Isogeny Graphs in Cryptography

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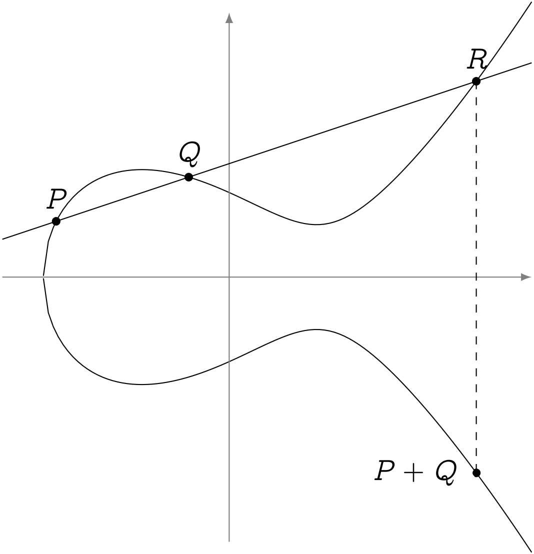
Elliptic curves

Let $E : y^2 = x^3 + ax + b$ be an elliptic curve…



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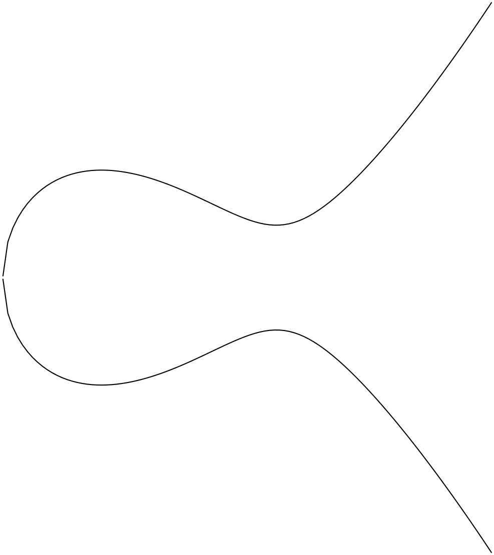

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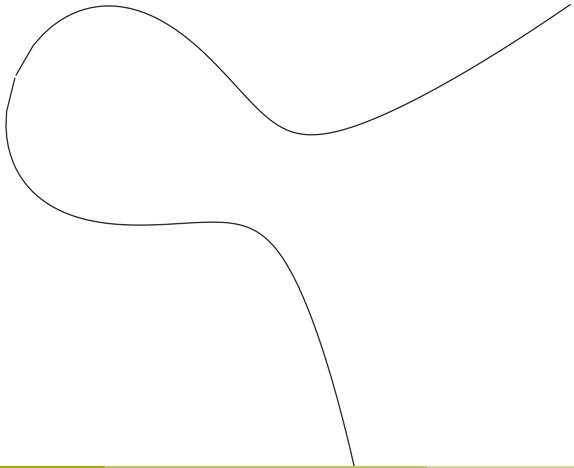

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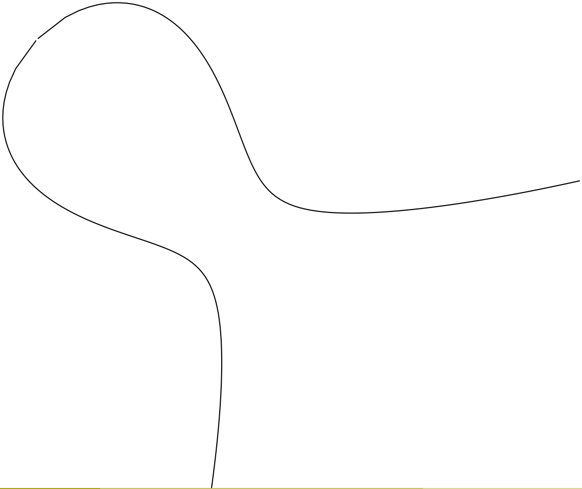

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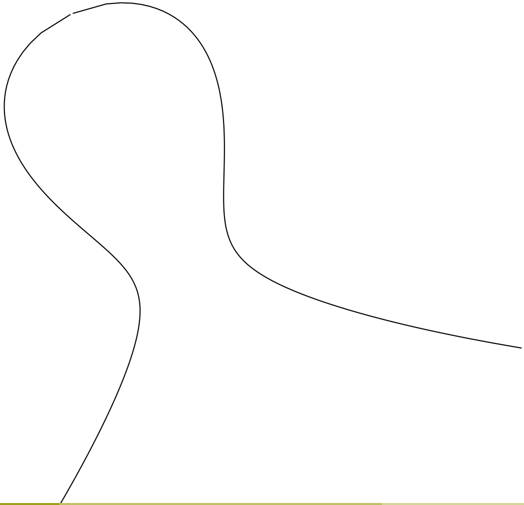

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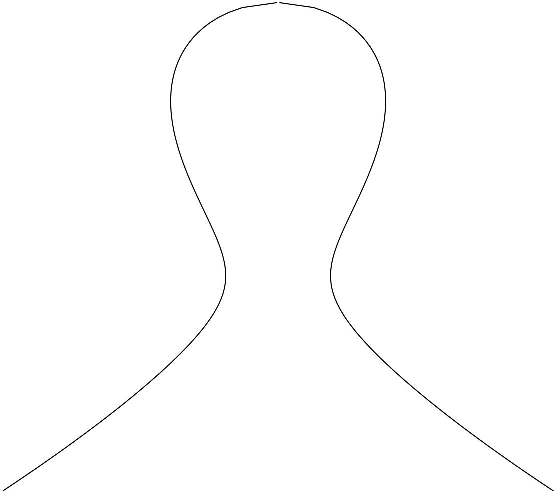
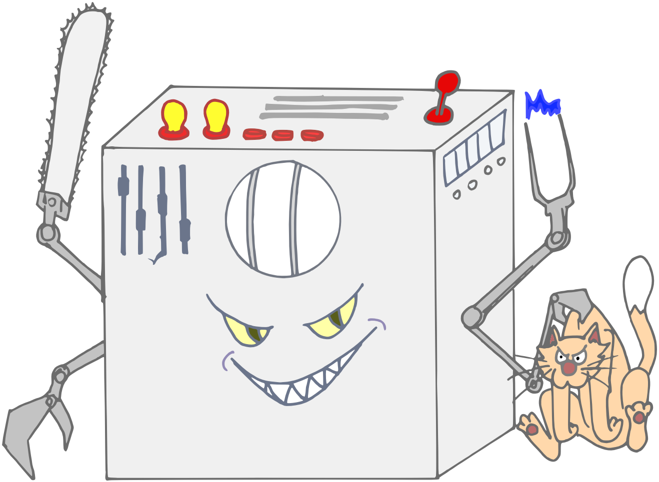
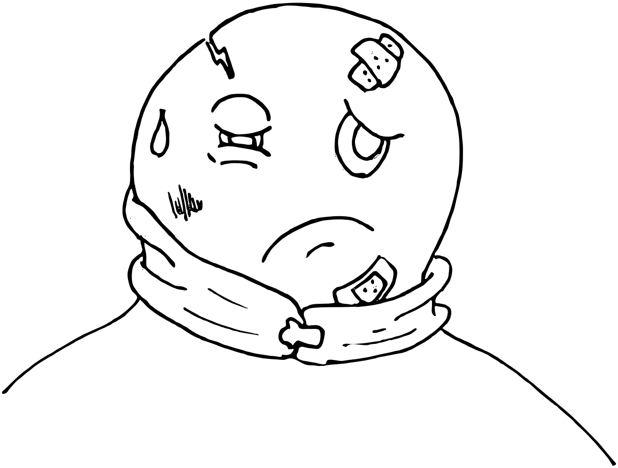


