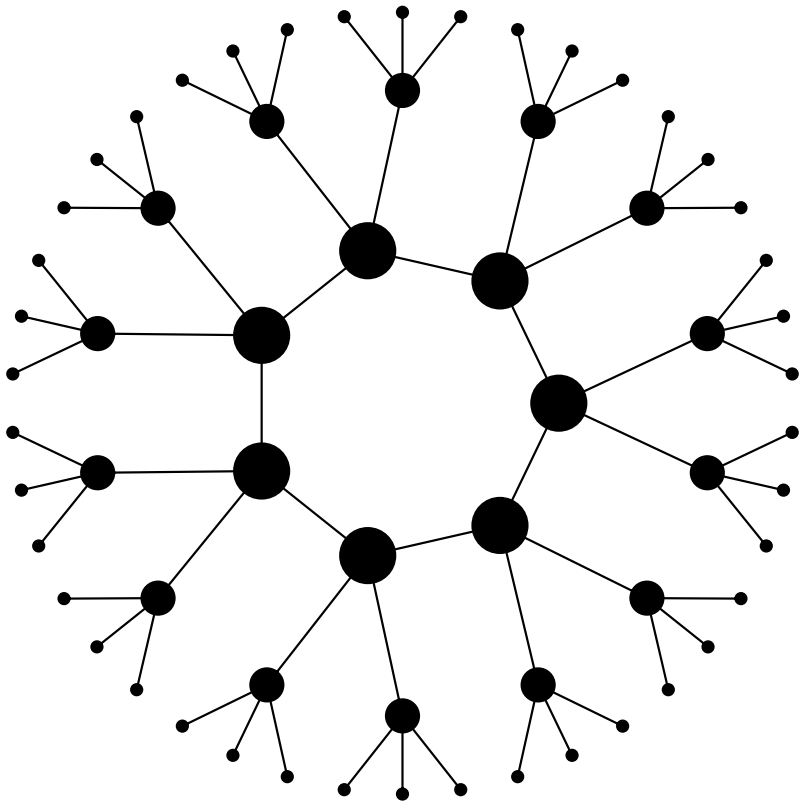
1. $\hat{\phi}$ is defined over k if and only if ϕ is;
2. $\widehat{\psi \circ \phi} = \hat{\phi} \circ \hat{\psi}$ for any isogeny $\psi : E' \to E''$;
3. $\widehat{\psi + \phi} = \hat{\psi} + \hat{\phi}$ for any isogeny $\psi : E \to E'$;
4. $\deg \phi = \deg \hat{\phi}$;
5. $\hat{\hat{\phi}} = \phi$.

Isogeny graphs

We look at the graph of elliptic curves with isogenies up to isomorphism. We say two isogenies ϕ, ϕ' are isomorphic if:



Example: Finite field, ordinary case, graph of isogenies of degree 3.



What do isogeny graphs look like?

Torsion subgroups (ℓ prime)

In an algebraically closed field:

$$E[\ell] = \langle P, Q \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

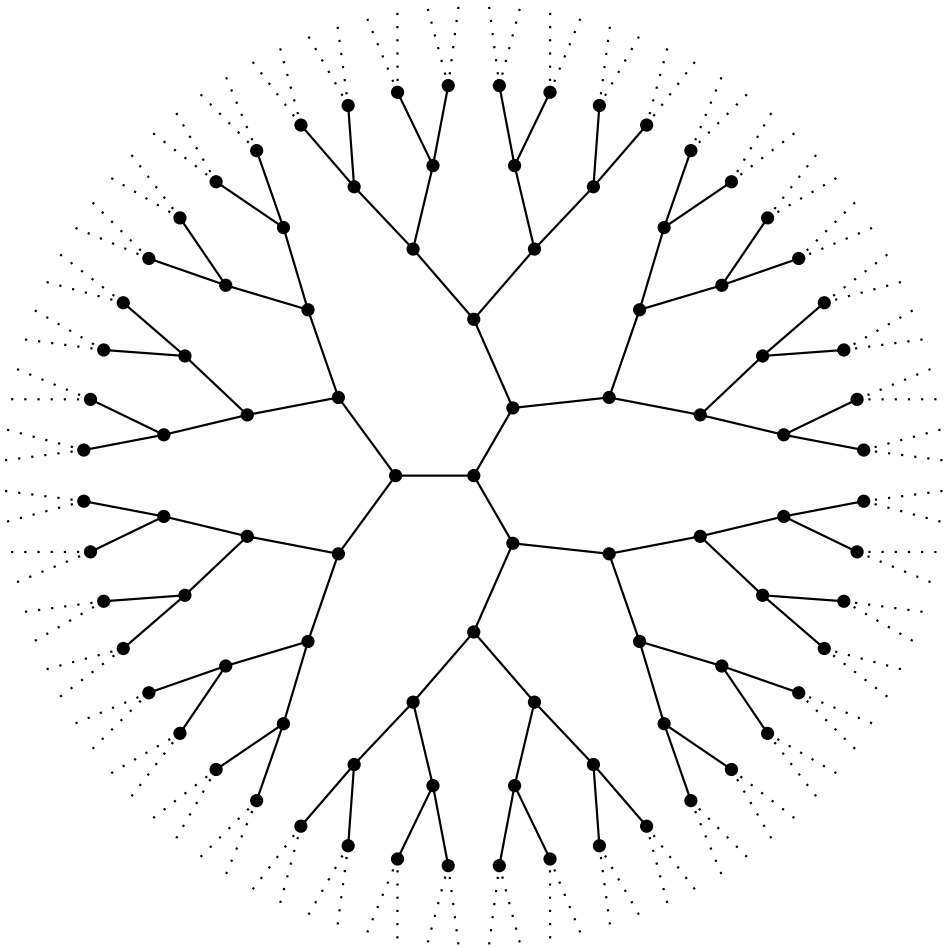
$$\Downarrow$$

There are exactly $\ell + 1$ cyclic subgroups $H \subset E$ of order ℓ:

$$\langle P + Q \rangle, \langle P + 2Q \rangle, \ldots, \langle P \rangle, \langle Q \rangle$$

$$\Downarrow$$

There are exactly $\ell + 1$ distinct isogenies of degree ℓ.



(non-CM) 2-isogeny graph over $\mathbb{C}$

What happens over a finite field $\mathbb{F}_p$?

Rational isogenies ($\ell \neq p$)

In the algebraic closure $\bar{\mathbb{F}}_p$

$$E[\ell] = \langle P, Q \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

However, an isogeny is defined over $\mathbb{F}_p$ only if its kernel is Galois invariant.

Enter the Frobenius map

$$\begin{aligned} \pi : E &\longrightarrow E \\ (x, y) &\longmapsto (x^p, y^p) \end{aligned}$$

E is seen here as a curve over $\bar{\mathbb{F}}_p$.

The Frobenius action on $E[\ell]$

$$\begin{aligned} \pi(P) &= aP + bQ \\ \pi(Q) &= cP + dQ \end{aligned}$$

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$$\pi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod \ell$$

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The Frobenius action on $E[\ell]$

$$\pi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod \ell$$

We identify $\pi | E[\ell]$ to a conjugacy class in $\mathrm{GL}(\mathbb{Z}/\ell\mathbb{Z})$.

What happens over a finite field $\mathbb{F}_p$?

Galois invariant subgroups of $E[\ell]$

=

eigenspaces of $\pi \in \mathrm{GL}(\mathbb{Z}/\ell\mathbb{Z})$

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rational isogenies of degree ℓ

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How many Galois invariant subgroups?

- $\pi|E[\ell] \sim \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ $\rightarrow \ell + 1$ isogenies
- $\pi|E[\ell] \sim \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ with $\lambda \neq \mu$ $\rightarrow$ two isogenies
- $\pi|E[\ell] \sim \left(\begin{smallmatrix} \lambda & * \\ 0 & \lambda \end{smallmatrix}\right)$ $\rightarrow$ one isogeny
- $\pi|E[\ell]$ is not diagonalizable $\rightarrow$ no isogeny

Weil pairing

Let $(N, p) = 1$, fix any basis $E[N] = \langle R, S \rangle$. For any points $P, Q \in E[N]$

$$P = aR + bS$$
$$Q = cR + dS$$

the form $\det_N(P, Q) = \det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = ad - bc \in \mathbb{Z}/N\mathbb{Z}$
is bilinear, non-degenerate, and independent from the choice of basis.

