Isogeny graphs in cryptography

Luca De Feo

Université Paris Saclay, UVSQ

July 29, 2019
Cryptography meets Graph Theory
Würzburg, Germany

Plan

1. Elliptic curves, isogenies, complex multiplication,
2. Isogeny graphs,
3. Key exchange,
4. Signatures and whatnot.

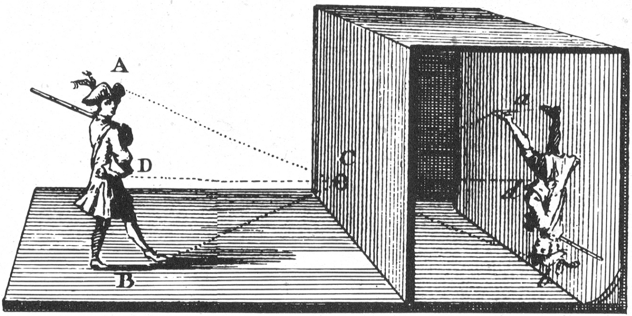
Projective space

Definition (Projective space)

Let $\bar{k}$ an algebraically closed field, the projective space $\mathbb{P}^n(\bar{k})$ is the set of non-null $(n+1)$-tuples $(x_0, \ldots, x_n) \in \bar{k}^n$ modulo the equivalence relation

$$(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n) \qquad \text{with } \lambda \in \bar{k} \setminus \{0\}.$$

A class is denoted by $(x_0 : \cdots : x_n)$.

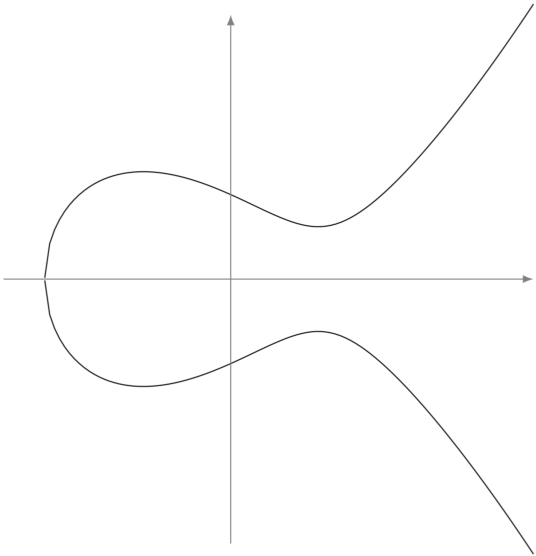


Weierstrass equations

Let k be a field of characteristic $\neq 2, 3$.
An elliptic curve *defined over* k is the locus in $\mathbb{P}^2(\bar{k})$ of an equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3,$$

where $a, b \in k$ and $4a^3 + 27b^2 \neq 0$.



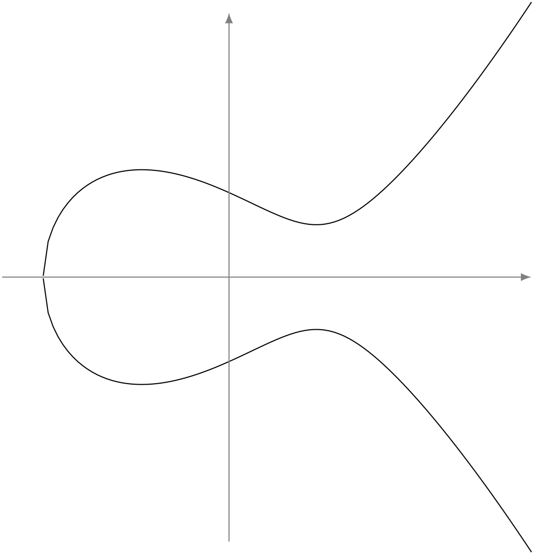
Weierstrass equations

Let k be a field of characteristic $\neq 2, 3$.
An elliptic curve *defined over* k is the locus in $\mathbb{P}^2(\bar{k})$ of an equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3,$$

where $a, b \in k$ and $4a^3 + 27b^2 \neq 0$.

- $\mathcal{O} = (0 : 1 : 0)$ is the point at infinity;



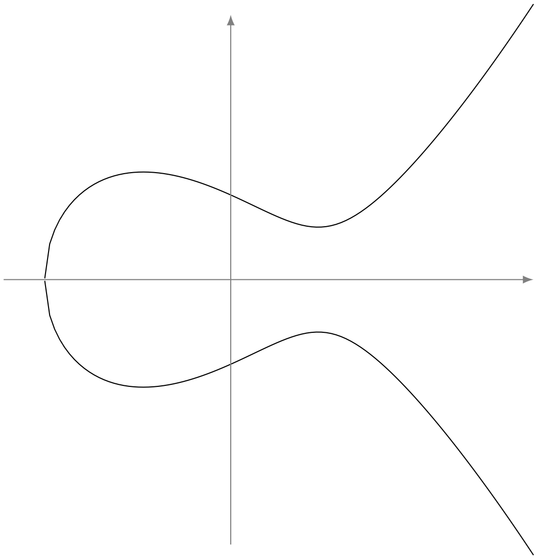
Weierstrass equations

Let k be a field of characteristic $\neq 2, 3$.
An elliptic curve *defined over* k is the locus in $\mathbb{P}^2(\bar{k})$ of an equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3,$$

where $a, b \in k$ and $4a^3 + 27b^2 \neq 0$.

- $\mathcal{O} = (0 : 1 : 0)$ is the point at infinity;
- $y^2 = x^3 + ax + b$ is the affine equation.

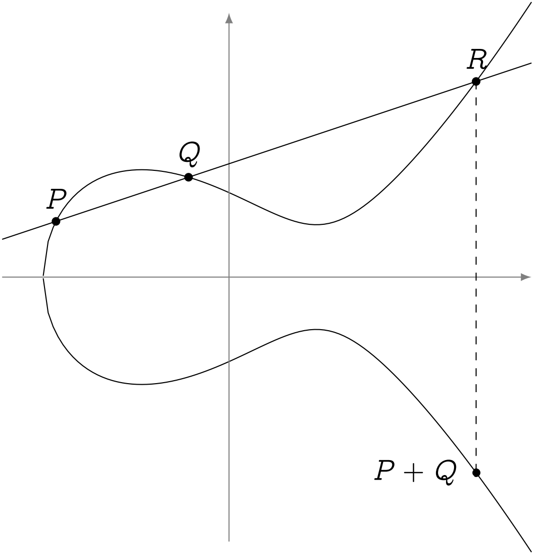


The group law

Bezout's theorem

Every line cuts E in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.



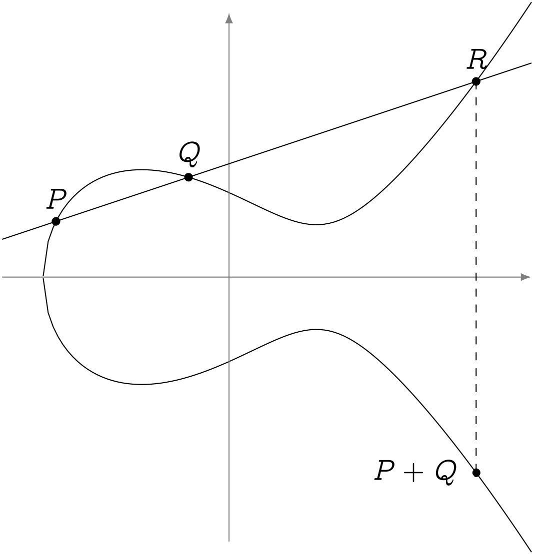
The group law

Bezout's theorem

Every line cuts E in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.

- The law is algebraic (it has *formulas*);



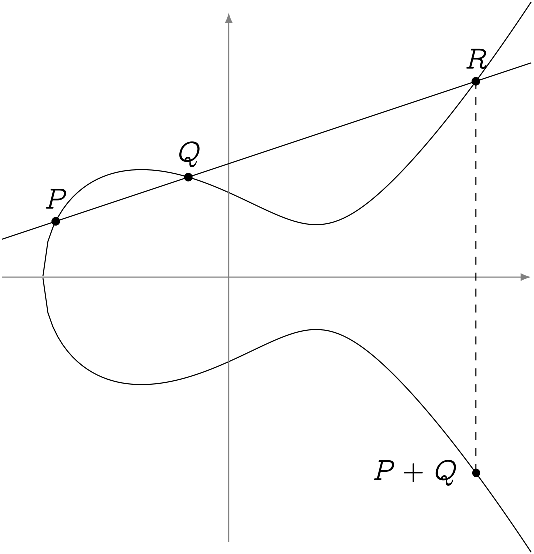
The group law

Bezout's theorem

Every line cuts E in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.

- The law is algebraic (it has *formulas*);
- The law is commutative;
- $\mathcal{O}$ is the group identity;
- Opposite points have the same x-value.



Group structure

Torsion structure

Let E be defined over an algebraically closed field $\bar{k}$ of characteristic p.

$$E[m] \simeq \quad \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \qquad \text{if } p \nmid m,$$

$$E[p^e] \simeq \begin{cases} \mathbb{Z}/p^e\mathbb{Z} & \text{ordinary case,} \\ \{\mathcal{O}\} & \text{supersingular case.} \end{cases}$$

Free part

Let E be defined over a number field k, the group of k-rational points $E(k)$ is finitely generated.

Maps: isomorphisms

Isomorphisms

The only invertible algebraic maps between elliptic curves are of the form

$$(x, y) \mapsto (u^2 x, u^3 y)$$

for some $u \in \bar{k}$.
They are group isomorphisms.

j-Invariant

Let $E : y^2 = x^3 + ax + b$, its j-invariant is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Two elliptic curves E, E' are isomorphic if and only if $j(E) = j(E')$.

Maps: isogenies

Theorem

Let $\phi : E \to E'$ be a map between elliptic curves. These conditions are equivalent:

- *ϕ is a surjective group morphism,*
- *ϕ is a group morphism with finite kernel,*
- *ϕ is a non-constant algebraic map of projective varieties sending the point at infinity of E onto the point at infinity of E'.*

If they hold ϕ is called an isogeny.

