EEE 321-Signals and Systems

-Lab 06-

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Section 001
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Part 1

part 1) $y[n] = \sum_{l=1}^{N} a[l]\,y[n-l] + \sum_{k=0}^{M} b[k]\,x[n-k]$

assuming $x[n] = y[n] = 0$ for $n<0$

$y[n] = a[1]y[n-1] + a[2]y[n-2] + \dots + a[N]y[n-N] + b[0]x[n] + b[1]x[n-1] + b[2]x[n-2] \dots$
$b[M]x[n-M]$

* $y[0] = a[1]\underbrace{\cancel{y[-1]}}_{0} + a[2]\underbrace{\cancel{y[-2]}}_{0} + \dots + a[N]\underbrace{\cancel{y[-N]}}_{0} + b[0]x[0] + b[1]\underbrace{\cancel{x[-1]}}_{0} + \dots + b[M]\underbrace{\cancel{x[-M]}}_{0}$

since $y[n]$ and $x[n] = 0$ for $n<0$.

$\boxed{y[0] = b[0]x[0]}$

* $y[1] = a[1]y[0] + a[2]\overset{0}{\cancel{y[-1]}} + a[3]\overset{0}{\cancel{y[-1]}} + \cancel{\dots} + b[0]x[1] + b[1]x[0] + b[2]\overset{0}{\cancel{x[-1]}} + \cancel{\dots} + \cancel{b[M]x[1-M]}$

$\boxed{y[1] = a[1]y[0] + b[0]x[1] + b[1]x[0]}$

↳ substituting $y[0] = b[0]x[0]$

$y[1] = a[1]b[0]x[0] + b[0]x[1] + b[1]x[0]$

* Z transform of $x[n] \Rightarrow \sum_{n=-\infty}^{\infty} x[n] z^{-n} \Rightarrow Y(z) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{N} a[l]\,y[n-l]\,z^{-n} + \sum_{n=-\infty}^{\infty} \sum_{k=0}^{M} b[k]\,x[n-k]\,z^{-n}$

$x[n] \rightarrow X(z)$

$x[n-l] \rightarrow z^{-l} X(z)$

↳ $\sum_{n=-\infty}^{\infty} x[n-l] z^{-n}$

Shifting property of z transform

$$\boxed{Y(z) = \sum_{l=1}^{N} a[l] z^{-l} Y(z) + \sum_{k=0}^{M} b[k] z^{-k} X(z)}$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b[k] z^{-k}}{1 - \sum_{l=1}^{N} a[l] z^{-l}}}$$

$p = k$

$cn[p] = b[k]$

$q = l$

$ca[q] = a[l]$

Function DTLTI:

```
function [y]=DTLTI(a,b,x,Ny)
l= length(a)+1;
wextend('1D','zpd',a,1,'l');
y =zeros(1,Ny);
for n = 1:Ny
for k = 1:min(n,length(b))
y(n) = y(n) + b(k)*x(n-k+1);
end
for l = 1:min(n,length(a))
y(n) = y(n) + a(l)*y(n-l+1);
end
end
end
```

Part 2

The impulse response of the filter:

