EEE 321-Signals and Systems

-Lab 02-

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Section 001
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Part A

The corresponding Matlab code for the function SUMCS.:

```
function [xs] = SUMCS(t,A,omega)
xs = (zeros(size(t)));
for i= 1:length(A)
xs=xs+A(i)*exp(1j*omega(i).*t);
end
end
```

After creating the function the code for computing and plotting different features of $x_s(t)$:

```
N=mod(22102167, 41); %N=10
t=[0:0.001:1];
omega=pi*rand(1,N);
A=3*rand(1,N,"like",1j);
xs= SUMCS(t,A,omega); %calling the subroutine
rxs= real(xs);%real part extraction
ixs= imag(xs);%imaginary part extraction
mxs= abs(xs); %finding the magnitude of each complex value stored in xs
pxs= angle(xs);%finding the phase of each complex value stored in xs

%plotting
figure
subplot(2,2,1);
plot (t,rxs,'r');
ylabel ({'$\Re(xs)$'},'Interpreter','latex');
xlabel ({'$Time$'},'Interpreter','latex');
box on
grid, grid minor
title({'$\Re(xs)$ vs t'},'Interpreter','latex')

subplot(2,2,2);
plot (t,ixs,'m');
ylabel ({'$\Im(xs)$'},'Interpreter','latex');
xlabel ({'$Time$'},'Interpreter','latex');
box on
grid, grid minor
title({'$\Im(xs)$ vs t'},'Interpreter','latex')

subplot(2,2,3);
plot (t,mxs,'b');
ylabel ({'$|xs|$'},'Interpreter','latex');
xlabel ({'$Time$'},'Interpreter','latex');
box on
grid, grid minor
title({'$|xs|$ vs t'},'Interpreter','latex')

subplot(2,2,4);
plot (t,pxs,'k');
ylabel ({'$\angle(xs)$'},'Interpreter','latex');
xlabel ({'$Time$'},'Interpreter','latex');
box on
grid, grid minor
title({'$\angle(xs)$ vs t'},'Interpreter','latex')
```

Here are the results:

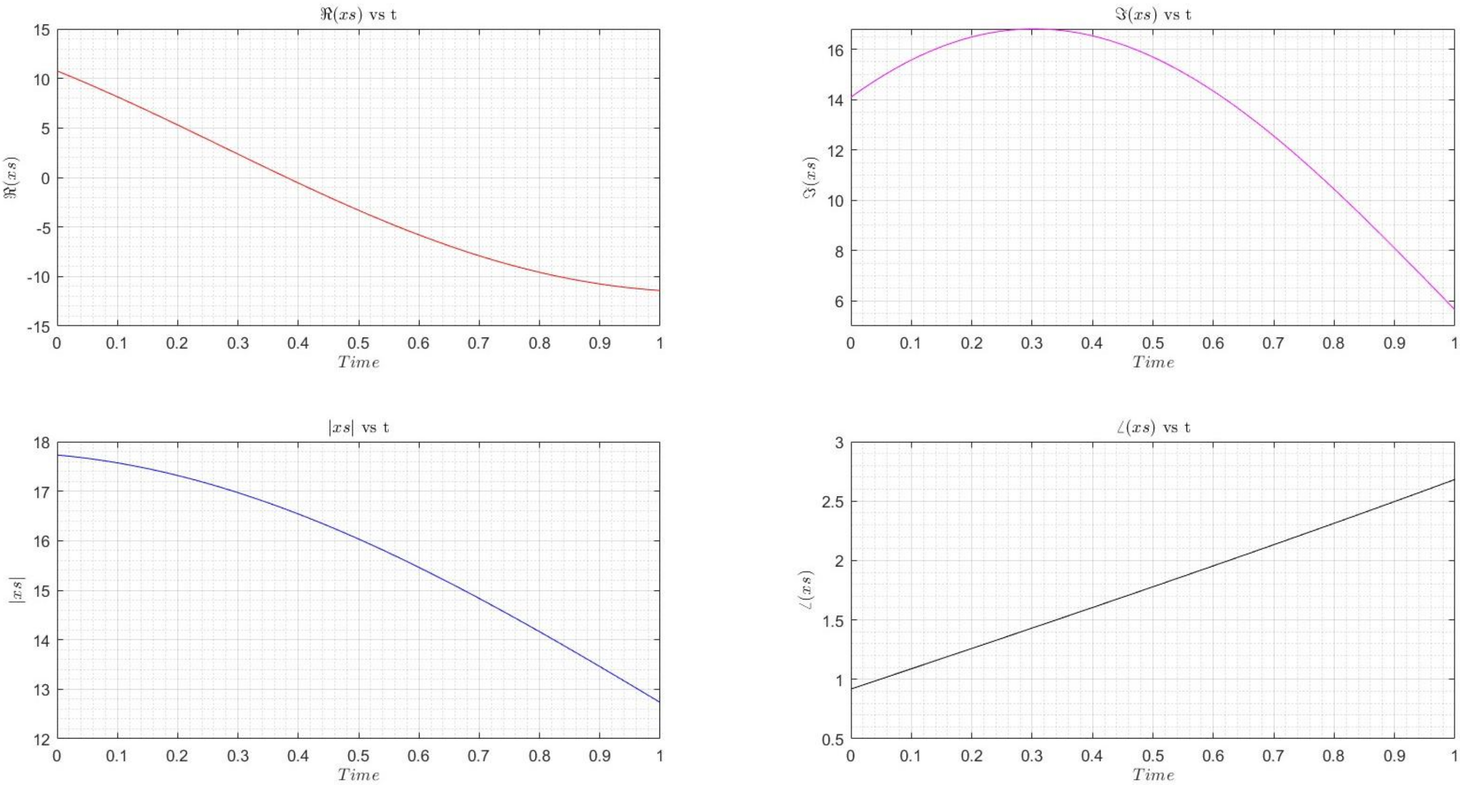


Fig.1. Plots of Part A

Part B

The sketch of the function x(t) can be seen in Fig.2.

