Isogeny graphs in cryptography

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Cryptography meets Graph Theory
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Plan

1. Elliptic curves, isogenies, complex multiplication
2. Isogeny graphs
3. Key exchange

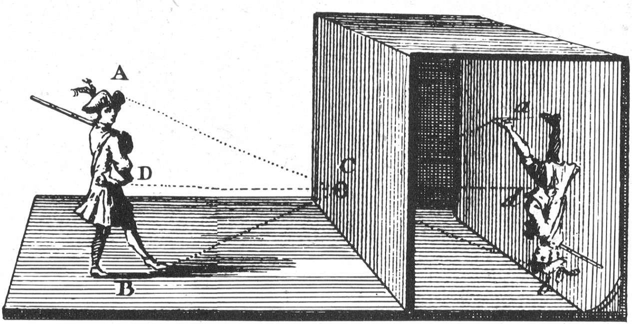
Projective space

Definition (Projective space)

Let $\bar{k}$ an algebraically closed field, the projective space $\mathbb{P}^n(\bar{k})$ is the set of non-null $(n+1)$-tuples $(x_0, \dots, x_n) \in \bar{k}^n$ modulo the equivalence relation

$$(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n) \qquad \text{with } \lambda \in \bar{k} \setminus \{0\}.$$

A class is denoted by $(x_0 : \cdots : x_n)$.

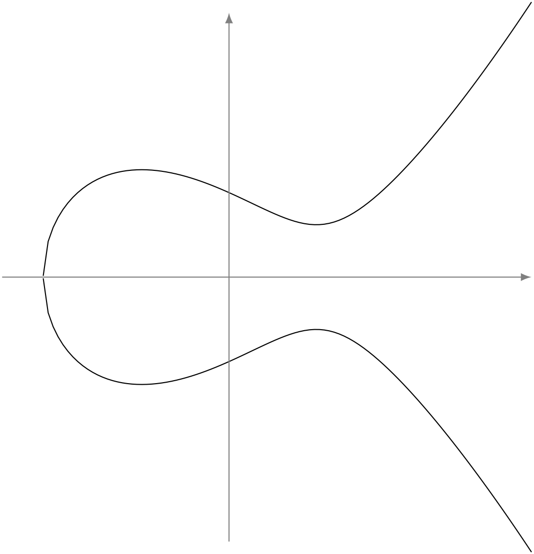


Weierstrass equations

Let k be a field of characteristic $\neq 2, 3$.
An elliptic curve *defined over* k is the locus in $\mathbb{P}^2(\bar{k})$ of an equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3,$$

where $a, b \in k$ and $4a^3 + 27b^2 \neq 0$.



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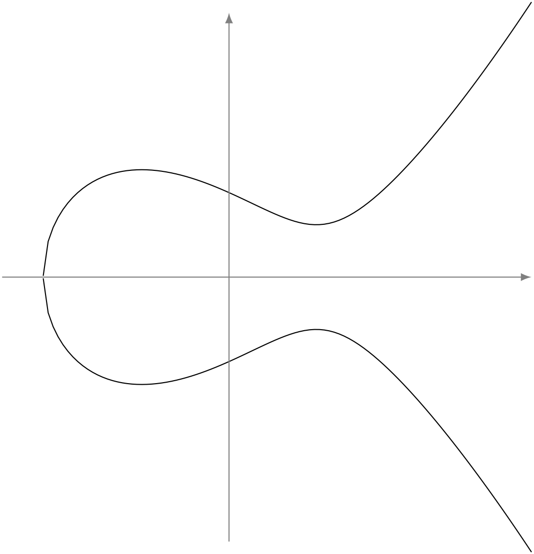

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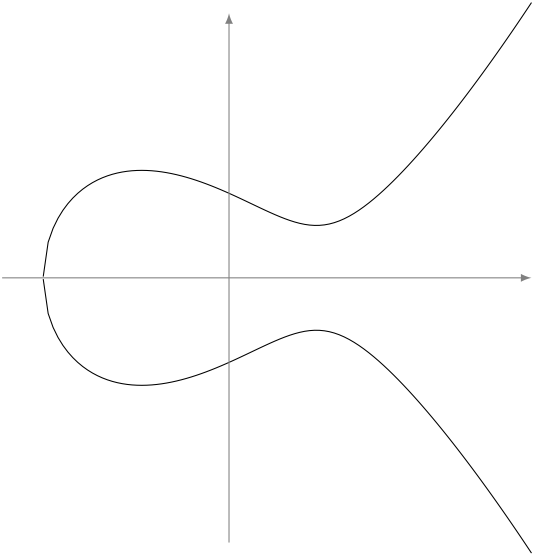
- $\mathcal{O} = (0 : 1 : 0)$ is the point at infinity;
- $y^2 = x^3 + ax + b$ is the affine equation.

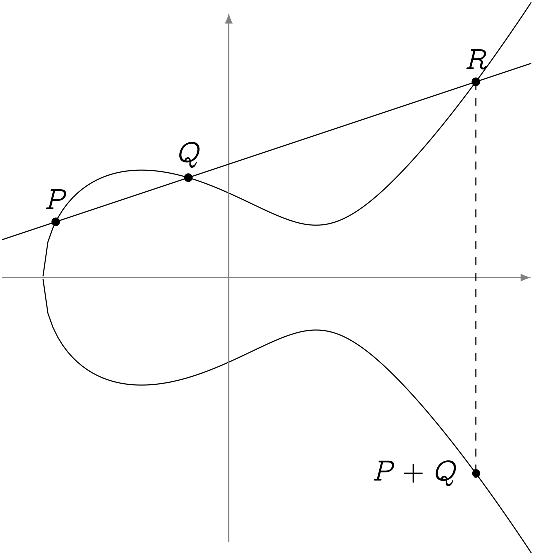


The group law

Bezout's theorem

Every line cuts E in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.



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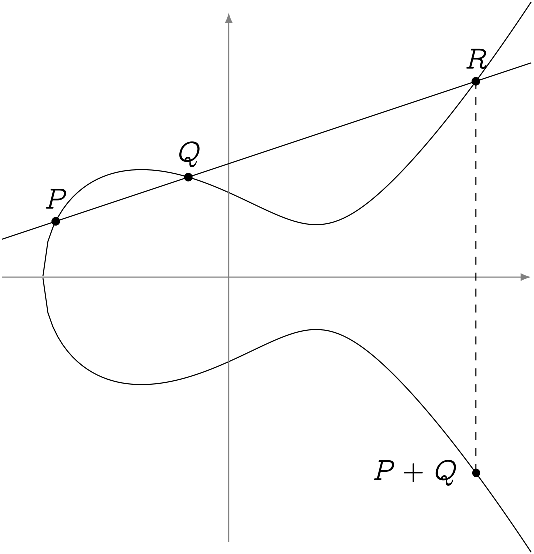

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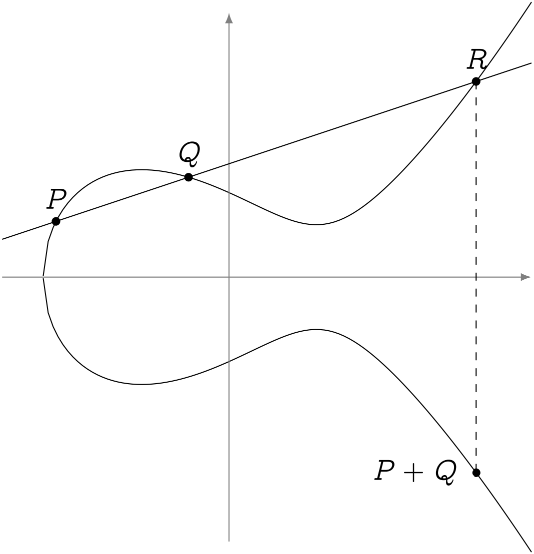
Define a group law such that any three colinear points add up to zero.

- The law is algebraic (it has *formulas*);
- The law is commutative;
- $\mathcal{O}$ is the group identity;
- Opposite points have the same x-value.



Group structure

Torsion structure

Let E be defined over an algebraically closed field $\bar{k}$ of characteristic p.

$$E[m] \simeq \quad \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \qquad \text{if } p \nmid m,$$

$$E[p^e] \simeq \begin{cases} \mathbb{Z}/p^e\mathbb{Z} & \text{ordinary case,} \\ \{\mathcal{O}\} & \text{supersingular case.} \end{cases}$$

Free part

Let E be defined over a number field k, the group of k-rational points $E(k)$ is finitely generated.

Maps: isomorphisms

Isomorphisms

The only invertible algebraic maps between elliptic curves are of the form

$$(x, y) \mapsto (u^2 x, u^3 y)$$

for some $u \in \bar{k}$.
They are group isomorphisms.

j-Invariant

Let $E : y^2 = x^3 + ax + b$, its j-invariant is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Two elliptic curves E, E' are isomorphic if and only if $j(E) = j(E')$.

Maps: isogenies

Theorem

Let $\phi : E \to E'$ be a map between elliptic curves. These conditions are equivalent:

- *ϕ is a surjective group morphism,*
- *ϕ is a group morphism with finite kernel,*
- *ϕ is a non-constant algebraic map of projective varieties sending the point at infinity of E onto the point at infinity of E'.*

If they hold ϕ is called an isogeny.

Two curves are called isogenous if there exists an isogeny between them.

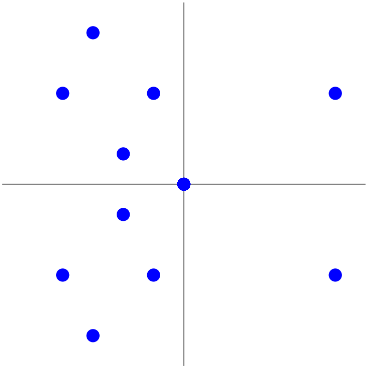
Example: Multiplication-by-m

On any curve, an isogeny from E to itself (i.e., an endomorphism):

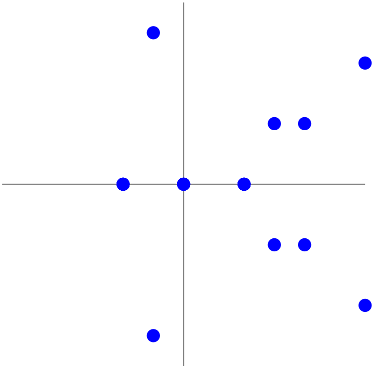
$$\begin{aligned}[m] \;:\; E &\to E,\\ P &\mapsto [m]P.\end{aligned}$$

Isogenies: an example over $\mathbb{F}_{11}$

$E \;:\; y^2 = x^3 + x$



$E' \;:\; y^2 = x^3 - 4x$



$$\phi(x,y) = \left(\frac{x^2+1}{x},\quad y\frac{x^2-1}{x^2}\right)$$

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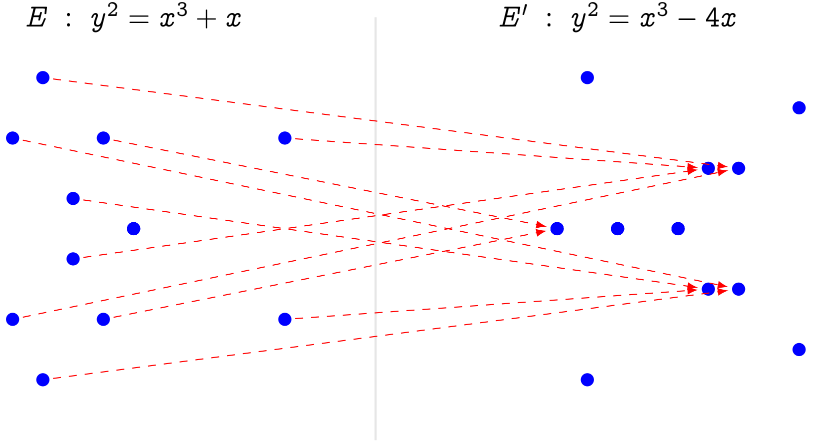

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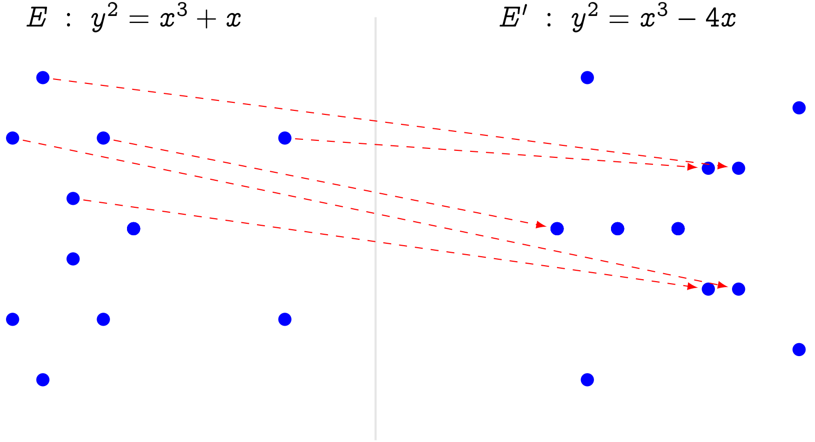

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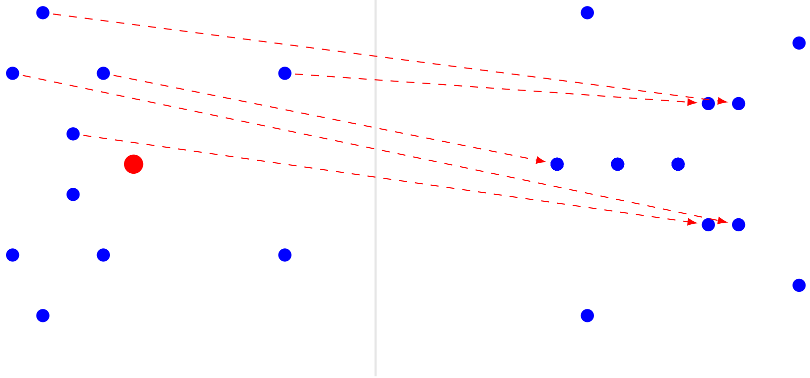

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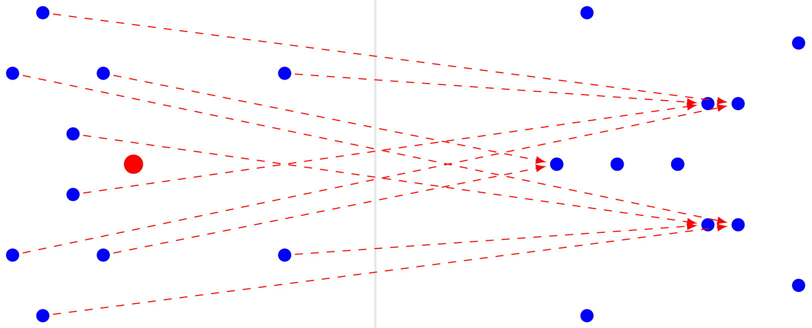

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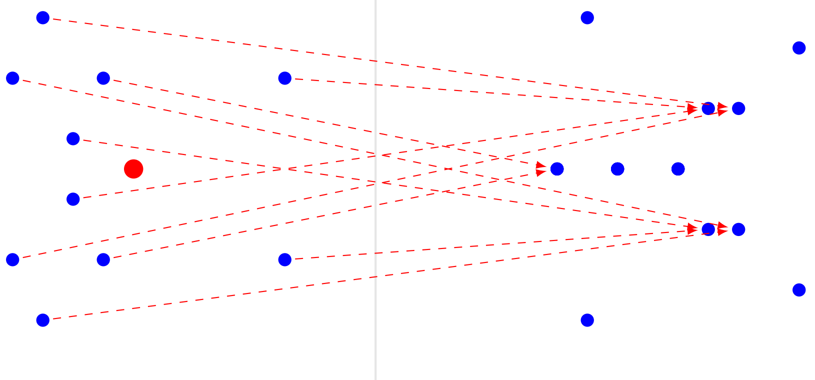


$$\phi(x,y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2}\right)$$

- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

Curves over finite fields

Frobenius endomorphism

Let E be defined over $\mathbb{F}_q$. The Frobenius endomorphism of E is the map

$$\pi \;:\; (X : Y : Z) \mapsto (X^q : Y^q : Z^q).$$

Hasse's theorem

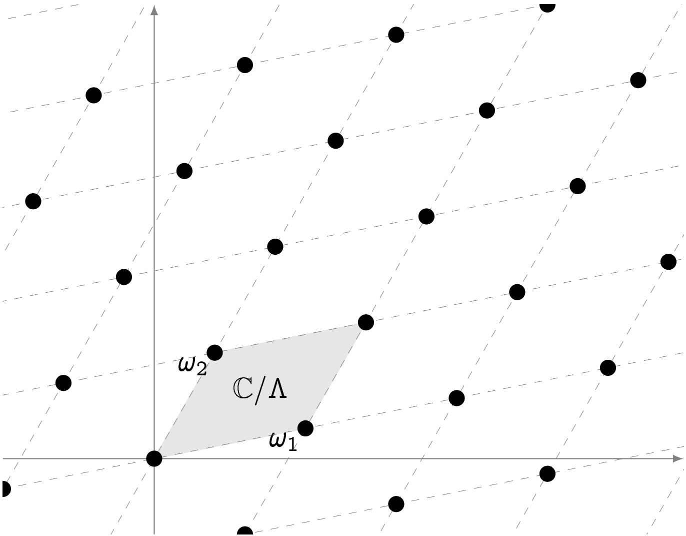
Let E be defined over $\mathbb{F}_q$, then

$$|\#E(k) - q - 1| \leq 2\sqrt{q}.$$

Serre-Tate theorem

Two elliptic curves E, E' defined over a finite field k are isogenous over k if and only if $\#E(k) = \#E'(k)$.

Complex tori

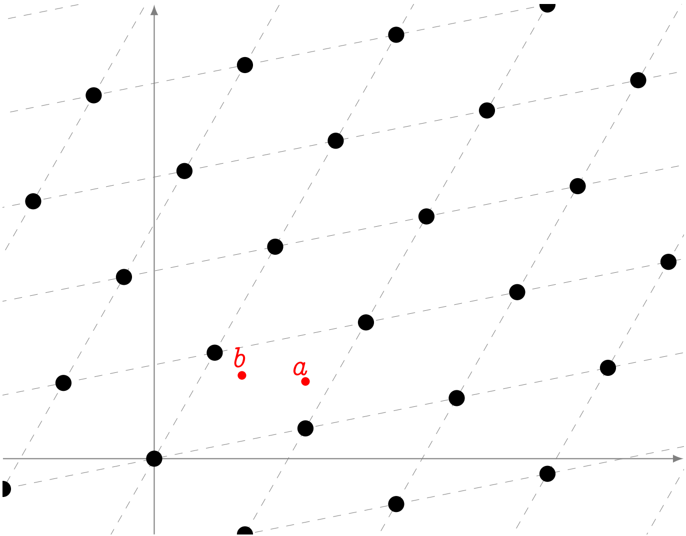


Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent complex numbers. Set

$$\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$$

$\mathbb{C}/\Lambda$ is a complex torus.

