The isogeny cycle seminar

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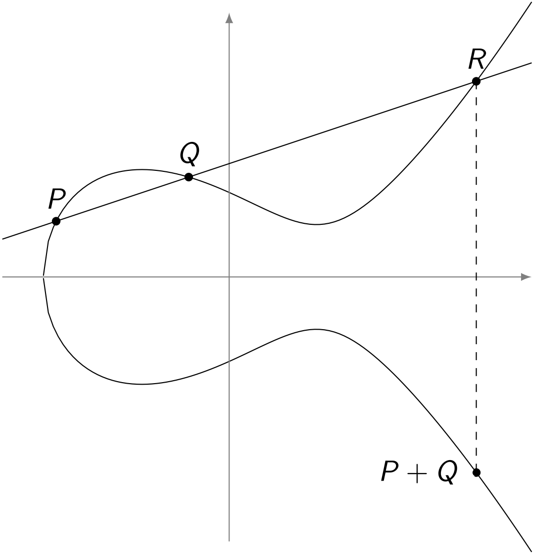
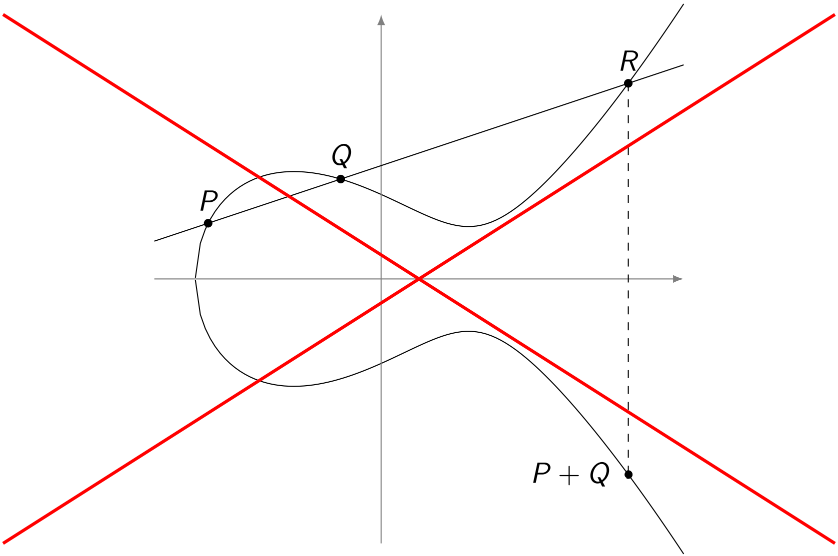
Elliptic curves

Let $E : y^2 = x^3 + ax + b$ be an elliptic curve...



Elliptic curves

Let $E : y^2 = x^3 + ax + b$ be an elliptic curve... forget it!



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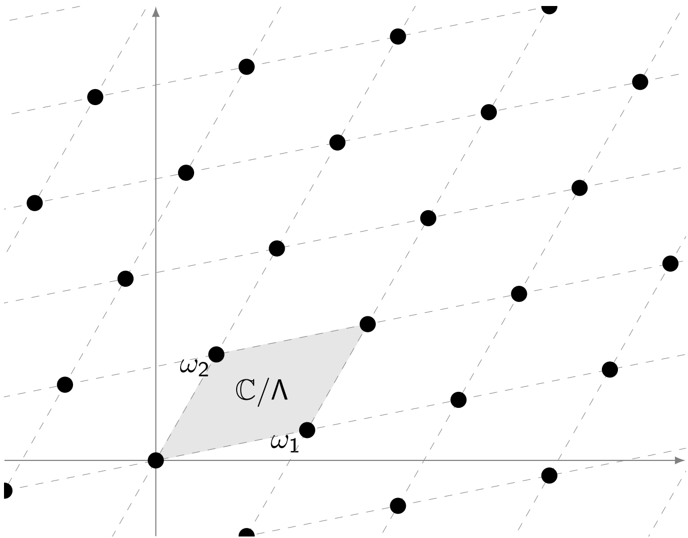
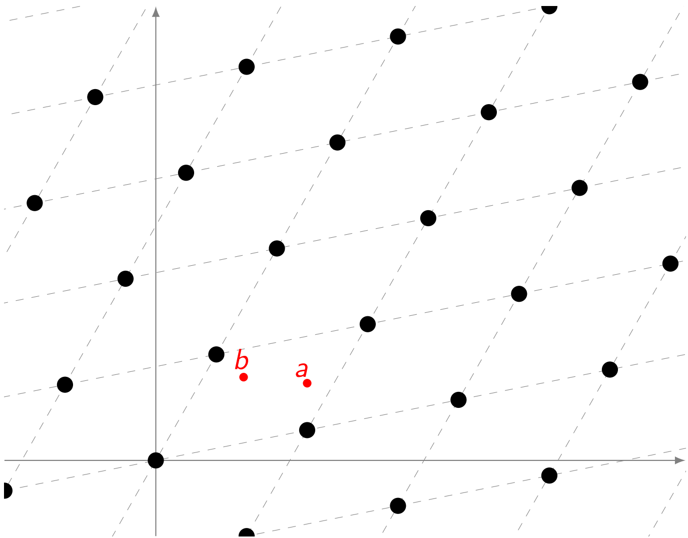


Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent complex numbers. Set

$$\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$$

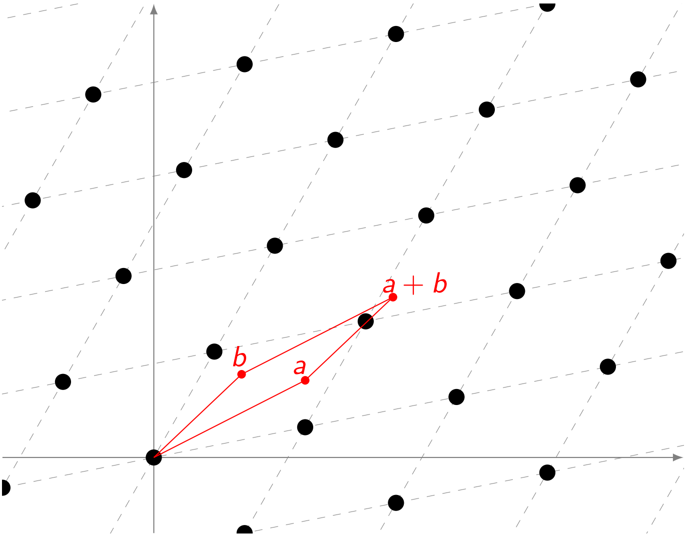
$\mathbb{C}/\Lambda$ is an elliptic curve.