Theorem

Let $E/\mathbb{F}_q$ be a curve, there exists a Galois invariant bilinear map

$$e_N : E[N] \times E[N] \longrightarrow \mu_N \subset \bar{\mathbb{F}}_q,$$

called the Weil pairing of order N, and a primitive N-th root of unity $\zeta \in \bar{\mathbb{F}}_q$ such that

$$e_N(P, Q) = \zeta^{\det_N(P,Q)}.$$

The degree k of the smallest extension such that $\zeta \in \mathbb{F}_{q^k}$ is called the embedding degree of the pairing.

Weil pairing and isogenies

Note

The Weil pairing is Galois invariant $\Leftrightarrow$ $\det(\pi | E[N]) = q$.

Theorem

Let $\phi : E \to E'$ be an isogeny and $\hat{\phi} : E' \to E$ its dual.
Let e_N be the Weil pairing of E and e'_N that of E'. Then, for

$$e_N(P, \hat{\phi}(Q)) = e_N(\phi(P), Q),$$

for any $P \in E[N]$ and $Q \in E'[N]$.

Corollary

$$e'_N(\phi(P), \phi(Q)) = e_N(P, Q)^{\deg \phi}.$$

From local to global

Theorem (Hasse)

Let E be defined over a finite field $\mathbb{F}_q$. Its Frobenius map π satisfies a quadratic equation

$$\pi^2 - t\pi + q = 0$$

for some $|t| \leq 2\sqrt{q}$, called the trace of π. The trace t is coprime to q if and only if E is ordinary.

Endomorphisms

An isogeny $E \to E$ is also called an endomorphism. Examples:

- scalar multiplication $[n]$,
- Frobenius map π.

With addition and composition, the endomorphisms form a ring $\mathrm{End}(E)$.

The endomorphism ring

Theorem (Deuring)

Let E be an ordinary elliptic curve defined over a finite field $\mathbb{F}_q$.
Let π be its Frobenius endomorphism, and $D_\pi = t^2 - 4q < 0$ the discriminant of its minimal polynomial.
Then $\mathrm{End}(E)$ is isomorphic to an order $\mathcal{O}$ of the quadratic imaginary field $\mathbb{Q}(\sqrt{D_\pi})$.[a]

[a] An order is a subring that is a $\mathbb{Z}$-module of rank 2 (equiv., a 2-dimensional $\mathbb{R}$-lattice).

In this case, we say that E has complex multiplication (CM) by $\mathcal{O}$.

Theorem (Serre-Tate)

CM elliptic curves E, E' are isogenous iff $\mathrm{End}(E) \otimes \mathbb{Q} \simeq \mathrm{End}(E') \otimes \mathbb{Q}$.

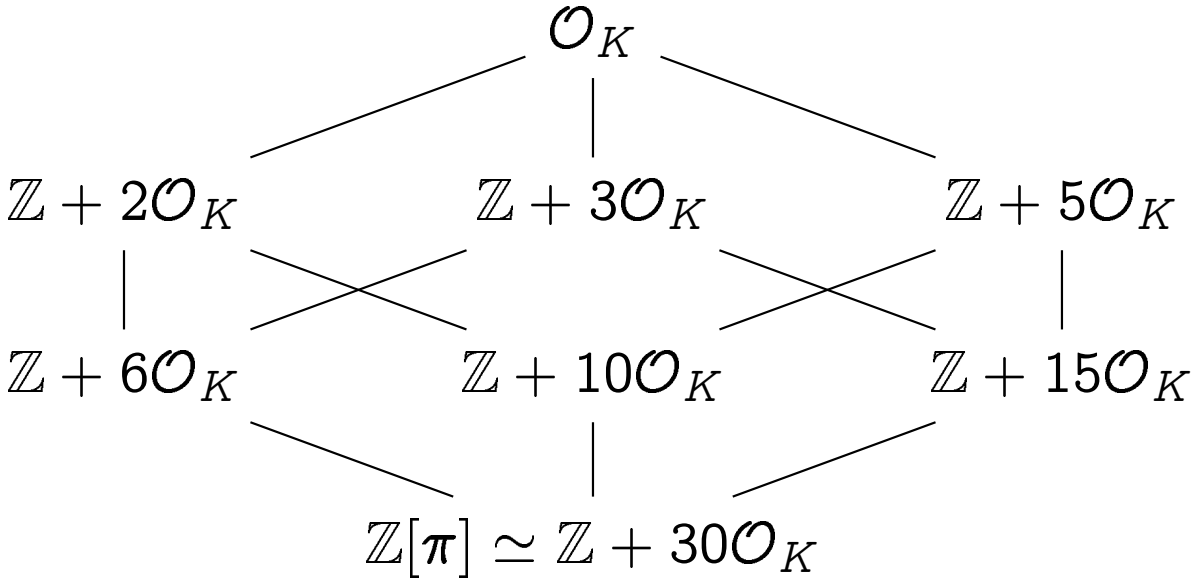
Corollary: $E/\mathbb{F}_p$ and $E'/\mathbb{F}_p$ are isogenous iff $\#E(\mathbb{F}_p) = \#E'(\mathbb{F}_p)$.

Endomorphism rings of ordinary curves

Classifying quadratic orders

Let K be a quadratic number field, and let $\mathcal{O}_K$ be its ring of integers.

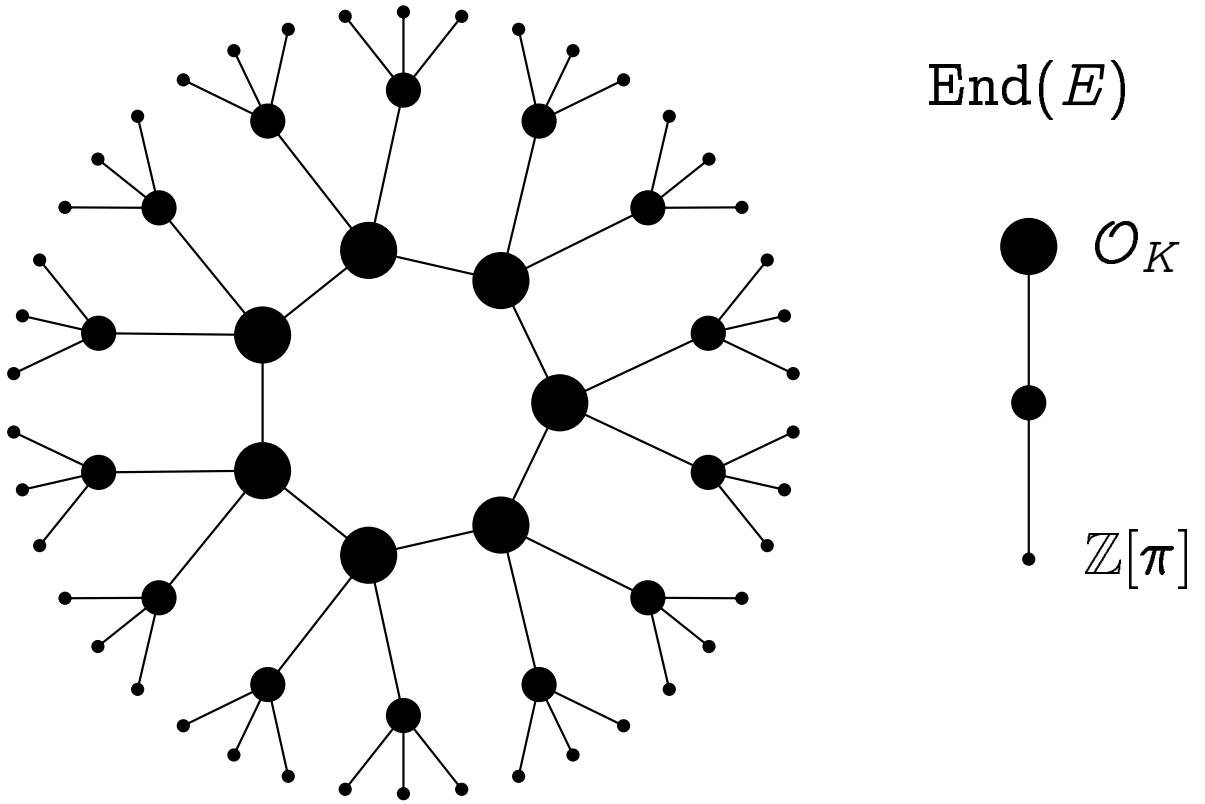
- Any order $\mathcal{O} \subset K$ can be written as $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ for an integer f, called the conductor of $\mathcal{O}$, denoted by $[\mathcal{O}_K : \mathcal{O}]$.
- If D_K is the discriminant of K, the discriminant of $\mathcal{O}$ is $f^2 D_K$.
- If $\mathcal{O}, \mathcal{O}'$ are two orders with discriminants D, D', then $\mathcal{O} \subset \mathcal{O}'$ iff $D' \mid D$.