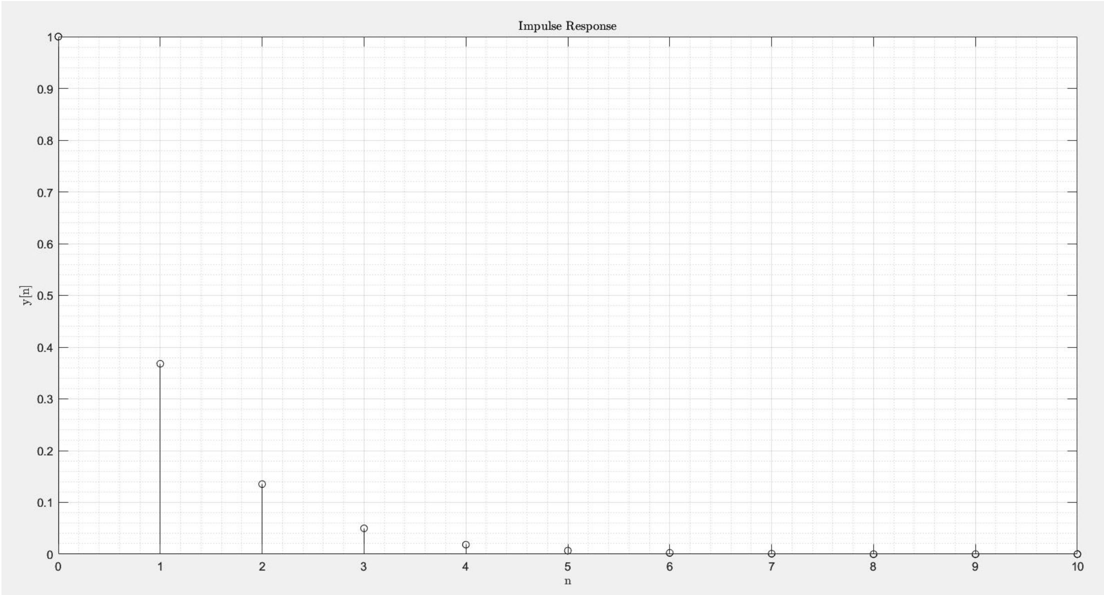


Fig.1. Impulse response

The impulse response contains 8 nonzero elements;

y =

1.0000 0.3679 0.1353 0.0498 0.0183 0.0067 0.0025 0.0009 0 0 0

The relation between these nonzero elements and the coefficients b[k] is explained as follows:

$$y[n] = \sum_{l=1}^{N} a[l]\,y[n-l] + \sum_{k=0}^{M} b[k]\,x[n-k]$$

$a[l] = 0$ for all $1 \leq l \leq N$

$b[k] = e^{-k}$ for all $0 \leq k \leq M-1$ where $M = 5 + (ID \bmod 4)$

$M = 5 + 3 = 8$

$$y[n] = \sum_{l=1}^{N} 0\, \cancel{y[n-l]}^{\,0} + \sum_{k=0}^{8} b[k]\,x[n-k]$$

$$h[n] = \sum_{k=0}^{7} b[k]\,\delta[n-k] = \underbrace{b[0]\,\delta[n-0]}_{b[0]} + \underbrace{b[1]\,\delta[n-1]}_{b[1]} \cdots \cdots$$

$$h[n] = \underbrace{e^{-0}}_{1} + e^{-1} + e^{-2} + e^{-3} + e^{-4} + e^{-5} + e^{-6} + e^{-7} +$$

therefore the nonzero values of $h[n]$ are basically $b[k]$ values for $0 \leq k \leq M-1 \rightarrow M$ values, since for my case $M = 8$ there are 8 nonzero values.

Fig.2. Part 2.b,c explanation

Since the impulse response is finite length in other words h[n] goes to zero as n goes to infinity, the system is FIR. We have only 8 nonzero values.

The z-transform and discrete-time fourier transform of the impulse response is found below:

d) $$H(z) = \frac{\sum_{k=0}^{M-1} b[k] z^{-k}}{1-\sum_{l=1}^{N} a[l] z^{-l}} = \sum_{k=0}^{M-1} e^{-k} z^{-k} = \sum_{k=0}^{M-1} (ez)^{-k} = \frac{1-(ez)^{-M}}{1-(ez)^{-1}}$$

$$H(e^{j\omega}) = \sum_{k=0}^{M-1} (e \cdot e^{j\omega})^{-k} = \sum_{k=0}^{M-1} e^{-k-kj\omega} = \frac{1-e^{-M-Mj\omega}}{1-e^{-1-j\omega}}$$

Fig.3. Part 2.d explanation

Using the fourier transform of the impulse response, the magnitude response is plotted as follows:

