Isogeny graphs in cryptography

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Cryptography meets Graph Theory
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Plan

1. Elliptic curves, isogenies, complex multiplication
2. Isogeny graphs
3. Key exchange
4. Signatures and whatnot

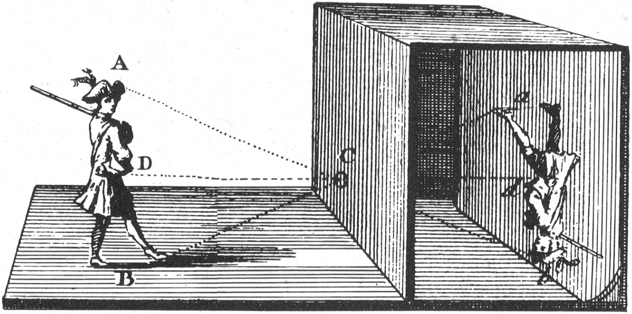
Projective space

Definition (Projective space)

Let $\bar{k}$ an algebraically closed field, the projective space $\mathbb{P}^n(\bar{k})$ is the set of non-null $(n+1)$-tuples $(x_0, \dots, x_n) \in \bar{k}^n$ modulo the equivalence relation

$$(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n) \qquad \text{with } \lambda \in \bar{k} \setminus \{0\}.$$

A class is denoted by $(x_0 : \cdots : x_n)$.

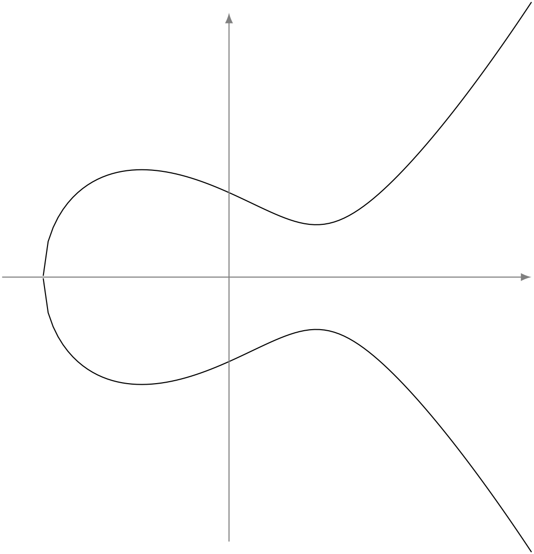


Weierstrass equations

Let k be a field of characteristic $\neq 2, 3$.
An elliptic curve *defined over* k is the locus in $\mathbb{P}^2(\bar{k})$ of an equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3,$$

where $a, b \in k$ and $4a^3 + 27b^2 \neq 0$.



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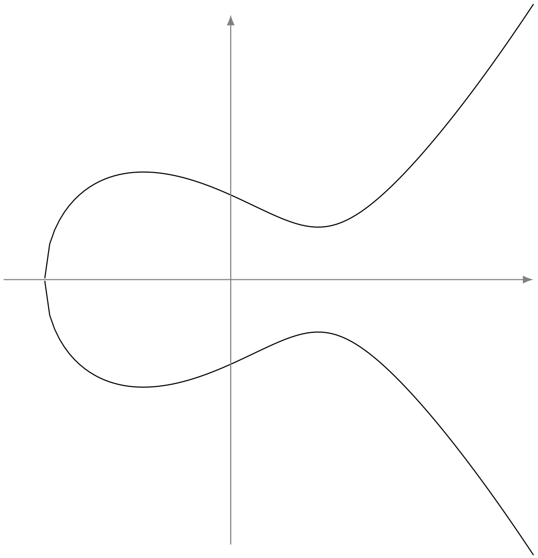

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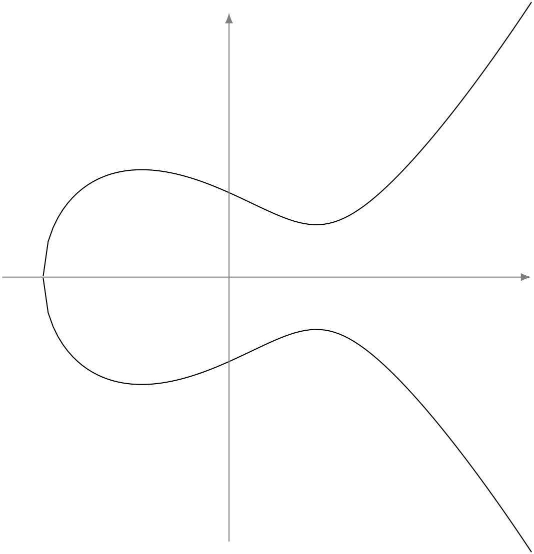
- $\mathcal{O} = (0 : 1 : 0)$ is the point at infinity;
- $y^2 = x^3 + ax + b$ is the affine equation.

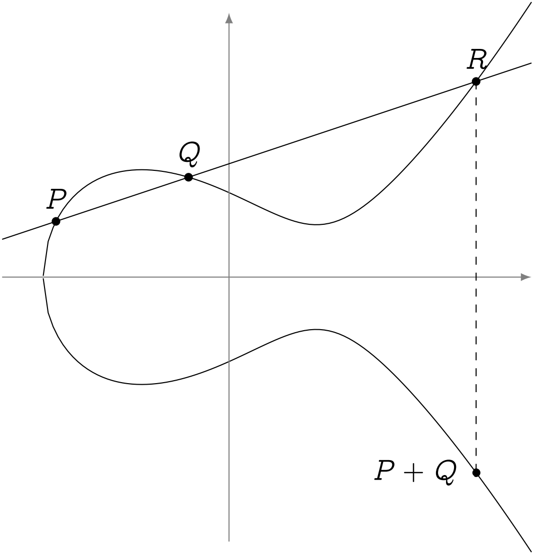


The group law

Bezout's theorem

Every line cuts E in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.



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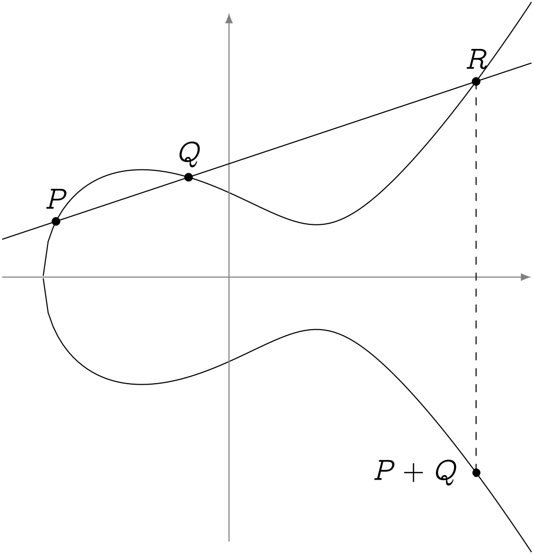

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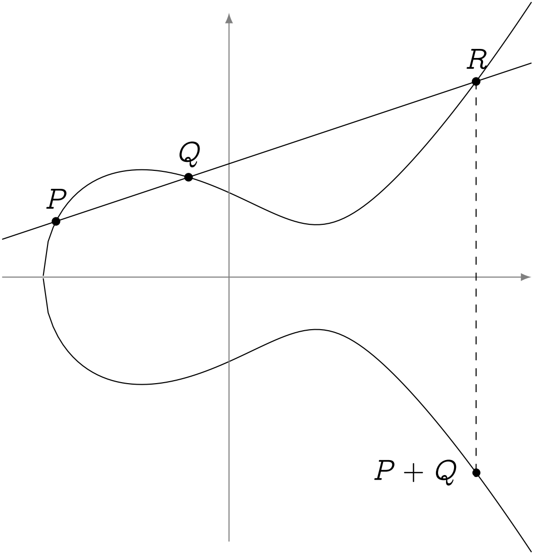
Define a group law such that any three colinear points add up to zero.

- The law is algebraic (it has *formulas*);
- The law is commutative;
- $\mathcal{O}$ is the group identity;
- Opposite points have the same x-value.



Group structure

Torsion structure

Let E be defined over an algebraically closed field $\bar{k}$ of characteristic p.

$$E[m] \simeq \quad \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \qquad \text{if } p \nmid m,$$

$$E[p^e] \simeq \begin{cases} \mathbb{Z}/p^e\mathbb{Z} & \text{ordinary case,} \\ \{\mathcal{O}\} & \text{supersingular case.} \end{cases}$$

Free part

Let E be defined over a number field k, the group of k-rational points $E(k)$ is finitely generated.

Maps: isomorphisms

Isomorphisms

The only invertible algebraic maps between elliptic curves are of the form

$$(x, y) \mapsto (u^2 x, u^3 y)$$

for some $u \in \bar{k}$.
They are group isomorphisms.

j-Invariant

Let $E \ : \ y^2 = x^3 + ax + b$, its j-invariant is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Two elliptic curves E, E' are isomorphic if and only if $j(E) = j(E')$.

Maps: isogenies

Theorem

Let $\phi : E \to E'$ be a map between elliptic curves. These conditions are equivalent:

- *ϕ is a surjective group morphism,*
- *ϕ is a group morphism with finite kernel,*
- *ϕ is a non-constant algebraic map of projective varieties sending the point at infinity of E onto the point at infinity of E'.*

If they hold ϕ is called an isogeny.

Two curves are called isogenous if there exists an isogeny between them.

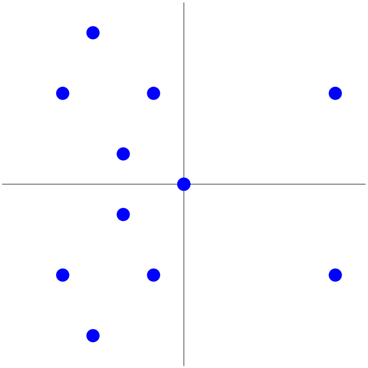
Example: Multiplication-by-m

On any curve, an isogeny from E to itself (i.e., an endomorphism):

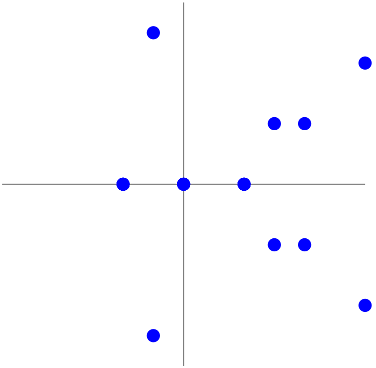
$$\begin{aligned}[m] \;:\; E &\to E, \\ P &\mapsto [m]P.\end{aligned}$$

Isogenies: an example over $\mathbb{F}_{11}$

$$E \ : \ y^2 = x^3 + x$$



$$E' \ : \ y^2 = x^3 - 4x$$



$$\phi(x,y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2}\right)$$

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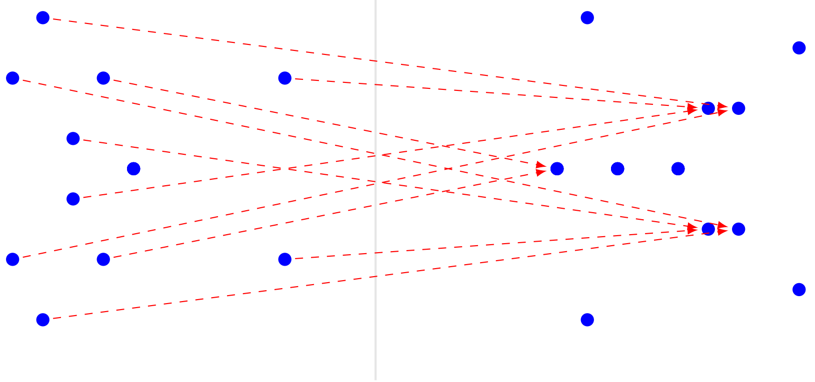

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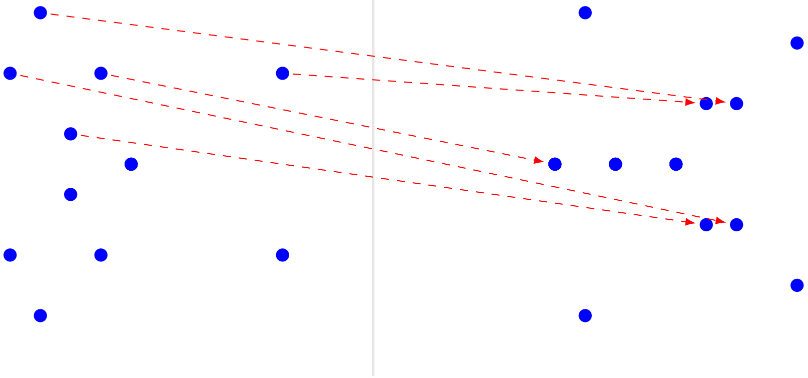

