Isogeny Based Cryptography: an Introduction

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Slides online at `https://defeo.lu/docet`

Why isogenies?

Six families still in NIST post-quantum competition:

Lattices	9 encryption	3 signature
Codes	7 encryption	
Multivariate		4 signature
Isogenies	1 encryption	
Hash-based		1 signature
MPC		1 signature

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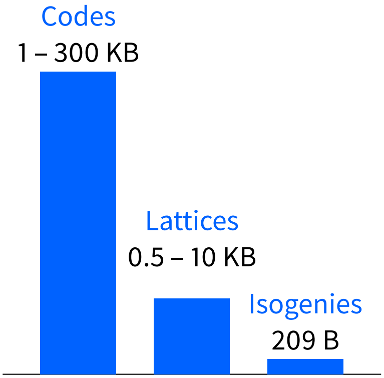

Public key size

NIST-1 level (AES128)

(not to scale)

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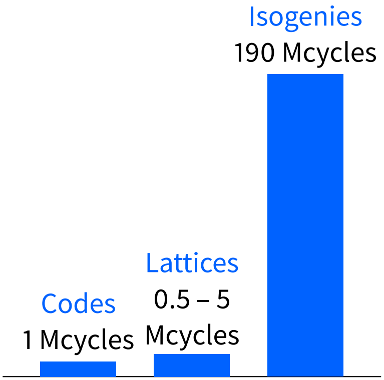


Encryption performance

NIST-1 level (AES128)

(not to scale)

*“We found that CECPQ2 ([NTRU] the ostrich) outperformed CECPQ2b ([SIKE] the turkey), for the majority of connections in the experiment, indicating that **fast algorithms with large keys may be more suitable for TLS than slow algorithms with small keys**. However, **we observed the opposite**—that CECPQ2b outperformed CECPQ2—**for the slowest connections on some devices**, including Windows computers and Android mobile devices. One possible explanation for this is packet fragmentation and packet loss.”*

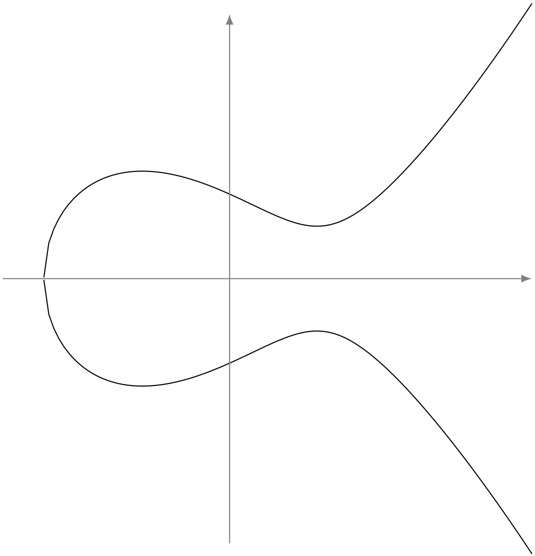
— K. Kwiatkowski, L. Valenta (Cloudflare)
The TLS Post-Quantum Experiment
`https://blog.cloudflare.com/the-tls-post-quantum-experiment/`

Weierstrass equations

Let k be a field of characteristic $\neq 2, 3$.
An elliptic curve *defined over* k is the locus in $\mathbb{P}^2(\bar{k})$ of an equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3,$$

where $a, b \in k$ and $4a^3 + 27b^2 \neq 0$.



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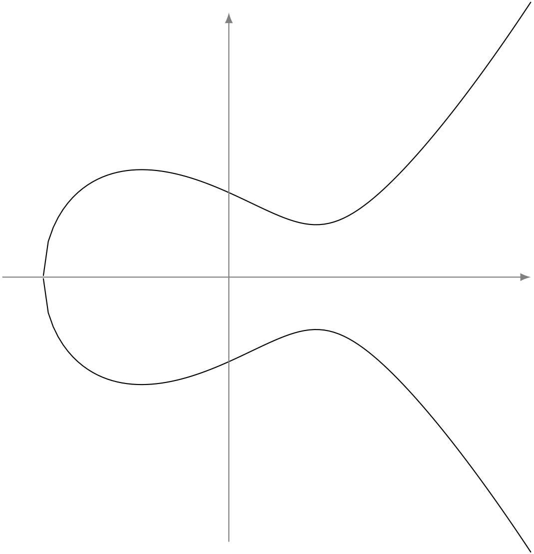
- $\mathcal{O} = (0 : 1 : 0)$ is the point at infinity;



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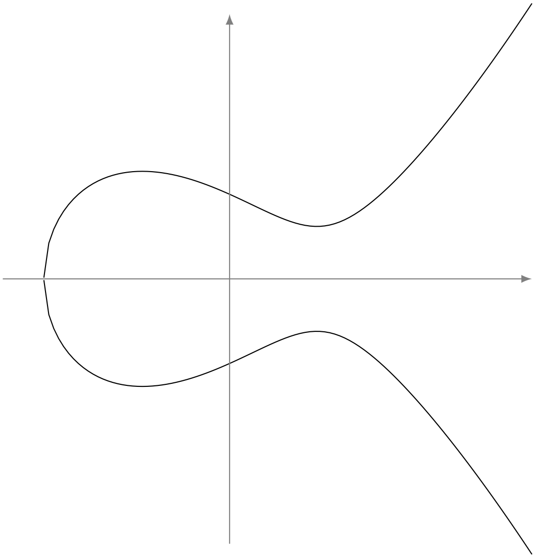
- $\mathcal{O} = (0 : 1 : 0)$ is the point at infinity;
- $y^2 = x^3 + ax + b$ is the affine equation.

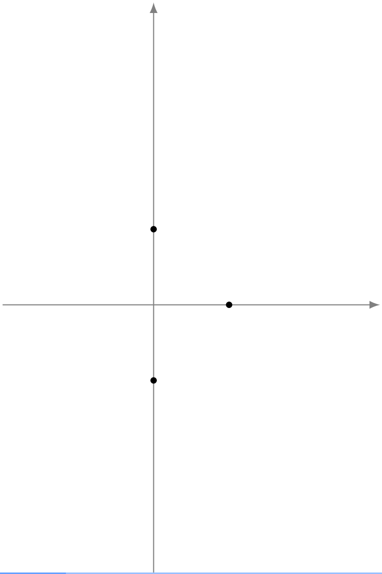


Attention: arithmetic geometry!

$$E \ : \ y^2 = x^3 - 2x + 1$$

Rational points:

- $E(\mathbb{Q}) = \{(1,0), (0,1), (0,-1), \mathcal{O}\}$,



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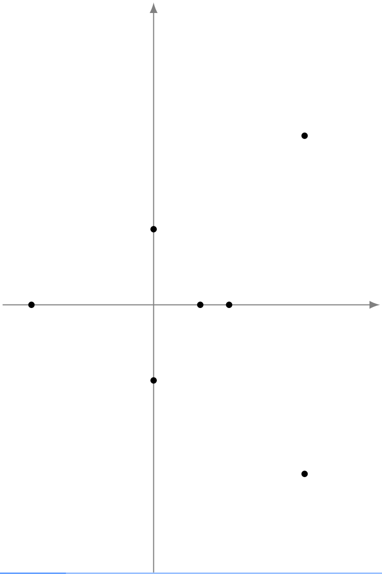

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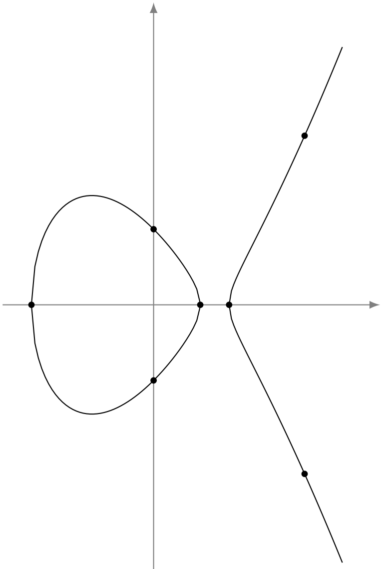

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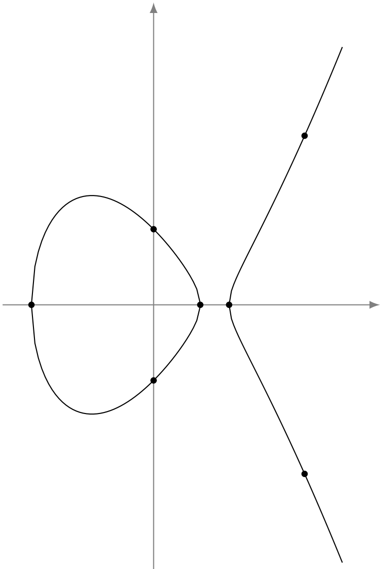

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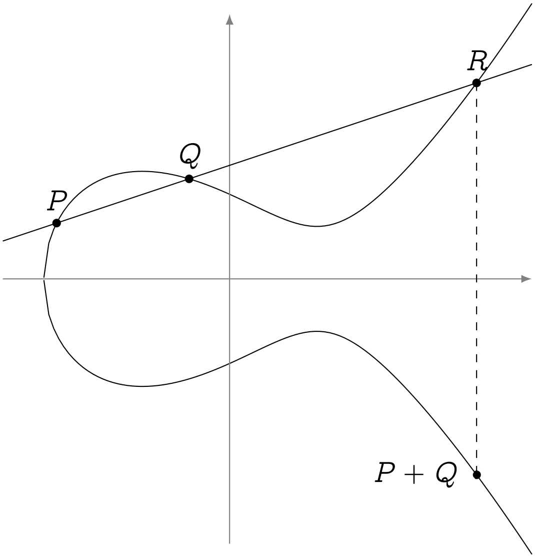
- $\#E(\mathbb{C}) = \infty$.

The group law

Bezout's theorem

Every line cuts E in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.



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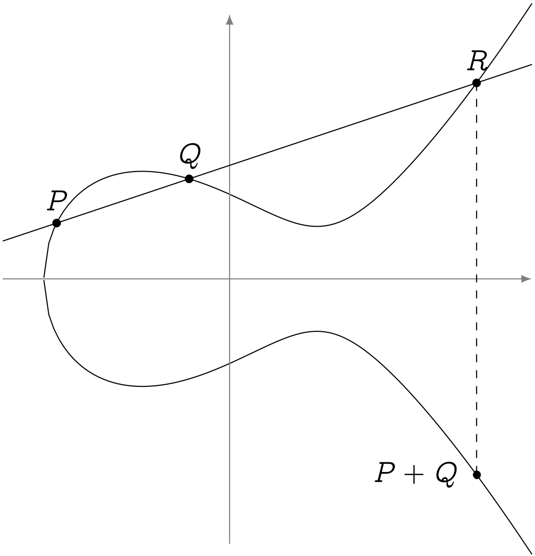

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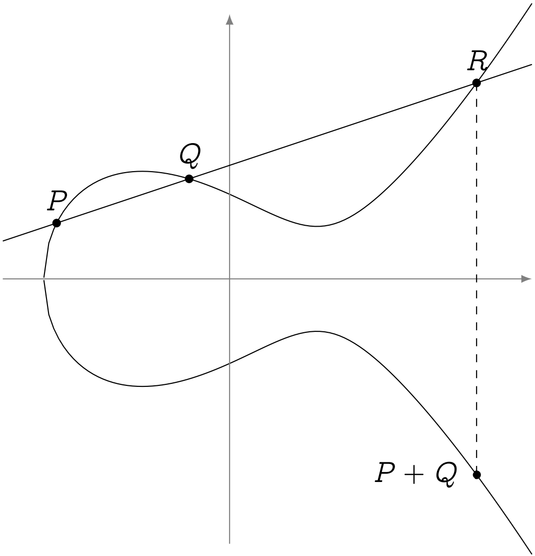
Define a group law such that any three colinear points add up to zero.

- The law is algebraic (it has *formulas*);
- The law is commutative;
- $\mathcal{O}$ is the group identity;
- Opposite points have the same x-value.



Maps: isomorphisms

Isomorphisms

The only invertible algebraic maps between elliptic curves are of the form

$$(x, y) \mapsto (u^2 x, u^3 y)$$

for some $u \in \bar{k}$.
They are group isomorphisms.

j-Invariant

Let $E \ : \ y^2 = x^3 + ax + b$, its j-invariant is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Two elliptic curves E, E' are isomorphic if and only if $j(E) = j(E')$.

Group structure

Torsion structure

Let E be defined over an algebraically closed field $\bar{k}$ of characteristic p.

$$E[m] \simeq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \qquad \text{if } p \nmid m,$$

$$E[p^e] \simeq \begin{cases} \mathbb{Z}/p^e\mathbb{Z} & \text{ordinary case,} \\ \{\mathcal{O}\} & \text{supersingular case.} \end{cases}$$

Finite fields (Hasse's theorem)

Let E be defined over a finite field $\mathbb{F}_q$, then

$$|\#E(\mathbb{F}_q) - q - 1| \le 2\sqrt{q}.$$

In particular, there exist integers n_1 and $n_2 | \gcd(n_1, q-1)$ such that

$$E(\mathbb{F}_q) \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}.$$

Maps: what's scalar multiplication?

$$[n] \;:\; P \mapsto \underbrace{P + P + \cdots + P}_{n \text{ times}}$$

- A map $E \to E$,
- a group morphism,
- with finite kernel
  (the torsion group $E[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$),
- surjective (in the algebraic closure),
- given by rational maps of degree n^2.

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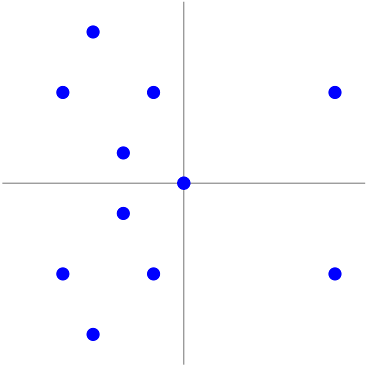
(Separable) isogenies $\Leftrightarrow$ finite subgroups:

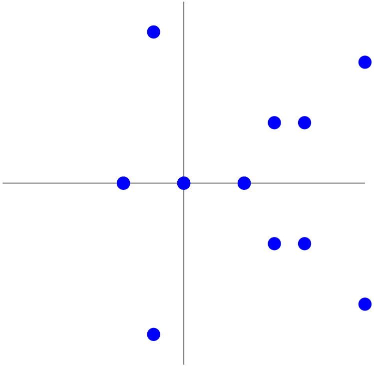
$$0 \to H \to E \xrightarrow{\phi} E' \to 0$$

Isogenies: an example over $\mathbb{F}_{11}$

$$E \; : \; y^2 = x^3 + x$$

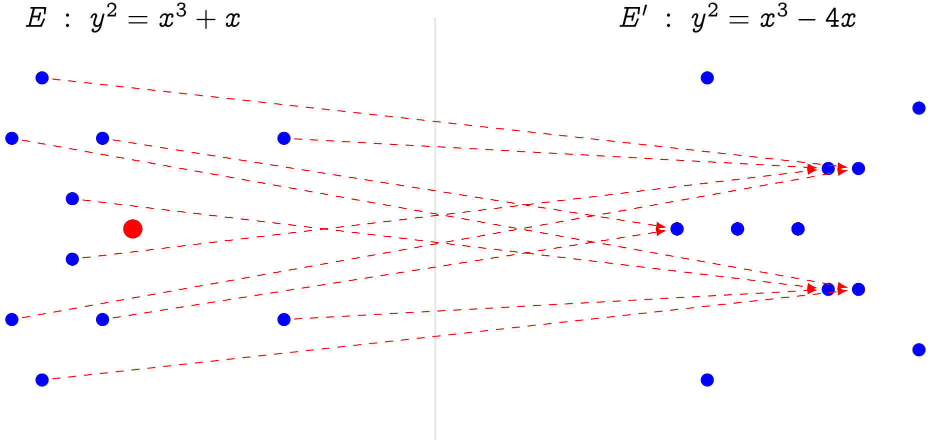


$$E' \; : \; y^2 = x^3 - 4x$$



$$\phi(x, y) = \left(\frac{x^2 + 1}{x}, \quad y \frac{x^2 - 1}{x^2} \right)$$

Isogenies: an example over $\mathbb{F}_{11}$



$$\phi(x, y) = \left(\frac{x^2 + 1}{x}, \quad y \frac{x^2 - 1}{x^2} \right)$$

- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

Maps: isogenies

Theorem

Let $\phi : E \to E'$ be a map between elliptic curves. These conditions are equivalent:

- *ϕ is a surjective group morphism,*
- *ϕ is a group morphism with finite kernel,*
- *ϕ is a non-constant algebraic map of projective varieties sending the point at infinity of E onto the point at infinity of E'.*

If they hold ϕ is called an isogeny.