Elliptic curves



Addition law induced by addition on $\mathbb{C}$.

Elliptic curves



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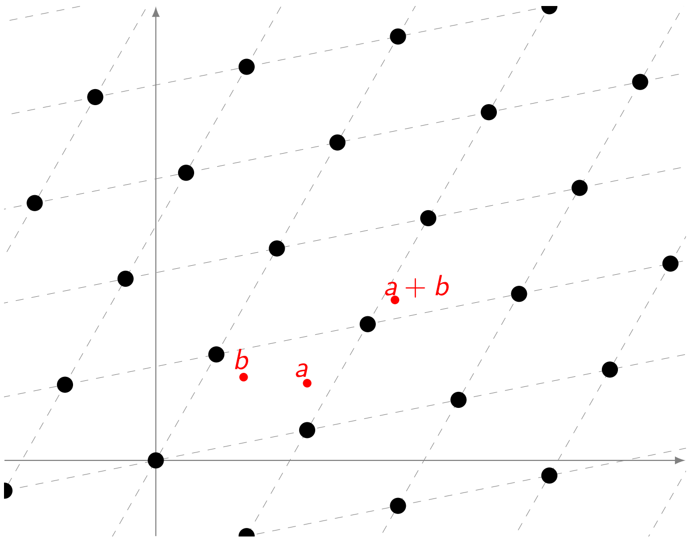

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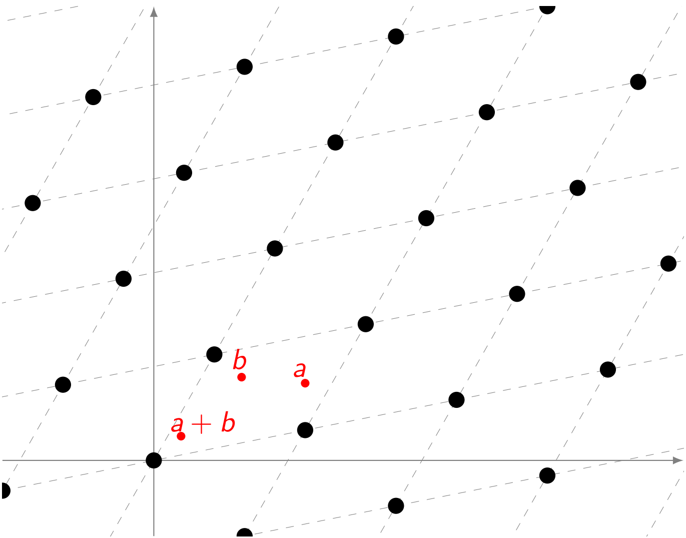

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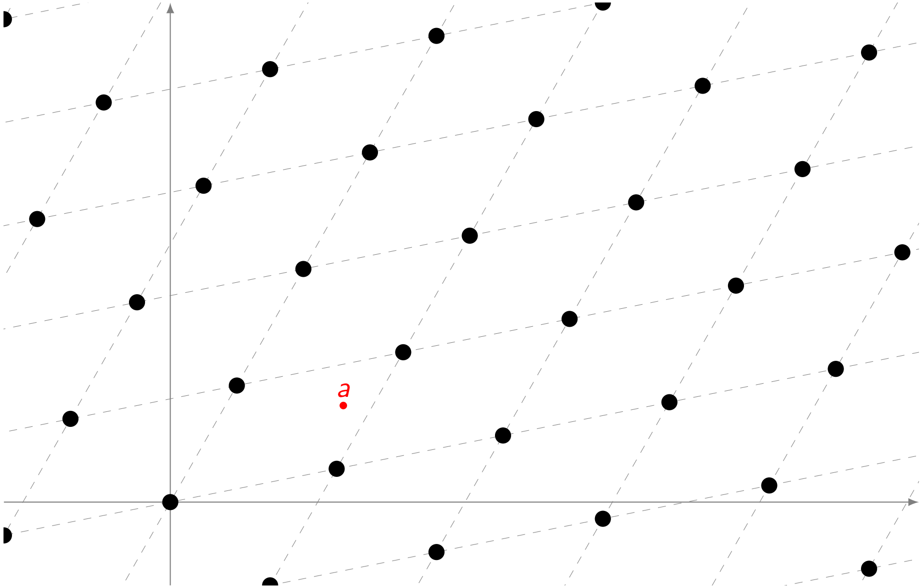
Multiplication



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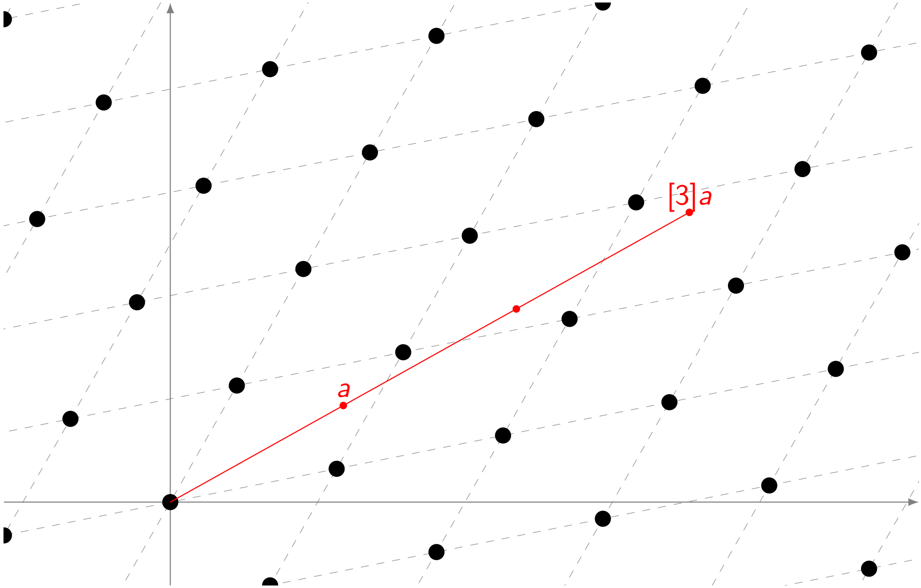

