Panel Data Models and Diff-In-Diff Methods

Zhiyuan Chen

RMEB
Remin Business School

April 2022

Outline

- Panel Data Models
  - Panel data
  - Random Effects vs. Fixed Effects
  - Fixed Effects vs. First Differencing
- Difference-In-Differences Models
  - Two-By-Two Case
  - Dynamic Treatment Effects and Event Study
  - Inference
- Staggered Difference-In-Differences
- Recent Developments in DID Models

Panel Data

- Now we have data for N units (people, products, firms, counties, cities, provinces, countries ...)
- For each unit we have T_i (≥ 1) observations
  - T_i typically refers to time periods
  - Could be products of a firm, students in a classroom, counties in a province

We assume that:

  - N is large; consistency exists as N grows
  - T_i is small; T_i stays constant as N grows

Now let's consider the regression model:

$$Y_{it} = X_{it}'\beta + u_{it}$$

- *Assumption 1*: assume independence **across individuals** but **not across time**
- *Assumption 2:* $\mathbf{E}(X_{it}u_{it}) = 0$

Consistency:

$$\begin{aligned}\hat{\beta} &= \beta + \left(\frac{\sum_{i=1}^{N}\sum_{t=1}^{T_i} X_{it}X_{it}'}{\sum_{i=1}^{N} T_i}\right)^{-1}\left(\frac{\sum_{i=1}^{N}\sum_{t=1}^{T_i} X_{it}u_{it}}{\sum_{i=1}^{N} T_i}\right) \\ &\to_p \beta\end{aligned}$$

Serial Correlation

- The assumption of NO serial correlation is problematic so the asymptotic variance from before is not right
- Error terms are uncorrelated across individuals, but not within individuals
- Formally speaking, we assume that:

$$\mathbf{E}\left(u_{it}u_{jt}\right) = 0, \forall\, i \neq j$$
$$\mathbf{E}\left(u_{it}X_{it}\right) = 0$$

- However, the assumption of OLS is not satisfied because:

$$\mathbf{E}\left(u_{it}u_{is}\right) = 0$$

  is a CRAZY assumption!
- Ignoring the serial correlation tends to **underestimate** the size of standard errors

Clustered Standard Errors

- Recall that

$$\hat{\beta}-\beta=\left(\sum_{i}^{N}\sum_{t}^{T_i}X_{it}X_{it}'\right)^{-1}\sum_{i}^{N}\sum_{t}^{T_i}X_{it}u_{it}$$

  - Define $\eta_i \equiv \sum_{t=1}^{T_i} X_{it}u_{it}$, then η_i is iid
  - $var(\eta_i) = V_\eta$
- Then we get

$$\sqrt{N}\left(\hat{\beta}-\beta\right)=\left(\frac{1}{N}\sum_{i}\sum_{t}X_{it}X_{it}'\right)^{-1}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\eta_i\right)$$

- Using the Central Limit Theorem we know

$$\sqrt{N}\left(\hat{\beta}-\beta\right)\sim\mathcal{N}\left(0,\left[\mathbf{E}\left(\sum_{t=1}^{T_i}X_{it}X_{it}'\right)^{-1}\right]V_\eta\left[\mathbf{E}\left(\sum_{t=1}^{T_i}X_{it}X_{it}'\right)\right]^{-1}\right)$$

We can approximate the variance of $\hat{\beta}$ as

$$Var\left(\hat{\beta}\right) \approx \left(\sum_i \sum_t X_{it} X_{it}'\right)^{-1} \left[\sum_i \left(\sum_t X_{it}\hat{u}_{it}\right)\left(\sum_t X_{it}'\hat{u}_{it}\right)\right]\left(\sum_i \sum_t X_{it} X_{it}'\right)^{-1}$$

- This is a generalization of the heteroskedastic robust standard errors
- We are allowing $[X_{i1}u_{i1}, X_{i2}u_{i2}, \cdots, X_{iT_i}u_{iT_i}]$ to have an arbitrary variance-covariance matrix

In STATA, this is done by coding `reg y x, cluster(i)`

- i is the id of group in which the errors are serially correlated

Random Effects vs. Fixed Effects

- Panel data enable us to take care of the idiosyncratic component in the error term
- Write the linear model as

$$Y_{it} = X_{it}'\beta + \theta_i + \varepsilon_{it}$$

  - In both of the models, we assume that

$$\mathbf{E}\left(\varepsilon_{it} X_{it}\right) = 0$$

  - In the Random Effects (RE) model, we assume that

$$\mathbf{E}\left(\theta_i X_{it}\right) = 0$$

  - In the Fixed Effects (FE) model, we do not assume anything about the relationship between θ_i and X_i! θ_i is an idiosyncratic constant.

