# Mathematics of Isogeny-based Cryptography


Luca De Feo

IBM Research, Zürich



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Slides online at https://defeo.lu/docet

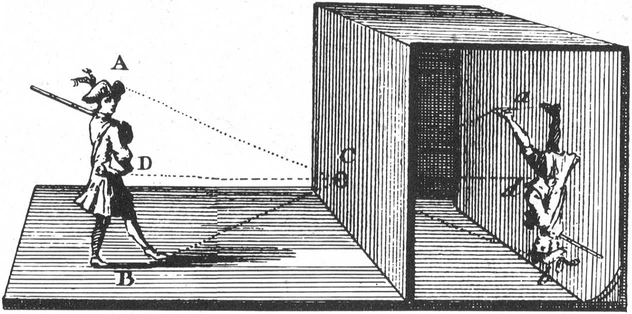
# Projective space

## Definition (Projective space)

Let $\bar{k}$ an algebraically closed field, the projective space $\mathbb{P}^n(\bar{k})$ is the set of non-null $(n+1)$-tuples $(x_0, \dots, x_n) \in \bar{k}^n$ modulo the equivalence relation

$$(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n) \qquad \text{with } \lambda \in \bar{k} \setminus \{0\}.$$

A class is denoted by $(x_0 : \cdots : x_n)$.

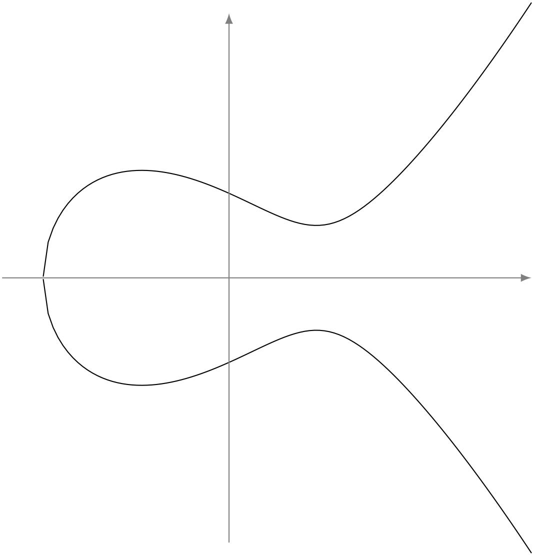


# Weierstrass equations

Let $k$ be a field of characteristic $\neq 2, 3$.
An elliptic curve *defined over* $k$ is the locus in $\mathbb{P}^2(\bar{k})$ of an equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3,$$

where $a, b \in k$ and $4a^3 + 27b^2 \neq 0$.



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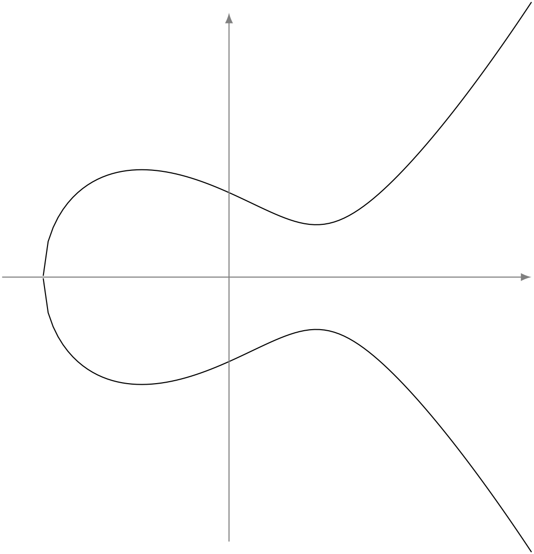

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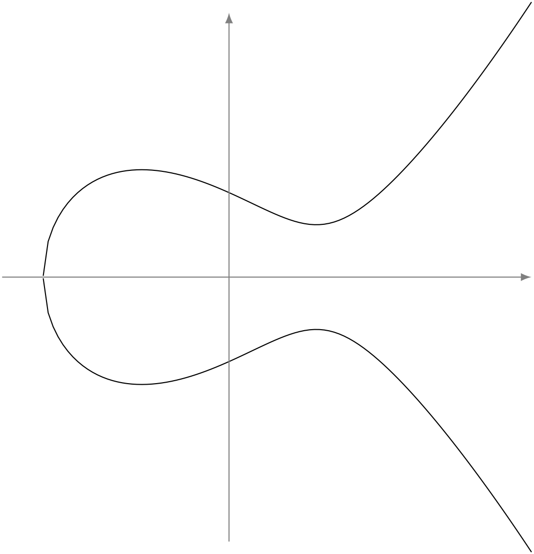
- $\mathcal{O} = (0 : 1 : 0)$ is the point at infinity;
- $y^2 = x^3 + ax + b$ is the affine equation.

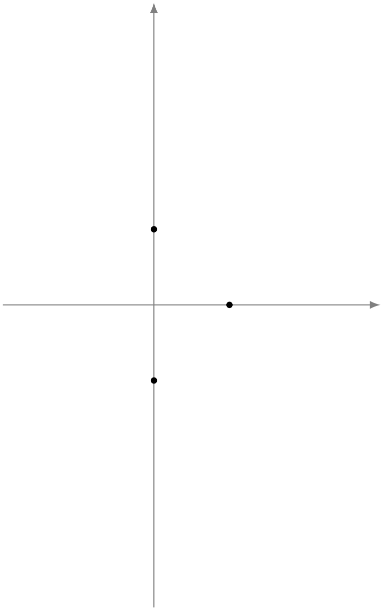


# Attention: arithmetic geometry!

$$E : y^2 = x^3 - 2x + 1$$

Rational points:

- $E(\mathbb{Q}) = \{(1,0), (0,1), (0,-1), \mathcal{O}\}$,



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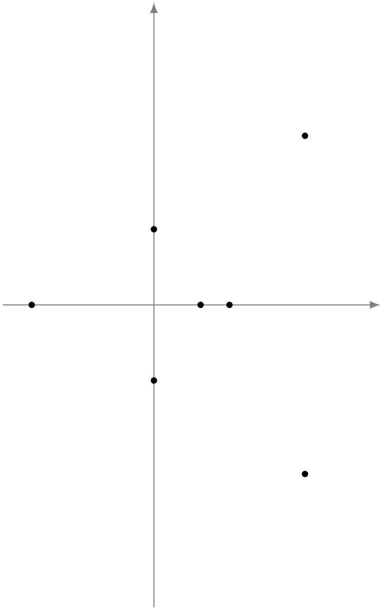

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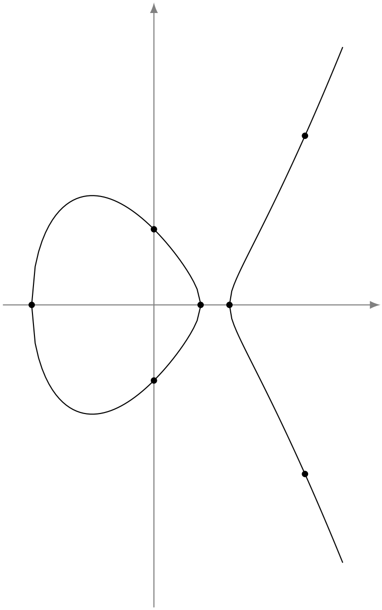

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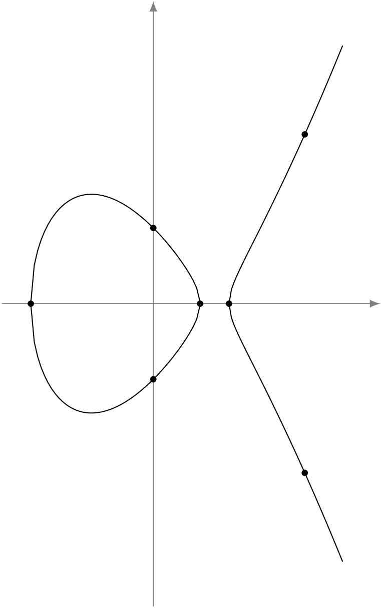
- $\#E(\mathbb{C}) = \infty$.

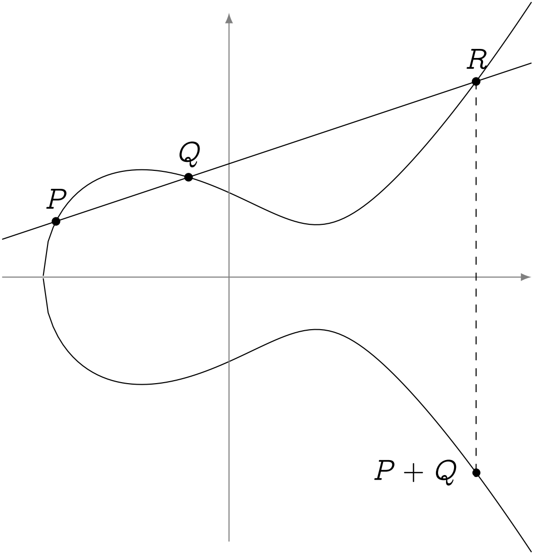


# The group law

## Bezout's theorem

Every line cuts $E$ in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.



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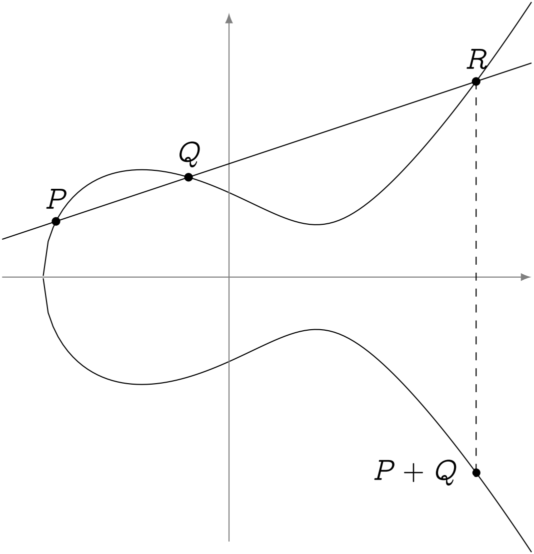


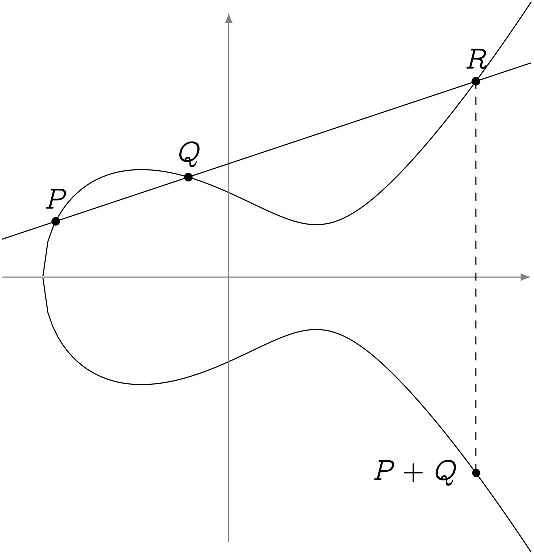
# The group law

**Bezout's theorem**

Every line cuts $E$ in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.