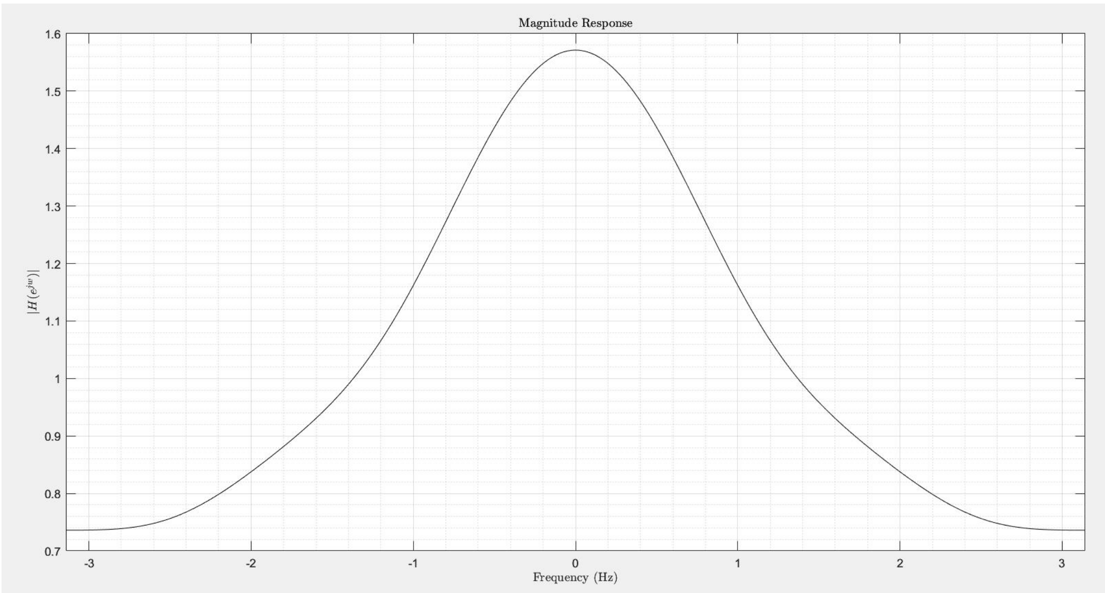


Fig.4. Magnitude response

As can be seen from the figure above, the magnitude decreases with the increasing frequencies. Therefore since higher frequencies are eliminated this is a LPF.

The bandwidth of a filter can be defined as the frequency range that is allowed to pass through the filter with minimal attenuation. Normally to be able to see the magnitude or phase response in a bigger frequency spectrum dB conversion is used. Decibel is a logarithmic unite of measurement therefore we can observe the frequency response to 10 Hz and to 1MHz in the same logarithmic plot. When this magnitude response is converted to a logarithmic plot, the frequency level at which the maximum value of the magnitude response decreases by 3dB are called the 3dB cut-off frequencies. Hence 3dB bandwidth means the range of these 3dB cut-off frequencies (Fig.5).

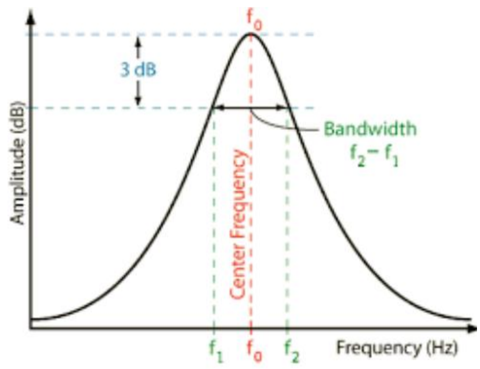


Fig.5. 3dB bandwidth of a filter

3dB lower values in dB plot corresponds to the values where the magnitude is

$$\frac{max\,|H(e^{jw})|}{\sqrt{2}}$$

The peak value of the plot in Fig.4. is 1.57. Dividing this value by $\sqrt{2}$, we have;

$$\frac{1.57}{\sqrt{2}} = 1.11$$

The cut-off frequencies for 1.11 and the maximum magnitude are shown:

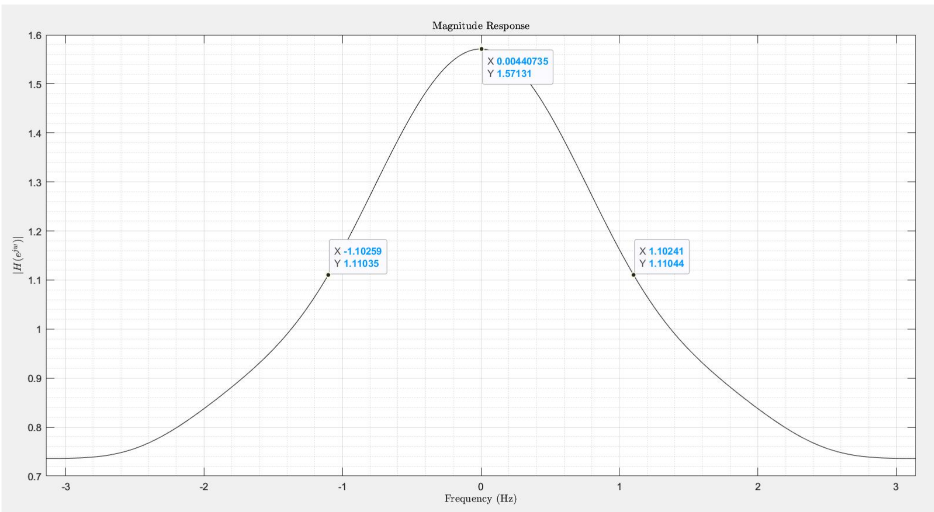


Fig.6. Max value and cut-off frequencies.