Two curves are called isogenous if there exists an isogeny between them.

Example: Multiplication-by-m

On any curve, an isogeny from E to itself (i.e., an endomorphism):

$$\begin{aligned}[m] \ :\ E &\to E,\\ P &\mapsto [m]P.\end{aligned}$$

Isogeny lexicon

Degree

- $\approx$ degree of the rational fractions defining the isogeny;
- Rough measure of the information needed to encode it.

Separable, inseparable, cyclic

An isogeny ϕ is separable iff $\deg \phi = \# \ker \phi$.

- Given $H \subset E$ finite, write $\phi : E \to E/H$ for the unique separable isogeny s.t. $\ker \phi = H$.
- ϕ inseparable $\Rightarrow$ p divides $\deg \phi$.
- Cyclic isogeny $\equiv$ separable isogeny with cyclic kernel.
  - Non-example: the multiplication map $[m] : E \to E$.

Rationality

Given E defined over k, an isogeny ϕ is rational if $\ker \phi$ is Galois invariant.

$\Rightarrow$ ϕ is represented by rational fractions with coefficients in k.

The dual isogeny

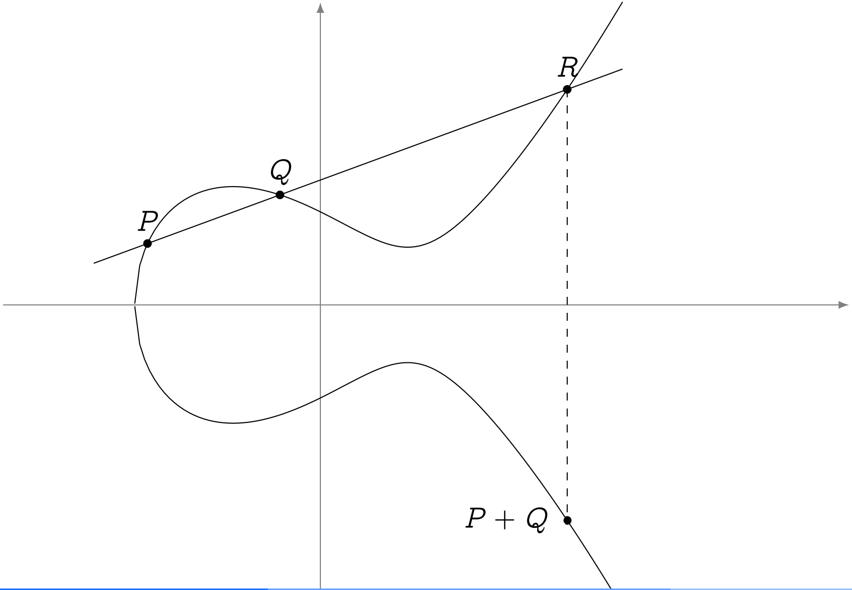
Let $\phi : E \to E'$ be an isogeny of degree m. There is a unique isogeny $\hat{\phi} : E' \to E$ such that

$$\hat{\phi} \circ \phi = [m]_E, \quad \phi \circ \hat{\phi} = [m]_{E'}.$$

$\hat{\phi}$ is called the dual isogeny of ϕ; it has the following properties:

1. $\hat{\phi}$ is defined over k if and only if ϕ is;
2. $\widehat{\psi \circ \phi} = \hat{\phi} \circ \hat{\psi}$ for any isogeny $\psi : E' \to E''$;
3. $\widehat{\psi + \phi} = \hat{\psi} + \hat{\phi}$ for any isogeny $\psi : E \to E'$;
4. $\deg \phi = \deg \hat{\phi}$;
5. $\hat{\hat{\phi}} = \phi$.

Up to isomorphism



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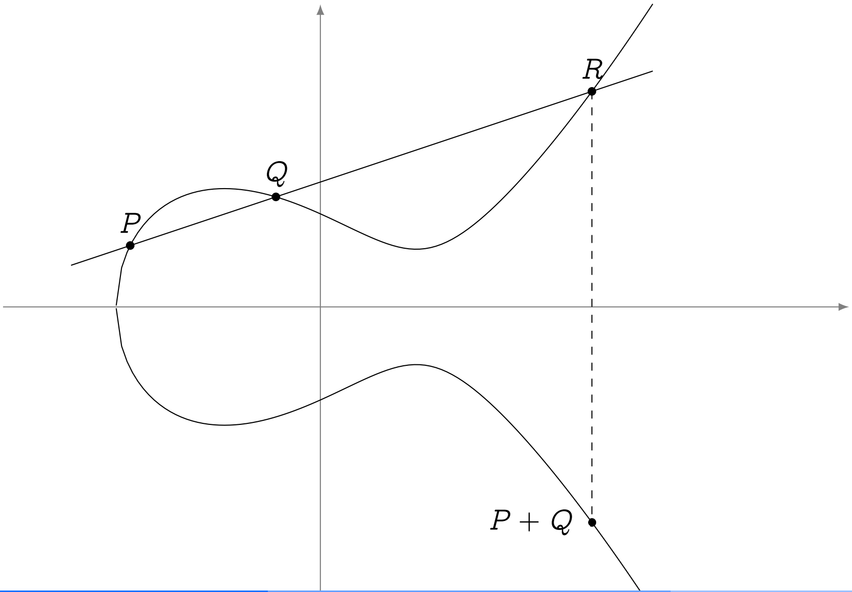

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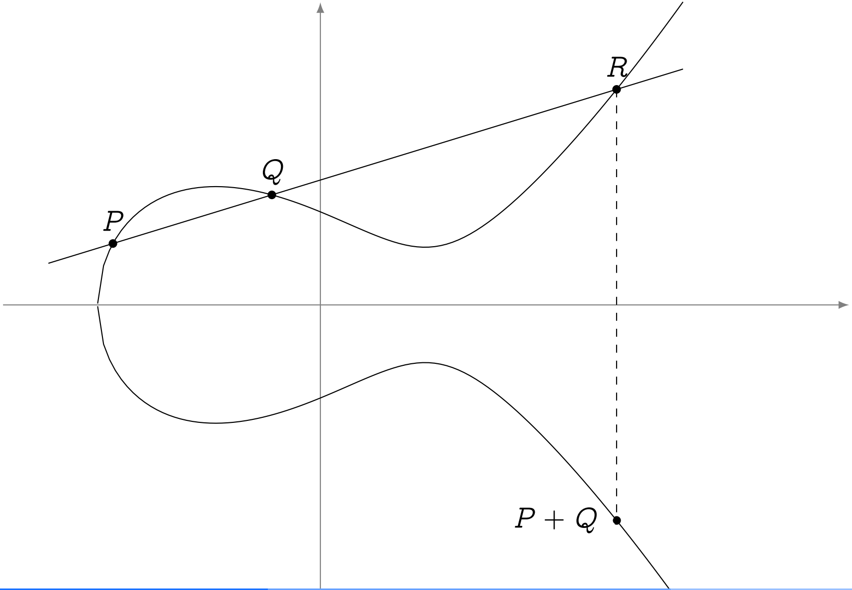

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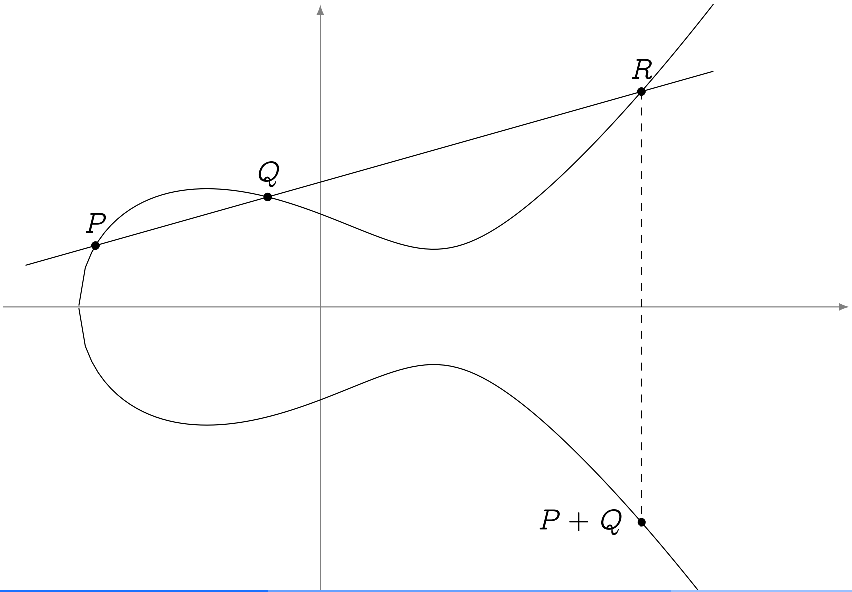

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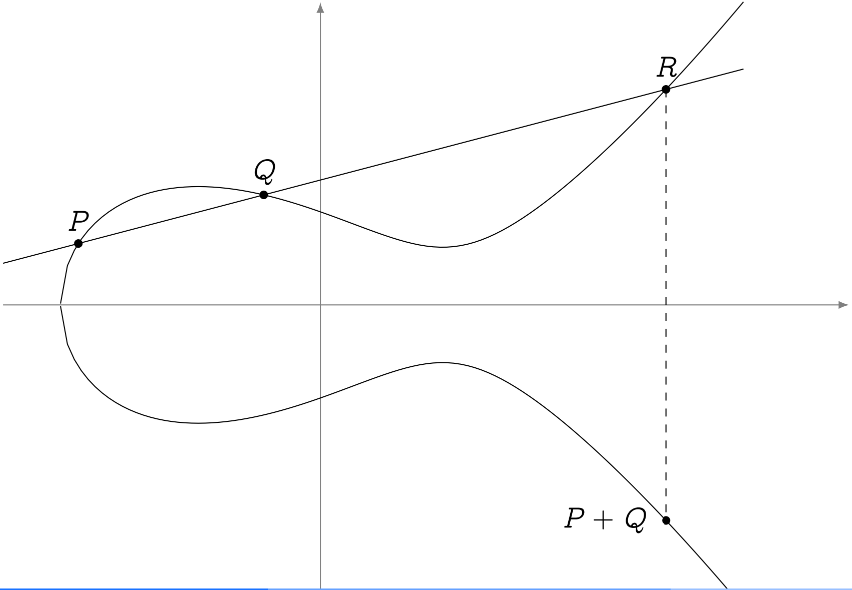

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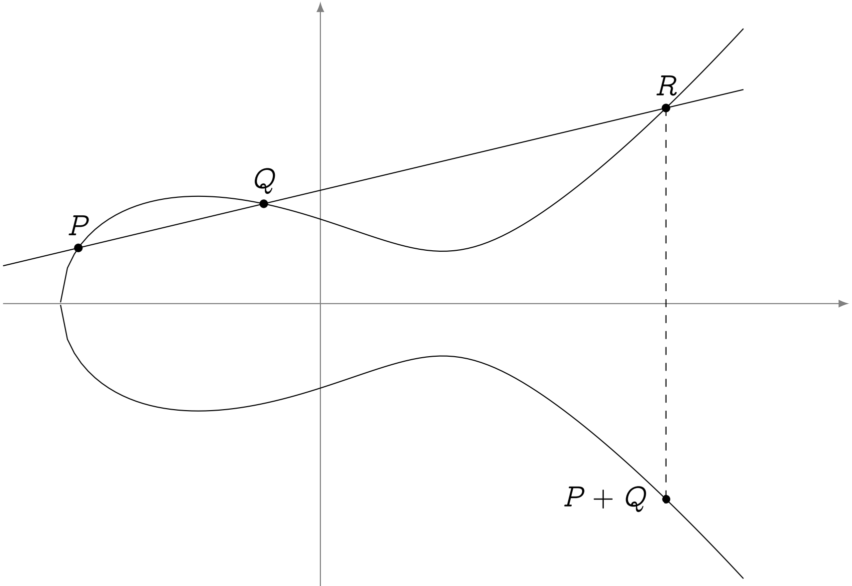

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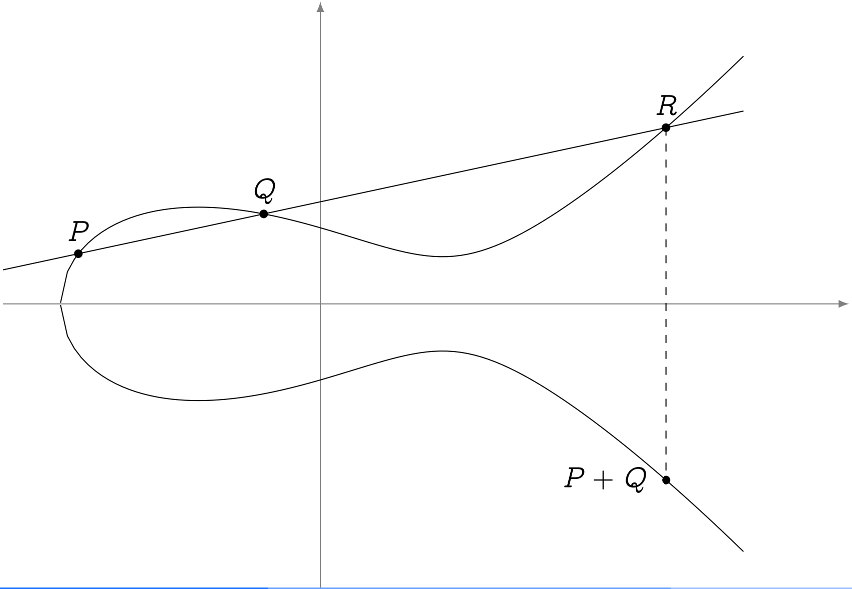

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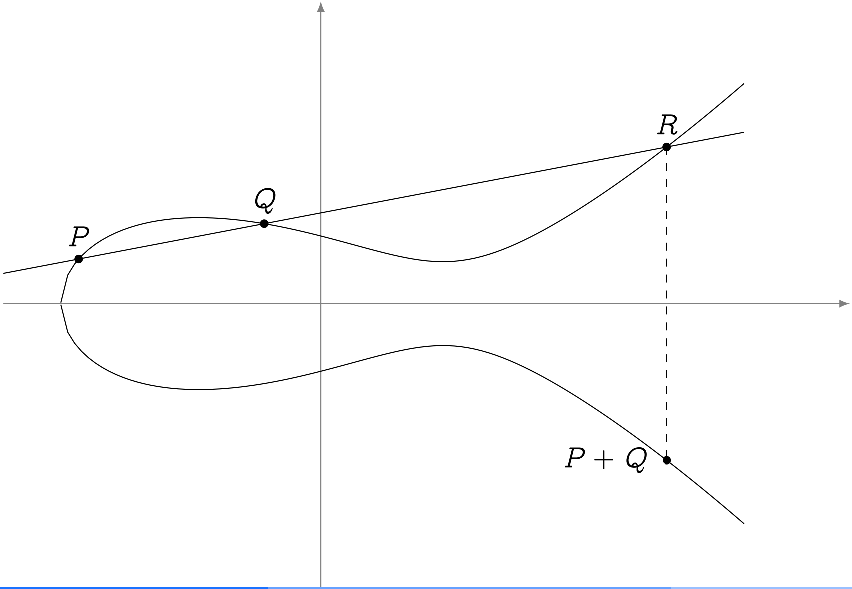