Volcanology (Kohel 1996)

Let E, E' be curves with respective endomorphism rings $\mathcal{O}, \mathcal{O}' \subset K$.
Let $\phi : E \to E'$ be an isogeny of prime degree ℓ, then:

if $\mathcal{O} = \mathcal{O}'$, ϕ is horizontal;
if $[\mathcal{O}' : \mathcal{O}] = \ell$, ϕ is ascending;
if $[\mathcal{O} : \mathcal{O}'] = \ell$, ϕ is descending.

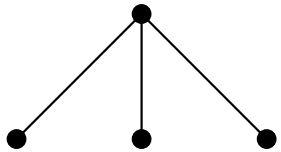


Ordinary isogeny volcano of degree $\ell = 3$.

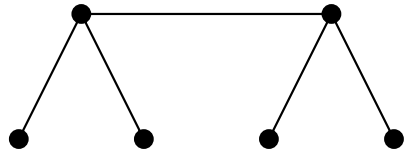
Volcanology (Kohel 1996)

Let E be ordinary,
$\mathrm{End}(E) \subset K$.

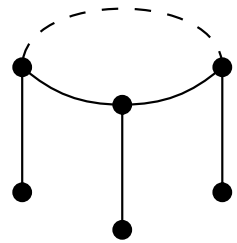
$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.



$\left(\frac{D_K}{\ell}\right) = -1$



$\left(\frac{D_K}{\ell}\right) = 0$



$\left(\frac{D_K}{\ell}\right) = +1$

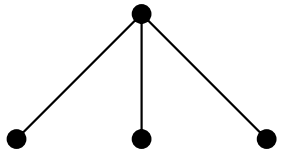
		Horizontal	**Ascending**	**Descending**
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		$\ell - \left(\frac{D_K}{\ell}\right)$
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	ℓ
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

Volcanology (Kohel 1996)

Let E be ordinary,
$\mathrm{End}(E) \subset K$.

$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.

Height $= v_\ell([\mathcal{O}_K : \mathbb{Z}[\pi]])$.



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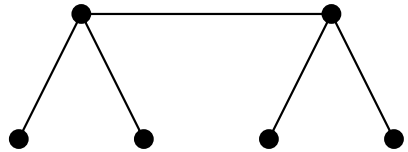
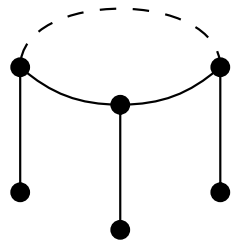


$\left(\frac{D_K}{\ell}\right) = 0$



$\left(\frac{D_K}{\ell}\right) = +1$

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$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		$\ell - \left(\frac{D_K}{\ell}\right)$
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$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

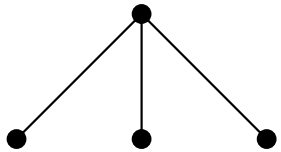
Volcanology (Kohel 1996)

Let E be ordinary,
$\mathrm{End}(E) \subset K$.

$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.

Height $= v_\ell([\mathcal{O}_K : \mathbb{Z}[\pi]])$.

How large is the crater?



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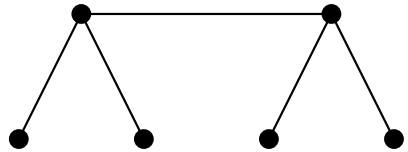

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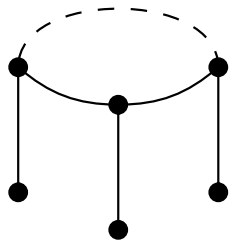

$\left(\frac{D_K}{\ell}\right) = +1$

		Horizontal	Ascending	Descending
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		
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$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	ℓ
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

How large is the crater of a volcano?

Let $\mathrm{End}(E) = \mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$. Define

- $\mathcal{I}(\mathcal{O})$, the group of invertible fractional ideals,
- $\mathcal{P}(\mathcal{O})$, the group of principal ideals,

The class group

The class group of $\mathcal{O}$ is

$$\mathrm{Cl}(\mathcal{O}) = \mathcal{I}(\mathcal{O})/\mathcal{P}(O).$$

- It is a finite abelian group.
- Its order $h(\mathcal{O})$ is called the class number of $\mathcal{O}$.
- It arises as the Galois group of an abelian extension of $\mathbb{Q}(\sqrt{-D})$.

Complex multiplication

The $\mathfrak{a}$-torsion

- Let $\mathfrak{a} \subset \mathcal{O}$ be an (integral invertible) ideal of $\mathcal{O}$;
- Let $E[\mathfrak{a}]$ be the subgroup of E annihilated by $\mathfrak{a}$:

$$E[\mathfrak{a}] = \{P \in E \mid \alpha(P) = 0 \text{ for all } \alpha \in \mathfrak{a}\};$$

- Let $\phi : E \to E_{\mathfrak{a}}$, where $E_{\mathfrak{a}} = E/E[\mathfrak{a}]$.

Then $\mathrm{End}(E_{\mathfrak{a}}) = \mathcal{O}$ (i.e., ϕ is horizontal).

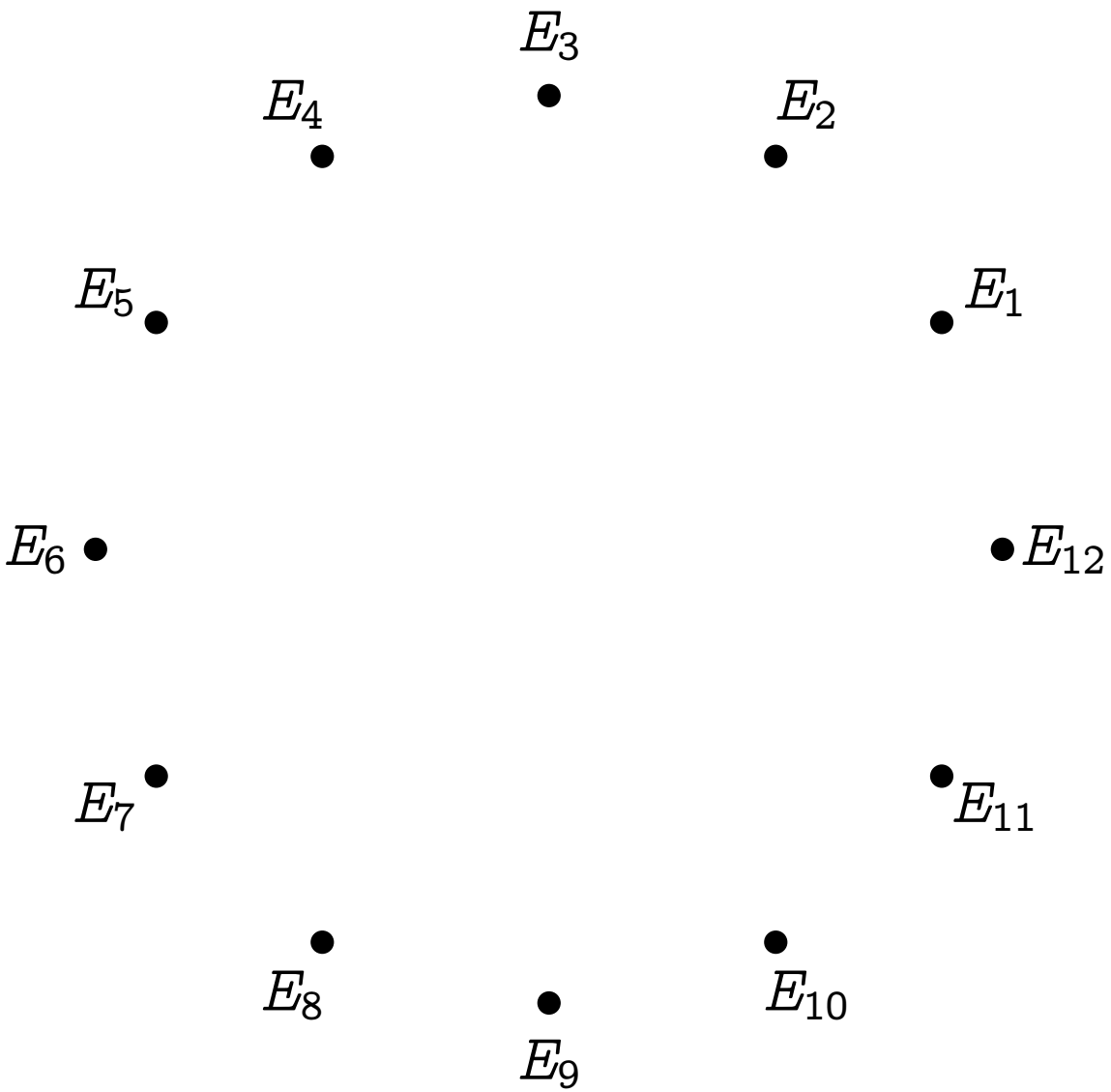
Theorem (Complex multiplication)