Subtracting the two frequencies, 3dB bandwidth is found as 2.2 Hz.

Here is the response to the system to a chirp function for different sampling rates:

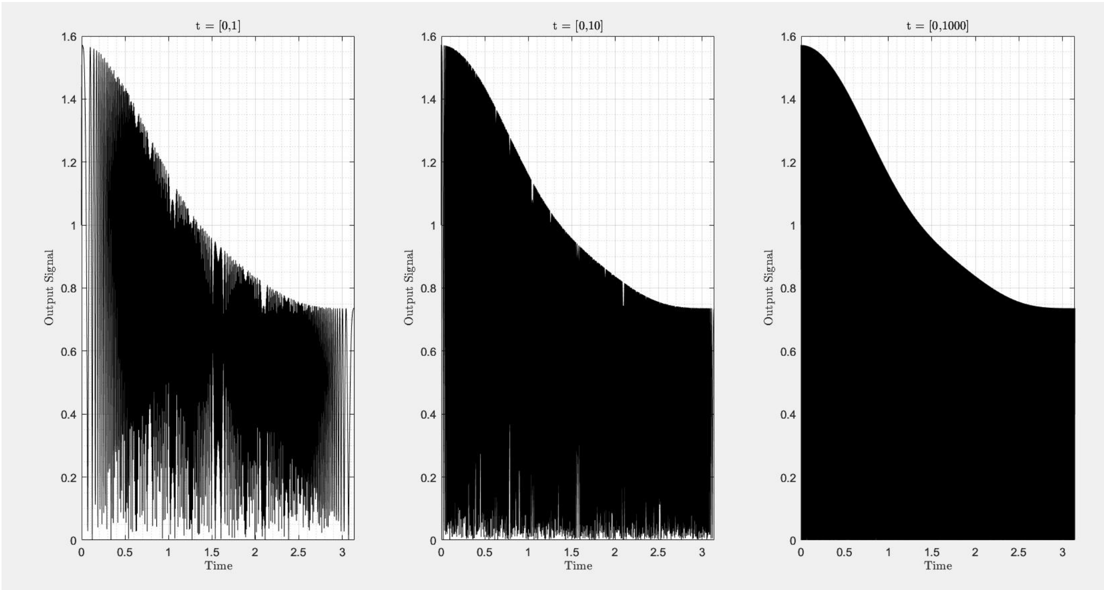


Fig.7. Response to Chirp

The chirp is a sinusoidal signal which the frequency increases with time therefore with this function we can see the frequency response of this system even in time domain. As can be seen from Fig.7 as time increases in another words as the frequency increases the output decreases. This proves that our filter is a LPF. Comparing this with the plot in Fig.4. we can say that the chirp response is the positive side of the frequency response of the filter. The LPF acts as an envelope over the output. As we increase the sampling rate the response becomes more ideal. Less jumps occur.

Part 3

Here are Part 3 explanations:

Part3

ID = 22102167

$n_1 = 2+2=4$
$n_2 = 2+2=4$
$n_3 = 2+1=3$
$n_4 = 2+0=2$
$n_5 = 2+5=7$
$n_6 = 2+1=3$
$n_7 = 2+6=8$
$n_8 = 2+7=9$

$$z_1 = \frac{n_2 + j n_3}{\sqrt{n_2^2 + n_3^2}} = \frac{4+3j}{5}$$

$$p_1 = \frac{n_1 + j n_5}{\sqrt{1 + n_1^2 + n_5^2}} = \frac{4+7j}{\sqrt{1+4^2+7^2}}$$

$$p_2 = \frac{n_8 + j n_6}{\sqrt{1 + n_8^2 + n_6^2}} = \frac{9+3j}{\sqrt{1+9^2+3^2}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z_1 z^{-1}}{(1-p_1 z^{-1})(1-p_2 z^{-1})}$$

$$Y(z) - Y(z) p_1 z^{-1} + p_1 p_2 z^{-2} Y(z) = X(z) - z_1 z^{-1} X(z)$$

$$y[n] - p_1 y[n-1] + p_1 p_2 y[n-2] = x[n] - z_1 x[n-1]$$

$$y[n] = \sum_{l=1}^{2} a[l]\, y[n-l] + \sum_{k=0}^{1} b[k]\, x[n-k]$$

$a[1] = p_1 + p_2$

$a[2] = -p_1 p_2$

$b[0] = 1$

$b[1] = -z_1$

Fig.8. Z transform of the filter

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z_1 z^{-1}}{(1-p_1 z^{-1})(1-p_2 z^{-1})} = \frac{A}{(1-p_1 z^{-1})} + \frac{B}{(1-p_2 z^{-1})}$$

