Une théorie locale des polylogarithmes.

Local continuation of Li.

V.C. Bùi, G.H.E. Duchamp,
Hoang Ngoc Minh, Q.H. Ngô et al.
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N. Behr, K. A. Penson, C. Tollu.

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Plan

Introduction

The aim of this quick talk is to explain how to extend polylogarithms

$$\mathrm{Li}(s_1, \dots s_r) = \sum_{n_1 > n_2 > \dots n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \dots n_r^{s_r}} \tag{1}$$

They are a priori coded by lists $(s_1, \dots s_r)$ but, when $s_i \in \mathbb{N}_+$, they admit an *iterated integral representation* and are better coded by words with letters in $X = \{x_0, x_1\}$. We will use the one-to-one correspondences.

$$(s_1, \dots, s_r) \in \mathbb{N}_+^r \leftrightarrow x_0^{s_1 - 1} x_1 \dots x_0^{s_r - 1} x_1 \in X^* x_1 \leftrightarrow y_{s_1} \dots y_{s_r} \in Y^* \tag{2}$$

- $\mathrm{Li}(s)[z]$ is Jonquière and, for $\Re(s) > 1$, one has $\mathrm{Li}(s)[1] = \zeta(s)$
- Completed by $Li(x_0^n) = \frac{log^n(z)}{n!}$ this provides a family of independant functions admitting an analytic continuation on the cleft plane $\mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ or $\widetilde{\mathbb{C} \setminus \{0,1\}}$.

Explicit construction of Li

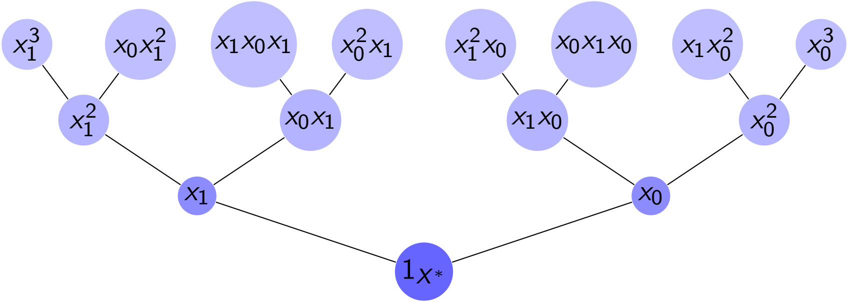
Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

$$\alpha_0^z(w) = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_0^z \alpha_0^s(u) \frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \int_1^z \alpha_0^s(u) \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \\ \int_0^z \alpha_0^s(u) \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0\ . \end{cases} \qquad (3)$$

Of course, the third line of this recursion implies

$$\alpha_0^z(x_0^n) = \frac{log(z)^n}{n!}$$

one can check that (a) all the integrals (although improper for the fourth line) are well defined (b) the series $S = \sum_{w \in X^*} \alpha_0^z(w)\, w$ satisfies (4). We then have $\alpha_0^z = \mathrm{Li}$.



As an example, we compute some coefficients

$$\langle \mathrm{Li} \mid x_0^n \rangle = \frac{log(z)^n}{n!} \quad ; \quad \langle \mathrm{Li} \mid x_1^n \rangle = \frac{(-log(1-z))^n}{n!}$$

$$\langle \mathrm{Li} \mid x_0x_1 \rangle = \mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2} \quad ; \quad \langle \mathrm{Li} \mid x_1x_0 \rangle = \langle \mathrm{Li} \mid x_1 \shuffle x_0 - x_0x_1 \rangle(z)$$

$$\langle \mathrm{Li} \mid x_0^2x_1 \rangle = \mathrm{Li}_3(z) = \sum_{n\geq 1} \frac{z^n}{n^3} \quad ; \quad \langle \mathrm{Li} \mid x_1x_0 \rangle = (-log(1-z))log(z) - \mathrm{Li}_2(z)$$

$$\langle \mathrm{Li} \mid x_0^{r-1}x_1 \rangle = \mathrm{Li}_r(z) = \sum_{n\geq 1} \frac{z^n}{n^r} \quad ; \quad \langle \mathrm{Li} \mid x_1^2x_0 \rangle = \langle \mathrm{Li} \mid \frac{1}{2}(x_1 \shuffle x_1 \shuffle x_0) - (x_1 \shuffle x_0x_1) + x_0x_1^2 \rangle$$

Li From Noncommutative Diff. Eq.