- The law is algebraic (it has *formulas*);
- The law is commutative;
- $\mathcal{O}$ is the group identity;
- Opposite points have the same $x$-value.



# What are elliptic curves?

## For mathematicians

- The smooth projective curves of genus 1 (with a distinguished point);
- The simplest abelian varieties (dimension 1);
- Finitely generated abelian groups of mysterious free rank (aka BSD conjecture);
- What you use to make examples.

# What are elliptic curves?

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## For cryptographers

- Finite abelian groups (often cyclic);
- Easy to compute the order;
- “2-dimensional” generalizations of $\mu_k$ (the roots of unity of $k$)…
- …with bilinear maps (aka pairings)!

# Maps: isomorphisms

## Isomorphisms

The only invertible algebraic maps between elliptic curves are of the form

$$(x, y) \mapsto (u^2 x, u^3 y)$$

for some $u \in \bar{k}$.
They are group isomorphisms.

## $j$-Invariant

Let $E : y^2 = x^3 + ax + b$, its $j$-invariant is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Two elliptic curves $E$, $E'$ are isomorphic if and only if $j(E) = j(E')$.

# Group structure

## Torsion structure

Let $E$ be defined over an algebraically closed field $\bar{k}$ of characteristic $p$.

$$E[m] \simeq \quad \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \qquad \text{if } p \nmid m,$$

$$E[p^e] \simeq \begin{cases} \mathbb{Z}/p^e\mathbb{Z} & \text{ordinary case,} \\ \{\mathcal{O}\} & \text{supersingular case.} \end{cases}$$

## Finite fields (Hasse's theorem)

Let $E$ be defined over a finite field $\mathbb{F}_q$, then

$$|\#E(\mathbb{F}_q) - q - 1| \leq 2\sqrt{q}.$$

In particular, there exist integers $n_1$ and $n_2 | \gcd(n_1, q-1)$ such that

$$E(\mathbb{F}_q) \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}.$$

# Maps: what's scalar multiplication?

$$[n] \;:\; P \mapsto \underbrace{P + P + \cdots + P}_{n \text{ times}}$$

- A map $E \to E$,
- a group morphism,
- with finite kernel
  (the torsion group $E[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$),
- surjective (in the algebraic closure),
- given by rational maps of degree $n^2$.

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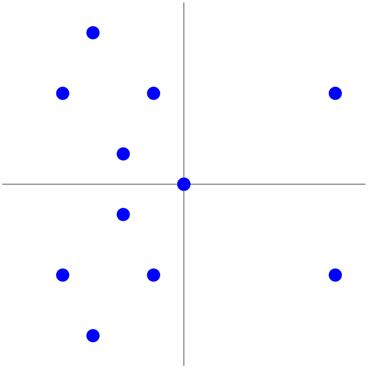
(Separable) isogenies $\Leftrightarrow$ finite subgroups:

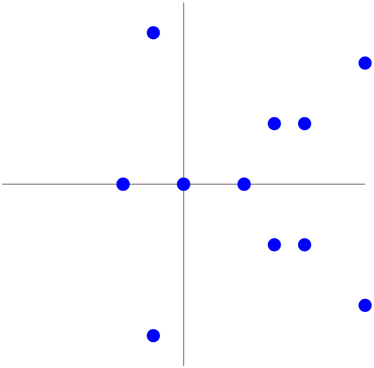
$$0 \to H \to E \xrightarrow{\phi} E' \to 0$$

# Isogenies: an example over $\mathbb{F}_{11}$

$$E \;:\; y^2 = x^3 + x$$

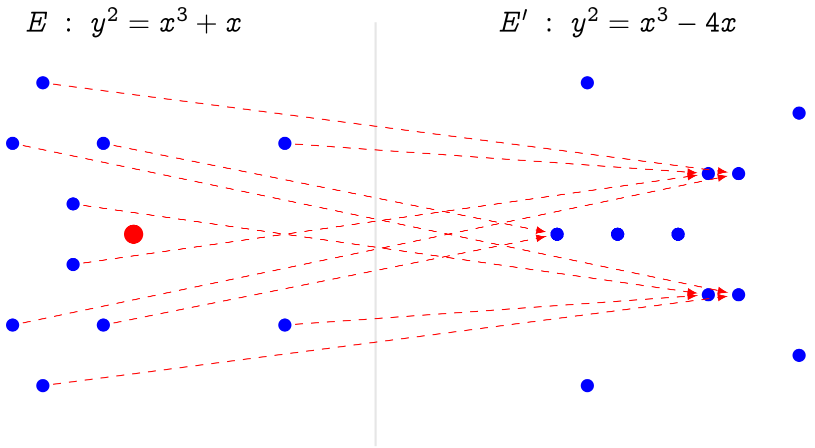


$$E' \;:\; y^2 = x^3 - 4x$$



$$\phi(x, y) = \left( \frac{x^2 + 1}{x}, \quad y\frac{x^2 - 1}{x^2} \right)$$

# Isogenies: an example over $\mathbb{F}_{11}$




$$\phi(x,y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2}\right)$$

- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

# Maps: isogenies

## Theorem

*Let $\phi : E \to E'$ be a map between elliptic curves. These conditions are equivalent:*

- *$\phi$ is a surjective group morphism,*
- *$\phi$ is a group morphism with finite kernel,*
- *$\phi$ is a non-constant algebraic map of projective varieties sending the point at infinity of $E$ onto the point at infinity of $E'$.*

*If they hold $\phi$ is called an isogeny.*

Two curves are called isogenous if there exists an isogeny between them.

## Example: Multiplication-by-$m$

On any curve, an isogeny from $E$ to itself (i.e., an endomorphism):

$$\begin{aligned}[m]\ :\ E &\to E,\\ P &\mapsto [m]P.\end{aligned}$$

# Isogeny lexicon

## Degree

- $\approx$ degree of the rational fractions defining the isogeny;
- Rough measure of the information needed to encode it.

## Separable, inseparable, cyclic

An isogeny $\phi$ is separable iff $\deg\phi = \ker\phi$.

- Given $H \subset E$ finite, write $\phi : E \to E/H$ for the unique separable isogeny s.t. $\ker\phi = H$.
- $\phi$ inseparable $\Rightarrow$ $p$ divides $\deg\phi$.
- Cyclic isogeny $\equiv$ separable isogeny with cyclic kernel.
  - Non-example: the multiplication map $[m] : E \to E$.

## Rationality

Given $E$ defined over $k$, an isogeny $\phi$ is rational if $\ker\phi$ is Galois invariant.

$\Rightarrow$ $\phi$ is represented by rational fractions with coefficients in $k$.

# The dual isogeny

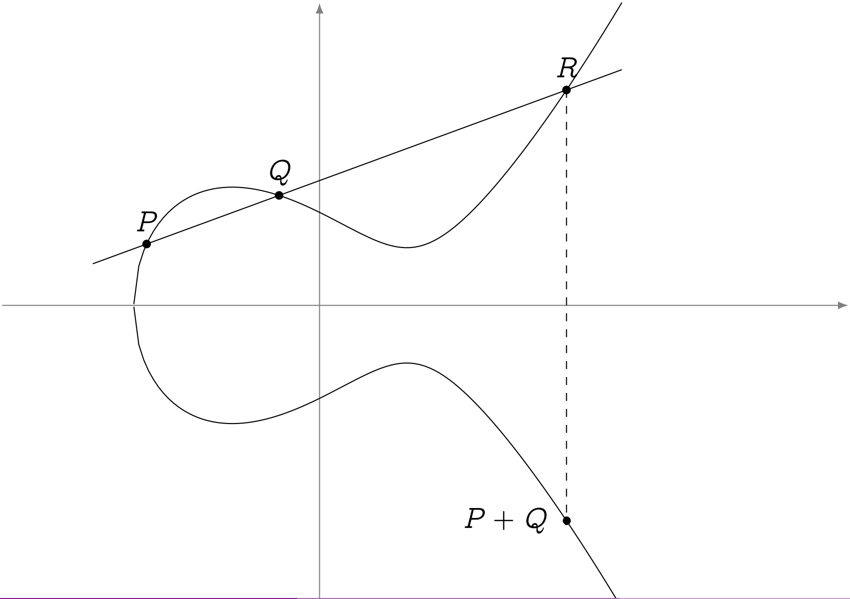
Let $\phi : E \to E'$ be an isogeny of degree $m$. There is a unique isogeny $\hat\phi : E' \to E$ such that

$$\hat\phi \circ \phi = [m]_E, \quad \phi \circ \hat\phi = [m]_{E'}.$$

$\hat\phi$ is called the dual isogeny of $\phi$; it has the following properties:

1. $\hat\phi$ is defined over $k$ if and only if $\phi$ is;
2. $\widehat{\psi \circ \phi} = \hat\phi \circ \hat\psi$ for any isogeny $\psi : E' \to E''$;
3. $\widehat{\psi + \phi} = \hat\psi + \hat\phi$ for any isogeny $\psi : E \to E'$;
4. $\deg \phi = \deg \hat\phi$;
5. $\hat{\hat\phi} = \phi$.