$$x(t) = \begin{cases} 1-2t^2 & \text{if } -\frac{W}{2} < t < \frac{W}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

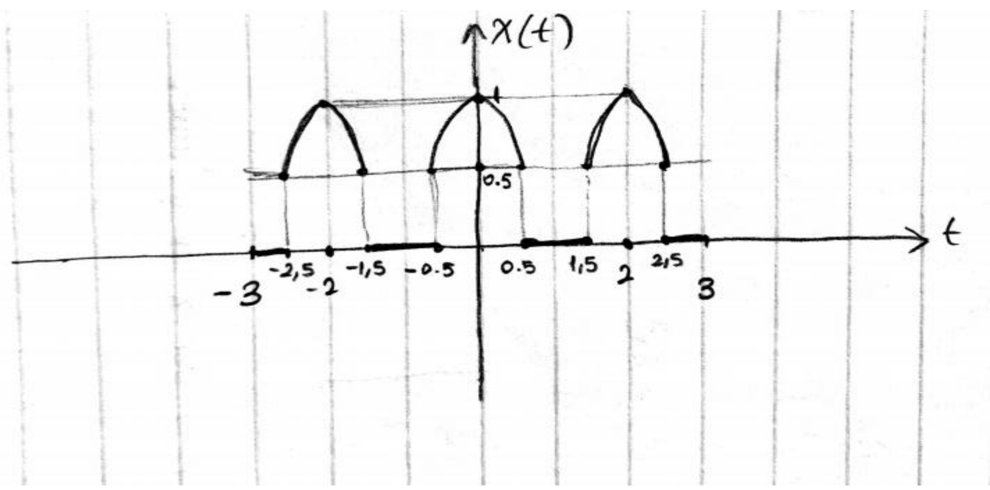
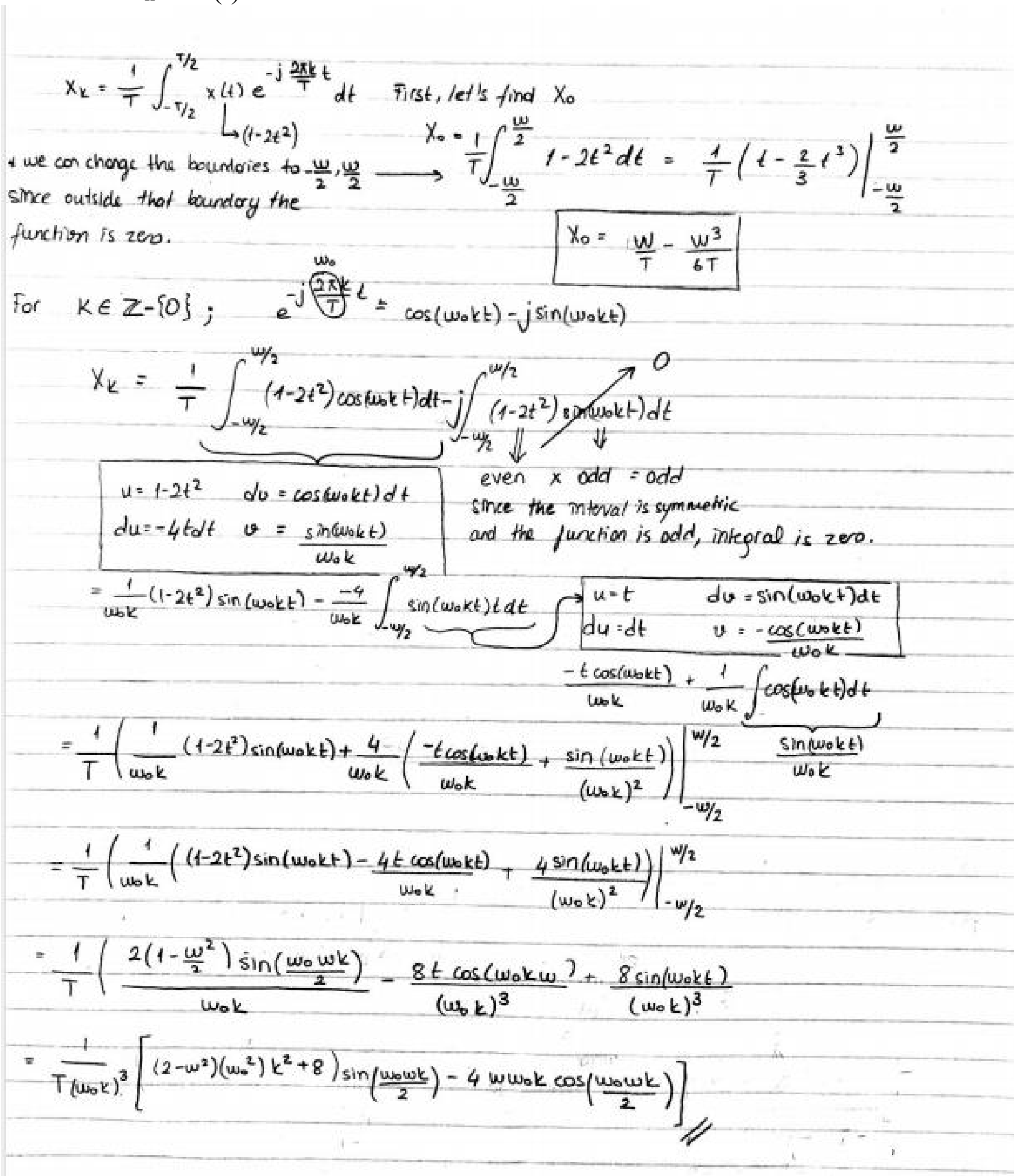


Fig.2. Rough sketch of x(t)

By evaluating the coefficient integral we can find the fourier series expansion coefficients X_k of x(t) as follows:



$$X_k = \frac{1}{T}\int_{-T/2}^{T/2} \underbrace{x(t)}_{(1-2t^2)} e^{-j\frac{2\pi k}{T}t}\,dt$$

First, let's find X_0

* we can change the boundaries to $-\frac{W}{2}, \frac{W}{2}$ since outside that boundary the function is zero.