Consistent Estimators for FE models

- Include individual dummies and run the regression:

$$Y_{it} = X_{it}\beta + D'_{it}\theta + \varepsilon_{it}$$

where $D_{it} = \left[D_{it}^{(j)}\right]$, $D_{it}^{(j)} = 1$ if $i = j$ and zero otherwise

  - Conceptually this is a different model, but technically it delivers consistent estimates for β

- **Standard FE (Mean-Differencing)** model:

$$(Y_{it} - \bar{Y}_i) = (X_{it} - \bar{X}_i)'\beta + (\varepsilon_{it} - \bar{\varepsilon}_i)$$

where $\bar{Z}_i \equiv \frac{1}{T_i}\sum_{t=1}^{T_i} Z_{it}$

- **First-Differencing (FD)** model:

$$Y_{it} - Y_{it-1} = (X_{it} - X_{it-1})'\beta + \varepsilon_{it} - \varepsilon_{it-1}$$

  - With 2 periods, this is equivalent to the standard FE model (verify it by yourself)

Fixed Effects vs. First Differencing

More generally, we consider the model

$$Y_{it} = \beta\tau_{it} + \theta_i + \varepsilon_{it}$$

Assume that $T_i = T$ for everyone, for everyone the only regressor τ_{it} is given by

$$\tau_{it} = \begin{cases} 0 & t \leq t_0 \\ 1 & t > t_0 \end{cases}$$

- τ_{it} can represent some macro-level program (trade liberalization, 5-year plan, expansion of college enrollment...) begins at $t_0 + 1$
- The FE and FD estimators are:

$$FE : \hat{\beta}_{FE} = \frac{\sum_{i=1}^{N}\sum_{t=t_0+1}^{T} Y_{it}}{N\left(T-t_0\right)} - \frac{\sum_{i=1}^{N}\sum_{t=1}^{t_0} Y_{it}}{Nt_0}$$

$$FD : \hat{\beta}_{FD} = \frac{\sum_{i=1}^{N}\left(Y_{it_0+1} - Y_{it_0}\right)}{N}$$

Fixed Effects vs. Pooled Regression

The fixed effects estimator is NOT always better than pooled OLS regression

- The sample variance of the data is:

$$\begin{aligned} ss\left(X_{it}\right) &= \sum_i \sum_t \left(X_{it} - \bar{X}\right)^2 \\ &= \underbrace{\sum_i \sum_t \left(X_{it} - \bar{X}_i\right)^2}_{within\,variance} + \underbrace{\sum_i T_i \left(\bar{X}_i - \bar{X}\right)^2}_{between\,variance} \end{aligned}$$

FE estimator only uses the within variance of X_i:

1. It is inefficient. Standard errors are very large when X_{it} does not change
2. It can even a worse bias:

$$\hat{\beta}_{POLS} = \beta + \frac{cov\left(X_{it},\, \theta_i + \varepsilon_{it}\right)}{var\left(X_{it}\right)}$$

$$\hat{\beta}_{FE} = \beta + \frac{cov\left(X_{it} - \bar{X}_i,\, \varepsilon_{it} - \bar{\varepsilon}_i\right)}{var\left(X_{it} - \bar{X}_i\right)}$$

Example: Almond, Chay, and Lee (QJE, 2005)

- Their goal is to estimate the effects of birth weight on health:

$$h_{ij} = \alpha + \beta bw_{ij} + X_i' \gamma + a_i + \varepsilon_{ij}$$

  where

  - h_{ij}: health of newborn j of mother i
  - bw_{ij}: birth weight
  - a_i: mother specific effect