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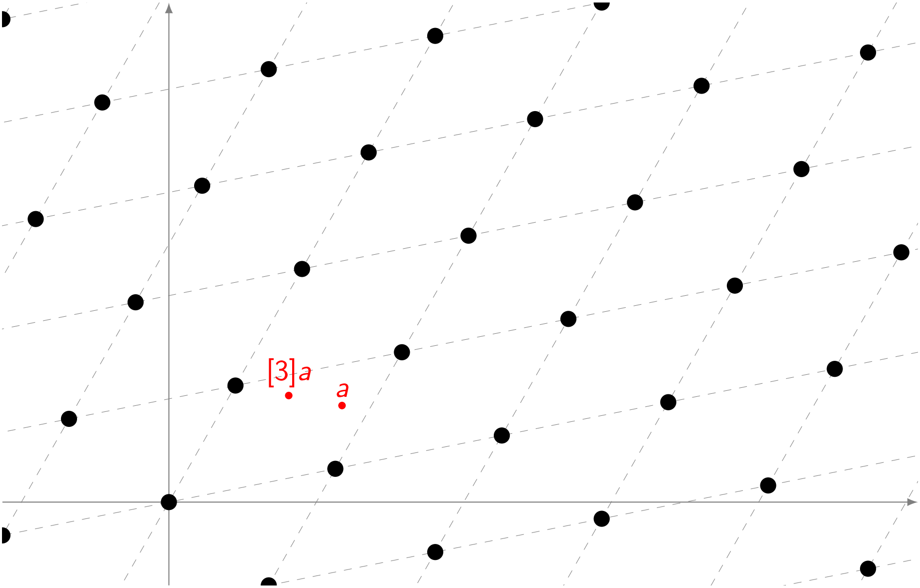
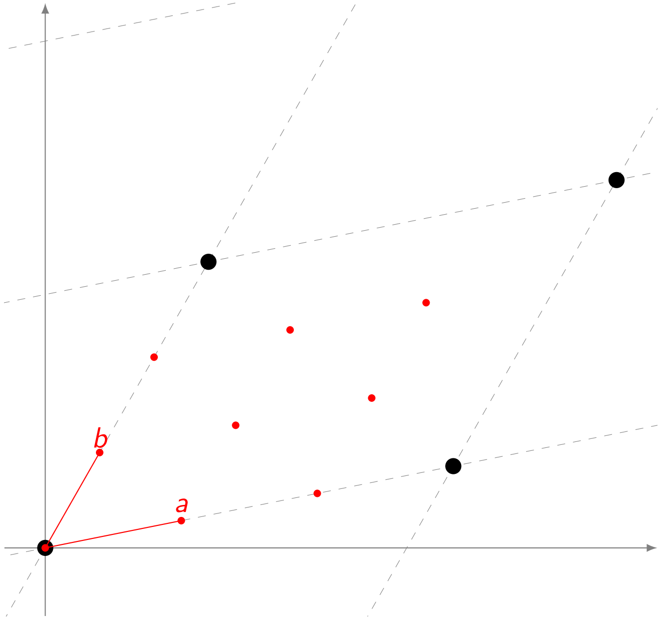


Torsion subgroups



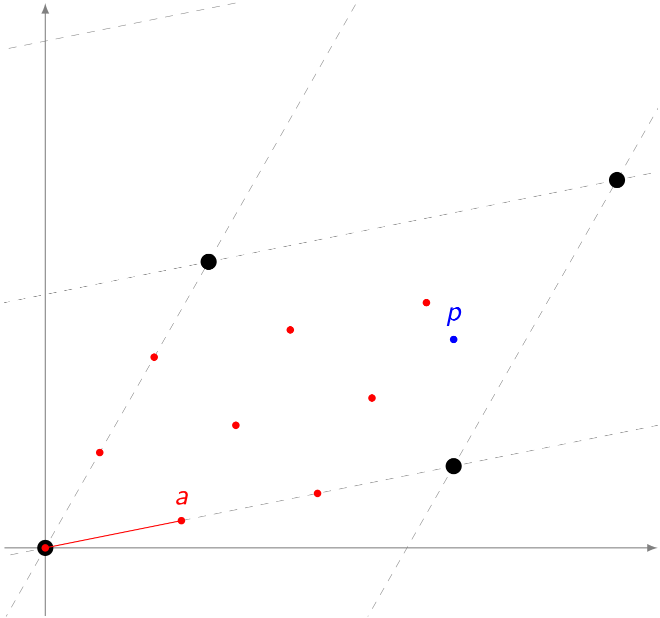
The ℓ-torsion subgroup is made up by the points

$$\left(\frac{i\omega_1}{\ell}, \frac{j\omega_2}{\ell}\right)$$

It is a group of rank two

$$E[\ell] = \langle a, b \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

Isogenies



Let $a \in \mathbb{C}/\Lambda_1$ be an ℓ-torsion point, and let

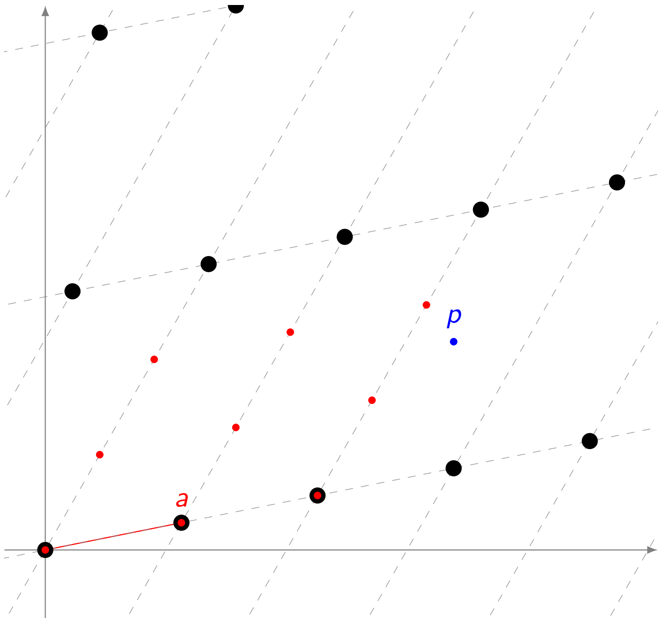
$$\Lambda_2 = a\mathbb{Z} \oplus \Lambda_1$$

Then $\Lambda_1 \subset \Lambda_2$ and we define a degree ℓ cover

$$\phi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$$

ϕ is a morphism of complex Lie groups and is called an isogeny.