$$X_0 = \frac{1}{T}\int_{-\frac{W}{2}}^{\frac{W}{2}} 1-2t^2\,dt = \frac{1}{T}\left(t-\frac{2}{3}t^3\right)\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}}$$

$$\boxed{X_0 = \frac{W}{T} - \frac{W^3}{6T}}$$

For $k \in \mathbb{Z}-\{0\}$; $e^{-j\overbrace{\frac{2\pi k}{T}}^{w_0}t} = \cos(w_0kt) - j\sin(w_0kt)$

$$X_k = \frac{1}{T}\left[\int_{-W/2}^{W/2}(1-2t^2)\cos(w_0kt)\,dt - j\int_{-W/2}^{W/2}(1-2t^2)\sin(w_0kt)\,dt\right] \qquad \rightarrow 0$$

even x odd = odd

since the interval is symmetric and the function is odd, integral is zero.

$u = 1-2t^2 \quad dv = \cos(w_0kt)\,dt$

$du = -4t\,dt \quad v = \frac{\sin(w_0kt)}{w_0k}$

$$= \frac{1}{w_0k}(1-2t^2)\sin(w_0kt) - \frac{-4}{w_0k}\int_{-W/2}^{W/2}\sin(w_0kt)\,t\,dt$$

$u = t \quad dv = \sin(w_0kt)\,dt$

$du = dt \quad v = -\frac{\cos(w_0kt)}{w_0k}$

$$\frac{-t\cos(w_0kt)}{w_0k} + \frac{1}{w_0k}\underbrace{\int\cos(w_0kt)\,dt}_{\frac{\sin(w_0kt)}{w_0k}}$$

$$= \frac{1}{T}\left(\frac{1}{w_0k}(1-2t^2)\sin(w_0kt) + \frac{4}{w_0k}\left(\frac{-t\cos(w_0kt)}{w_0k} + \frac{\sin(w_0kt)}{(w_0k)^2}\right)\right)\Bigg|_{-W/2}^{W/2}$$

$$= \frac{1}{T}\left(\frac{1}{w_0k}\left((1-2t^2)\sin(w_0kt) - \frac{4t\cos(w_0kt)}{w_0k} + \frac{4\sin(w_0kt)}{(w_0k)^2}\right)\right)\Bigg|_{-W/2}^{W/2}$$

$$= \frac{1}{T}\left(\frac{2\left(1-\frac{W^2}{2}\right)\sin\left(\frac{w_0Wk}{2}\right)}{w_0k} - \frac{8t\cos(w_0kW)}{(w_0k)^3} + \frac{8\sin(w_0kt)}{(w_0k)^3}\right)$$

$$= \frac{1}{T(w_0k)^3}\left[\left((2-W^2)(w_0^2)k^2+8\right)\sin\left(\frac{w_0Wk}{2}\right) - 4Ww_0k\cos\left(\frac{w_0Wk}{2}\right)\right]$$

Fig.3. Finding fourier series expansion coefficients X_k

Part C

The code for function FSWave can be found below:

```
function [xt] = FSWave(t,K,T,W)

A=zeros(1,2*K+1);
omega=zeros(1,2*K+1);
fundfreq= (2*pi)/T;
for k= -K:K
A(k+K+1)=(1/(T*fundfreq^3*k^3))*((((2-
W^2)*fundfreq^2*k^2)+8)*sin(fundfreq*W*k/2)-
4*W*fundfreq*k*cos(fundfreq*W*k/2));
%finding the coefficients and storing them to array A
```

```
omega(k+K+1)= k*((2*pi)/T);
end
A(K+1)=(W-W^3/6)/T;
xt= SUMCS(t,A,omega);
end
```

Using this function the plots are generated by the code below:

```
D11 = mod(22102167, 11); %=10
D5 = mod(22102167, 5); %=2
T=2;
W=1;
K=20+D11; %K=30
t=[-5:0.001:5];
xt = FSWave(t,K,T,W);
rxt= real(xt);%real part extraction
ixt= imag(xt);%imaginary part extraction
%plotting
figure
subplot(1,2,1);
plot (t,rxt,'r'); %plot of the real part versus t
ylabel ({'$\Re(xt)$'},'Interpreter','latex');
xlabel ({'$Time$'},'Interpreter','latex');
box on
grid, grid minor
title({'$\Re(xt)$ vs t'},'Interpreter','latex')

subplot(1,2,2);
plot (t,ixt,'m'); %plot of the imaginary part versus t
ylabel ({'$\Im(xt)$'},'Interpreter','latex');
xlabel ({'$Time$'},'Interpreter','latex');
box on
grid, grid minor
title({'$\Im(xt)$ vs t'},'Interpreter','latex')
```

Here are the results:

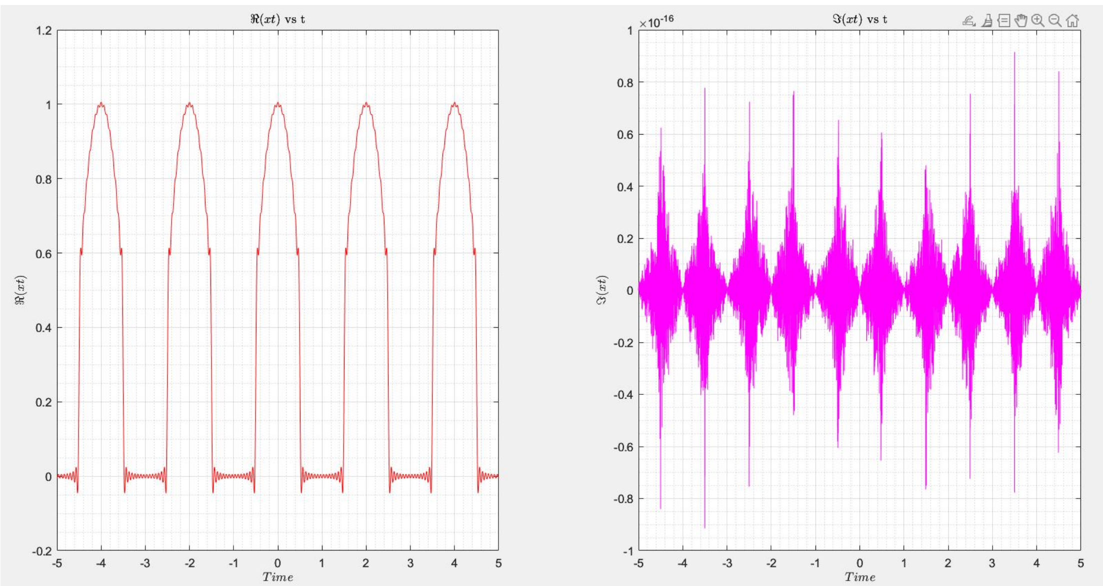


Fig.4. Plots of real and imaginary parts of $\tilde{x}(t)$

As can be seen from the plot of real(xt) in Fig.4. the real part of of xt resembles a lot to the sketched function in Fig.2. Since we are representing a periodic function as a sum of sinusoids at different magnitudes and frequencies some features of the represented function is hard to express as sinusoidal waves, the discontinuities are one of them. The lowest points of xt occurs when xt=0 which causes a jump from 0.5 to 0 as expressed in Fig.2. In fourier series these discontinuities correspond to lower points in the real xt plot. Summing the sinusoids produce large peaks around the discontinuities which overshoot and undershoot the actual function values. This phenomenon is known as the Gibbs Phenomenon.

The maximum value of the actual function in Fig.2. is 1 and as can be seen from the real xt plot of the fourier series expansion in Fig.4., the maximum value of the fourier series is also oscillating around 1. Since there is no jumps near the local maximum points in the actual function, the oscillations around 1 are very small compared to local minimum points where there are discontinuities.

The actual function in Fig.2. has no imaginary components however in Fourier approximation we get an imaginary signal as well as seen in Fig.4. This is due to Matlab's rounding algorithm, in the imaginary component plot we get values changing between $9.14\text{x}10^{-17}$ and $-9.14\text{x}10^{-17}$ which are very close to zero but not exactly zero. Similary when calculating $\sin(\pi/6)$ - 0.5 although the results must be zero, Matlab gives the result as -.5.5…$\text{x}10^{-17}$. Since these values are very close to zero, we can ignore this round-off error.

Again by doing the same computation for different K values, $K=2+D_5$, $K=7+D_5$, $K=15+D_5$, $K=50+D_5$, $K=100+D_5$ the resulting plots can be seen below.