The generating series $S = \sum_{w \in X^*} Li(w)$ satisfies (and is unique to do so)

$$\left\{ \begin{array}{l} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S \\ \\ \lim_{\substack{z \to 0 \\ z \in \Omega}} S(z) e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle} \end{array} \right. \qquad (4)$$

with $X = \{x_0, x_1\}$. This is, up to the sign of x_1, the solution G_0 of Drinfel'd [1] for KZ3. We define this unique solution as Li. All Li_w are $\mathbb{C}$- and even $\mathbb{C}(z)$-linearly independant (see CAP 17 *Linear independance without monodromy*).

1. V. Drinfel'd, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.

Domain of Li (definition)

In order to extend Li to series, we define $Dom(Li;\Omega)$ (or $Dom(Li)$) if the context is clear) as the set of series $S=\sum_{n\geq 0}S_n$ (decomposition by homogeneous components) such that $\sum_{n\geq 0}Li_{S_n}(z)$ converges for the compact convergence in Ω. One sets

$$Li_S(z):=\sum_{n\geq 0}Li_{S_n}(z) \tag{5}$$

Starting the ladder

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle,\sqcup\!\sqcup,1_{X^*}) & \xhookrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\big\downarrow & & \big\downarrow \\
(\mathbb{C}\langle X\rangle,\sqcup\!\sqcup,1_{X^*})[x_0^*,(-x_0)^*,x_1^*] & \xrightarrow{\mathrm{Li}^{(1)}_\bullet} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*}
\end{array}$$

Examples

$$Li_{x_0^*}(z)=z,\quad Li_{x_1^*}(z)=(1-z)^{-1},\ Li_{\alpha x_0^*+\beta x_1^*}(z)=z^{\alpha}(1-z)^{-\beta}$$

Properties of the extended Li

Proposition

With this definition, we have

1. $Dom(Li)$ is a shuffle subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ and then so is $Dom^{rat}(Li) := Dom(Li) \cap \mathbb{C}^{rat}\langle\langle X\rangle\rangle$
2. For $S, T \in Dom(Li)$, we have
$$\mathrm{Li}_{S \sqcup\!\sqcup T} = \mathrm{Li}_S \,.\, \mathrm{Li}_T$$

Examples and counterexamples

For $|t| < 1$, one has $(tx_0)^* x_1 \in Dom(Li, D)$ (D being the open unit slit disc and $Dom(Li, D)$ defined similarly), whereas $x_0^* x_1 \notin Dom(Li, D)$. Indeed, we have to examine the convergence of $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z)$, but, for $z \in]0, 1[$, one has $0 < z < \mathrm{Li}_{x_0^n x_1}(z) \in \mathbb{R}$ and therefore, for these values $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z) = +\infty$. One can show that, for $|t| < 1$
$$\mathrm{Li}_{(tx_0)^* x_1}(z) = \sum_{n\geq 1} \frac{z^n}{t-n}$$

Passing to harmonic sums $\mathrm{H}_w, \; w \in Y^*$

Polylogarithms having a removable singularity at zero

The following proposition helps us characterize their indices.

Proposition

Let $f(z) = \langle \mathrm{L} \mid P \rangle = \sum_{w \in X^*} \langle P \mid w \rangle \, \mathrm{Li}_w$. *The following conditions are equivalent*

i) *f can be analytically extended around zero*

ii) $P \in \mathbb{C}\langle X \rangle x_1 \oplus \mathbb{C}.1_{X^*}$

TODO recall the formula with $\frac{\mathrm{Li}(z)}{1-z}$

Global and local domains

This proposition and the lemma lead us to the following definitions.