Complex tori



Addition law induced by addition on $\mathbb{C}$.

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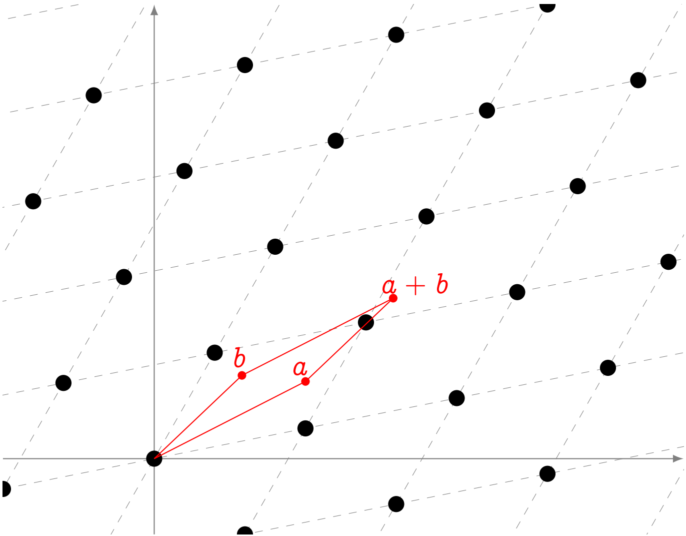

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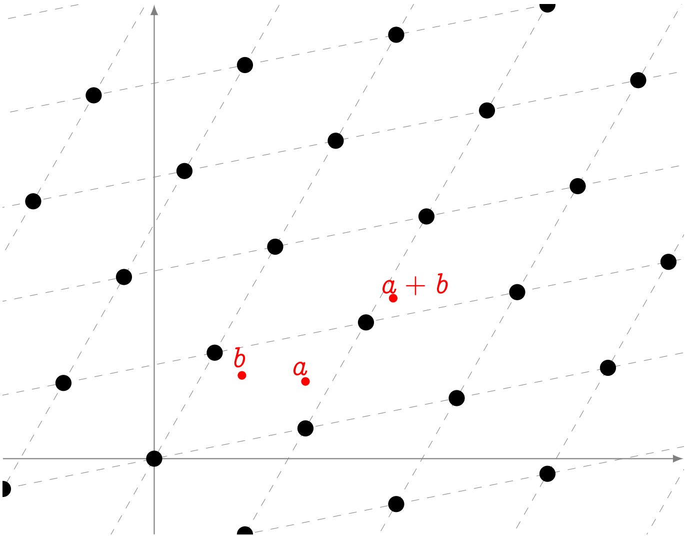

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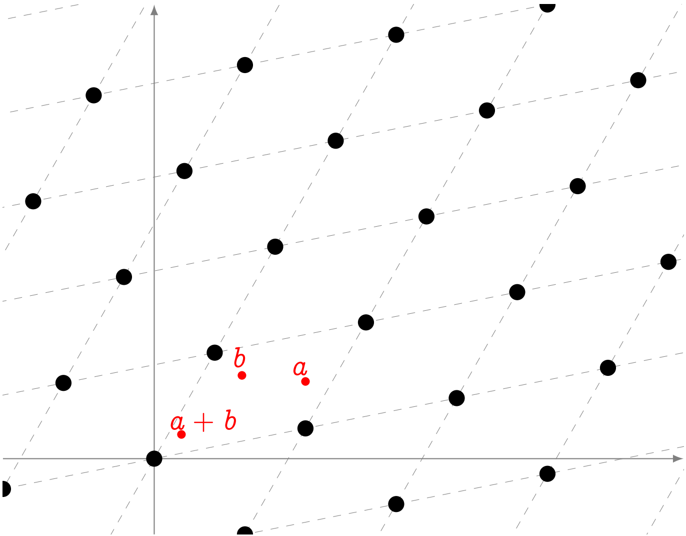

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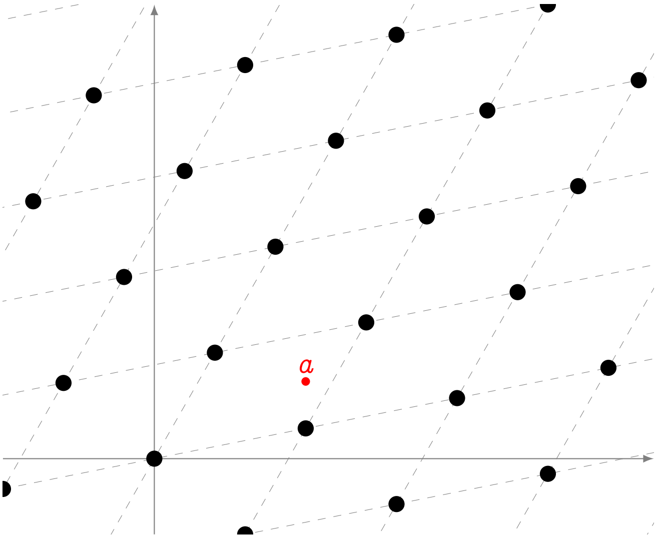
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

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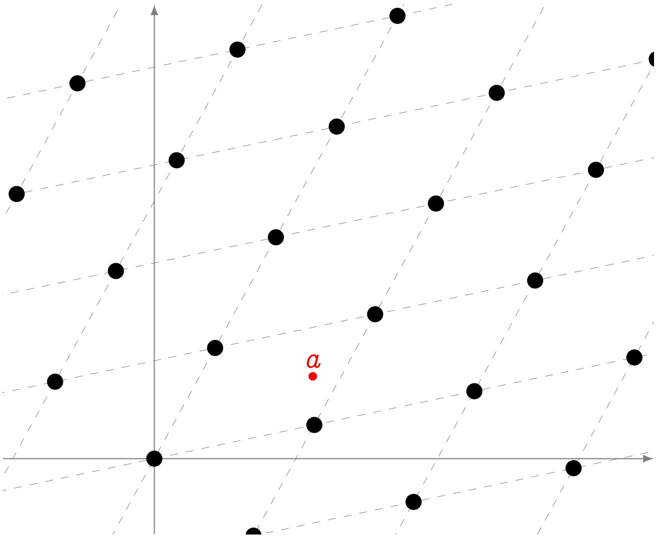

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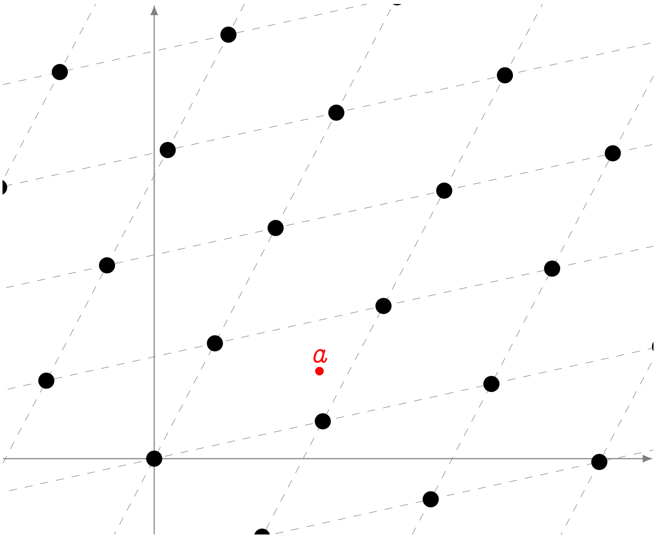

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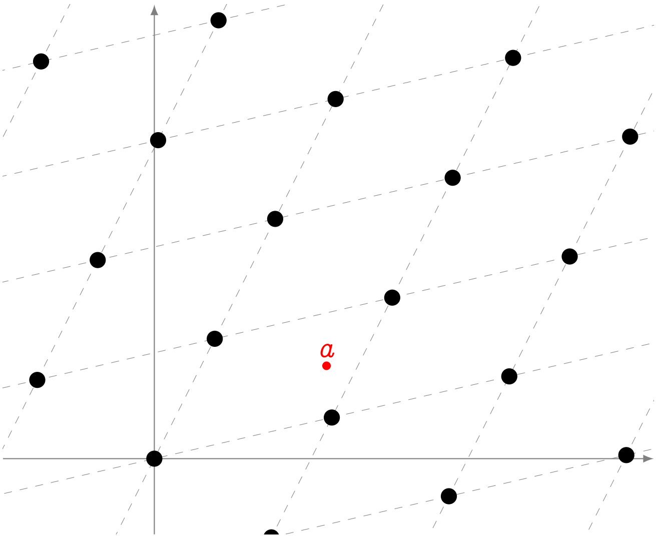

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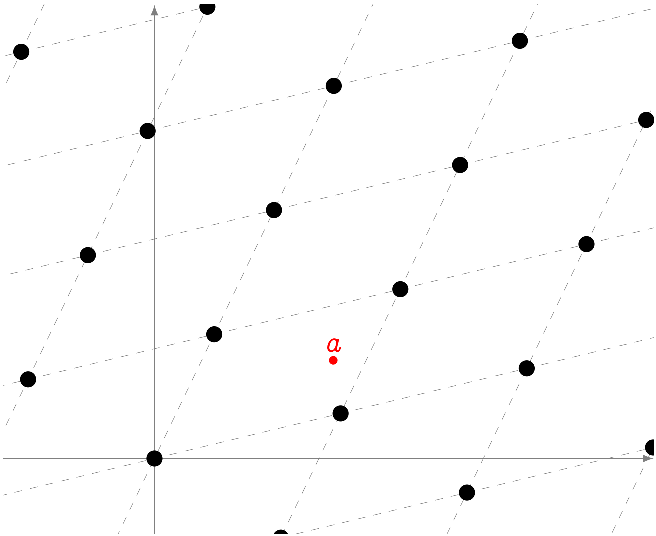

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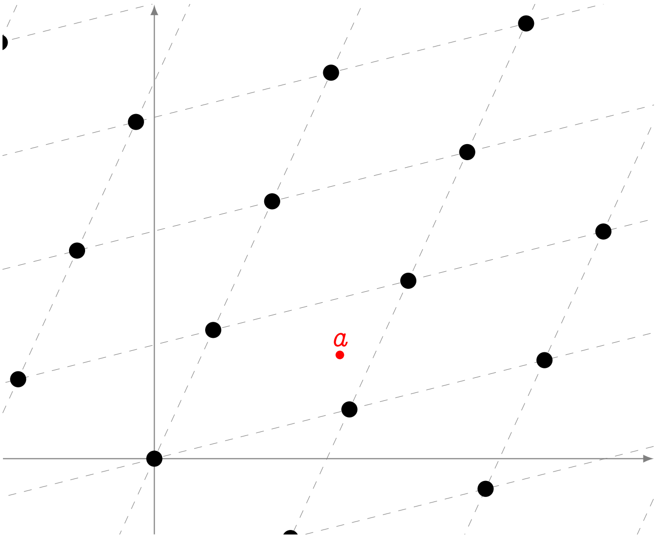

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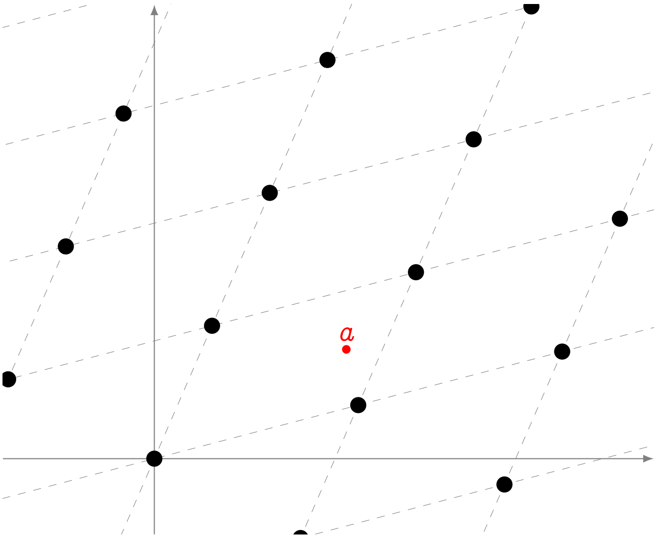

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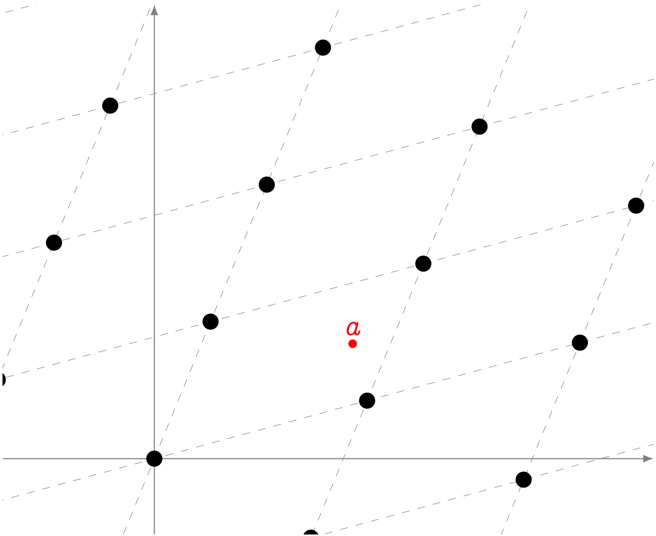

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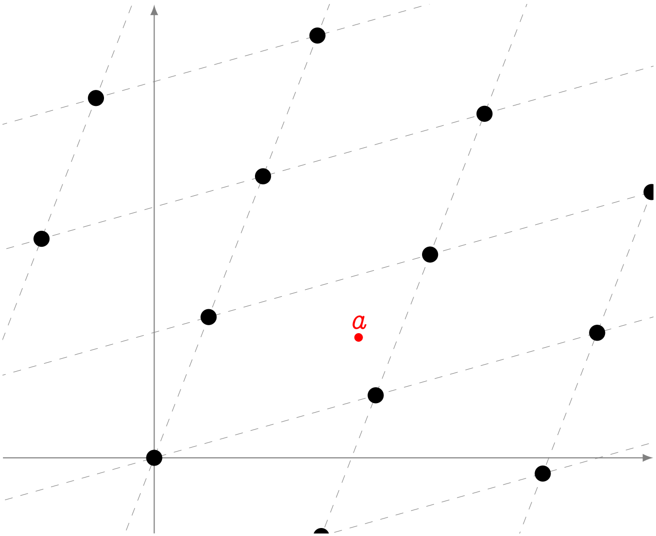

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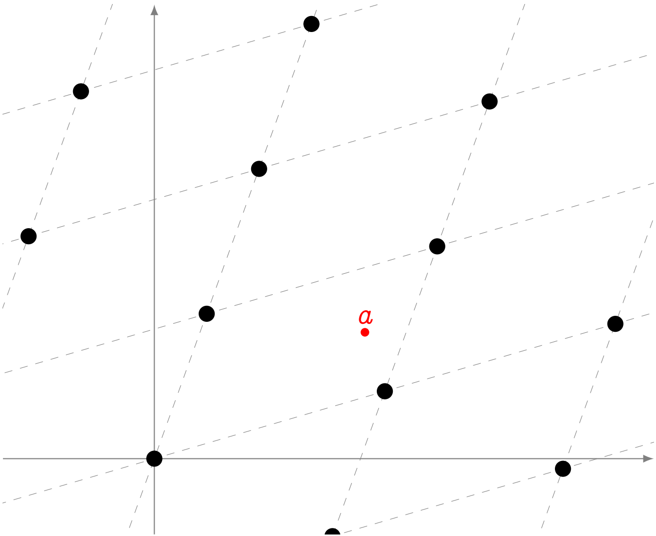

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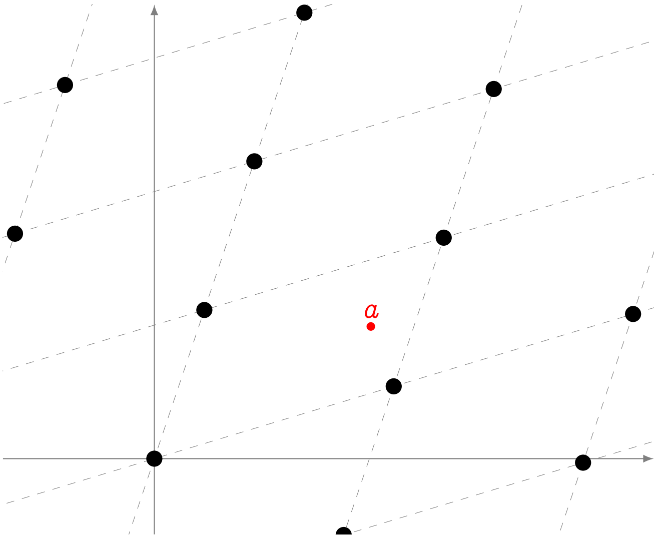

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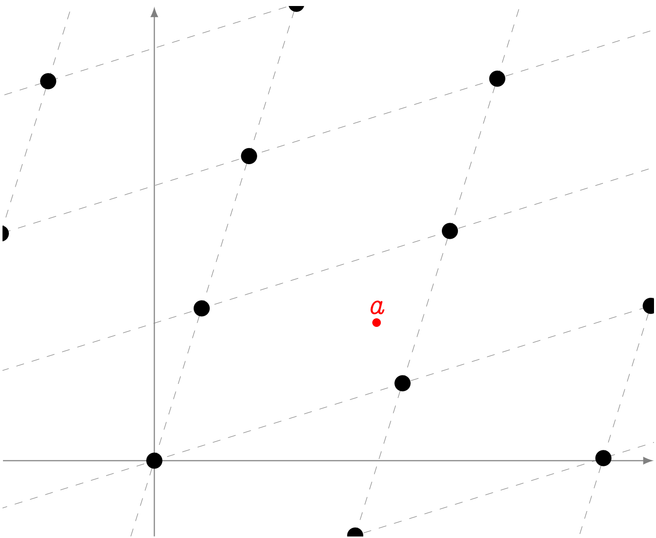

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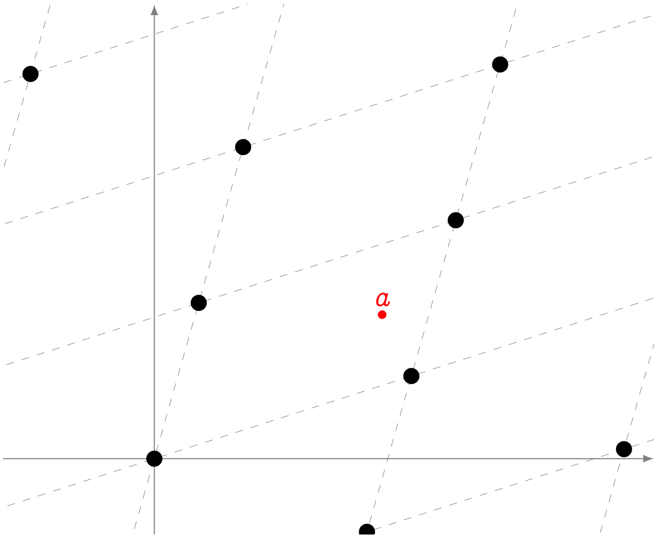