Elliptic curves

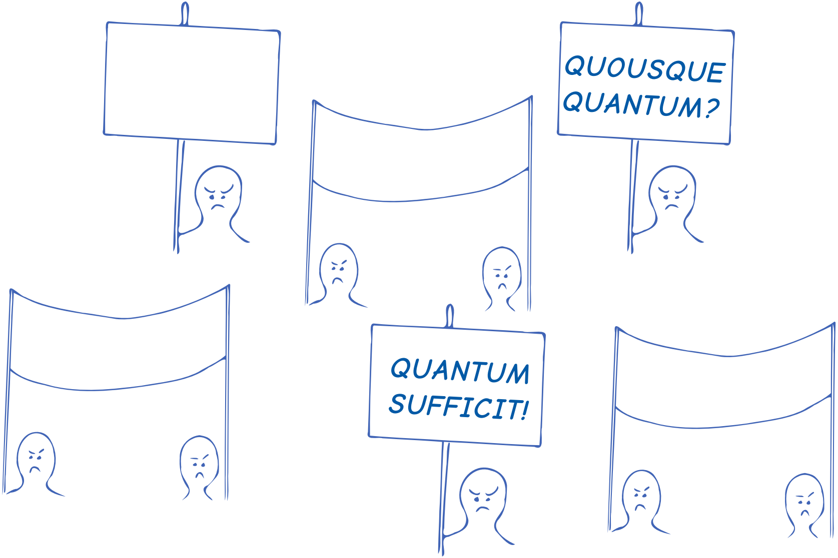
The QUANTHOM Menace



Post-quantum cryptographer?



Elliptic curves of the world, UNITE!



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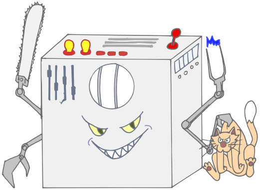

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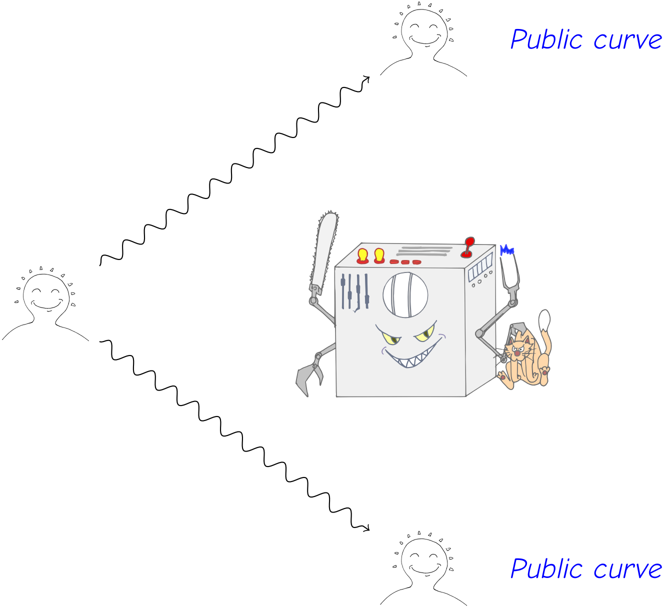

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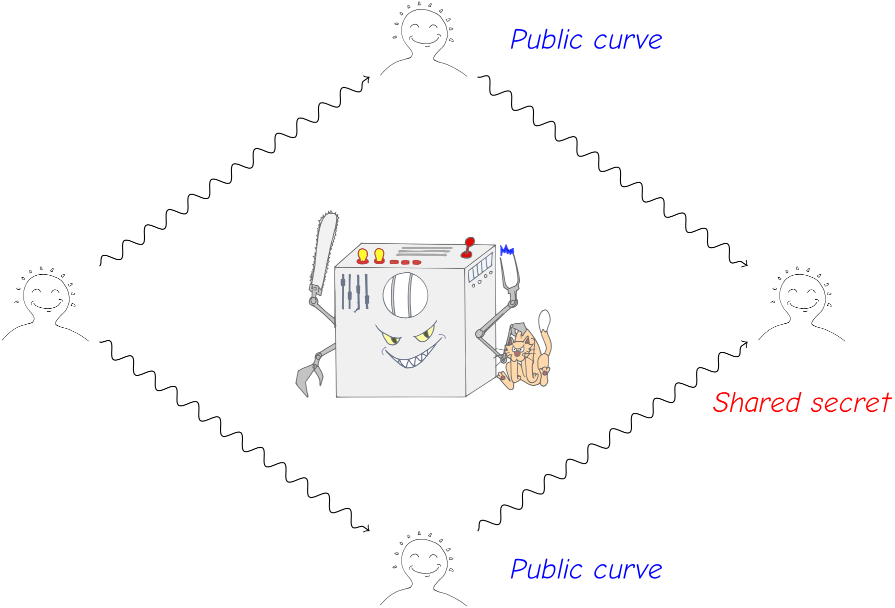


What’s an isogeny?