- The pooled OLS regression of h_{ij}on bw_{ij} gives us

$$\hat{\beta}_{POLS} = \beta + \frac{\operatorname{cov}(bw_{ij}, X_i'\gamma)}{\operatorname{var}(bw_{ij})} + \frac{\operatorname{cov}(bw_{ij}, a_i)}{\operatorname{var}(bw_{ij})}$$

- Their clever solution is to **use twins**:
    - Twins share the same mother, so a_i effectively controls for race, age, education, family background...
    - Estimate the model as

$$\Delta h_{ij} = \beta \Delta bw_{ij} + \Delta \varepsilon_{ij}$$

where they assume that $\operatorname{cov}(\Delta bw_{ij}, \Delta \varepsilon_{ij}) = 0$

Difference-In-Differences

Difference-In-Differences: The Two-by-Two Case

Simple Before-After policy evaluation:

- Data on individuals right before and after the policy intervention: *Pre* and *Post*
- Two years dated 0 and 1 and that the policy is enacted in between
- We can simply identify the effect as:

$$\hat{\alpha} = \bar{Y}_1 - \bar{Y}_0$$

We could justify this using a FE model:

$$Y_{it} = \alpha_0 + \alpha\tau_{it} + \theta_i + \varepsilon_{it}$$

where

$$\tau_{it} = \begin{cases} 0 & t = 0 \\ 1 & t = 1 \end{cases}$$

We assume that $\mathbf{E}(\tau_{it}\varepsilon_{it}) = 0$ but no particularly assumption on θ_i

In the two-period case, FE model is equivalent to FD model, which is

$$Y_{i1} - Y_{i0} = \alpha + u_{i1} - u_{i0}$$

The estimator for the policy effect is

$$\hat{\alpha} = \bar{Y}_{i1} - \bar{Y}_{i0}$$

- We assume no other changes in time between and attribute whatever that is to the program
- Add time dummies into the model the treatment effect is not separated from the policy's impact

After adding time dummy variables, the model becomes

$$Y_{it} = \alpha_0 + \alpha \tau_{it} + \delta t + \theta_i + \varepsilon_{it}$$

Then the difference estimator delivers

$$\begin{aligned}\mathbf{E}\left(Y_{i1} - Y_{i0}\right) &= \mathbf{E}\left[\left(\alpha_0 + \alpha + \delta + \theta_i + \varepsilon_{i1}\right) - \left(\alpha_0 + \theta_i + \varepsilon_{i0}\right)\right] \\ &= \alpha + \delta\end{aligned}$$

- The Diff-in-Diff estimator is designed to solve this problem

Now we have two periods for two groups:

1. People who are affected by the policy changes (treated): Y_{it}^{1}
2. People who are not affected by the policy changes (controls): Y_{it}^{0}

We can think:

1. Using the treated to pick up the time changes and policy effects:

$$\hat{\alpha} + \hat{\delta} = \bar{Y}_{i1}^{1} - \bar{Y}_{i0}^{1}$$

2. Under the **common trend** assumption controls pick up the time changes:

$$\hat{\delta} = \bar{Y}_{i1}^{0} - \bar{Y}_{i0}^{0}$$

We can then estimate the policy effect as a difference in difference

$$\hat{\alpha} = \left(\bar{Y}_{1}^{1} - \bar{Y}_{0}^{1}\right) - \left(\bar{Y}_{1}^{0} - \bar{Y}_{0}^{0}\right)$$

Formally, we can write the DGP (data generating process) for the DID estimator as

$$Y_{it} = \alpha_0 + \alpha \tau_{it}^{g_i} + \delta t + \theta_i + \varepsilon_{it}$$

where

$$\tau_{it}^{g} = \begin{cases} 1 & t = 1, g_i = 1 \\ 0 & otherwise \end{cases}$$

τ_{it}^{g} is usually written as $\tau_{it} \times t$

- Now we run a fixed effect regression and get a consistent estimate of the treatment effects α