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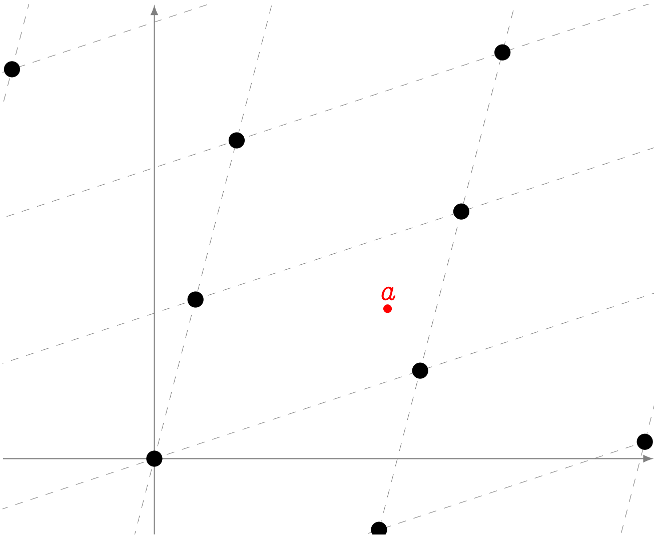

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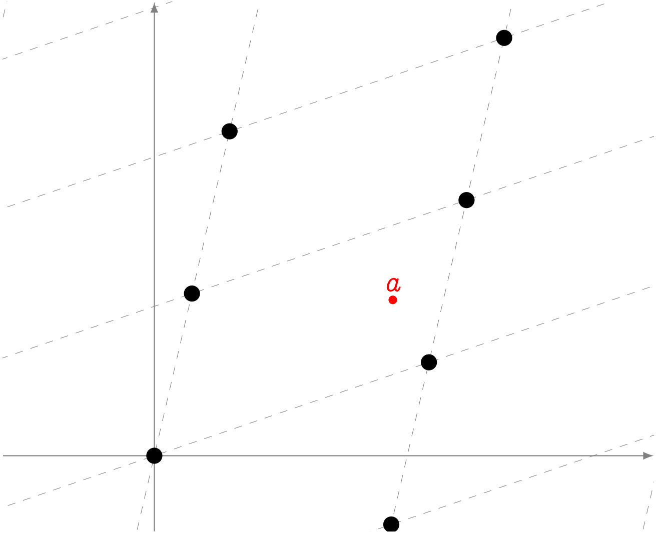

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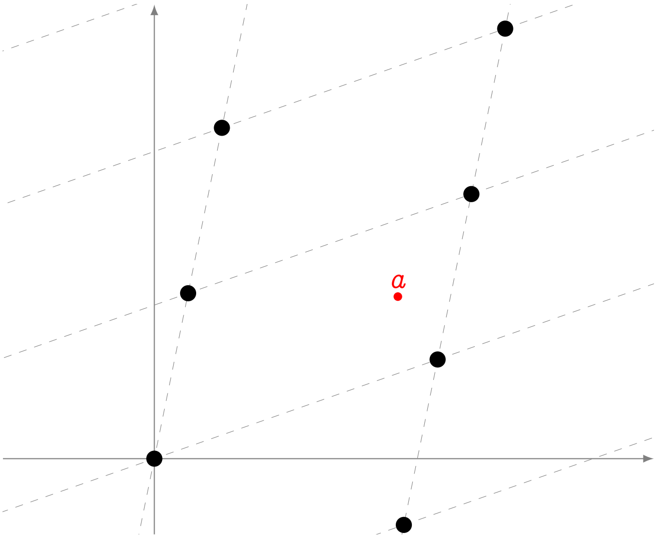

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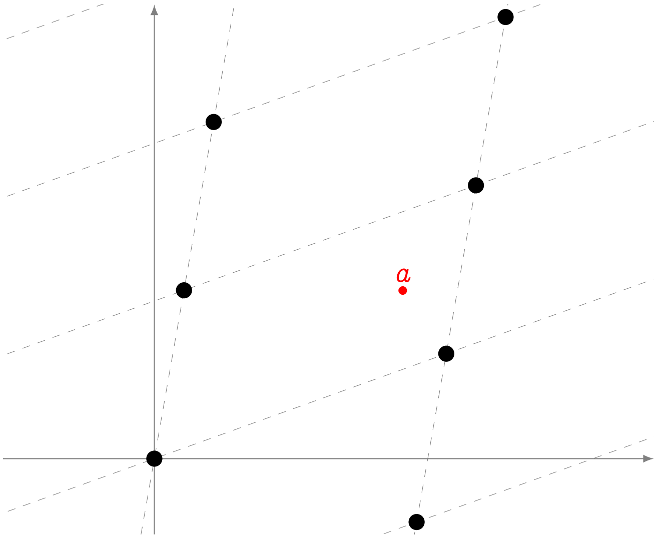

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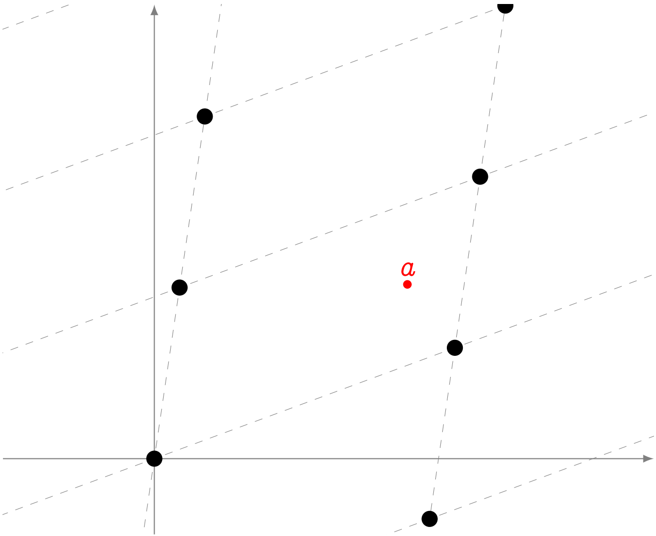

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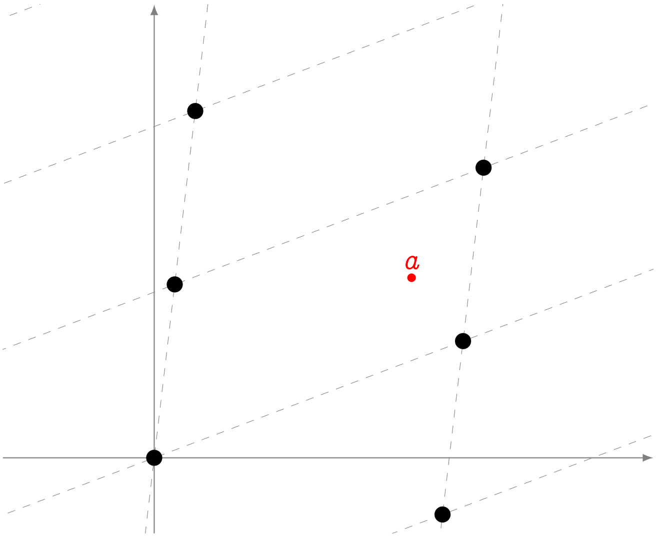

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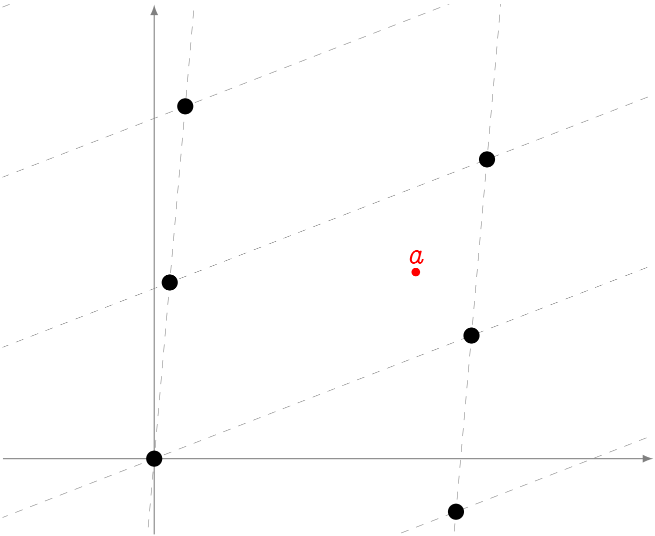

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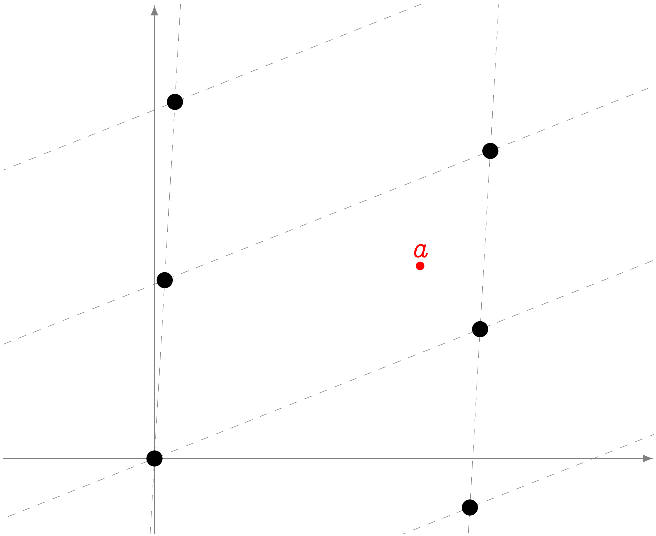

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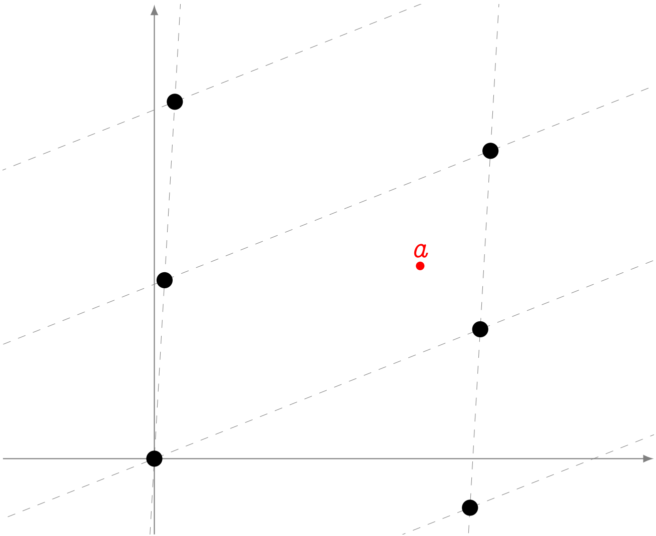

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The j-invariant

We want to classify complex lattices/tori up to homothety.

Eisenstein series

Let Λ be a complex lattice. For any integer $k > 0$ define

$$G_{2k}(\Lambda) = \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-2k}.$$

Also set

$$g_2(\Lambda) = 60\, G_4(\Lambda), \qquad g_3(\Lambda) = 140\, G_6(\Lambda).$$

Modular j-invariant

Let Λ be a complex lattice, the modular j-invariant is

$$j(\Lambda) = 1728 \frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27 g_3(\Lambda)^2}.$$

Two lattices Λ, Λ' are homothetic if and only if $j(\Lambda) = j(\Lambda')$.

Elliptic curves over $\mathbb{C}$

Weierstrass $\wp$ function

Let Λ be a complex lattice, the Weierstrass $\wp$ function associated to Λ is the series

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

Fix a lattice Λ, then $\wp$ and its derivative $\wp'$ are elliptic functions:

$$\wp(z+\omega) = \wp(z), \qquad \wp'(z+\omega) = \wp'(z)$$

for all $\omega \in \Lambda$.

Uniformization theorem

Let Λ be a complex lattice. The curve

$$E \ : \ y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$$

is an elliptic curve over $\mathbb{C}$. The map

$$\begin{aligned} \mathbb{C}/\Lambda &\to E(\mathbb{C}), \\ 0 &\mapsto (0 : 1 : 0), \\ z &\mapsto (\wp(z) : \wp'(z) : 1) \end{aligned}$$

is an isomorphism of Riemann surfaces and a group morphism.

Conversely, for any elliptic curve

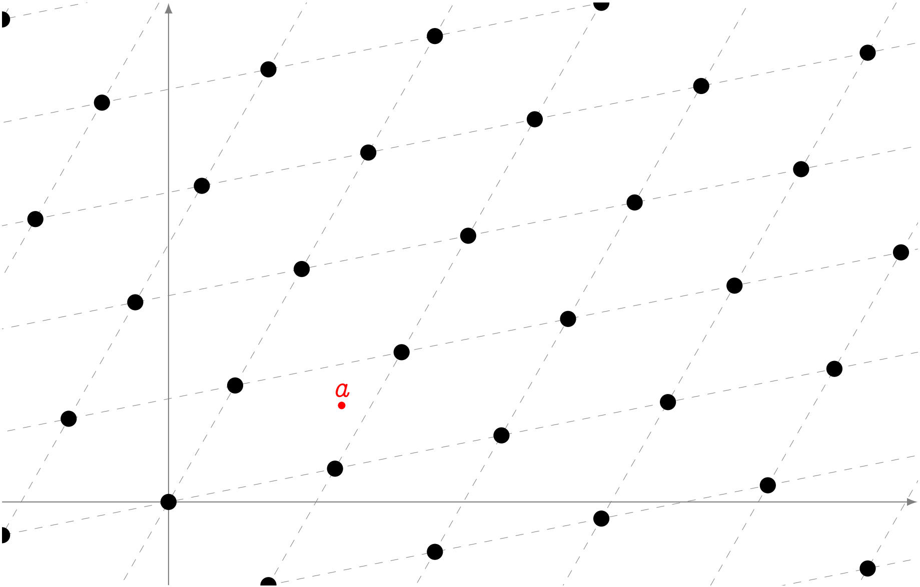
$$E \ : \ y^2 = x^3 + ax + b$$

there is a unique complex lattice Λ such that

$$g_2(\Lambda) = -4a, \qquad g_3(\Lambda) = -4b.$$

Moreover $j(\Lambda) = j(E)$.

Multiplication



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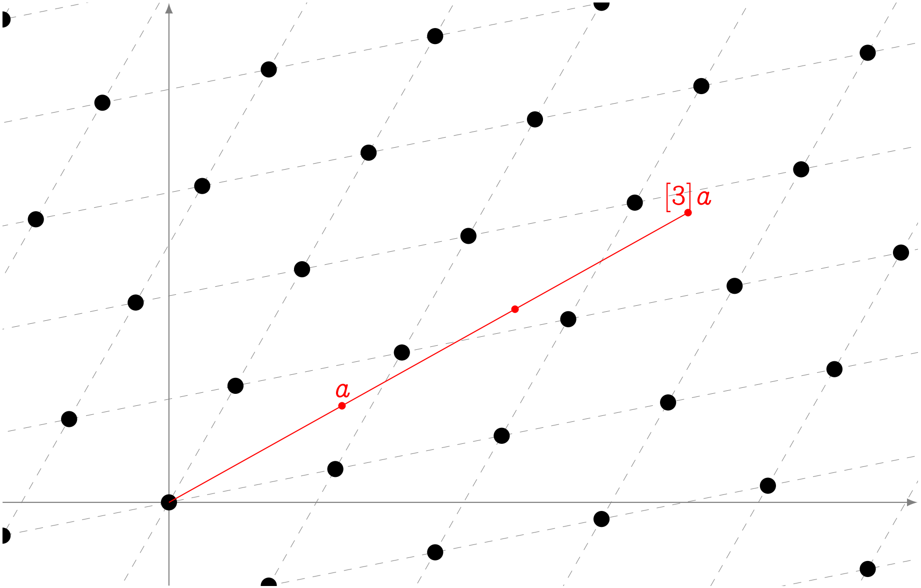
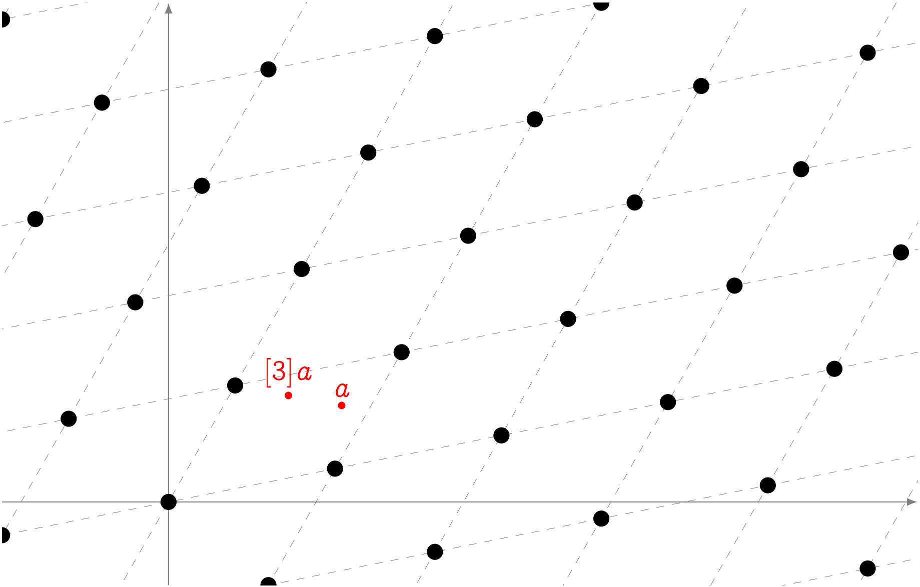
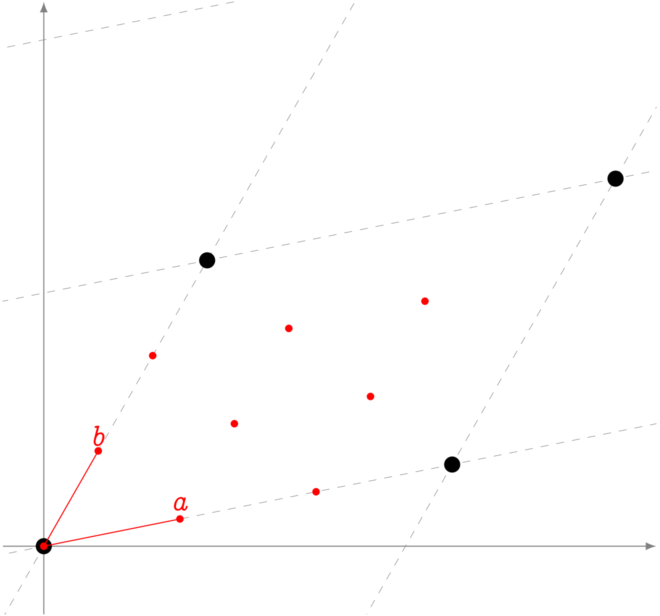


Multiplication



Torsion subgroups



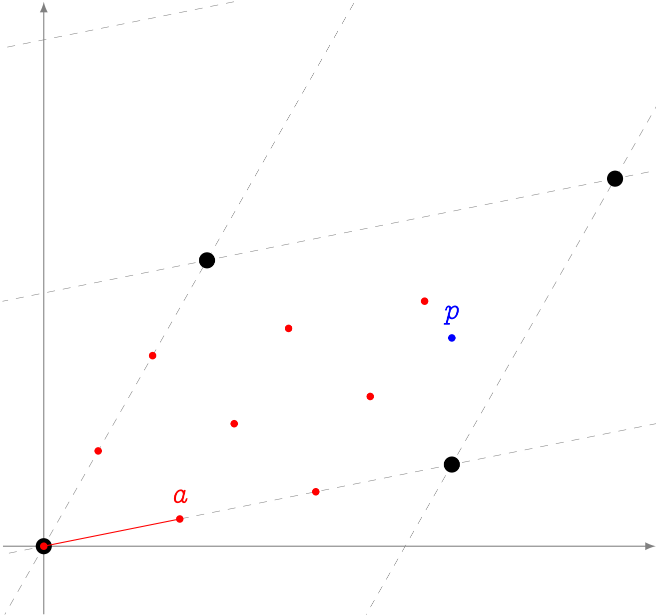
The ℓ-torsion subgroup is made up by the points

$$\left(\frac{i\omega_1}{\ell}, \frac{j\omega_2}{\ell}\right)$$

It is a group of rank two

$$E[\ell] = \langle a, b \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

Isogenies



Let $a \in \mathbb{C}/\Lambda_1$ be an ℓ-torsion point, and let

$$\Lambda_2 = a\mathbb{Z} \oplus \Lambda_1$$

Then $\Lambda_1 \subset \Lambda_2$ and we define a degree ℓ cover

$$\phi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$$

ϕ is a morphism of complex Lie groups and is called an isogeny.