$$A + A p_2 z^{-1} + B - p_1 z^{-1} B = 1 - z_1 z^{-1} \Rightarrow \begin{array}{l} A + B = 1 \\ A p_2 + B p_1 = z_1 \end{array}$$

$$A = \frac{p_1 - z_1}{p_1 - p_2}, \quad B = \frac{z_1 - p_2}{p_1 - p_2}$$

$$h[n] = A(p_1)^n u[n] + B(p_2)^n u[n],$$

$$h[n] = \left(\frac{p_1 - z_1}{p_1 - p_2}\right)(p_1)^n u[n] + B\left(\frac{z_1 - p_2}{p_1 - p_2}\right)(p_2)^n u[n]$$

Fig.9. Impulse Response

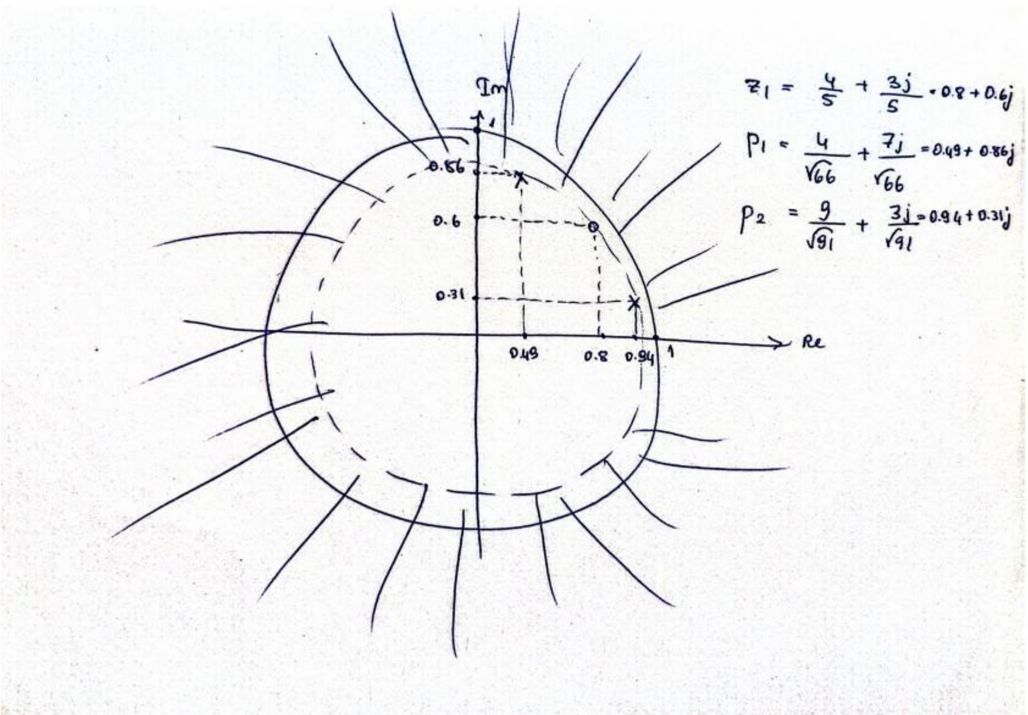


Fig.10. ROC

The system is stable since all poles are stated inside the unit circle. It is an IIR filter since the response depends on past values of y[n] it is recursive hence does not converge to zero.