1. *Global domains*.–
   Let $\emptyset \neq \Omega \subset \widetilde{B}$, we define $Dom_{\Omega}(Li) \subset \mathbb{C}\langle\langle X\rangle\rangle$ to be the set of series $S = \sum_{n\geq 0} S_n$ (with $S_n = \sum_{|w|=n}\langle S \mid w\rangle\, w$ each homogeneous component) such that $\sum_{n\in\mathbb{N}} Li_{S_n}$ is unconditionally convergent for the compact convergence (UCC) [3].
   As examples, we have Ω_1, the doubly cleft plane then $Dom(\mathrm{Li}) := Dom_{\Omega_1}(\mathrm{Li})$ or $\Omega_2 = \widetilde{B}$

2. *Local domains around zero (fit with H-theory)*.–
   Here, we consider series $S \in (\mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}\, 1_{X^*})$ (i.e. $supp(S) \cap Xx_0 = \emptyset$).
   We consider radii $0 < R \leq 1$, the corresponding open discs $D_R = \{z \in \mathbb{C} \mid |z| < R\}$ and define

$$\begin{aligned} Dom_R(\mathrm{Li}) &:= \{S = \Sigma_{n\geq 0} S_n \in (\mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{\Omega}) \mid \sum_{n\in\mathbb{N}} Li_{S_n} \text{ (UCC) in } D_R\} \\ Dom_{loc}(\mathrm{Li}) &:= \cup_{0<R\leq 1} Dom_R(\mathrm{Li}). \end{aligned} \tag{6}$$

Properties of the domains

Theorem A

1. For all $\emptyset \neq \Omega \subset \widetilde{B}$, $Dom_{\Omega}(\mathrm{Li})$ is a shuffle subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ and so are the $Dom_R(\mathrm{Li})$.
2. $R \mapsto Dom_R(\mathrm{Li})$ is strictly decreasing for $R \in]0,1]$.
3. All $Dom_R(\mathrm{Li})$ and $Dom_{loc}(\mathrm{Li})$ are shuffle subalgebras of $\mathbb{C}\langle\langle X\rangle\rangle$ and $\pi_Y(Dom_{loc}(\mathrm{Li}))$ is a stuffle subalgebra of $\mathbb{C}\langle\langle Y\rangle\rangle$.
4. Let $T(z) = \sum_{N\geq 0} a_N z^N$ be a Taylor series i.e. such that $\limsup_{N\to+\infty} |a_N|^{1/n} = B < +\infty$, then the series

$$S = \sum_{N\geq 0} a_N(-(-x_1)^+)^{\sqcup\!\sqcup\, N} \tag{7}$$

is summable in $\mathbb{C}\langle\langle X\rangle\rangle$ (with sum in $\mathbb{C}\langle\langle x_1\rangle\rangle$) and $S \in Dom_R(Li)$ with $R = \frac{1}{B+1}$ and $\mathrm{Li}_S = T(z)$.

Theorem A/2

5. Let $S \in Dom_R(\mathrm{Li})$ and $S = \sum_{n\geq 0} S_n$ (homogeneous decomposition), we define $N \mapsto \mathrm{H}_{\pi_Y(S)}(N) by$[a]

$$\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{\pi_Y(S)}(N) z^N \ . \tag{8}$$

Moreover, for all $r \in]0, R[$,

$$\sum_{n,N\geq 0} |\mathrm{H}_{\pi_Y(S_n)} r^N| < +\infty, \tag{9}$$

in particular, for all $N \in \mathbb{N}$ the series (of complex numbers) $\sum_{n\geq 0} \mathrm{H}_{\pi_Y(S_n)}(N)$ converges absolutely to $\mathrm{H}_{\pi_Y(S)}(N)$.

[a] This definition is compatible with the old one when S is a polynomial.

Theorem A/3

6. Conversely, let $Q \in \mathbb{C}\langle\langle Y \rangle\rangle$ with $Q = \sum_{n \geq 0} Q_n$ (decomposition by weights), we suppose that it exists $r \in]0, 1]$ such that

$$\sum_{n, N \geq 0} |\mathrm{H}_{Q_n}(N) r^N| < +\infty. \tag{10}$$

in particular, for all $N \in \mathbb{N}$, $\sum_{n \geq 0} \mathrm{H}_{Q_n}(N) = \ell(N) \in \mathbb{C}$ unconditionally.
Under such circumstances, $\pi_X(Q) \in Dom_r(\mathrm{Li})$ and, for all $|z| \leq r$

$$\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N \geq 0} \ell(N) z^N, \tag{11}$$