# Up to isomorphism



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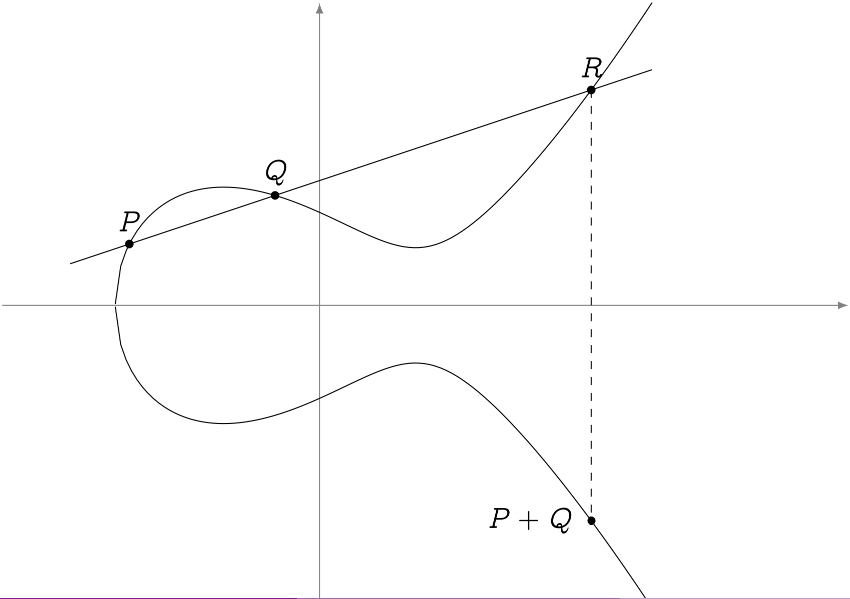

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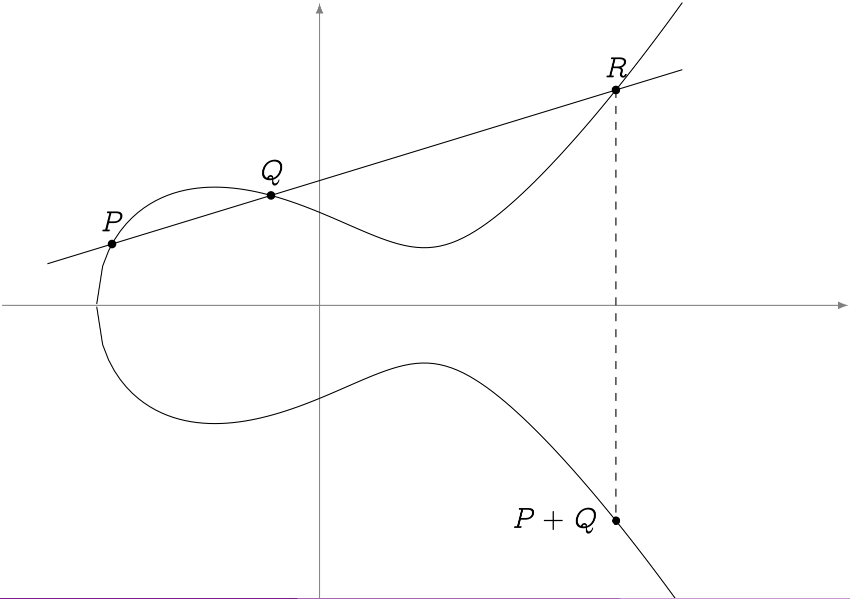

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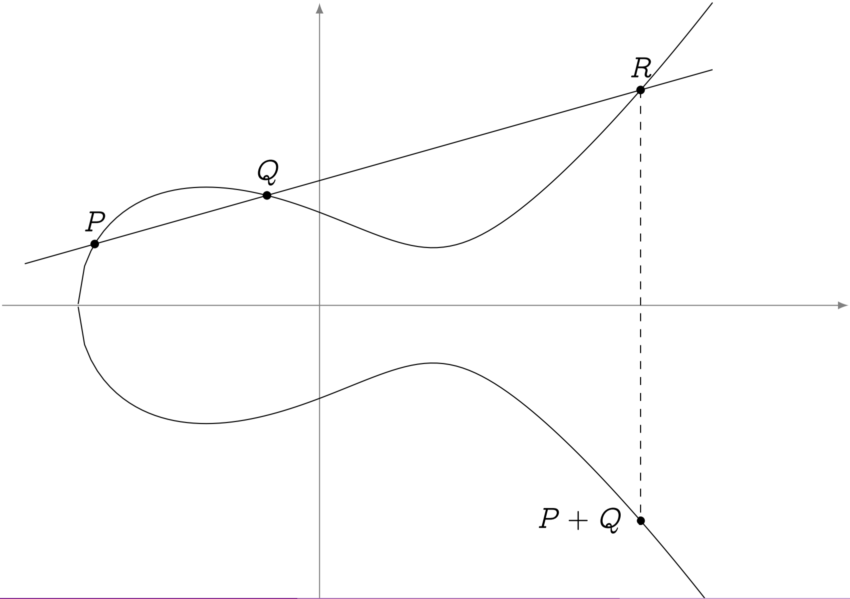

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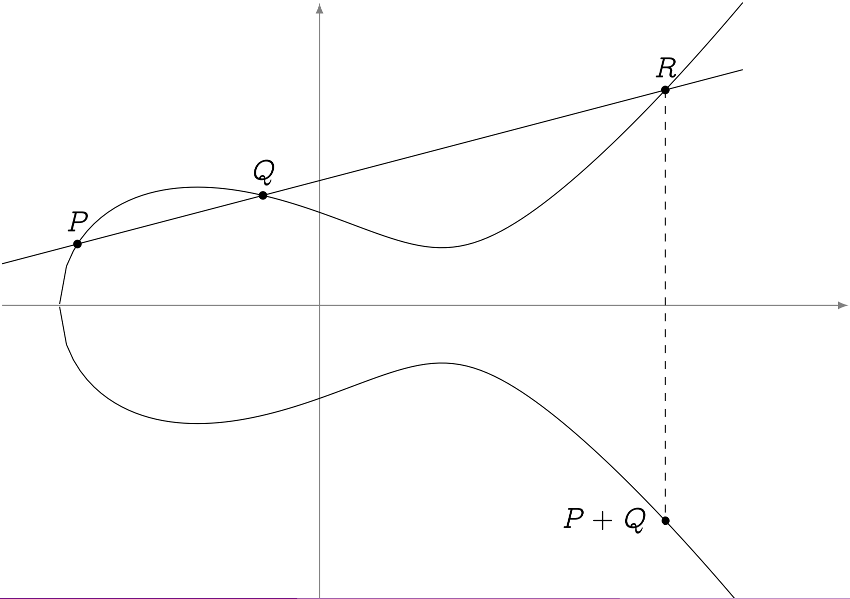

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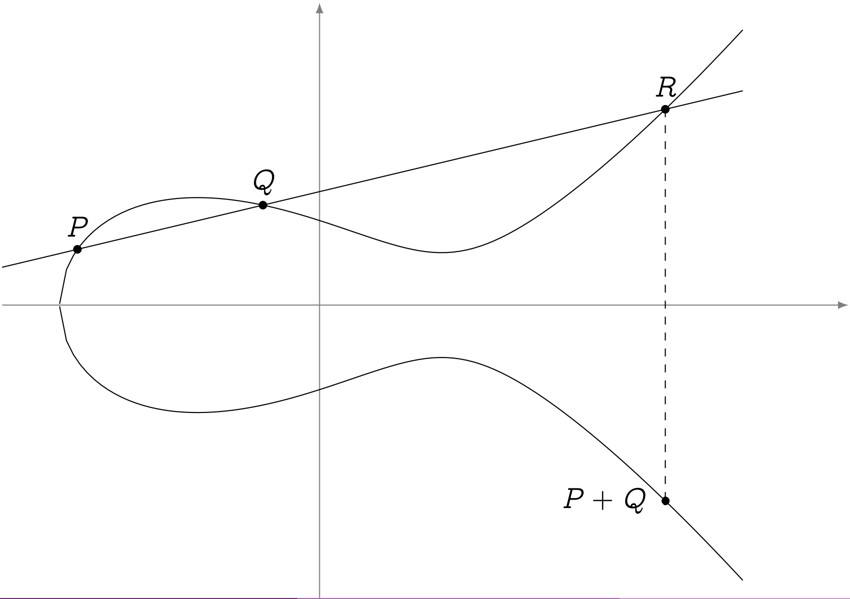

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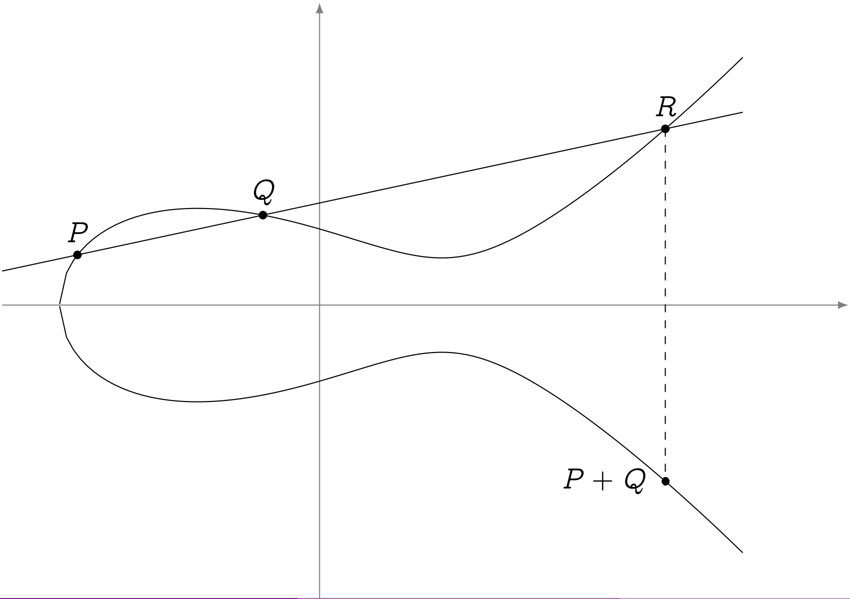