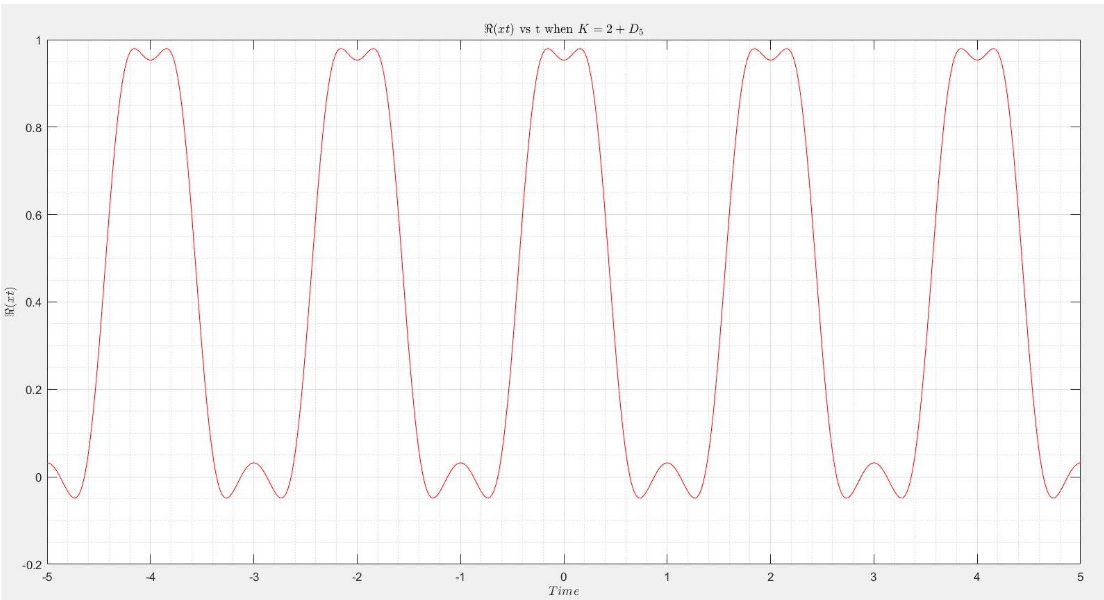


Fig.5. Plot of Real $\tilde{x}(t)$ when K=4

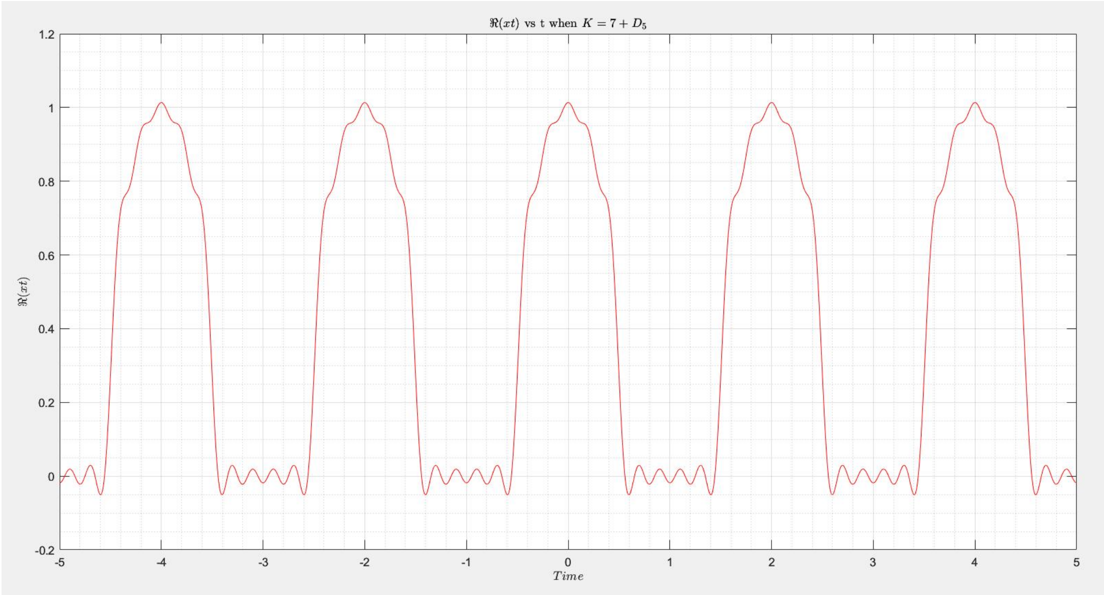


Fig.6. Plot of Real $\tilde{x}(t)$when K=9

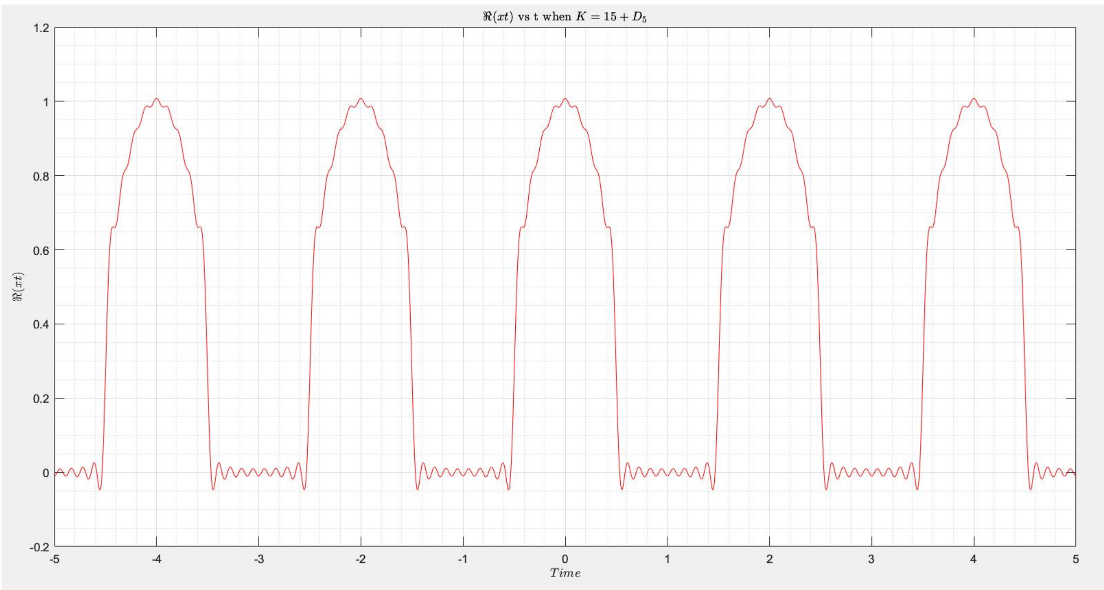


Fig.7. Plot of Real $\tilde{x}(t)$ when K=17

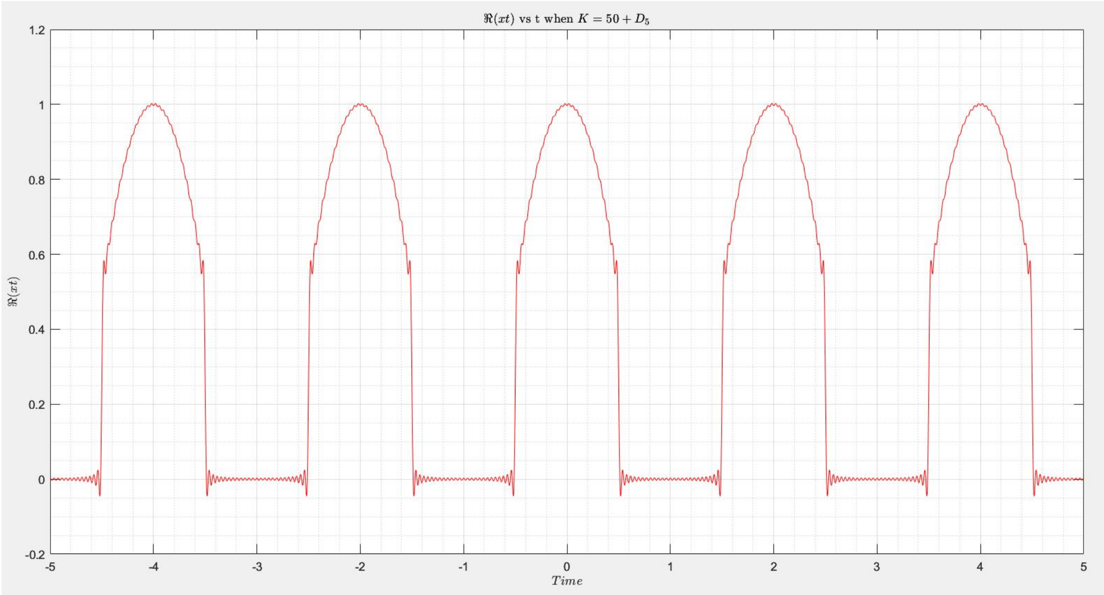


Fig.8. Plot of Real $\tilde{x}(t)$when K=52

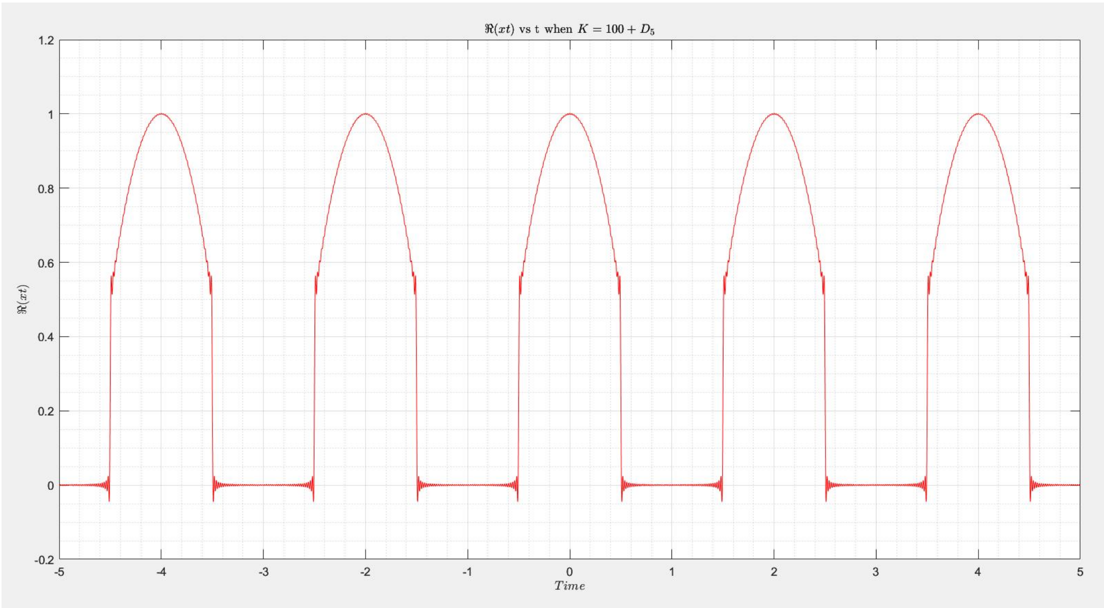


Fig.9. Plot of Real $\tilde{x}(t)$ when K=102

We can see that as K gets larger the $\tilde{x}(t)$ gets closer to the actual x(t). Since at the higher K values we are summing up more sinusoids the approximation becomes more detailed and more like the actual function. Ideally if the K value goes to infinity $\tilde{x}(t)$ becomes the same as x(t). However even though the continuous parts gets better the error at the discontinuities is still present.

Part 4

a) In order to compute the desired y(t) the calculations in Fig.10. are made. From Fig.10. we can see that changing the sign of k is equal to applying time reversal to x(t) e.g. obtaining x(-t). Hence for computing y(t), the function FSWave is changed by putting a minus sign in front of "t" value when calling the SUMCS function inside FSWave.