Two curves are called isogenous if there exists an isogeny between them.

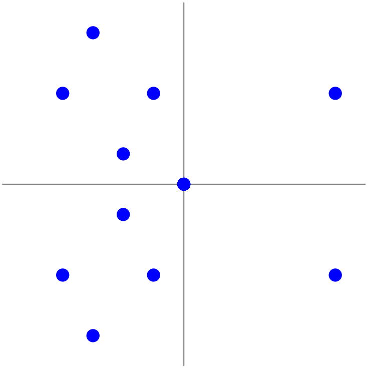
Example: Multiplication-by-m

On any curve, an isogeny from E to itself (i.e., an endomorphism):

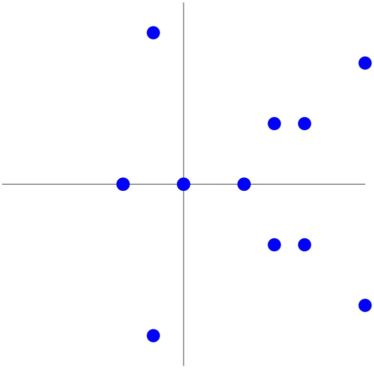
$$\begin{aligned}[m] \; : \; E &\to E, \\ P &\mapsto [m]P.\end{aligned}$$

Isogenies: an example over $\mathbb{F}_{11}$

$$E \;:\; y^2 = x^3 + x$$



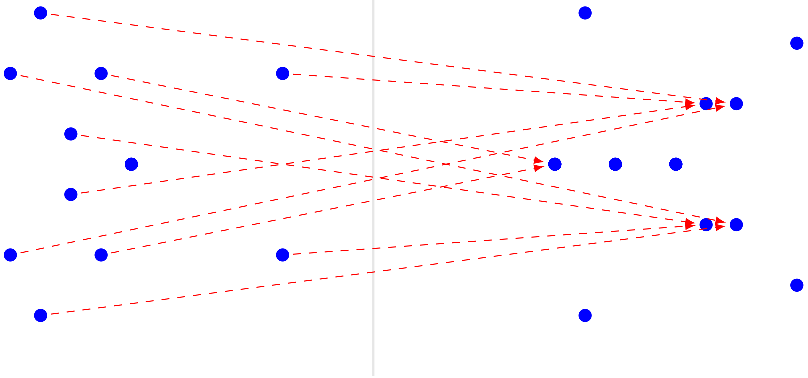
$$E' \;:\; y^2 = x^3 - 4x$$



$$\phi(x,y) = \left(\frac{x^2+1}{x},\quad y\frac{x^2-1}{x^2}\right)$$

Isogenies: an example over $\mathbb{F}_{11}$

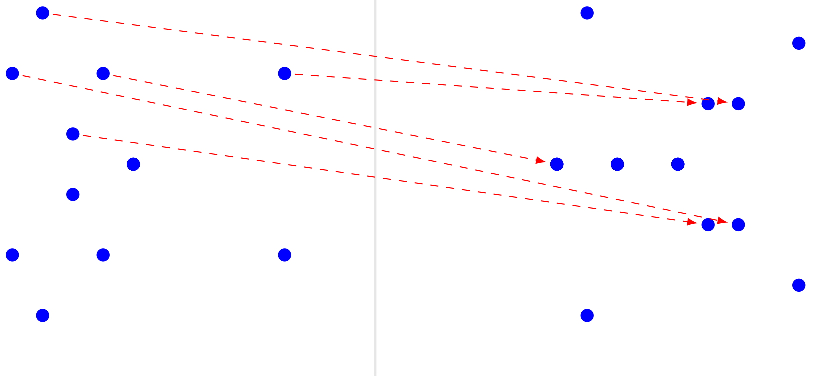
$E : y^2 = x^3 + x$ $\quad\quad$ $E' : y^2 = x^3 - 4x$



$$\phi(x, y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2} \right)$$

Isogenies: an example over $\mathbb{F}_{11}$

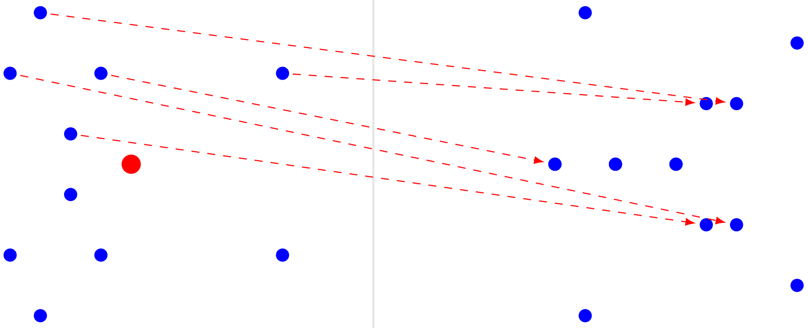
$E \ : \ y^2 = x^3 + x$ $\qquad$ $E' \ : \ y^2 = x^3 - 4x$



$$\phi(x,y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2}\right)$$

Isogenies: an example over $\mathbb{F}_{11}$

$E \; : \; y^2 = x^3 + x$ $\qquad$ $E' \; : \; y^2 = x^3 - 4x$

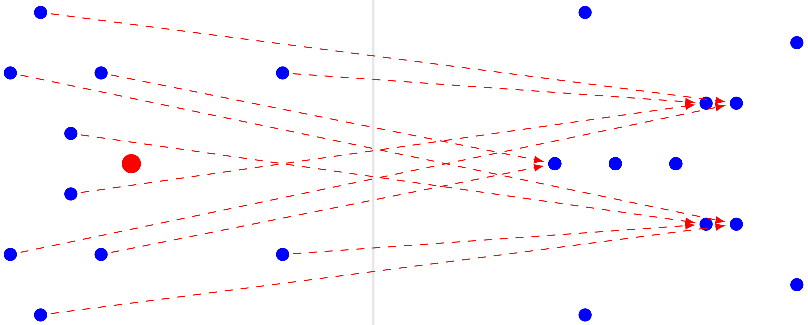


$$\phi(x, y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2} \right)$$

- Kernel generator in red.

Isogenies: an example over $\mathbb{F}_{11}$

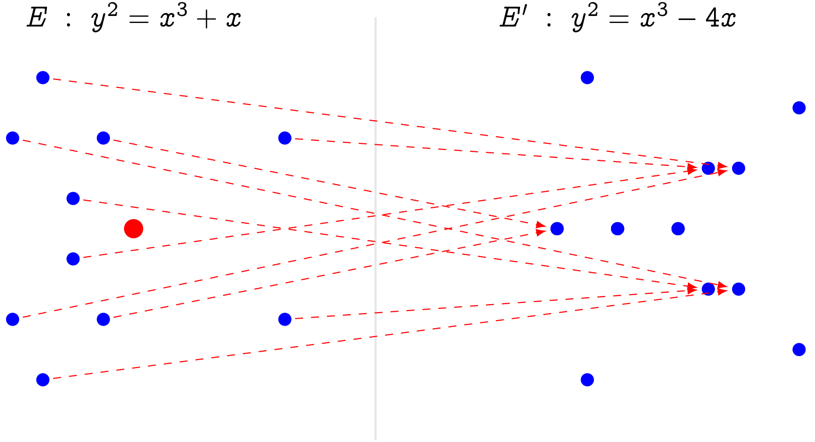
$E \;:\; y^2 = x^3 + x$ $\qquad\qquad$ $E' \;:\; y^2 = x^3 - 4x$



$$\phi(x, y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2} \right)$$

- Kernel generator in red.
- This is a degree 2 map.

Isogenies: an example over $\mathbb{F}_{11}$



$$\phi(x, y) = \left(\frac{x^2 + 1}{x}, \quad y\frac{x^2 - 1}{x^2} \right)$$

- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

Curves over finite fields

Frobenius endomorphism

Let E be defined over $\mathbb{F}_q$. The Frobenius endomorphism of E is the map

$$\pi : (X : Y : Z) \mapsto (X^q : Y^q : Z^q).$$

Hasse's theorem

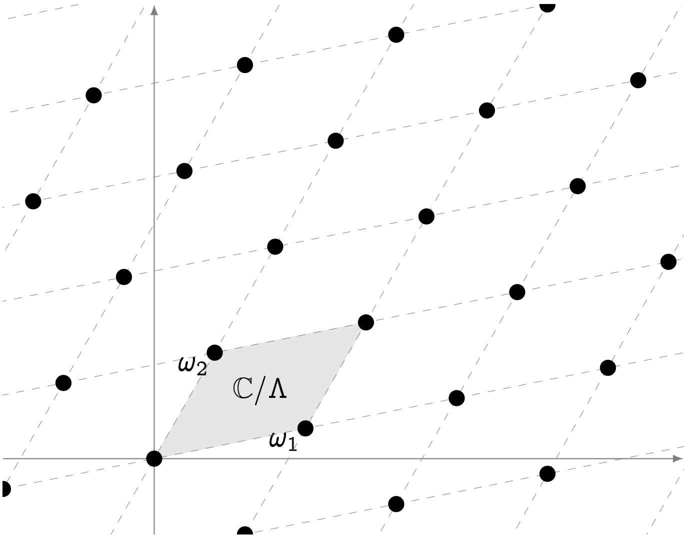
Let E be defined over $\mathbb{F}_q$, then

$$|\#E(k) - q - 1| \leq 2\sqrt{q}.$$

Serre-Tate theorem

Two elliptic curves E, E' defined over a finite field k are isogenous over k if and only if $\#E(k) = \#E'(k)$.