Recall that the FE estimator for two-periods data is equivalent to the FD estimator:

$$\hat{\alpha} = \frac{\sum_{i=1}^{N}\left[\left(\tau_{i1}^{g_i} - \tau_{i0}^{g_i}\right) - \overline{\left(\tau_{i1}^{g_i} - \tau_{i0}^{g_i}\right)}\right]\left(Y_{i1} - Y_{i0}\right)}{\sum_{i=1}^{N}\left[\left(\tau_{i1}^{g_i} - \tau_{i0}^{g_i}\right) - \overline{\left(\tau_{i1}^{g_i} - \tau_{i0}^{g_i}\right)}\right]^2}$$
$$= \left(\bar{Y}_1^1 - \bar{Y}_0^1\right) - \left(\bar{Y}_1^0 - \bar{Y}_0^0\right)$$

- Notice that

$$\overline{\left(\tau_{i1}^{g_i} - \tau_{i0}^{g_i}\right)} = \frac{1}{N_1 + N_0}\sum_i \left(\tau_{i1}^{g_i} - \tau_{i0}^{g_i}\right)$$
$$= \frac{1}{N_1 + N_0}\left[\sum_{\{i:\, g_i=1\}} \left(\tau_{i1}^{1} - \tau_{i0}^{1}\right) + \sum_{\{i:\, g_i=0\}} \left(\tau_{i1}^{0} - \tau_{i0}^{0}\right)\right]$$
$$= \frac{N_1}{N_1 + N_0}$$

In principle, you don't need panel data to implement DID regression; repeated cross section data are fine!

- In general, we can write the regression as

$$Y_i = \alpha_0 + \alpha \tau_{t_i}^{g_i} + \delta t_i + \gamma g_i + \varepsilon_i$$

  - where g_i is the group indicator of person i and t_i is the time period in which i exists in the data
  - $\tau_{t_i}^{g_i} = g_i \times t_i$
- We have 4 categories of people: (before, treated), (before, controls), (after, treated), (after, controls)
- We end up with having 4 groups of *moment conditions*:

$$\begin{aligned} \mathbf{E}(\varepsilon_i) &= 0 \\ \mathbf{E}\left(\tau_{t_i}^{g_i} \varepsilon_i\right) &= 0 \\ \mathbf{E}(t_i \varepsilon_i) &= 0 \\ \mathbf{E}(g_i \varepsilon_i) &= 0 \end{aligned}$$

Using

$$Y_i = \hat{\alpha}_0 + \hat{\alpha}\tau_{t_i}^{g_i} + \hat{\delta}t_i + \hat{\gamma}g_i + \hat{\varepsilon}_i$$

We can rewrite the *sample analog* of the moment conditions as

$$\begin{aligned}
\bar{Y}_0^1 &= \hat{\alpha}_0 + \hat{\gamma} \\
\bar{Y}_1^1 &= \hat{\alpha}_0 + \hat{\alpha} + \hat{\delta} + \hat{\gamma} \\
\bar{Y}_0^0 &= \hat{\alpha}_0 \\
Y_1^0 &= \hat{\alpha}_0 + \hat{\delta}
\end{aligned}$$

We can solve for the parameters as

$$\begin{aligned}
\hat{\alpha}_0 &= \bar{Y}_0^0 \\
\hat{\gamma} &= \bar{Y}_0^1 - \bar{Y}_0^0 \\
\hat{\delta} &= \bar{Y}_0^1 - \bar{Y}_0^0 \\
\hat{\alpha} &= \left(Y_1^1 - \bar{Y}_0^1\right) - \left(\bar{Y}_1^0 - \bar{Y}_0^0\right)
\end{aligned}$$

More generally, we can add more control variables and specify the DID model as

$$Y_i = \alpha \tau_{t_i}^{g_i} + X_{it}'\beta + \delta_{t_i} + \theta_{g_i} + \varepsilon_i$$

This setting is simple yet powerful. There are many papers that do this basic stuff.