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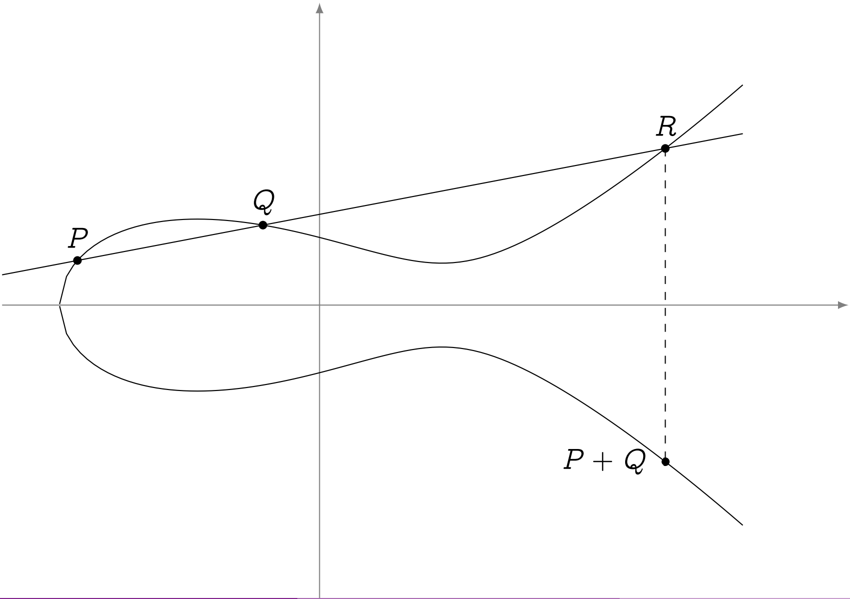

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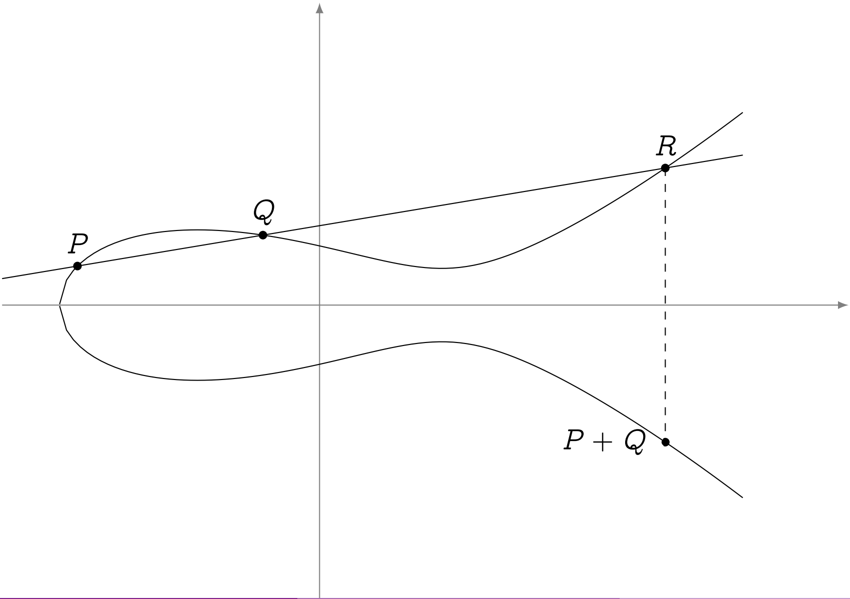

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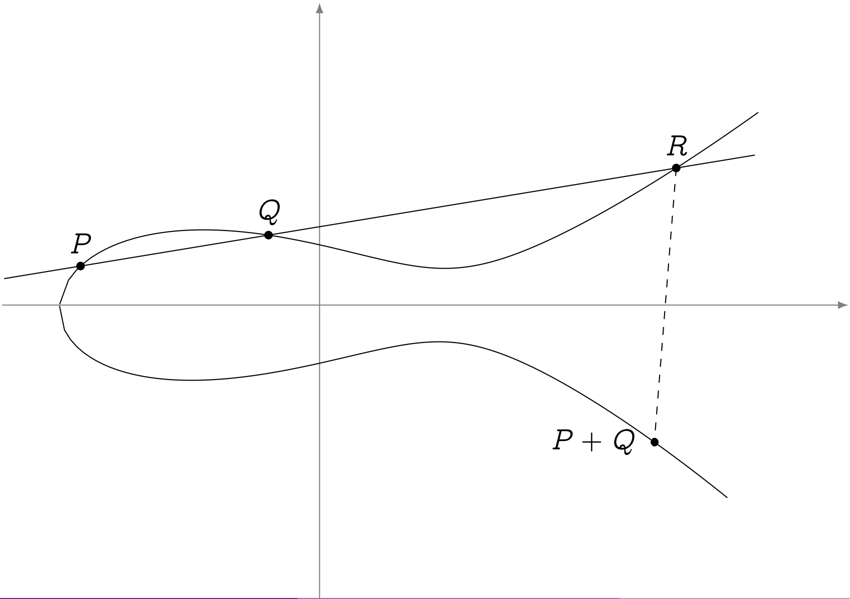

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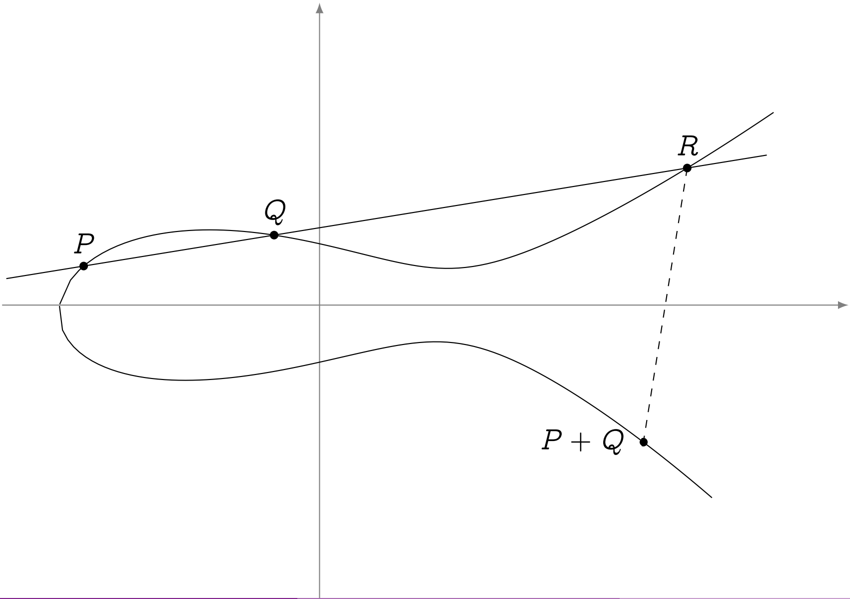

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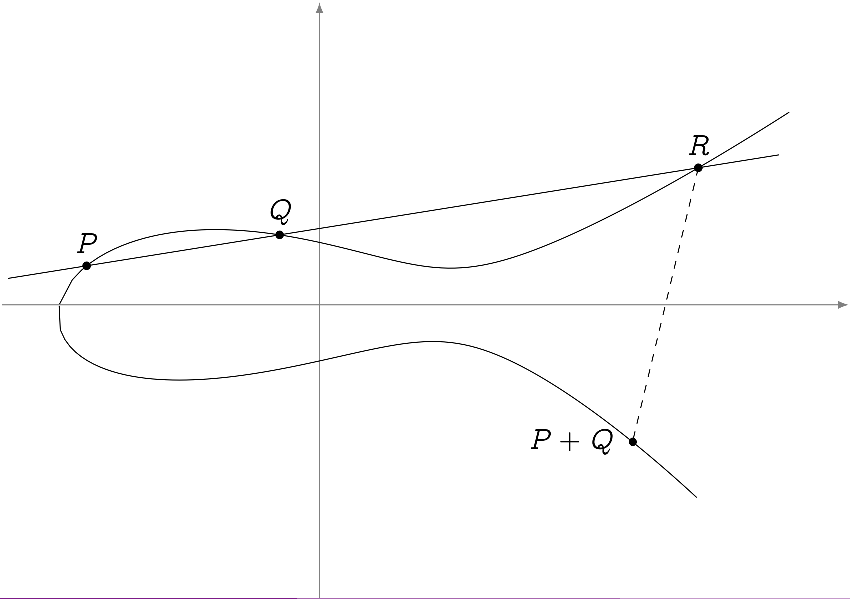

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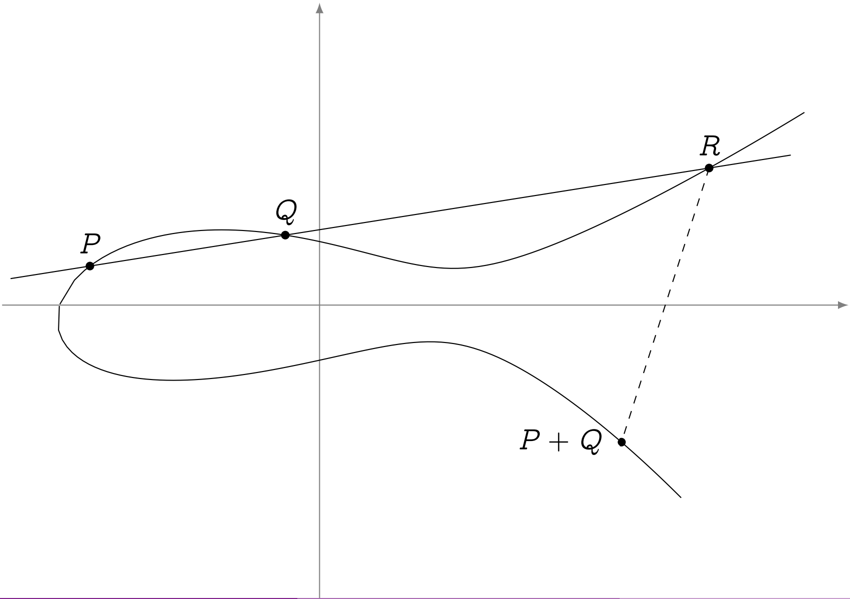