The action on the set of elliptic curves with complex multiplication by $\mathcal{O}$ defined by $\mathfrak{a} * j(E) = j(E_{\mathfrak{a}})$ factors through $\mathrm{Cl}(\mathcal{O})$, is faithful and transitive.

Corollary

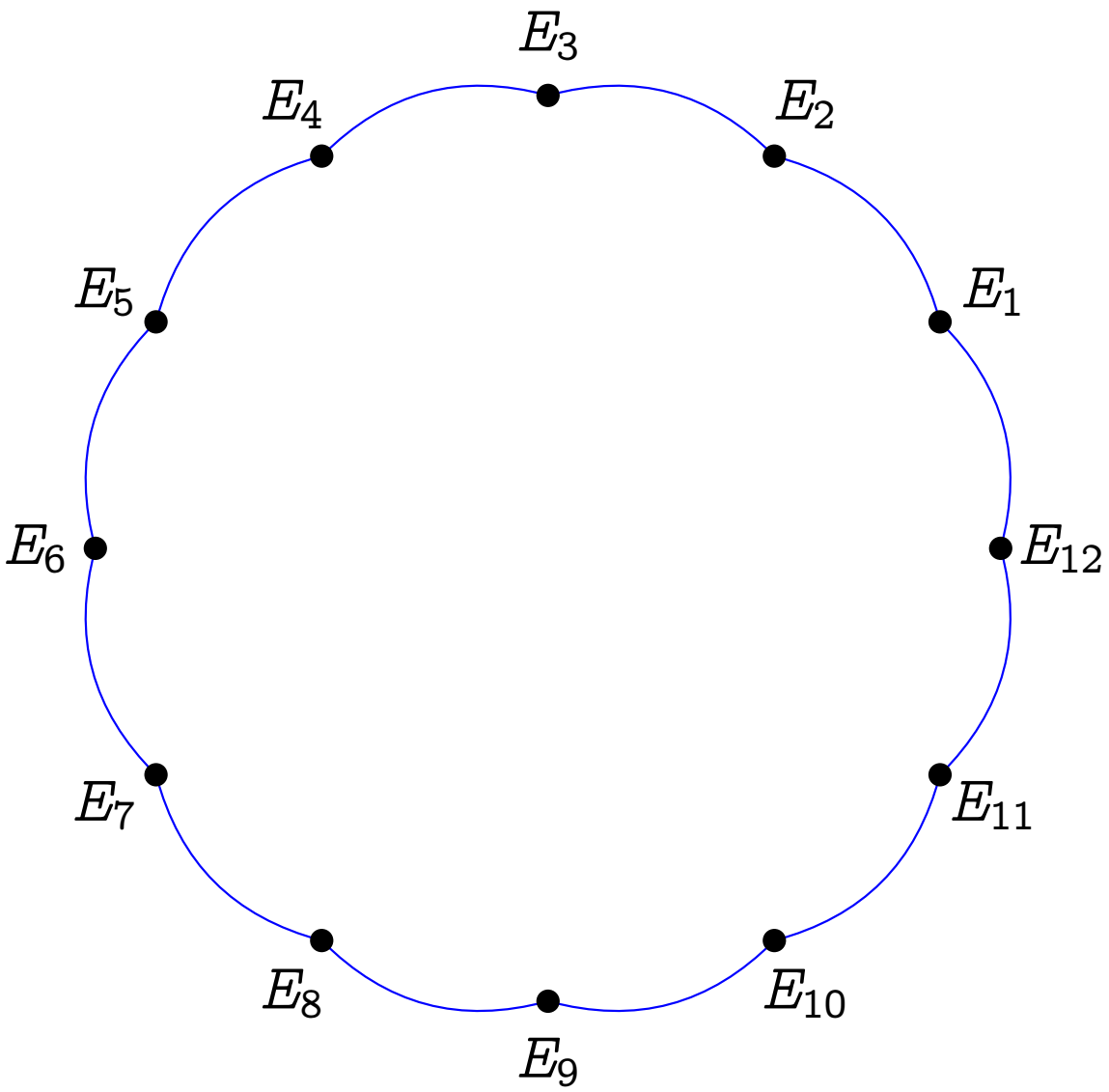
Let $\mathrm{End}(E)$ have discriminant D. Assume that $\left(\frac{D}{\ell}\right) = 1$, then E is on a crater of an ℓ-volcano, and the crater contains $h(\mathrm{End}(E))$ curves.

Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

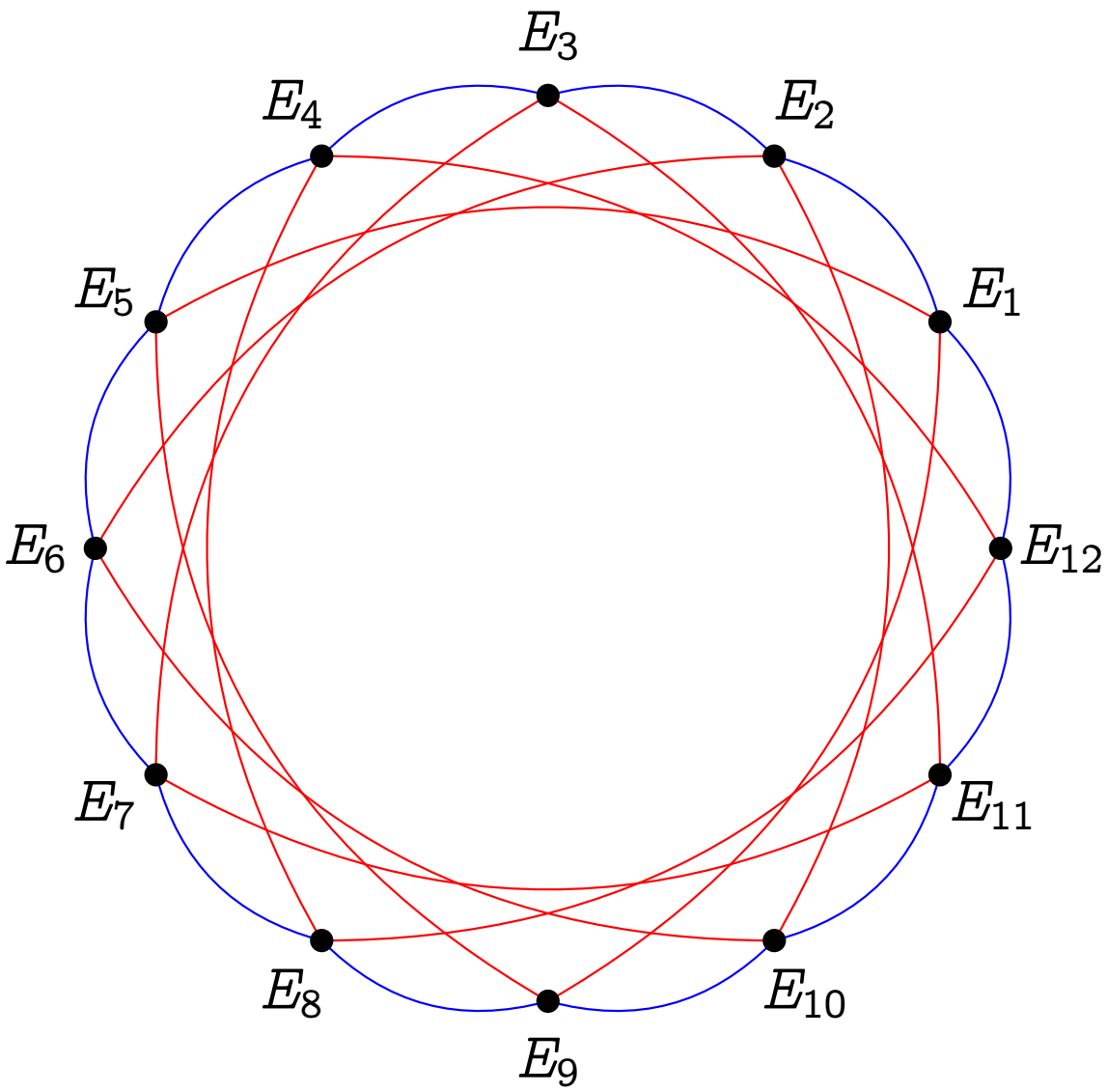
Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).
Edges are horizontal isogenies of bounded prime degree.