Complex tori

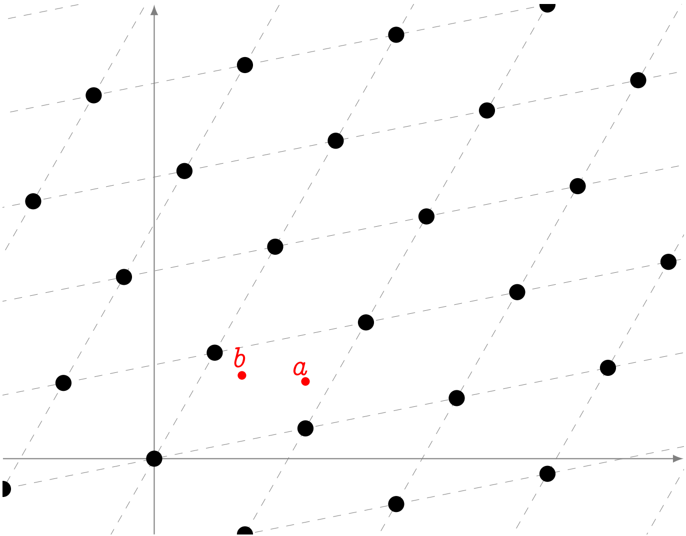


Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent complex numbers. Set

$$\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$$

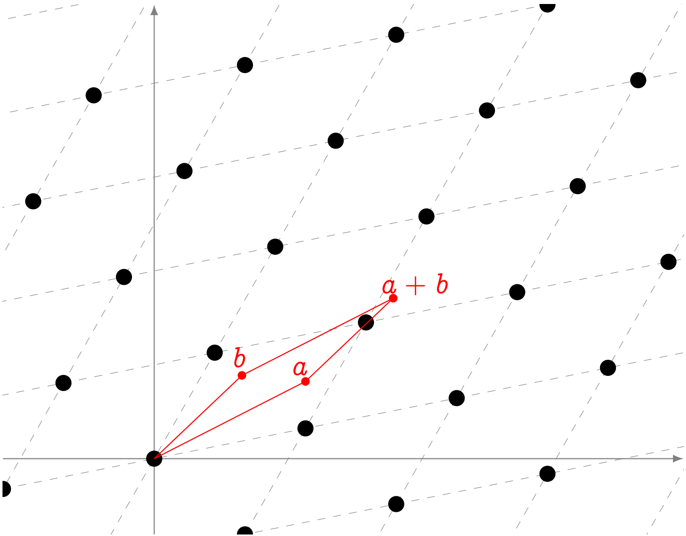
$\mathbb{C}/\Lambda$ is a complex torus.

Complex tori



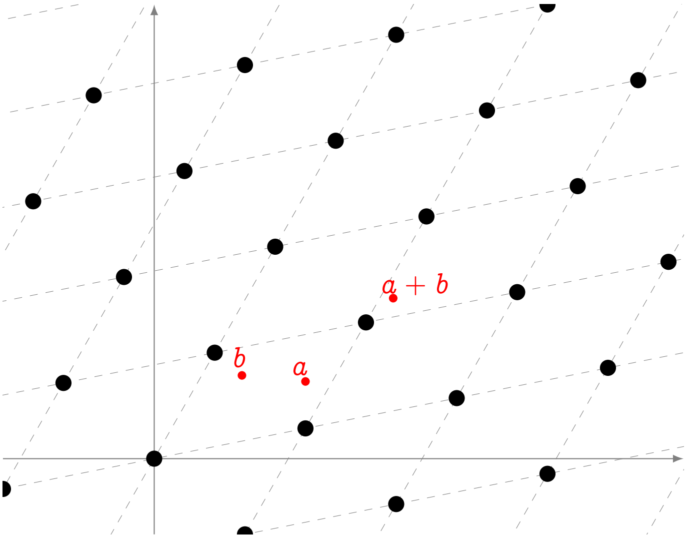
Addition law induced by addition on $\mathbb{C}$.

Complex tori



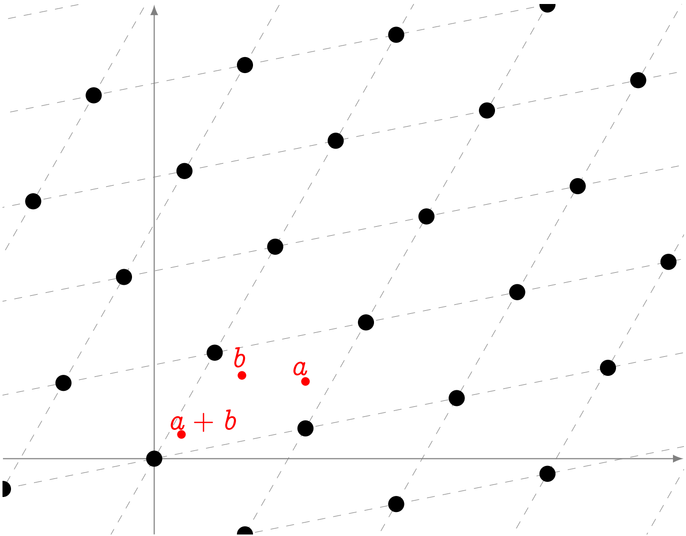
Addition law induced by addition on $\mathbb{C}$.

Complex tori



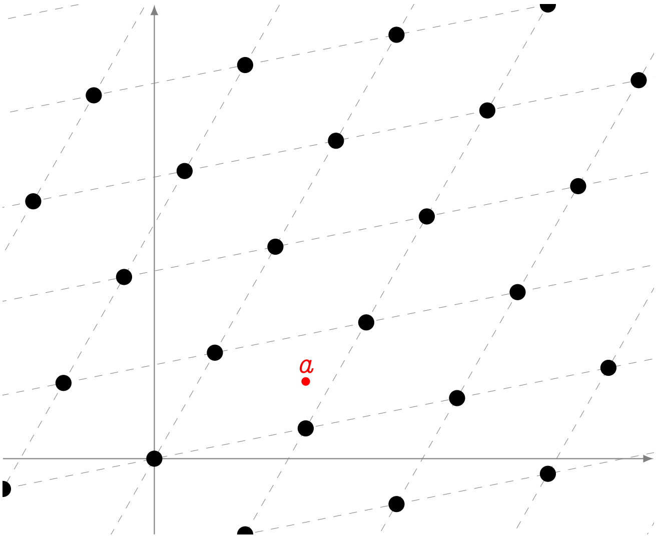
Addition law induced by addition on $\mathbb{C}$.

Complex tori



Addition law induced by addition on $\mathbb{C}$.

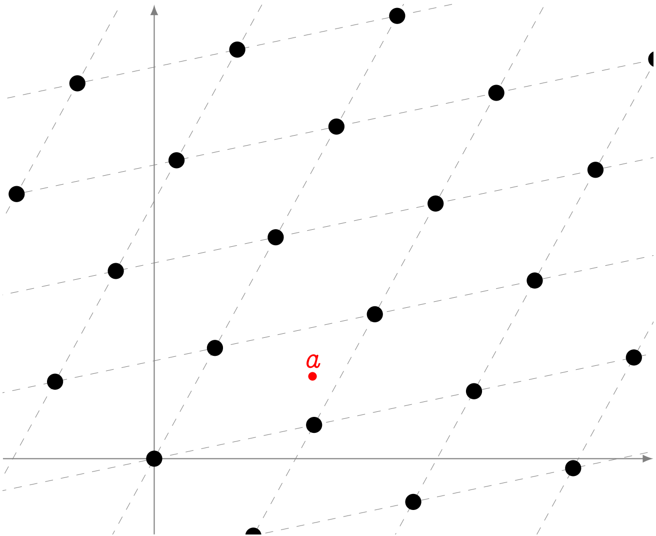
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

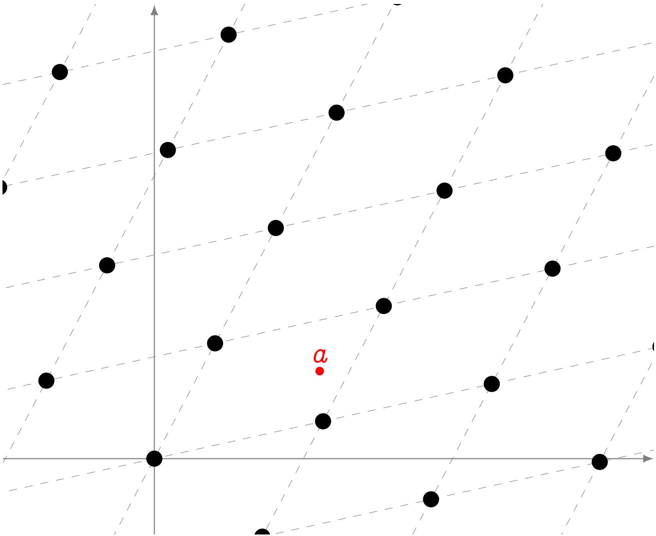
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

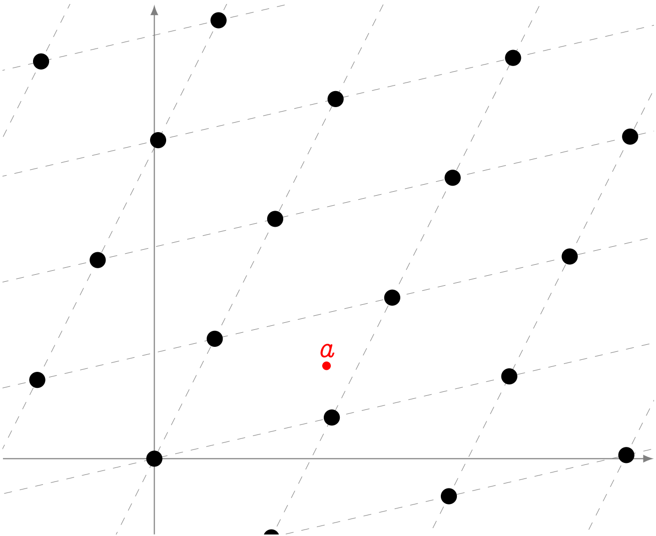
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