Rebus: 1-3-7-3-8-6

Isogenies

Isogenies are just the right notion™ of morphism for elliptic curves

- Surjective group morphisms.
- Algebraic maps (i.e., defined by polynomials).

(Separable) isogenies $\Leftrightarrow$ finite subgroups:

$$0 \to H \to E \xrightarrow{\phi} E' \to 0$$

The kernel H determines the image curve E' up to isomorphism

$$E/H \stackrel{\text{def}}{=} E'.$$

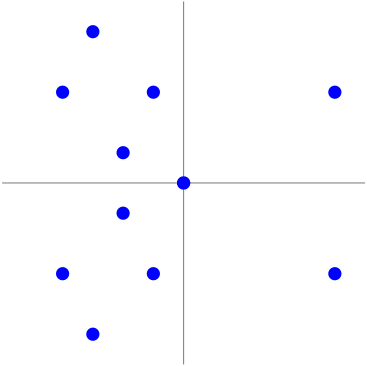
Isogeny degree

Neither of these definitions is quite correct, but they *nearly* are:

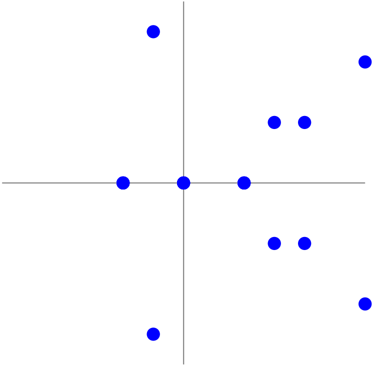
- The degree of ϕ is the cardinality of $\ker\phi$.
- (Bisson) the degree of ϕ is the time needed to compute it.

Isogenies: an example over $\mathbb{F}_{11}$

$$E \;:\; y^2 = x^3 + x$$



$$E' \;:\; y^2 = x^3 - 4x$$



$$\phi(x,y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2}\right)$$

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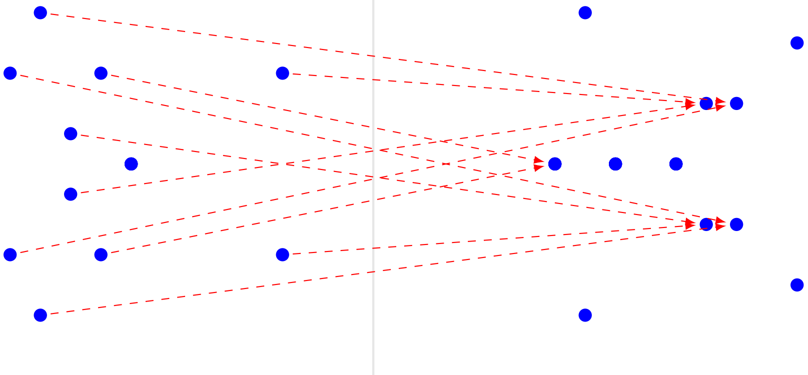

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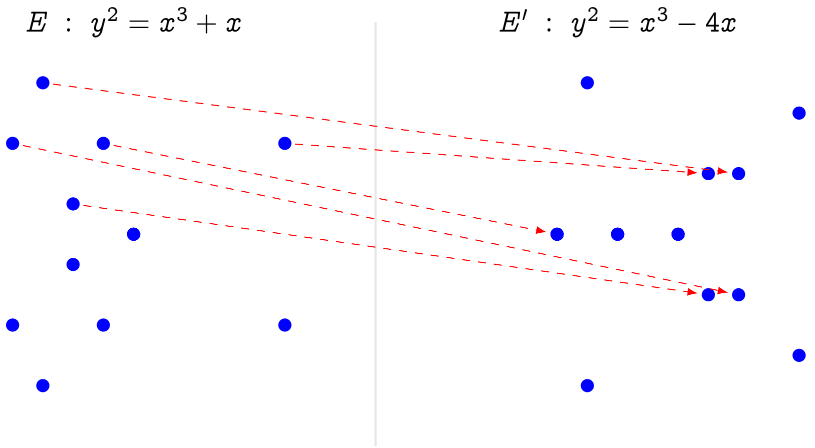

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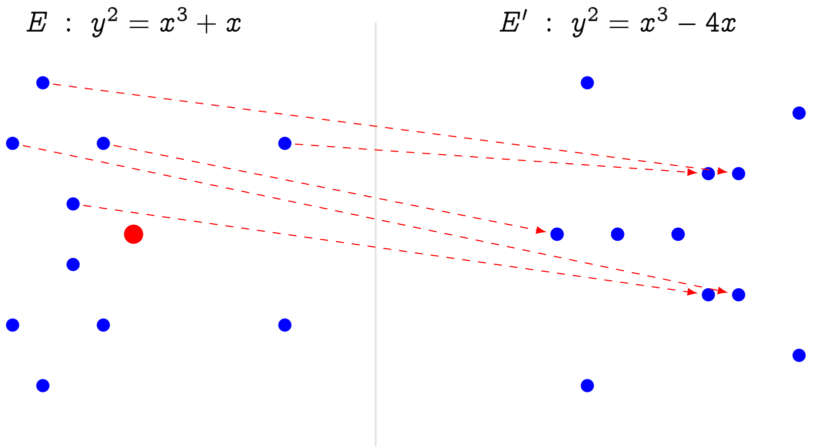

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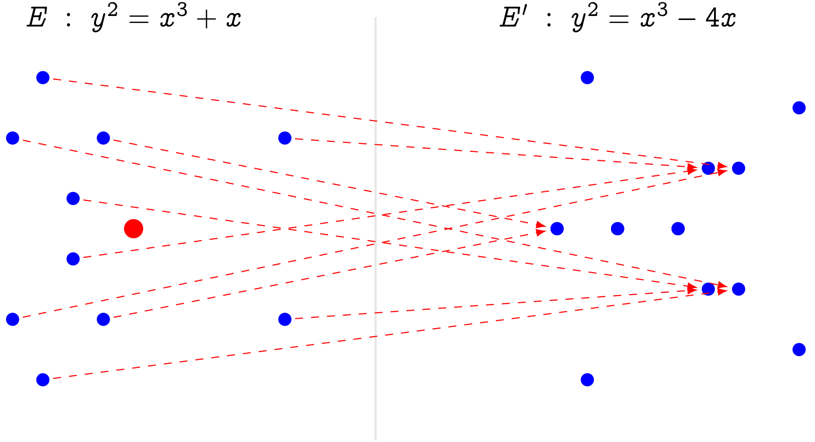