Isogenies



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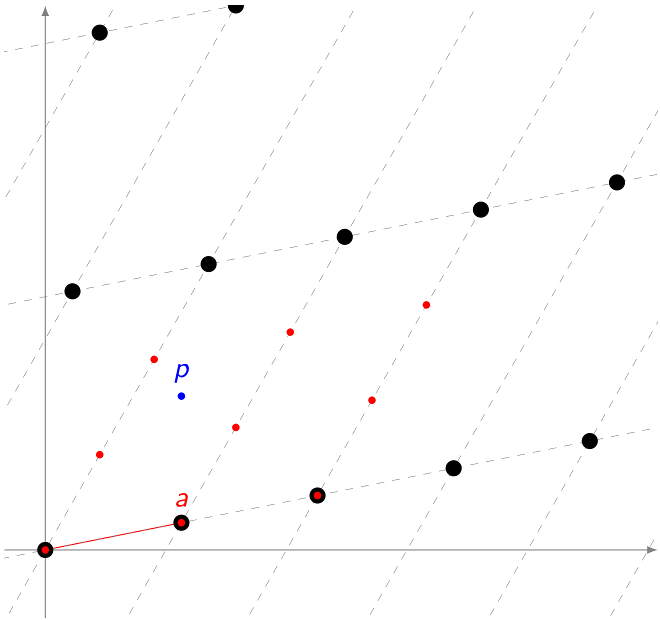

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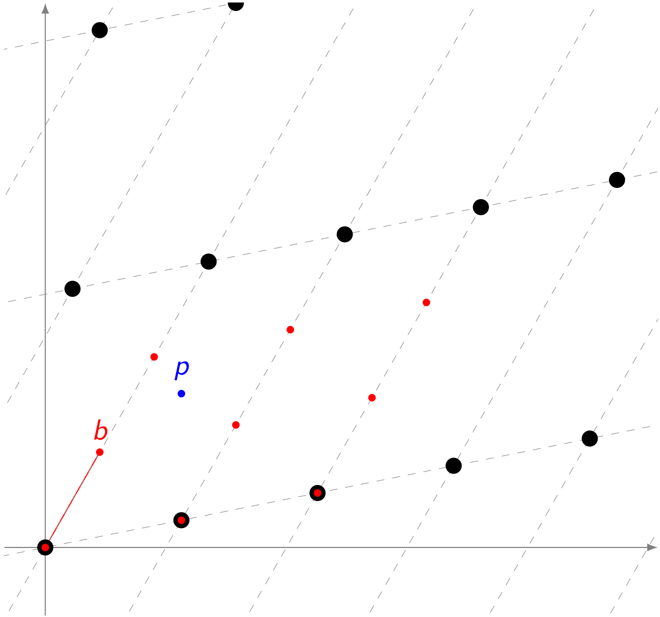
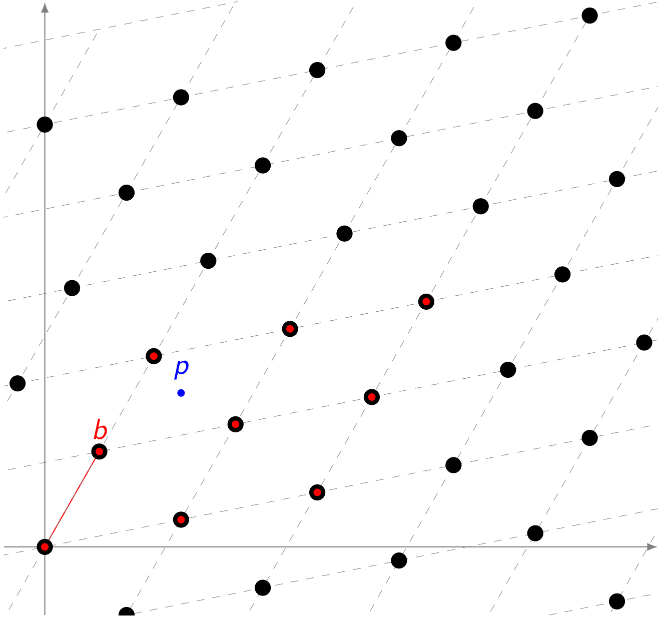


Taking a point b not in the kernel of ϕ, we obtain a new degree ℓ cover

$$\hat{\phi} : \mathbb{C}/\Lambda_2 \to \mathbb{C}/\Lambda_3$$

The composition $\hat{\phi} \circ \phi$ has degree ℓ^2 and is homothetic to the multiplication by ℓ map.
$\hat{\phi}$ is called the dual isogeny of ϕ.

Isogenies



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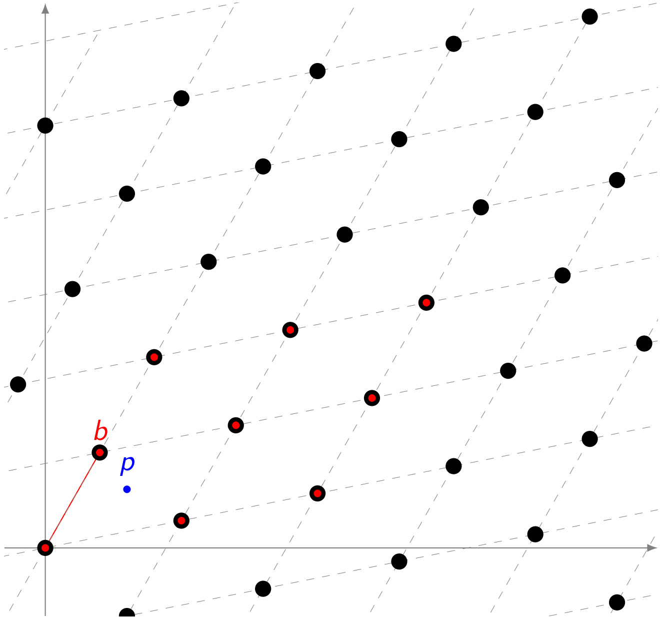

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Isogenies over arbitrary fields

Isogenies are just the right notion of morphism for elliptic curves

- Surjective group morphisms.
- Algebraic maps (i.e., defined by polynomials).

(Separable) isogenies $\Leftrightarrow$ finite subgroups:

$$0 \to H \to E \xrightarrow{\phi} E' \to 0$$

The kernel H determines the image curve E' up to isomorphism

$$E/H \stackrel{\text{def}}{=} E'.$$

Isogeny degree

Neither of these definitions is quite correct, but they *nearly* are:

- The degree of ϕ is the cardinality of $\ker \phi$.
- (Bisson) the degree of ϕ is the time needed to compute it.