— degree 2

Complex multiplication graphs

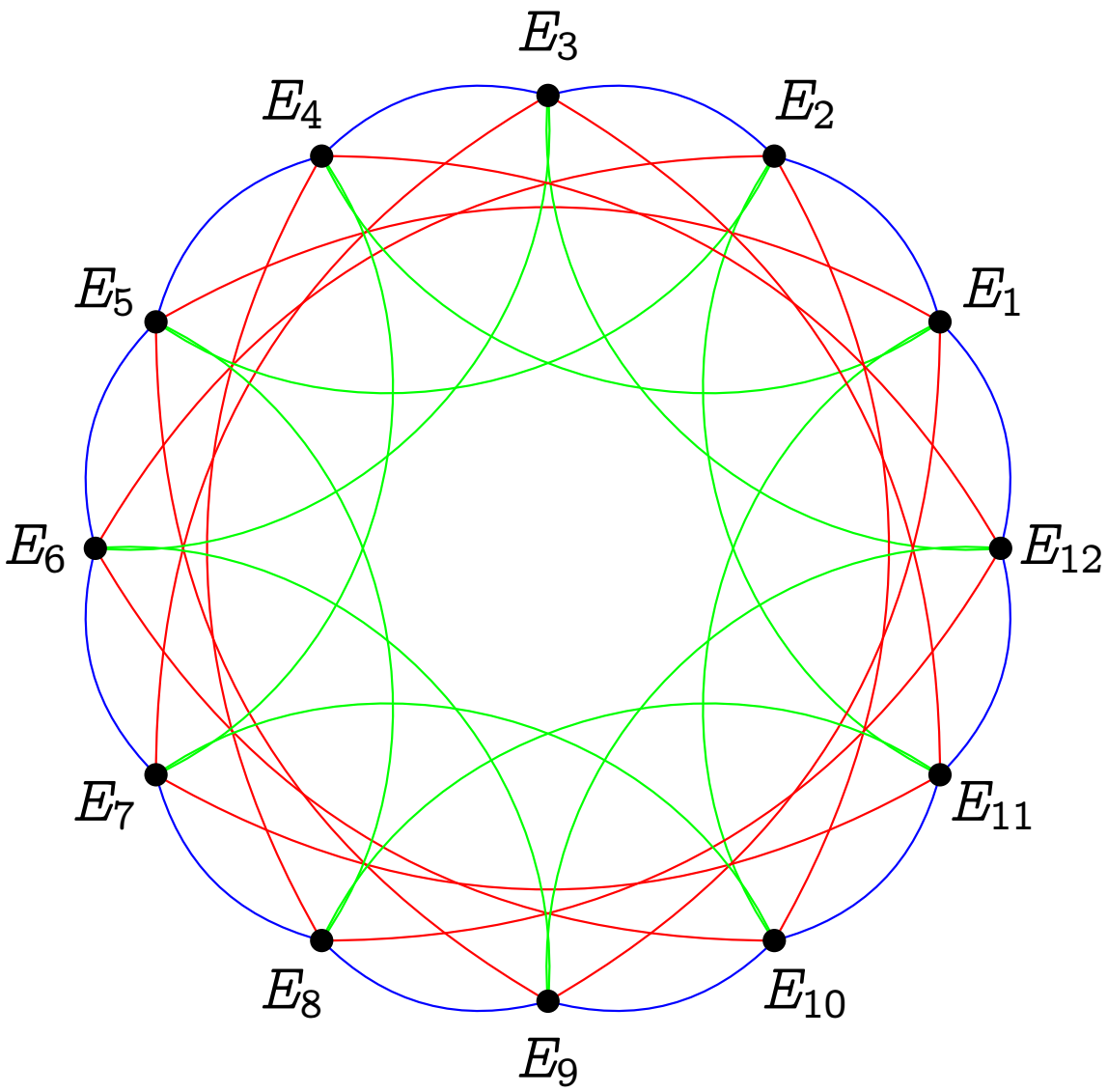


Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).
Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

Complex multiplication graphs



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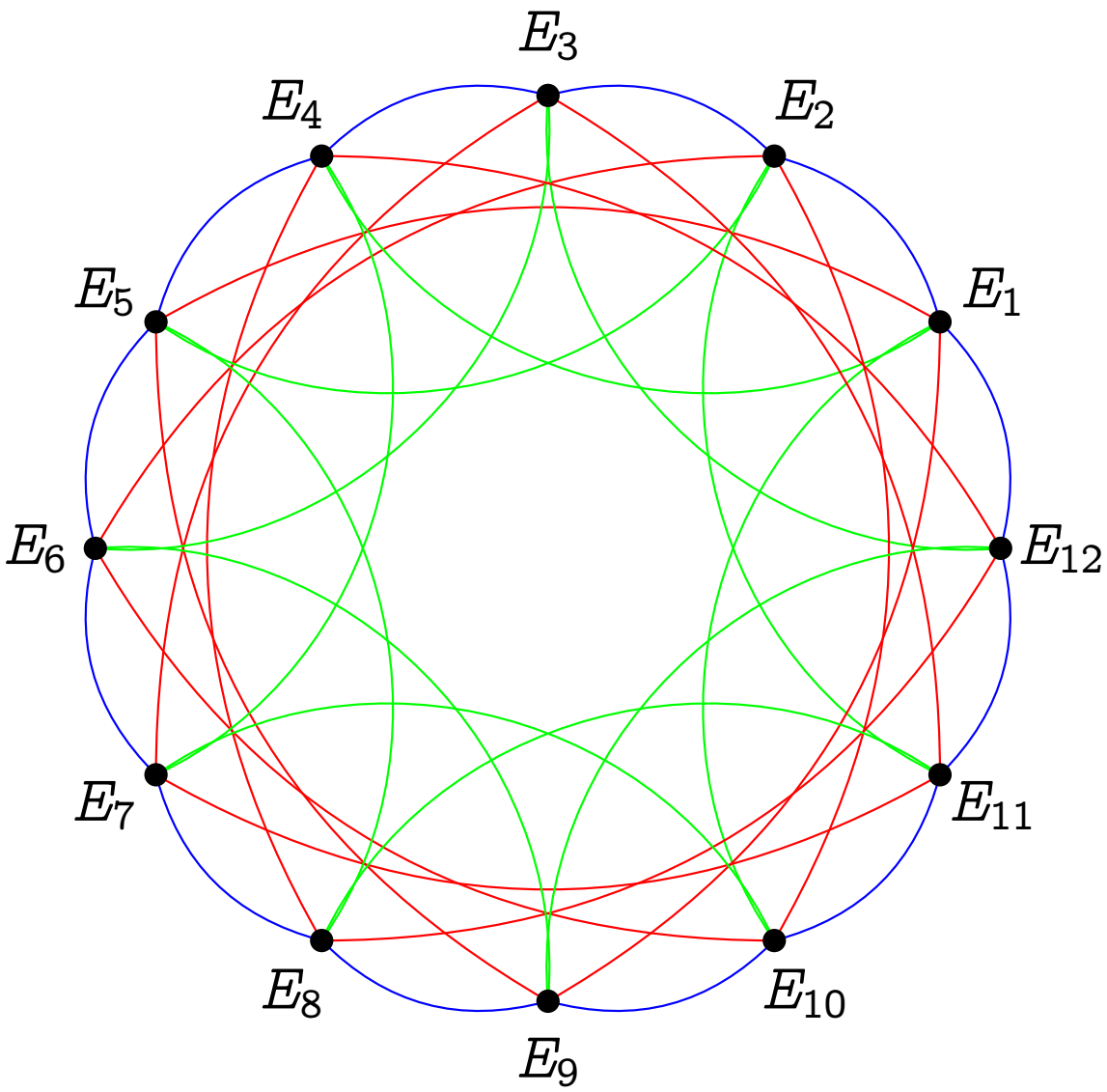
Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