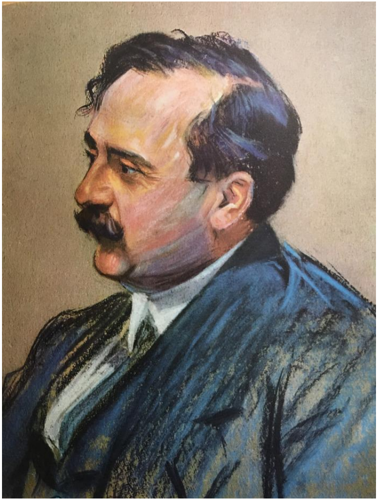


Figure: Jacques Hadamard and Paul Montel.

Continuing the ladder

$$
\begin{tikzcd}
(\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*}) \arrow[r, hook, two heads, "\mathrm{Li}_\bullet"] \arrow[d, hook] & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \arrow[d, hook] \\
(\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] \arrow[r, two heads, "\mathrm{Li}_\bullet^{(1)}"] \arrow[d, hook] & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \arrow[d, hook] \\
\mathbb{C}\langle X\rangle \shuffle \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \shuffle \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \arrow[r, "\mathrm{Li}_\bullet^{(2)}"] & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\mathbb{C}\langle X\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \arrow[u, two heads] \arrow[ur, dashed] &
\end{tikzcd}
$$

We have, after a theorem by Leopold Kronecker,

$$\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \left\{ \frac{P}{Q} \right\}_{\substack{P, Q \in \mathbb{C}[x] \\ Q(0)\neq 0}} \tag{12}$$

On the right: freeness without monodromy

Theorem (Deneufchâtel, GHED,Minh & Solomon, 2011 [1])

Let $(\mathcal{A},\partial)$ be a k-commutative associative differential algebra with unit and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $\partial(\mathcal{C})\subset\mathcal{C}$). We suppose that $k=ker(\partial)$ and that $S\in\mathcal{A}\langle\langle X\rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S)=MS\ ;\ \langle S\mid 1\rangle=1 \text{ with } M=\sum_{x\in X}u_x x\in\mathcal{C}\langle\langle X\rangle\rangle \tag{13}$$

(i.e. M is a homogeneous series of degree 1)
The following conditions are equivalent :

1. *The family $(\langle S\mid w\rangle)_{w\in X^*}$ of coefficients of S is (linearly) free over $\mathcal{C}$.*
2. *The family of coefficients $(\langle S\mid x\rangle)_{x\in X\cup\{1_{X^*}\}}$ is (linearly) free over $\mathcal{C}$.*
3. *The family $(u_x)_{x\in X}$ is such that, for $f\in\mathcal{C}$ et $\alpha_x\in k$*

$$\partial(f)=\sum_{x\in X}\alpha_x u_x\Longrightarrow(\forall x\in X)(\alpha_x=0).$$

A useful property

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Independence of characters with respect to polynomials

5

1

I came across the following property :

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U}=\mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra and ϵ, its counit, is the only character of $\mathcal{U}\to k$ which vanishes on $\mathfrak{g}$.

Set $\mathcal{U}_+=ker(\epsilon)$. We build the following filtrations $(N\geq 1)$

$$\mathcal{U}_N=\mathcal{U}_+^N=\underbrace{\mathcal{U}_+\ldots\ldots\mathcal{U}_+}_{N\text{ times}} \qquad (1)$$

and

$$\mathcal{U}_N^*=\mathcal{U}_{N+1}^{\perp}=\{f\in\mathcal{U}^*|(\forall u\in\mathcal{U}_{N+1})(f(u)=0)\} \qquad (2)$$

the first one is decreasing and the second one increasing. One shows easily that (with $\diamond$ as the convolution product)

$$\mathcal{U}_p^*\diamond\mathcal{U}_q^*\subset\mathcal{U}_{p+q}^*$$

so that $\mathcal{U}_\infty^*=\cup_{n\geq 1}\mathcal{U}_n^*$ is a convolution subalgebra of $\mathcal{U}^*$.