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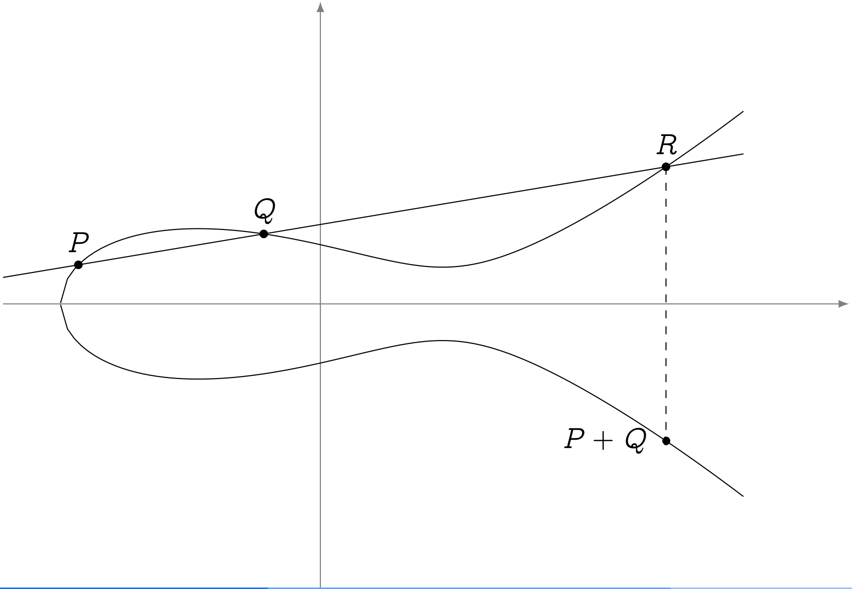

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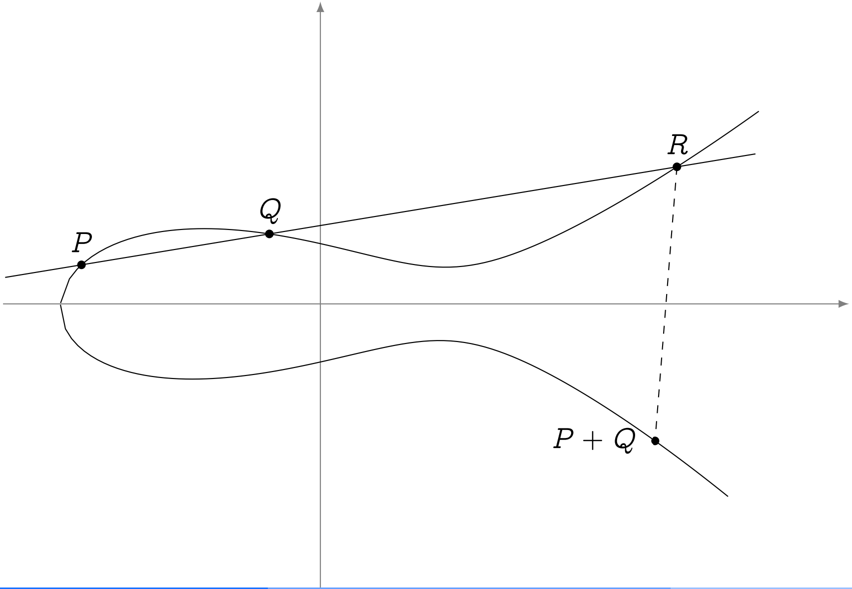

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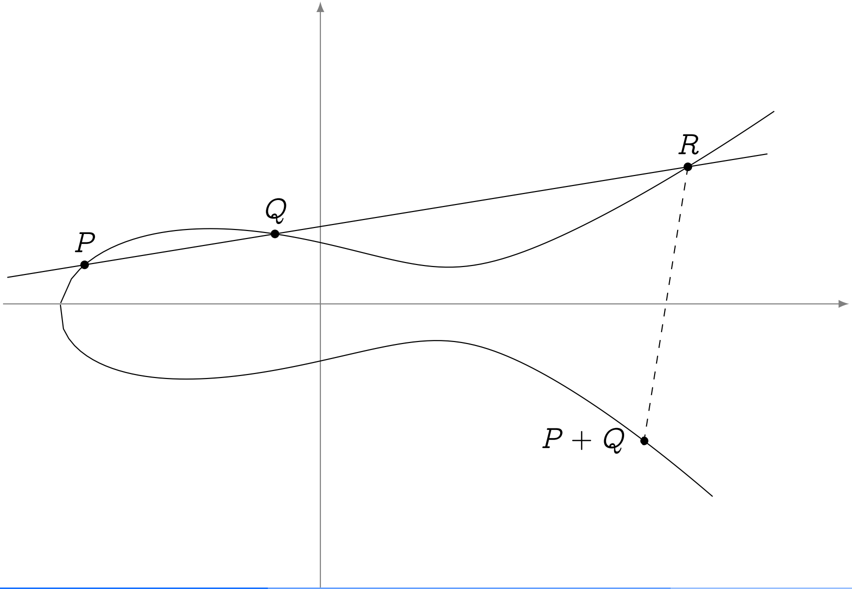

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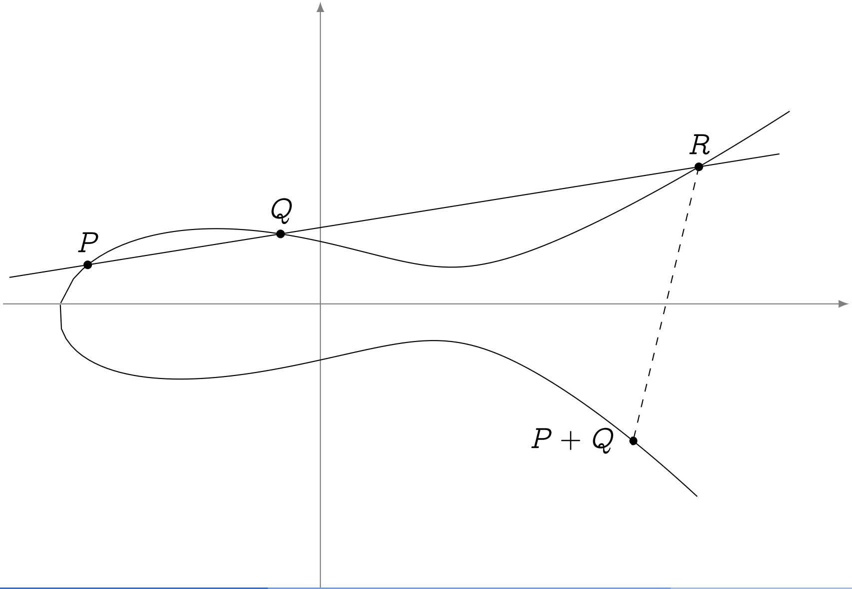

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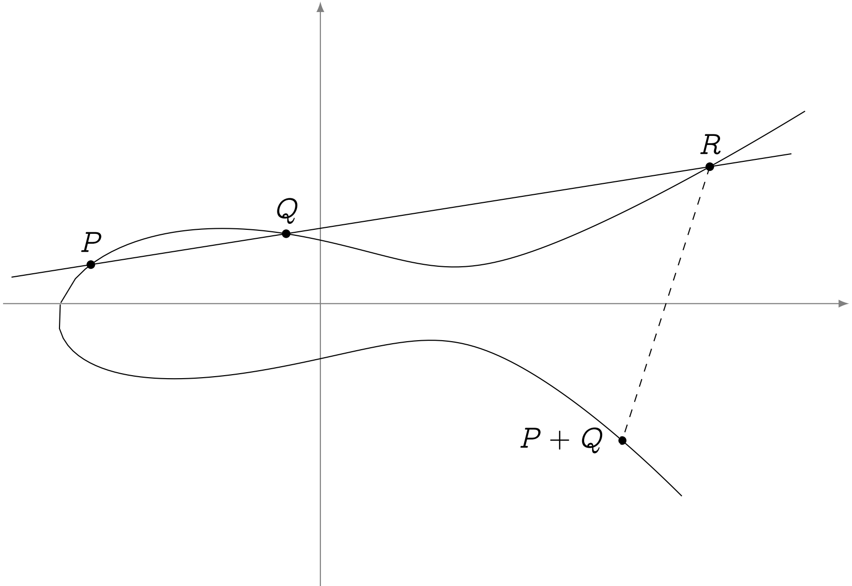

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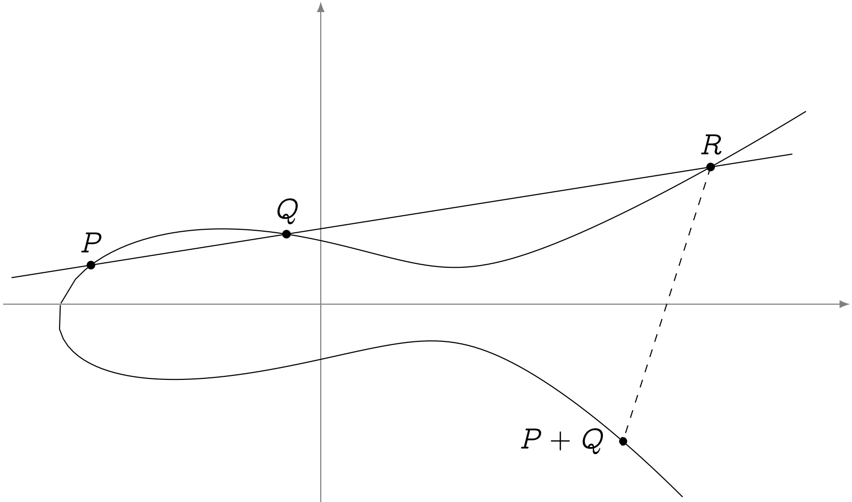

$$y^2 = x^3 + ax + b \quad \longrightarrow \quad j = 1728\frac{4a^3}{4a^3+27b^2}$$

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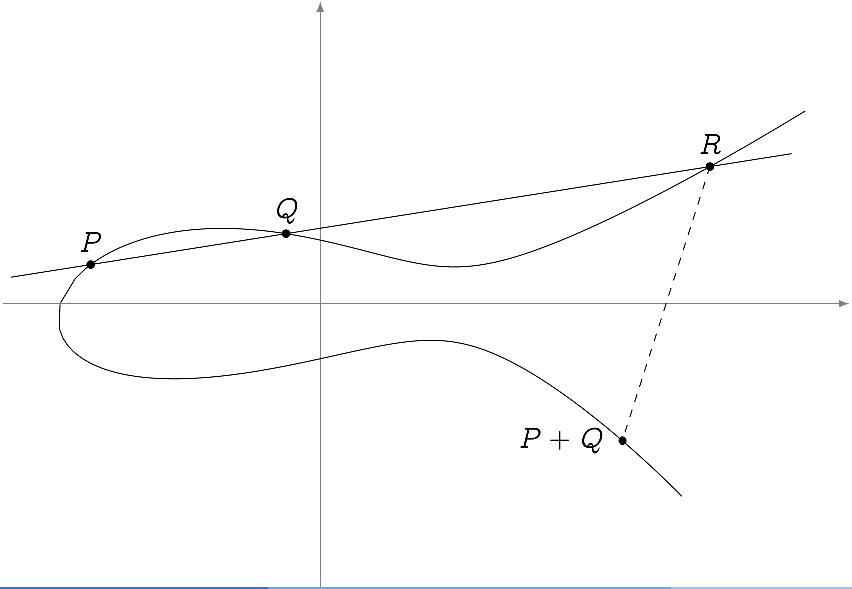

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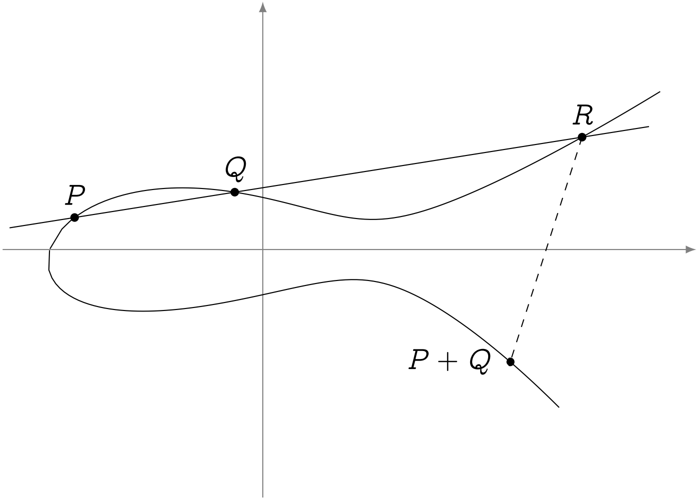

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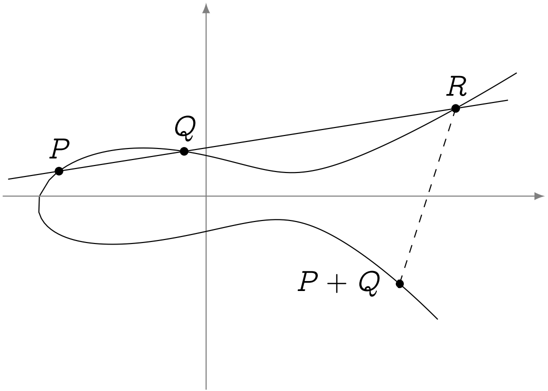

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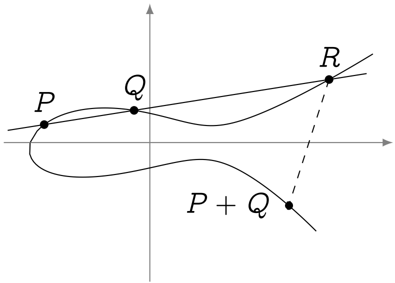

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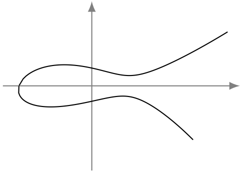

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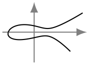

Up to isomorphism

$$j = 1728$$

Up to isomorphism

$$j = 1728 \xrightarrow{\phi} j = 287496$$

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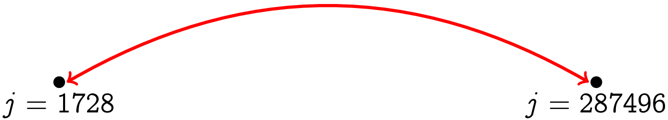


Isogeny graphs

Serre-Tate theorem

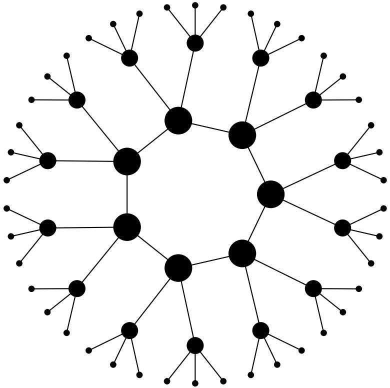
Two elliptic curves E, E' defined over a finite field $\mathbb{F}_q$ are isogenous (over $\mathbb{F}_q$) iff $\#E(\mathbb{F}_q) = \#E'(\mathbb{F}_q)$.