$$y(t) = \sum_{k=-K}^{K} Y_k \, e^{j\frac{2\pi k t}{T}} = \sum_{k=-K}^{K} X_{-k} \, e^{j\frac{2\pi k t}{T}}$$

$$-k = u$$
$$k = -u$$
$$\hookrightarrow \sum_{u=-K}^{K} X_u \, e^{-j\frac{2\pi u t}{T}}$$

Fig.10. Calculations for Y_k=X_{-k}

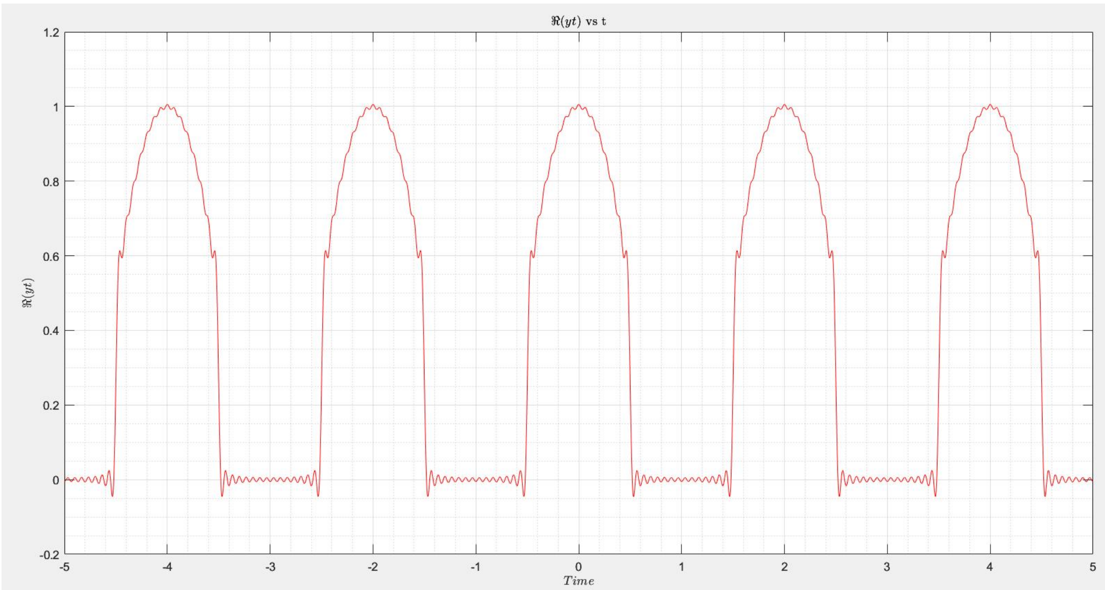


Fig.11. Plot of Real $\tilde{x}(t)$ when Y_k=X_{-k}

Fig.11 shows that when we change k to (-k) nothing changes, the result is the same as in Fig.4. The reason for that is as follows:

Since $x(t)$ is real valued $x(t) = x^*(t)$

$$x(t) = \sum_{k=-K}^{K} X_k e^{j\frac{2\pi k}{T}t} = \underbrace{\sum_{k=-K}^{K} X_k^* e^{-j\frac{2\pi k}{T}t}}_{x^*(t)}$$

Replacing k with $-k$ in $x^*(t)$ summation:

$$x(t) = \sum_{k=-K}^{K} X_{-k}^* e^{j\frac{2\pi k}{T}t} \quad \text{this requires that}$$

$$X_{-k}^* = X_k \quad \text{or} \quad X_k^* = X_{-k}$$

since $x(t)$ is real and even function, X_k's are real and even as well therefore

$$X_k^* = X_K \quad \text{which results in}$$

$$X_K = X_{-k}$$

Nothing changes when we flip the sign of K.