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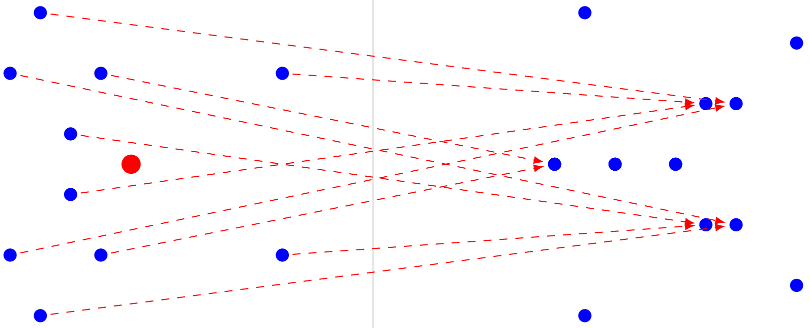

$$\phi(x, y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2}\right)$$

- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

Easy and hard problems

In practice: an isogeny ϕ is just a pair of rational fractions

$$\frac{N(x)}{D(x)} = \frac{x^n + \cdots + n_1 x + n_0}{x^{n-1} + \cdots + d_1 x + d_0} \in k(x), \qquad \text{with } n = \deg \phi,$$

and $D(x)$ vanishes on $\ker \phi$.

Vélu's formulas $\tilde{O}(n)$

Input: A generator of the kernel H of the isogeny.

Output: The curve E/H and the rational fraction N/D.

The explicit isogeny problem

Input: The curves E and E/H, the degree n.

Output: The rational fraction N/D.

Algorithms[a]

- Elkies' algorithm (and variants); $\tilde{O}(n)$
- Couveignes' algorithm (and variants). $\tilde{O}(n^2)$

[a] Elkies 1998; Couveignes 1996.

Easy and hard problems

Isogeny evaluation

Input: A *description* of the isogeny ϕ, a point $P \in E(k)$.

Output: The curve E/H and $\phi(P)$.

Examples

- Input = rational fraction; $O(n)$
- Input = composition of *low degree* isogenies; $\tilde{O}(\log n)$

The isogeny walk problem $O(??)$

Input: Isogenous curves E, E'.

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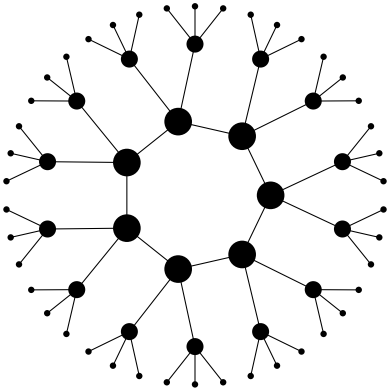
Exponential separation...Crypto happens!

Isogeny graphs

We look at the graph of elliptic curves with isogenies up to isomorphism. We say two isogenies ϕ, ϕ' are isomorphic if:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & E' \arrow[d, leftrightarrow, "\wr"] \\ & E' \end{tikzcd}$$

Example: Finite field, ordinary case, graph of isogenies of degree 3.



Structure of the graph[1]

Theorem (Serre-Tate)

Two curves are isogenous over a finite field k if and only if they have the same number of points on k.

The graph of isogenies of prime degree $\ell \neq p$

Ordinary case (isogeny volcanoes)

- Nodes can have degree $0, 1, 2$ or $\ell + 1$.
  - For $\sim 50\%$ of the primes ℓ, graphs are just isolated points;
  - For other $\sim 50\%$, graphs are 2-regular;
  - other cases only happen for finitely many ℓ's.

Supersingular case (algebraic closure)

- The graph is $\ell + 1$-regular.
- There is a unique (finite) connected component made of all supersingular curves with the same number of points.

[1] Deuring 1941; Kohel 1996; Fouquet and Morain 2002.

Expander graphs from isogenies

Expander graphs

An infinite family of connected k-regular graphs on n vertices is an expander family if there exists an $\epsilon > 0$ such that all non-trivial eigenvalues satisfy $|\lambda| \leq (1-\epsilon)k$ for n large enough.

- Expander graphs have short diameter ($O(\log n)$);
- Random walks mix rapidly (after $O(\log n)$ steps, the induced distribution on the vertices is close to uniform).

Supersingular Let ℓ be fixed, the graphs of all supersingular curves with ℓ-isogenies are expanders;[2]

Ordinary* Let $\mathcal{O} \subset \mathbb{Q}[\sqrt{-D}]$ be an order in a quadratic imaginary field. The graphs of all curves over $\mathbb{F}_q$ with complex multiplication by $\mathcal{O}$, with isogenies of prime degree bounded by $(\log q)^{2+\delta}$, are expanders.[3]

*(may contain traces of GRH)

[2] Pizer 1990, 1998.

[3] Jao, Miller, and Venkatesan 2009.

The first 10 years of isogeny based cryptography

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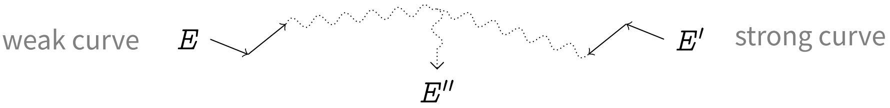
Ok. Let’s move on to the next 10 years!

Isogeny walks and cryptanalysis[5]

Fact: Having a weak DLP is not (always) isogeny invariant.



Fourth root attacks

- Start two random walks from the two curves and wait for a collision.
- Over $\mathbb{F}_q$, the average size of an isogeny class is $h_\Delta \sim \sqrt{q}$.
- A collision is expected after $O(\sqrt{h_\Delta}) = O(q^{\frac{1}{4}})$ steps.

Note: Can be used to build trapdoor systems[4].

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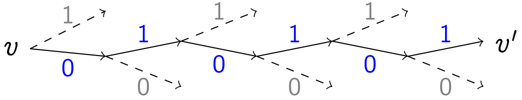
[4] Teske 2006.

[5] Galbraith 1999; Galbraith, Hess, and Smart 2002; Bisson and Sutherland 2011.

Random walks and hash functions

Any expander graph gives rise to a hash function.



$H(010101) = v'$

- Fix a starting vertex v;
- The value to be hashed determines a random path to v';
- v' is the hash.