Isogeny graphs

- Vertices are curves up to isomorphism,
- Edges are isogenies up to isomorphism.

Isogeny volcanoes

- Curves are ordinary,
- Isogenies all have degree a prime ℓ.



The endomorphism ring

The endomorphism ring $\mathrm{End}(E)$ of an elliptic curve E is the ring of all isogenies $E \to E$ (plus the null map) with addition and composition.

Theorem (Deuring)

Let E be an elliptic curve defined over a field k of characteristic p.
$\mathrm{End}(E)$ is isomorphic to one of the following:

- $\mathbb{Z}$, only if $p = 0$

  E is ordinary.
- An order $\mathcal{O}$ in a quadratic imaginary field:

  E is ordinary with complex multiplication by $\mathcal{O}$.
- Only if $p > 0$, a maximal order in a quaternion algebra[a]:

  E is supersingular.

[a] (ramified at p and ∞)

Algebras, orders

- A quadratic imaginary number field is an extension of $\mathbb{Q}$ of the form $Q(\sqrt{-D})$ for some non-square $D > 0$.
- A quaternion algebra is an algebra of the form $\mathbb{Q} + \alpha\mathbb{Q} + \beta\mathbb{Q} + \alpha\beta\mathbb{Q}$, where the generators satisfy the relations

$$\alpha^2, \beta^2 \in \mathbb{Q}, \quad \alpha^2 < 0, \quad \beta^2 < 0, \quad \beta\alpha = -\alpha\beta.$$

Orders

Let K be a finitely generated $\mathbb{Q}$-algebra. An order $\mathcal{O} \subset K$ is a subring of K that is a finitely generated $\mathbb{Z}$-module of maximal dimension. An order that is not contained in any other order of K is called a maximal order.

Examples:

- $\mathbb{Z}$ is the only order contained in $\mathbb{Q}$,
- $\mathbb{Z}[i]$ is the only maximal order of $\mathbb{Q}(i)$,
- $\mathbb{Z}[\sqrt{5}]$ is a non-maximal order of $\mathbb{Q}(\sqrt{5})$,
- The ring of integers of a number field is its only maximal order,
- In general, maximal orders in quaternion algebras are not unique.

The finite field case

Theorem (Hasse)

Let E be defined over a finite field. Its Frobenius endomorphism π satisfies a quadratic equation

$$\pi^2 - t\pi + q = 0$$

in $\mathrm{End}(E)$ for some $|t| \leq 2\sqrt{q}$, called the trace of π. The trace t is coprime to q if and only if E is ordinary.

Suppose E is ordinary, then $D_\pi = t^2 - 4q < 0$ is the discriminant of $\mathbb{Z}[\pi]$.

- $K = \mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{D_\pi})$ is the endomorphism algebra of E.
- Denote by $\mathcal{O}_K$ its ring of integers, then

$$\mathbb{Z} \neq \mathbb{Z}[\pi] \subset \mathrm{End}(E) \subset \mathcal{O}_K.$$

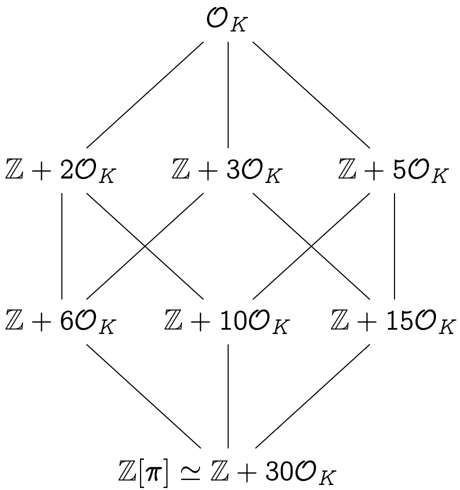
In the supersingular case, π may or may not be in $\mathbb{Z}$, depending on q.

Endomorphism rings of ordinary curves

Classifying quadratic orders

Let K be a quadratic number field, and let $\mathcal{O}_K$ be its ring of integers.

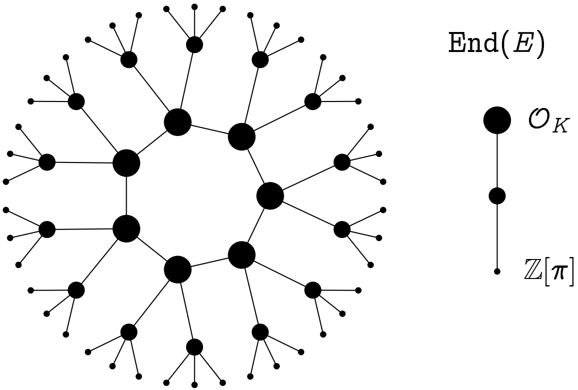
- Any order $\mathcal{O} \subset K$ can be written as $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ for an integer f, called the conductor of $\mathcal{O}$, denoted by $[\mathcal{O}_k : \mathcal{O}]$.
- If d_K is the discriminant of K, the discriminant of $\mathcal{O}$ is $f^2 d_K$.
- If $\mathcal{O}, \mathcal{O}'$ are two orders with discriminants d, d', then $\mathcal{O} \subset \mathcal{O}'$ iff $d' | d$.



Volcanology (Kohel 1996)

Let E, E' be curves with respective endomorphism rings $\mathcal{O}, \mathcal{O}' \subset K$.
Let $\phi : E \to E'$ be an isogeny of prime degree ℓ, then:

if $\mathcal{O} = \mathcal{O}'$, ϕ is horizontal;
if $[\mathcal{O}' : \mathcal{O}] = \ell$, ϕ is ascending;
if $[\mathcal{O} : \mathcal{O}'] = \ell$, ϕ is descending.

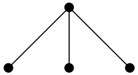


Ordinary isogeny volcano of degree $\ell = 3$.

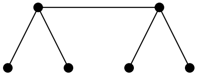
Volcanology (Kohel 1996)

Let E be ordinary, $\mathtt{End}(E) \subset K$.

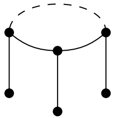
$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.



$\left(\frac{D_K}{\ell}\right) = -1$



$\left(\frac{D_K}{\ell}\right) = 0$



$\left(\frac{D_K}{\ell}\right) = +1$

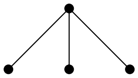
		Horizontal	**Ascending**	**Descending**
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		$\ell - \left(\frac{D_K}{\ell}\right)$
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	ℓ
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

Volcanology (Kohel 1996)

Let E be ordinary, $\mathrm{End}(E) \subset K$.

$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.

Height $= v_\ell([\mathcal{O}_K : \mathbb{Z}[\pi]])$.



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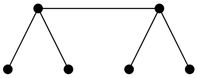

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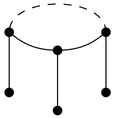

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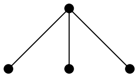
How large is the crater?



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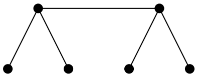

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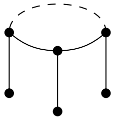

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How large is the crater of a volcano?

Let $\mathrm{End}(E) = \mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$. Define

- $\mathcal{I}(\mathcal{O})$, the group of invertible fractional ideals,
- $\mathcal{P}(\mathcal{O})$, the group of principal ideals,

The class group

The class group of $\mathcal{O}$ is

$$\mathrm{Cl}(\mathcal{O}) = \mathcal{I}(\mathcal{O})/\mathcal{P}(\mathcal{O}).$$

- It is a finite abelian group.
- Its order $h(\mathcal{O})$ is called the class number of $\mathcal{O}$.
- It arises as the Galois group of an abelian extension of $\mathbb{Q}(\sqrt{-D})$.

Complex multiplication

The $\mathfrak{a}$-torsion

Let $\mathfrak{a} \subset \mathcal{O}$ be an (integral invertible) ideal of $\mathcal{O}$; Let $E[\mathfrak{a}]$ be the subgroup of E annihilated by $\mathfrak{a}$:

$$E[\mathfrak{a}] = \{P \in E \mid \alpha(P) = 0 \text{ for all } \alpha \in \mathfrak{a}\};$$

Let $\phi : E \to E_{\mathfrak{a}}$, where $E_{\mathfrak{a}} = E/E[\mathfrak{a}]$. Then $\mathrm{End}(E_{\mathfrak{a}}) = \mathcal{O}$ (i.e., ϕ is horizontal).

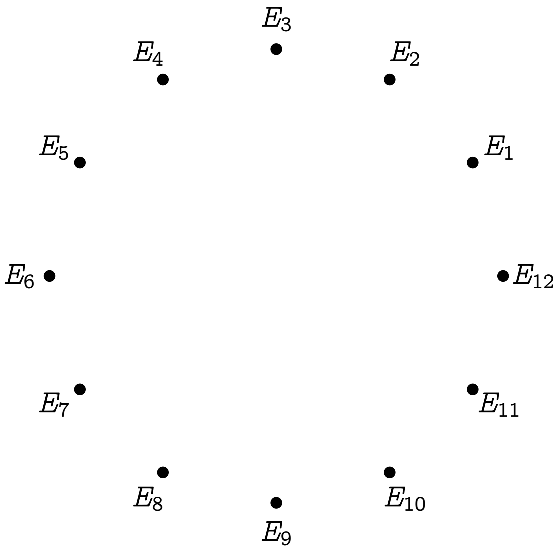
Theorem (Complex multiplication)

The action on the set of elliptic curves with complex multiplication by $\mathcal{O}$ defined by $\mathfrak{a} * j(E) = j(E_{\mathfrak{a}})$ factors through $\mathrm{Cl}(\mathcal{O})$, is faithful and transitive.

Corollary

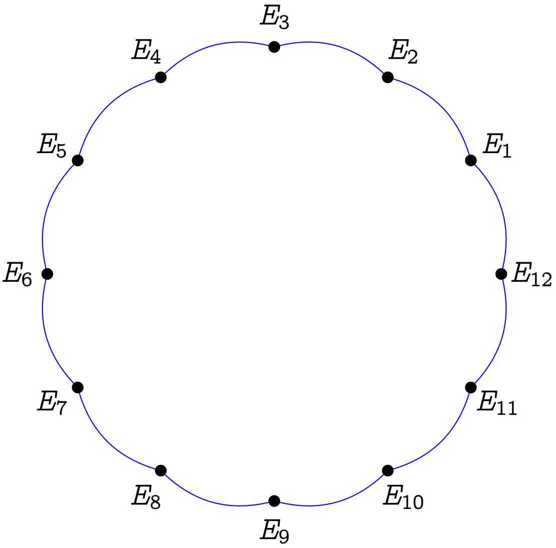
Let $\mathrm{End}(E)$ have discriminant D. Assume that $\left(\frac{D}{\ell}\right) = 1$, then E is on a crater of size N of an ℓ-volcano, and $N \mid h(\mathrm{End}(E))$.

Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

Complex multiplication graphs

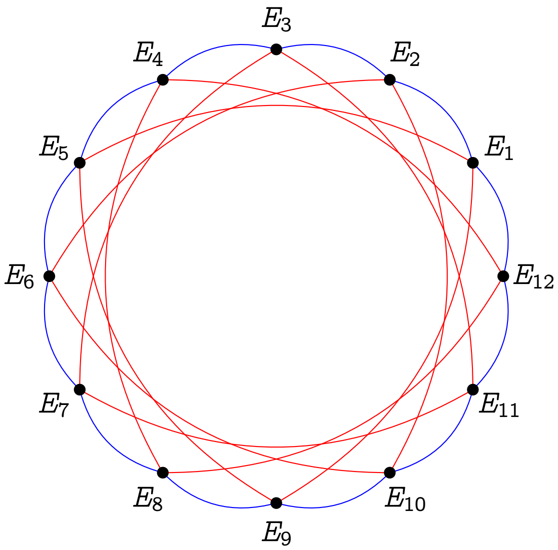


Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

Edges are horizontal isogenies of bounded prime degree.

— degree 2

Complex multiplication graphs



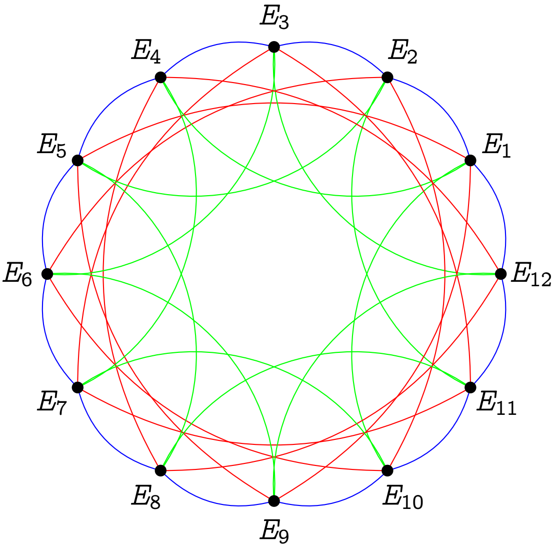
Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).
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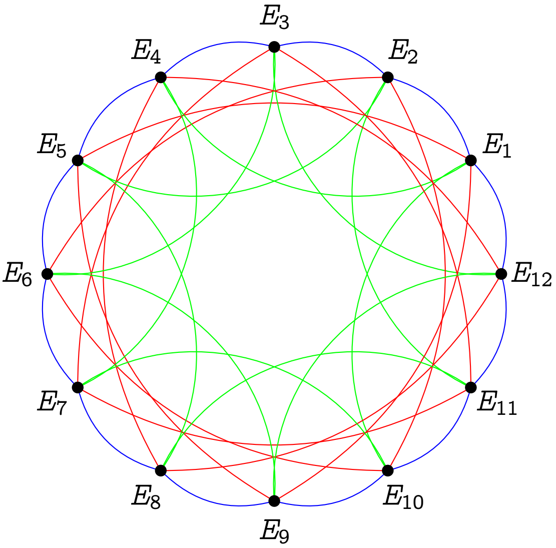
— degree 2

— degree 3

— degree 5

Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).
Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

— degree 5

Isomorphic to a Cayley graph of $\mathrm{Cl}(\mathcal{O}_K)$.

Supersingular endomorphisms

Recall, a curve E over a field $\mathbb{F}_q$ of characteristic p is supersingular iff

$$\pi^2 - t\pi + q = 0$$

with $t = 0 \mod p$.

Case: $t = 0 \quad\Rightarrow\quad D_\pi = -4q$

- Only possibility for $E/\mathbb{F}_p$,
- $E/\mathbb{F}_p$ has CM by an order of $\mathbb{Q}(\sqrt{-p})$, similar to the ordinary case.

Case: $t = \pm 2\sqrt{q} \quad\Rightarrow\quad D_\pi = 0$

- General case for $E/\mathbb{F}_q$, when q is an even power.
- $\pi = \pm\sqrt{q} \in \mathbb{Z}$, hence no complex multiplication.

We will ignore marginal cases: $t = \pm\sqrt{q}, \pm\sqrt{2q}, \pm\sqrt{3q}$.

Supersingular complex multiplication

Let $E/\mathbb{F}_p$ be a supersingular curve, then $\pi^2 = -p$.

Theorem (Delfs, Galbraith 2016)

Let $\mathrm{End}_{\mathbb{F}_p}(E)$ denote the ring of $\mathbb{F}_p$-rational endomorphisms of E. Then

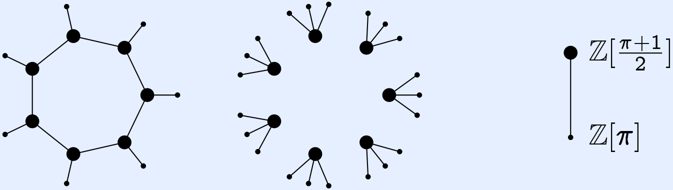
$$\mathbb{Z}[\pi] \subset \mathrm{End}_{\mathbb{F}_p}(E) \subset \mathbb{Q}(\sqrt{-p}).$$

Orders of $\mathbb{Q}(\sqrt{-p})$

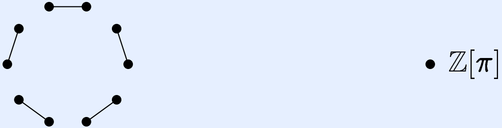
- If $p = 1 \mod 4$, then $\mathbb{Z}[\pi]$ is the maximal order.
- If $p = -1 \mod 4$, then $\mathbb{Z}[\frac{\pi+1}{2}]$ is the maximal order, and $[\mathbb{Z}[\frac{\pi+1}{2}] : \mathbb{Z}[\pi]] = 2$.