Fig.12. Explanation of Part 4-a

b) In order to make the required changes the calculations in Fig.13 are made. We can see that multiplying the coefficients by a complex exponential results in a time shift $x(t-t_0)$. Hence for computing $y(t)$, the function FSWave is changed by replacing the "t" by "$t-t_0$" when calling the SUMCS function inside FSWave.

b $$y(t) = \sum_{k=-k}^{k} Y_k e^{j\frac{2\pi k t}{T}} = \sum_{k=-k}^{k} X_k e^{-j\frac{2\pi k t_0}{T}} e^{j\frac{2\pi k t}{T}}$$

$$= \sum_{k=-k}^{k} X_k e^{j\frac{2\pi k}{T}(t - t_0)}$$

Fig.13. Calculations for Part4-b

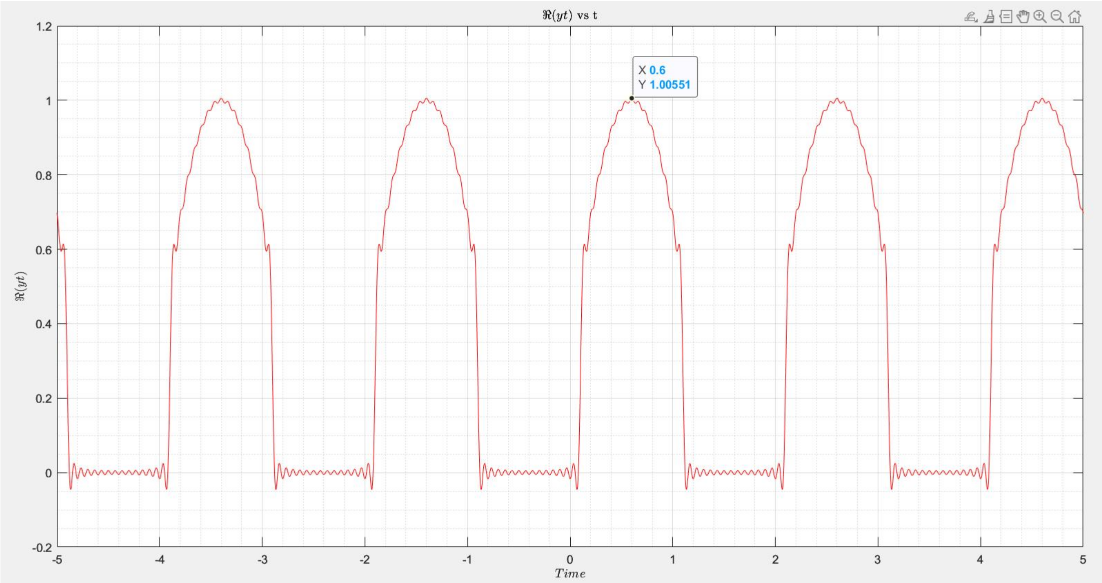


Fig.14. Plot of Part4-b

As seen in Fig.14. the function is shifted by 0.6 to the right since we multiplied with an complex exponential that has t_0=0.6.

c) For modifying the coefficients as wanted, we can add the highlighted multiplication term inside the for loop of the function FSWave.

```
A(k+K+1)=(1j*fundfreq*k)*(1/(T*fundfreq^3*k^3))*((((2-
W^2)*fundfreq^2*k^2)+8)*sin(fundfreq*W*k/2)-4*W*fundfreq*k*cos(fundfreq*W*k/2));
```

Multiplying the coefficients by j2πk/T in frequency domain results in differentiation in time domain as explained in Fig.15.

$$y(t) = \sum_{k=-K}^{K} Y_k e^{\frac{j2\pi k t}{T}} = \sum_{k=-K}^{K} \left(jk\frac{2\pi}{T}\right) X_k e^{\frac{j2\pi k t}{T}} \Bigg) = y(t) = \underline{\frac{dx(t)}{dt}}$$

$$\frac{dx}{dt} = \sum_{k=-K}^{K} X_k \frac{d\left(e^{\frac{j2\pi k}{T}t}\right)}{dt} = \sum_{k=-K}^{K} X_k \left(j\frac{2\pi k}{T}\right) e^{j\frac{2\pi k t}{T}}$$