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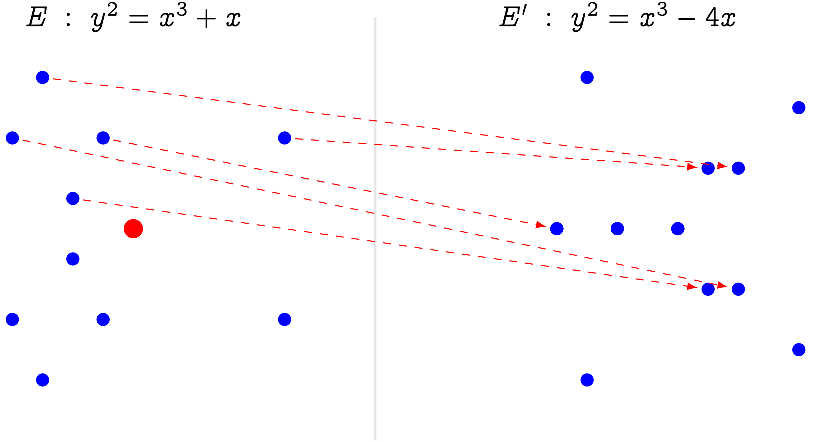

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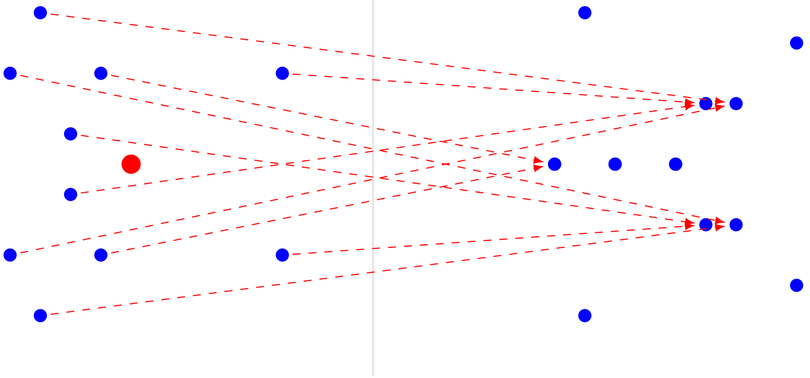

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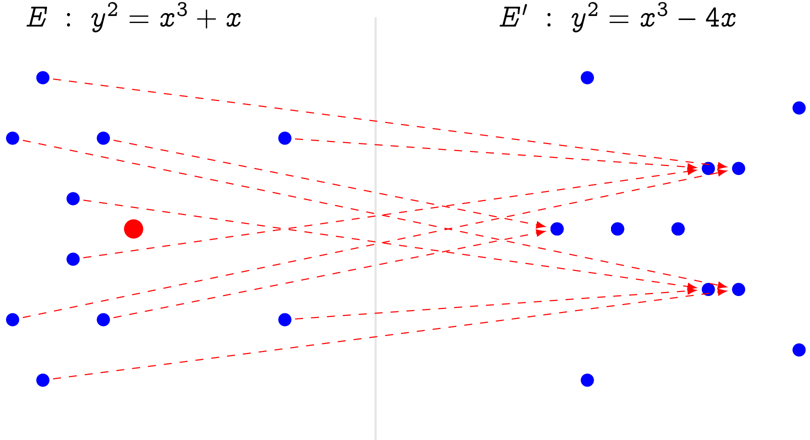


$$\phi(x, y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2}\right)$$

- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

Curves over finite fields

Frobenius endomorphism

Let E be defined over $\mathbb{F}_q$. The Frobenius endomorphism of E is the map

$$\pi : (X : Y : Z) \mapsto (X^q : Y^q : Z^q).$$

Hasse's theorem

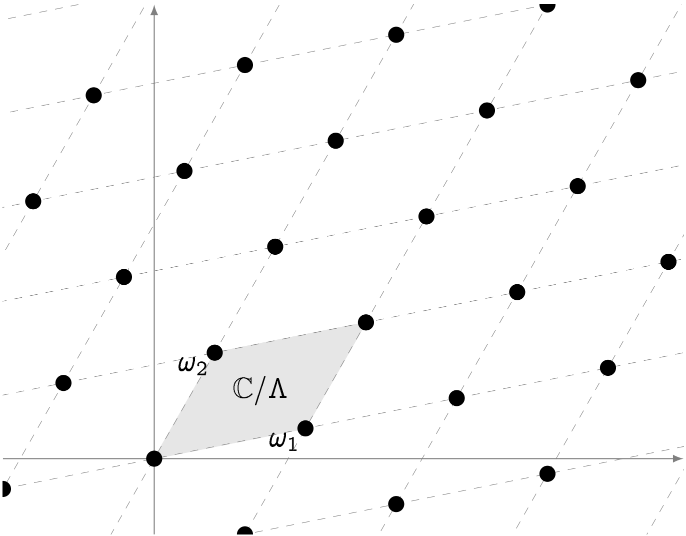
Let E be defined over $\mathbb{F}_q$, then

$$|\#E(k) - q - 1| \leq 2\sqrt{q}.$$

Serre-Tate theorem

Two elliptic curves E, E' defined over a finite field k are isogenous over k if and only if $\#E(k) = \#E'(k)$.

Complex tori

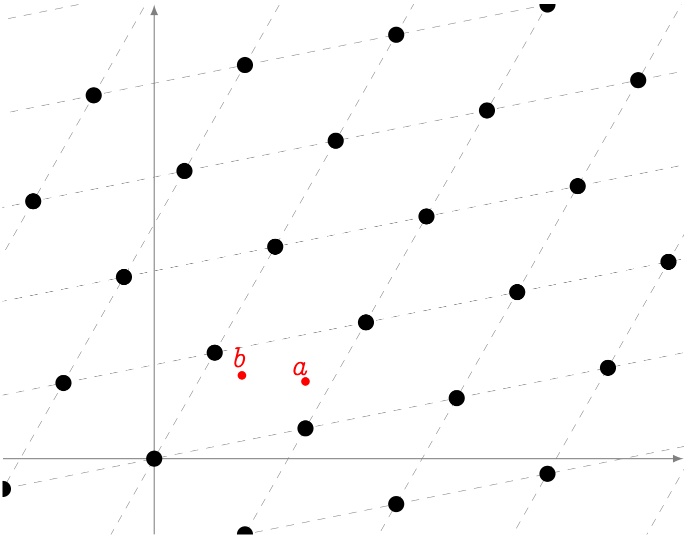


Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent complex numbers. Set

$$\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$$

$\mathbb{C}/\Lambda$ is a complex torus.

Complex tori



Addition law induced by addition on $\mathbb{C}$.

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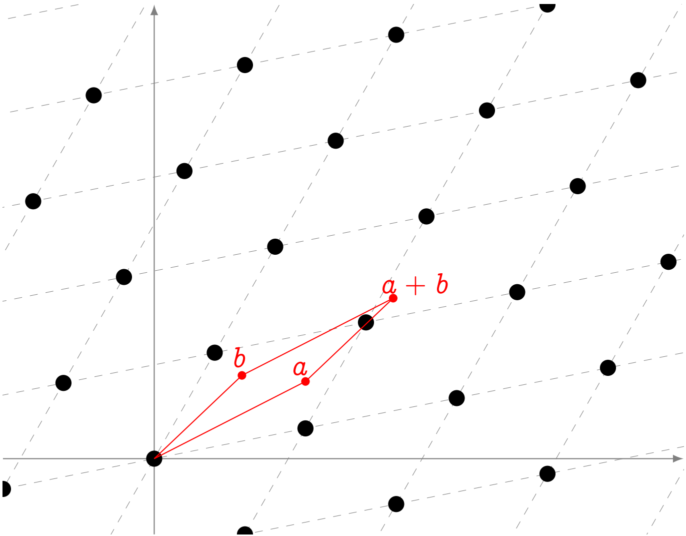

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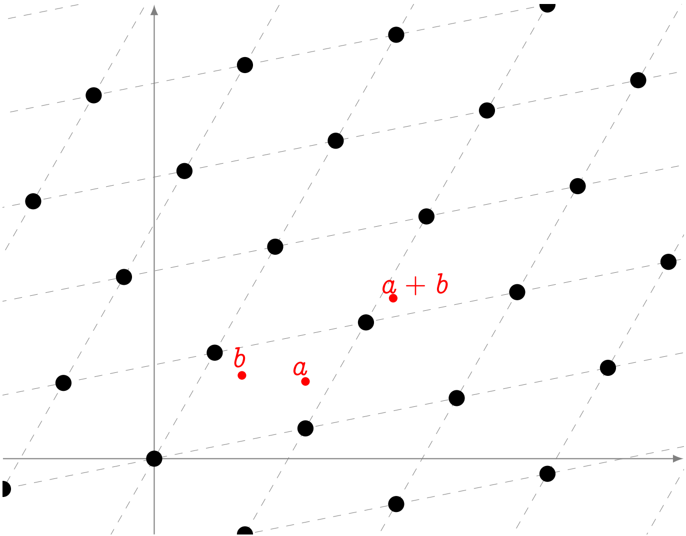

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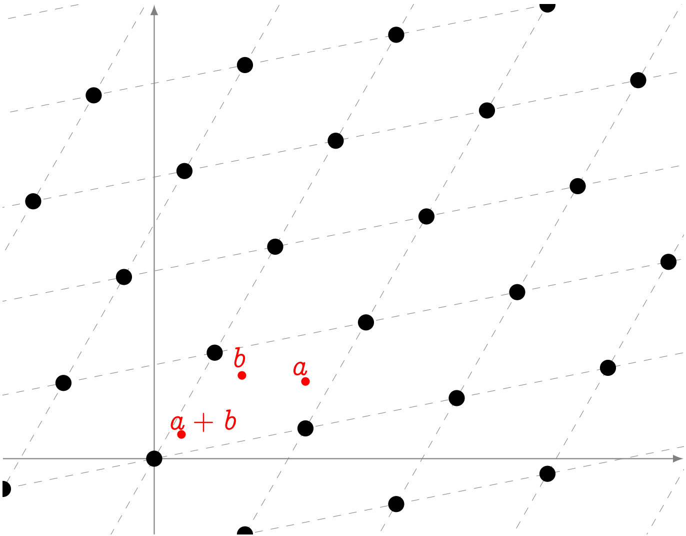

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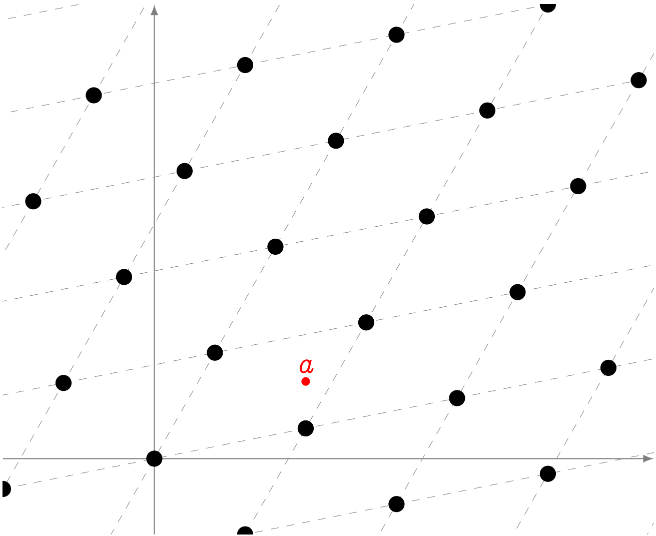
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

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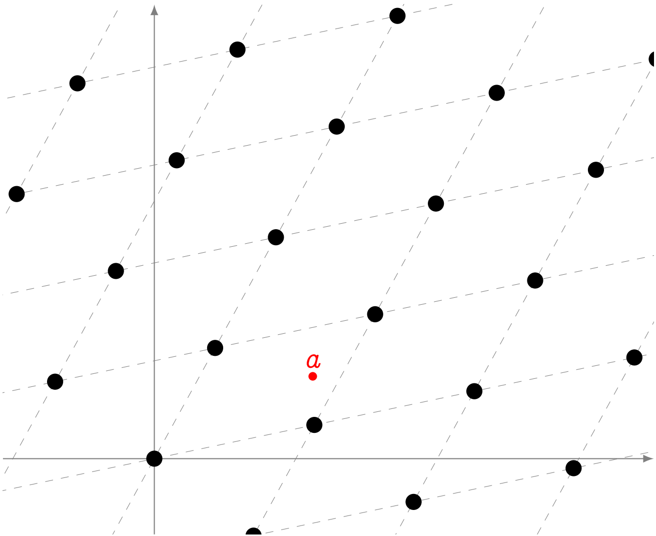