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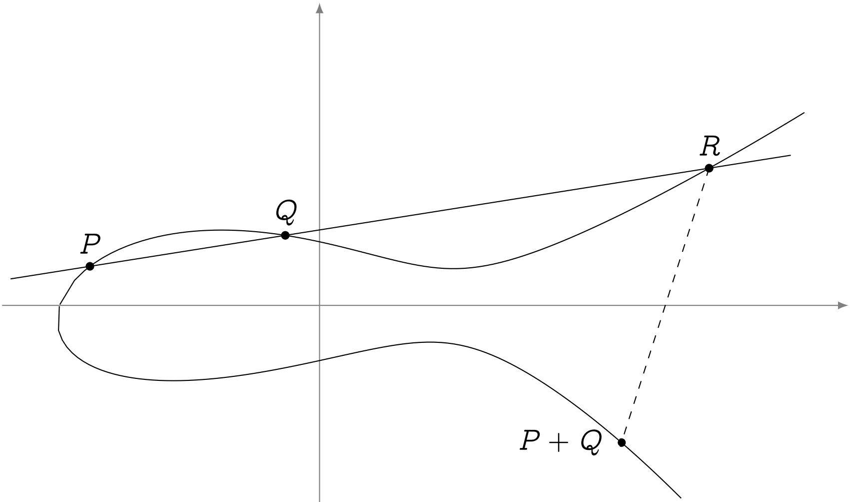


$$y^2 = x^3 + ax + b \quad \longrightarrow \quad j \equiv 1728\frac{4a^3}{4a^3+27b^2}$$

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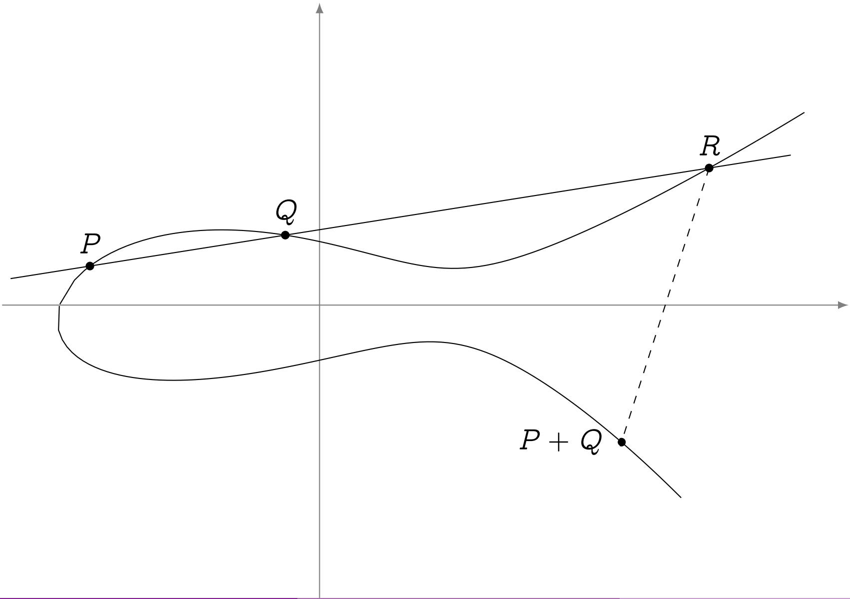

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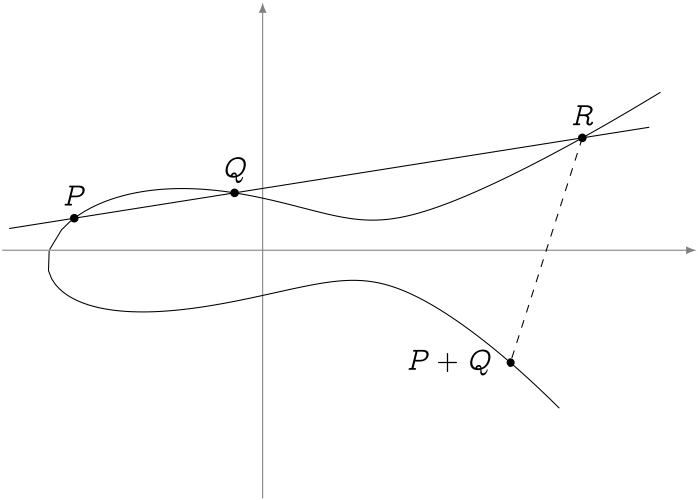

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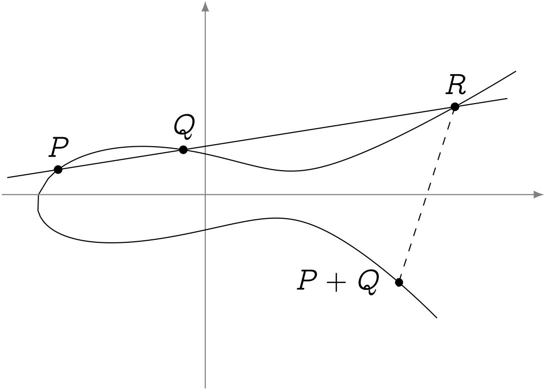

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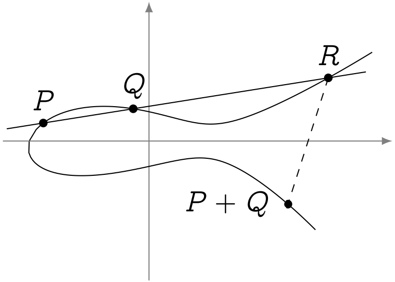

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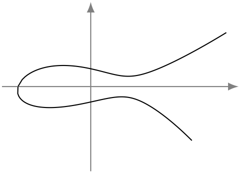

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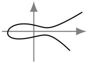

# Up to isomorphism

$j = 1728$

# Up to isomorphism

$$j = 1728 \xrightarrow{\phi} j = 287496$$

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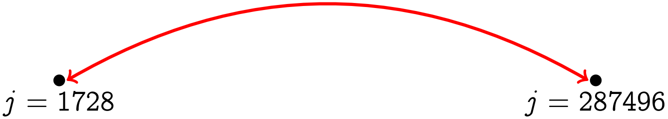


# Isogeny graphs

## Serre-Tate theorem

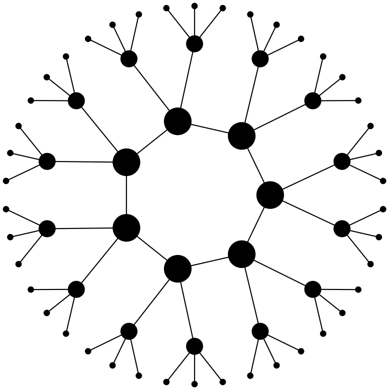
Two elliptic curves $E$, $E'$ defined over a finite field $\mathbb{F}_q$ are isogenous (over $\mathbb{F}_q$) iff $\#E(\mathbb{F}_q) = \#E'(\mathbb{F}_q)$.

### Isogeny graphs

- Vertices are curves up to isomorphism,
- Edges are isogenies up to isomorphism.

### Isogeny volcanoes

- Curves are ordinary,
- Isogenies all have degree a prime $\ell$.



# What do isogeny graphs look like?

## Torsion subgroups ($\ell$ prime)

In an algebraically closed field:

$$E[\ell] = \langle P, Q \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

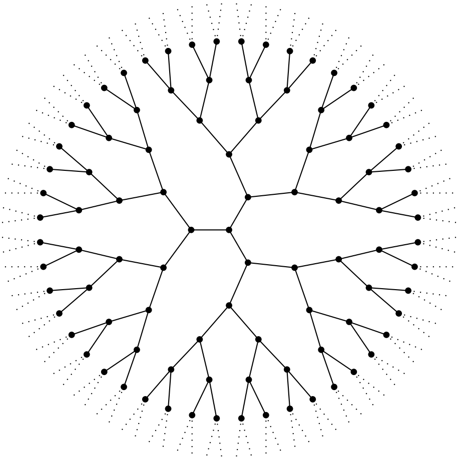
$$\Downarrow$$

There are exactly $\ell + 1$ cyclic subgroups $H \subset E$ of order $\ell$:

$$\langle P + Q \rangle, \langle P + 2Q \rangle, \dots, \langle P \rangle, \langle Q \rangle$$

$$\Downarrow$$

There are exactly $\ell + 1$ distinct isogenies of degree $\ell$.



(non-CM) 2-isogeny graph over $\mathbb{C}$

# What happens over a finite field $\mathbb{F}_p$?

## Rational isogenies ($\ell \neq p$)

In the algebraic closure $\bar{\mathbb{F}}_p$

$$E[\ell] = \langle P, Q \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

However, an isogeny is defined over $\mathbb{F}_p$ only if its kernel is Galois invariant.

Enter the Frobenius map

$$\begin{aligned} \pi : E &\longrightarrow E \\ (x, y) &\longmapsto (x^p, y^p) \end{aligned}$$

$E$ is seen here as a curve over $\bar{\mathbb{F}}_p$.

## The Frobenius action on $E[\ell]$

$$\begin{aligned} \pi(P) &= \quad aP + bQ \\ \pi(Q) &= \quad cP + dQ \end{aligned}$$

# What happens over a finite field $\mathbb{F}_p$?

## Rational isogenies ($\ell \neq p$)

In the algebraic closure $\bar{\mathbb{F}}_p$

$$E[\ell] = \langle P, Q\rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

However, an isogeny is defined over $\mathbb{F}_p$ only if its kernel is Galois invariant.

Enter the Frobenius map

$$\begin{aligned} \pi : E &\longrightarrow E \\ (x, y) &\longmapsto (x^p, y^p) \end{aligned}$$

$E$ is seen here as a curve over $\bar{\mathbb{F}}_p$.

## The Frobenius action on $E[\ell]$

$$aP + bQ$$

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## The Frobenius action on $E[\ell]$

$$\pi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod \ell$$

We identify $\pi|E[\ell]$ to a conjugacy class in $\mathrm{GL}(\mathbb{Z}/\ell\mathbb{Z})$.

# What happens over a finite field $\mathbb{F}_p$?

Galois invariant proper subgroups of $E[\ell]$

=

eigenspaces of $\pi \in \mathrm{GL}(\mathbb{Z}/\ell\mathbb{Z})$

=

rational isogenies of degree $\ell$

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## How many Galois invariant subgroups?