— degree 5

Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).
Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

— degree 5

Isomorphic to a Cayley graph of $\mathrm{Cl}(\mathcal{O}_K)$.

Supersingular endomorphisms

Recall, a curve E over a field $\mathbb{F}_q$ of characteristic p is supersingular iff

$$\pi^2 - t\pi + q = 0$$

with $t = 0 \mod p$.

Case: $t = 0 \Rightarrow D_\pi = -4q$

- Only possibility for $E/\mathbb{F}_p$,
- E/F_p has CM by an order of $\mathbb{Q}(\sqrt{-p})$, similar to the ordinary case.

Case: $t = \pm 2\sqrt{q} \Rightarrow D_\pi = 0$

- General case for $E/\mathbb{F}_{p^2}$,
- $\pi = \pm q$, hence no complex multiplication.

We will ignore marginal cases: $t = \pm\sqrt{q}, \pm\sqrt{2q}, \pm\sqrt{3q}$.

Supersingular complex multiplication

Let $E/\mathbb{F}_p$ be a supersingular curve, then $\pi^2 = -p$, and

$$\pi = \begin{pmatrix} \sqrt{-p} & 0 \\ 0 & -\sqrt{-p} \end{pmatrix} \mod \ell$$

for any ℓ s.t. $\left(\frac{-p}{\ell}\right) = 1$.

Theorem (Delfs and Galbraith 2016)

Let $\mathrm{End}_{\mathbb{F}_p}(E)$ denote the ring of $\mathbb{F}_p$-rational endomorphisms of E. Then

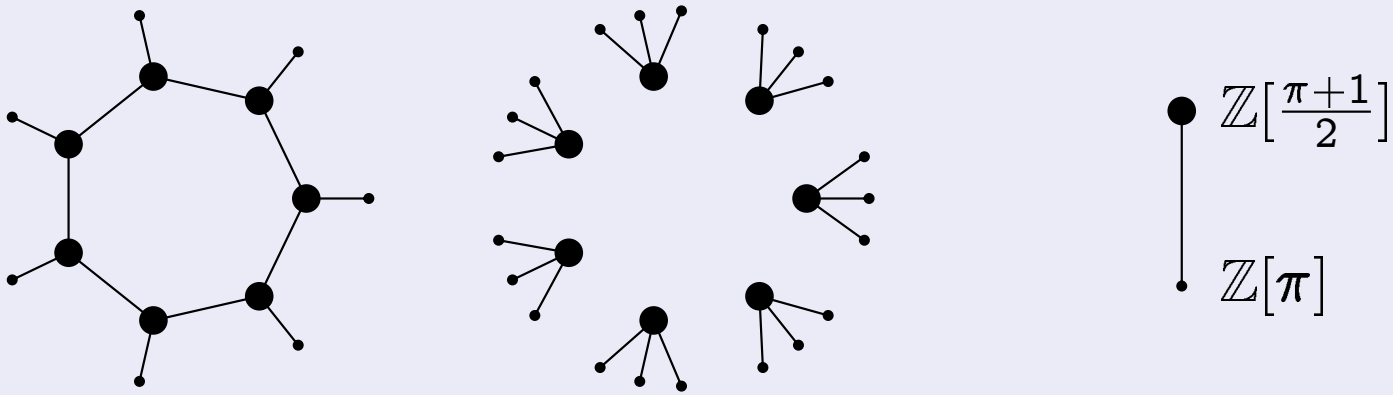
$$\mathbb{Z}[\pi] \subset \mathrm{End}_{\mathbb{F}_p}(E) \subset \mathbb{Q}(\sqrt{-p}).$$

Orders of $\mathbb{Q}(\sqrt{-p})$

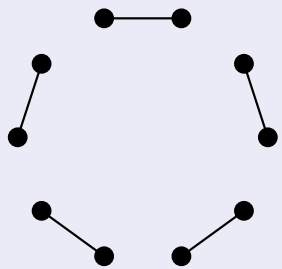
- If $p = 1 \mod 4$, then $\mathbb{Z}[\pi]$ is the maximal order.
- If $p = -1 \mod 4$, then $\mathbb{Z}[\frac{\pi+1}{2}]$ is the maximal order, and $[\mathbb{Z}[\frac{\pi+1}{2}] : \mathbb{Z}[\pi]] = 2$.

Supersingular CM graphs

2-volcanoes, $p = -1 \bmod 4$



2-graphs, $p = 1 \bmod 4$



- $\mathbb{Z}[\pi]$

All other ℓ-graphs are cycles of horizontal isogenies iff $\left(\frac{-p}{\ell}\right) = 1$.

The full endomorphism ring

Theorem (Deuring)

Let E be a supersingular elliptic curve, then

- E is isomorphic to a curve defined over $\mathbb{F}_{p^2}$;
- Every isogeny of E is defined over $\mathbb{F}_{p^2}$;
- Every endomorphism of E is defined over $\mathbb{F}_{p^2}$;
- $\mathrm{End}(E)$ is isomorphic to a maximal order in a quaternion algebra ramified at p and ∞.

In particular:

- If E is defined over $\mathbb{F}_p$, then $\mathrm{End}_{\mathbb{F}_p}(E)$ is strictly contained in $\mathrm{End}(E)$.
- Some endomorphisms do not commute!

An example

The curve of j-invariant 1728

$$E : y^2 = x^3 + x$$

is supersingular over $\mathbb{F}_p$ iff $p = -1 \mod 4$.

Endomorphisms

$\mathrm{End}(E) = \mathbb{Z}\langle \iota, \pi \rangle$, with:

- π the Frobenius endomorphism, s.t. $\pi^2 = -p$;
- ι the map

$$\iota(x, y) = (-x, iy),$$

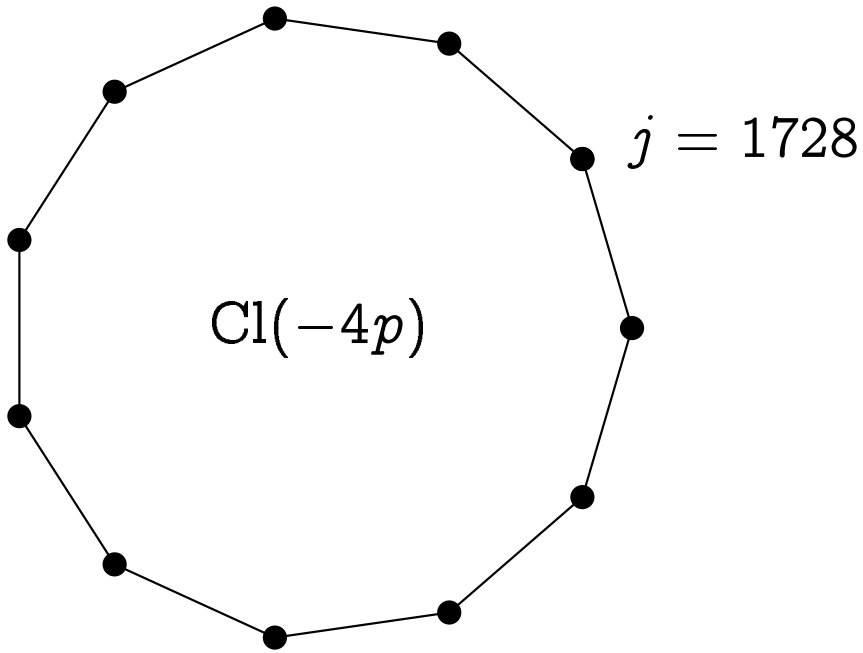
where $i \in \mathbb{F}_{p^2}$ is a 4-th root of unity. Clearly, $\iota^2 = -1$.