Supersingular CM graphs

2-volcanoes, $p = -1 \bmod 4$



2-graphs, $p = 1 \bmod 4$



All other ℓ-graphs are cycles of horizontal isogenies iff $\left(\frac{-p}{\ell}\right) = 1$.

The full endomorphism ring

Theorem (Deuring)

Let E be a supersingular elliptic curve, then

- E is isomorphic to a curve defined over $\mathbb{F}_{p^2}$;
- Every isogeny of E is defined over $\mathbb{F}_{p^2}$;
- Every endomorphism of E is defined over $\mathbb{F}_{p^2}$;
- $\mathrm{End}(E)$ is isomorphic to a maximal order in a quaternion algebra ramified at p and ∞.

In particular:

- If E is defined over $\mathbb{F}_p$, then $\mathrm{End}_{\mathbb{F}_p}(E)$ is strictly contained in $\mathrm{End}(E)$.
- Some endomorphisms do not commute!

An example

The curve of j-invariant 1728

$$E : y^2 = x^3 + x$$

is supersingular over $\mathbb{F}_p$ iff $p = -1 \mod 4$.

Endomorphisms

$\mathrm{End}(E) = \mathbb{Z}\langle\iota, \pi\rangle$, with:

- π the Frobenius endomorphism, s.t. $\pi^2 = -p$;
- ι the map

$$\iota(x, y) = (-x, iy),$$

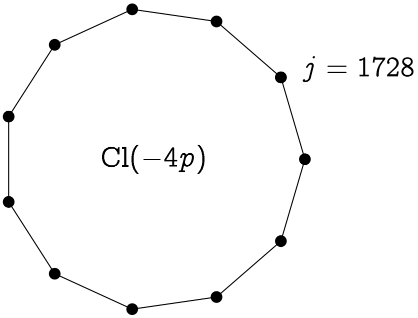
  where $i \in \mathbb{F}_{p^2}$ is a 4-th root of unity. Clearly, $\iota^2 = -1$.

And $\iota\pi = -\pi\iota$.

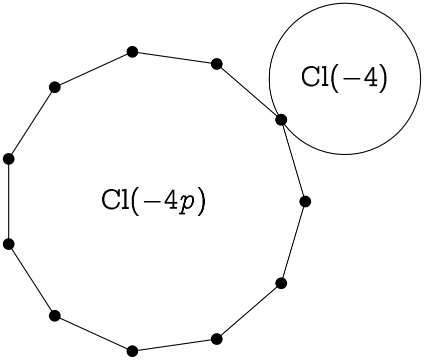
Class group action party

$j = 1728$

Class group action party



Class group action party



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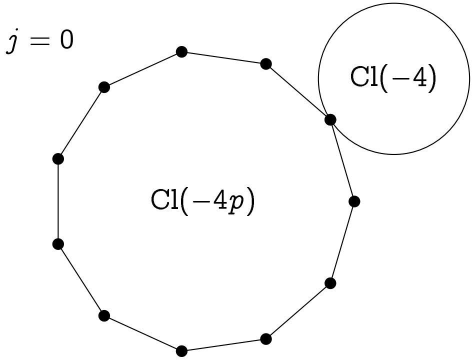

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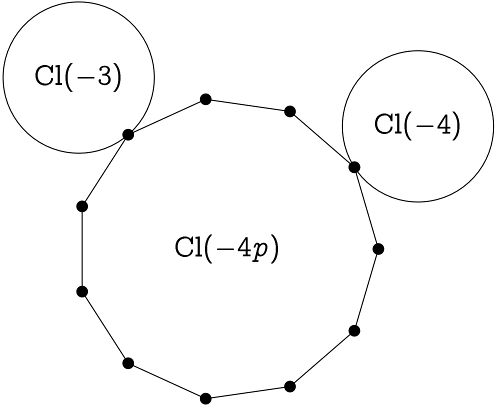

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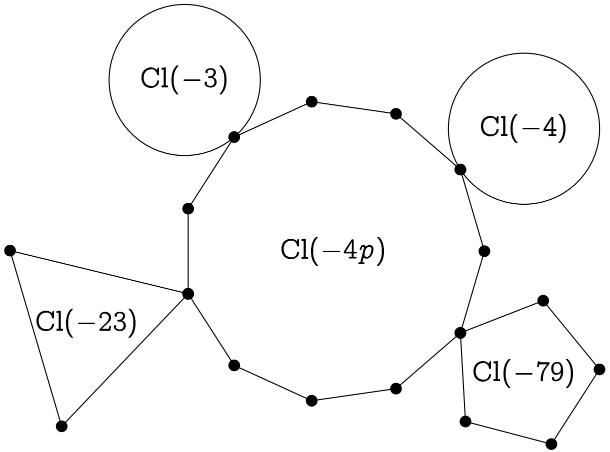


Supersingular graphs

- Quaternion algebras have many maximal orders.
- For every maximal order type of $B_{p,\infty}$ there are 1 or 2 curves over $\mathbb{F}_{p^2}$ having endomorphism ring isomorphic to it.
- There is a unique isogeny class of supersingular curves over $\bar{\mathbb{F}}_p$ of size $\approx p/12$.
- Left ideals act on the set of maximal orders like isogenies.
- The graph of ℓ-isogenies is $(\ell+1)$-regular.

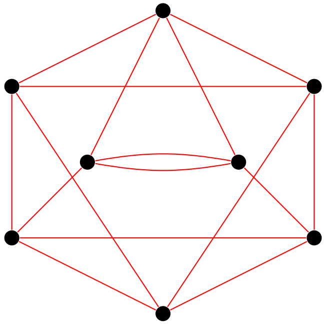


Figure: 3-isogeny graph on $\mathbb{F}_{97^2}$.

Graphs lexicon

Degree: Number of (outgoing/ingoing) edges.

k-regular: All vertices have degree k.

Connected: There is a path between any two vertices.

Distance: The length of the shortest path between two vertices.

Diameter: The longest distance between two vertices.

$\lambda_1 \geq \cdots \geq \lambda_n$: The (ordered) eigenvalues of the adjacency matrix.

Expander graphs

Proposition

If G is a k-regular graph, its largest and smallest eigenvalues satisfy

$$k = \lambda_1 \geq \lambda_n \geq -k.$$

Expander families

An infinite family of connected k-regular graphs on n vertices is an expander family if there exists an $\epsilon > 0$ such that all non-trivial eigenvalues satisfy $|\lambda| \leq (1-\epsilon)k$ for n large enough.

- Expander graphs have short diameter: $O(\log n)$;
- Random walks mix rapidly: after $O(\log n)$ steps, the induced distribution on the vertices is close to uniform.

Expander graphs from isogenies

Theorem (Pizer)

Let ℓ be fixed. The family of graphs of supersingular curves over $\mathbb{F}_{p^2}$ with ℓ-isogenies, as $p \to \infty$, is an expander family[a].

[a] Even better, it has the Ramanujan property.

Theorem (Jao, Miller, Venkatesan)

Let $\mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$ be an order in a quadratic imaginary field. The graphs of all curves over $\mathbb{F}_q$ with complex multiplication by $\mathcal{O}$, with isogenies of prime degree bounded[a] by $(\log q)^{2+\delta}$, are expanders.

[a] May contain traces of GRH.

Executive summary

- Separable ℓ-isogeny = finite kernel = subgroup of $E[\ell]$ (= ideal of norm ℓ),
- Isogeny graphs have j-invariants for vertices and "some" isogenies for edges.
- By varying the choices for the vertex and the isogeny set, we obtain graphs with different properties.
- ℓ-isogeny graphs of ordinary curves are volcanoes, (full) ℓ-isogeny graphs of supersingular curves are finite $(\ell + 1)$-regular.
- CM theory naturally leads to define graphs of horizontal isogenies (both in the ordinary and the supersingular case) that are isomorphic to Cayley graphs of class groups.
- CM graphs are expanders. Supersingular full ℓ-isogeny graphs are Ramanujan.

Isogeny Based Cryptography: an Introduction

Luca De Feo

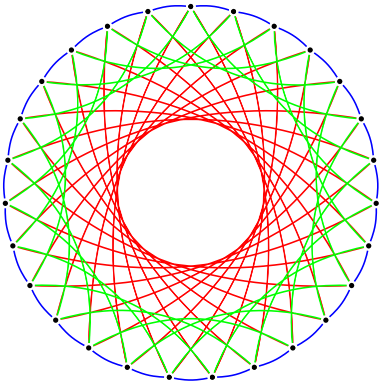
IBM Research Zürich

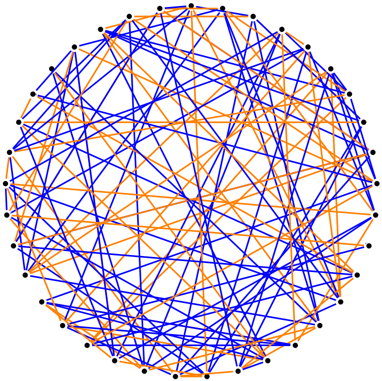
November 18, 2019
Simula UiB, Bergen

Slides online at `https://defeo.lu/docet`

The beauty and the beast (credit: Lorenz Panny)

Components of particular isogeny graphs look like this:





Which of these is good for crypto?

The beauty and the beast (credit: Lorenz Panny)

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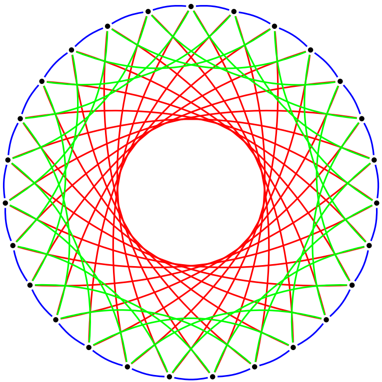
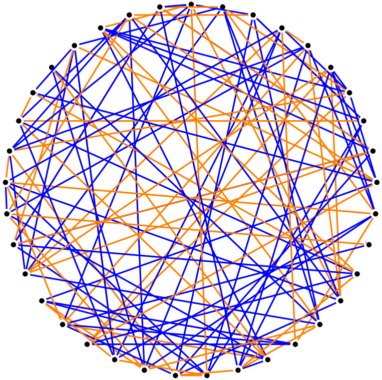
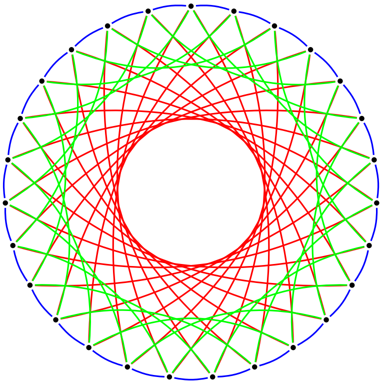




Which of these is good for crypto? **Both.**

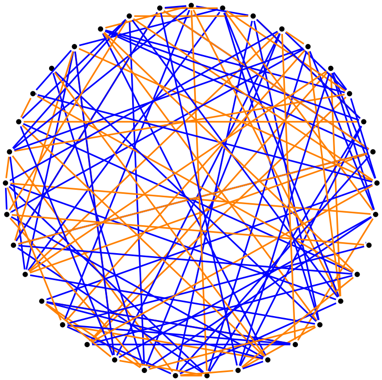
At this time, there are <u>two distinct families</u> of systems:



$\mathbb{F}_p$

CSIDH [pron.: sea-side]

`https://csidh.isogeny.org`



$\mathbb{F}_{p^2}$

SIDH

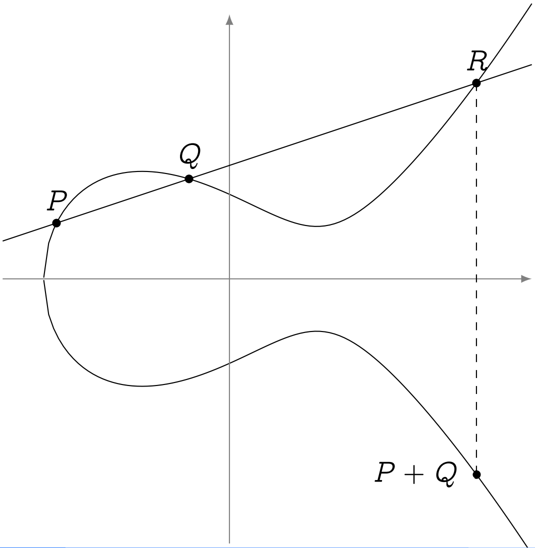
`https://sike.org`

Brief history of isogeny-based cryptography

- **1997** Couveignes introduces the Hard Homogeneous Spaces framework. His work stays unpublished for 10 years.
- **2006** Rostovtsev & Stolbunov independently rediscover Couveignes ideas, suggest isogeny-based Diffie–Hellman as a quantum-resistant primitive.
- **2006-2010** Other isogeny-based protocols by Teske and Charles, Goren & Lauter.
- **2011-2012** D., Jao & Plût introduce SIDH, an efficient post-quantum key exchange inspired by Couveignes, Rostovtsev, Stolbunov, Charles, Goren, Lauter.
- **2017** SIDH is submitted to the NIST competition (with the name SIKE, only isogeny-based candidate).
- **2018** D., Kieffer & Smith *resurrect* the Couveignes–Rostovtsev–Stolbunov protocol, Castryck, Lange, Martindale, Panny & Renes create an efficient variant named CSIDH.
- **2019** The year of proofs of isogeny knowledge: SeaSign (D. & Galbraith; Decru, Panny & Vercauteren), CSI-FiSh (Beullens, Kleinjung & Vercauteren), VDF (D., Masson, Petit & Sanso), threshold (D. & Meyer).