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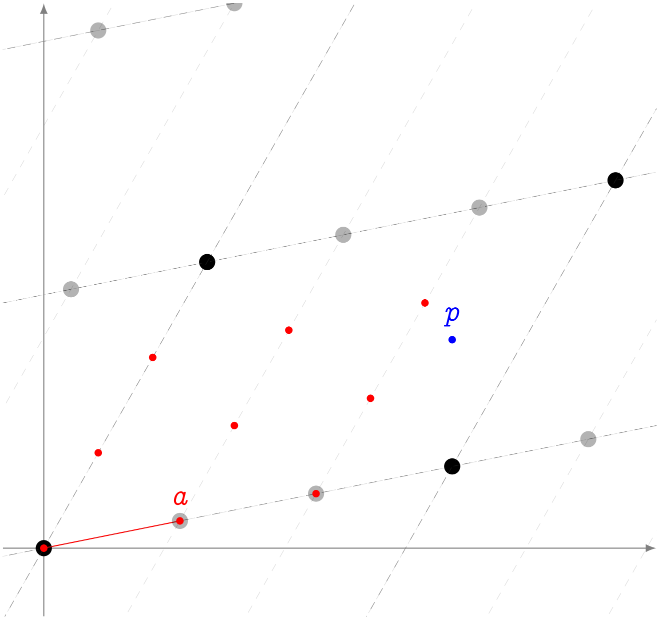

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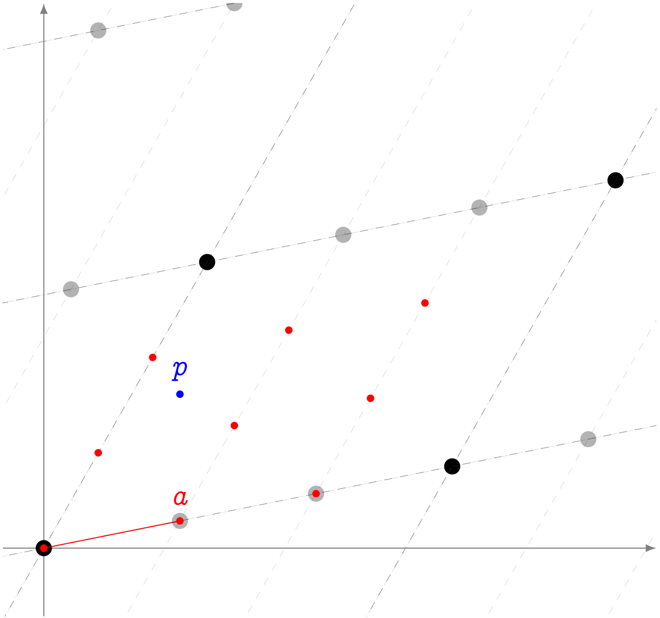

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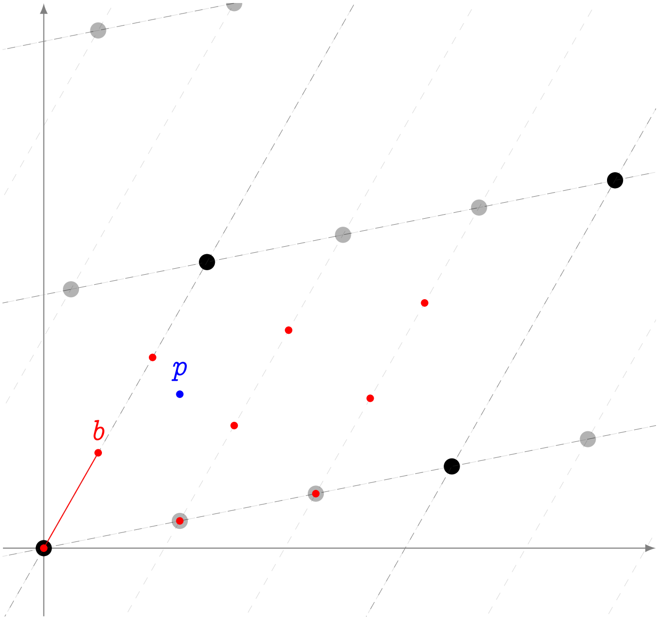


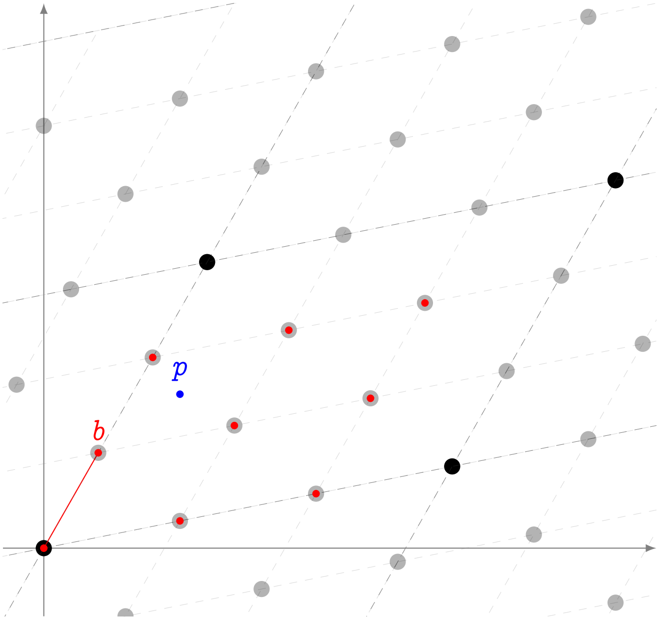
Taking a point b not in the kernel of ϕ, we obtain a new degree ℓ cover

$$\hat{\phi} : \mathbb{C}/\Lambda_2 \to \mathbb{C}/\Lambda_3$$

The composition $\hat{\phi} \circ \phi$ has degree ℓ^2 and is homothetic to the multiplication by ℓ map.

$\hat{\phi}$ is called the dual isogeny of ϕ.

Isogenies



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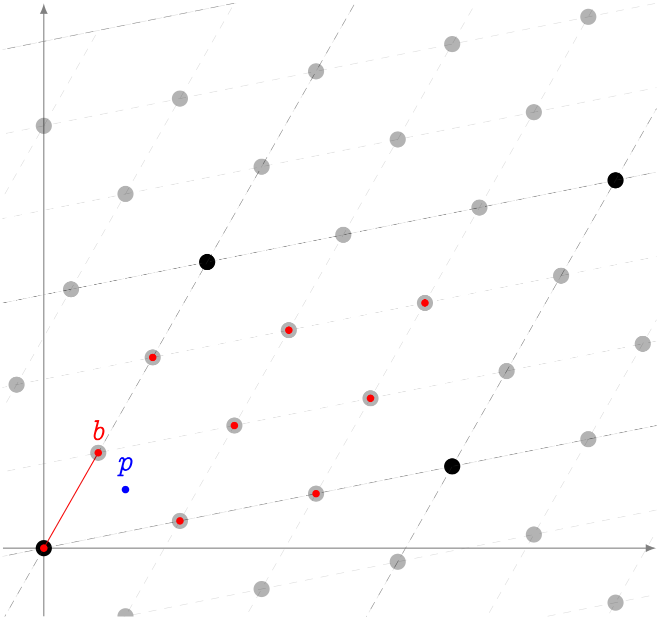

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Isogenies: back to algebra

Let $\phi : E \to E'$ be an isogeny defined over a field k of characteristic p.

- $k(E)$ is the field of all rational functions from E to k;
- $\phi^* k(E')$ is the subfield of $k(E)$ defined as

$$\phi^* k(E') = \{f \circ \phi \mid f \in k(E')\}.$$

Degree, separability

1. The degree of ϕ is $\deg \phi = [k(E) : \phi^* k(E')]$. It is always finite.
2. ϕ is said to be separable, inseparable, or purely inseparable if the extension of function fields is.
3. If ϕ is separable, then $\deg \phi = \# \ker \phi$.
4. If ϕ is purely inseparable, then $\ker \phi = \{\mathcal{O}\}$ and $\deg \phi$ is a power of p.
5. Any isogeny can be decomposed as a product of a separable and a purely inseparable isogeny.

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Isogenies: separable vs inseparable

Purely inseparable isogenies

Examples:

- The Frobenius endomorphism is purely inseparable of degree q.
- All purely inseparable maps in characteristic p are of the form $(X : Y : Z) \mapsto (X^{p^e} : Y^{p^e} : Z^{p^e})$.

Separable isogenies

Let E be an elliptic curve, and let G be a finite subgroup of E. There are a unique elliptic curve E' and a unique separable isogeny ϕ, such that $\ker\phi = G$ and $\phi : E \to E'$.
The curve E' is called the quotient of E by G and is denoted by E/G.

The dual isogeny

Let $\phi : E \to E'$ be an isogeny of degree m. There is a unique isogeny $\hat{\phi} : E' \to E$ such that

$$\hat{\phi} \circ \phi = [m]_E, \quad \phi \circ \hat{\phi} = [m]_{E'}.$$

$\hat{\phi}$ is called the dual isogeny of ϕ; it has the following properties:

1. $\hat{\phi}$ is defined over k if and only if ϕ is;
2. $\widehat{\psi \circ \phi} = \hat{\phi} \circ \hat{\psi}$ for any isogeny $\psi : E' \to E''$;
3. $\widehat{\psi + \phi} = \hat{\psi} + \hat{\phi}$ for any isogeny $\psi : E \to E'$;
4. $\deg \phi = \deg \hat{\phi}$;
5. $\hat{\hat{\phi}} = \phi$.

Algebras, orders

- A quadratic imaginary number field is an extension of $\mathbb{Q}$ of the form $Q[\sqrt{-D}]$ for some non-square $D > 0$.
- A quaternion algebra is an algebra of the form $\mathbb{Q} + \alpha\mathbb{Q} + \beta\mathbb{Q} + \alpha\beta\mathbb{Q}$, where the generators satisfy the relations

$$\alpha^2, \beta^2 \in \mathbb{Q}, \quad \alpha^2 < 0, \quad \beta^2 < 0, \quad \beta\alpha = -\alpha\beta.$$

Orders

Let K be a finitely generated $\mathbb{Q}$-algebra. An order $\mathcal{O} \subset K$ is a subring of K that is a finitely generated $\mathbb{Z}$-module of maximal dimension. An order that is not contained in any other order of K is called a maximal order.

Examples:

- $\mathbb{Z}$ is the only order contained in $\mathbb{Q}$,
- $\mathbb{Z}[i]$ is the only maximal order of $\mathbb{Q}(i)$,
- $\mathbb{Z}[\sqrt{5}]$ is a non-maximal order of $\mathbb{Q}(\sqrt{5})$,
- The ring of integers of a number field is its only maximal order,
- In general, maximal orders in quaternion algebras are not unique.

The endomorphism ring

The endomorphism ring $\mathrm{End}(E)$ of an elliptic curve E is the ring of all isogenies $E \to E$ (plus the null map) with addition and composition.

Theorem (Deuring)

Let E be an elliptic curve defined over a field k of characteristic p.
$\mathrm{End}(E)$ is isomorphic to one of the following:

- $\mathbb{Z}$, only if $p = 0$

  E is ordinary.

- An order $\mathcal{O}$ in a quadratic imaginary field:

  E is ordinary with complex multiplication by $\mathcal{O}$.

- Only if $p > 0$, a maximal order in a quaternion algebra[a]:

  E is supersingular.

[a](ramified at p and ∞)

The finite field case

Theorem (Hasse)

Let E be defined over a finite field. Its Frobenius endomorphism π satisfies a quadratic equation

$$\pi^2 - t\pi + q = 0$$

in $\mathrm{End}(E)$ for some $|t| \leq 2\sqrt{q}$, called the trace of π. The trace t is coprime to q if and only if E is ordinary.

Suppose E is ordinary, then $D_\pi = t^2 - 4q < 0$ is the discriminant of $\mathbb{Z}[\pi]$.

- $K = \mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{D_\pi})$ is the endomorphism algebra of E.
- Denote by $\mathcal{O}_K$ its ring of integers, then

$$\mathbb{Z} \neq \mathbb{Z}[\pi] \subset \mathrm{End}(E) \subset \mathcal{O}_K.$$

In the supersingular case, π may or may not be in $\mathbb{Z}$, depending on q.

Endomorphism rings of ordinary curves

Classifying quadratic orders

Let K be a quadratic number field, and let $\mathcal{O}_K$ be its ring of integers.

- Any order $\mathcal{O} \subset K$ can be written as $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ for an integer f, called the conductor of $\mathcal{O}$, denoted by $[\mathcal{O}_k : \mathcal{O}]$.
- If d_K is the discriminant of K, the discriminant of $\mathcal{O}$ is $f^2 d_K$.
- If $\mathcal{O}, \mathcal{O}'$ are two orders with discriminants d, d', then $\mathcal{O} \subset \mathcal{O}'$ iff $d' | d$.

$$\begin{tikzcd}
 & \mathcal{O}_K \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Z}+2\mathcal{O}_K \arrow[d, dash] \arrow[dr, dash] & \mathbb{Z}+3\mathcal{O}_K \arrow[dl, dash] \arrow[dr, dash] & \mathbb{Z}+5\mathcal{O}_K \arrow[dl, dash] \arrow[d, dash] \\
\mathbb{Z}+6\mathcal{O}_K \arrow[dr, dash] & \mathbb{Z}+10\mathcal{O}_K \arrow[d, dash] & \mathbb{Z}+15\mathcal{O}_K \arrow[dl, dash] \\
 & \mathbb{Z}[\pi] \simeq \mathbb{Z}+30\mathcal{O}_K &
\end{tikzcd}$$

Ideal lattices

Fractional ideals

Let $\mathcal{O}$ be an order of a number field K. A (fractional) $\mathcal{O}$-ideal $\mathfrak{a}$ is a finitely generated non-zero $\mathcal{O}$-submodule of K.

When K is imaginary quadratic:

- Fractional ideals are complex lattices,
- Any lattice $\Lambda \subset K$ is a fractional ideal,
- The order of a lattice Λ is

$$\mathcal{O}_\Lambda = \{\alpha \in K \mid \alpha\Lambda \subset \Lambda\}$$

Complex multiplication

Let $\Lambda \subset K$, the elliptic curve associated to $\mathbb{C}/\Lambda$ has complex multiplication by $\mathcal{O}_\Lambda$.

The class group

Let $\mathrm{End}(E) = \mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$. Define

- $\mathcal{I}(\mathcal{O})$, the group of invertible fractional ideals,
- $\mathcal{P}(\mathcal{O})$, the group of principal ideals,

The class group

The class group of $\mathcal{O}$ is

$$\mathrm{Cl}(\mathcal{O}) = \mathcal{I}(\mathcal{O})/\mathcal{P}(\mathcal{O}).$$

- It is a finite abelian group.
- Its order $h(\mathcal{O})$ is called the class number of $\mathcal{O}$.
- It arises as the Galois group of an abelian extension of $\mathbb{Q}(\sqrt{-D})$.

Complex multiplication

Fundamental theorem of CM

Let $\mathcal{O}$ be an order of a number field K, and let $\mathfrak{a}_1, \dots, \mathfrak{a}_{h(\mathcal{O})}$ be representatives of $\mathrm{Cl}(\mathcal{O})$. Then:

- $K(j(\mathfrak{a}_i))$ is an Abelian extension of K;
- The $j(\mathfrak{a}_i)$ are all conjugate over K;
- The Galois group of $K(j(\mathfrak{a}_i))$ is isomorphic to $\mathrm{Cl}(\mathcal{O})$;
- $[\mathbb{Q}(j(\mathfrak{a}_i)) : \mathbb{Q}] = [K(j(\mathfrak{a}_i)) : K] = h(\mathcal{O})$;
- The $j(\mathfrak{a}_i)$ are integral, their minimal polynomial is called the Hilbert class polynomial of $\mathcal{O}$.

Lifting

Deuring's lifting theorem

Let E_0 be an elliptic curve in characteristic p, with an endomorphism ω_o which is not trivial. Then there exists an elliptic curve E defined over a number field L, an endomorphism ω of E, and a non-singular reduction of E at a place $\mathfrak{p}$ of L lying above p, such that E_0 is isomorphic to $E(\mathfrak{p})$, and ω_0 corresponds to $\omega(\mathfrak{p})$ under the isomorphism.

Executive summary

- Elliptic curves are algebraic groups;
- Isogenies are the natural notion of morphism for EC: both group and projective variety morphism;
- We can understand most things about isogenies by looking only at endomorphisms;
- Isogenies of curves over $\mathbb{C}$ are especially simple to describe;
- It is easy to construct curves over $\mathbb{C}$ with prescribed complex multiplication;
- Most of what happens in positive characteristic can be understood by:
  - looking at the Frobenius endomorphism, and/or
  - looking at reductions of curves in characteristic 0.