And $\iota\pi = -\pi\iota$.

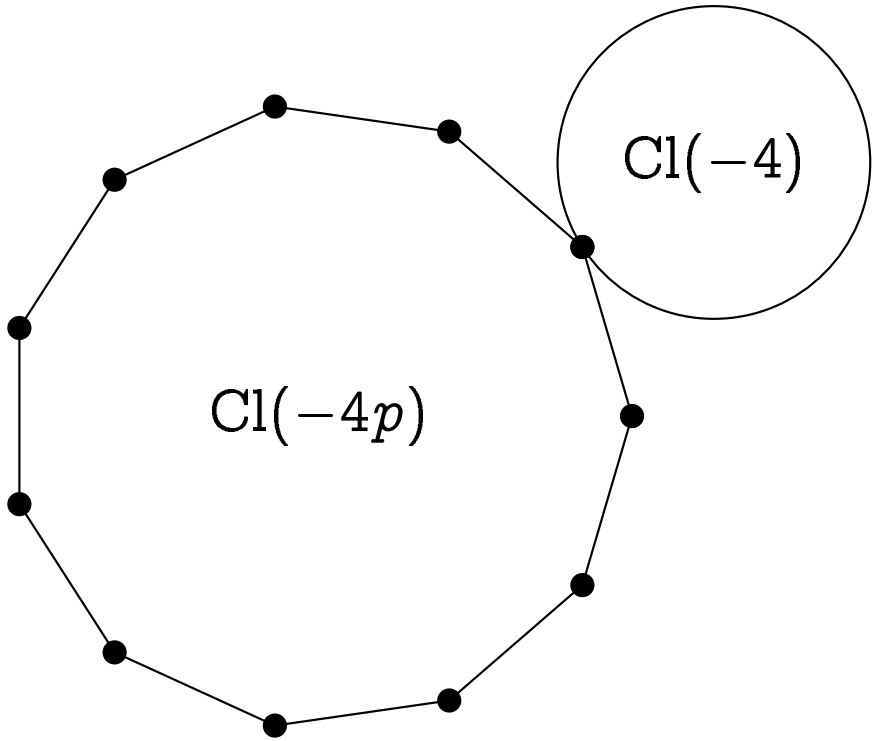
Class group action party

- $j = 1728$

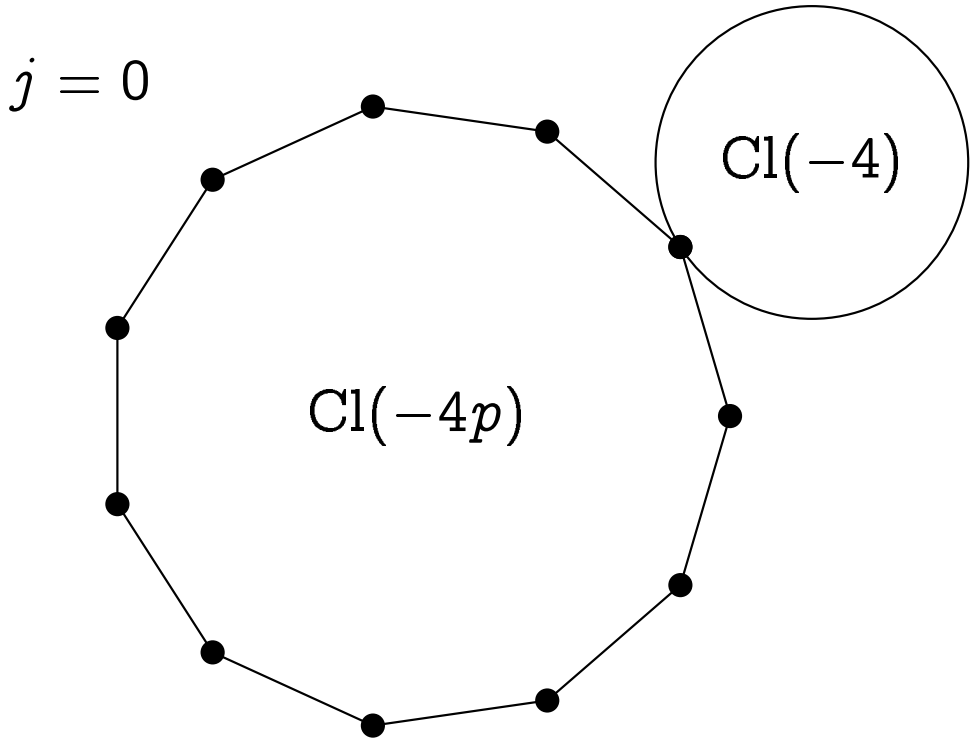
Class group action party



Class group action party



Class group action party



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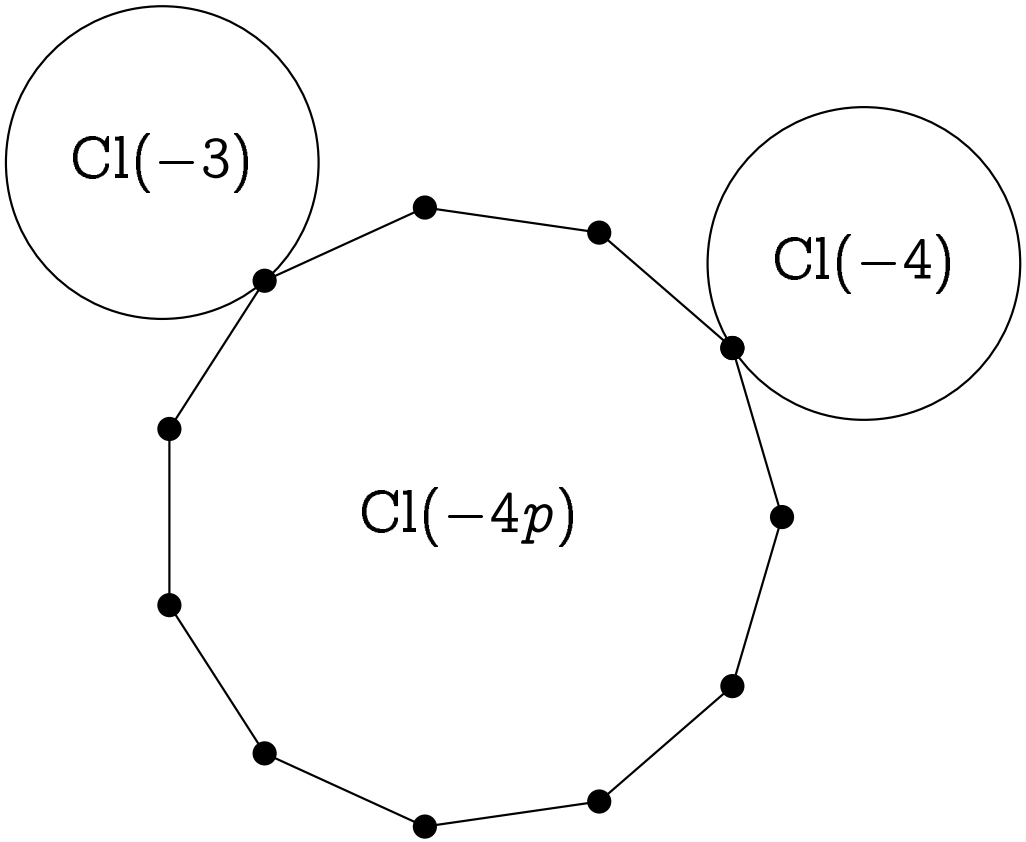
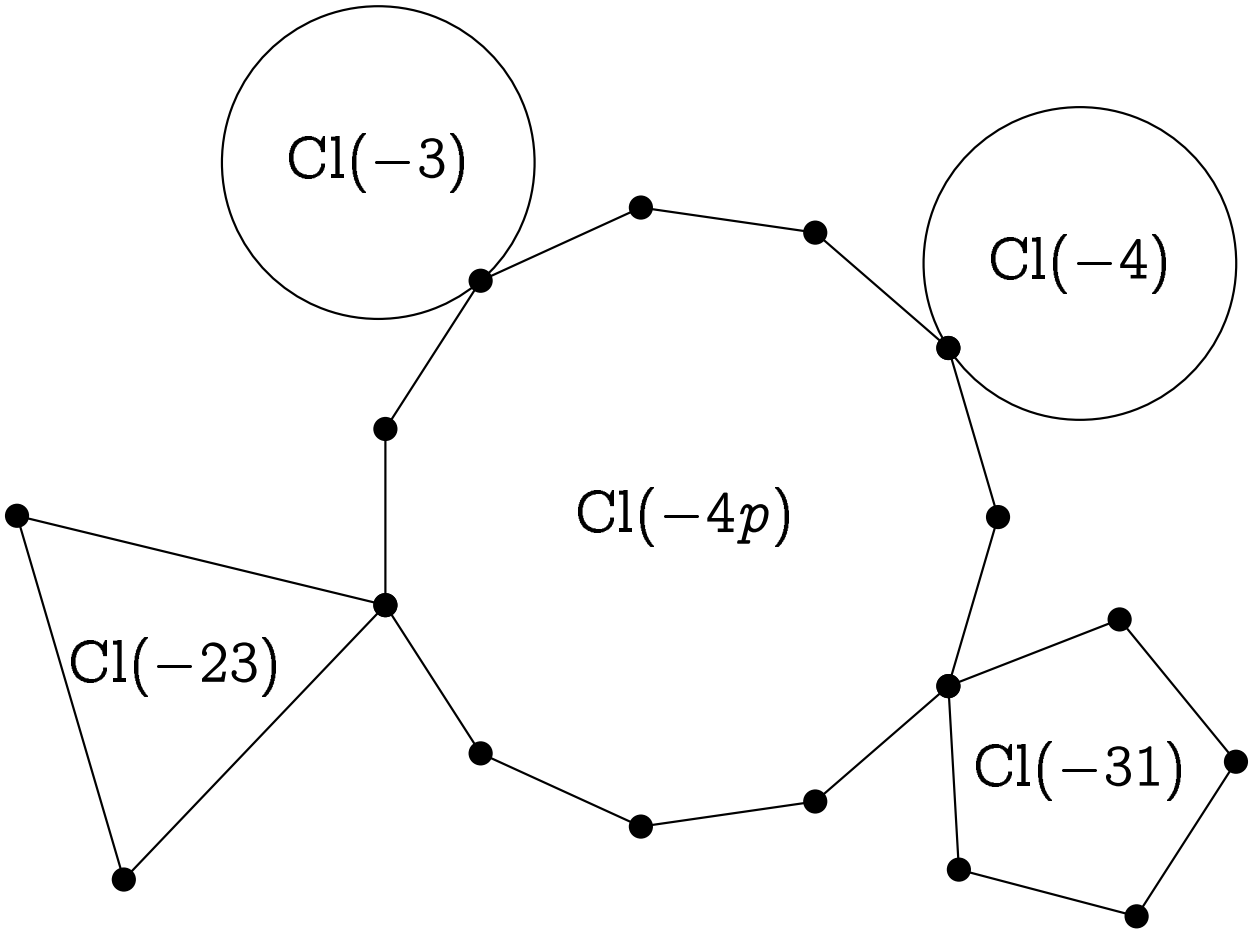


Class group action party



Quaternion algebra?! WTF?[2]

The quaternion algebra $B_{p,\infty}$ is:

- A 4-dimensional $\mathbb{Q}$-vector space with basis $(1, i, j, k)$;
- A non-commutative division algebra[1] $B_{p,\infty} = \mathbb{Q}\langle i, j\rangle$ with the relations:
$$i^2 = a, \quad j^2 = -p, \quad ij = -ji = k,$$
for some $a < 0$ (depending on p).
- All elements of $B_{p,\infty}$ are quadratic algebraic numbers;
- $B_{p,\infty} \otimes \mathbb{Q}_\ell \simeq \mathcal{M}_{2\times 2}(\mathbb{Q}_\ell)$ for all $\ell \neq p$.
I.e., endomorphisms restricted to $E[\ell^e]$ are just 2×2 matrices $\bmod \ell^e$.
- $B_{p,\infty} \otimes \mathbb{R} \simeq \mathcal{M}_{2\times 2}(\mathbb{R})$.

[1] All elements have inverses.

[2] What The Field?

Supersingular graphs

- Quaternion algebras have many maximal orders.
- For every maximal order type of $B_{p,\infty}$ there are 1 or 2 curves over $\mathbb{F}_{p^2}$ having endomorphism ring isomorphic to it.
- There is a unique isogeny class of supersingular curves over $\bar{\mathbb{F}}_p$ of size $\approx p/12$.
- Left ideals act on the set of maximal orders like isogenies.
- The graph of ℓ-isogenies is $(\ell+1)$-regular.

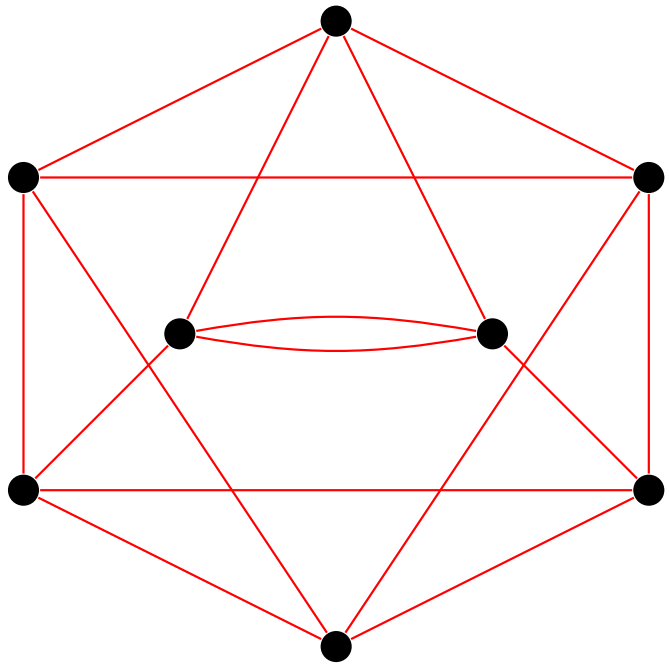


Figure: 3-isogeny graph on $\mathbb{F}_{97^2}$.