Elliptic curves

Let $E : y^2 = x^3 + ax + b$ be an elliptic curve…



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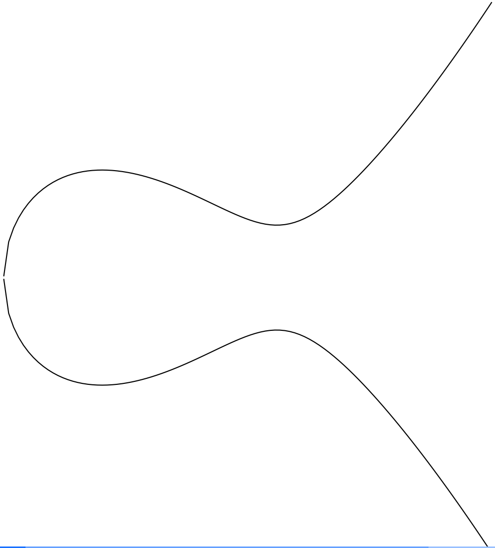

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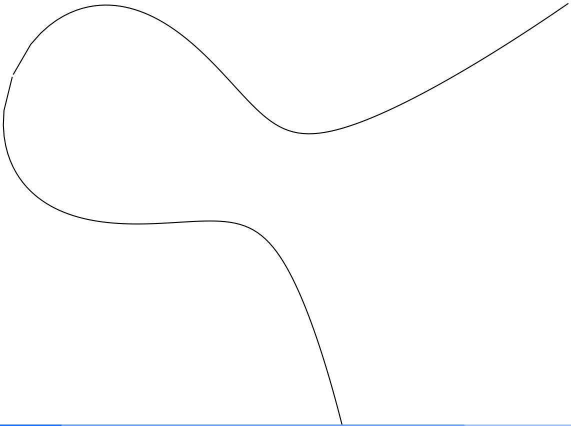

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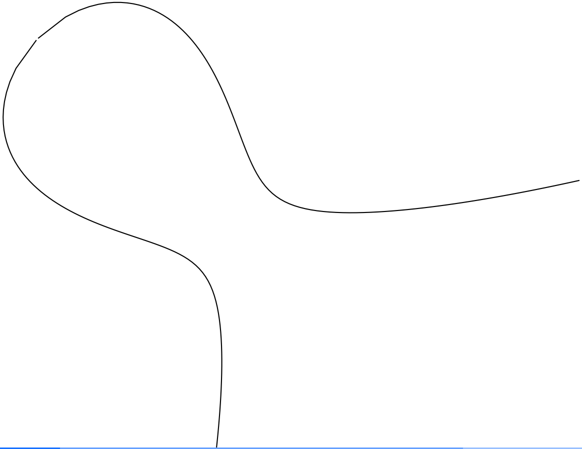

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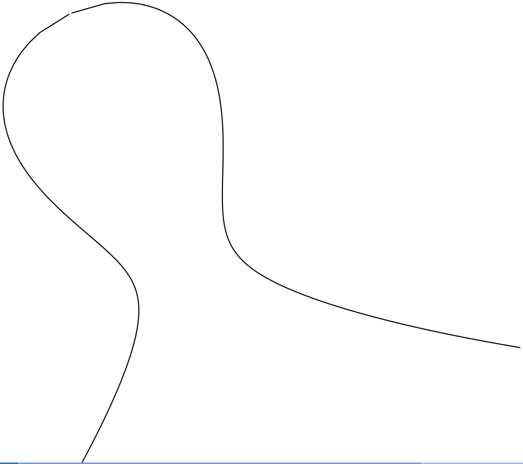

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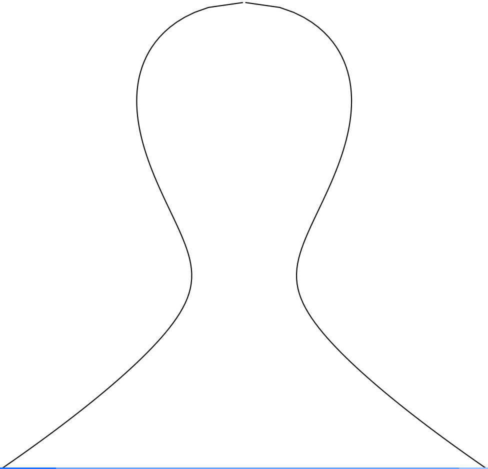
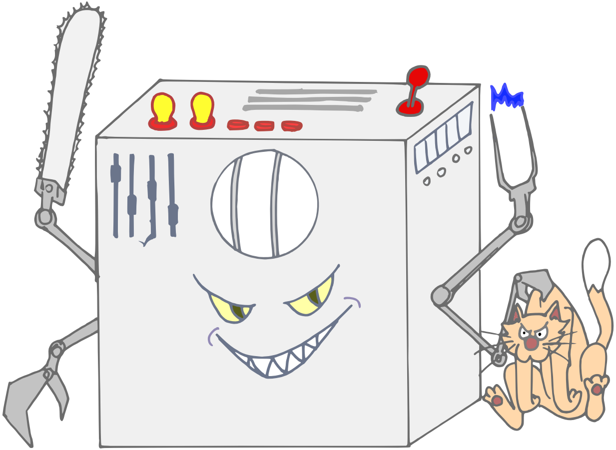
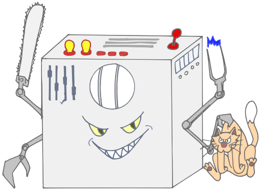


Elliptic curves

The QUANTHOM Menace



Basically every isogeny-based key-exchange...



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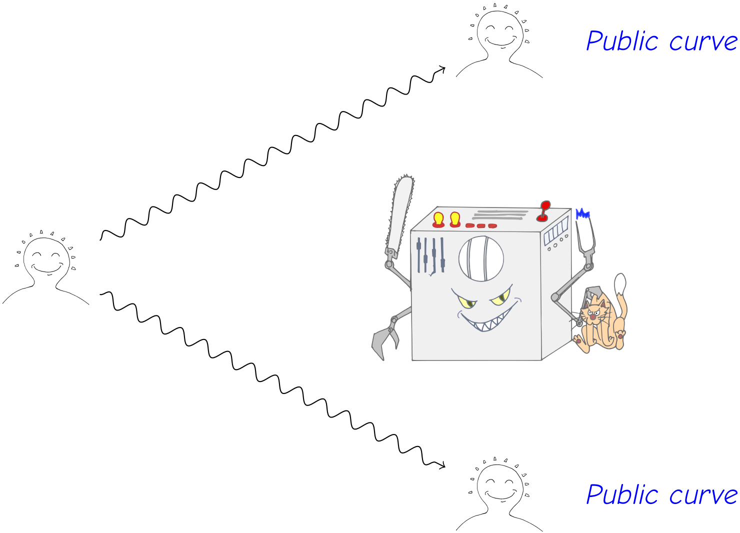

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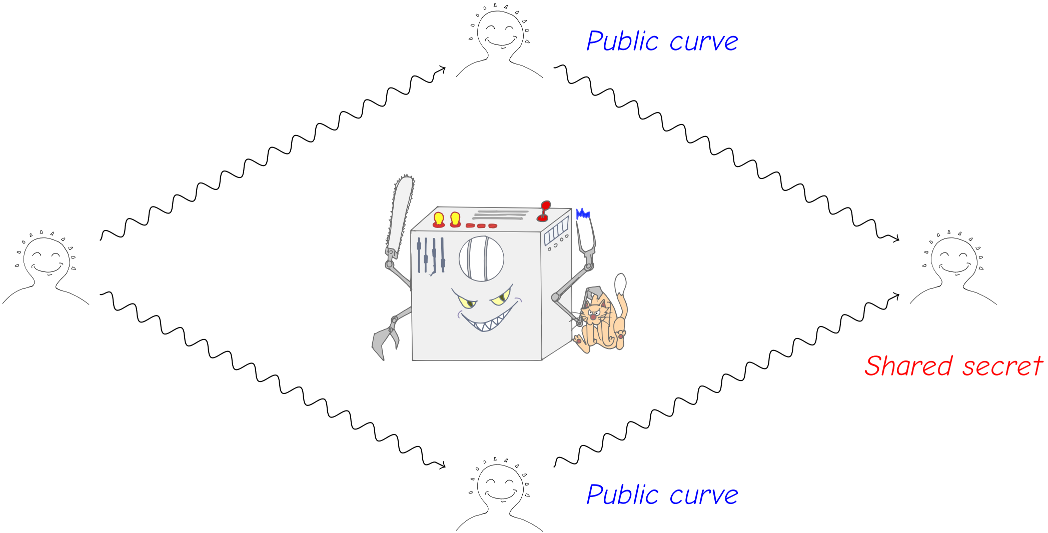


Hard Homogeneous Spaces[1]

Principal Homogeneous Space

$\mathcal{G} \circlearrowleft \mathcal{E}$: A (finite) set $\mathcal{E}$ acted upon by a group $\mathcal{G}$ faithfully and transitively:

$$\begin{aligned} * : \mathcal{G} \times \mathcal{E} &\longrightarrow \mathcal{E} \\ \mathfrak{g} * E &\longmapsto E' \end{aligned}$$

Compatibility: $\mathfrak{g}' * (\mathfrak{g} * E) = (\mathfrak{g}'\mathfrak{g}) * E$ for all $\mathfrak{g}, \mathfrak{g}' \in \mathcal{G}$ and $E \in \mathcal{E}$;

Identity: $\mathfrak{e} * E = E$ if and only if $\mathfrak{e} \in \mathcal{G}$ is the identity element;

Transitivity: for all $E, E' \in \mathcal{E}$ there exist a unique $\mathfrak{g} \in \mathcal{G}$ such that $\mathfrak{g} * E' = E$.

Example: the set of elliptic curves with complex multiplication by $\mathcal{O}$ is a PHS for the class group $\mathrm{Cl}(\mathcal{O})$.

[1] Couveignes 2006.

Hard Homogeneous Spaces

Hard Homogeneous Space (HHS)

A Principal Homogeneous Space $\mathcal{G} \circlearrowright \mathcal{E}$ such that:

- Evaluating $E' = \mathfrak{g} * E$ is easy;
- Inverting the action is hard.

Discrete logarithms in $\mathcal{G} = \langle \mathfrak{g} \rangle$ are easy $\Leftrightarrow$ there is an effective isomorphism

$$\begin{aligned} \mathbb{Z}/N\mathbb{Z} &\longleftrightarrow \mathcal{G} \\ a &\longmapsto \mathfrak{g}^a \end{aligned}$$

Then we like to see $\mathcal{E}$ as an HHS for $\mathbb{Z}/N\mathbb{Z}$:

$$\begin{aligned} \mathbb{Z}/N\mathbb{Z} \times \mathcal{E} &\longrightarrow \mathcal{E} \\ [a]E &\longmapsto \mathfrak{g}^a * E \end{aligned}$$

Warning: $[a][b]E = [a+b]E$!!!

HHS Diffie–Hellman

Goal: Alice and Bob have never met before. They are chatting over a public channel, and want to agree on a shared secret to start a private conversation.

Setup: They agree on a (large) HHS $\langle g \rangle \circlearrowleft \mathcal{E}$ of order N.

Alice		**Bob**
pick random $a \in \mathbb{Z}/N\mathbb{Z}$		pick random $b \in \mathbb{Z}/N\mathbb{Z}$
compute $E_A = [a]E_0$		compute $E_B = [b]E_0$
	$\xrightarrow{\quad E_A \quad}$	
	$\xleftarrow[\quad E_B \quad]{}$	

Shared secret is $[a]E_B = [a+b]E_0 = [b]E_A$

HHSDH from complex multiplication

Obstacles:

- We don't want to wait for a quantum computer for solving discrete logs in $\mathrm{Cl}(\mathcal{O})$!
- Until then, even the group size of $\mathrm{Cl}(\mathcal{O})$ is unknown.
- Only ideals of small norm (isogenies of small degree) are efficient to evaluate.

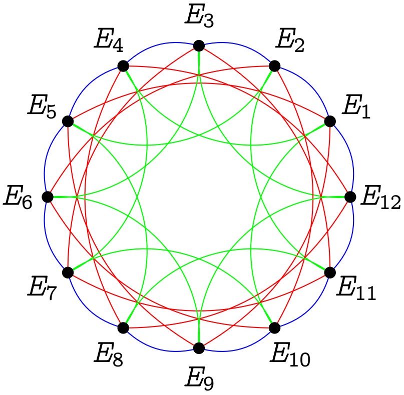
Solution:

- Restrict to elements of $\mathrm{Cl}(\mathcal{O})$ of the form

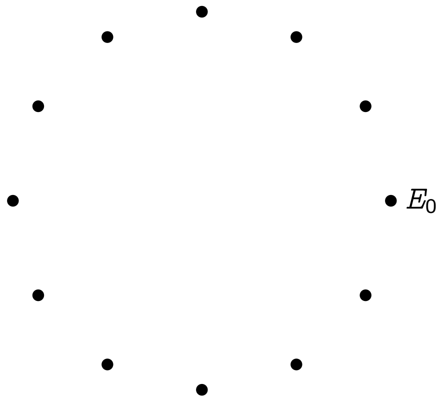
$$\mathfrak{g} = \prod \mathfrak{a}_i^{e_i}$$

for a basis of $\mathfrak{a}_i$ of small norm.

- Equivalent to doing isogeny walks of smooth degree.



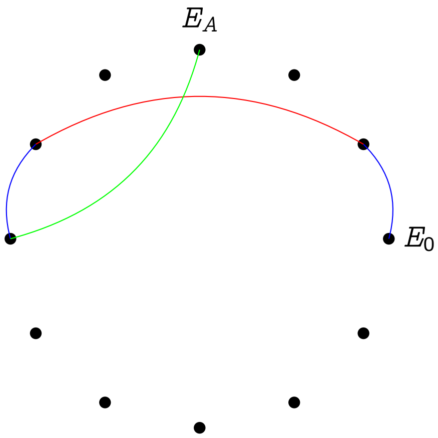
CSIDH key exchange



Public parameters:

- A supersingular curve $E_0/\mathbb{F}_p$;
- A set of small prime degree isogenies.

CSIDH key exchange



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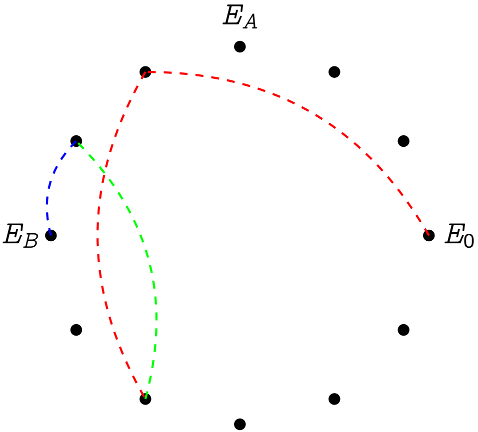

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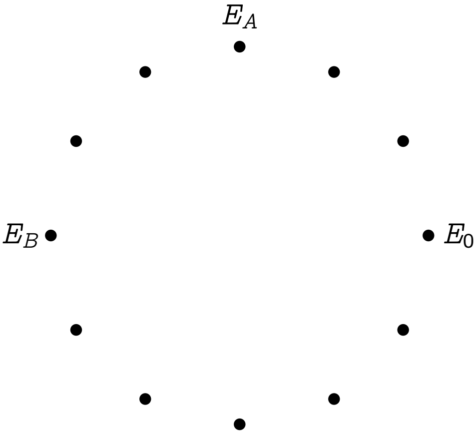

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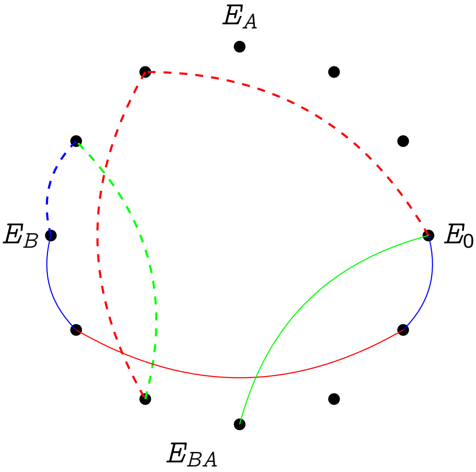

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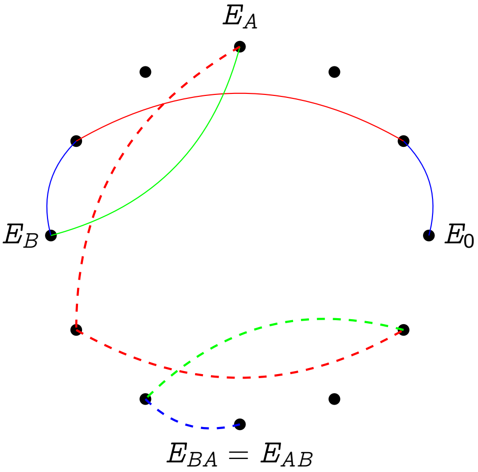
4. **Alice** repeats her secret walk ϕ_A starting from E_B.