Fig.15. Calculations for Part4-c

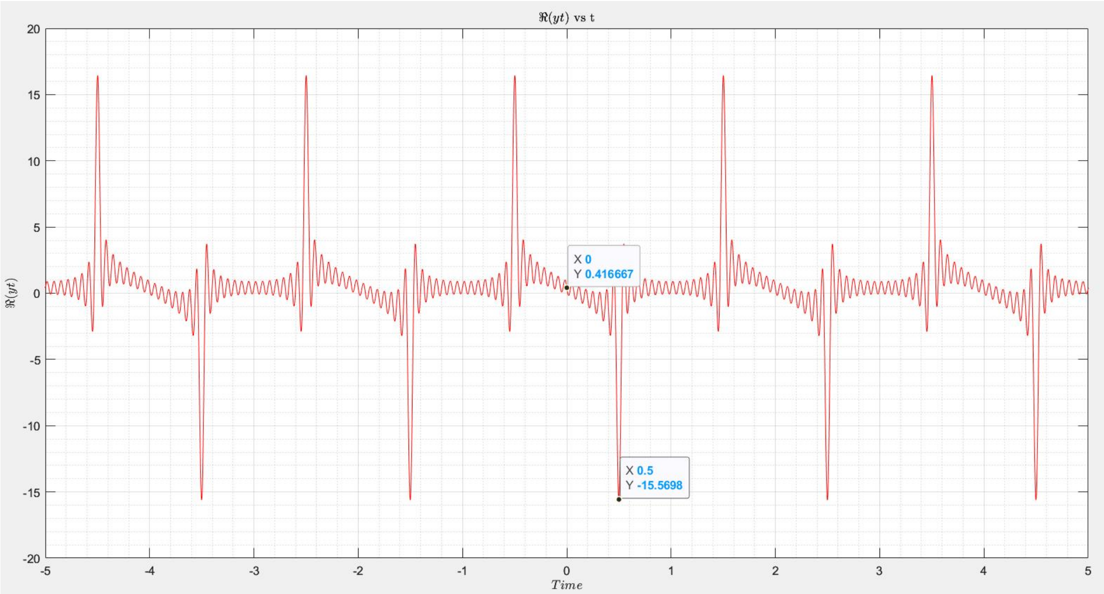


Fig.16. Plot of y(t) for Part4-c

Since we are now plotting the differentiation of x(t) versus time graph, at the points t=2n, n=0,1,2,3… where the local maximums occur,the slope of x(t) approaches to zero hence at those points the values of y(t) is close to zero. When we approach to the discontinuities of x(t) from one side the slope is very high however from the other side the slope is zero since the function is constant. Therefore we have high peaks in y(t) showing the instant changes in the slope of x(t).

d) In order to make the required changes in the signal the FSWave function is modified by adding conditional statements for k values.

```
function [xt] = FSWave(t,K,T,W)
A=zeros(1,2*K+1);
omega=zeros(1,2*K+1);
fundfreq= (2*pi)/T;
for k= -K:K
A(k+K+1)=(1/(T*fundfreq^3*k^3))*((((2-
W^2)*fundfreq^2*k^2)+8)*sin(fundfreq*W*k/2)-
4*W*fundfreq*k*cos(fundfreq*W*k/2));
%finding the coefficients and storing them to array A
if k>0
omega(k+K+1)= (-k+K+1)*((2*pi)/T);
elseif k<0
omega(k+K+1)= (-k-K-1)*((2*pi)/T);

else
omega(k+K+1)= k*((2*pi)/T);
end
end
A(K+1)=(W-W^3/6)/T;
xt= SUMCS(t,A,omega);
end
```

For understanding the meaning of this transformation we have to investigate the function in three parts as in Fig.17. For k= 1,2,3…K values the modifications correspond to a time reversal and frequency shift towards higher frequencies for x(t), for k=0 the function stays the same and for negative values of k they correspond to time reversal and frequency shift towards lower values.

$$y_k = \begin{cases} x_{K+1-k} & \text{if } k = 1, 2, \dots K \quad ① \\ y_k & \text{if } k = 0 \\ y_{-(K+1+k)} & \text{if } k = -K, \dots -1 \quad ② \end{cases}$$

① $\sum_{k=-K}^{K} x_{K+1-k} \, e^{j\frac{2\pi k}{T}t}$

$\downarrow$ $K+1-k=u$ $\quad k = K+1-u$

$$\sum_{u=1}^{2K+1} x_u \, e^{j\frac{(K+1-u)2\pi}{T}t} = \sum_{k=1}^{2K+1} x_k \, e^{j\frac{(K+1-k)2\pi}{T}t}$$

$$= \sum_{k=1}^{2K+1} x_k \, e^{-j\frac{2\pi k}{T}t} \cdot e^{j\frac{2\pi(K+1)}{T}t}$$

↳ $\sum_{k=-K}^{K}$ due to periodicity

$y(t) = x(-t) \cdot e^{j\frac{2\pi(K+1)}{T}t} \Rightarrow$ frequency shifting & time reversal

② $\sum_{k=-K}^{K} x_{-(K+1+k)} \, e^{j\frac{2\pi k t}{T}}$

$\downarrow$ $-K-1-k=u$ $\quad k = -K-1-u$

$$\sum_{u=-2K-1}^{-1} x_u \, e^{j\frac{2\pi-(K+1+u)}{T}t} = \sum_{k=-2K-1}^{-1} x_k \, e^{j\frac{2\pi-(K+1+k)}{T}t}$$

$$= \sum_{k=-2K-1}^{-1} x_k \, e^{-j\frac{2\pi k}{T}t} \, e^{-j\frac{2\pi(K+1)}{T}t}$$

↳ $\sum_{k=-K}^{K}$ due to periodicity

$y(t) = x(-t) \, e^{-j\frac{2\pi(K+1)}{T}t} \Rightarrow$ frequency shifting & time reversal

Fig.17. Calculations of y(t) for Part4-d

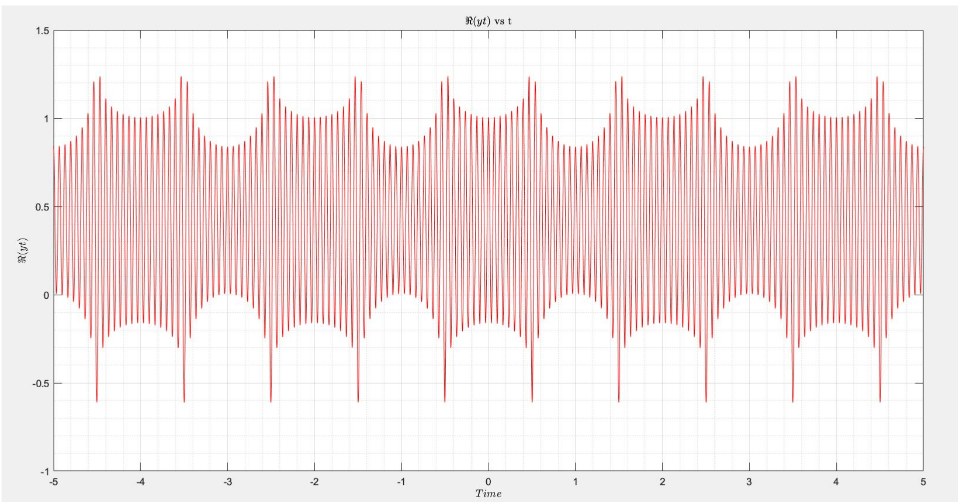


Fig.18. Plot of y(t) for Part4-d