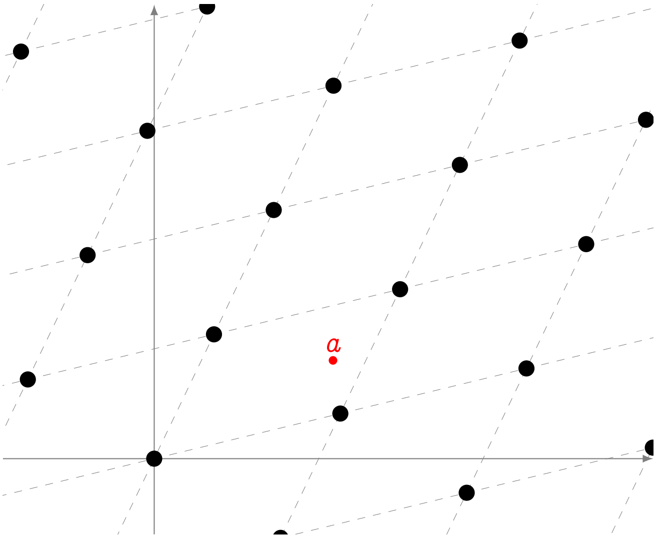
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

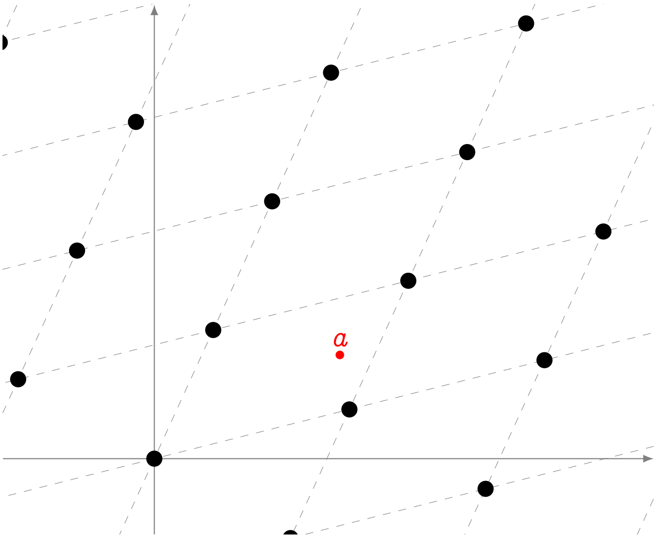
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

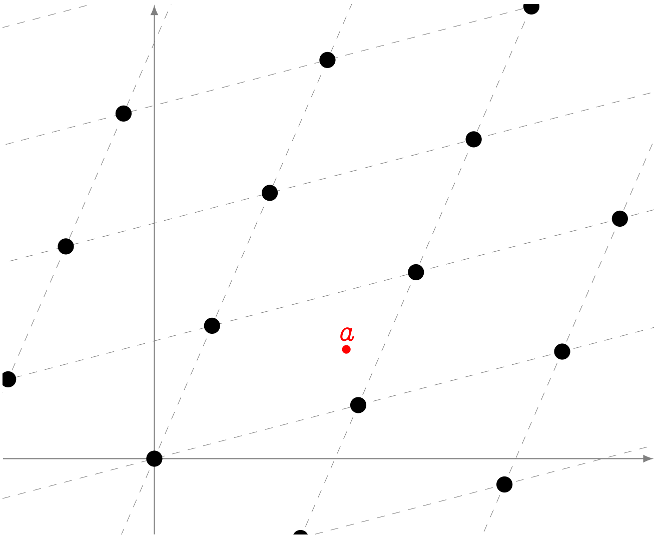
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

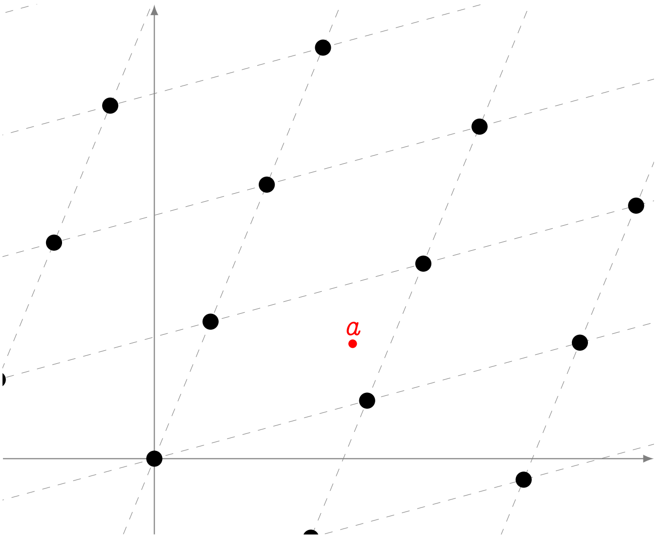
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

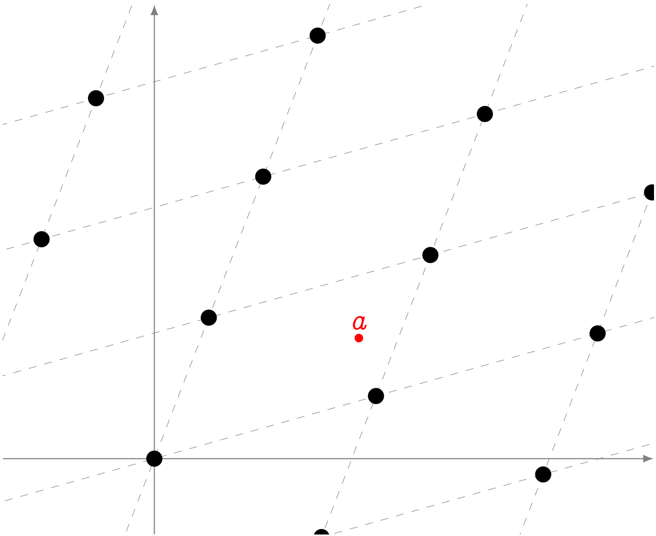
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

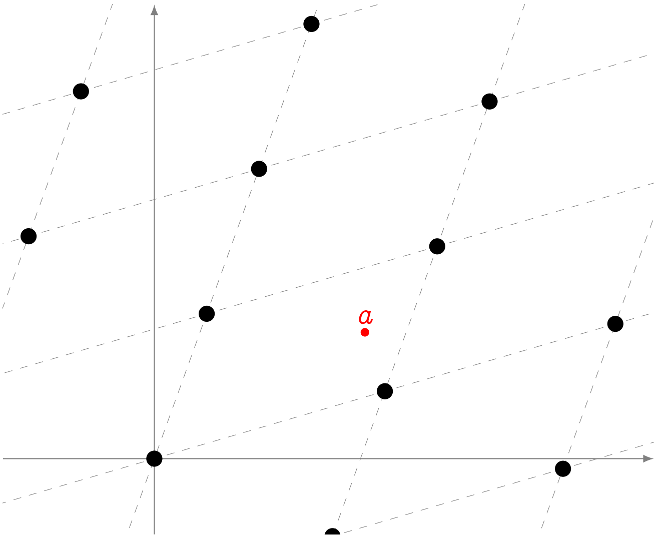
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

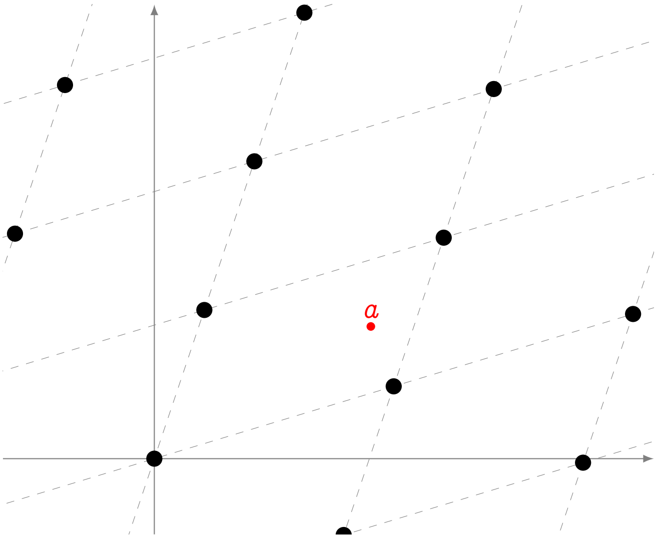
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

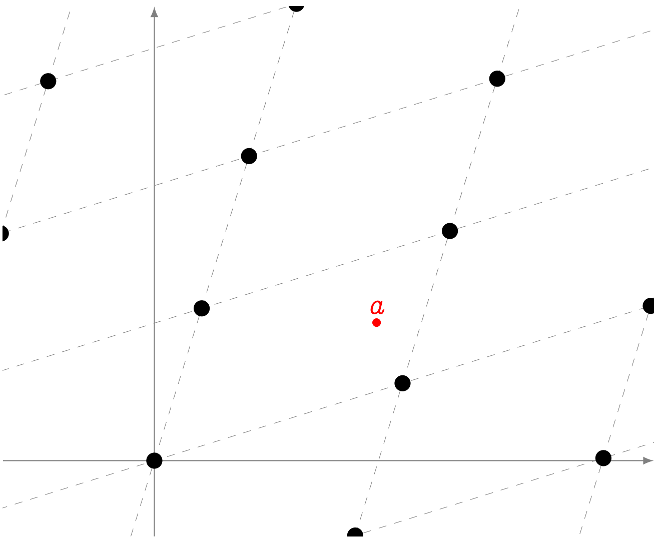
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

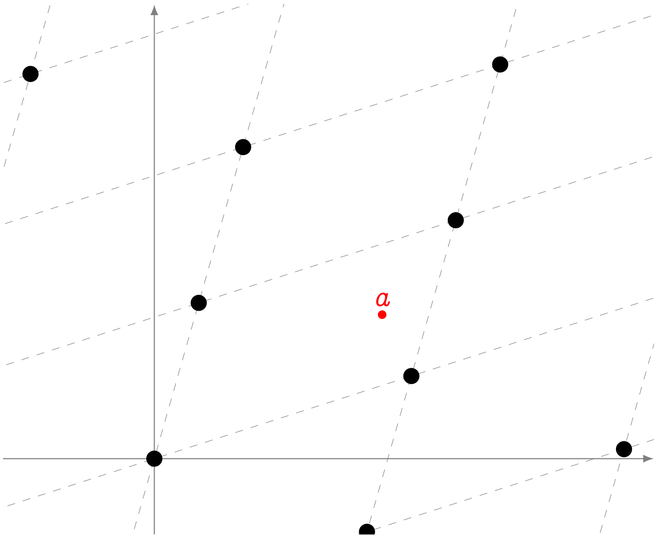
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha \Lambda_1 = \Lambda_2$$