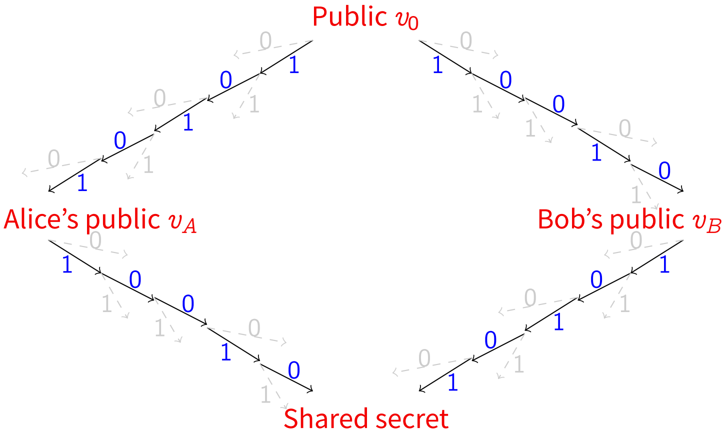
Provably secure hash functions

- Use the expander graph of supersingular 2-isogenies;[a]
- Collision resistance = hardness of finding cycles in the graph;
- Preimage resistance = hardness of finding a path from v to v'.

[a] Charles, Lauter, and Goren 2009.

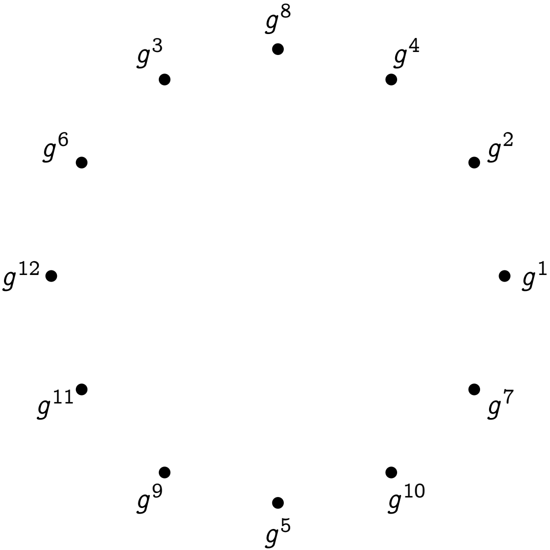
Random walks and key exchange

Let's try something harder...



...is this even possible?

Expander graphs from groups



Let $G = \langle g \rangle$ be a cyclic group of order p.

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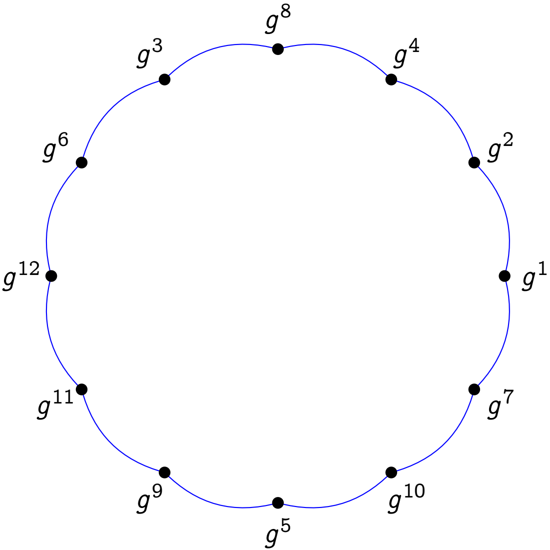

Let $G = \langle g \rangle$ be a cyclic group of order p. Let $S \subset (\mathbb{Z}/p\mathbb{Z})^\times$ s.t. $S^{-1} \subset S$.

The Schreier graph of $(S, G \setminus \{1\})$ is (usually) an expander.

— $x \mapsto x^2$

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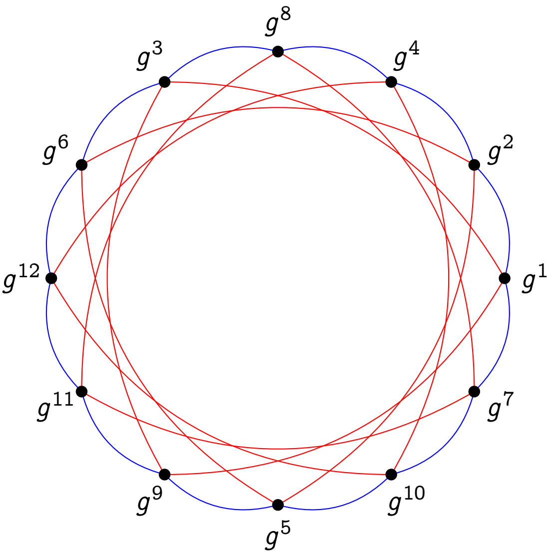

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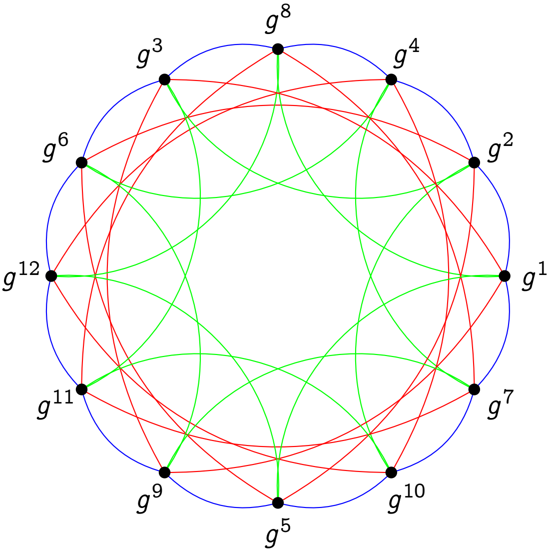

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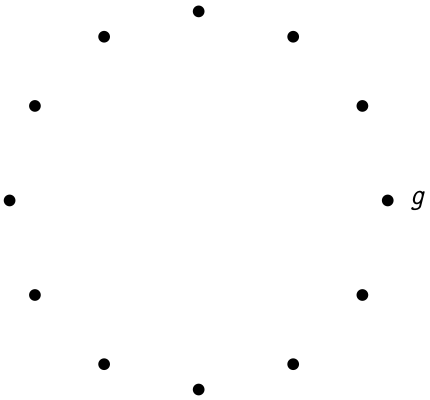
— $x \mapsto x^2$