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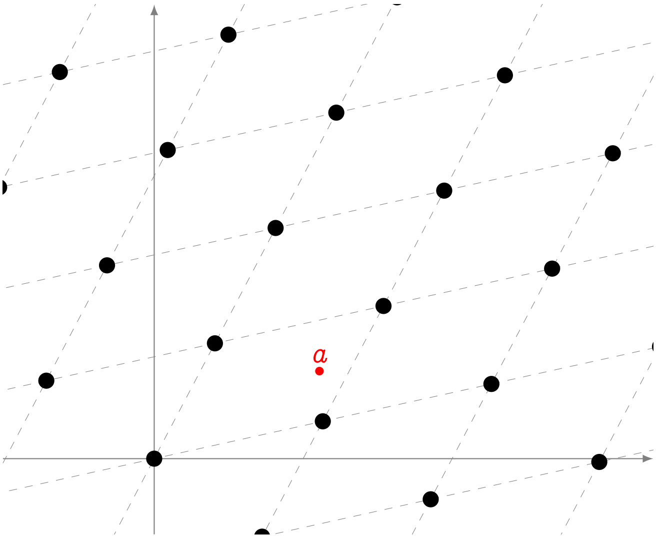

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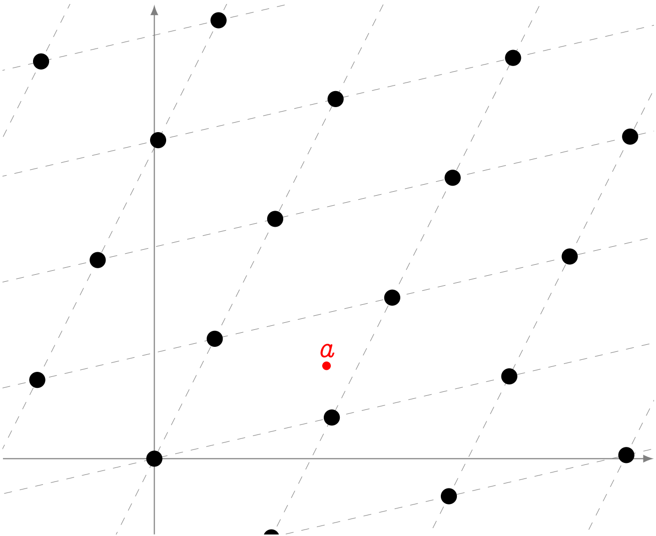

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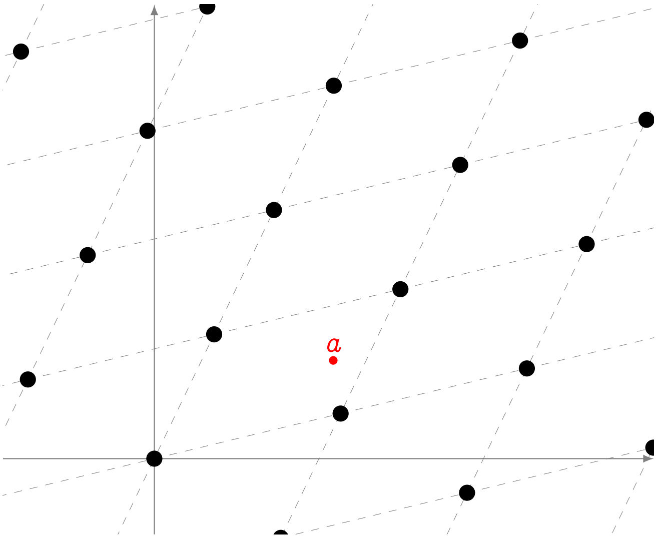

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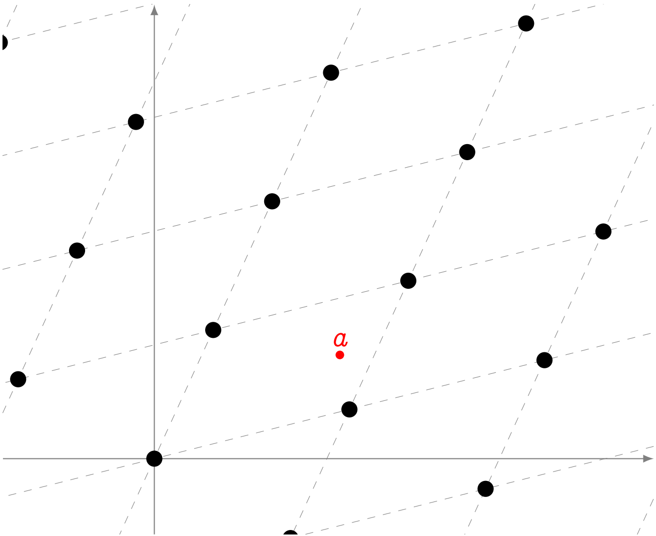

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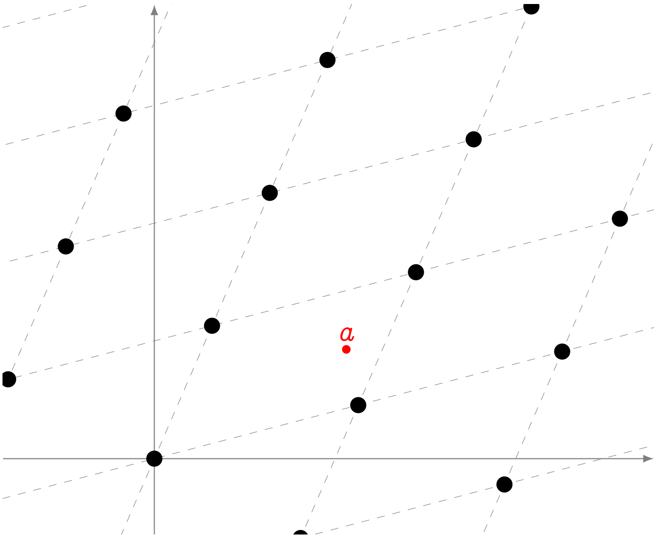

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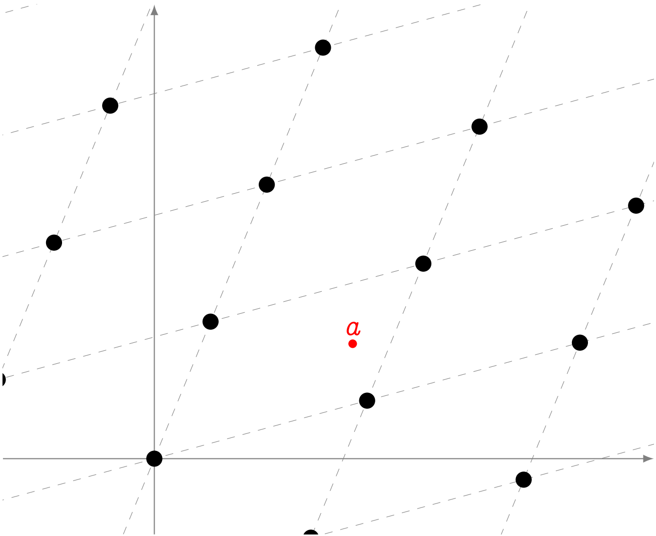

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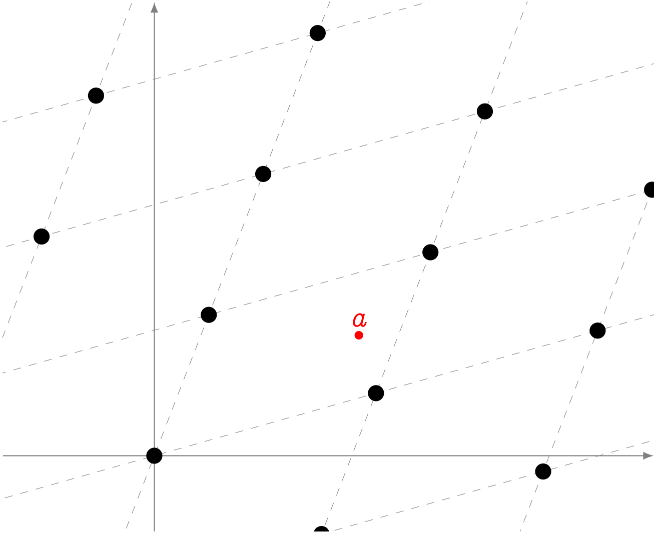

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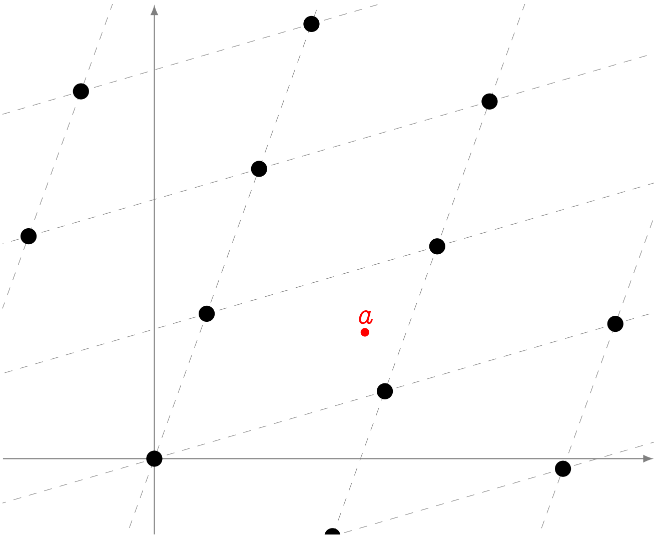

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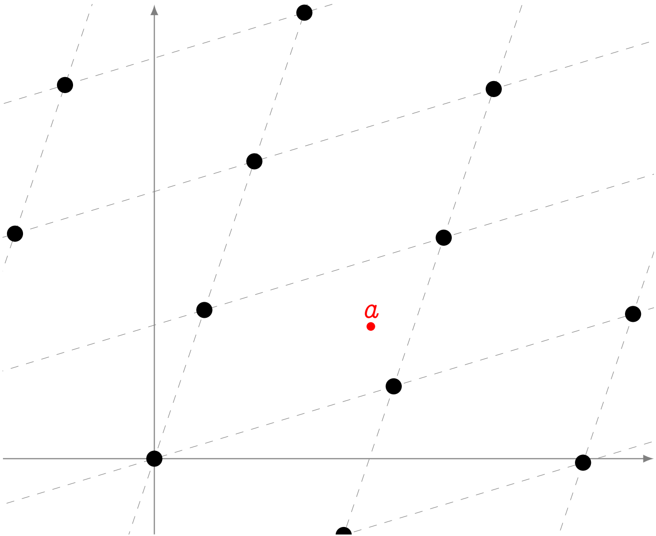

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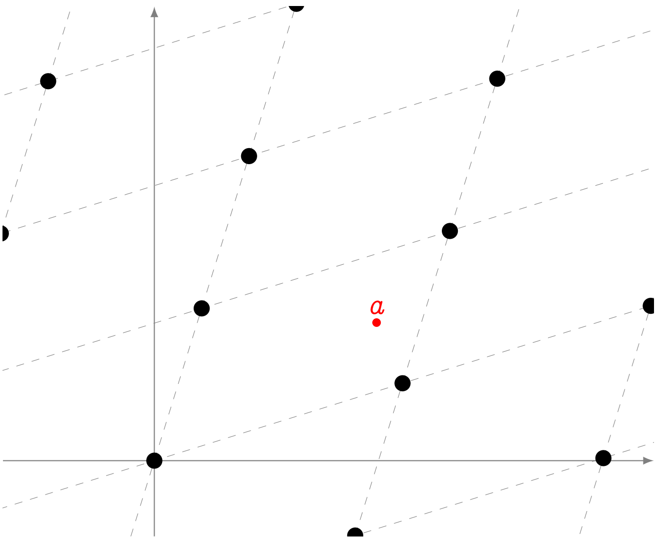

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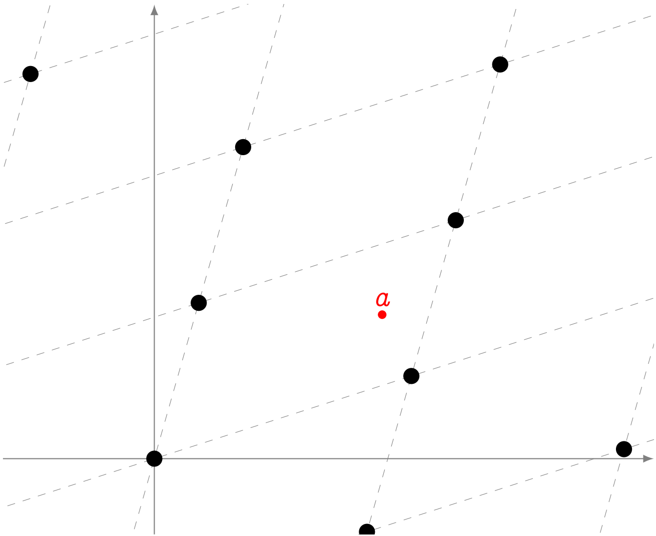