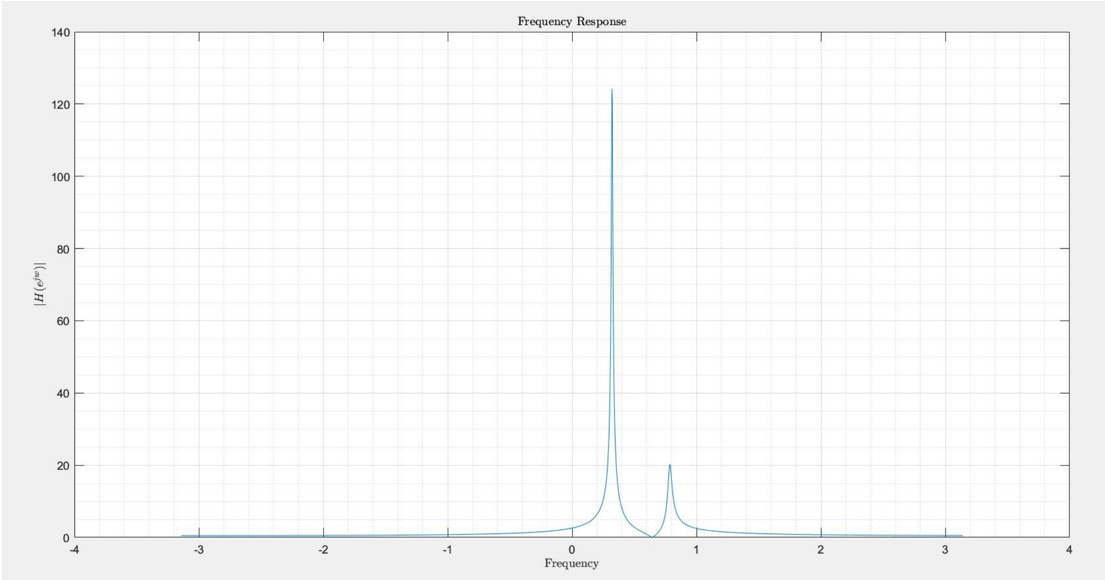
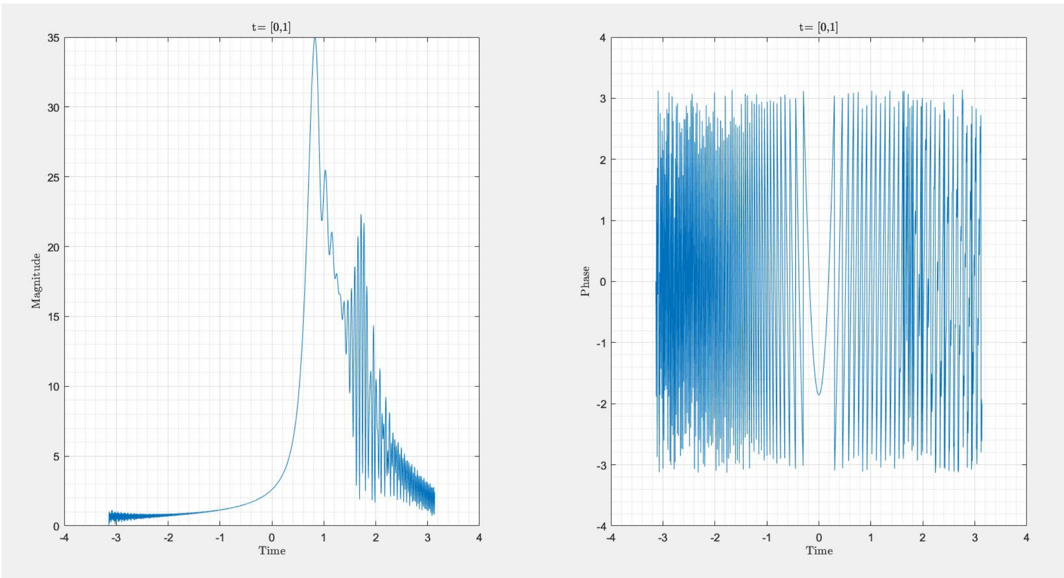