— $x \mapsto x^3$

— $x \mapsto x^5$

Key exchange from Schreier graphs



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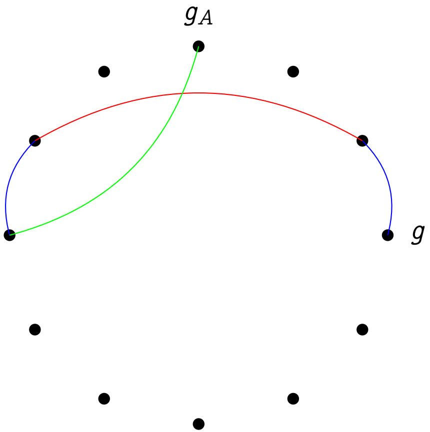

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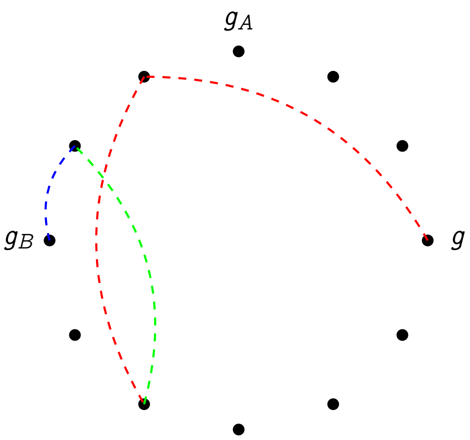

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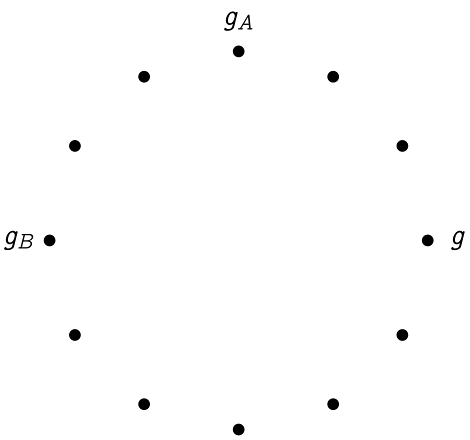

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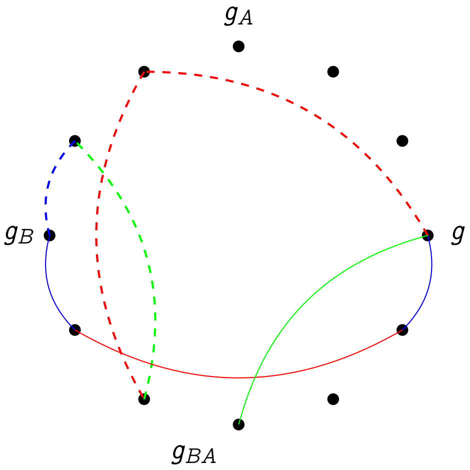

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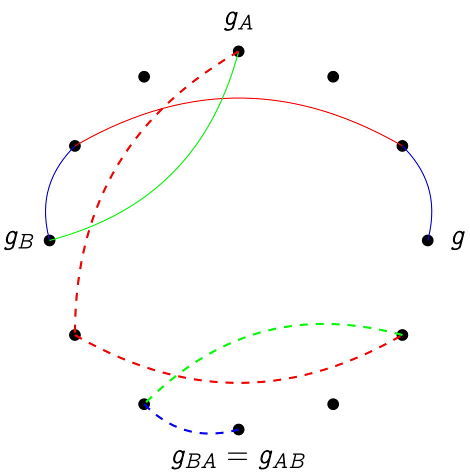

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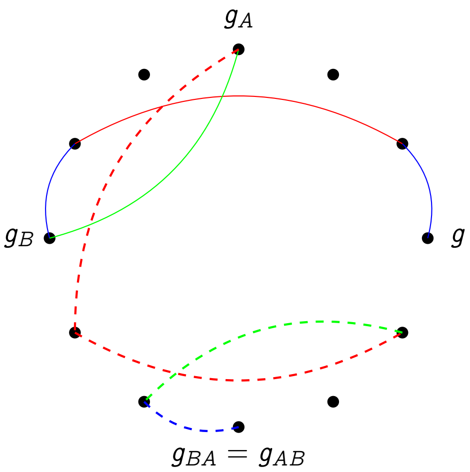

Why does this work?

$$g_A = g^{2\cdot 3\cdot 2\cdot 5},$$
$$g_B = g^{3^2\cdot 5\cdot 2},$$
$$g_{BA} = g_{AB} = g^{2^3\cdot 3^3\cdot 5^2};$$

and g_A, g_B, g_{AB} are (nearly) uniformly distributed in G…

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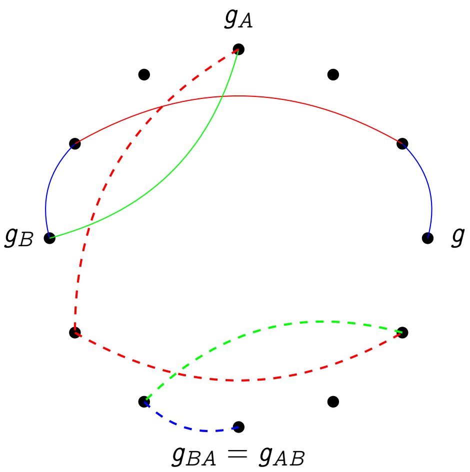

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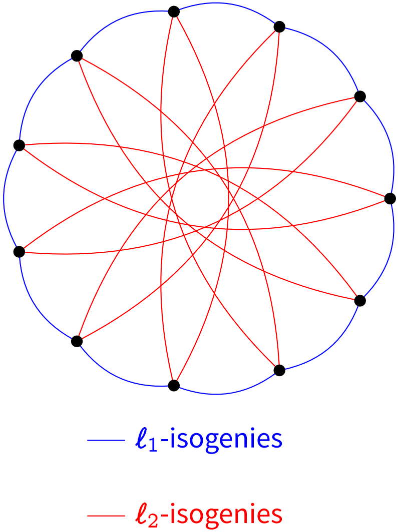
and g_A, g_B, g_{AB} are (nearly) uniformly distributed in G…

…Indeed, this is just a twisted presentation of the classical Diffie-Hellman protocol!