CSIDH key exchange



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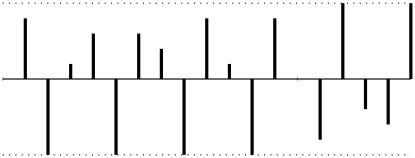
5. **Bob** repeats his secret walk ϕ_B starting from E_A.

CSIDH data flow

Your secret: a vector of number of isogeny steps for each degree

$$(5, 1, -4, \dots)$$



Your public key: (the j-invariant of) a supersingular elliptic curve

$$j = \texttt{0x23baf75419531a44f3b97cc9d8291a275047fcdae0c9a0c0ebb993964f821f2}$$
$$\texttt{0c11058a4200ff38c4a85e208345300033b0d3119ff4a7c1be0acd62a622002a9}$$

Quantum security

Fact: Shor's algorithm does not apply to Diffie-Hellman protocols from group actions.

Subexponential attack $\exp(\sqrt{\log p \log\log p})$

- Reduction to the hidden shift problem by evaluating the class group action in quantum superspositio[a] (subexpoential cost);
- Well known reduction from the hidden shift to the dihedral (non-abelian) hidden subgroup problem;
- Kuperberg's algorithm[b] solves the dHSP with a subexponential number of class group evaluations.
- Recent work[c] suggests that 2^{64}-qbit security is achieved somewhere in $512 < \log p < 1024$.

[a] Childs, Jao, and Soukharev 2014.

[b] Kuperberg 2005; Regev 2004; Kuperberg 2013.

[c] Bonnetain and Naya-Plasencia 2018; Bonnetain and Schrottenloher 2018; Biasse, Jacobson Jr, and Iezzi 2018; Jao, LeGrow, Leonardi, and Ruiz-Lopez 2018; Bernstein, Lange, Martindale, and Panny 2018.

Key exchange with supersingular curves (2011)

Good news: there is no action of a commutative class group.

Bad news: there is no action of a commutative class group.

Idea: Let Alice and Bob walk in two different isogeny graphs on the same vertex set.

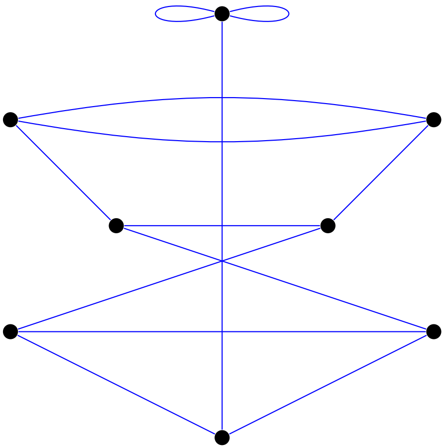


Figure: 2- and 3-isogeny graphs on $\mathbb{F}_{97^2}$.

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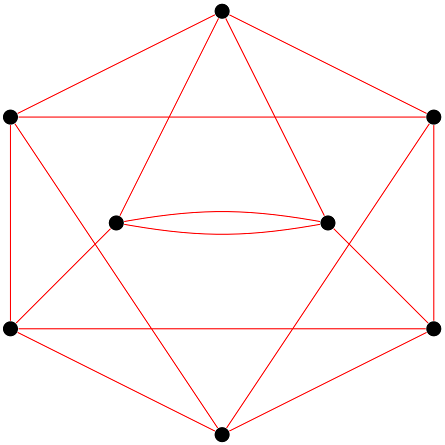

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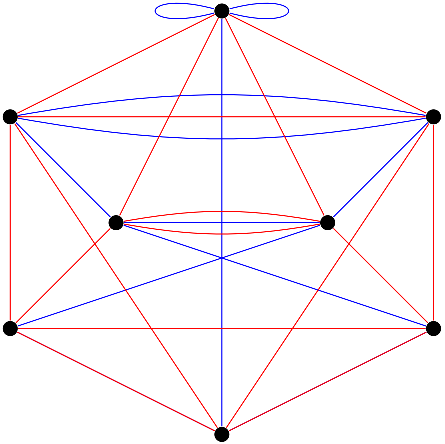

Figure: 2- and 3-isogeny graphs on $\mathbb{F}_{97^2}$.

Key exchange with supersingular curves (2011)

- Fix small primes ℓ_A, ℓ_B;
- No canonical labeling of the ℓ_A- and ℓ_B-isogeny graphs; however…

Walk of length e_A

$=$

Isogeny of degree $\ell_A^{e_A}$

$=$

Kernel $\langle P\rangle \subset E[\ell_A^{e_A}]$

$$\ker\phi = \langle P\rangle \subset E[\ell_A^{e_A}]$$

$$\ker\psi = \langle Q\rangle \subset E[\ell_B^{e_B}]$$

$$\ker\phi' = \langle\psi(P)\rangle$$

$$\ker\psi' = \langle\phi(Q)\rangle$$

$$\begin{tikzcd} E \arrow[r, rightsquigarrow, "\phi"] \arrow[d, rightsquigarrow, "\psi"'] & E/\langle P\rangle \arrow[d, rightsquigarrow, "\psi'"] \\ E/\langle Q\rangle \arrow[r, rightsquigarrow, "\phi'"'] & E/\langle P, Q\rangle \end{tikzcd}$$

Supersingular Isogeny Diffie-Hellman[2]

Parameters:

- Prime p such that $p+1=\ell_A^a\ell_B^b$;
- Supersingular curve $E\simeq(\mathbb{Z}/(p+1)\mathbb{Z})^2$;
- $E[\ell_A^a]=\langle P_A,Q_A\rangle$;
- $E[\ell_B^b]=\langle P_B,Q_B\rangle$.

Secret data:

- $R_A=m_AP_A+n_AQ_A$,
- $R_B=m_BP_B+n_BQ_B$,

$$\begin{tikzcd}
& E \arrow[dl, rightsquigarrow, "\phi"'] \arrow[dr, rightsquigarrow, "\psi"] & \\
E/\langle R_A\rangle \arrow[dr, rightsquigarrow, "\psi'"'] & & E/\langle R_B\rangle \arrow[dl, rightsquigarrow, "\phi'"] \\
\frac{E/\langle R_A\rangle}{\phi(R_B)} \simeq & E/\langle R_A, R_B\rangle & \simeq \frac{E/\langle R_B\rangle}{\psi(R_A)}
\end{tikzcd}$$

[2] Jao and De Feo 2011; De Feo, Jao, and Plût 2014.

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Secret data:

- $R_A = m_A P_A + n_A Q_A$,
- $R_B = m_B P_B + n_B Q_B$,

$$
\begin{tikzcd}
 & E \arrow[dl, squiggly, "\phi"'] \arrow[dr, squiggly, "\psi"] & \\
\begin{array}{l} E/\langle R_A \rangle \\ \phi(P_B) \\ \phi(Q_B) \end{array} \arrow[dr, squiggly, "\psi'"'] & & \begin{array}{l} E/\langle R_B \rangle \\ \psi(P_A) \\ \psi(Q_A) \end{array} \arrow[dl, squiggly, "\phi'"] \\
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 & E/\langle R_A, R_B \rangle &
\end{tikzcd}$$

$\phi(P_B)$, $\phi(Q_B)$ $\qquad$ $\phi(R_B)$ $\qquad$ $\psi(R_A)$ $\qquad$ $\psi(P_A)$, $\psi(Q_A)$

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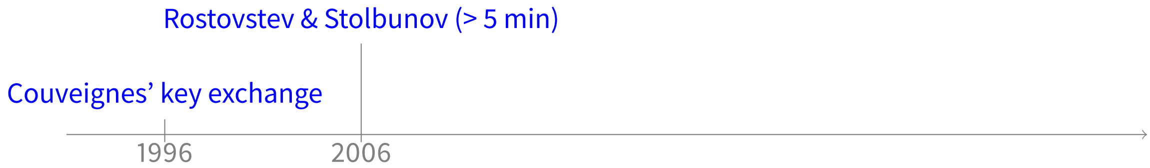
From 10 minutes to 10ms in 20 years

Couveignes' key exchange

1996

From 10 minutes to 10ms in 20 years



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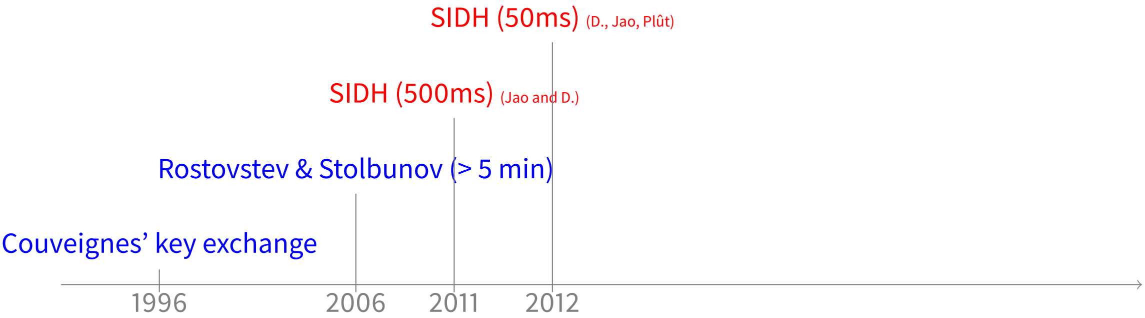

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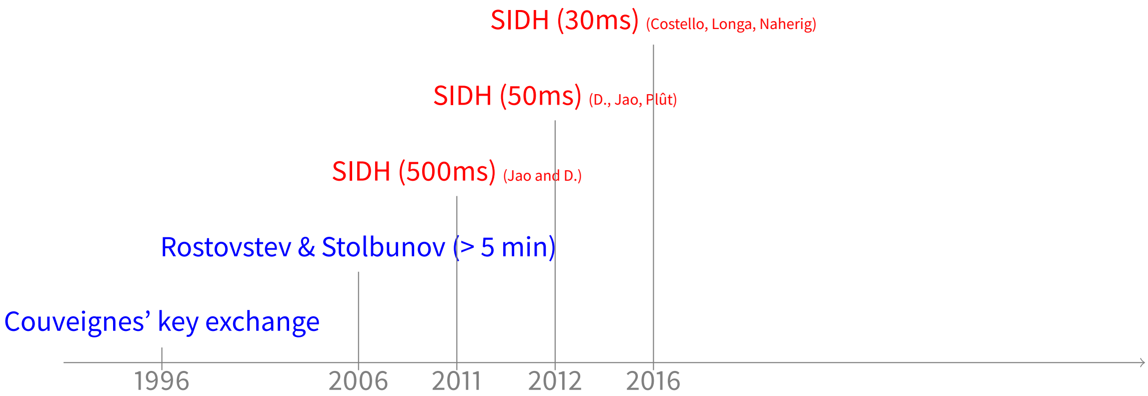

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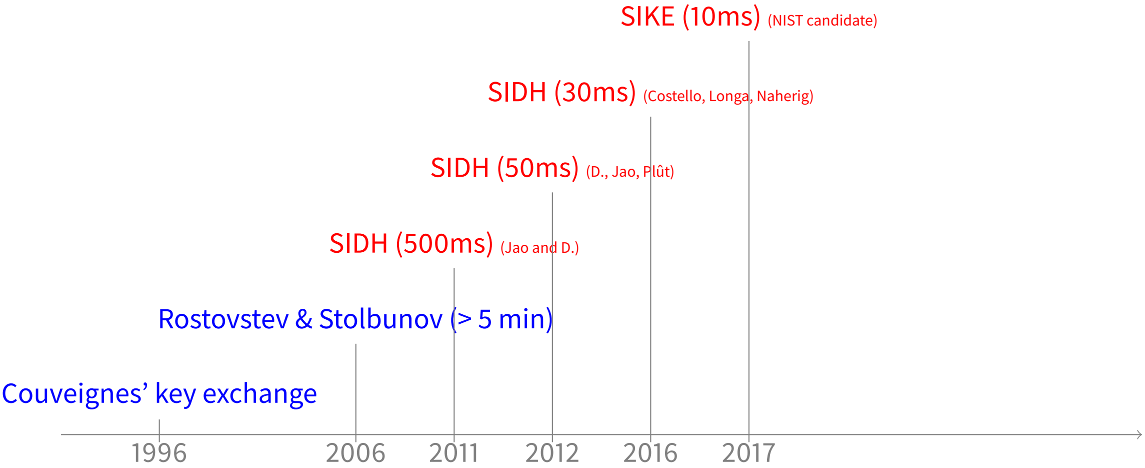

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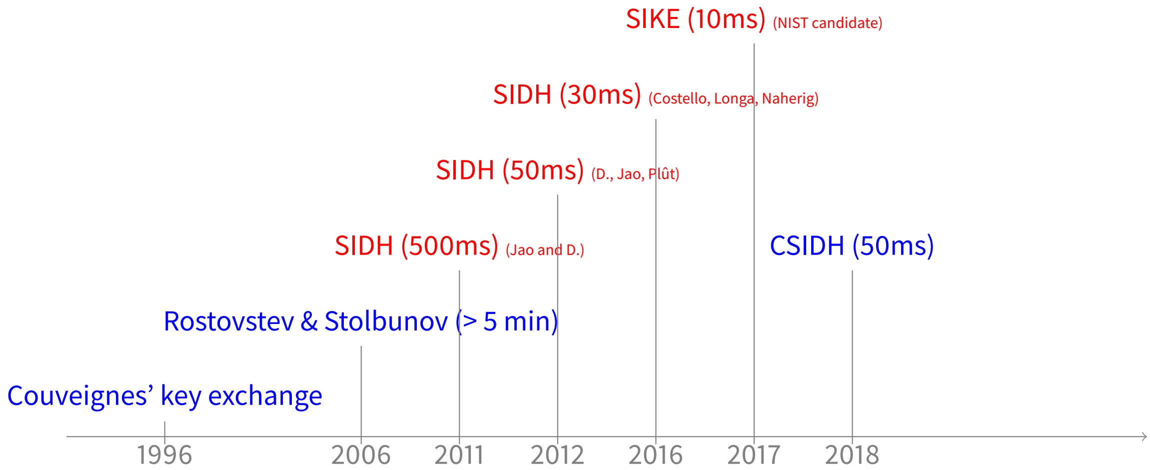

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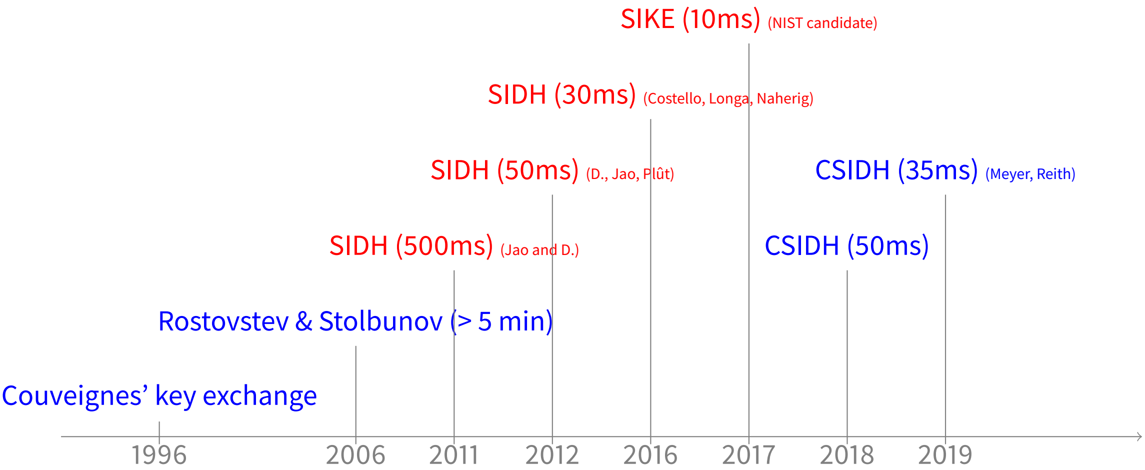

CSIDH vs SIDH

	CSIDH	SIDH
Speed (on x64 arch., NIST 1)	$\sim$ 35ms	$\sim$ 6ms
Public key size (NIST 1)	64B	346B
Key compression		
↳ speed		$\sim$ 11ms
↳ size		209B
Submitted to NIST	no	yes
TRL	4	6
Best classical attack	$p^{1/4}$	$p^{1/4}$ $(p^{3/8})$
Best quantum attack	$\tilde{O}\left(3^{\sqrt{\log_3 p}}\right)$	$p^{1/6}$ $(p^{3/8})$
Key size scales	quadratically	linearly
CPA security	yes	yes
CCA security	yes	Fujisaki-Okamoto
Constant time	it's complicated	yes
Non-interactive key exchange	yes	no
Signatures	short but (slow | do not scale)	big and slow

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Why prove a secret isogeny?