- $\pi|E[\ell] \sim \left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right)$ → $\ell + 1$ isogenies
- $\pi|E[\ell] \sim \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ with $\lambda \neq \mu$ → two isogenies
- $\pi|E[\ell] \sim \left(\begin{smallmatrix}\lambda & *\\ 0 & \lambda\end{smallmatrix}\right)$ → one isogeny
- $\pi|E[\ell]$ has no eigenvalues in $\mathbb{Z}/\ell\mathbb{Z}$ → no isogeny

# Algebras, orders

- A quadratic imaginary number field is an extension of $\mathbb{Q}$ of the form $Q[\sqrt{-D}]$ for some non-square $D > 0$.
- A quaternion algebra is an algebra of the form $\mathbb{Q} + \alpha\mathbb{Q} + \beta\mathbb{Q} + \alpha\beta\mathbb{Q}$, where the generators satisfy the relations

$$\alpha^2, \beta^2 \in \mathbb{Q}, \quad \alpha^2 < 0, \quad \beta^2 < 0, \quad \beta\alpha = -\alpha\beta.$$

### Orders

Let $K$ be a finitely generated $\mathbb{Q}$-algebra. An order $\mathcal{O} \subset K$ is a subring of $K$ that is a finitely generated $\mathbb{Z}$-module of maximal dimension. An order that is not contained in any other order of $K$ is called a maximal order.

Examples:

- $\mathbb{Z}$ is the only order contained in $\mathbb{Q}$,
- $\mathbb{Z}[i]$ is the only maximal order of $\mathbb{Q}(i)$,
- $\mathbb{Z}[\sqrt{5}]$ is a non-maximal order of $\mathbb{Q}(\sqrt{5})$,
- The ring of integers of a number field is its only maximal order,
- In general, maximal orders in quaternion algebras are not unique.

# The endomorphism ring

The endomorphism ring $\mathrm{End}(E)$ of an elliptic curve $E$ is the ring of all isogenies $E \to E$ (plus the null map) with addition and composition.

## Theorem (Deuring)

Let $E$ be an elliptic curve defined over a field $k$ of characteristic $p$. $\mathrm{End}(E)$ is isomorphic to one of the following:

- $\mathbb{Z}$, only if $p = 0$

  $E$ is ordinary.

- An order $\mathcal{O}$ in a quadratic imaginary field:

  $E$ is ordinary with complex multiplication by $\mathcal{O}$.

- Only if $p > 0$, a maximal order in a quaternion algebra[a]:

  $E$ is supersingular.

[a] (ramified at $p$ and $\infty$)

# The finite field case

**Theorem (Hasse)**

Let $E$ be defined over a finite field. Its Frobenius endomorphism $\pi$ satisfies a quadratic equation

$$\pi^2 - t\pi + q = 0$$

in $\mathrm{End}(E)$ for some $|t| \leq 2\sqrt{q}$, called the trace of $\pi$. The trace $t$ is coprime to $q$ if and only if $E$ is ordinary.

Suppose $E$ is ordinary, then $D_\pi = t^2 - 4q < 0$ is the discriminant of $\mathbb{Z}[\pi]$.

- $K = \mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{D_\pi})$ is the endomorphism algebra of $E$.
- Denote by $\mathcal{O}_K$ its ring of integers, then

$$\mathbb{Z} \neq \mathbb{Z}[\pi] \subset \mathrm{End}(E) \subset \mathcal{O}_K.$$

In the supersingular case, $\pi$ may or may not be in $\mathbb{Z}$, depending on $q$.

# Endomorphism rings of ordinary curves

## Classifying quadratic orders

Let $K$ be a quadratic number field, and let $\mathcal{O}_K$ be its ring of integers.

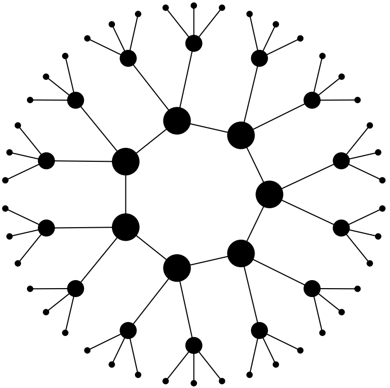
- Any order $\mathcal{O} \subset K$ can be written as $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ for an integer $f$, called the conductor of $\mathcal{O}$, denoted by $[\mathcal{O}_k : \mathcal{O}]$.
- If $d_K$ is the discriminant of $K$, the discriminant of $\mathcal{O}$ is $f^2 d_K$.
- If $\mathcal{O}, \mathcal{O}'$ are two orders with discriminants $d, d'$, then $\mathcal{O} \subset \mathcal{O}'$ iff $d' | d$.

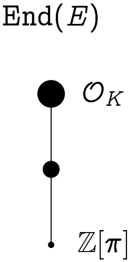
$$
\begin{tikzcd}
 & \mathcal{O}_K \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
\mathbb{Z} + 2\mathcal{O}_K \arrow[d, no head] \arrow[dr, no head] & \mathbb{Z} + 3\mathcal{O}_K \arrow[dl, no head] \arrow[dr, no head] & \mathbb{Z} + 5\mathcal{O}_K \arrow[dl, no head] \arrow[d, no head] \\
\mathbb{Z} + 6\mathcal{O}_K \arrow[dr, no head] & \mathbb{Z} + 10\mathcal{O}_K \arrow[d, no head] & \mathbb{Z} + 15\mathcal{O}_K \arrow[dl, no head] \\
 & \mathbb{Z}[\pi] \simeq \mathbb{Z} + 30\mathcal{O}_K &
\end{tikzcd}
$$

# Volcanology (Kohel 1996)

Let $E, E'$ be curves with respective endomorphism rings $\mathcal{O}, \mathcal{O}' \subset K$.
Let $\phi : E \to E'$ be an isogeny of prime degree $\ell$, then:

if $\mathcal{O} = \mathcal{O}'$, $\phi$ is horizontal;
if $[\mathcal{O}' : \mathcal{O}] = \ell$, $\phi$ is ascending;
if $[\mathcal{O} : \mathcal{O}'] = \ell$, $\phi$ is descending.



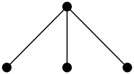



Ordinary isogeny volcano of degree $\ell = 3$.

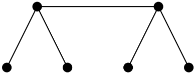
# Volcanology (Kohel 1996)

Let $E$ be ordinary,
$\mathtt{End}(E) \subset K$.

$\mathcal{O}_K$: maximal order of $K$,
$D_K$: discriminant of $K$.



$\left(\frac{D_K}{\ell}\right) = -1$



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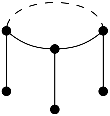

$\left(\frac{D_K}{\ell}\right) = +1$

| | | Horizontal | Ascending | Descending |
|---|---|---|---|---|
| $\ell \nmid [\mathcal{O}_K : \mathcal{O}]$ | $\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$ | $1 + \left(\frac{D_K}{\ell}\right)$ | | |
| $\ell \nmid [\mathcal{O}_K : \mathcal{O}]$ | $\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$ | $1 + \left(\frac{D_K}{\ell}\right)$ | | $\ell - \left(\frac{D_K}{\ell}\right)$ |
| $\ell \mid [\mathcal{O}_K : \mathcal{O}]$ | $\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$ | | $1$ | $\ell$ |
| $\ell \mid [\mathcal{O}_K : \mathcal{O}]$ | $\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$ | | $1$ | |

# Volcanology (Kohel 1996)

Let $E$ be ordinary,
$\mathrm{End}(E) \subset K$.

$\mathcal{O}_K$: maximal order of $K$,
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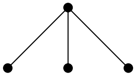
Height $= v_\ell([\mathcal{O}_K : \mathbb{Z}[\pi]])$.



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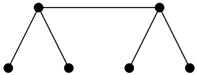

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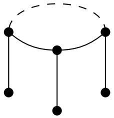

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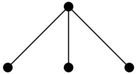
How large is the crater?



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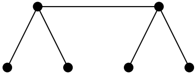

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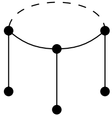

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| $\ell \mid [\mathcal{O}_K : \mathcal{O}]$ | $\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$ | | 1 | |

# How large is the crater of a volcano?

Let $\mathtt{End}(E) = \mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$. Define

- $\mathcal{I}(\mathcal{O})$, the group of invertible fractional ideals,
- $\mathcal{P}(\mathcal{O})$, the group of principal ideals,

### The class group

The class group of $\mathcal{O}$ is

$$\mathrm{Cl}(\mathcal{O}) = \mathcal{I}(\mathcal{O})/\mathcal{P}(\mathcal{O}).$$

- It is a finite abelian group.
- Its order $h(\mathcal{O})$ is called the class number of $\mathcal{O}$.
- It arises as the Galois group of an abelian extension of $\mathbb{Q}(\sqrt{-D})$.

# Complex multiplication

## The $\mathfrak{a}$-torsion

- Let $\mathfrak{a} \subset \mathcal{O}$ be an (integral invertible) ideal of $\mathcal{O}$;
- Let $E[\mathfrak{a}]$ be the subgroup of $E$ annihilated by $\mathfrak{a}$:

$$E[\mathfrak{a}] = \{P \in E \mid \alpha(P) = 0 \text{ for all } \alpha \in \mathfrak{a}\};$$

- Let $\phi : E \to E_{\mathfrak{a}}$, where $E_{\mathfrak{a}} = E/E[\mathfrak{a}]$.

Then $\mathrm{End}(E_{\mathfrak{a}}) = \mathcal{O}$ (i.e., $\phi$ is horizontal).