The computational point of view

In practice: an isogeny ϕ is just a rational fraction (or maybe two)

$$\frac{N(x)}{D(x)} = \frac{x^n + \cdots + n_1 x + n_0}{x^{n-1} + \cdots + d_1 x + d_0} \in k(x), \qquad \text{with } n = \deg \phi,$$

and $D(x)$ vanishes on $\ker \phi$.

The explicit isogeny problem

Input: A *description* of the isogeny (e.g, its kernel).

Output: The curve E/H and the rational fraction N/D.

Lower bound: $\Omega(n)$.

The isogeny evaluation problem

Input: A *description* of the isogeny ϕ, a point $P \in E(k)$.

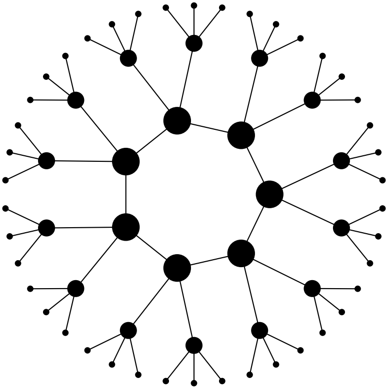
Output: The curve E/H and $\phi(P)$.

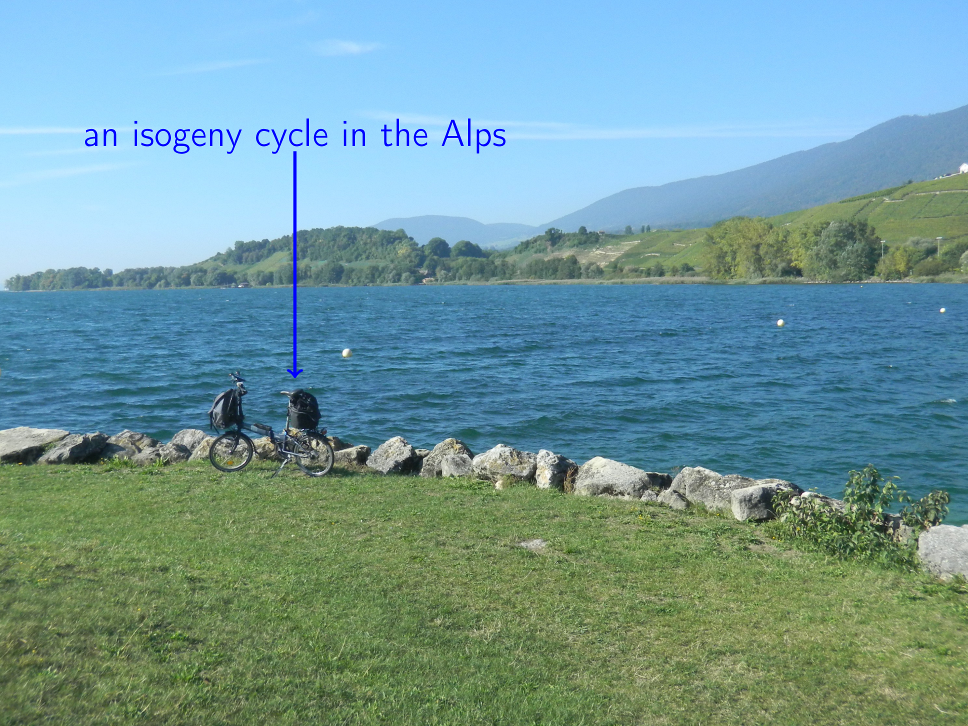
Isogeny graphs

We want to study the graph of elliptic curves with isogenies up to isomorphism. We say two isogenies ϕ, ϕ' are isomorphic if:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & E' \arrow[d, leftrightarrow, "\wr"] \\ & E' \end{tikzcd}$$

Example: Finite field, ordinary case, graph of isogenies of degree 3.




an isogeny cycle in the Alps

Structure of the graph[1]

Theorem (Serre-Tate)

Two curves are isogenous over a finite field k if and only if they have the same number of points on k.

The graph of isogenies of prime degree $\ell \neq p$

Ordinary case

- Nodes can have degree $0, 1, 2$ or $\ell + 1$.
- Connected components form so called volcanoes.

Supersingular case

- The graph is $\ell + 1$-regular.
- There is a unique connected component made of all supersingular curves with the same number of points.

[1] Kohel 1996; Fouquet and Morain 2002.

Expander graphs

Let G be a finite undirected k-regular graph.

- k is the trivial eigenvalue of the adjacency matrix of G.
- G is called an expander if all non-trivial eigenvalues satisfy $|\lambda| \leq (1-\delta)k$.
- It is called a Ramanujan graph if $|\lambda| \leq 2\sqrt{k-1}$. This is optimal.

In practice, in an expander graph random walks of length $O(\frac{1}{\delta}\log|G|)$ land anywhere in the graph with probability distribution close to uniform.

Isogeny graphs and expansion

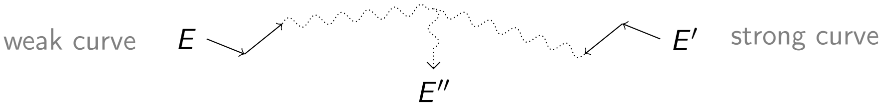
- The graph of ordinary isogenies of degree less than $(\log 4q)^B$ is an expander if $B > 2$.[a]
- The graph of supersingular isogenies of prime degree $\ell \neq p$ is Ramanujan.[b]

[a] Jao, Miller, and Venkatesan 2009.

[b] Pizer 1990, 1998.

Isogeny walks and cryptanalysis[3]

Recall: Having a weak DLP is not isogeny invariant.



Fourth root attacks

- Start two random walks from the two curves and wait for a collision.
- Over $\mathbb{F}_q$, the average size of an isogeny class is $h_\Delta \sim \sqrt{q}$.
- A collision is expected after $O(\sqrt{h_\Delta}) = O(q^{\frac{1}{4}})$ steps.

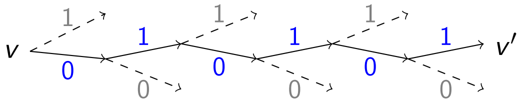
Note: Can be used to build trapdoor systems[2].

[2]Teske 2006.

[3]Steven D. Galbraith 1999; Steven D. Galbraith, Hess, and Smart 2002; Charles, K. E. Lauter, and Goren 2009; Bisson and Sutherland 2011.