Plan

1. Elliptic curves, isogenies, complex multiplication
2. Isogeny graphs
3. Key exchange

Isogeny graphs

Serre-Tate theorem reloaded

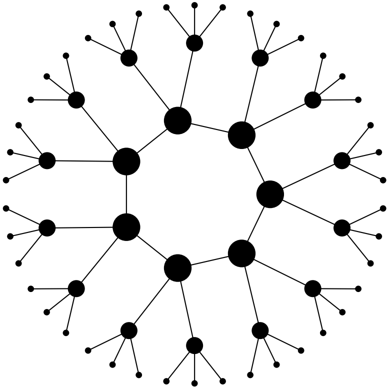
Two elliptic curves E, E' defined over a finite field are isogenous iff their endomorphism algebras $\mathrm{End}(E) \otimes \mathbb{Q}$ and $\mathrm{End}(E') \otimes \mathbb{Q}$ are isomorphic.

Isogeny graphs

- Vertices are curves up to isomorphism,
- Edges are isogenies up to isomorphism.

Isogeny volcanoes

- Curves are ordinary,
- Isogenies all have degree a prime ℓ.



What do isogeny graphs look like?

Torsion subgroups (ℓ prime)

In an algebraically closed field:

$$E[\ell] = \langle P, Q \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

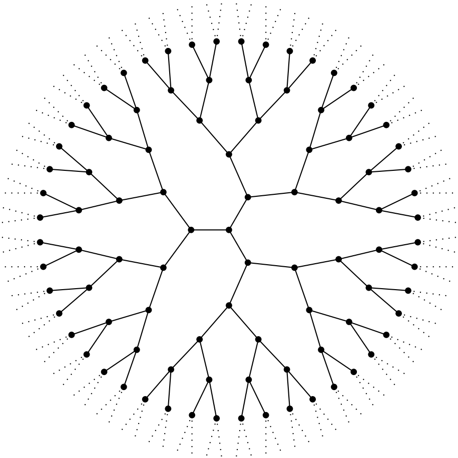
$$\Downarrow$$

There are exactly $\ell + 1$ cyclic subgroups $H \subset E$ of order ℓ:

$$\langle P + Q \rangle, \langle P + 2Q \rangle, \dots, \langle P \rangle, \langle Q \rangle$$

$$\Downarrow$$

There are exactly $\ell + 1$ distinct isogenies of degree ℓ.



(non-CM) 2-isogeny graph over $\mathbb{C}$

What happens over a finite field $\mathbb{F}_p$?

Rational isogenies ($\ell \neq p$)

In the algebraic closure $\bar{\mathbb{F}}_p$

$$E[\ell] = \langle P, Q \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

However, an isogeny is defined over $\mathbb{F}_p$ only if its kernel is Galois invariant.

Enter the Frobenius map

$$\begin{aligned} \pi : E &\longrightarrow E \\ (x, y) &\longmapsto (x^p, y^p) \end{aligned}$$

E is seen here as a curve over $\bar{\mathbb{F}}_p$.

The Frobenius action on $E[\ell]$

$$\begin{aligned} \pi(P) &= \quad aP + bQ \\ \pi(Q) &= \quad cP + dQ \end{aligned}$$

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The Frobenius action on $E[\ell]$

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We identify $\pi|E[\ell]$ to a conjugacy class in $\mathrm{GL}(\mathbb{Z}/\ell\mathbb{Z})$.

What happens over a finite field $\mathbb{F}_p$?

Galois invariant subgroups of $E[\ell]$

=

eigenspaces of $\pi \in \mathrm{GL}(\mathbb{Z}/\ell\mathbb{Z})$

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rational isogenies of degree ℓ

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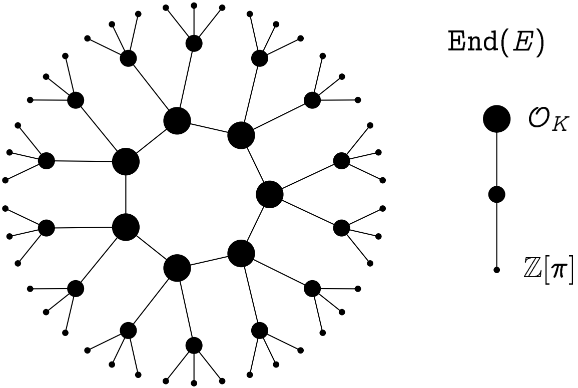
How many Galois invariant subgroups?

- $\pi|E[\ell] \sim \left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right)$ $\rightarrow$ $\ell + 1$ isogenies
- $\pi|E[\ell] \sim \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ with $\lambda \neq \mu$ $\rightarrow$ two isogenies
- $\pi|E[\ell] \sim \left(\begin{smallmatrix}\lambda & *\\ 0 & \lambda\end{smallmatrix}\right)$ $\rightarrow$ one isogeny
- $\pi|E[\ell]$ is not diagonalizable over $\mathbb{Z}/\ell\mathbb{Z}$ $\rightarrow$ no isogeny

Volcanology (Kohel 1996)

Let E, E' be curves with respective endomorphism rings $\mathcal{O}, \mathcal{O}' \subset K$.
Let $\phi : E \to E'$ be an isogeny of prime degree ℓ, then:

if $\mathcal{O} = \mathcal{O}'$, ϕ is horizontal;
if $[\mathcal{O}' : \mathcal{O}] = \ell$, ϕ is ascending;
if $[\mathcal{O} : \mathcal{O}'] = \ell$, ϕ is descending.

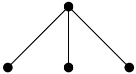


Ordinary isogeny volcano of degree $\ell = 3$.

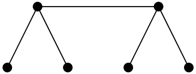
Volcanology (Kohel 1996)

Let E be ordinary,
$\mathrm{End}(E) \subset K$.

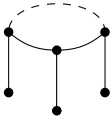
$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.



$\left(\frac{D_K}{\ell}\right) = -1$



$\left(\frac{D_K}{\ell}\right) = 0$



$\left(\frac{D_K}{\ell}\right) = +1$

		Horizontal	Ascending	Descending
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		
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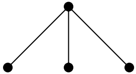
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	ℓ
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

Volcanology (Kohel 1996)

Let E be ordinary,
$\mathtt{End}(E) \subset K$.

$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.

Height $= v_\ell([\mathcal{O}_K : \mathbb{Z}[\pi]])$.



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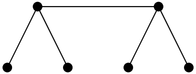

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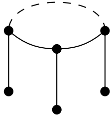

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$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		$\ell - \left(\frac{D_K}{\ell}\right)$
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	ℓ
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

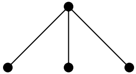
Volcanology (Kohel 1996)

Let E be ordinary,
$\mathrm{End}(E) \subset K$.

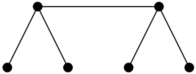
$\mathcal{O}_K$: maximal order of K,
D_K: discriminant of K.

Height $= v_\ell([\mathcal{O}_K : \mathbb{Z}[\pi]])$.

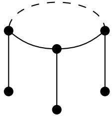
How large is the crater?



$\left(\frac{D_K}{\ell}\right) = -1$



$\left(\frac{D_K}{\ell}\right) = 0$



$\left(\frac{D_K}{\ell}\right) = +1$

		Horizontal	Ascending	Descending
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		
$\ell \nmid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$	$1 + \left(\frac{D_K}{\ell}\right)$		$\ell - \left(\frac{D_K}{\ell}\right)$
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \mid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	ℓ
$\ell \mid [\mathcal{O}_K : \mathcal{O}]$	$\ell \nmid [\mathcal{O} : \mathbb{Z}[\pi]]$		1	

How large is the crater of a volcano?

Let $\mathrm{End}(E) = \mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$. Define

- $\mathcal{I}(\mathcal{O})$, the group of invertible fractional ideals,
- $\mathcal{P}(\mathcal{O})$, the group of principal ideals,

The class group

The class group of $\mathcal{O}$ is

$$\mathrm{Cl}(\mathcal{O}) = \mathcal{I}(\mathcal{O})/\mathcal{P}(\mathcal{O}).$$

- It is a finite abelian group.
- Its order $h(\mathcal{O})$ is called the class number of $\mathcal{O}$.
- It arises as the Galois group of an abelian extension of $\mathbb{Q}(\sqrt{-D})$.

Complex multiplication

The $\mathfrak{a}$-torsion

- Let $\mathfrak{a} \subset \mathcal{O}$ be an (integral invertible) ideal of $\mathcal{O}$;
- Let $E[\mathfrak{a}]$ be the subgroup of E annihilated by $\mathfrak{a}$:

$$E[\mathfrak{a}] = \{P \in E \mid \alpha(P) = 0 \text{ for all } \alpha \in \mathfrak{a}\};$$

- Let $\phi : E \to E_{\mathfrak{a}}$, where $E_{\mathfrak{a}} = E/E[\mathfrak{a}]$.

Then $\mathrm{End}(E_{\mathfrak{a}}) = \mathcal{O}$ (i.e., ϕ is horizontal).

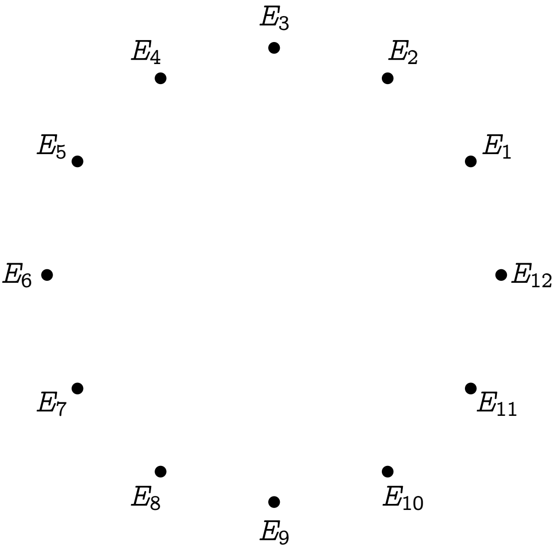
Theorem (Complex multiplication)

The action on the set of elliptic curves with complex multiplication by $\mathcal{O}$ defined by $\mathfrak{a} * j(E) = j(E_{\mathfrak{a}})$ factors through $\mathrm{Cl}(\mathcal{O})$, is faithful and transitive.

Corollary

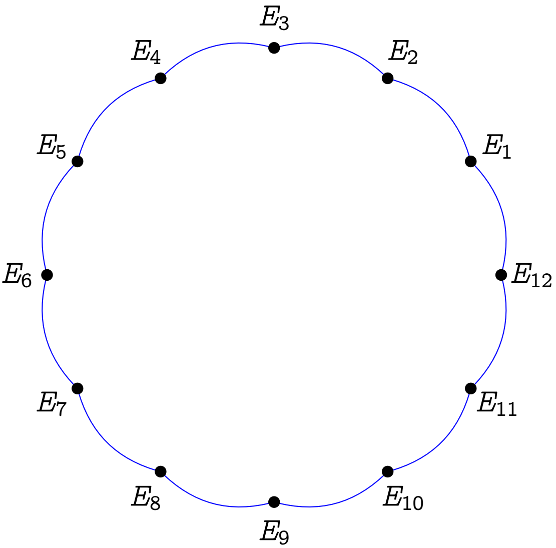
Let $\mathrm{End}(E)$ have discriminant D. Assume that $\left(\frac{D}{\ell}\right) = 1$, then E is on a crater of size N of an ℓ-volcano, and $N | h(\mathrm{End}(E))$

Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathtt{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

Complex multiplication graphs

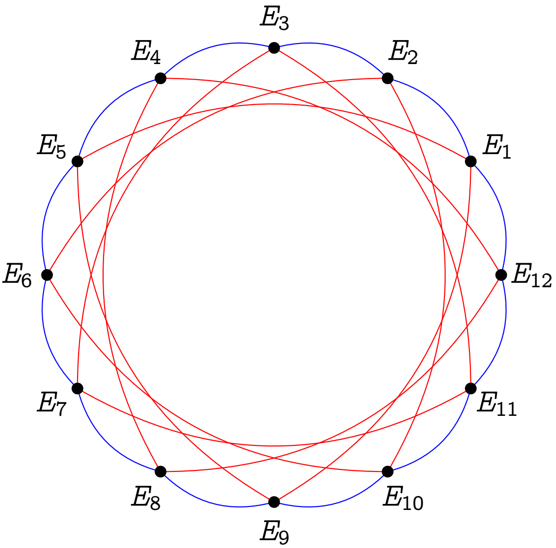


Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

Edges are horizontal isogenies of bounded prime degree.

— degree 2

Complex multiplication graphs



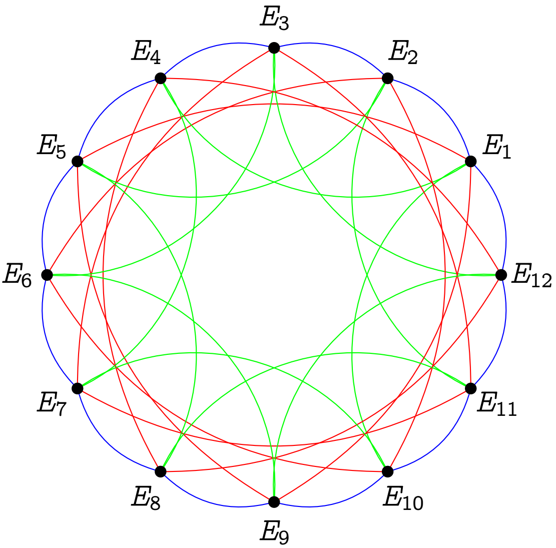
Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathrm{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathtt{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

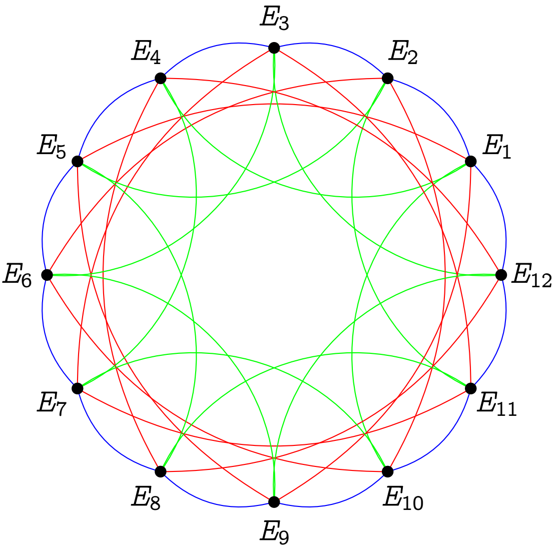
Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

— degree 5

Complex multiplication graphs



Vertices are elliptic curves with complex multiplication by $\mathcal{O}_K$ (i.e., $\mathtt{End}(E) \simeq \mathcal{O}_K \subset \mathbb{Q}(\sqrt{-D})$).

Edges are horizontal isogenies of bounded prime degree.

— degree 2

— degree 3

— degree 5

Isomorphic to a Cayley graph of $\mathtt{Cl}(\mathcal{O}_K)$.

Supersingular endomorphisms

Recall, a curve E over a field $\mathbb{F}_q$ of characteristic p is supersingular iff

$$\pi^2 - t\pi + q = 0$$

with $t = 0 \mod p$.

Case: $t = 0 \quad \Rightarrow \quad D_\pi = -4q$

- Only possibility for $E/\mathbb{F}_p$,
- $E/\mathbb{F}_p$ has CM by an order of $\mathbb{Q}(\sqrt{-p})$, similar to the ordinary case.

Case: $t = \pm 2\sqrt{q} \quad \Rightarrow \quad D_\pi = 0$

- General case for $E/\mathbb{F}_q$, when q is an even power.
- $\pi = \pm\sqrt{q}$, hence no complex multiplication.

We will ignore marginal cases: $t = \pm\sqrt{q}, \pm\sqrt{2q}, \pm\sqrt{3q}$.

Supersingular complex multiplication

Let $E/\mathbb{F}_p$ be a supersingular curve, then $\pi^2 = -p$, and

$$\pi = \begin{pmatrix} \sqrt{-p} & 0 \\ 0 & -\sqrt{-p} \end{pmatrix} \mod \ell$$

for any ℓ s.t. $\left(\frac{-p}{\ell}\right) = 1$.

Theorem (Delfs and Galbraith 2016)

Let $\mathrm{End}_{\mathbb{F}_p}(E)$ denote the ring of $\mathbb{F}_p$-rational endomorphisms of E. Then

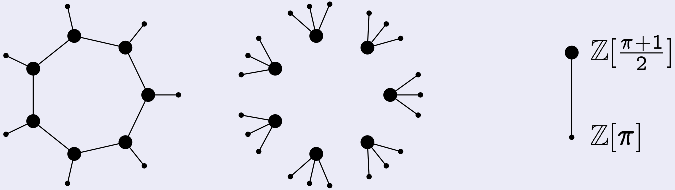
$$\mathbb{Z}[\pi] \subset \mathrm{End}_{\mathbb{F}_p}(E) \subset \mathbb{Q}(\sqrt{-p}).$$

Orders of $\mathbb{Q}(\sqrt{-p})$

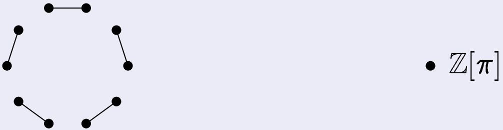
- If $p = 1 \mod 4$, then $\mathbb{Z}[\pi]$ is the maximal order.
- If $p = -1 \mod 4$, then $\mathbb{Z}[\frac{\pi+1}{2}]$ is the maximal order, and $[\mathbb{Z}[\frac{\pi+1}{2}] : \mathbb{Z}[\pi]] = 2$.

Supersingular CM graphs

2-volcanoes, $p = -1 \bmod 4$



2-graphs, $p = 1 \bmod 4$



All other ℓ-graphs are cycles of horizontal isogenies iff $\left(\frac{-p}{\ell}\right) = 1$.

The full endomorphism ring

Theorem (Deuring)

Let E be a supersingular elliptic curve, then

- E is isomorphic to a curve defined over $\mathbb{F}_{p^2}$;
- Every isogeny of E is defined over $\mathbb{F}_{p^2}$;
- Every endomorphism of E is defined over $\mathbb{F}_{p^2}$;
- $\mathrm{End}(E)$ is isomorphic to a maximal order in a quaternion algebra ramified at p and ∞.

In particular:

- If E is defined over $\mathbb{F}_p$, then $\mathrm{End}_{\mathbb{F}_p}(E)$ is strictly contained in $\mathrm{End}(E)$.
- Some endomorphisms do not commute!

An example

The curve of j-invariant 1728

$$E : y^2 = x^3 + x$$

is supersingular over $\mathbb{F}_p$ iff $p = -1 \mod 4$.