Group action on isogeny graphs



- There is a group action of the ideal class group $\mathrm{Cl}(\mathcal{O})$ on the set of ordinary curves with complex multiplication by $\mathcal{O}$.
- Its Schreier graph is an isogeny graph (and an expander if we take enough generators)


Class Group Action

Key exchange in graphs of ordinary isogenies[6] (CRS)

Parameters:

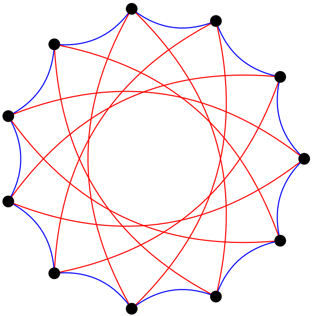
- $E/\mathbb{F}_p$ ordinary elliptic curve,
- (small) primes ℓ_1,ℓ_2,...such that $\left(\frac{D_\pi}{\ell_i}\right) = 1$.
- elements $\mathfrak{f}_1 = (\ell_1, \pi - \lambda_1)$,$\mathfrak{f}_2 = (\ell_2, \pi - \lambda_2)$ in $\mathrm{Cl}(\mathcal{O})$.

Secret data: Random walks $\mathfrak{a}, \mathfrak{b} \in \mathrm{Cl}(\mathcal{O})$ in the isogeny graph.

$$\begin{tikzcd}
 & E \arrow[dl, "(\mathfrak{f}_1)^{a_1}(\mathfrak{f}_2)^{a_2}\cdots = \mathfrak{a}"'] \arrow[dr, "\mathfrak{b} = (\mathfrak{f}_1)^{b_1}(\mathfrak{f}_2)^{b_2}\cdots"] & \\
\mathfrak{a} * E \arrow[dr] & & \mathfrak{b} * E \arrow[dl] \\
 & \mathfrak{ab} * E = \mathfrak{ba} * E &
\end{tikzcd}$$

[6] Couveignes 2006; Rostovtsev and Stolbunov 2006.

CRS key exchange



Key generation: compose small degree isogenies
polynomial in the lenght of the random walk.

Attack: find an isogeny between two curves
polynomial in the degree, exponential in the length.

In practice[7]: 5 minutes for a key exchange at 128-bits security level…

[7] Feo, Kieffer, and Smith 2018.

CSIDH *(pron.: Seaside)*[8]

One walk step in CRS: the explicit isogeny problem

Input: Curves E and E/H, an isogeny degree ℓ_i.

Output: The rational fraction N/D.

Algorithm: Elkies' algorithm (very expensive). $\tilde{O}(n)$

CSIDH: Key observations

1. If we know the kernel H in advance, we can apply Vélu's formulas (much faster than Elkies).
2. If the curves are supersingular, it is very easy to control the kernels.
3. If we restrict to supersingular isogenies defined over $\mathbb{F}_p$, the isogeny graph structure is identical to CRS![a]

[a] Delfs and Galbraith 2016.

Result: Same security as CRS in less than 100ms!

[8] Castryck, Lange, Martindale, Panny, and Renes 2018.

CRS and CSIDH: quantum security

Fact: Shor's algorithm does not apply to Diffie-Hellman protocols from group actions.

Subexponential attack $\exp(\sqrt{\log p \log \log p})$

- Reduction to the hidden shift problem by evaluating the class group action in quantum superspositon[a] (subexpoential cost);
- Well known reduction from the hidden shift to the dihedral (non-abelian) hidden subgroup problem;
- Kuperberg's algorithm[b] solves the dHSP with a subexponential number of class group evaluations.

[a] Childs, Jao, and Soukharev 2010.
[b] Kuperberg 2005; Regev 2004; Kuperberg 2013.

Key exchange in the full supersingular graph

Good news: there is no action of a commutative class group.

Bad news: there is no action of a commutative class group.

However: an algebraic structure is still acting on supersingular graphs: ideals of maximal orders of a quaternion algebra.

$$\begin{tikzcd}
E \arrow[r, "\mathfrak{a}"] \arrow[d, "\mathfrak{b}"'] & E' \arrow[d, "\mathfrak{b}_{\mathfrak{a}}"] \\
E'' \arrow[r, "\mathfrak{a}_{\mathfrak{b}}"'] & E'''
\end{tikzcd}$$

- The action is not commutative, we cannot use the same technique;
- We let instead Alice and Bob walk in two different isogeny graphs on the same vertex set.

Key exchange with supersingular curves

In practice, we fix:

- Small primes ℓ_A, ℓ_B;
- A large prime p such that $p+1 = \ell_A^{e_A}\ell_B^{e_B}$;
- A supersingular curve E over $\mathbb{F}_{p^2}$, such that

$$E \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2 = (\mathbb{Z}/\ell_A^{e_A}\mathbb{Z})^2 \oplus (\mathbb{Z}/\ell_B^{e_B}\mathbb{Z})^2,$$

- We use isogenies of degrees $\ell_A^{e_A}$ and $\ell_B^{e_B}$ with cyclic rational kernels;
- The diagram below can be constructed in time poly($e_A + e_B$).

$$\ker\phi = \langle P\rangle \subset E[\ell_A^{e_A}]$$

$$\ker\psi = \langle Q\rangle \subset E[\ell_B^{e_B}]$$

$$\ker\phi' = \langle \psi(P)\rangle$$

$$\ker\psi' = \langle \phi(Q)\rangle$$

$$\begin{tikzcd}
E \arrow[r, "\phi"] \arrow[d, "\psi"'] & E/\langle P\rangle \arrow[d, "\psi'"] \\
E/\langle Q\rangle \arrow[r, "\phi'"'] & E/\langle P, Q\rangle
\end{tikzcd}$$

Supersingular Isogeny Diffie-Hellman[9]

Parameters:

- Prime p such that $p+1=\ell_A^a\ell_B^b$;
- Supersingular curve $E \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2$;
- $E[\ell_A^a] = \langle P_A, Q_A\rangle$;
- $E[\ell_B^b] = \langle P_B, Q_B\rangle$.