Random walks and hash functions

Any expander graph gives rise to a hash function.



$H(010101) = v'$

- Fix a starting vertex v;
- The value to be hashed determines a random path to v';
- v' is the hash.

Provably secure hash functions

- Use the Ramanujan graph of supersingular 2-isogenies;[a]
- Collision resistance = hardness of finding cycles in the graph;
- Preimage resistance = hardness of finding a path from v to v'.

[a] Charles, K. E. Lauter, and Goren 2009.

The endomorphism ring

- An endomorphism is an isogeny $\phi : E \to E$.
- The endomorphisms form a ring denoted $\mathrm{End}_k(E)$.

Theorem

$\mathbb{Q} \otimes \mathrm{End}_{\bar{k}}(E)$ is isomorphic to one of the following

ordinary case: $\mathbb{Q}$ *(only possible if* char $k = 0$*),*

ordinary case (complex multiplication): *an imaginary quadratic field,*

supersingular case: *a quaternion algebra (only possible if* char $k \neq 0$*).*

Corollary

$\mathrm{End}(E)$ *is isomorphic to an order* $\mathcal{O} \subset \mathbb{Q} \otimes \mathrm{End}(E)$.

Isogenies and endomorphisms

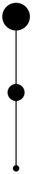
Theorem (Serre-Tate)

Two elliptic curves E, E' are isogenous if and only if

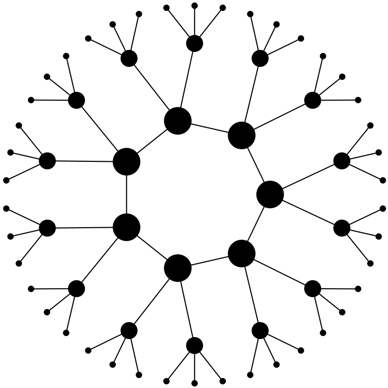
$$\mathbb{Q} \otimes \mathrm{End}(E) \simeq \mathbb{Q} \otimes \mathrm{End}(E').$$

Example: Finite field, ordinary case, 3-isogeny graph.

$\mathrm{End}(E)$



bigger node = bigger $\mathrm{End}(E)$



The ordinary case

Let $\mathrm{End}(E) = \mathcal{O} \subset \mathbb{Q}(\sqrt{d})$ be the endomorphism ring of E. Define

- $\mathcal{I}(\mathcal{O})$, the group of invertible fractional ideals,
- $\mathcal{P}(\mathcal{O})$, the group of principal ideals,

Definition (The class group)

The class group of $\mathcal{O}$ is

$$\mathrm{Cl}(\mathcal{O}) = \mathcal{I}(\mathcal{O})/\mathcal{P}(O).$$

- It is a finite abelian group.
- It arises as the Galois group of an abelian extension of $\mathbb{Q}(\sqrt{d})$.
- Isogeny (classes) = ideal (classes): The class group acts faithfully and transitively on the isogeny graph.

DH-like key exchange based on (semi)-group actions

Let G be an abelian group acting (faithfully and transitively) on a set X.

$$\begin{tikzcd}
 & x_0 \arrow[dl, "g"'] \arrow[dr, "h"] & \\
g \cdot x_0 \arrow[dr, "h"'] & & h \cdot x_0 \arrow[dl, "g"] \\
 & gh \cdot x_0 = hg \cdot x_0 &
\end{tikzcd}$$

Hidden Subgroup Problem

Let G be a group, X a set and $f : G \to X$. We say that f hides a subgroup $H \subset G$ if

$$f(g_1) = f(g_2) \Leftrightarrow g_1 H = g_2 H.$$

Definition (Hidden Subgroup Problem (HSP))

Input: G, X as above, an oracle computing f.

Output: generators of H.

Theorem (Schorr, Josza)

If G is abelian, then

- $HSP \in poly_{BQP}(\log |G|)$,
- *using $poly(\log |G|)$ queries to the oracle.*

Post-Quantum cryptography

Known reductions

- Discrete Log on G of size $p \to$ HSP on $(\mathbb{Z}/p\mathbb{Z})^2$,
- hence DH, ECDH, etc. are broken by quantum computers.
- Semigroup-DH on $G \to$ HSP on the dihedral group $G \ltimes \mathbb{Z}/2\mathbb{Z}$.

Quantum algorithms for dihedral HSP

Kuperberg[a]: $2^{O(\sqrt{\log|G|})}$ quantum time, space and query complexity.

Regev[b]: $L_{|G|}(\frac{1}{2}, \sqrt{2})$ quantum time and query complexity, $\mathrm{poly}(\log(|G|)$ quantum space.

[a]Kuperberg 2005.
[b]Regev 2004.

Remark (Regev): certain lattice-based cryptosystems are also vulnerable to the HSP for dihedral groups.

DH using class groups[4]

Public data:

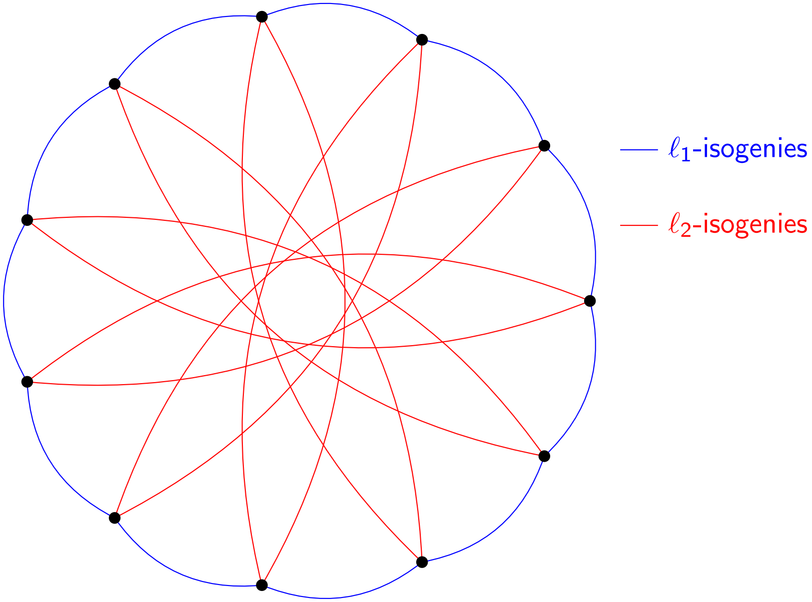
- $E/\mathbb{F}_p$ ordinary elliptic curve with complex multiplication field $\mathbb{K}$,
- primes ℓ_1,ℓ_2 not dividing $\mathrm{Disc}(E)$ and s.t. $\left(\frac{D_{\mathbb{K}}}{\ell_i}\right) = 1$.
- A *direction* on the isogeny graph (i.e. an element of the class group).