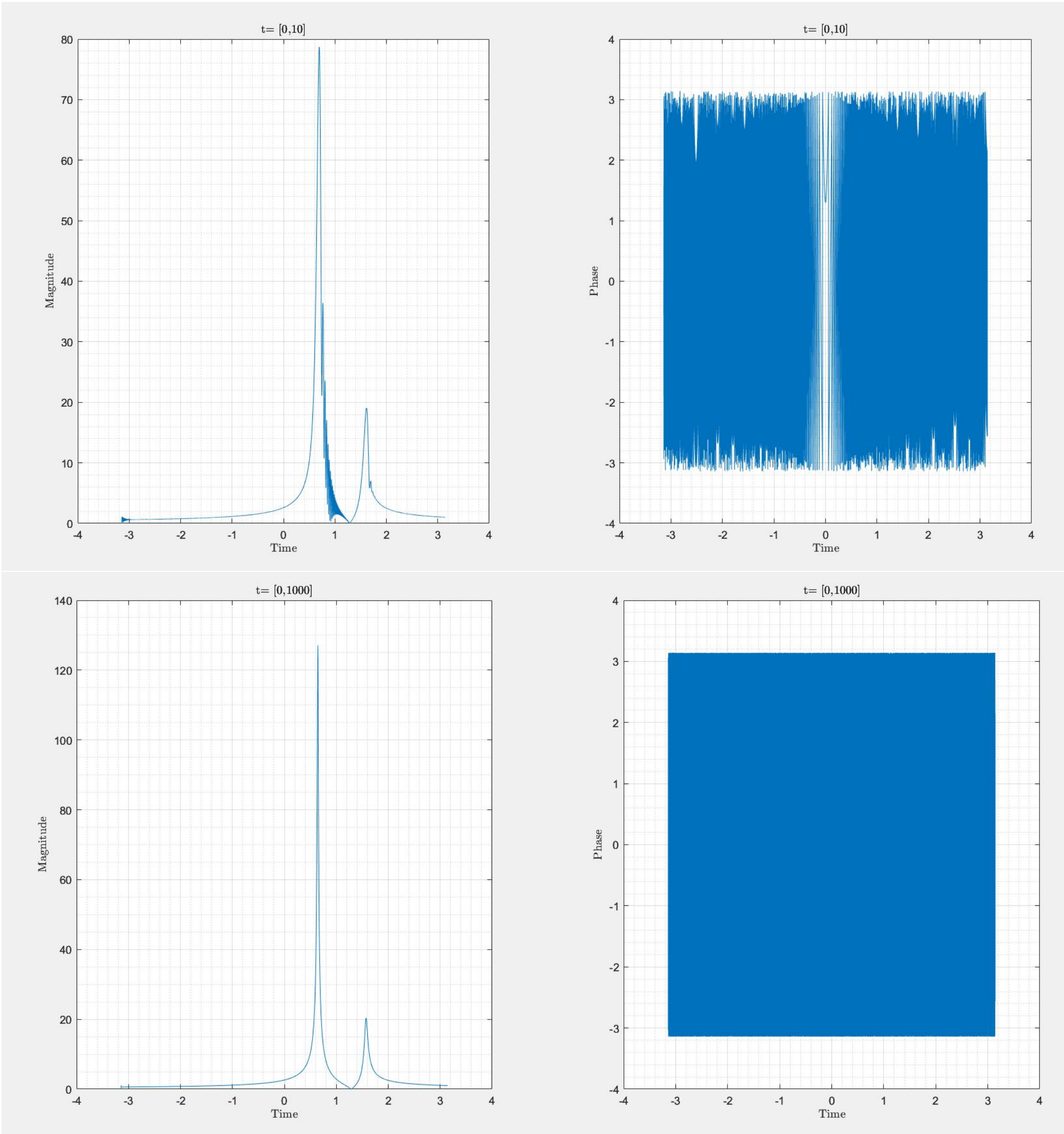