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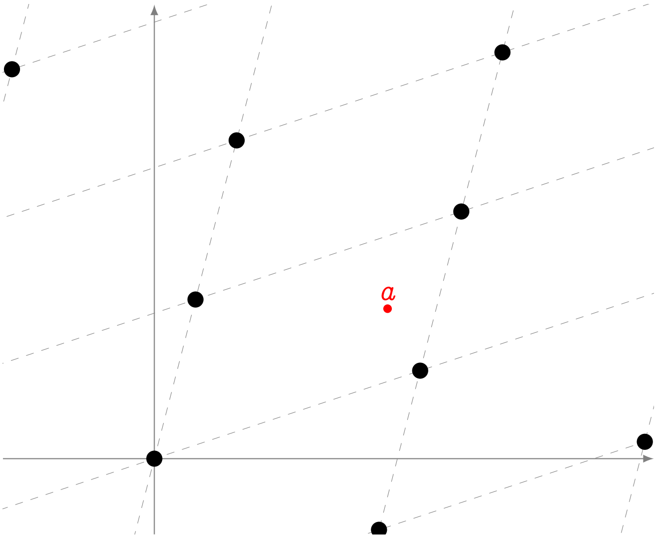

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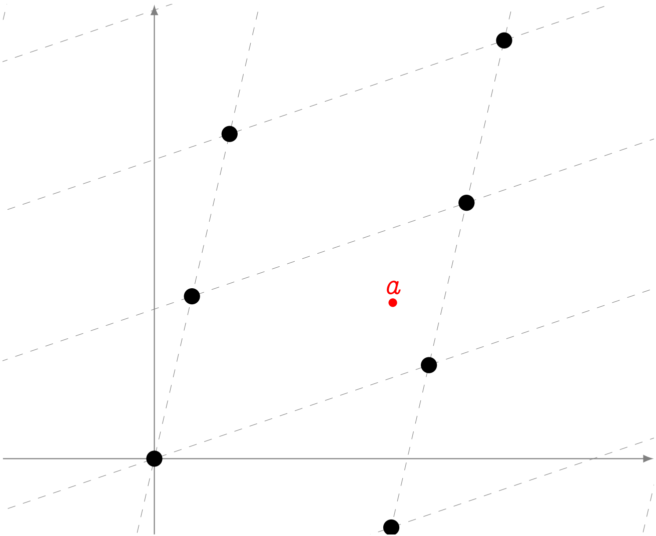

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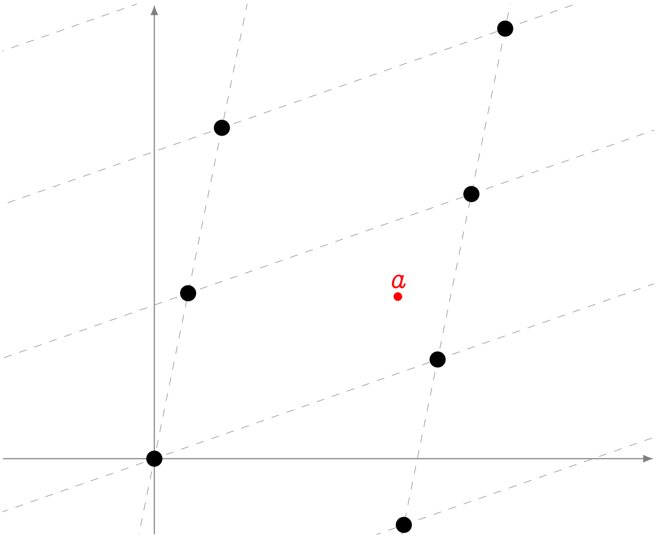

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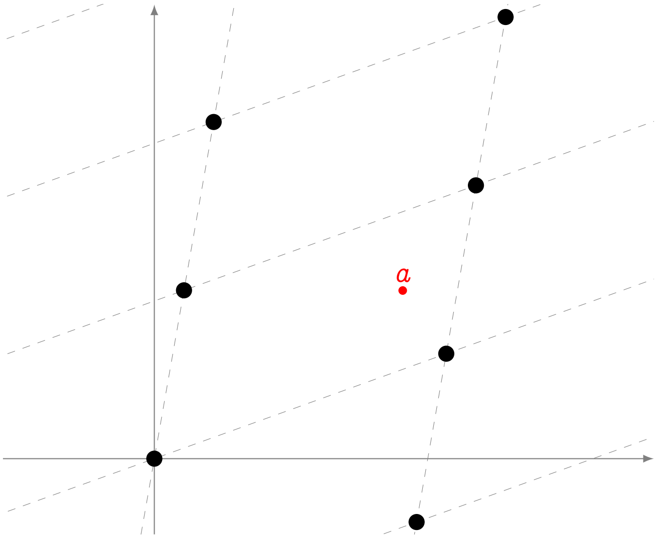

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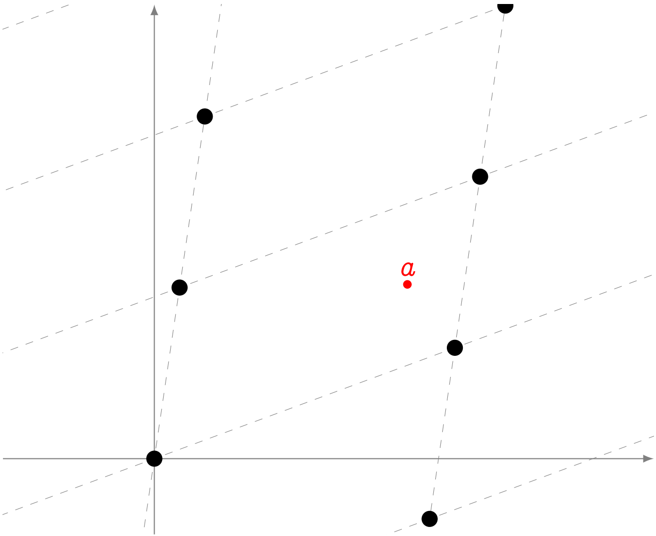

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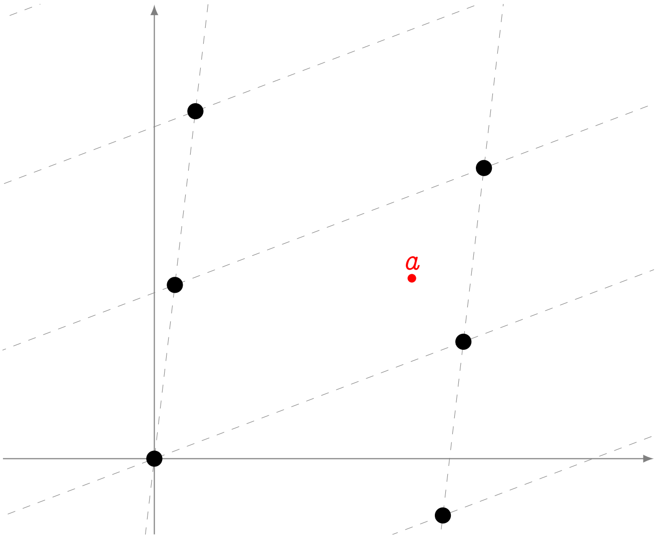

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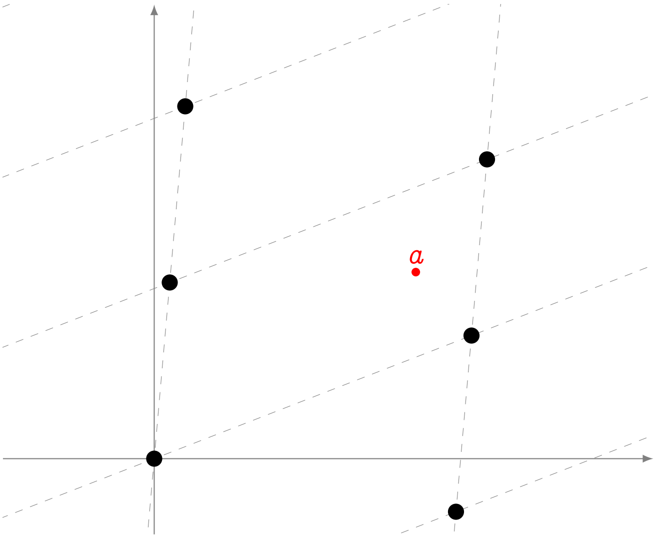

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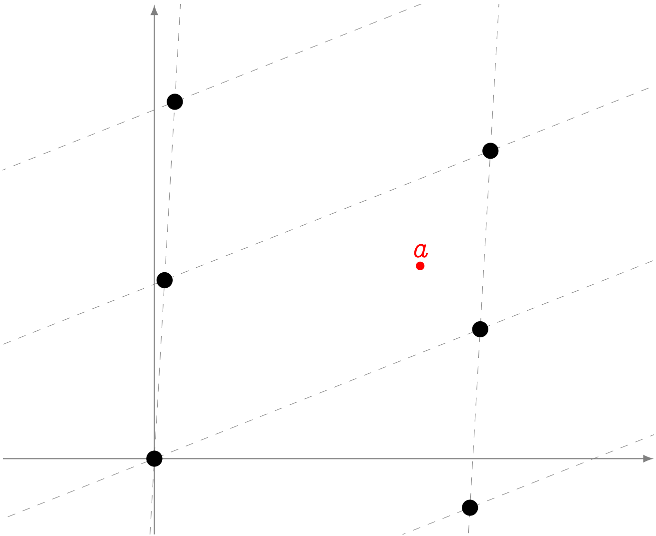

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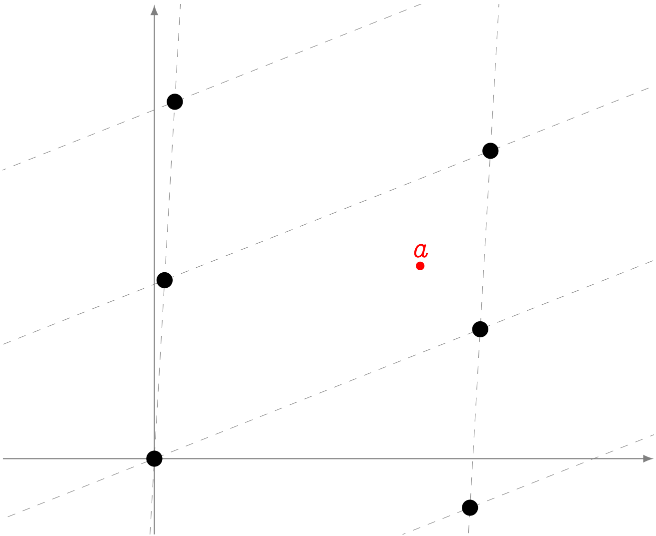

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The j-invariant

We want to classify complex lattices/tori up to homothety.

Eisenstein series

Let Λ be a complex lattice. For any integer $k > 0$ define

$$G_{2k}(\Lambda) = \sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-2k}.$$

Also set

$$g_2(\Lambda) = 60\,G_4(\Lambda), \qquad g_3(\Lambda) = 140\,G_6(\Lambda).$$

Modular j-invariant

Let Λ be a complex lattice, the modular j-invariant is

$$j(\Lambda) = 1728\frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27g_3(\Lambda)^2}.$$

Two lattices Λ, Λ' are homothetic if and only if $j(\Lambda) = j(\Lambda')$.

Elliptic curves over $\mathbb{C}$

Weierstrass $\wp$ function

Let Λ be a complex lattice, the Weierstrass $\wp$ function associated to Λ is the series

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

Fix a lattice Λ, then $\wp$ and its derivative $\wp'$ are elliptic functions:

$$\wp(z+\omega) = \wp(z), \qquad \wp'(z+\omega) = \wp'(z)$$

for all $\omega \in \Lambda$.

Uniformization theorem

Let Λ be a complex lattice. The curve

$$E \;:\; y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$$

is an elliptic curve over $\mathbb{C}$. The map

$$\begin{aligned} \mathbb{C}/\Lambda &\to E(\mathbb{C}), \\ 0 &\mapsto (0:1:0), \\ z &\mapsto (\wp(z) : \wp'(z) : 1) \end{aligned}$$

is an isomorphism of Riemann surfaces and a group morphism.