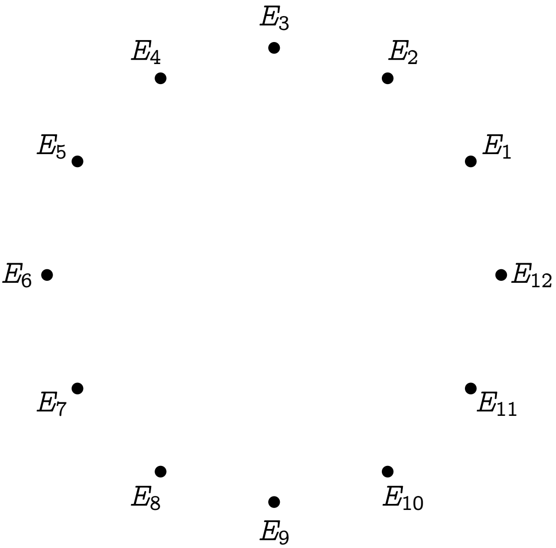
## Theorem (Complex multiplication)

*The action on the set of elliptic curves with complex multiplication by $\mathcal{O}$ defined by $\mathfrak{a} * j(E) = j(E_{\mathfrak{a}})$ factors through $\mathrm{Cl}(\mathcal{O})$, is faithful and transitive.*

## Corollary

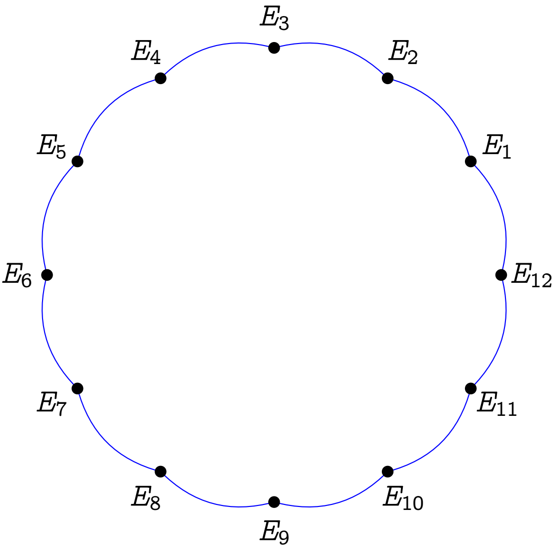
*Let $\mathrm{End}(E)$ have discriminant $D$. Assume that $\left(\frac{D}{\ell}\right) = 1$, then $E$ is on a crater of size $N$ of an $\ell$-volcano, and $N \mid h(\mathrm{End}(E))$*

# Complex multiplication graphs




Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathtt{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

# Complex multiplication graphs

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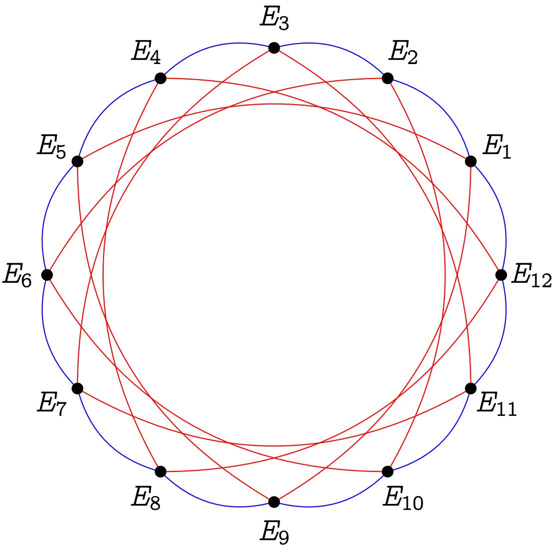
Edges are horizontal isogenies of bounded prime degree.

— degree 2

# Complex multiplication graphs

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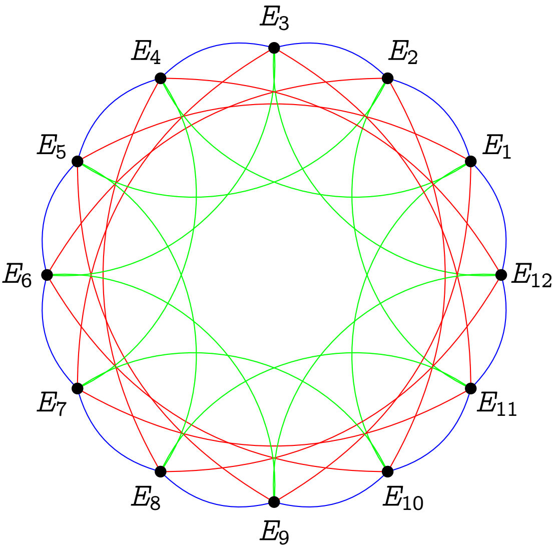
— degree 2

— degree 3

# Complex multiplication graphs

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— degree 2

— degree 3

— degree 5

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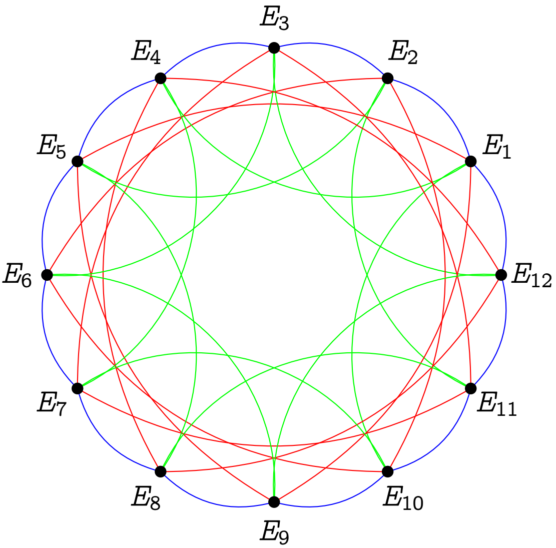


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Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

— degree 5

Isomorphic to a Cayley graph of $\mathrm{Cl}(\mathcal{O}_K)$.

# Supersingular endomorphisms

Recall, a curve $E$ over a field $\mathbb{F}_q$ of characteristic $p$ is supersingular iff

$$\pi^2 - t\pi + q = 0$$

with $t = 0 \mod p$.

**Case:** $t = 0 \quad \Rightarrow \quad D_\pi = -4q$

- Only possibility for $E/\mathbb{F}_p$,
- $E/\mathbb{F}_p$ has CM by an order of $\mathbb{Q}(\sqrt{-p})$, similar to the ordinary case.

**Case:** $t = \pm 2\sqrt{q} \quad \Rightarrow \quad D_\pi = 0$

- General case for $E/\mathbb{F}_q$, when $q$ is an even power.
- $\pi = \pm\sqrt{q}$, hence no complex multiplication.

We will ignore marginal cases: $t = \pm\sqrt{q}, \pm\sqrt{2q}, \pm\sqrt{3q}$.

# Supersingular complex multiplication

Let $E/\mathbb{F}_p$ be a supersingular curve, then $\pi^2 = -p$, and

$$\pi = \begin{pmatrix} \sqrt{-p} & 0 \\ 0 & -\sqrt{-p} \end{pmatrix} \mod \ell$$

for any $\ell$ s.t. $\left(\frac{-p}{\ell}\right) = 1$.

**Theorem (Delfs, Galbraith 2016)**

Let $\mathrm{End}_{\mathbb{F}_p}(E)$ denote the ring of $\mathbb{F}_p$-rational endomorphisms of $E$. Then

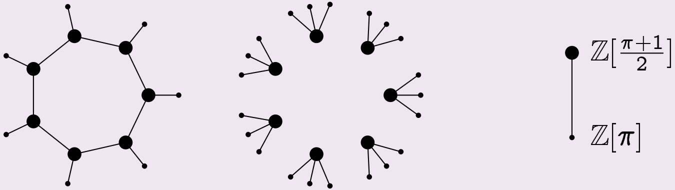
$$\mathbb{Z}[\pi] \subset \mathrm{End}_{\mathbb{F}_p}(E) \subset \mathbb{Q}(\sqrt{-p}).$$

**Orders of $\mathbb{Q}(\sqrt{-p})$**

- If $p = 1 \mod 4$, then $\mathbb{Z}[\pi]$ is the maximal order.
- If $p = -1 \mod 4$, then $\mathbb{Z}[\frac{\pi+1}{2}]$ is the maximal order, and $[\mathbb{Z}[\frac{\pi+1}{2}] : \mathbb{Z}[\pi]] = 2$.

# Supersingular CM graphs

## 2-volcanoes, $p = -1 \bmod 4$




## 2-graphs, $p = 1 \bmod 4$


All other $\ell$-graphs are cycles of horizontal isogenies iff $\left(\frac{-p}{\ell}\right) = 1$.

# The full endomorphism ring

## Theorem (Deuring)

Let $E$ be a supersingular elliptic curve, then

- $E$ is isomorphic to a curve defined over $\mathbb{F}_{p^2}$;
- Every isogeny of $E$ is defined over $\mathbb{F}_{p^2}$;
- Every endomorphism of $E$ is defined over $\mathbb{F}_{p^2}$;
- $\mathrm{End}(E)$ is isomorphic to a maximal order in a quaternion algebra ramified at $p$ and $\infty$.

In particular:

- If $E$ is defined over $\mathbb{F}_p$, then $\mathrm{End}_{\mathbb{F}_p}(E)$ is strictly contained in $\mathrm{End}(E)$.
- Some endomorphisms do not commute!

# An example

The curve of $j$-invariant $1728$

$$E : y^2 = x^3 + x$$

is supersingular over $\mathbb{F}_p$ iff $p = -1 \mod 4$.

## Endomorphisms

$\mathrm{End}(E) = \mathbb{Z}\langle \iota, \pi \rangle$, with:

- $\pi$ the Frobenius endomorphism, s.t. $\pi^2 = -p$;
- $\iota$ the map

  $$\iota(x, y) = (-x, iy),$$

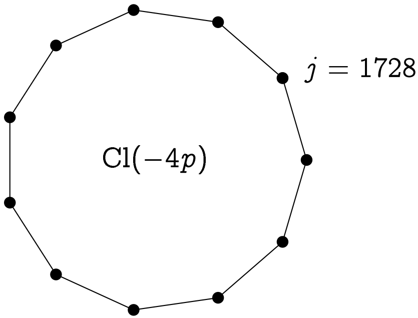
  where $i \in \mathbb{F}_{p^2}$ is a 4-th root of unity. Clearly, $\iota^2 = -1$.

And $\iota\pi = -\pi\iota$.