Secret data:

- $R_A = m_A P_A + n_A Q_A$,
- $R_B = m_B P_B + n_B Q_B$,

$$
\begin{tikzcd}
 & E \arrow[dl, dash, "\phi"'] \arrow[dr, dash, "\psi"] & \\
E/\langle R_A\rangle \arrow[dr, dash, "\psi'"'] & & E/\langle R_B\rangle \arrow[dl, dash, "\phi'"] \\
 & E/\langle R_A, R_B\rangle &
\end{tikzcd}
$$

$$\frac{E/\langle R_A\rangle}{\phi(R_B)} \simeq E/\langle R_A, R_B\rangle \simeq \frac{E/\langle R_B\rangle}{\psi(R_A)}$$

[9] Jao and De Feo 2011; De Feo, Jao, and Plût 2014.

Supersingular Isogeny Diffie-Hellman[9]

Parameters:

- Prime p such that $p+1 = \ell_A^a \ell_B^b$;
- Supersingular curve $E \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2$;
- $E[\ell_A^a] = \langle P_A, Q_A \rangle$;
- $E[\ell_B^b] = \langle P_B, Q_B \rangle$.

Secret data:

- $R_A = m_A P_A + n_A Q_A$,
- $R_B = m_B P_B + n_B Q_B$,

$$\begin{tikzcd}
 & E \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
\begin{array}{l} E/\langle R_A \rangle \\ \phi(P_B) \\ \phi(Q_B) \end{array} \arrow[dr, "\psi'"'] & & \begin{array}{l} E/\langle R_B \rangle \\ \psi(P_A) \\ \psi(Q_A) \end{array} \arrow[dl, "\phi'"] \\
 & \frac{E/\langle R_A \rangle}{\phi(R_B)} \simeq E/\langle R_A, R_B \rangle \simeq \frac{E/\langle R_B \rangle}{\psi(R_A)} &
\end{tikzcd}$$

[9] Jao and De Feo 2011; De Feo, Jao, and Plût 2014.

Supersingular Isogeny Diffie-Hellman[9]

Parameters:

- Prime p such that $p+1=\ell_A^a\ell_B^b$;
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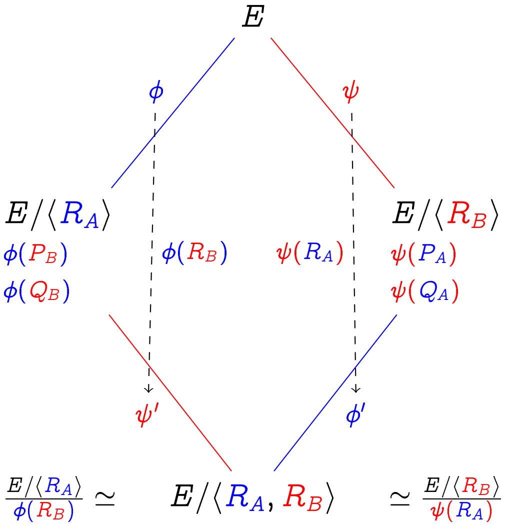
- $E[\ell_A^a] = \langle P_A, Q_A\rangle$;
- $E[\ell_B^b] = \langle P_B, Q_B\rangle$.

Secret data:

- $R_A = m_A P_A + n_A Q_A$,
- $R_B = m_B P_B + n_B Q_B$,



[9] Jao and De Feo 2011; De Feo, Jao, and Plût 2014.

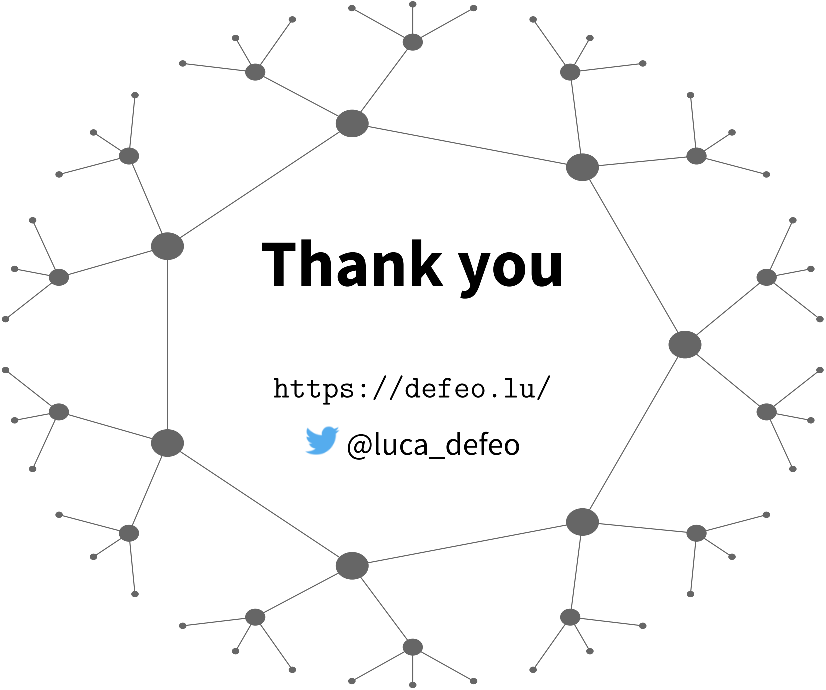
CSIDH vs SIDH

	CSIDH	**SIDH**
Speed @128b	<100ms	10ms
Public key size @128b	64B	378B
Submitted to NIST	no	yes
Best classical attack	$p^{1/4}$	$p^{1/4}$
Best quantum attack	subexponential	$p^{1/6}$
Key size scales	quadratically	linearly
Security assumption	isogeny walk problem	ad hoc
CPA security	yes	yes
CCA security	yes	Fujsaki-Okamoto
Non-interactive key ex.	yes	no
Signatures	unclear	very slow

SIKE: Supersingular Isogeny Key Encapsulation

- Submission to the NIST PQ competition:
  - SIKE.PKE: El Gamal-type system with IND-CPA security proof,
  - SIKE.KEM: generically transformed system with IND-CCA security proof.
- Security levels 1, 3 and 5.
- Smallest communication complexity among all proposals in each level.
- Slowest among all benchmarked proposals in each level.
- A team of 14 submitters, from 8 universities and companies.
- Visit `https://sike.org/`.

	p	cl. security	q. security	speed	comm.
SIKEp503	$2^{250}3^{159} - 1$	126 bits	84 bits	10ms	0.4KB
SIKEp751	$2^{372}3^{239} - 1$	188 bits	125 bits	30ms	0.6KB
SIKEp964	$2^{486}3^{301} - 1$	241 bits	161 bits		0.8KB



Thank you

`https://defeo.lu/`

@luca_defeo

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