Conversely, for any elliptic curve

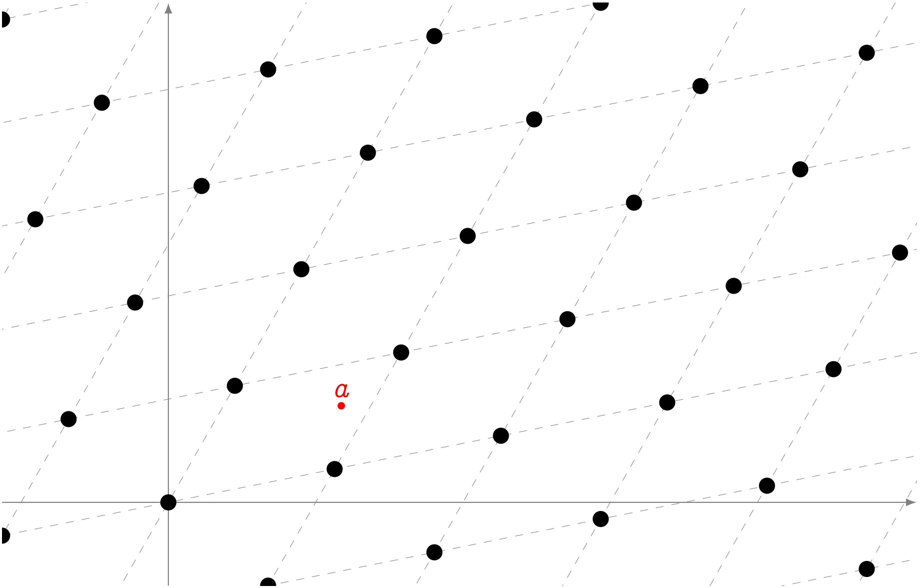
$$E \;:\; y^2 = x^3 + ax + b$$

there is a unique complex lattice Λ such that

$$g_2(\Lambda) = -4a, \qquad g_3(\Lambda) = -4b.$$

Moreover $j(\Lambda) = j(E)$.

Multiplication



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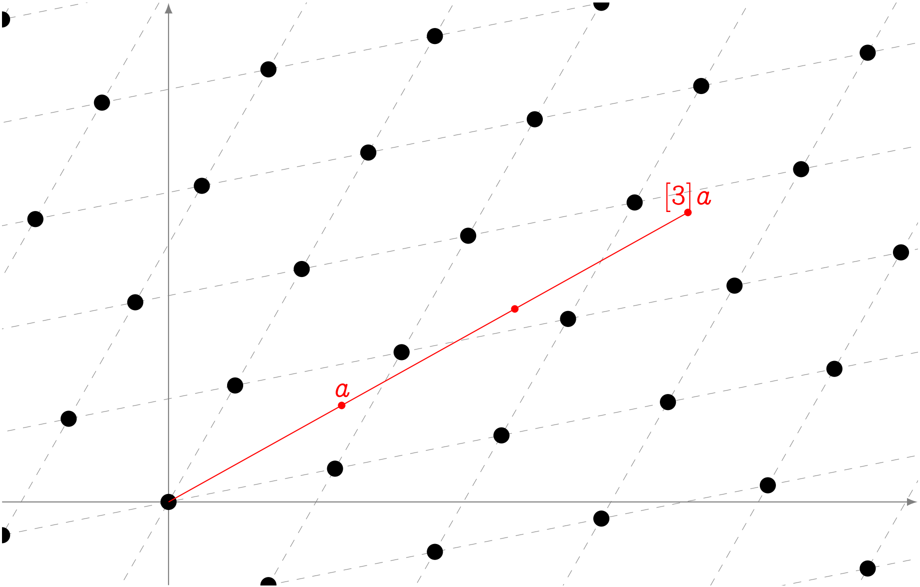
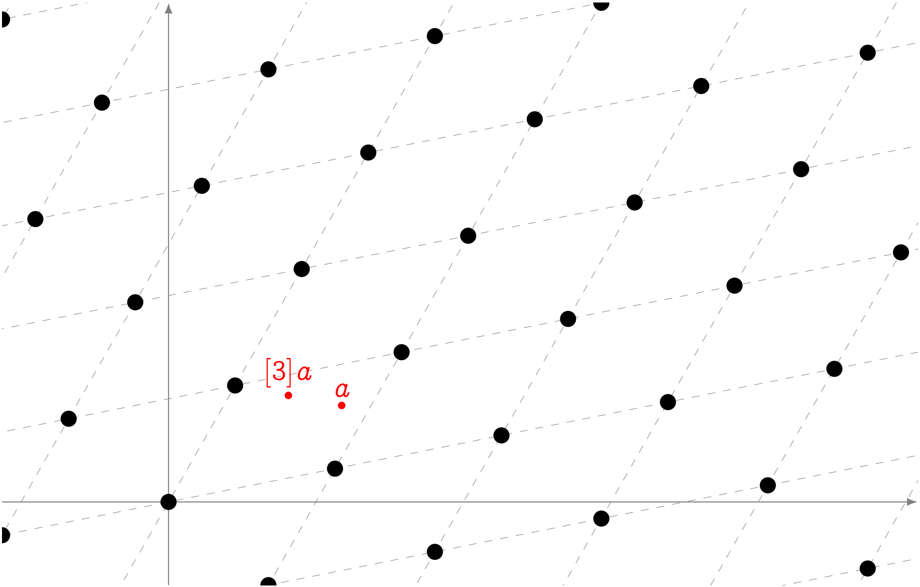
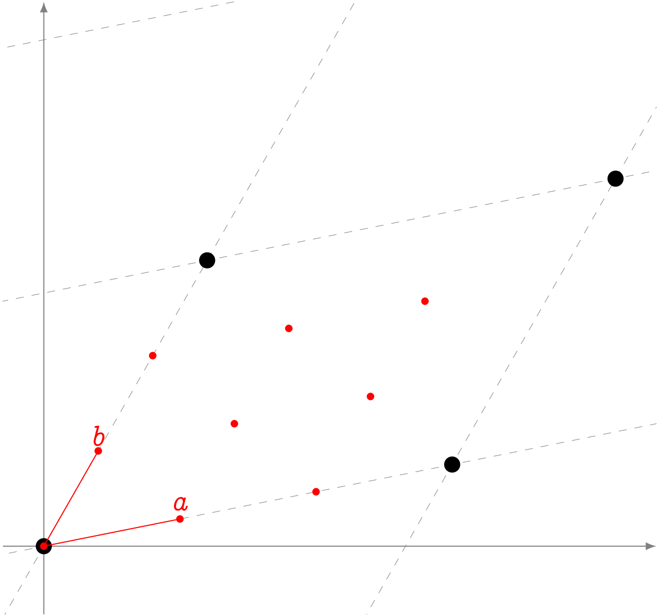


Multiplication



Torsion subgroups



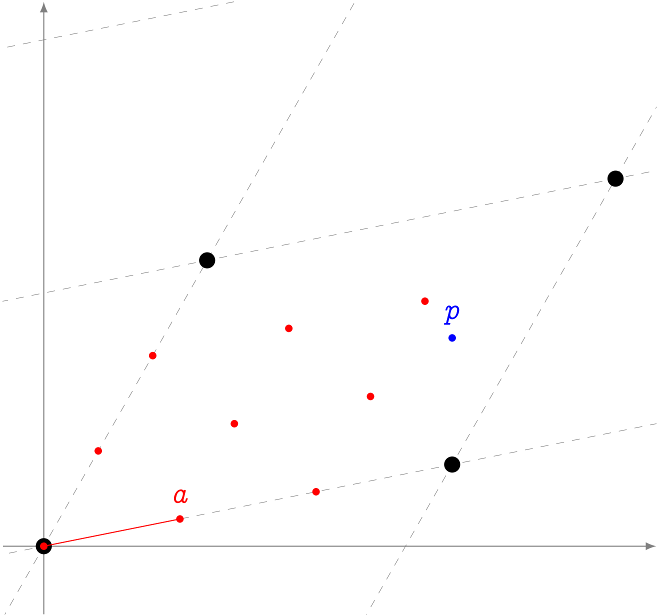
The ℓ-torsion subgroup is made up by the points

$$\left(\frac{i\omega_1}{\ell}, \frac{j\omega_2}{\ell}\right)$$

It is a group of rank two

$$E[\ell] = \langle a, b\rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

Isogenies



Let $a \in \mathbb{C}/\Lambda_1$ be an ℓ-torsion point, and let

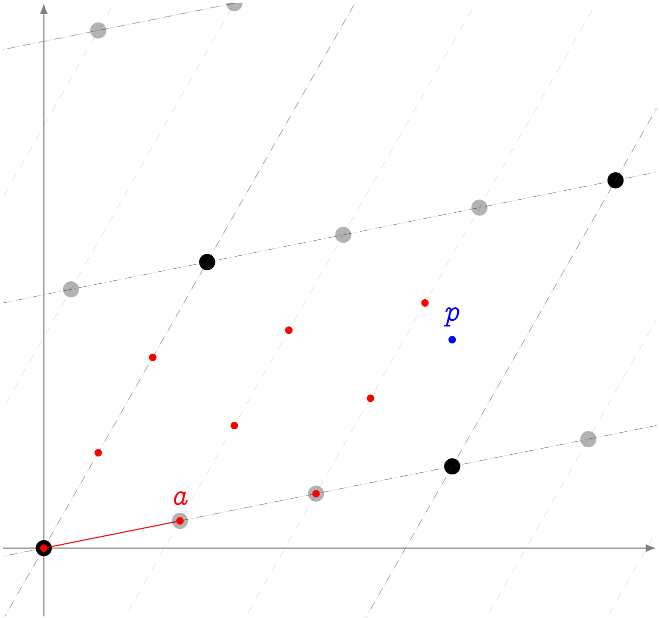
$$\Lambda_2 = a\mathbb{Z} \oplus \Lambda_1$$

Then $\Lambda_1 \subset \Lambda_2$ and we define a degree ℓ cover

$$\phi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$$

ϕ is a morphism of complex Lie groups and is called an isogeny.

Isogenies



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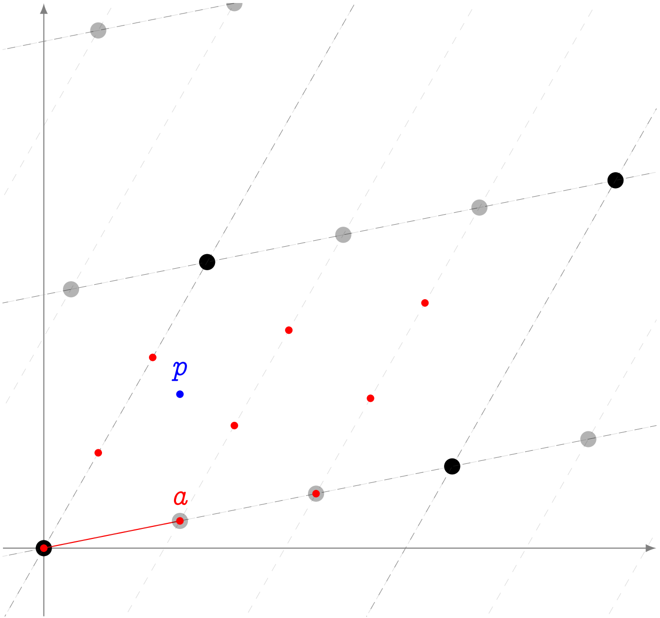

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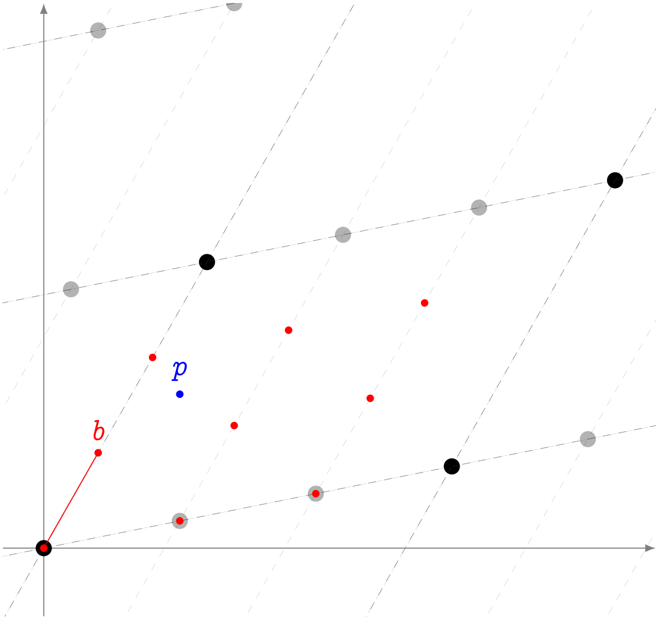


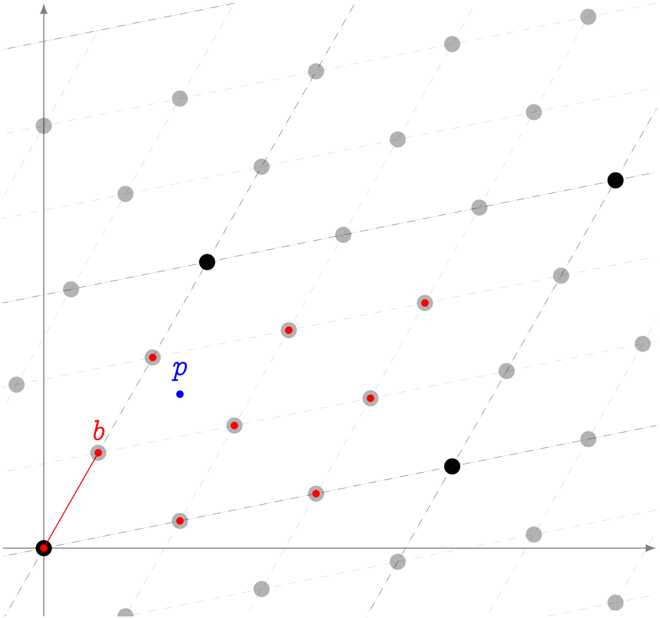
Taking a point b not in the kernel of ϕ, we obtain a new degree ℓ cover

$$\hat{\phi} : \mathbb{C}/\Lambda_2 \to \mathbb{C}/\Lambda_3$$

The composition $\hat{\phi} \circ \phi$ has degree ℓ^2 and is homothetic to the multiplication by ℓ map.