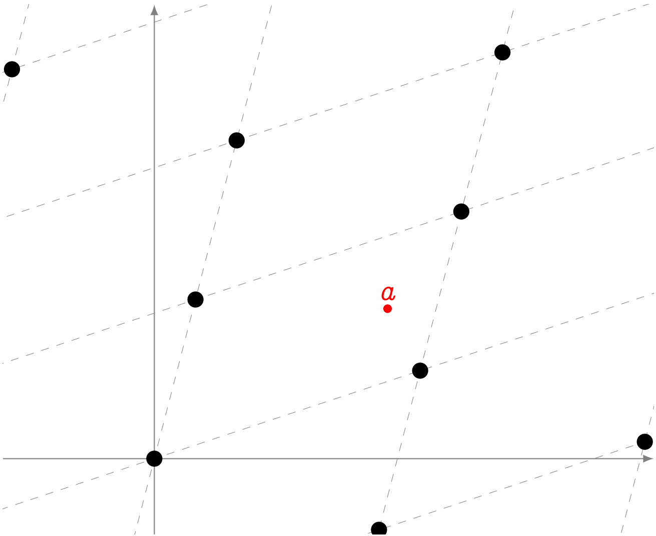
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

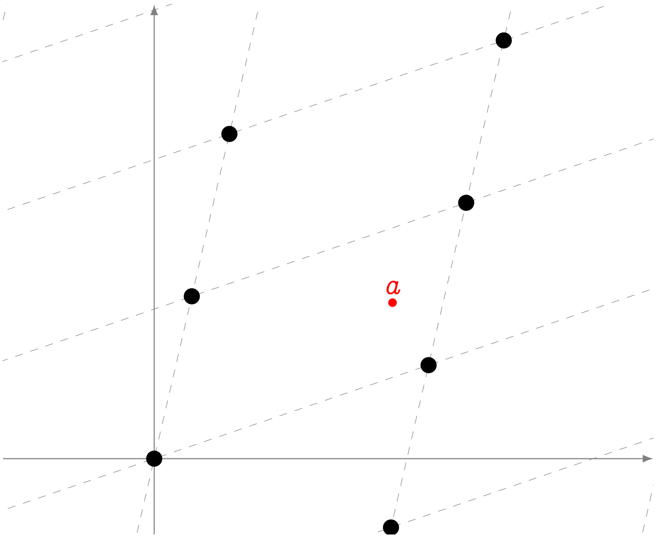
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

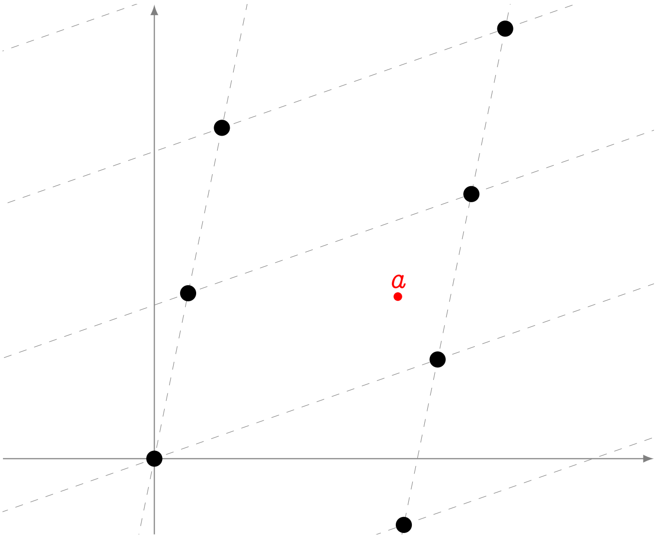
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

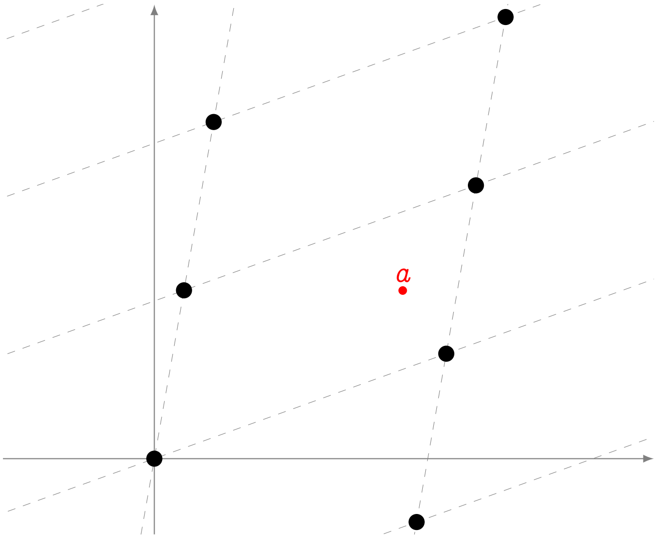
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

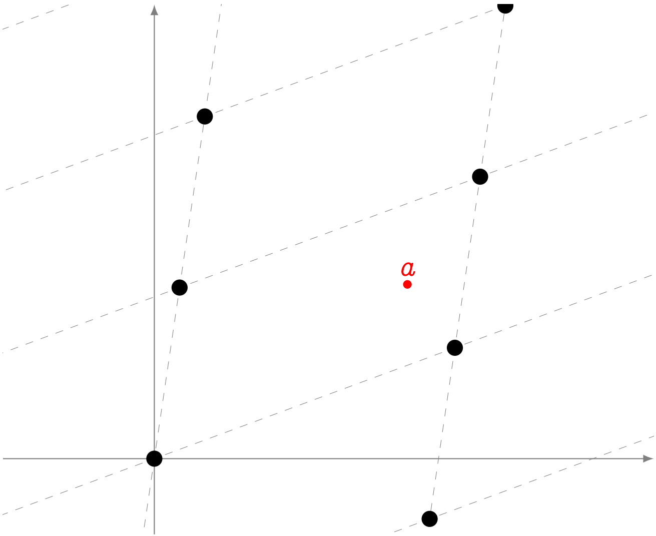
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha \Lambda_1 = \Lambda_2$$

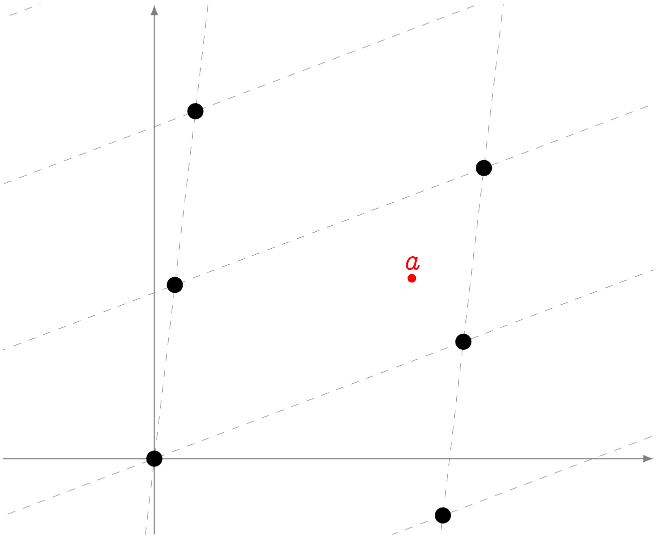
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

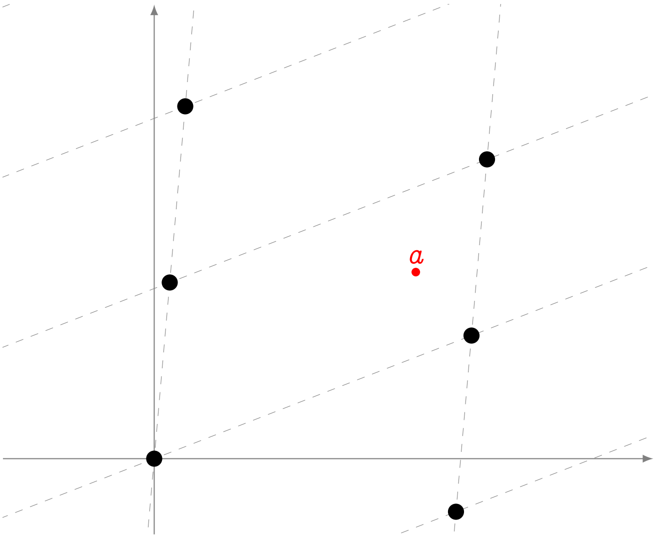
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

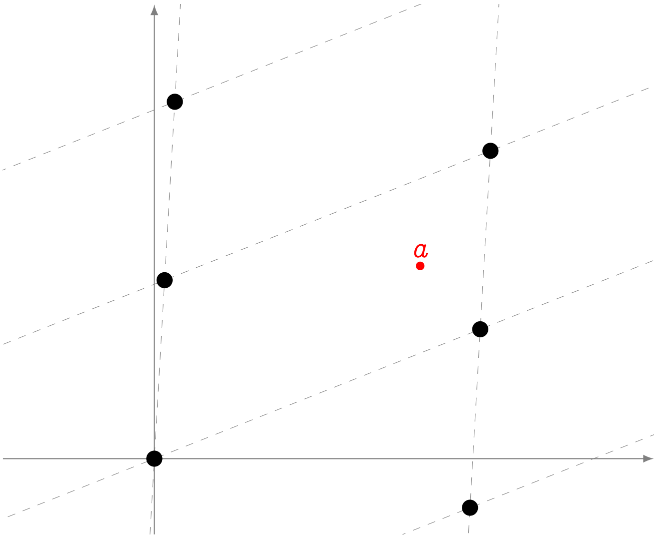
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

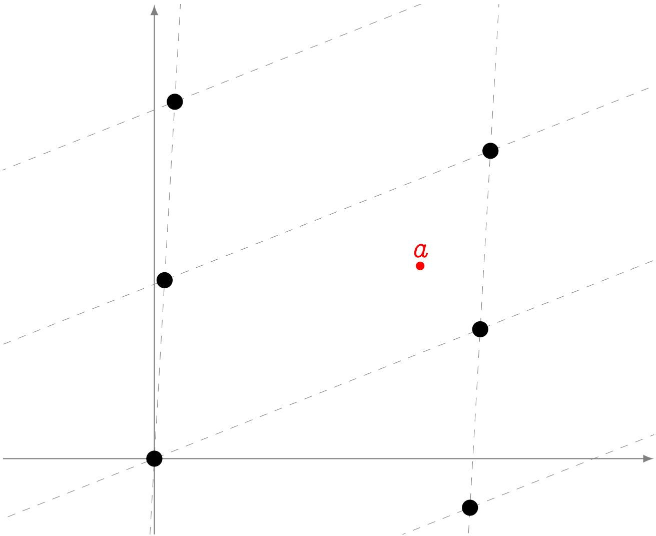
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha \Lambda_1 = \Lambda_2$$

Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

The j-invariant

We want to classify complex lattices/tori up to homothety.