Endomorphisms

$\mathrm{End}(E) = \mathbb{Z}\langle\iota, \pi\rangle$, with:

- π the Frobenius endomorphism, s.t. $\pi^2 = -p$;
- ι the map

  $$\iota(x, y) = (-x, iy),$$

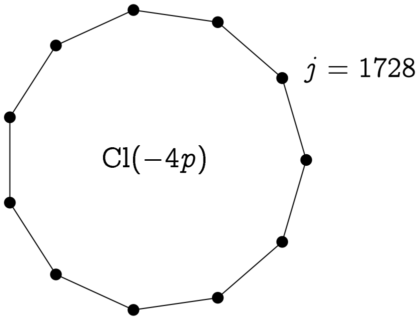
  where $i \in \mathbb{F}_{p^2}$ is a 4-th root of unity. Clearly, $\iota^2 = -1$.

And $\iota\pi = -\pi\iota$.

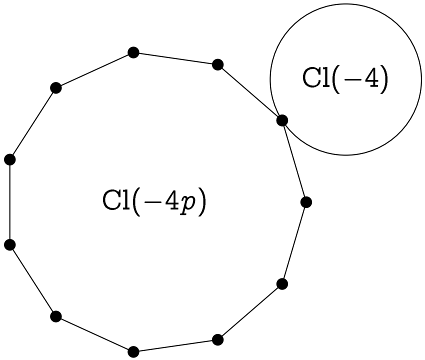
Class group action party

• $j = 1728$

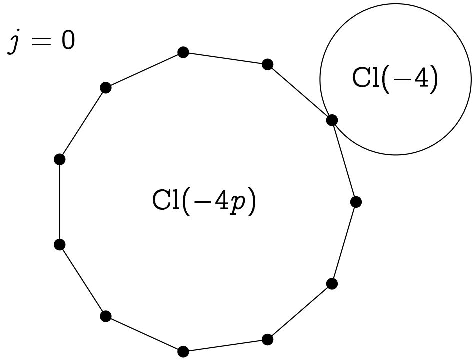
Class group action party



Class group action party



Class group action party



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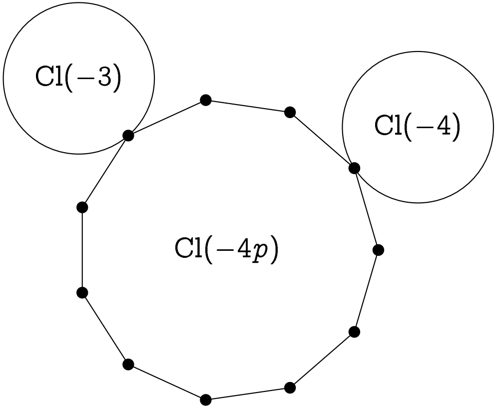
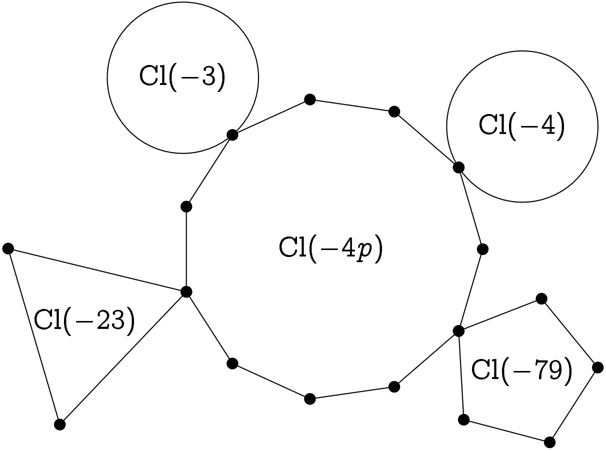


Class group action party



Quaternion algebra?! WTF?[2]

The quaternion algebra $B_{p,\infty}$ is:

- A 4-dimensional $\mathbb{Q}$-vector space with basis $(1, i, j, k)$.
- A non-commutative division algebra[1] $B_{p,\infty} = \mathbb{Q}\langle i, j\rangle$ with the relations:
$$i^2 = a, \quad j^2 = -p, \quad ij = -ji = k,$$
for some $a < 0$ (depending on p).
- All elements of $B_{p,\infty}$ are quadratic algebraic numbers.
- $B_{p,\infty} \otimes \mathbb{Q}_\ell \simeq \mathcal{M}_{2\times 2}(\mathbb{Q}_\ell)$ for all $\ell \neq p$.
I.e., endomorphisms restricted to $E[\ell^e]$ are just 2×2 matrices mod ℓ^e.
- $B_{p,\infty} \otimes \mathbb{R}$ is isomorphic to Hamilton's quaternions.
- $B_{p,\infty} \otimes \mathbb{Q}_p$ is a division algebra.

[1] All elements have inverses.
[2] What The Field?

Supersingular graphs

- Quaternion algebras have many maximal orders.
- For every maximal order type of $B_{p,\infty}$ there are 1 or 2 curves over $\mathbb{F}_{p^2}$ having endomorphism ring isomorphic to it.
- There is a unique isogeny class of supersingular curves over $\bar{\mathbb{F}}_p$ of size $\approx p/12$.
- Left ideals act on the set of maximal orders like isogenies.
- The graph of ℓ-isogenies is $(\ell + 1)$-regular.

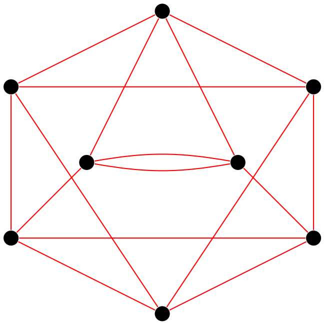


Figure: 3-isogeny graph on $\mathbb{F}_{97^2}$.

Graphs lexicon

Degree: Number of (outgoing/ingoing) edges.

k-regular: All vertices have degree k.

Connected: There is a path between any two vertices.

Distance: The length of the shortest path between two vertices.

Diamater: The longest distance between two vertices.

$\lambda_1 \geq \cdots \geq \lambda_n$: The (ordered) eigenvalues of the adjacency matrix.

Expander graphs

Proposition

If G is a k-regular graph, its largest and smallest eigenvalues satisfy

$$k = \lambda_1 \geq \lambda_n \geq -k.$$

Expander families

An infinite family of connected k-regular graphs on n vertices is an expander family if there exists an $\epsilon > 0$ such that all non-trivial eigenvalues satisfy $|\lambda| \leq (1 - \epsilon)k$ for n large enough.

- Expander graphs have short diameter ($O(\log n)$);
- Random walks mix rapidly (after $O(\log n)$ steps, the induced distribution on the vertices is close to uniform).

Expander graphs from isogenies

Theorem (Pizer 1990, 1998)

Let ℓ be fixed. The family of graphs of supersingular curves over $\mathbb{F}_{p^2}$ with ℓ-isogenies, as $p \to \infty$, is an expander family[a].

[a] Even better, it has the Ramanujan property.

Theorem (Jao, Miller, and Venkatesan 2009)

Let $\mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$ be an order in a quadratic imaginary field. The graphs of all curves over $\mathbb{F}_q$ with complex multiplication by $\mathcal{O}$, with isogenies of prime degree bounded[a] by $(\log q)^{2+\delta}$, are expanders.

[a] May contain traces of GRH.

Executive summary

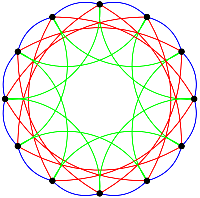
- Separable ℓ-isogeny = finite kernel = subgroup of $E[\ell]$,
  - eigenspace of π iff $\mathbb{F}_q$-rational,
  - distinct eigenvalues $\lambda \neq \mu$ define distinct directions on the crater.
- Isogeny graphs have j-invariants for vertices and “some” isogenies for edges.
- By varying the choices for the vertex and the isogeny set, we obtain graphs with different properties.
- ℓ-isogeny graphs of ordinary curves are volcanoes, (full) ℓ-isogeny graphs of supersingular curves are finite $(\ell + 1)$-regular.
- CM theory naturally leads to define graphs of horizontal isogenies (both in the ordinary and the supersingular case) that are isomorphic to Cayley graphs of class groups.
- CM graphs are expanders. Supseringular full ℓ-isogeny graphs are Ramanujan.

Plan

1. Elliptic curves, isogenies, complex multiplication
2. Isogeny graphs
3. Key exchange

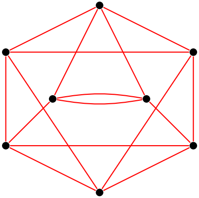
Isogeny graphs taxonomy

Complex Multiplication (CM) graphs



- Ordinary / Supersingular ($\mathbb{F}_p$)
- Superposition of isogeny cycles (one color per degree)
- Isomorphic to Cayley graph of a quadratic class group
- Large automorphism group
- Typical size $O(\sqrt{p})$
- Used in: **CSIDH**

Full **supersingular graphs**



- Supersingular ($\mathbb{F}_{p^2}$)
- One isogeny degree
- $(\ell + 1)$-regular
- Tiny automorphism group
- Size $\approx p/12$
- Used in: **SIDH**

Diffie–Hellman key exchange

Goal: Alice and Bob have never met before. They are chatting over a public channel, and want to agree on a shared secret to start a private conversation.

Setup: They agree on a (large) cyclic group $G = \langle g \rangle$ of order N.

Alice		**Bob**
pick random $a \in \mathbb{Z}/N\mathbb{Z}$		pick random $b \in \mathbb{Z}/N\mathbb{Z}$
compute $A = g^a$		compute $B = g^b$
	$\xrightarrow{\quad A \quad}$	
	$\xleftarrow[\quad B \quad]{}$	

Shared secret is $B^a = g^{ab} = A^b$

Brief history of DH key exchange

1976 Diffie & Hellman publish New directions in cryptography, suggest using $G = \mathbb{F}_p^*$.

1978 Pollard publishes his discrete logarithm algorithm ($O(\sqrt{\#G})$ complexity).

1980 Miller and Koblitz independently suggest using elliptic curves $G = E(\mathbb{F}_p)$.

1994 Shor publishes his quantum discrete logarithm / factoring algorithm.

2005 NSA standardizes elliptic curve key agreement (ECDH) and signatures ECDSA.

2017 $\sim$ 70% of web traffic is secured by ECDH and/or ECDSA.

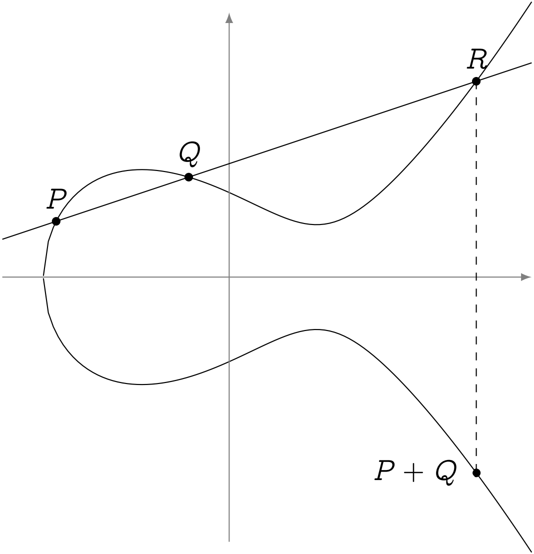
2017 NIST launches post-quantum competition, says “not to bother moving to elliptic curves, if you haven’t yet”.

History of isogeny-based cryptography

- **1996** Couveignes introduces the Hard Homogeneous Spaces. His work stays unpublished for 10 years.
- **2006** Rostovtsev & Stolbunov independently rediscover Couveignes ideas, suggest isogeny-based Diffie–Hellman as a quantum-resistant primitive.
- **2006-2010** Other isogeny-based protocols by Teske and Charles, Goren & Lauter.
- **2011-2012** D., Jao & Plût introduce SIDH, an efficient post-quantum key exchange inspired by Couveignes, Rostovtsev, Stolbunov, Charles, Goren, Lauter.
- **2017** SIDH is submitted to the NIST competition (with the name SIKE, only isogeny-based candidate).
- **2018** D., Kieffer & Smith *resurrect* the Couveignes–Rostovtsev–Stolbunov protocol, Castryck, Lange, Martindale, Panny & Renes publish an efficient variant named CSIDH.

Elliptic curves

Let $E\ :\ y^2 = x^3 + ax + b$ be an elliptic curve...



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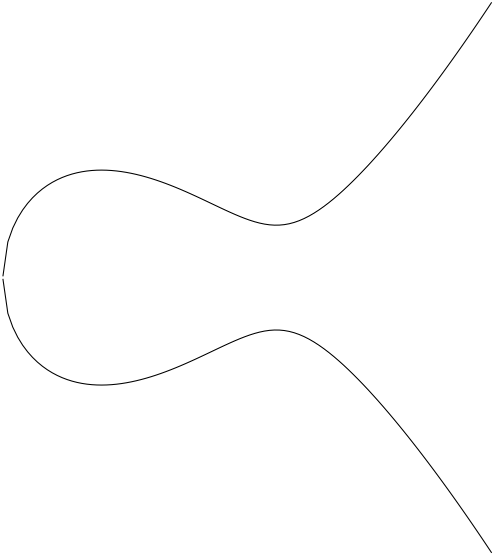

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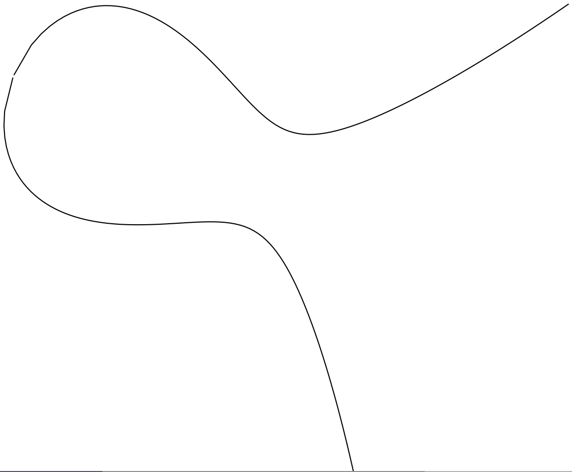

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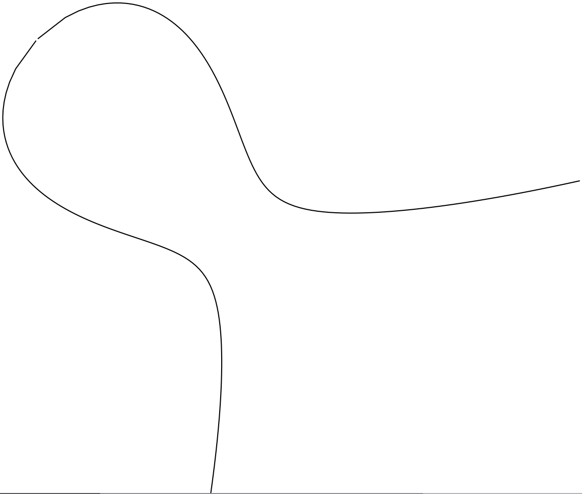

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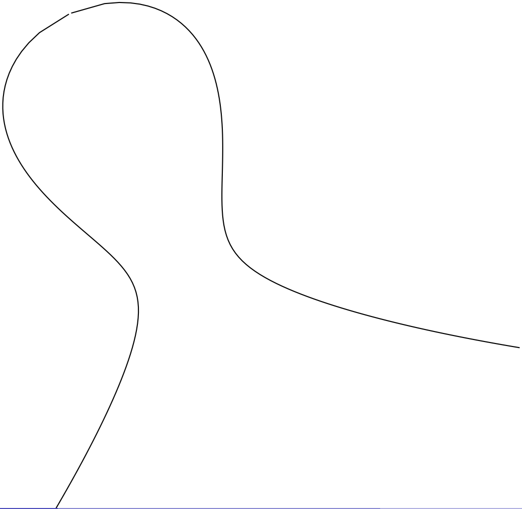

Elliptic curves

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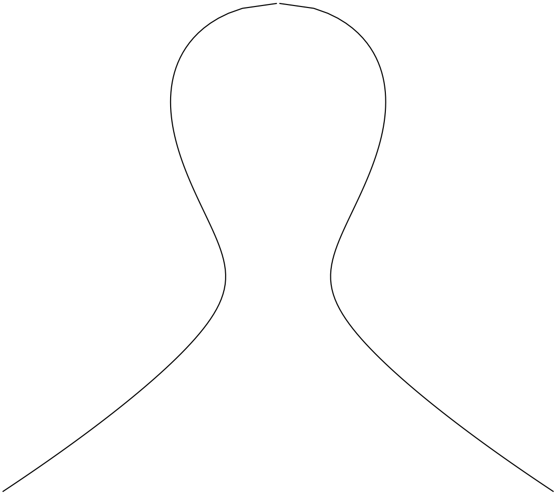
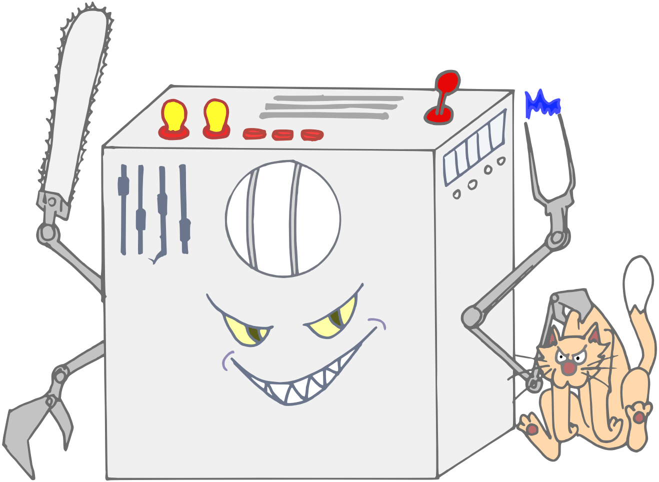
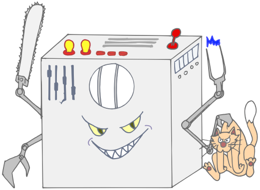


Elliptic curves

The QUANTHOM Menace



Basically every isogeny-based protocol...