$\hat{\phi}$ is called the dual isogeny of ϕ.

Isogenies



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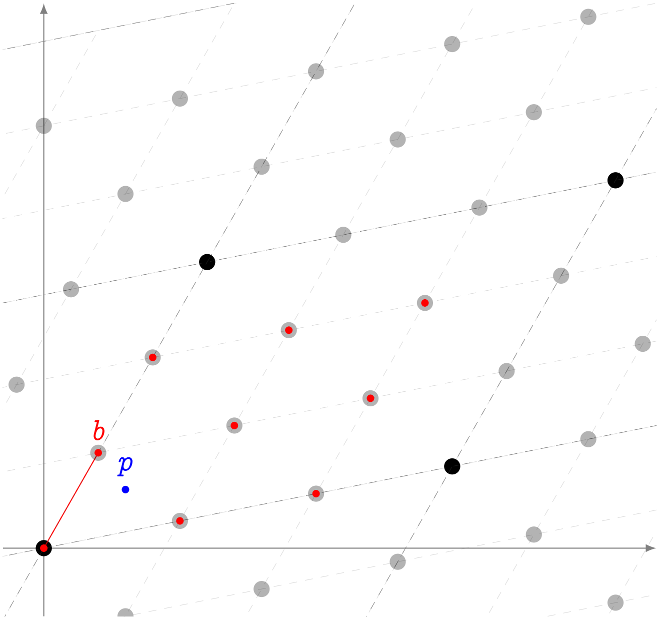

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Isogenies: back to algebra

Let $\phi : E \to E'$ be an isogeny defined over a field k of characteristic p.

- $k(E)$ is the field of all rational functions from E to k;
- $\phi^* k(E')$ is the subfield of $k(E)$ defined as

$$\phi^* k(E') = \{f \circ \phi \mid f \in k(E')\}.$$

Degree, separability

1. The degree of ϕ is $\deg \phi = [k(E) : \phi^* k(E')]$. It is always finite.
2. ϕ is said to be separable, inseparable, or purely inseparable if the extension of function fields is.
3. If ϕ is separable, then $\deg \phi = \# \ker \phi$.
4. If ϕ is purely inseparable, then $\ker \phi = \{\mathcal{O}\}$ and $\deg \phi$ is a power of p.
5. Any isogeny can be decomposed as a product of a separable and a purely inseparable isogeny.

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Isogenies: separable vs inseparable

Purely inseparable isogenies

Examples:

- The Frobenius endomorphism is purely inseparable of degree q.
- All purely inseparable maps in characteristic p are of the form $(X : Y : Z) \mapsto (X^{p^e} : Y^{p^e} : Z^{p^e})$.

Separable isogenies

Let E be an elliptic curve, and let G be a finite subgroup of E. There are a unique elliptic curve E' and a unique separable isogeny ϕ, such that $\ker\phi = G$ and $\phi : E \to E'$.
The curve E' is called the quotient of E by G and is denoted by E/G.

The dual isogeny

Let $\phi : E \to E'$ be an isogeny of degree m. There is a unique isogeny $\hat{\phi} : E' \to E$ such that

$$\hat{\phi} \circ \phi = [m]_E, \quad \phi \circ \hat{\phi} = [m]_{E'}.$$

$\hat{\phi}$ is called the dual isogeny of ϕ; it has the following properties:

1. $\hat{\phi}$ is defined over k if and only if ϕ is;
2. $\widehat{\psi \circ \phi} = \hat{\phi} \circ \hat{\psi}$ for any isogeny $\psi : E' \to E''$;
3. $\widehat{\psi + \phi} = \hat{\psi} + \hat{\phi}$ for any isogeny $\psi : E \to E'$;
4. $\deg \phi = \deg \hat{\phi}$;
5. $\hat{\hat{\phi}} = \phi$.

Algebras, orders

- A quadratic imaginary number field is an extension of $\mathbb{Q}$ of the form $Q[\sqrt{-D}]$ for some non-square $D > 0$.
- A quaternion algebra is an algebra of the form $\mathbb{Q} + \alpha\mathbb{Q} + \beta\mathbb{Q} + \alpha\beta\mathbb{Q}$, where the generators satisfy the relations

$$\alpha^2, \beta^2 \in \mathbb{Q}, \quad \alpha^2 < 0, \quad \beta^2 < 0, \quad \beta\alpha = -\alpha\beta.$$

Orders

Let K be a finitely generated $\mathbb{Q}$-algebra. An order $\mathcal{O} \subset K$ is a subring of K that is a finitely generated $\mathbb{Z}$-module of maximal dimension. An order that is not contained in any other order of K is called a maximal order.

Examples:

- $\mathbb{Z}$ is the only order contained in $\mathbb{Q}$,
- $\mathbb{Z}[i]$ is the only maximal order of $\mathbb{Q}(i)$,
- $\mathbb{Z}[\sqrt{5}]$ is a non-maximal order of $\mathbb{Q}(\sqrt{5})$,
- The ring of integers of a number field is its only maximal order,
- In general, maximal orders in quaternion algebras are not unique.

The endomorphism ring

The endomorphism ring $\mathrm{End}(E)$ of an elliptic curve E is the ring of all isogenies $E \to E$ (plus the null map) with addition and composition.

Theorem (Deuring)

Let E be an elliptic curve defined over a field k of characteristic p. $\mathrm{End}(E)$ is isomorphic to one of the following:

- $\mathbb{Z}$, only if $p = 0$

  E is ordinary.
- An order $\mathcal{O}$ in a quadratic imaginary field:

  E is ordinary with complex multiplication by $\mathcal{O}$.
- Only if $p > 0$, a maximal order in a quaternion algebra[a]:

  E is supersingular.

[a] (ramified at p and ∞)

The finite field case

Theorem (Hasse)

Let E be defined over a finite field. Its Frobenius endomorphism π satisfies a quadratic equation

$$\pi^2 - t\pi + q = 0$$

in $\mathrm{End}(E)$ for some $|t| \leq 2\sqrt{q}$, called the trace of π. The trace t is coprime to q if and only if E is ordinary.

Suppose E is ordinary, then $D_\pi = t^2 - 4q < 0$ is the discriminant of $\mathbb{Z}[\pi]$.

- $K = \mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{D_\pi})$ is the endomorphism algebra of E.
- Denote by $\mathcal{O}_K$ its ring of integers, then

$$\mathbb{Z} \neq \mathbb{Z}[\pi] \subset \mathrm{End}(E) \subset \mathcal{O}_K.$$

In the supersingular case, π may or may not be in $\mathbb{Z}$, depending on q.

Endomorphism rings of ordinary curves

Classifying quadratic orders

Let K be a quadratic number field, and let $\mathcal{O}_K$ be its ring of integers.

- Any order $\mathcal{O} \subset K$ can be written as $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ for an integer f, called the conductor of $\mathcal{O}$, denoted by $[\mathcal{O}_k : \mathcal{O}]$.
- If d_K is the discriminant of K, the discriminant of $\mathcal{O}$ is $f^2 d_K$.
- If $\mathcal{O}, \mathcal{O}'$ are two orders with discriminants d, d', then $\mathcal{O} \subset \mathcal{O}'$ iff $d' | d$.

$$
\begin{tikzcd}[arrows=dash]
 & \mathcal{O}_K \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbb{Z} + 2\mathcal{O}_K \arrow[d] \arrow[dr] & \mathbb{Z} + 3\mathcal{O}_K \arrow[dl] \arrow[dr] & \mathbb{Z} + 5\mathcal{O}_K \arrow[dl] \arrow[d] \\
\mathbb{Z} + 6\mathcal{O}_K \arrow[dr] & \mathbb{Z} + 10\mathcal{O}_K \arrow[d] & \mathbb{Z} + 15\mathcal{O}_K \arrow[dl] \\
 & \mathbb{Z}[\pi] \simeq \mathbb{Z} + 30\mathcal{O}_K &
\end{tikzcd}
$$

Ideal lattices

Fractional ideals

Let $\mathcal{O}$ be an order of a number field K. A (fractional) $\mathcal{O}$-ideal $\mathfrak{a}$ is a finitely generated non-zero $\mathcal{O}$-submodule of K.

When K is imaginary quadratic:

- Fractional ideals are complex lattices,
- Any lattice $\Lambda \subset K$ is a fractional ideal,
- The order of a lattice Λ is

$$\mathcal{O}_\Lambda = \{\alpha \in K \mid \alpha\Lambda \subset \Lambda\}$$

Complex multiplication

Let $\Lambda \subset K$, the elliptic curve associated to $\mathbb{C}/\Lambda$ has complex multiplication by $\mathcal{O}_\Lambda$.

The class group

Let $\mathrm{End}(E) = \mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$. Define

- $\mathcal{I}(\mathcal{O})$, the group of invertible fractional ideals,
- $\mathcal{P}(\mathcal{O})$, the group of principal ideals,

The class group

The class group of $\mathcal{O}$ is

$$\mathrm{Cl}(\mathcal{O}) = \mathcal{I}(\mathcal{O})/\mathcal{P}(\mathcal{O}).$$

- It is a finite abelian group.
- Its order $h(\mathcal{O})$ is called the class number of $\mathcal{O}$.
- It arises as the Galois group of an abelian extension of $\mathbb{Q}(\sqrt{-D})$.

Complex multiplication

Fundamental theorem of CM

Let $\mathcal{O}$ be an order of a number field K, and let $\mathfrak{a}_1, \dots, \mathfrak{a}_{h(\mathcal{O})}$ be representatives of $\mathrm{Cl}(\mathcal{O})$. Then:

- $K(j(\mathfrak{a}_i))$ is an Abelian extension of K;
- The $j(\mathfrak{a}_i)$ are all conjugate over K;
- The Galois group of $K(j(\mathfrak{a}_i))$ is isomorphic to $\mathrm{Cl}(\mathcal{O})$;
- $[\mathbb{Q}(j(\mathfrak{a}_i)) : \mathbb{Q}] = [K(j(\mathfrak{a}_i)) : K] = h(\mathcal{O})$;
- The $j(\mathfrak{a}_i)$ are integral, their minimal polynomial is called the Hilbert class polynomial of $\mathcal{O}$.

Lifting

Deuring's lifting theorem

Let E_0 be an elliptic curve in characteristic p, with an endomorphism ω_o which is not trivial. Then there exists an elliptic curve E defined over a number field L, an endomorphism ω of E, and a non-singular reduction of E at a place $\mathfrak{p}$ of L lying above p, such that E_0 is isomorphic to $E(\mathfrak{p})$, and ω_0 corresponds to $\omega(\mathfrak{p})$ under the isomorphism.

Executive summary

- Elliptic curves are algebraic groups;
- Isogenies are the natural notion of morphism for EC: both group and projective variety morphism;
- We can understand most things about isogenies by looking only at endomorphisms;
- Isogenies of curves over $\mathbb{C}$ are especially simple to describe;
- It is easy to construct curves over $\mathbb{C}$ with prescribed complex multiplication;
- Most of what happens in positive characteristic can be understood by:
  - looking at the Frobenius endomorphism, and/or
  - looking at reductions of curves in characteristic 0.

Plan

1. Elliptic curves, isogenies, complex multiplication
2. Isogeny graphs
3. Key exchange
4. Signatures and whatnot

Isogeny graphs

Serre-Tate theorem reloaded

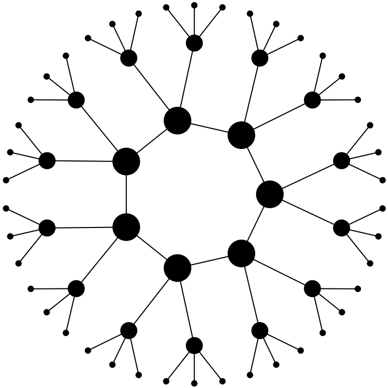
Two elliptic curves E, E' defined over a finite field are isogenous iff their endomorphism algebras $\mathrm{End}(E) \otimes \mathbb{Q}$ and $\mathrm{End}(E') \otimes \mathbb{Q}$ are isomorphic.