Public: Curves E, E'
Secret: An isogeny walk $E \to E'$

Why?

- For interactive identification;
- For signing messages;
- For validating public keys (esp. SIDH);
- More…

Some properties

	Zero knowledge			
	Statistical	Computational	Quantum resistance	Succinctness
CSIDH	✓		✓ /sort of	
SIDH		✓	✓	
Pairings				✓

Security assumptions in Isogeny-based Cryptography

Isogeny walk problem

Input Two isogenous elliptic curves E, E' over $\mathbb{F}_q$.

Output A path $E \to E'$ in an isogeny graph.

SIDH problem (1)

Input Elliptic curves E, E' over $\mathbb{F}_q$, isogenous of degree $\ell_A^{e_A}$.

Output The unique path $E \to E'$ of length e_A in the ℓ_A-isogeny graph.

SIDH problem (2)

Input
- Elliptic curves E, E' over $\mathbb{F}_q$, isogenous of degree $\ell_A^{e_A}$;
- The action of the isogeny on $E[\ell_B^{e_B}]$.

Output The unique path $E \to E'$ of length e_A in the ℓ_A-isogeny graph.

A Σ-protocol from Diffie–Hellman[3]

- A key pair (s, g^s);

$$g \xrightarrow{\quad s \quad} g^s$$

[3] Kids, do not try this at home! Use Schnorr!

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- A key pair (s, g^s);
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$$\begin{tikzcd} g \arrow[r, "s"] \arrow[dr, "r"'] & g^s \\ & g^r \end{tikzcd}$$

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- Challenge with bit $b \in \{0, 1\}$;

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- Respond with $c = r - b \cdot s \mod \#G$;

$$\begin{tikzcd}
g \arrow[rr, "s"] \arrow[dr, "r"'] & & g^s \arrow[dl, dashed, "r-s"] \\
& g^r &
\end{tikzcd}$$

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- Challenge with bit $b \in \{0, 1\}$;
- Respond with $c = r - b \cdot s \mod \#G$;
- Verify that $g^c(g^s)^b = g^r$.

$$\begin{tikzcd}
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- Respond with $c = r - b \cdot s \mod \#G$;
- Verify that $g^c (g^s)^b = g^r$.

$$
\begin{tikzcd}
g \arrow[rr, "s"] \arrow[dr, "r"'] & & g^s \arrow[dl, dashed, "r - s"] \\
& g^r &
\end{tikzcd}
$$

Zero-knowledge

Does not leak because:
c is uniformly distributed and independent from s.

[3] Kids, do not try this at home! Use Schnorr!

A Σ-protocol from Diffie–Hellman[3]

- A key pair (s, g^s);
- Commit to a random element g^r;
- Challenge with bit $b \in \{0, 1\}$;
- Respond with $c = r - b \cdot s \mod \#G$;
- Verify that $g^c (g^s)^b = g^r$.

$$\begin{tikzcd}
E_1 \arrow[rr, "g^s"] \arrow[dr, "g^r"'] & & E_s \arrow[dl, dashed, "g^{r-s}"] \\
& E_r &
\end{tikzcd}$$

Zero-knowledge

Does not leak because:
c is uniformly distributed and independent from s.

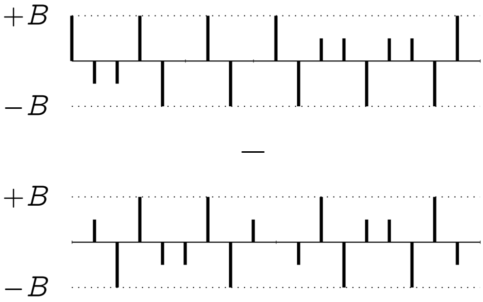
Unlike Schnorr, compatible with group action Diffie–Hellman.

[3] Kids, do not try this at home! Use Schnorr!

The trouble with groups of unknown structure

In CSIDH secrets look like: $g^{\vec{s}} = g_2^{s_2} g_3^{s_3} g_5^{s_5} \cdots$

- the elements g_i are fixed,
- the secret is the exponent vector $\vec{s} = (s_2, s_3, \dots) \in [-B, B]^n$,
- secrets must be sampled in a box $[-B, B]^n$ "large enough"…



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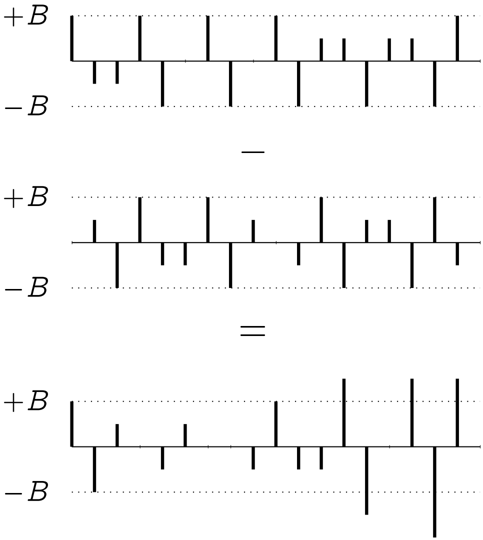


The leakage

With $\vec{s}, \vec{r} \xleftarrow{\$} [-B, B]^n$, the distribution of $\vec{r} - \vec{s}$ depends on the long term secret $\vec{s}$!

The two fixes

Do like the lattice people

SeaSign: D. and Galbraith 2019

- Use Fiat–Shamir with aborts (Lyubashevsky 2009).
- – Huge increase in signature size and time.
- Compromise signature size/time with public key size (still slow).

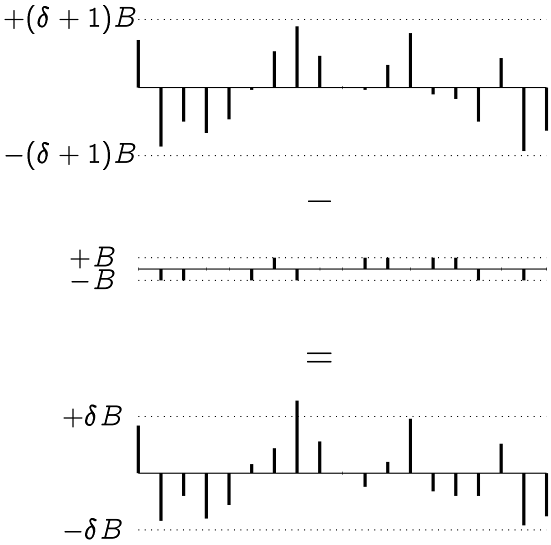
Compute the group structure and stop whining

CSI-FiSh: Beullens, Kleinjung and Vercauteren 2019

- Already suggested by Couveignes (1996) and Stolbunov (2006).
- Computationally intensive (subexponential parameter generation).
- Decent parameters, e.g.: 263 bytes, 390 ms, @NIST-1.
- – Technically not post-quantum (signing requires solving ApproxCVP).

Rejection sampling

- Sample long term secret $\vec{s}$ in the usual box $[-B, B]^n$,
- Sample ephemeral $\vec{r}$ in a larger box $[-(\delta+1)B, (\delta+1)B]^n$,
- Throw away $\vec{r} - \vec{s}$ if it is out of the box $[-\delta B, \delta B]^n$.



Zero-knowledge

Theorem: $\vec{r} - \vec{s}$ is uniformly distributed in $[-\delta B, \delta B]^n$.

Problem: set δ so that rejection probability is low.

SeaSign Performance (NIST-1)

	$t = 1$ **bit challenges**	$t = 16$ **bits challenges**	**PK compression**
Sig size	20 KiB	978 B	3136 B
PK size	64 B	4 MiB	32 B
SK size	32 B	16 B	1 MiB
Est. keygen time	30 ms	30 mins	30 mins
Est. sign time	30 hours	6 mins	6 mins
Est. verify time	10 hours	2 mins	2 mins
Asymptotic sig size	$O(\lambda^2 \log(\lambda))$	$O(\lambda t \log(\lambda))$	$O(\lambda^2 t)$

Speed/size compromises by Decru, Panny and Vercauteren 2019

Sig size	36 KiB	2 KiB	—
Est. sign time	30 mins	80 s	—
Est. verify time	20 mins	20 s	—

CSI-FiSh[4]

Table 3. Parameter choices and benchmark results for the "simple" variant of CSI-FiSh .

S	t	k	$\|\mathbf{sk}\|$	$\|\mathbf{pk}\|$	$\|\mathbf{sig}\|$	KeyGen	Sign	Verify
2^1	56	16	16 B	128 B	1880 B	100 ms	2.92 s	2.92 s
2^2	38	14	16 B	256 B	1286 B	200 ms	1.98 s	1.97 s
2^3	28	16	16 B	512 B	956 B	400 ms	1.48 s	1.48 s
2^4	23	13	16 B	1 KB	791 B	810 ms	1.20 s	1.19 s
2^6	16	16	16 B	4 KB	560 B	3.3 s	862 ms	859 ms
2^8	13	11	16 B	16 KB	461 B	13 s	671 ms	670 ms
2^{10}	11	7	16 B	64 KB	395 B	52 s	569 ms	567 ms
2^{12}	9	11	16 B	256 KB	329 B	3.5 m	471 ms	469 ms
2^{15}	7	16	16 B	2 MB	263 B	28 m	395 ms	393 ms

[4] Beullens, Kleinjung, and Vercauteren 2019.

A Σ-protocol for SIDH

$$E \xrightarrow{\phi} E/\langle S\rangle$$

$\frac{1}{3}$-soundness

Secret ϕ of degree $\ell_A^{e_A}$.

A Σ-protocol for SIDH

$$\begin{tikzcd}
E \arrow[r, dash, "\phi"] \arrow[d, dash, "?"'] & E/\langle S\rangle \arrow[d, dash, "?"] \\
E/\langle P\rangle \arrow[r, dash, "?"'] & E/\langle P, S\rangle
\end{tikzcd}$$

$\frac{1}{3}$-soundness

Secret ϕ of degree $\ell_A^{e_A}$.

1. Choose a random point $P \in E[\ell_B^{e_B}]$, compute the diagram;
2. Publish the curves $E/\langle P\rangle$ and $E/\langle P, S\rangle$;

A Σ-protocol for SIDH

$$\begin{tikzcd}
E \arrow[r, dash, "\phi"] \arrow[d, dash, "\psi"'] & E/\langle S \rangle \arrow[d, dash, "?"] \\
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3. The verifier challenges to reveal one out of the 3 sides
   - Isogenies ψ, ψ' (degree $\ell_B^{e_B}$) unrelated to secret;

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Improving to $\frac{1}{2}$-soundness

- Reveal ψ, ψ' simultaneously;
- Reveals action of ϕ on $E[\ell_B^{e_B}]$ $\Rightarrow$ Stronger security assumption.

SIDH signature performance (NIST-1)

According to Yoo, Azarderakhsh, Jalali, Jao and Vladimir Soukharev 2017:

Size: $\approx 100 KB$,

Time: seconds.

SIDH signature performance (NIST-1)

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Size: $\approx 100KB$,

Time: seconds.

Galbraith, Petit and Silva 2017

- Concept similar to CSI-FiSh: exploits known structure of endomorphism ring;
- Statistical zero knowledge (under heuristic assumptions);
- Based on the generic isogeny walk problem (requires special starting curve, though);
- Size/performance comparable to Yoo *et al.* (and possibly slower).

Weil pairing and isogenies

Theorem

Let $\phi : E \to E'$ be an isogeny and $\hat{\phi} : E' \to E$ its dual.
Let e_N be the Weil pairing of E and e'_N that of E'. Then, for

$$e_N(P, \hat{\phi}(Q)) = e'_N(\phi(P), Q),$$

for any $P \in E[N]$ and $Q \in E'[N]$.

Corollary

$$e'_N(\phi(P), \phi(Q)) = e_N(P, Q)^{\deg \phi}.$$

Pairing proofs: what for?

- Non-interactive, not post-quantum, not zero knowledge;

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- Useful for (partially) validating SIDH public keys;

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- Non-interactive, not post-quantum, not zero knowledge;
- Useful for (partially) validating SIDH public keys;
- Succinct: proof size, verification time independent of walk length!


#BLOCKCHAIN

Distributed lottery

Participants **A, B, . . . , Z** want to agree on a random winning ticket.

Flawed protocol

- Each participant x broadcasts a random string s_x;
- Winning ticket is $H(s_A, \dots, s_Z)$.

Distributed lottery

Participants **A, B, ..., Z** want to agree on a random winning ticket.

Flawed protocol

- Each participant x broadcasts a random string s_x;
- Winning ticket is $H(s_A, \dots, s_Z)$.

Fixes

- Make the hash function **slooooooooooooooooooooooooooooow**;

Distributed lottery

Participants **A, B, . . . , Z** want to agree on a random winning ticket.

Flawed protocol

- Each participant x broadcasts a random string s_x;
- Winning ticket is $H(s_A, \dots, s_Z)$.

Fixes

- Make the hash function **sloooooooooooooooooooooooooooow**;
- Make it possible to verify $w = H(s_A, \dots, s_Z)$ **fast**.

Verifiable Delay Functions (Boneh, Bonneau, Bünz, Fisch 2018)

Wanted

Function (family) $f : X \to Y$ s.t.:

- Evaluating $f(x)$ takes long time:
  - uniformly long time,
  - on almost all random inputs x,
  - even after having seen many values of $f(x')$,
  - even given massive number of processors;
- Verifying $y = f(x)$ is efficient:
  - ideally, exponential separation between evaluation and verification.

Sequentiality

Ideal functionality:

$$y = f(x) = \underbrace{H(H(\cdots(H(x))))}_{T\text{ times}}$$

- Sequential assuming hash output “unpredictability”,
- but how do you verify?

Isogeny VDF ($\mathbb{F}_p$-version)

(Trusted) Setup

- Pairing friendly supersingular curve $E/\mathbb{F}_p$ with unknown endomorphism ring
- Isogeny $\phi : E \to E'$ of degree 2^T,
- Point $P \in E[(N, \pi - 1)]$, image $\phi(P)$.

Evaluation

Input: random $Q \in E'[(N, \pi + 1)]$,

Output: $\hat{\phi}(Q)$.

Verification

$$e_N(P, \hat{\phi}(Q)) \quad \stackrel{?}{=} \quad e_N(\phi(P), Q).$$

Conclusion

- Repeat with me: I need isogeny-based crypto!
- ...
- Different isogeny graphs enable different styles of proofs, different security assumptions.
- Post-quantum isogeny signatures are still far from practical.
- Practical isogeny signatures do exists (CSI-FiSh); you can start using them now if you are an isogeny hippie, but they do not scale.
- Pairing-based proofs are usable, but not interesting for signatures: look into succinctness, instead!
- Proofs can be chained easily: useful for multi-party supersingular curve generation (work in progress with J. Burdges).

Thank you

`https://defeo.lu/`

@luca_defeo

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