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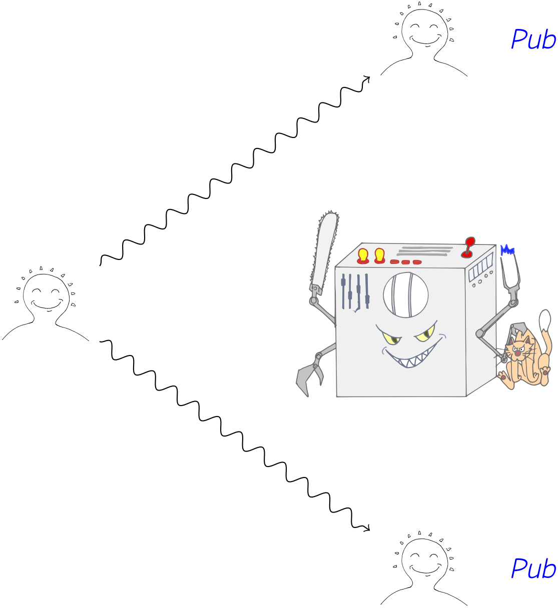
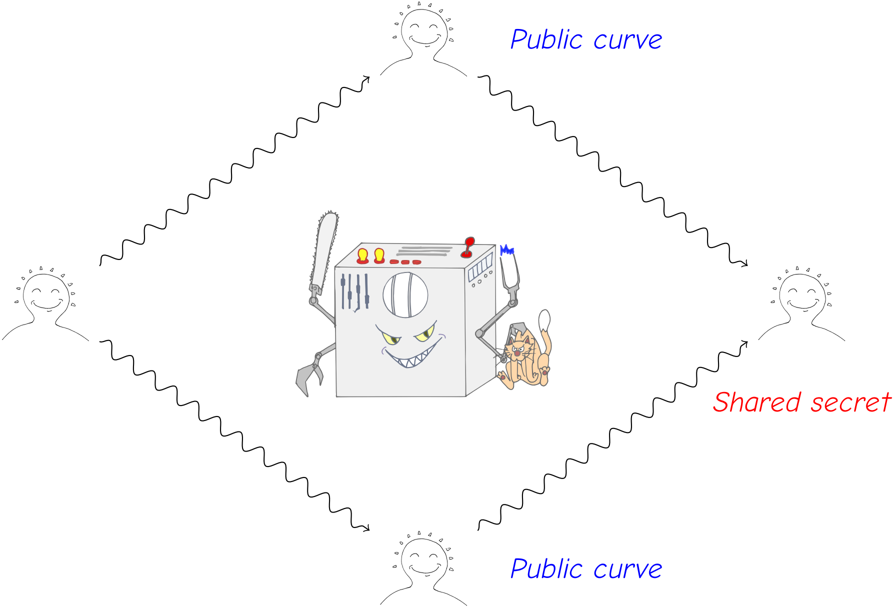


Public curve

Public curve

Basically every isogeny-based protocol...



Computing Isogenies

Vélu's formulas

Input: A subgroup $H \subset E$,

Output: The isogeny $\phi : E \to E/H$.

Complexity: $O(\ell)$ — Vélu 1971, …

Why?
- Evaluate isogeny on points $P \in E$;
- Walk in isogeny graphs.

Computing Isogenies

Vélu's formulas

Input: A subgroup $H \subset E$,

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Why?
- Evaluate isogeny on points $P \in E$;
- Walk in isogeny graphs.

Explicit Isogeny Problem

Input: Curve E, (prime) integer ℓ

Output: All subgroups $H \subset E$ of order ℓ.

Complexity: $\tilde{O}(\ell^2)$ — Elkies 1992

Why?
- List all isogenies of given degree;
- Count points of elliptic curves;
- Compute endomorphism rings of elliptic curves;
- Walk in isogeny graphs.

Computing Isogenies

Explicit Isogeny Problem (2)

Input: Curves E, E', isogenous of degree ℓ.

Output: The isogeny $\phi : E \to E'$ of degree ℓ.

Complexity: $O(\ell^2)$ — Elkies 1992; Couveignes 1996; Lercier and Sirvent 2008; De Feo 2011; De Feo, Hugounenq, Plût, and Schost 2016; Lairez and Vaccon 2016, …

Why?

- Count points of elliptic curves.

Computing Isogenies

Explicit Isogeny Problem (2)

Input: Curves E, E', isogenous of degree ℓ.

Output: The isogeny $\phi : E \to E'$ of degree ℓ.

Complexity: $O(\ell^2)$ — Elkies 1992; Couveignes 1996; Lercier and Sirvent 2008; De Feo 2011; De Feo, Hugounenq, Plût, and Schost 2016; Lairez and Vaccon 2016, …

Why?
- Count points of elliptic curves.

Isogeny Walk Problem

Input: Isogenous curves E, E'.

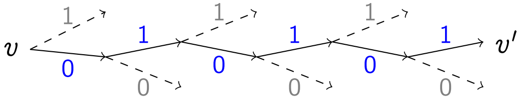
Output: An isogeny $\phi : E \to E'$ of smooth degree.

Complexity: Generically hard — Galbraith, Hess, and Smart 2002, …

Why?
- Cryptanalysis (ECC);
- Foundational problem for isogeny-based cryptography.

Random walks and hash functions (circa 2006)

Any expander graph gives rise to a hash function.



$$H(010101) = v'$$

- Fix a starting vertex v;
- The value to be hashed determines a random path to v';
- v' is the hash.

(Charles, K. E. Lauter, and Goren 2009) hash function (CGL)

- Use the expander graph of supersingular 2-isogenies;
- $\left.\begin{array}{r}\text{Collision resistance}\\ \text{2nd preimage resistance}\end{array}\right\} =$ hardness of finding cycles in the graph;
- Preimage resistance = hardness of finding a path from v to v'.

Hardness of CGL

Finding cycles

- Analogous to finding endomorphisms…
- …very bad idea to start from a curve with known endomorphism ring!
- Translation algortihm: elements of $B_{p,\infty} \leftrightarrow$ isogeny loops
  Doable in $\mathrm{polylog}(p)$.[a]

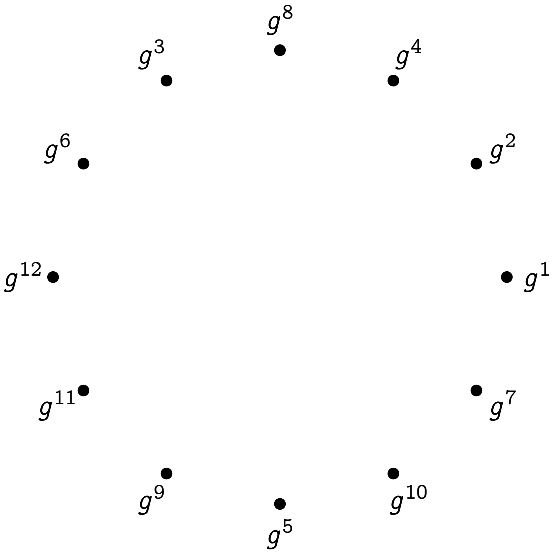
[a] Kohel, K. Lauter, Petit, and Tignol 2014; Eisenträger, Hallgren, K. Lauter, Morrison, and Petit 2018.

Finding paths $E \to E'$

- Analogous to finding connecting ideals between two maximal orders $\mathcal{O}, \mathcal{O}'$ (i.e. a left ideal $I \subset \mathcal{O}$ that is a right ideal of $\mathcal{O}'$).
- Poly-time equivalent to computing $\mathrm{End}(E)$ and $\mathrm{End}(E')$.[a]
- Best known algorithm to compute $\mathrm{End}(E)$ takes $\mathrm{poly}(p)$.[b]

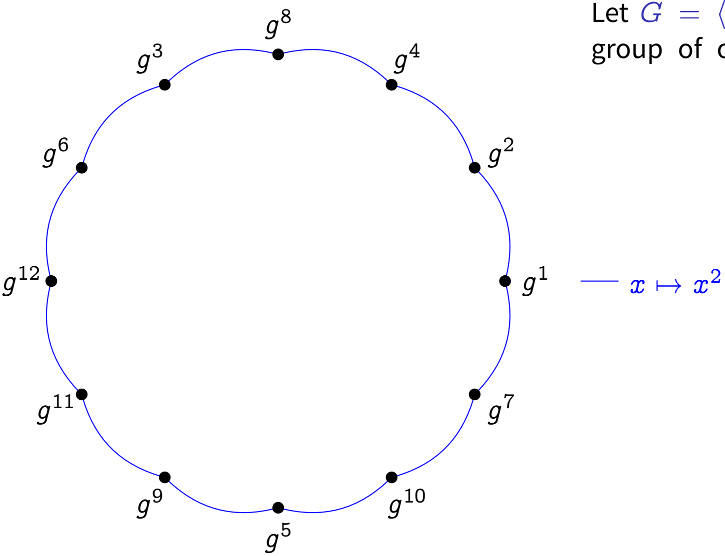
[a] Eisenträger, Hallgren, K. Lauter, Morrison, and Petit 2018.
[b] Kohel 1996; Cerviño 2004.

Expander graphs from groups



Let $G = \langle g \rangle$ be a cyclic group of order p.

Expander graphs from groups



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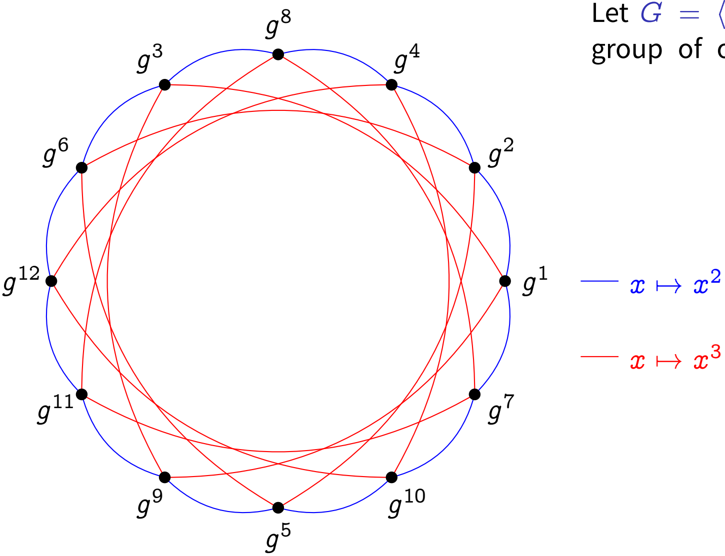

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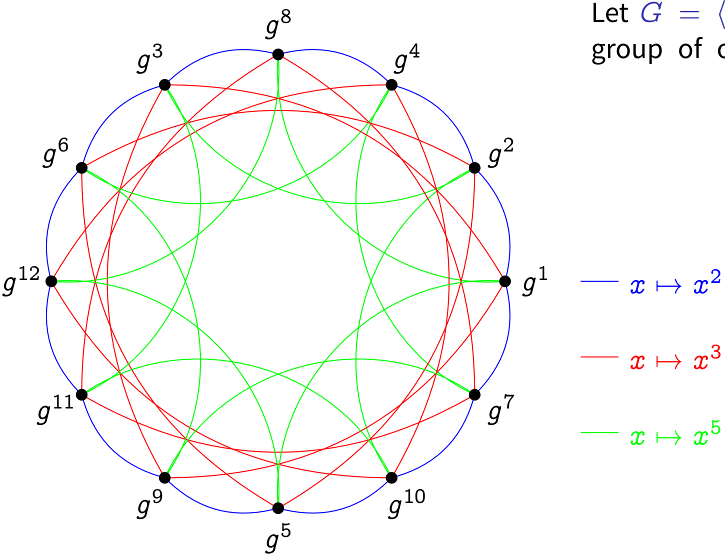

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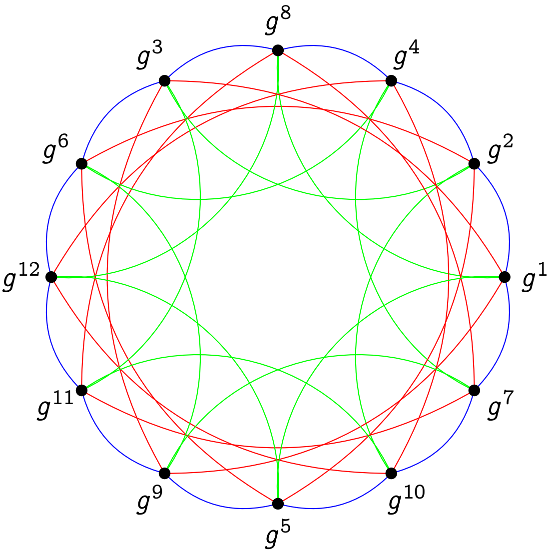


Let $G = \langle g \rangle$ be a cyclic group of order p. Let $S \subset (\mathbb{Z}/p\mathbb{Z})^\times$ s.t. $S^{-1} \subset S$.

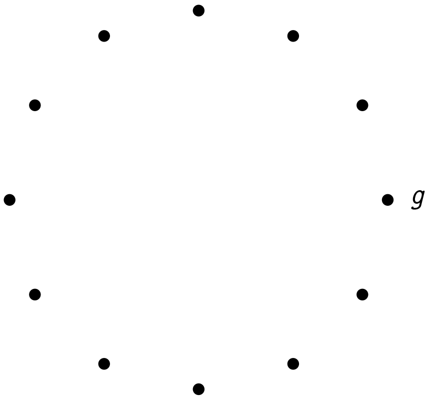
The Schreier graph of $(S, G \setminus \{1\})$ is (usually) an expander.

— $x \mapsto x^2$

— $x \mapsto x^3$

— $x \mapsto x^5$

Key exchange from Schreier graphs



Public parameters:

- A group $G = \langle g \rangle$ of order p;
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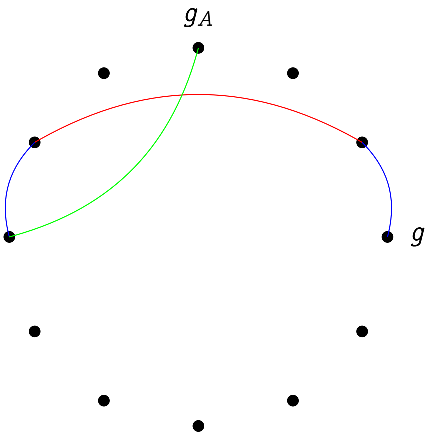

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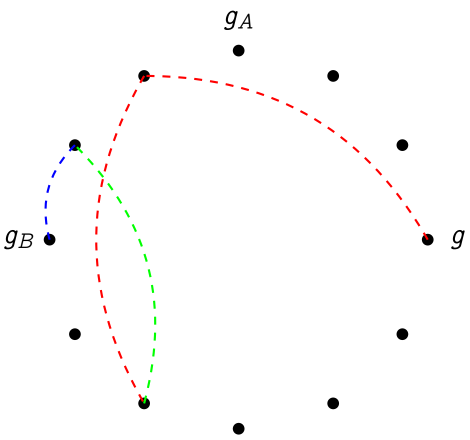

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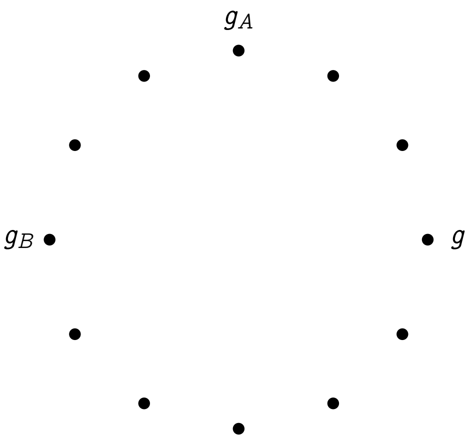

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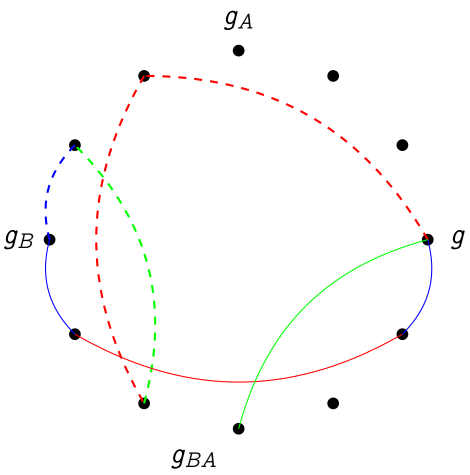

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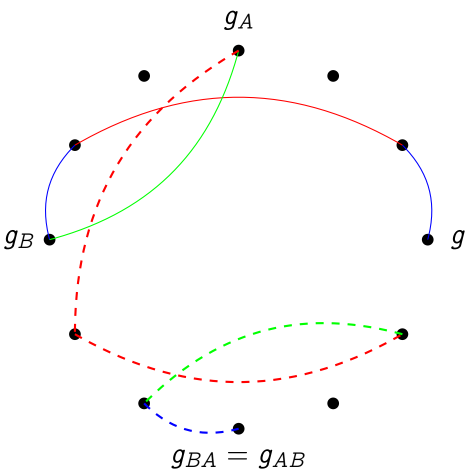

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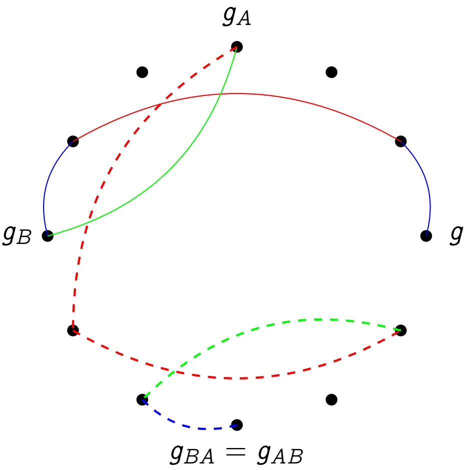

Why does this work?

$$g_A = g^{2\cdot 3\cdot 2\cdot 5},$$
$$g_B = g^{3^2\cdot 5\cdot 2},$$
$$g_{BA} = g_{AB} = g^{2^3\cdot 3^3\cdot 5^2};$$

and g_A, g_B, g_{AB} are uniformly distributed in G…

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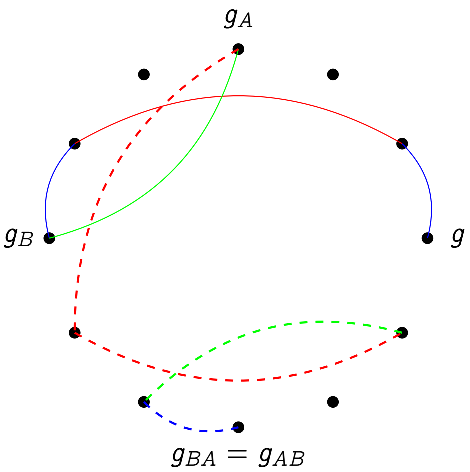

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…Indeed, this is just a twisted presentation of the classical Diffie-Hellman protocol!

Key exchange in graphs of ordinary isogenies[3] (CRS)

Parameters:

- $E/\mathbb{F}_p$ ordinary elliptic curve, with Frobenius endomorphism $\pi \in \mathcal{O}$.
- (small) primes ℓ_1,ℓ_2,…such that $\left(\frac{D_\pi}{\ell_i}\right) = 1$.
- elements $\mathfrak{f}_1 = (\ell_1, \pi - \lambda_1)$, $\mathfrak{f}_2 = (\ell_2, \pi - \lambda_2)$,…in $\mathrm{Cl}(\mathcal{O})$.

Secret data: Random walks $\mathfrak{a}, \mathfrak{b} \in \mathrm{Cl}(\mathcal{O})$ in the isogeny graph.

$$\begin{tikzcd}
 & E \arrow[dl, "\mathfrak{f}_1^{a_1}\mathfrak{f}_2^{a_2}\cdots = \mathfrak{a}"'] \arrow[dr, "\mathfrak{b} = \mathfrak{f}_1^{b_1}\mathfrak{f}_2^{b_2}\cdots"] & \\
\mathfrak{a} * E \arrow[dr] & & \mathfrak{b} * E \arrow[dl] \\
 & \mathfrak{a}\mathfrak{b} * E = \mathfrak{b}\mathfrak{a} * E &
\end{tikzcd}$$

[3] Couveignes 2006; Rostovtsev and Stolbunov 2006.