Graphs lexicon

Degree:	Number of (outgoing/ingoing) edges.
k-regular:	All vertices have degree k.
Connected:	There is a path between any two vertices.
Distance:	The length of the shortest path between two vertices.
Diamater:	The longest distance between two vertices.
$\lambda_1 \geq \cdots \geq \lambda_n$:	The (ordered) eigenvalues of the adjacency matrix.

Expander graphs

Proposition

If G is a k-regular graph, its largest and smallest eigenvalues satisfy

$$k = \lambda_1 \geq \lambda_n \geq -k.$$

Expander families

An infinite family of connected k-regular graphs on n vertices is an expander family if there exists an $\epsilon > 0$ such that all non-trivial eigenvalues satisfy $|\lambda| \leq (1 - \epsilon)k$ for n large enough.

- Expander graphs have short diameter ($O(\log n)$);
- Random walks mix rapidly (after $O(\log n)$ steps, the induced distribution on the vertices is close to uniform).

Expander graphs from isogenies

Theorem (Pizer 1990, 1998)

Let ℓ be fixed. The family of graphs of supersingular curves over $\mathbb{F}_{p^2}$ with ℓ-isogenies, as $p \to \infty$, is an expander family[a].

[a] Even better, it has the Ramanujan property.

Theorem (Jao, Miller, and Venkatesan 2009)

Let $\mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$ be an order in a quadratic imaginary field. The graphs of all curves over $\mathbb{F}_q$ with complex multiplication by $\mathcal{O}$, with isogenies of prime degree bounded[a] by $(\log q)^{2+\delta}$, are expanders.

[a] May contain traces of GRH.

Thank you

https://defeo.lu/

@luca_defeo

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