Eisenstein series

Let Λ be a complex lattice. For any integer $k > 0$ define

$$G_{2k}(\Lambda) = \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-2k}.$$

Also set

$$g_2(\Lambda) = 60\, G_4(\Lambda), \qquad g_3(\Lambda) = 140\, G_6(\Lambda).$$

Modular j-invariant

Let Λ be a complex lattice, the modular j-invariant is

$$j(\Lambda) = 1728 \frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27 g_3(\Lambda)^2}.$$

Two lattices Λ, Λ' are homothetic if and only if $j(\Lambda) = j(\Lambda')$.

Elliptic curves over $\mathbb{C}$

Weierstrass $\wp$ function

Let Λ be a complex lattice, the Weierstrass $\wp$ function associated to Λ is the series

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

Fix a lattice Λ, then $\wp$ and its derivative $\wp'$ are elliptic functions:

$$\wp(z+\omega) = \wp(z), \qquad \wp'(z+\omega) = \wp'(z)$$

for all $\omega \in \Lambda$.

Uniformization theorem

Let Λ be a complex lattice. The curve

$$E \ : \ y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$$

is an elliptic curve over $\mathbb{C}$. The map

$$\begin{aligned} \mathbb{C}/\Lambda &\to E(\mathbb{C}), \\ 0 &\mapsto (0:1:0), \\ z &\mapsto (\wp(z):\wp'(z):1) \end{aligned}$$

is an isomorphism of Riemann surfaces and a group morphism.

Conversely, for any elliptic curve

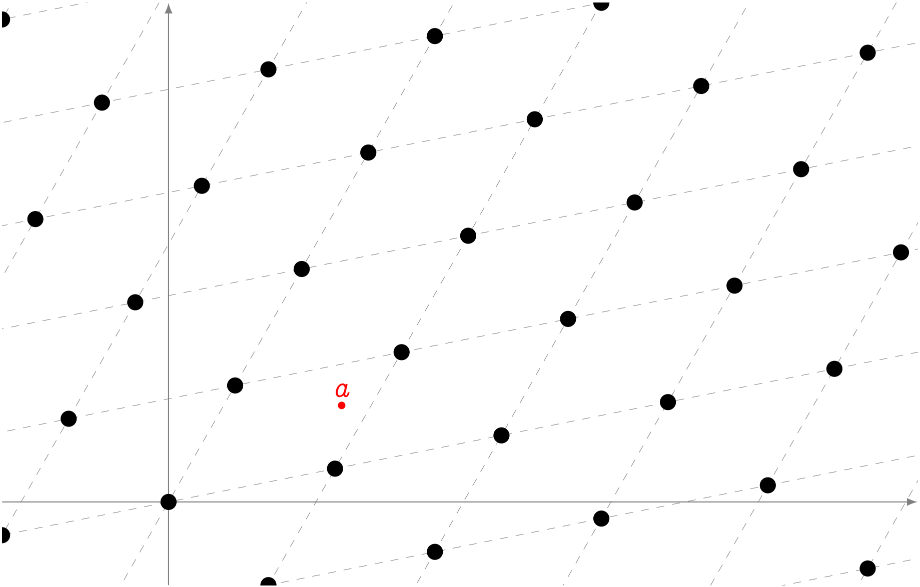
$$E \ : \ y^2 = x^3 + ax + b$$

there is a unique complex lattice Λ such that

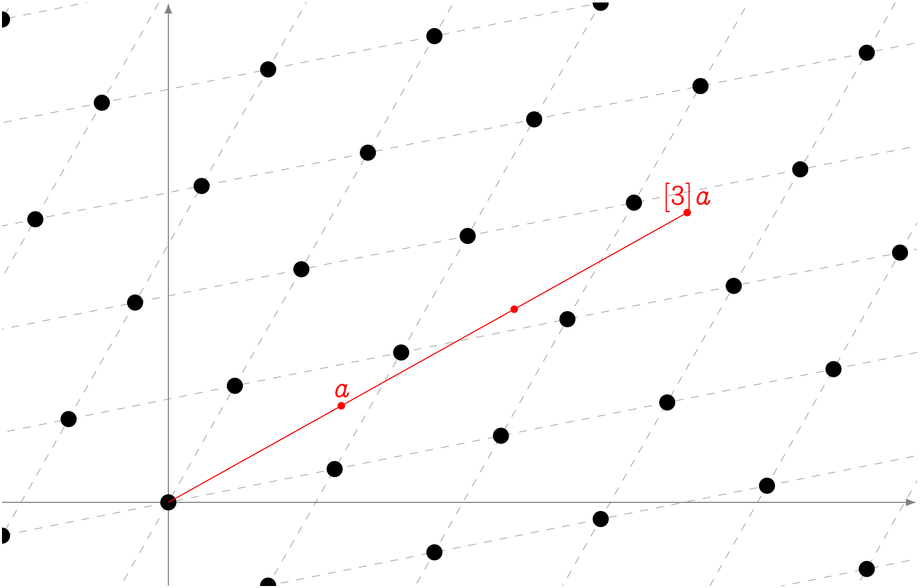
$$g_2(\Lambda) = -4a, \qquad g_3(\Lambda) = -4b.$$

Moreover $j(\Lambda) = j(E)$.

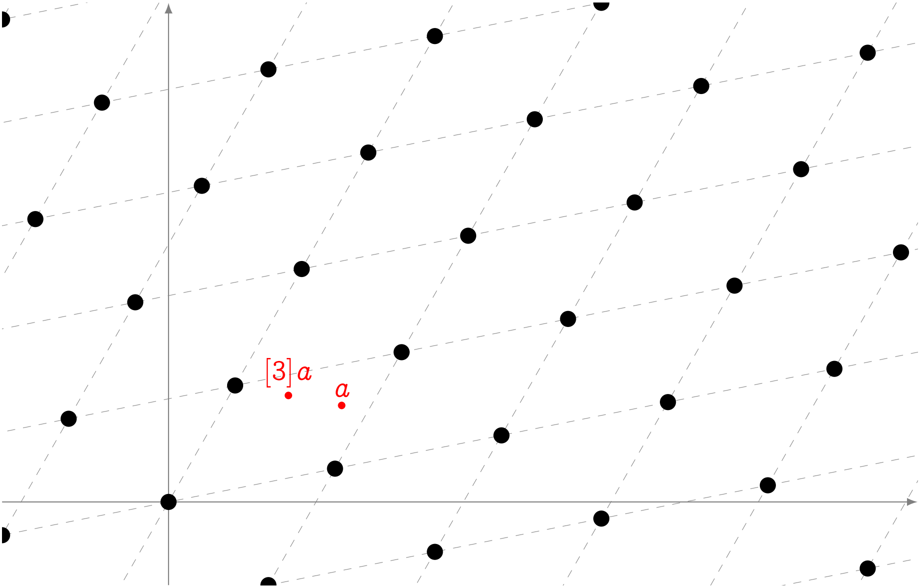
Multiplication



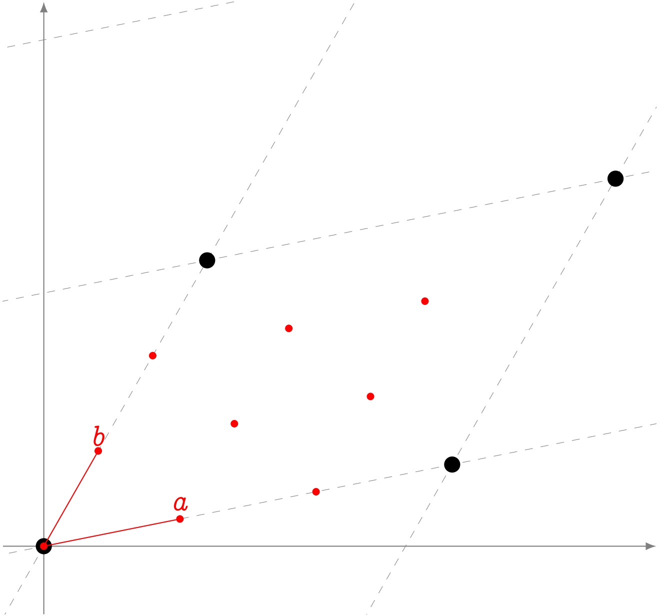
Multiplication



Multiplication



Torsion subgroups



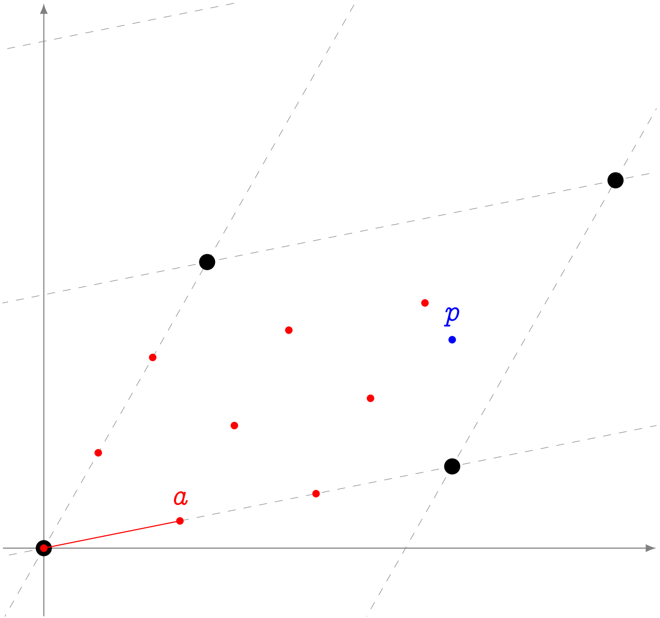
The ℓ-torsion subgroup is made up by the points

$$\left(\frac{i\omega_1}{\ell}, \frac{j\omega_2}{\ell}\right)$$

It is a group of rank two

$$E[\ell] = \langle a, b \rangle$$
$$\simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