Now, we can state the

Theorem : The set of characters of $(\mathcal{U},.,1_{\mathcal{U}})$ is linearly free w.r.t. $\mathcal{U}_\infty^*$.

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Left and then right: the arrow $\mathrm{Li}_\bullet^{(1)}$

Proposition

i. The family $\{x_0^*, x_1^*\}$ is algebraically independent over $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$ within $(\mathbb{C}\langle\langle X\rangle\rangle^{\mathrm{rat}}, ш, 1_{X^*})$.

ii. $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ is a free module over $\mathbb{C}\langle X\rangle$, the family $\{(x_0^*)^{ш\, k} ш (x_1^*)^{ш\, l}\}_{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ is a $\mathbb{C}\langle X\rangle$-basis of it.

iii. As a consequence, $\{w ш (x_0^*)^{ш\, k} ш (x_1^*)^{ш\, l}\}_{\substack{w\in X^* \\ (k,l)\in\mathbb{Z}\times\mathbb{N}}}$ is a $\mathbb{C}$-basis of it.

iv. $\mathrm{Li}_\bullet^{(1)}$ is the unique morphism from $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*]$ to $\mathcal{H}(\Omega)$ such that

$$x_0^* \to z,\ (-x_0)^* \to z^{-1} \text{ and } x_1^* \to (1-z)^{-1}$$

v. $\mathrm{Im}(\mathrm{Li}_\bullet^{(1)}) = \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*}$.

vi. $\ker(\mathrm{Li}_\bullet^{(1)})$ is the (shuffle) ideal generated by $x_0^* ш x_1^* - x_1^* + 1_{X^*}$.

Sketch of the proof (pictorial)

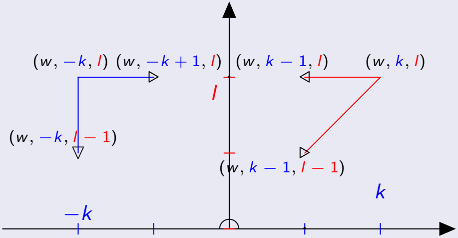

Figure: Rewriting mod $\mathcal{J}$ of $\{w \sqcup\!\sqcup (x_0^*)^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, l}\}_{k\in\mathbb{Z}, l\in\mathbb{N}, w\in X^*}$.

Concluding remarks

- Extending the domain of polylogarithms to (some) rational series permits the projection of rational identities. Such as

$$(\alpha x)^* ш (\beta y)^* = (\alpha x + \beta y)^*$$

- The theory developed here allows to pursue, for the Harmonic sums, this investigation such as

$$(\alpha y_i)^* ⧢ (\beta y_j)^* = (\alpha y_i + \beta y_j + \alpha\beta y_{i+j})^*$$

- More in Minh's talk.

[1] M. Deneufchâtel, GHED, Hoang Ngoc Minh, A. I. Solomon.– *Independence of hyperlogarithms over function fields via algebraic combinatorics*, Lecture Notes in Computer Science (2011), Volume 6742 (2011), 127-139.

[2] J. Hadamard, *Théorème sur les séries entières*, Acta Math., Vol 22 (1899), 55-63.

[3] P. Montel.– *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Gauthier-Villars (1927)

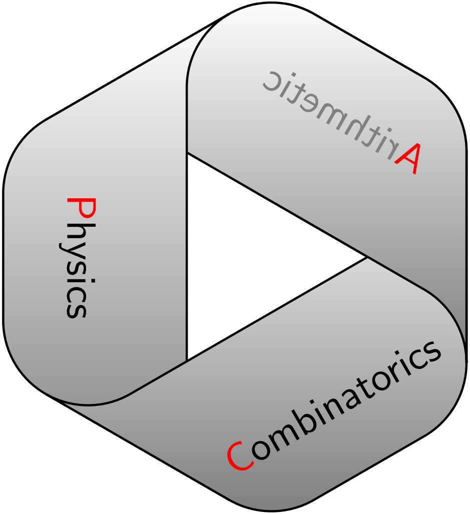

Figure: ... and a lot of (machine) computations.

THANK YOU FOR YOUR ATTENTION !