Isogeny graphs

- Vertices are curves up to isomorphism,
- Edges are isogenies up to isomorphism.

Isogeny volcanoes

- Curves are ordinary,
- Isogenies all have degree a prime ℓ.



What do isogeny graphs look like?

Torsion subgroups (ℓ prime)

In an algebraically closed field:

$$E[\ell] = \langle P, Q \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

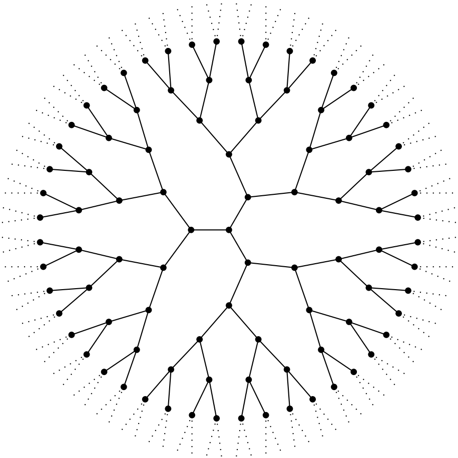
$$\Downarrow$$

There are exactly $\ell + 1$ cyclic subgroups $H \subset E$ of order ℓ:

$$\langle P + Q \rangle, \langle P + 2Q \rangle, \dots, \langle P \rangle, \langle Q \rangle$$

$$\Downarrow$$

There are exactly $\ell + 1$ distinct isogenies of degree ℓ.



(non-CM) 2-isogeny graph over $\mathbb{C}$

What happens over a finite field $\mathbb{F}_p$?

Rational isogenies ($\ell \neq p$)

In the algebraic closure $\bar{\mathbb{F}}_p$

$$E[\ell] = \langle P, Q\rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

However, an isogeny is defined over $\mathbb{F}_p$ only if its kernel is Galois invariant.

Enter the Frobenius map

$$\begin{aligned} \pi : E &\longrightarrow E \\ (x, y) &\longmapsto (x^p, y^p) \end{aligned}$$

E is seen here as a curve over $\bar{\mathbb{F}}_p$.

The Frobenius action on $E[\ell]$

$$\begin{aligned} \pi(P) &= aP + bQ \\ \pi(Q) &= cP + dQ \end{aligned}$$

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The Frobenius action on $E[\ell]$

$$\pi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod \ell$$

We identify $\pi|E[\ell]$ to a conjugacy class in $\mathrm{GL}(\mathbb{Z}/\ell\mathbb{Z})$.

What happens over a finite field $\mathbb{F}_p$?

Galois invariant subgroups of $E[\ell]$

=

eigenspaces of $\pi \in \mathrm{GL}(\mathbb{Z}/\ell\mathbb{Z})$

=

rational isogenies of degree ℓ

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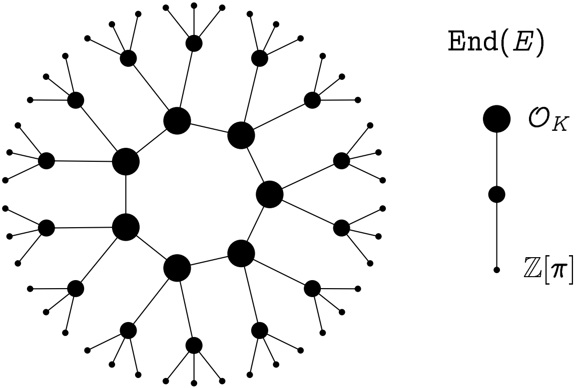
How many Galois invariant subgroups?

- $\pi|E[\ell] \sim \left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right)$ $\rightarrow$ $\ell + 1$ isogenies
- $\pi|E[\ell] \sim \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ with $\lambda \neq \mu$ $\rightarrow$ two isogenies
- $\pi|E[\ell] \sim \left(\begin{smallmatrix}\lambda & *\\ 0 & \lambda\end{smallmatrix}\right)$ $\rightarrow$ one isogeny
- $\pi|E[\ell]$ is not diagonalizable over $\mathbb{Z}/\ell\mathbb{Z}$ $\rightarrow$ no isogeny

Volcanology (Kohel 1996)

Let E, E' be curves with respective endomorphism rings $\mathcal{O}, \mathcal{O}' \subset K$.
Let $\phi : E \to E'$ be an isogeny of prime degree ℓ, then:

if $\mathcal{O} = \mathcal{O}'$, ϕ is horizontal;
if $[\mathcal{O}' : \mathcal{O}] = \ell$, ϕ is ascending;
if $[\mathcal{O} : \mathcal{O}'] = \ell$, ϕ is descending.

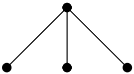


Ordinary isogeny volcano of degree $\ell = 3$.

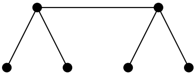
Volcanology (Kohel 1996)

Let E be ordinary,
$\mathtt{End}(E) \subset K$.

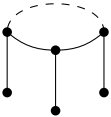
$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.



$\left(\frac{D_K}{\ell}\right) = -1$



$\left(\frac{D_K}{\ell}\right) = 0$



$\left(\frac{D_K}{\ell}\right) = +1$

		Horizontal	**Ascending**	**Descending**
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		
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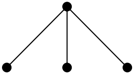
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	ℓ
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

Volcanology (Kohel 1996)

Let E be ordinary,
$\mathtt{End}(E) \subset K$.

$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.

Height $= v_\ell([\mathcal{O}_K : \mathbb{Z}[\pi]])$.



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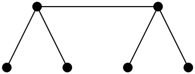

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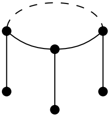

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$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		$\ell - \left(\frac{D_K}{\ell}\right)$
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	ℓ
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

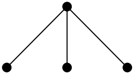
Volcanology (Kohel 1996)

Let E be ordinary,
$\mathrm{End}(E) \subset K$.

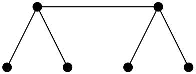
$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.

Height $= v_\ell([\mathcal{O}_K : \mathbb{Z}[\pi]])$.

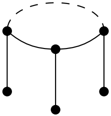
How large is the crater?



$\left(\frac{D_K}{\ell}\right) = -1$



$\left(\frac{D_K}{\ell}\right) = 0$



$\left(\frac{D_K}{\ell}\right) = +1$

		Horizontal	Ascending	Descending
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		$\ell - \left(\frac{D_K}{\ell}\right)$
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	ℓ
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

How large is the crater of a volcano?

Let $\mathrm{End}(E) = \mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$. Define

- $\mathcal{I}(\mathcal{O})$, the group of invertible fractional ideals,
- $\mathcal{P}(\mathcal{O})$, the group of principal ideals,

The class group

The class group of $\mathcal{O}$ is

$$\mathrm{Cl}(\mathcal{O}) = \mathcal{I}(\mathcal{O})/\mathcal{P}(\mathcal{O}).$$

- It is a finite abelian group.
- Its order $h(\mathcal{O})$ is called the class number of $\mathcal{O}$.
- It arises as the Galois group of an abelian extension of $\mathbb{Q}(\sqrt{-D})$.

Complex multiplication

The $\mathfrak{a}$-torsion

- Let $\mathfrak{a} \subset \mathcal{O}$ be an (integral invertible) ideal of $\mathcal{O}$;
- Let $E[\mathfrak{a}]$ be the subgroup of E annihilated by $\mathfrak{a}$:

$$E[\mathfrak{a}] = \{P \in E \mid \alpha(P) = 0 \text{ for all } \alpha \in \mathfrak{a}\};$$

- Let $\phi : E \to E_{\mathfrak{a}}$, where $E_{\mathfrak{a}} = E/E[\mathfrak{a}]$.

Then $\mathrm{End}(E_{\mathfrak{a}}) = \mathcal{O}$ (i.e., ϕ is horizontal).

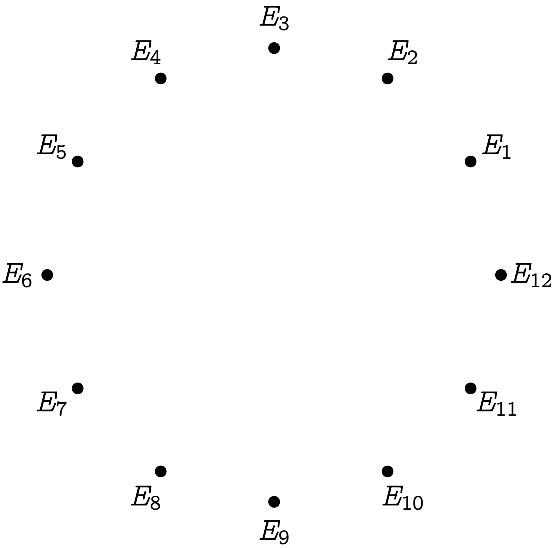
Theorem (Complex multiplication)

The action on the set of elliptic curves with complex multiplication by $\mathcal{O}$ defined by $\mathfrak{a} * j(E) = j(E_{\mathfrak{a}})$ factors through $\mathrm{Cl}(\mathcal{O})$, is faithful and transitive.

Corollary

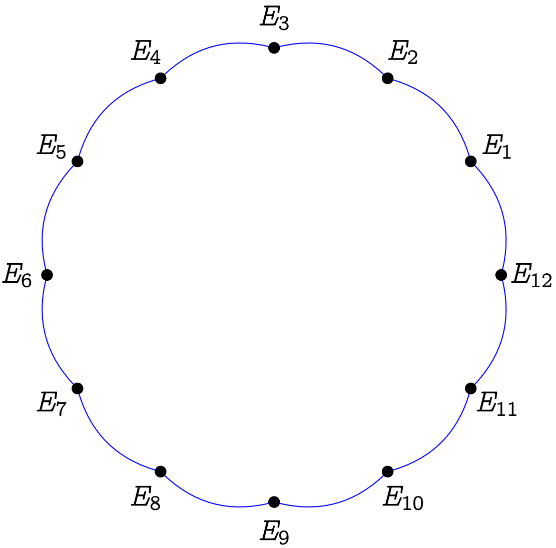
Let $\mathrm{End}(E)$ have discriminant D. Assume that $\left(\frac{D}{\ell}\right) = 1$, then E is on a crater of size N of an ℓ-volcano, and $N | h(\mathrm{End}(E))$

Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathtt{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

Complex multiplication graphs

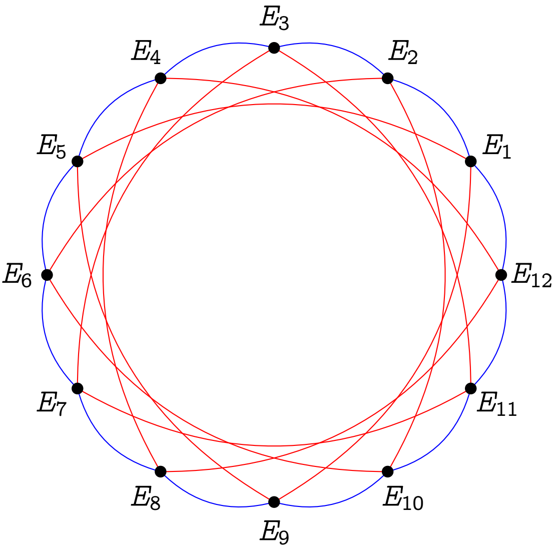


Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

Edges are horizontal isogenies of bounded prime degree.

— degree 2

Complex multiplication graphs



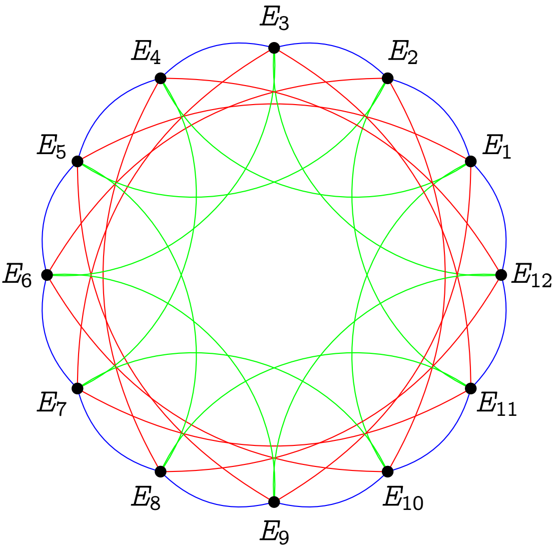
Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

Complex multiplication graphs



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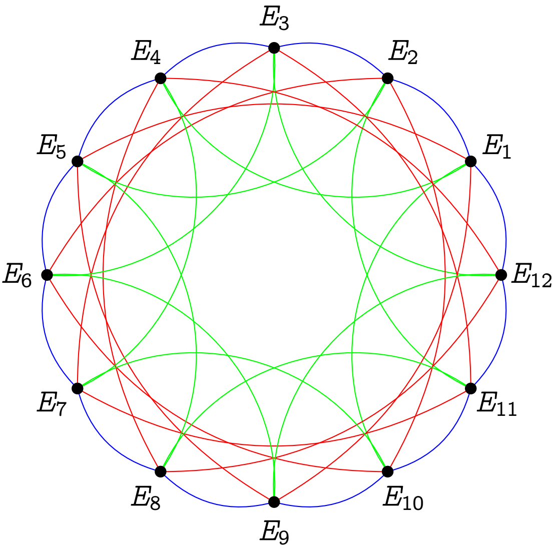
Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

— degree 5

Isomorphic to a Cayley graph of $\mathrm{Cl}(\mathcal{O}_K)$.