Isogenies



Let $a \in \mathbb{C}/\Lambda_1$ be an ℓ-torsion point, and let

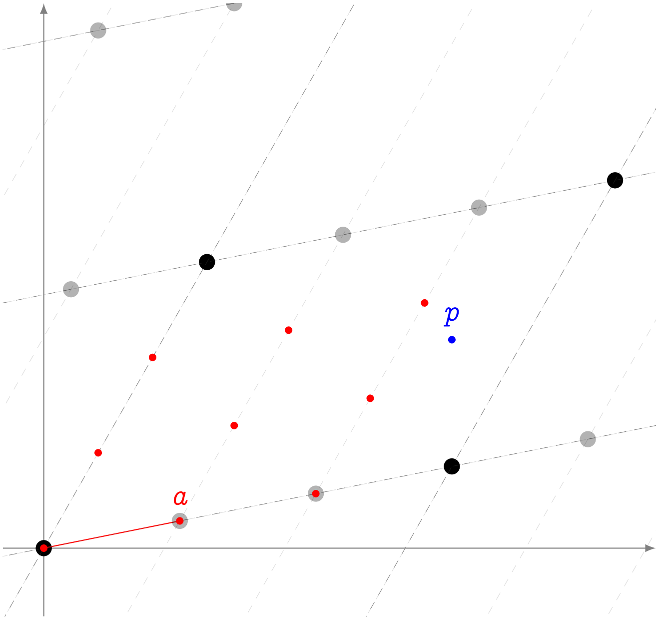
$$\Lambda_2 = a\mathbb{Z} \oplus \Lambda_1$$

Then $\Lambda_1 \subset \Lambda_2$ and we define a degree ℓ cover

$$\phi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$$

ϕ is a morphism of complex Lie groups and is called an isogeny.

Isogenies



Let $a \in \mathbb{C}/\Lambda_1$ be an ℓ-torsion point, and let

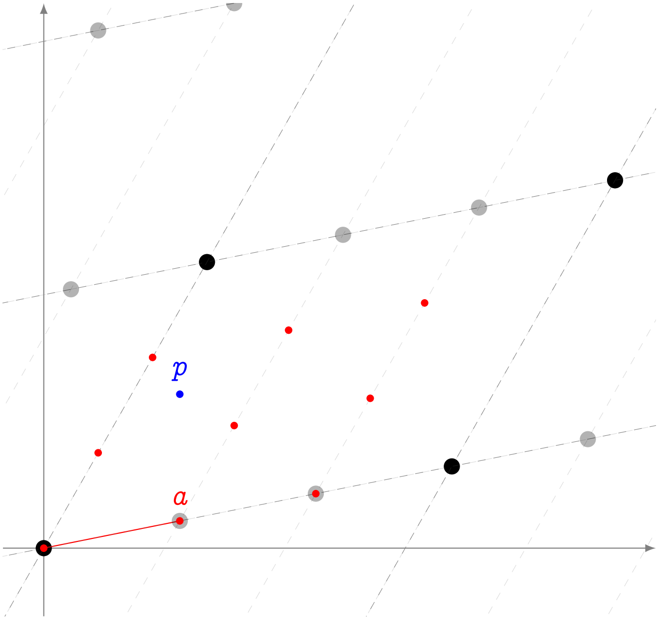
$$\Lambda_2 = a\mathbb{Z} \oplus \Lambda_1$$

Then $\Lambda_1 \subset \Lambda_2$ and we define a degree ℓ cover

$$\phi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$$

ϕ is a morphism of complex Lie groups and is called an isogeny.

Isogenies



Let $a \in \mathbb{C}/\Lambda_1$ be an ℓ-torsion point, and let

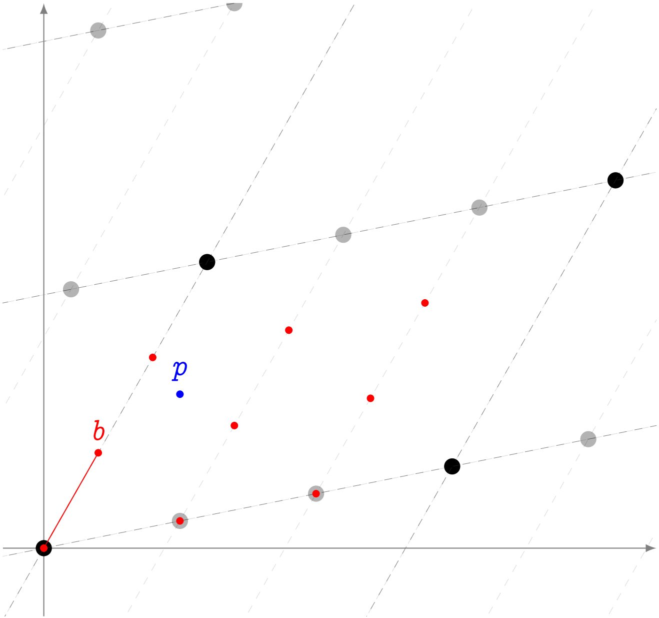
$$\Lambda_2 = a\mathbb{Z} \oplus \Lambda_1$$

Then $\Lambda_1 \subset \Lambda_2$ and we define a degree ℓ cover

$$\phi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$$

ϕ is a morphism of complex Lie groups and is called an isogeny.

Isogenies



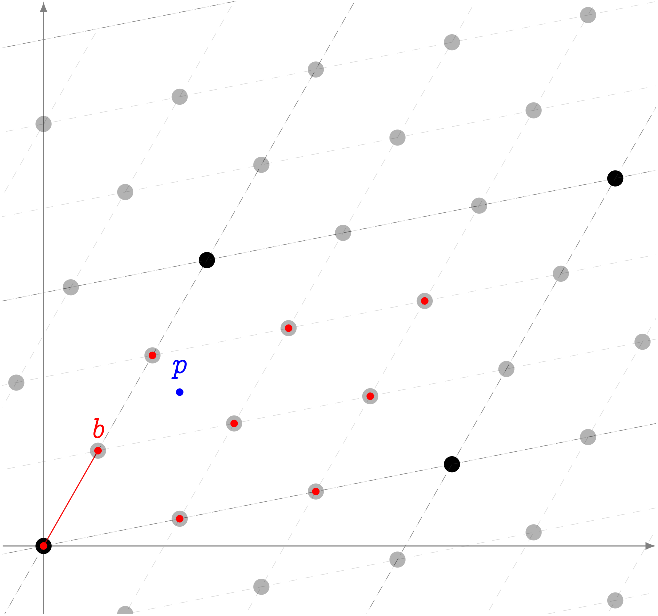
Taking a point b not in the kernel of ϕ, we obtain a new degree ℓ cover

$$\hat{\phi} : \mathbb{C}/\Lambda_2 \to \mathbb{C}/\Lambda_3$$

The composition $\hat{\phi} \circ \phi$ has degree ℓ^2 and is homothetic to the multiplication by ℓ map.

$\hat{\phi}$ is called the dual isogeny of ϕ.

Isogenies



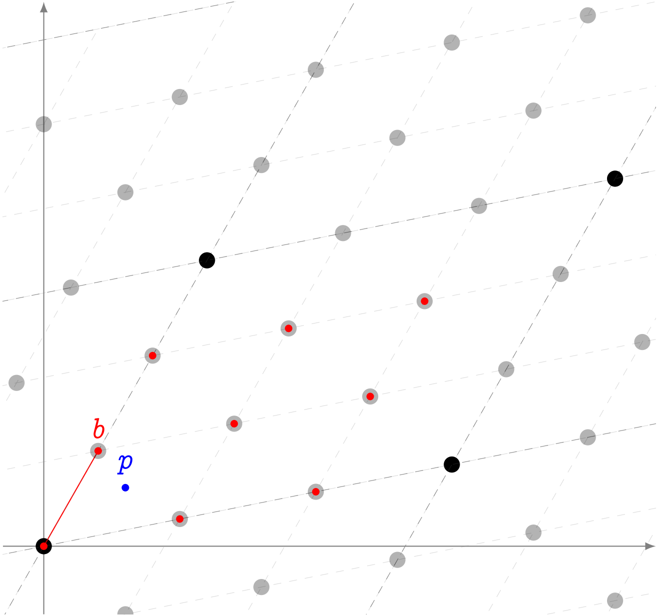
Taking a point b not in the kernel of ϕ, we obtain a new degree ℓ cover

$$\hat{\phi} : \mathbb{C}/\Lambda_2 \to \mathbb{C}/\Lambda_3$$

The composition $\hat{\phi} \circ \phi$ has degree ℓ^2 and is homothetic to the multiplication by ℓ map.

$\hat{\phi}$ is called the dual isogeny of ϕ.

Isogenies



Taking a point b not in the kernel of ϕ, we obtain a new degree ℓ cover

$$\hat{\phi} : \mathbb{C}/\Lambda_2 \to \mathbb{C}/\Lambda_3$$

The composition $\hat{\phi} \circ \phi$ has degree ℓ^2 and is homothetic to the multiplication by ℓ map.

$\hat{\phi}$ is called the dual isogeny of ϕ.