# Class group action party

$j = 1728$

# Class group action party



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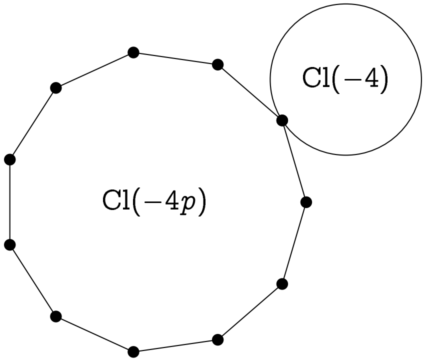

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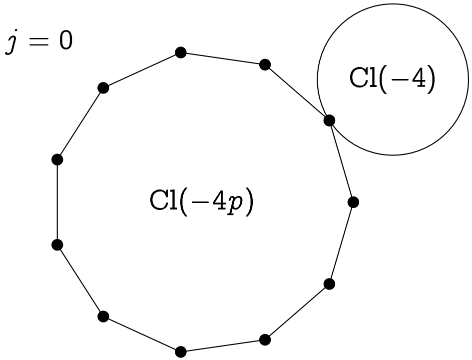

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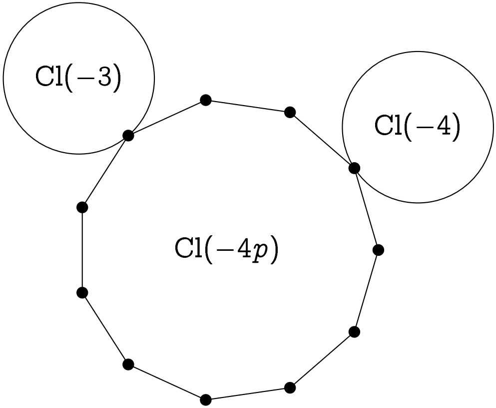

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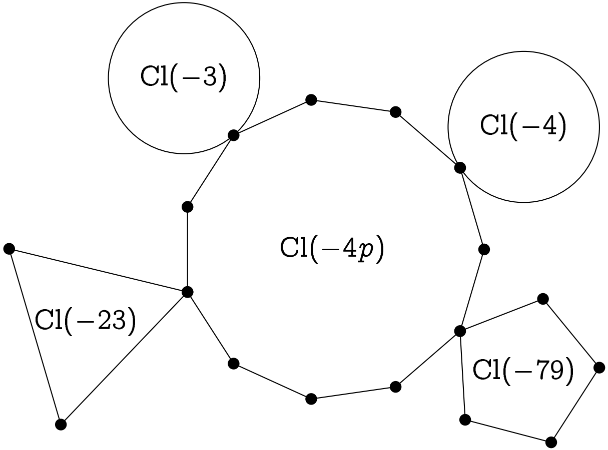


# Quaternion algebra?! WTF?[2]

The quaternion algebra $B_{p,\infty}$ is:

- A 4-dimensional $\mathbb{Q}$-vector space with basis $(1, i, j, k)$.
- A non-commutative division algebra[1] $B_{p,\infty} = \mathbb{Q}\langle i, j\rangle$ with the relations:
$$i^2 = a, \quad j^2 = -p, \quad ij = -ji = k,$$
for some $a < 0$ (depending on $p$).
- All elements of $B_{p,\infty}$ are quadratic algebraic numbers.
- $B_{p,\infty} \otimes \mathbb{Q}_\ell \simeq \mathcal{M}_{2\times 2}(\mathbb{Q}_\ell)$ for all $\ell \neq p$.
I.e., endomorphisms restricted to $E[\ell^e]$ are just $2 \times 2$ matrices $\bmod \ell^e$.
- $B_{p,\infty} \otimes \mathbb{R}$ is isomorphic to Hamilton's quaternions.
- $B_{p,\infty} \otimes \mathbb{Q}_p$ is a division algebra.

[1] All elements have inverses.
[2] What The Field?

# Supersingular graphs

- Quaternion algebras have many maximal orders.
- For every maximal order type of $B_{p,\infty}$ there are 1 or 2 curves over $\mathbb{F}_{p^2}$ having endomorphism ring isomorphic to it.
- There is a unique isogeny class of supersingular curves over $\bar{\mathbb{F}}_p$ of size $\approx p/12$.
- Left ideals act on the set of maximal orders like isogenies.
- The graph of $\ell$-isogenies is $(\ell+1)$-regular.

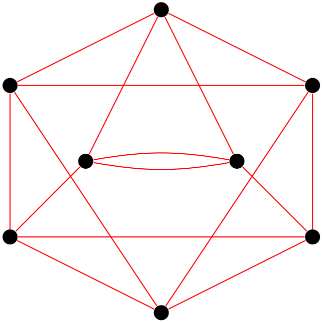


Figure: 3-isogeny graph on $\mathbb{F}_{97^2}$.

# Graphs lexicon

**Degree:** Number of (outgoing/ingoing) edges.

**$k$-regular:** All vertices have degree $k$.

**Connected:** There is a path between any two vertices.

**Distance:** The length of the shortest path between two vertices.

**Diameter:** The longest distance between two vertices.

**$\lambda_1 \geq \cdots \geq \lambda_n$:** The (ordered) eigenvalues of the adjacency matrix.

# Expander graphs

## Proposition

If $G$ is a $k$-regular graph, its largest and smallest eigenvalues satisfy

$$k = \lambda_1 \geq \lambda_n \geq -k.$$

## Expander families

An infinite family of connected $k$-regular graphs on $n$ vertices is an expander family if there exists an $\epsilon > 0$ such that all non-trivial eigenvalues satisfy $|\lambda| \leq (1-\epsilon)k$ for $n$ large enough.

- Expander graphs have short diameter ($O(\log n)$);
- Random walks mix rapidly (after $O(\log n)$ steps, the induced distribution on the vertices is close to uniform).

# Expander graphs from isogenies

## Theorem (Pizer)

Let $\ell$ be fixed. The family of graphs of supersingular curves over $\mathbb{F}_{p^2}$ with $\ell$-isogenies, as $p \to \infty$, is an expander family[a].

[a] Even better, it has the Ramanujan property.

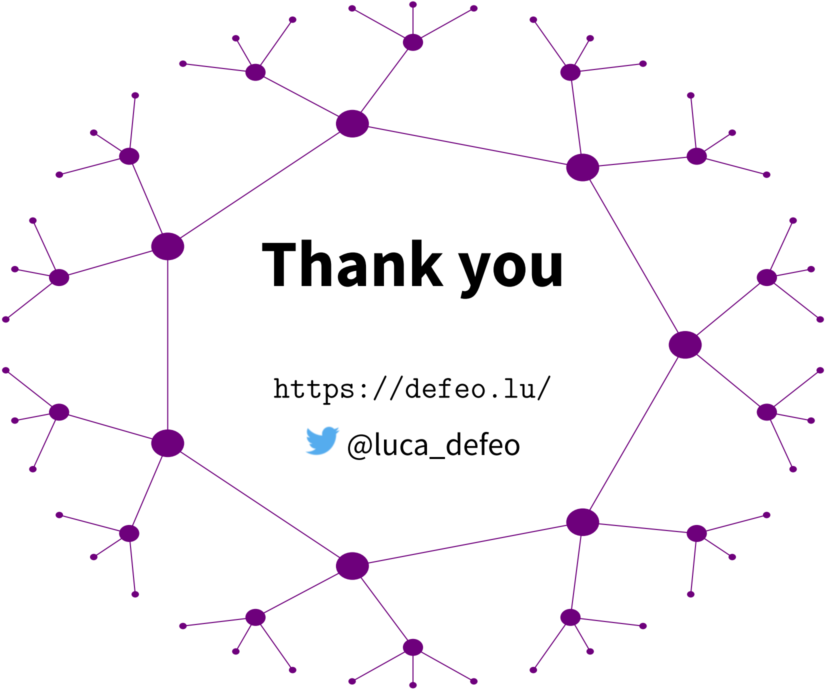
## Theorem (Jao, Miller, Venkatesan)

Let $\mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$ be an order in a quadratic imaginary field. The graphs of all curves over $\mathbb{F}_q$ with complex multiplication by $\mathcal{O}$, with isogenies of prime degree bounded[a] by $(\log q)^{2+\delta}$, are expanders.

[a] May contain traces of GRH.

# Executive summary

- Separable $\ell$-isogeny = finite kernel = subgroup of $E[\ell]$,
  - eigenspace of $\pi$ iff $\mathbb{F}_q$-rational,
  - distinct eigenvalues $\lambda \neq \mu$ define distinct directions on the crater.
- Isogeny graphs have $j$-invariants for vertices and “some” isogenies for edges.
- By varying the choices for the vertex and the isogeny set, we obtain graphs with different properties.
- $\ell$-isogeny graphs of ordinary curves are volcanoes, (full) $\ell$-isogeny graphs of supersingular curves are finite $(\ell + 1)$-regular.
- CM theory naturally leads to define graphs of horizontal isogenies (both in the ordinary and the supersingular case) that are isomorphic to Cayley graphs of class groups.
- CM graphs are expanders. Supersingular full $\ell$-isogeny graphs are Ramanujan.



# Thank you

https://defeo.lu/

@luca_defeo

# Weil pairing

Let $(N, p) = 1$, fix any basis $E[N] = \langle R, S \rangle$. For any points $P, Q \in E[N]$

$$P = aR + bS$$
$$Q = cR + dS$$

the form $\det_N(P, Q) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \in \mathbb{Z}/N\mathbb{Z}$
is bilinear, non-degenerate, and independent from the choice of basis.

## Theorem

Let $E/\mathbb{F}_q$ be a curve, there exists a Galois invariant bilinear map

$$e_N : E[N] \times E[N] \longrightarrow \mu_N \subset \bar{\mathbb{F}}_q,$$

called the Weil pairing of order $N$, and a primitive $N$-th root of unity $\zeta \in \bar{\mathbb{F}}_q$ such that

$$e_N(P, Q) = \zeta^{\det_N(P,Q)}.$$

The degree $k$ of the smallest extension such that $\zeta \in \mathbb{F}_{q^k}$ is called the embedding degree of the pairing.

# Weil pairing and isogenies

## Note

The Weil pairing is Galois invariant $\Leftrightarrow$ $\det(\pi|E[N]) = q$.

## Theorem

Let $\phi : E \to E'$ be an isogeny and $\hat{\phi} : E' \to E$ its dual.
Let $e_N$ be the Weil pairing of $E$ and $e'_N$ that of $E'$. Then, for

$$e_N(P, \hat{\phi}(Q)) = e'_N(\phi(P), Q),$$

for any $P \in E[N]$ and $Q \in E'[N]$.

## Corollary

$$e'_N(\phi(P), \phi(Q)) = e_N(P, Q)^{\deg \phi}.$$