Supersingular endomorphisms

Recall, a curve E over a field $\mathbb{F}_q$ of characteristic p is supersingular iff

$$\pi^2 - t\pi + q = 0$$

with $t = 0 \mod p$.

Case: $t = 0 \quad \Rightarrow \quad D_\pi = -4q$

- Only possibility for $E/\mathbb{F}_p$,
- $E/\mathbb{F}_p$ has CM by an order of $\mathbb{Q}(\sqrt{-p})$, similar to the ordinary case.

Case: $t = \pm 2\sqrt{q} \quad \Rightarrow \quad D_\pi = 0$

- General case for $E/\mathbb{F}_q$, when q is an even power.
- $\pi = \pm\sqrt{q}$, hence no complex multiplication.

We will ignore marginal cases: $t = \pm\sqrt{q}, \pm\sqrt{2q}, \pm\sqrt{3q}$.

Supersingular complex multiplication

Let $E/\mathbb{F}_p$ be a supersingular curve, then $\pi^2 = -p$, and

$$\pi = \begin{pmatrix} \sqrt{-p} & 0 \\ 0 & -\sqrt{-p} \end{pmatrix} \mod \ell$$

for any ℓ s.t. $\left(\frac{-p}{\ell}\right) = 1$.

Theorem (Delfs and Galbraith 2016)

Let $\mathrm{End}_{\mathbb{F}_p}(E)$ denote the ring of $\mathbb{F}_p$-rational endomorphisms of E. Then

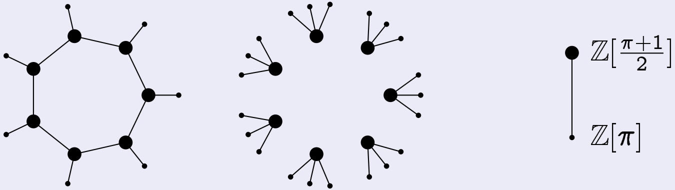
$$\mathbb{Z}[\pi] \subset \mathrm{End}_{\mathbb{F}_p}(E) \subset \mathbb{Q}(\sqrt{-p}).$$

Orders of $\mathbb{Q}(\sqrt{-p})$

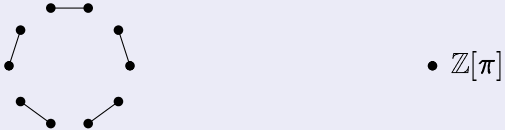
- If $p = 1 \mod 4$, then $\mathbb{Z}[\pi]$ is the maximal order.
- If $p = -1 \mod 4$, then $\mathbb{Z}[\frac{\pi+1}{2}]$ is the maximal order, and $[\mathbb{Z}[\frac{\pi+1}{2}] : \mathbb{Z}[\pi]] = 2$.

Supersingular CM graphs

2-volcanoes, $p = -1 \bmod 4$



2-graphs, $p = 1 \bmod 4$



All other ℓ-graphs are cycles of horizontal isogenies iff $\left(\frac{-p}{\ell}\right) = 1$.

The full endomorphism ring

Theorem (Deuring)

Let E be a supersingular elliptic curve, then

- E is isomorphic to a curve defined over $\mathbb{F}_{p^2}$;
- Every isogeny of E is defined over $\mathbb{F}_{p^2}$;
- Every endomorphism of E is defined over $\mathbb{F}_{p^2}$;
- $\mathrm{End}(E)$ is isomorphic to a maximal order in a quaternion algebra ramified at p and ∞.

In particular:

- If E is defined over $\mathbb{F}_p$, then $\mathrm{End}_{\mathbb{F}_p}(E)$ is strictly contained in $\mathrm{End}(E)$.
- Some endomorphisms do not commute!

An example

The curve of j-invariant 1728

$$E : y^2 = x^3 + x$$

is supersingular over $\mathbb{F}_p$ iff $p = -1 \mod 4$.

Endomorphisms

$\mathrm{End}(E) = \mathbb{Z}\langle\iota, \pi\rangle$, with:

- π the Frobenius endomorphism, s.t. $\pi^2 = -p$;
- ι the map

$$\iota(x, y) = (-x, iy),$$

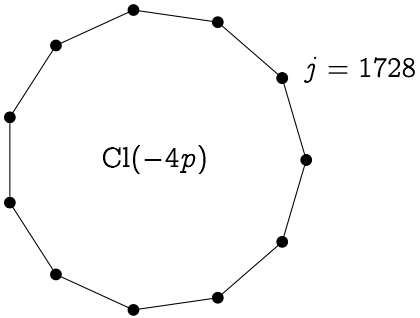
  where $i \in \mathbb{F}_{p^2}$ is a 4-th root of unity. Clearly, $\iota^2 = -1$.

And $\iota\pi = -\pi\iota$.

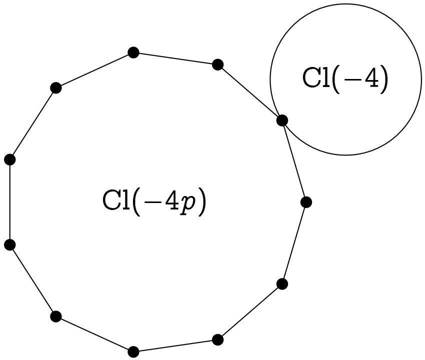
Class group action party

- $j = 1728$

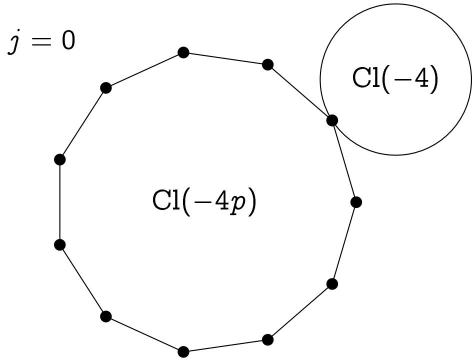
Class group action party



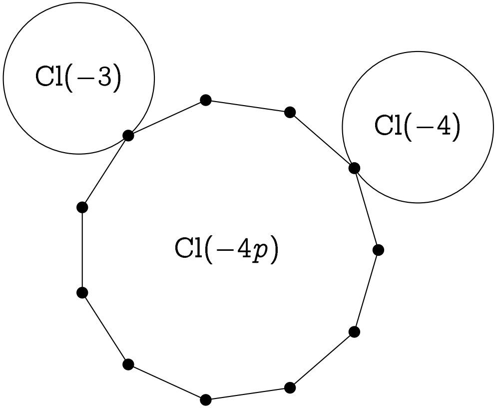
Class group action party



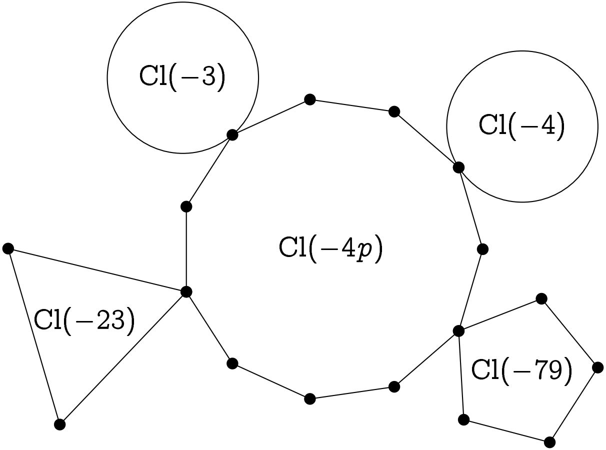
Class group action party



Class group action party



Class group action party



Quaternion algebra?! WTF?[2]

The quaternion algebra $B_{p,\infty}$ is:

- A 4-dimensional $\mathbb{Q}$-vector space with basis $(1, i, j, k)$.
- A non-commutative division algebra[1] $B_{p,\infty} = \mathbb{Q}\langle i, j\rangle$ with the relations:

$$i^2 = a, \quad j^2 = -p, \quad ij = -ji = k,$$

  for some $a < 0$ (depending on p).
- All elements of $B_{p,\infty}$ are quadratic algebraic numbers.
- $B_{p,\infty} \otimes \mathbb{Q}_\ell \simeq \mathcal{M}_{2\times 2}(\mathbb{Q}_\ell)$ for all $\ell \neq p$.
  I.e., endomorphisms restricted to $E[\ell^e]$ are just 2×2 matrices mod ℓ^e.
- $B_{p,\infty} \otimes \mathbb{R}$ is isomorphic to Hamilton's quaternions.
- $B_{p,\infty} \otimes \mathbb{Q}_p$ is a division algebra.

[1] All elements have inverses.

[2] What The Field?

Supersingular graphs

- Quaternion algebras have many maximal orders.
- For every maximal order type of $B_{p,\infty}$ there are 1 or 2 curves over $\mathbb{F}_{p^2}$ having endomorphism ring isomorphic to it.
- There is a unique isogeny class of supersingular curves over $\bar{\mathbb{F}}_p$ of size $\approx p/12$.
- Left ideals act on the set of maximal orders like isogenies.
- The graph of ℓ-isogenies is $(\ell+1)$-regular.

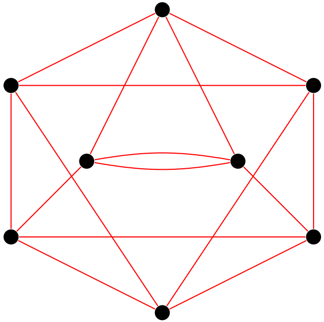


Figure: 3-isogeny graph on $\mathbb{F}_{97^2}$.

Graphs lexicon

Degree: Number of (outgoing/ingoing) edges.

k-regular: All vertices have degree k.

Connected: There is a path between any two vertices.

Distance: The length of the shortest path between two vertices.

Diamater: The longest distance between two vertices.

$\lambda_1 \geq \cdots \geq \lambda_n$: The (ordered) eigenvalues of the adjacency matrix.

Expander graphs

Proposition

If G is a k-regular graph, its largest and smallest eigenvalues satisfy

$$k = \lambda_1 \geq \lambda_n \geq -k.$$

Expander families

An infinite family of connected k-regular graphs on n vertices is an expander family if there exists an $\epsilon > 0$ such that all non-trivial eigenvalues satisfy $|\lambda| \leq (1-\epsilon)k$ for n large enough.

- Expander graphs have short diameter ($O(\log n)$);
- Random walks mix rapidly (after $O(\log n)$ steps, the induced distribution on the vertices is close to uniform).

Expander graphs from isogenies

Theorem (Pizer 1990, 1998)

Let ℓ be fixed. The family of graphs of supersingular curves over $\mathbb{F}_{p^2}$ with ℓ-isogenies, as $p \to \infty$, is an expander family[a].

[a] Even better, it has the Ramanujan property.

Theorem (Jao, Miller, and Venkatesan 2009)

Let $\mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$ be an order in a quadratic imaginary field. The graphs of all curves over $\mathbb{F}_q$ with complex multiplication by $\mathcal{O}$, with isogenies of prime degree bounded[a] by $(\log q)^{2+\delta}$, are expanders.

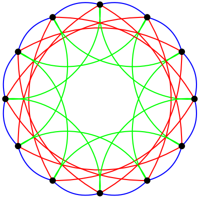
[a] May contain traces of GRH.

Plan

1. Elliptic curves, isogenies, complex multiplication
2. Isogeny graphs
3. Key exchange
4. Signatures and whatnot

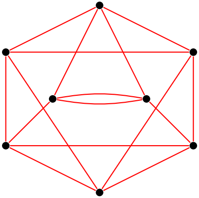
Isogeny graphs taxonomy

Complex Multiplication (CM) graphs



- Ordinary / Supersingular ($\mathbb{F}_p$)
- Superposition of isogeny cycles (one color per degree)
- Isomorphic to Cayley graph of a quadratic class group
- Large automorphism group
- Typical size $O(\sqrt{p})$
- Used in: **CSIDH**

***Full* supersingular graphs**



- Supersingular ($\mathbb{F}_{p^2}$)
- One isogeny degree
- $(\ell + 1)$-regular
- Tiny automorphism group
- Size $\approx p/12$
- Used in: **SIDH**

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