Isogenies: back to algebra

Let $\phi : E \to E'$ be an isogeny defined over a field k of characteristic p.

- $k(E)$ is the field of all rational functions from E to k;
- $\phi^* k(E')$ is the subfield of $k(E)$ defined as

$$\phi^* k(E') = \{f \circ \phi \mid f \in k(E')\}.$$

Degree, separability

1. The degree of ϕ is $\deg \phi = [k(E) : \phi^* k(E')]$. It is always finite.
2. ϕ is said to be separable, inseparable, or purely inseparable if the extension of function fields is.
3. If ϕ is separable, then $\deg \phi = \# \ker \phi$.
4. If ϕ is purely inseparable, then $\ker \phi = \{\mathcal{O}\}$ and $\deg \phi$ is a power of p.
5. Any isogeny can be decomposed as a product of a separable and a purely inseparable isogeny.

Isogenies: back to algebra

Let $\phi : E \to E'$ be an isogeny defined over a field k of characteristic p.

- $k(E)$ is the field of all rational functions from E to k;
- $\phi^* k(E')$ is the subfield of $k(E)$ defined as

$$\phi^* k(E') = \{f \circ \phi \mid f \in k(E')\}.$$

Degree, separability

1. The degree of ϕ is $\deg \phi = [k(E) : \phi^* k(E')]$. It is always finite.
2. ϕ is said to be separable, inseparable, or purely inseparable if the extension of function fields is.
3. If ϕ is separable, then $\deg \phi = \# \ker \phi$.
4. If ϕ is purely inseparable, then $\ker \phi = \{\mathcal{O}\}$ and $\deg \phi$ is a power of p.
5. Any isogeny can be decomposed as a product of a separable and a purely inseparable isogeny.

Isogenies: separable vs inseparable

Purely inseparable isogenies

Examples:

- The Frobenius endomorphism is purely inseparable of degree q.
- All purely inseparable maps in characteristic p are of the form $(X : Y : Z) \mapsto (X^{p^e} : Y^{p^e} : Z^{p^e})$.

Separable isogenies

Let E be an elliptic curve, and let G be a finite subgroup of E. There are a unique elliptic curve E' and a unique separable isogeny ϕ, such that $\ker \phi = G$ and $\phi : E \to E'$.
The curve E' is called the quotient of E by G and is denoted by E/G.

The dual isogeny

Let $\phi : E \to E'$ be an isogeny of degree m. There is a unique isogeny $\hat{\phi} : E' \to E$ such that

$$\hat{\phi} \circ \phi = [m]_E, \quad \phi \circ \hat{\phi} = [m]_{E'}.$$

$\hat{\phi}$ is called the dual isogeny of ϕ; it has the following properties:

1. $\hat{\phi}$ is defined over k if and only if ϕ is;
2. $\widehat{\psi \circ \phi} = \hat{\phi} \circ \hat{\psi}$ for any isogeny $\psi : E' \to E''$;
3. $\widehat{\psi + \phi} = \hat{\psi} + \hat{\phi}$ for any isogeny $\psi : E \to E'$;
4. $\deg \phi = \deg \hat{\phi}$;
5. $\hat{\hat{\phi}} = \phi$.

Algebras, orders

- A quadratic imaginary number field is an extension of $\mathbb{Q}$ of the form $Q[\sqrt{-D}]$ for some non-square $D > 0$.
- A quaternion algebra is an algebra of the form $\mathbb{Q} + \alpha\mathbb{Q} + \beta\mathbb{Q} + \alpha\beta\mathbb{Q}$, where the generators satisfy the relations

$$\alpha^2, \beta^2 \in \mathbb{Q}, \quad \alpha^2 < 0, \quad \beta^2 < 0, \quad \beta\alpha = -\alpha\beta.$$

Orders

Let K be a finitely generated $\mathbb{Q}$-algebra. An order $\mathcal{O} \subset K$ is a subring of K that is a finitely generated $\mathbb{Z}$-module of maximal dimension. An order that is not contained in any other order of K is called a maximal order.

Examples:

- $\mathbb{Z}$ is the only order contained in $\mathbb{Q}$,
- $\mathbb{Z}[i]$ is the only maximal order of $\mathbb{Q}(i)$,
- $\mathbb{Z}[\sqrt{5}]$ is a non-maximal order of $\mathbb{Q}(\sqrt{5})$,
- The ring of integers of a number field is its only maximal order,
- In general, maximal orders in quaternion algebras are not unique.

The endomorphism ring

The endomorphism ring $\mathtt{End}(E)$ of an elliptic curve E is the ring of all isogenies $E \to E$ (plus the null map) with addition and composition.

Theorem (Deuring)

Let E be an elliptic curve defined over a field k of characteristic p. $\mathtt{End}(E)$ is isomorphic to one of the following:

- $\mathbb{Z}$, only if $p = 0$

  E is ordinary.

- An order $\mathcal{O}$ in a quadratic imaginary field:

  E is ordinary with complex multiplication by $\mathcal{O}$.

- Only if $p > 0$, a maximal order in a quaternion algebra[a]:

  E is supersingular.

[a] (ramified at p and ∞)

The finite field case

Theorem (Hasse)

Let E be defined over a finite field. Its Frobenius endomorphism π satisfies a quadratic equation

$$\pi^2 - t\pi + q = 0$$

in $\mathrm{End}(E)$ for some $|t| \leq 2\sqrt{q}$, called the trace of π. The trace t is coprime to q if and only if E is ordinary.

Suppose E is ordinary, then $D_\pi = t^2 - 4q < 0$ is the discriminant of $\mathbb{Z}[\pi]$.

- $K = \mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{D_\pi})$ is the endomorphism algebra of E.
- Denote by $\mathcal{O}_K$ its ring of integers, then

$$\mathbb{Z} \neq \mathbb{Z}[\pi] \subset \mathrm{End}(E) \subset \mathcal{O}_K.$$

In the supersingular case, π may or may not be in $\mathbb{Z}$, depending on q.

Endomorphism rings of ordinary curves

Classifying quadratic orders

Let K be a quadratic number field, and let $\mathcal{O}_K$ be its ring of integers.

- Any order $\mathcal{O} \subset K$ can be written as $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ for an integer f, called the conductor of $\mathcal{O}$, denoted by $[\mathcal{O}_k : \mathcal{O}]$.
- If d_K is the discriminant of K, the discriminant of $\mathcal{O}$ is $f^2 d_K$.
- If $\mathcal{O}, \mathcal{O}'$ are two orders with discriminants d, d', then $\mathcal{O} \subset \mathcal{O}'$ iff $d' | d$.

$$\begin{tikzcd}
 & \mathcal{O}_K \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Z} + 2\mathcal{O}_K \arrow[d, dash] \arrow[dr, dash] & \mathbb{Z} + 3\mathcal{O}_K \arrow[dl, dash] \arrow[dr, dash] & \mathbb{Z} + 5\mathcal{O}_K \arrow[dl, dash] \arrow[d, dash] \\
\mathbb{Z} + 6\mathcal{O}_K \arrow[dr, dash] & \mathbb{Z} + 10\mathcal{O}_K \arrow[d, dash] & \mathbb{Z} + 15\mathcal{O}_K \arrow[dl, dash] \\
 & \mathbb{Z}[\pi] \simeq \mathbb{Z} + 30\mathcal{O}_K &
\end{tikzcd}$$

Ideal lattices

Fractional ideals

Let $\mathcal{O}$ be an order of a number field K. A (fractional) $\mathcal{O}$-ideal $\mathfrak{a}$ is a finitely generated non-zero $\mathcal{O}$-submodule of K.

When K is imaginary quadratic:

- Fractional ideals are complex lattices,
- Any lattice $\Lambda \subset K$ is a fractional ideal,
- The order of a lattice Λ is

$$\mathcal{O}_\Lambda = \{\alpha \in K \mid \alpha\Lambda \subset \Lambda\}$$

Complex multiplication

Let $\Lambda \subset K$, the elliptic curve associated to $\mathbb{C}/\Lambda$ has complex multiplication by $\mathcal{O}_\Lambda$.

The class group

Let $\mathrm{End}(E) = \mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$. Define

- $\mathcal{I}(\mathcal{O})$, the group of invertible fractional ideals,
- $\mathcal{P}(\mathcal{O})$, the group of principal ideals,

The class group

The class group of $\mathcal{O}$ is

$$\mathrm{Cl}(\mathcal{O}) = \mathcal{I}(\mathcal{O})/\mathcal{P}(\mathcal{O}).$$

- It is a finite abelian group.
- Its order $h(\mathcal{O})$ is called the class number of $\mathcal{O}$.
- It arises as the Galois group of an abelian extension of $\mathbb{Q}(\sqrt{-D})$.

Complex multiplication

Fundamental theorem of CM

Let $\mathcal{O}$ be an order of a number field K, and let $\mathfrak{a}_1, \dots, \mathfrak{a}_{h(\mathcal{O})}$ be representatives of $\mathrm{Cl}(\mathcal{O})$. Then:

- $K(j(\mathfrak{a}_i))$ is an Abelian extension of K;
- The $j(\mathfrak{a}_i)$ are all conjugate over K;
- The Galois group of $K(j(\mathfrak{a}_i))$ is isomorphic to $\mathrm{Cl}(\mathcal{O})$;
- $[\mathbb{Q}(j(\mathfrak{a}_i)) : \mathbb{Q}] = [K(j(\mathfrak{a}_i)) : K] = h(\mathcal{O})$;
- The $j(\mathfrak{a}_i)$ are integral, their minimal polynomial is called the Hilbert class polynomial of $\mathcal{O}$.

Lifting

Deuring's lifting theorem

Let E_0 be an elliptic curve in characteristic p, with an endomorphism ω_o which is not trivial. Then there exists an elliptic curve E defined over a number field L, an endomorphism ω of E, and a non-singular reduction of E at a place $\mathfrak{p}$ of L lying above p, such that E_0 is isomorphic to $E(\mathfrak{p})$, and ω_0 corresponds to $\omega(\mathfrak{p})$ under the isomorphism.