Secret data: Random walks $\mathfrak{a}, \mathfrak{b}$ in the ℓ_i-isogeny graphs.

$$\begin{tikzcd}
 & E \arrow[dl, "\ell_1^{a_1}\ell_2^{a_2} = \mathcal{N}(\mathfrak{a})"'] \arrow[dr, "\mathcal{N}(\mathfrak{b}) = \ell_1^{b_1}\ell_2^{b_2}"] & \\
\mathfrak{a} * E \arrow[dr] & & \mathfrak{b} * E \arrow[dl] \\
 & \mathfrak{a}\mathfrak{b} * E = \mathfrak{b}\mathfrak{a} * E &
\end{tikzcd}$$

[4] Rostovtsev and Stolbunov 2006.

R&S key exchange



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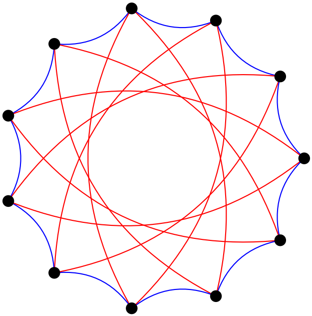

Key generation: compose small degree isogenies
polynomial in the lenght of the random walk.

Attack: find an isogeny between two curves
polynomial in the degree, exponential in the length.

Quantum[5]: HShP + isogeny evaluation
subexponential in the length of the walk.

[5] Childs, Jao, and Soukharev 2010.

Supersingular curves

$\mathbb{Q} \otimes \mathrm{End}(E)$ is a quaternion algebra (non-commutative)

Facts

- Every supersingular curve is defined over $\mathbb{F}_{p^2}$.
- $E(\mathbb{F}_{p^2}) \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2$ (up to twist, and overly simplifying!).
- There are $g(X_0(p)) + 1 \sim \frac{p+1}{12}$ supersingular curves up to isomorphism.
- For every maximal order type of the quaternion algebra $\mathbb{Q}_{p,\infty}$ there are 1 or 2 curves over $\mathbb{F}_{p^2}$ having endomorphism ring isomorphic to it.
- There is a unique isogeny class of supersingular curves over $\bar{\mathbb{F}}_p$ (there are two over any finite field).
- The graph of ℓ-isogenies is $\ell + 1$-regular.

R&S key exchange with supersingular curves

Good news: there is no action of a commutative class group.

Bad news: there is no action of a commutative class group.

However: left ideals of End(E) still act on the isogeny graph:

$$\begin{tikzcd} E \arrow[r, "\mathfrak{a}"] \arrow[d, "\mathfrak{b}"'] & E' \arrow[d, "\mathfrak{b}_{\mathfrak{a}}"] \\ E'' \arrow[r, "\mathfrak{a}_{\mathfrak{b}}"'] & E''' \end{tikzcd}$$

- The action factors through the right-isomorphism equivalence of ideals.
- Ideal classes form a groupoid (in other words, an undirected multigraph. . .).

From ideals back to isogenies

In practice, computations with ideals are hard. We fix, instead:

- Small primes ℓ_A, ℓ_B;
- A large prime p such that $p+1 = \ell_A^{e_A}\ell_B^{e_B}$;
- A supersingular curve E over $\mathbb{F}_{p^2}$, such that

$$E \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2 = (\mathbb{Z}/\ell_A^{e_A}\mathbb{Z})^2 \oplus (\mathbb{Z}/\ell_B^{e_B}\mathbb{Z})^2,$$

- We use isogenies of degrees $\ell_A^{e_A}$ and $\ell_B^{e_B}$ with cyclic rational kernels;
- The diagram below can be constructed in time $\mathrm{poly}(e_A + e_B)$.

$$\ker\phi = \langle P\rangle \subset E[\ell_A^{e_A}]$$

$$\ker\psi = \langle Q\rangle \subset E[\ell_B^{e_B}]$$

$$\ker\phi' = \langle\psi(P)\rangle$$

$$\ker\psi' = \langle\phi(Q)\rangle$$

$$\begin{tikzcd}
E \arrow[r, "\phi"] \arrow[d, "\psi"'] & E/\langle P\rangle \arrow[d, "\psi'"] \\
E/\langle Q\rangle \arrow[r, "\phi'"'] & E/\langle P, Q\rangle
\end{tikzcd}$$

Our proposal: SIDH[6]

Public data:

- Prime p such that $p+1 = \ell_A^a \ell_B^b$;
- Supersingular curve $E \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2$;
- $E[\ell_A^a] = \langle P_A, Q_A \rangle$;
- $E[\ell_B^b] = \langle P_B, Q_B \rangle$.

Secret data:

- $R_A = m_A P_A + n_A Q_A$,
- $R_B = m_B P_B + n_B Q_B$,

$$\begin{tikzcd} & E \arrow[dl, dash, "\phi"'] \arrow[dr, dash, "\psi"] & \\ E/\langle R_A\rangle \arrow[dr, dash, "\psi'"'] & & E/\langle R_B\rangle \arrow[dl, dash, "\phi'"] \\ & E/\langle R_A, R_B\rangle & \end{tikzcd}$$

$$\frac{E/\langle R_A\rangle}{\phi(R_B)} \simeq E/\langle R_A, R_B\rangle \simeq \frac{E/\langle R_B\rangle}{\psi(R_A)}$$

[6] Jao and De Feo 2011.

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Secret data:

- $R_A=m_AP_A+n_AQ_A$,
- $R_B=m_BP_B+n_BQ_B$,

$$\begin{tikzcd}
 & E \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
\begin{array}{l} E/\langle R_A\rangle \\ \phi(P_B) \\ \phi(Q_B) \end{array} \arrow[dr, "\psi'"'] & & \begin{array}{l} E/\langle R_B\rangle \\ \psi(P_A) \\ \psi(Q_A) \end{array} \arrow[dl, "\phi'"] \\
\frac{E/\langle R_A\rangle}{\phi(R_B)} \simeq & E/\langle R_A, R_B\rangle & \simeq \frac{E/\langle R_B\rangle}{\psi(R_A)}
\end{tikzcd}$$

[6] Jao and De Feo 2011.