- Eissa and Liebman (QJE, 1996): Estimate the effect of the earned income tax credit on labor supply of women
- Dohahue and Levitt (QJE, 2001): Impact of legalized abortion on crime

Event Studies

- The treatment effects could be dynamic, with short-run effects differing from long-run effects
- We can easily extend the baseline model to allow for this:

$$Y_i = \beta_0 + \sum_{j=0}^{J} \alpha_j \tau_{t_i - j}^{g_i} + \delta_{g_i} + \gamma_{t_i} + \varepsilon_i$$

where

$$\tau_t^g = \begin{cases} 1 & g_i = 1 \text{ and policy started in year } t \\ 0 & \text{otherwise} \end{cases}$$

Common Trend Assumption

- Recall that the model we specified for DID is:

$$Y_i = \beta_0 + \alpha \tau_{t_i}^{g_i} + \delta t_i + \gamma g_i + \varepsilon_i$$

The DID estimator is

$$\hat{\alpha}_{DID} = \alpha + \left(\bar{\varepsilon}_1^1 - \bar{\varepsilon}_0^1\right) - \left(\bar{\varepsilon}_1^0 - \bar{\varepsilon}_0^0\right)$$

To obtain a consistent estimator, we require that

$$\mathbf{E}\left[\left(\bar{\varepsilon}_1^1 - \bar{\varepsilon}_0^1\right) - \left(\bar{\varepsilon}_1^0 - \bar{\varepsilon}_0^0\right)\right] = 0$$

- The treated units can have different *levels* of the error term, but the *change* in the error term must be random
- Two approaches to validate the common trend assumption: **placebo policies** and **add group-specify time trends**

Placebo Policies

- A popular strategy for robustness check: if a policy was enacted in say 2010 you could pretend it was enacted in 2005 in the same place and then only use data through 2009
- The easiest (and most common) is in the Event framework: include leads as well as lags in the model (Bertrand, Duflo, Mullainathan, 2004, QJE)
- To implement it you can run a regression like:

$$Y_i = \beta_0 + \sum_{j=-\tilde{J}, j \neq -1}^{J} \alpha_j \tau_{t_i - j}^{g_i} + \delta_{g_i} + \rho_{t_i} + \varepsilon_i$$

where $t_i = -\tilde{J}, \cdots, J$ and the period of enactment to be zero

- $\tau_{t_i - j}^{1} = 1$ whenever $t_i - j = 0$
- Note $\sum_{j=-\tilde{J}}^{J} \tau_{t_i - j}^{g_i} = \delta_{g_i}$, so we drop one period to avoid perfect co-linearity

If we normalize $\alpha_{-1} = 0$, α_{-2} is estimated as

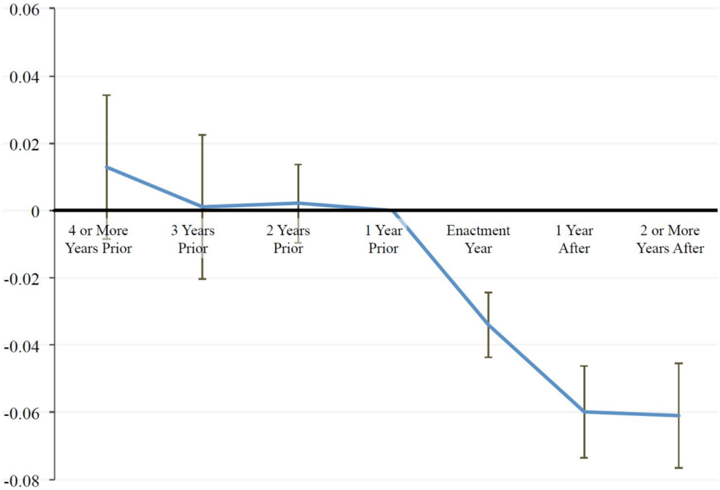
$$\hat{\alpha}_{-2} = \left(\bar{Y}^1_{-2} - \bar{Y}^1_{-1}\right) - \left(\bar{Y}^0_{-2} - \bar{Y}^0_{-1}\right)$$

where

$$\begin{aligned}
\bar{Y}^1_{-2} &= \hat{\beta}_0 + \hat{\alpha}_{-2} + \hat{\delta} + \hat{\rho}_{-2} \\
\bar{Y}^1_{-1} &= \hat{\beta}_0 + \hat{\delta} + \hat{\rho}_{-1} \\
\bar{Y}^0_{-2} &= \hat{\beta}_0 + \hat{\rho}_{-2} \\
\bar{Y}^0_{-1} &= \hat{\beta}_0 + \hat{\rho}_{-1}
\end{aligned}$$