Computing the action of $\mathrm{Cl}(\mathcal{O})$

Input: An ideal class $\mathfrak{a} = \mathfrak{f}_1^{a_1}\mathfrak{f}_2^{a_2}\cdots$.

Output: The elliptic curve $\mathfrak{a} * E$.

Algorithm: Let $\mathfrak{f}^n = (\ell, \pi - \lambda)^n$, repeat n times:

- Use **Elkies' algorithm** to find all (two) curves isogenous to E of degree ℓ,
- Choose the one such that $\ker\phi \subset \ker(\pi - \lambda)$.

Parameters size / performance

Adversary goal: Given E, $\mathfrak{a} * E$, find $\mathfrak{a}$;

Graph size: $\#\,\mathrm{Cl}(\mathcal{O}) \approx \sqrt{p}$;

Best (classical) attack: Meet-in-the-middle / Random-walk in $\sqrt{\#\,\mathrm{Cl}(\mathcal{O})}$;

For 2^{128} security: choose $\log p \sim 512$;

Time to evaluate the isogeny action[a]: Dozens of minutes!

[a] De Feo, Kieffer, and Smith 2018.

Vélu to the rescue?

Input: An ideal class $\mathfrak{a} = \mathfrak{f}_1^{a_1}\mathfrak{f}_2^{a_2}\cdots$.

Output: The elliptic curve $\mathfrak{a} * E$.

Algorithm: Let $\mathfrak{f}^n = (\ell, \pi - \lambda)^n$. Why not:

- *Presciently* find $H = E[\ell] \cap \ker(\pi - \lambda)$,
- Apply Vélu's formulas to H.

Speeding up the class group action

Problem: H must be in $E(\mathbb{F}_p)$ for Vélu's formulas to be efficient.

Idea[a]: Force
$$\begin{cases} p = -1 & \mod \ell, \\ \lambda = 1 & \mod \ell, \end{cases}$$
so that $E[\ell] = H \subset E(\mathbb{F}_p)$.

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Time to evaluate the isogeny action: Still 5 minutes!

[a] De Feo, Kieffer, and Smith 2018.

Supersingular to the rescue!

For all supersingular curves defined over $\mathbb{F}_p$,

$$\pi = \begin{pmatrix} \sqrt{-p} & 0 \\ 0 & -\sqrt{-p} \end{pmatrix} \mod \ell$$

CSIDH *(pron.: Seaside)*

Choose $p = -1 \mod \ell$ for many primes ℓ;

Hence, $\lambda = 1 \mod \ell$. Win!

Performance: Same security as CRS in less than 50ms![a]

[a] Castryck, Lange, Martindale, Panny, and Renes 2018.

Quantum security

Fact: Shor's algorithm does not apply to Diffie-Hellman protocols from group actions.

Subexponential attack $\exp(\sqrt{\log p \log\log p})$

- Reduction to the hidden shift problem by evaluating the class group action in quantum superspositio[a] (subexpoential cost);
- Well known reduction from the hidden shift to the dihedral (non-abelian) hidden subgroup problem;
- Kuperberg's algorithm[b] solves the dHSP with a subexponential number of class group evaluations.
- Recent work[c] suggests that 2^{64}-qbit security is achieved somewhere in $512 < \log p < 1024$.

[a] Childs, Jao, and Soukharev 2014.

[b] Kuperberg 2005; Regev 2004; Kuperberg 2013.

[c] Bonnetain and Naya-Plasencia 2018; Bonnetain and Schrottenloher 2018; Biasse, Jacobson Jr, and Iezzi 2018; Jao, LeGrow, Leonardi, and Ruiz-Lopez 2018; Bernstein, Lange, Martindale, and Panny 2018.

Key exchange with supersingular curves (2011)

Good news: there is no action of a commutative class group.

Bad news: there is no action of a commutative class group.

Idea: Let Alice and Bob walk in two different isogeny graphs on the same vertex set.

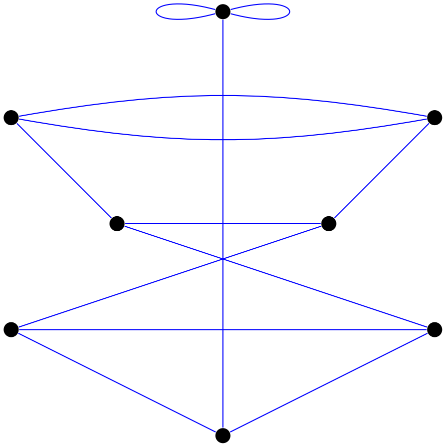

Figure: 2- and 3-isogeny graphs on $\mathbb{F}_{97^2}$.

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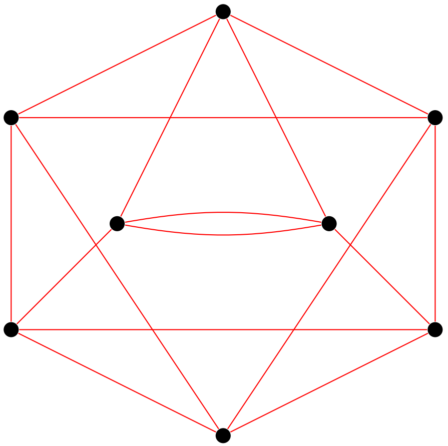

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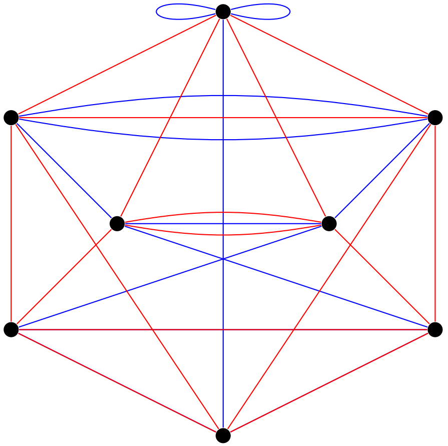

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Key exchange with supersingular curves (2011)

- Fix small primes ℓ_A, ℓ_B;
- No canonical labeling of the ℓ_A- and ℓ_B-isogeny graphs; however…

Walk of length e_A

$$=$$

Isogeny of degree $\ell_A^{e_A}$

$$=$$

Kernel $\langle P\rangle \subset E[\ell_A^{e_A}]$

$$\ker\phi = \langle P\rangle \subset E[\ell_A^{e_A}]$$

$$\ker\psi = \langle Q\rangle \subset E[\ell_B^{e_B}]$$

$$\ker\phi' = \langle \psi(P)\rangle$$

$$\ker\psi' = \langle \phi(Q)\rangle$$

$$\begin{tikzcd}
E \arrow[r, squiggly, "\phi"] \arrow[d, squiggly, "\psi"'] & E/\langle P\rangle \arrow[d, squiggly, "\psi'"] \\
E/\langle Q\rangle \arrow[r, squiggly, "\phi'"'] & E/\langle P, Q\rangle
\end{tikzcd}$$

Supersingular Isogeny Diffie-Hellman[4]

Parameters:

- Prime p such that $p+1 = \ell_A^a \ell_B^b$;
- Supersingular curve $E \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2$;
- $E[\ell_A^a] = \langle P_A, Q_A \rangle$;
- $E[\ell_B^b] = \langle P_B, Q_B \rangle$.

Secret data:

- $R_A = m_A P_A + n_A Q_A$,
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$$
\begin{tikzcd}
 & E \arrow[dl, squiggly, "\phi"'] \arrow[dr, squiggly, "\psi"] & \\
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 & E/\langle R_A, R_B \rangle &
\end{tikzcd}
$$

$$\frac{E/\langle R_A \rangle}{\phi(R_B)} \simeq E/\langle R_A, R_B \rangle \simeq \frac{E/\langle R_B \rangle}{\psi(R_A)}$$

[4] Jao and De Feo 2011; De Feo, Jao, and Plût 2014.

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\end{tikzcd}$$

$\phi(P_B)$ $\phi(Q_B)$ $\qquad$ $\phi(R_B)$ $\qquad$ $\psi(R_A)$ $\qquad$ $\psi(P_A)$ $\psi(Q_A)$

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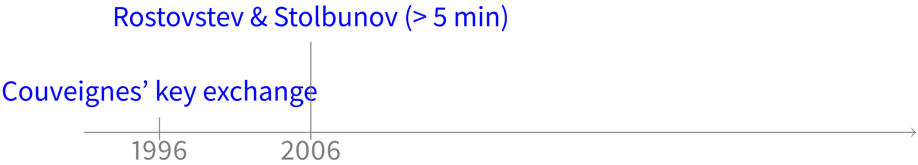
From 10 minutes to 10ms in 20 years

Couveignes’ key exchange

1996

From 10 minutes to 10ms in 20 years



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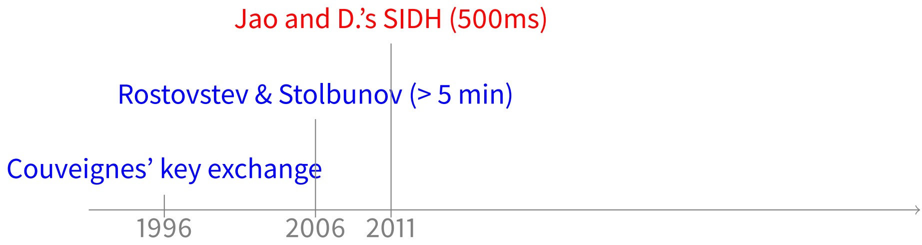

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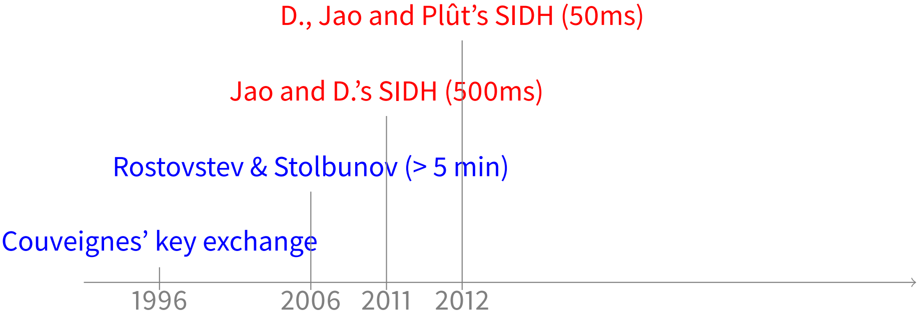

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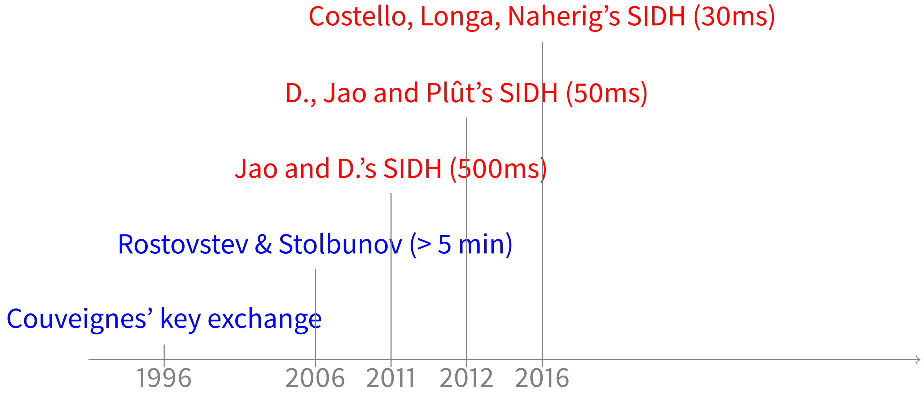

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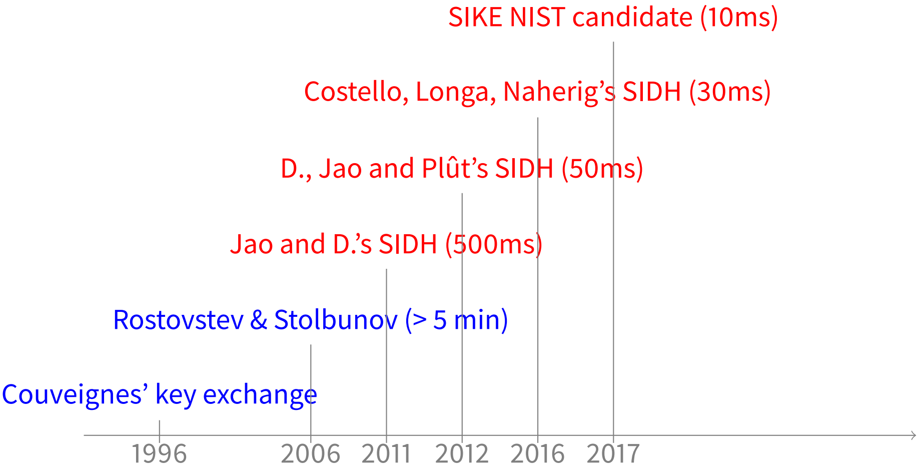

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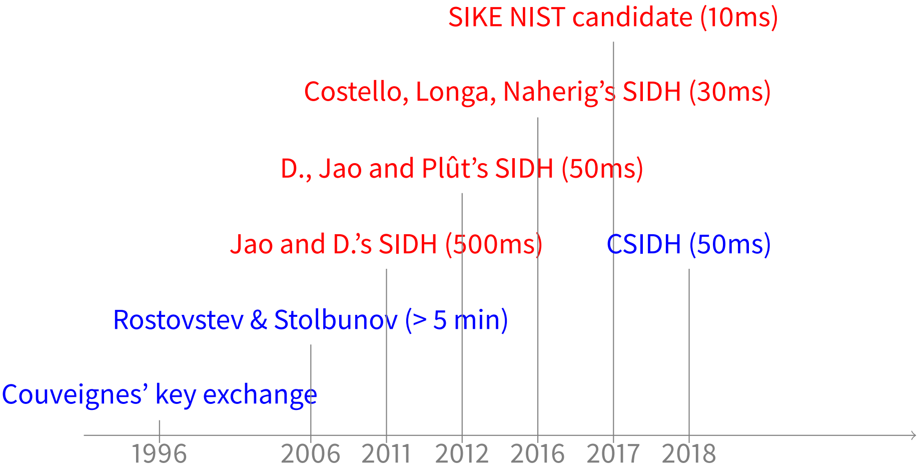
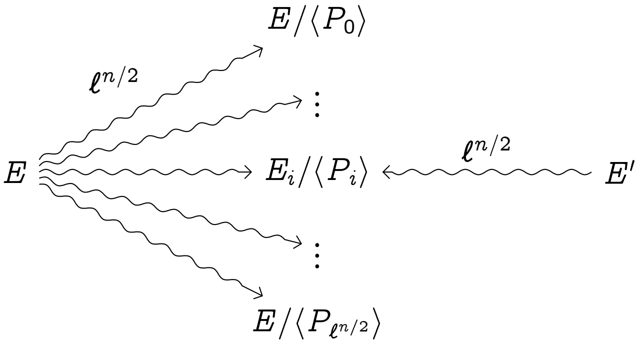


Generic attacks

Problem: Given E, E', isogenous of degree ℓ^n, find $\phi : E \to E'$.



- With high probability ϕ is the unique collision (or *claw*) $O(\ell^{n/2})$.
- A quantum claw finding[5] algorithm solves the problem in $O(\ell^{n/3})$.

[5] Tani 2009.

Security

The SIDH problem

Given E, Alice's public data $E/\langle R_A\rangle, \phi(P_B), \phi(Q_B)$, and Bob's public data $E/\langle R_B\rangle, \psi(P_A), \psi(Q_A)$, find the shared secret $E/\langle R_A, R_B\rangle$.

Under the SIDH assumption:

- The SIDH key exchange protocol is session-key secure.
- The derived El Gamal-type PKE is CPA secure.

Reductions

- SIDH $\to$ Isogeny Walk Problem;
- SIDH $\to$ Computing the endomorphism rings of E and $E/\langle R_A\rangle$.[a]

[a] Kohel, K. Lauter, Petit, and Tignol 2014; Galbraith, Petit, Shani, and Ti 2016.

Chosen ciphertext attack[6]

For simplicity, assume Alice's prime is $\ell = 2$.

Evil Bob

- Alice has a long-term secret $R = mP + nQ \in E[2^e]$;
- Bob produces an ephemeral secret ψ;
- Bob sends to Alice $\psi(P)$, $\psi(Q + 2^{e-1}P)$;
- Alice computes the shared secret correctly iff

$$\begin{aligned} R &= mP + nQ \\ &= mP + nQ + n2^{e-1}P, \end{aligned}$$

  i.e., iff n is even;
- Bob learns one bit of the secret key by checking that Alice gets the right shared secret.

- Bob repeats the queries in a similar fashion, learning one bit per query.
- Detecting Bob's faulty key seems to be as hard as breaking SIDH.

[6] Galbraith, Petit, Shani, and Ti 2016.

CSIDH vs SIDH

	CSIDH	**SIDH**
Speed (NIST 1)	<100ms	$\sim$ 10ms
Public key size (NIST 1)	64B	378B
Key compression[7]		
↳ speed		$\sim$ 15ms[8]
↳ size		222B
Constant time impl.	not yet	yes
Submitted to NIST	no	yes
Best classical attack	$p^{1/4}$	$p^{1/4}$
Best quantum attack	$\tilde{O}\left(3^{\sqrt{\log_3 p}}\right)$	$p^{1/6}$
Key size scales	quadratically	linearly
Security assumption	isogeny walk problem	ad hoc
CPA security	yes	yes
CCA security	yes	Fujisaki-Okamoto
Non-interactive key ex.	yes	no
Signatures	short but slooow!	big and slow

[7] Zanon, Simplicio, Pereira, Doliskani, and Barreto 2018.

[8] https://twitter.com/PatrickLonga/status/1002313366466015232?s=20

SIKE: Supersingular Isogeny Key Encapsulation

- Submission to the NIST PQ competition:

  SIKE.PKE: El Gamal-type system with IND-CPA security proof,
  SIKE.KEM: generically transformed system with IND-CCA security proof.

- Security levels 1, 3 and 5.
- Smallest communication complexity among all proposals in each level.
- Slowest among all benchmarked proposals in each level.
- A team of 14 submitters, from 8 universities and companies.
- Head to `https://sike.org`.

	p	cl. security	q. security	speed	comm.
SIKEp434	$2^{216}3^{137}-1$	NIST-1	NIST-1	–	–
SIKEp503	$2^{250}3^{159}-1$	126 bits	84 bits	10ms	0.4KB
SIKEp751	$2^{372}3^{239}-1$	188 bits	125 bits	30ms	0.6KB

Thank you

http://defeo.lu/

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