Fig.11. Frequency content

Since there are two peaks we can say that it is a bandpass filter.

Using the chirp function the following plots are obtained:





The magnitude plots are similar to the one found in part g. However as the sampling size increases the magnitude increases as well. The plots are not symmetrical w.r.t origin since the impulse function itself is not a real valued signal and the complex conjugates does not exist. If the frequency range is between -600 and 800 to meet the Nyquist criteria which states that the sampling frequency should be greater than twice the maximum frequency of the signal for an accurate sampling, we need to choose the maximum frequency which is 800Hz and obtain 1600 Hz.

Appendix

Part 2)

```
clear
D = [22102167];
Ny = 11;
M = 5 + mod(D, 4);
b = exp(-[0:M-1]);
a = zeros(1, M);
x=zeros(1,11);
x(1) = 1 ;
n = 0:Ny-1;
h = DTLTI(a, b, x, Ny);
freq = [-pi:1/1000:pi-1/1000];
DTFT = (1-exp(-M*(1i*freq+1)))./(1-exp(-(1i*freq+1)));

figure;
plot(freq,abs(DTFT),'k');
xlim([-pi,pi]);
ylabel ({'$|H(e^{jw})|$'},'Interpreter','latex');
xlabel ({'Frequency (Hz)'},'Interpreter','latex');
box on
grid, grid minor
title({'Magnitude Response'},'Interpreter','latex')
```

```
clear
D = [22102167];
Ny = 11;
M = 5 + mod(D, 4);
b = exp(-[0:M-1]);
a = zeros(1, M);
x = [1 zeros(1, Ny-1)];
h = DTLTI(a, b, x, Ny);
chirp_1 = chirpfunction(0,1000,1,2000);
freqs = linspace(0,pi,length(chirp_1));
y_1 = DTLTI(a,b,chirp_1,length(chirp_1));

figure;
subplot(1,3,1);
plot(freqs,abs(y_1),'k');
xlim([0,pi]);
ylabel ({'Output Signal'},'Interpreter','latex');
xlabel ({'Time'},'Interpreter','latex');
box on
grid, grid minor
title({'t = [0,1]'},'Interpreter','latex')



chirp_2 = chirpfunction(0,1000,10,2000);
freqs_2 = linspace(0,pi,length(chirp_2));
y_2 = DTLTI(a,b,chirp_2,length(chirp_2));

subplot(1,3,2);
plot(freqs_2,abs(y_2),'k');
ylabel ({'Output Signal'},'Interpreter','latex');
```

```
xlabel ({'Time'},'Interpreter','latex');
box on
grid, grid minor
title({'t = [0,10]'},'Interpreter','latex')
xlim([0,pi]);


chirp_3 = chirpfunction(0,1000,1000,2000);
freqs_3 = linspace(0,pi,length(chirp_3));
y_3 = DTLTI(a,b,chirp_3,length(chirp_3));

subplot(1,3,3);
plot(freqs_3,abs(y_3),'k');
ylabel ({'Output Signal'},'Interpreter','latex');
xlabel ({'Time'},'Interpreter','latex');
box on
grid, grid minor
title({'t = [0,1000]'},'Interpreter','latex')
xlim([0,pi]);


function [x] = chirpfunction(f0,endfreq, endtime, fs)
t = [0:1/fs:endtime-1/fs];
k = (endfreq-f0)/endtime;
x = cos(2*pi*(k*(t.^2)/2 + f0*t));
end
```

Part 3)

```
% Part 3 of the Lab


n1 = 2 + 2;
n2 = 2 + 2;
n3 = 2 + 1;
n4 = 2 + 0;
n5 = 2 + 2;
n6 = 2 + 1;
n7 = 2 + 6;
n8 = 2 + 7;

z1 = (n2+1i*n3)/(sqrt(n2^2+n3^2));
p1 = (n1+1i*n5)/(sqrt(1+n1^2+n5^2));
p2 = (n8+1i*n6)/(sqrt(1+n8^2+n6^2));

j = sqrt(-1);
H_ejw = @(w) (exp(j * w) - z1) ./ ((exp(j * w) - p1) .* (exp(j * w) - p2));

w = -pi:pi/1000:pi;
magH = abs(H_ejw(w));

figure(6)
plot(w, magH, 'LineWidth', 2)
ylabel('$|H(e^{jw})|$', 'Interpreter', 'latex');
xlabel('Frequency', 'Interpreter', 'latex');
box on
grid on, grid minor
title('Frequency Response', 'Interpreter', 'latex')
```

```matlab
fS = 1400;
t1 = 0:1/fS:1;
t2 = 0:1/fS:10;
t3 = 0:1/fS:1000;

k1 = 700;
f0 = -700;
a = [p1 + p2, -p1 * p2];
b = [0, 1, -z1];


x1 = exp(1i * 2 * pi * (k1 * t1.^2 - 700 * t1)/2);
y1 = DTLTI(a, b, x1);


x2 = exp(1i * 2 * pi * (f0 * t2 + k1 * t2.^2)/2);
y2 = DTLTI(a, b, x2);


x3 = exp(1i * 2 * pi * (f0 * t3 + 0.7 * t3.^2)/2);
y3 = DTLTI(a, b, x3);


plotTimeDomain(t1, y1, 't = [0, 1]');
plotTimeDomain(t2, y2, 't = [0, 10]');
plotTimeDomain(t3, y3, 't = [0, 1000]');


function plotTimeDomain(t, y, titleStr)
freq = -pi:2*pi/length(y):pi-2*pi/length(y);
figure;
subplot(1, 2, 1)
plot(freq, abs(y), 'LineWidth', 2)
ylabel('Magnitude', 'Interpreter', 'latex');
xlabel('Frequency', 'Interpreter', 'latex');
title(titleStr, 'Interpreter', 'latex');
grid on;
box on;

subplot(1, 2, 2)
plot(freq, angle(y), 'LineWidth', 2)
ylabel('Phase', 'Interpreter', 'latex');
xlabel('Frequency', 'Interpreter', 'latex');
title(titleStr, 'Interpreter', 'latex');
grid on;
box on;
end
```