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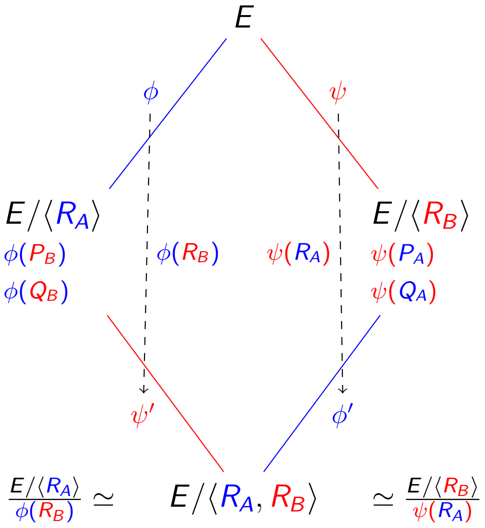

[6] Jao and De Feo 2011.

Other protocols based on SIDH

Non-interactive protocols

- El-Gamal encryption.

Interactive protocols

- Zero-knowledge proofs of identity[a],
- Undeniable signatures[b],
- Strong designated verifier signatures[c],
- Authenticated encryption[d].

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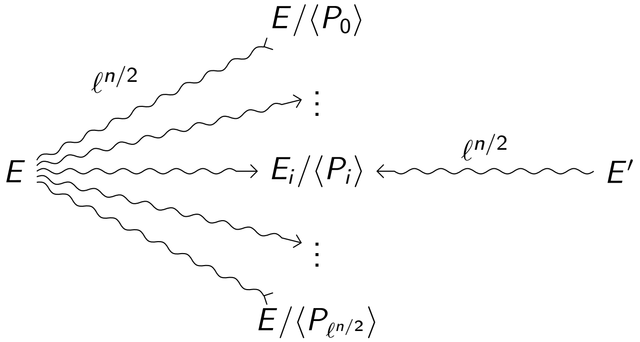
[a] De Feo, Jao, and Plût 2011.
[b] Jao and Soukharev 2014.
[c] Sun, Tian, and Wang 2012.
[d] Soukharev, Jao, and Seshadri 2016.

Missing: Classical signatures, ...

Generic attacks

Problem: Given E, E', isogenous of degree ℓ^n, find $\phi : E \to E'$.



- With high probability ϕ is the unique collision (or *claw*).
- A quantum claw finding[7] algorithm solves the problem in $O(\ell^{n/3})$.

[7] Tani 2008.

Other attacks

Ephemeral key recovery (total break)

Given E_0 and a public curve $E_0/\langle R\rangle$, find the kernel of the secret isogeny:

Subexponential $L_p(1/2, \sqrt{3}/2)$ when both curves are defined over $\mathbb{F}_p$.[a]

Polynomial isomorphic problem on quaternion algebras.[b]

Equivalent to computing the endomorphism rings of both E_0 and $E_0/\langle R_A\rangle$.[c]

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[a] Biasse, Jao, and Sankar 2014.

[b] Kohel, K. Lauter, Petit, and Tignol 2014.

[c] Steven D Galbraith, Petit, Shani, and Ti 2016.

Other attacks

Other security models

Active attack against long term keys, learns the full key with (close to) optimal number of oracle queries. Countermeasures are relatively expensive.[a]

Side channel Constant-time implementation available.[b]

Attack on partially leaked keys.[a]

[a] Steven D Galbraith, Petit, Shani, and Ti 2016.
[b] Costello and Longa 2015.

Recommended parameters

- For efficiency chose p such that $p+1 = 2^a 3^b$.
- For classical n-bit security, choose $2^a \sim 3^b \sim 2^{2n}$, hence $p \sim 2^{4n}$.
- For quantum n-bit security, choose $2^a \sim 3^b \sim 2^{3n}$, hence $p \sim 2^{6n}$.

Practical optimizations:

- Optimize arithmetic for $\mathbb{F}_p$.[a][b]
- -1 is a quadratic non-residue: $\mathbb{F}_{p^2} \simeq \mathbb{F}_p[X]/(X^2+1)$.
- E (or its twist) has a 4-torsion point: use Montgomery form.
- Avoid inversions by using *projective curve equations*.[a]
- Use $j = 0$ as starting curve.[a]

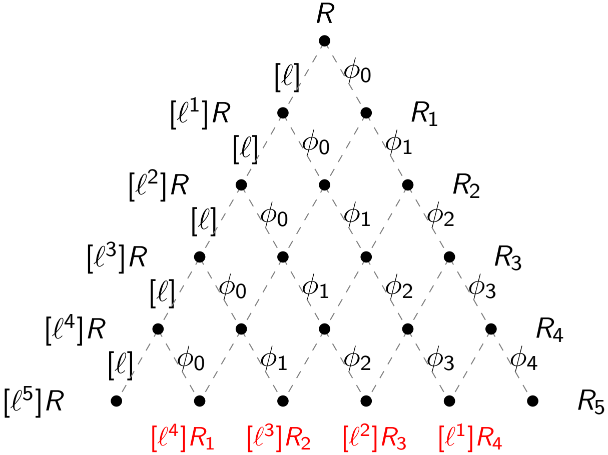
Fastest implementation[a]: 100Mcycles (Intel Haswell) @128bits quantum security level, 4512bits public key size.

[a] Costello and Longa 2015.

[b] Karmakar, Roy, Vercauteren, and Verbauwhede 2016.

Evaluating $\phi : E \to E/\langle R\rangle$ efficiently

$\mathrm{ord}(R) = \ell^a$ and $\phi = \phi_0 \circ \phi_1 \circ \cdots \circ \phi_{a-1}$, each of degree ℓ



For each i, one needs to compute $[\ell^{e-i}]R_i$ in order to compute ϕ_i.

What's the best strategy?

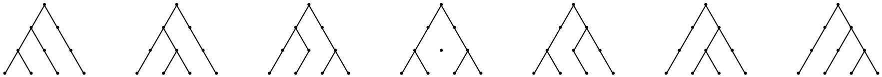


Figure: The seven well formed strategies for $e = 4$.

- Right edges are ℓ-isogeny evaluation;
- Left edges are multiplications by ℓ (about twice as expensive);

The best strategy can be precomputed offline and hardcoded in an embedded system.

A package to explore strategies:
`https://github.com/sidh-crypto/sidh-optimizer`.

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