- $\hat{\alpha}_{-2}$ should be zero under the common trend assumption
- $\hat{\alpha}_{-j} (j \geq 2)$ should also be zero under the common trend assumption

Figure 1: Typical event studies for the placebo policies



Group-Specific Time Trends

- One might be worried that units that are trending up or trending down are more likely to change policy
- One can include group×time dummy variables in the model to fix this problem
- Now let's consider the three-period case where the policy happens between period 1 and 2, the model becomes

$$Y_i = \beta_0 + \alpha \tau_{t_i}^{g_i} + \delta_0 t_i \times (1 - g_i) + \delta_1 t_i \times g_i + \delta_2 \mathbb{I}(t_i = 2) + \gamma g_i + \varepsilon_i$$

- We can write the estimator of α as a Triple Difference (DDD):

$$\begin{aligned}
\hat{\alpha}_{DDD} &= \left(\bar{Y}_2^1 - \bar{Y}_1^1\right) - \left(\bar{Y}_2^0 - \bar{Y}_1^0\right) - \left[\left(\bar{Y}_1^1 - \bar{Y}_0^1\right) - \left(\bar{Y}_1^0 - \bar{Y}_0^0\right)\right] \\
&\approx (\alpha + \delta_1 + \delta_2) - (\delta_0 + \delta_2) - (\delta_1 - \delta_0) \\
&= \alpha
\end{aligned}$$

If we do not add the group-specific time trends, the error term is a composite:

$$\tilde{\varepsilon}_i = \varepsilon_i + \delta_0 t_i \times (1 - g_i) + \delta_1 t_i \times g_i - \delta t_i$$

We go back to the common trend assumption and get

$$\begin{aligned}
\mathbf{E}\left(\left(\bar{\tilde{\varepsilon}}_1^1 - \bar{\tilde{\varepsilon}}_1^0\right) - \left(\bar{\tilde{\varepsilon}}_1^0 - \bar{\tilde{\varepsilon}}_0^0\right)\right) &= \mathbf{E}\left(\left(\bar{\varepsilon}_1^1 - \bar{\varepsilon}_1^0\right) - \left(\bar{\varepsilon}_1^0 - \bar{\varepsilon}_0^0\right)\right) \\
&\quad + (\delta_1 - \delta) - (\delta_0 - \delta) \\
&= \mathbf{E}\left(\left(\bar{\varepsilon}_1^1 - \bar{\varepsilon}_1^0\right) - \left(\bar{\varepsilon}_1^0 - \bar{\varepsilon}_0^0\right)\right) + \delta_1 - \delta_0 \\
&\approx \delta_1 - \delta_0
\end{aligned}$$

Inference: Get the Right Standard Errors

- In applications, we often have individual data (people, firms...) and the policy is enacted at more aggregated level (cities, provinces)
- It seems that we have more data than we need

Consider the individual-level model

$$Y_i = \alpha \tau_{t_i}^{g_i} + Z_i' \delta + X_{g_i t_i}' \beta + \theta_{g_i} + \gamma_{t_i} + u_i$$

One could aggregate the data by group and year:

$$\bar{Y}_{gt} = \alpha \bar{\tau}_{gt} + Z_{gt}' \delta + X_{gt}' \beta + \theta_g + \gamma_t + \bar{u}_{gt}$$

The second regression gives consistent estimates for the treatment effects

- HOWEVER, two regressions yield different standard errors: which one should we use?

- The error term for the individual level model is

$$u_i = \eta_{g_i t_i} + \varepsilon_i$$

  - The i.i.d assumption is violated is $\eta_{g_i t_i}$ exists
  - Standard solution: *Cluster the standard error by group×year* to allow for arbitrary correlation within a group
- Bertrand, Duflo, and Mullainathan (QJE, 2004):
  - If there is serial correlation in η_{gt}, cluster by group×year is problematic
- *Solutions*:
  - **# of groups is large**: block bootstrap by state works well
  - **Moderate # of states**: asymptotic approximation of the variance-covariance matrix
  - **Small # of states**: collapse the data into a "pre"- and "post"-period and account for the effective sample size
- A practioner's guide is offered by Cameron and Miller (2015, JHR)