Linear Algebra

OVERVIEW & PURPOSE

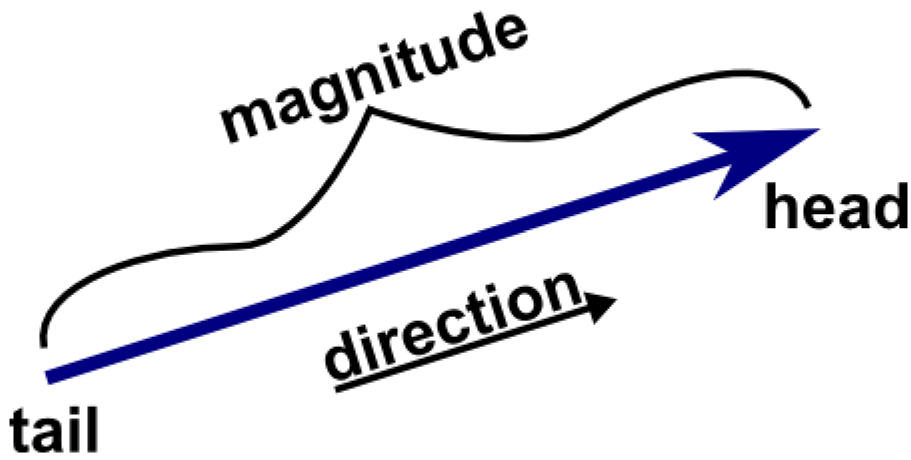
In this session, participants will have an Introduction to linear algebra for data sciences

OBJECTIVE

- vectors
- addition of vectors
- subtractions of vectors
- scaling of vectors
- independent vectors
- dependent vectors
- linear combination
- span of vectors
- dot product of two vectors
- norms of vectors
- distance between vectors
- angle between vectors
- correlation coefficient
- basis in vectors
- Matrix

- Matrix-vector multiplication
- Matrix-matrix multiplication

Vectors:

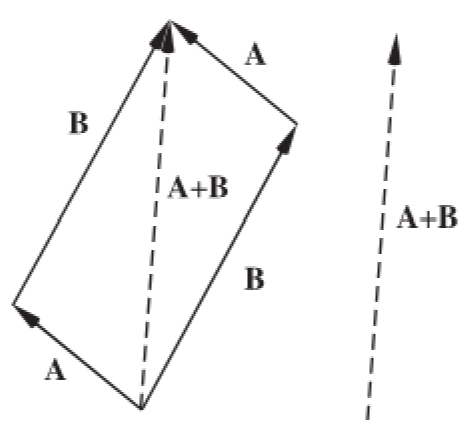


Description: A vector is an ordered list of numbers that can represent things like points in space or attributes of data samples.

Example: A point in a 2D plane might be represented by the vector [2, 5].

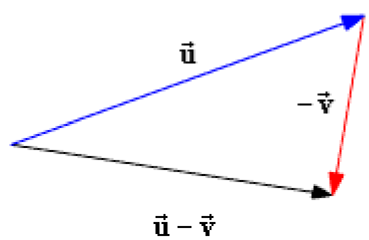
Importance: Vectors serve as the basic building blocks for data representation in many algorithms, encapsulating information in a structured manner.

Addition of Vectors:



Description: Combining two vectors by adding their corresponding elements.
Example: [1,2] + [3,4] = [4,6].
Importance: Useful for aggregating data or moving data points in various algorithms.

Subtraction of Vectors:

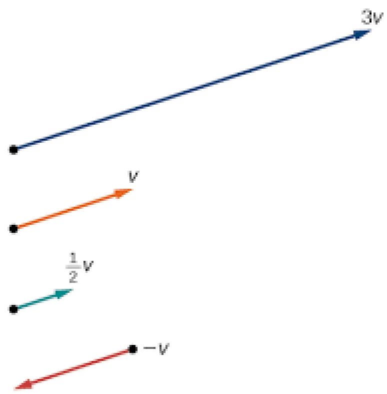


Description: Deducting one vector from another by subtracting corresponding elements.

Example: [5,4] - [2,3] = [3,1].

Importance: Essential for determining differences or changes between data points.

Scaling of Vectors:



Description: Multiplying each element of a vector by a scalar.

Example: 3 x [2,3] = [6,9].

Importance: Central in optimizing algorithms, transforming, and normalizing data.

Addition of Vectors:

Given vectors A = (3, 4) and B = (2, -1), find the sum A + B.
Vector C = (5, -2) is added to vector D = (-3, 7). What is the resultant vector?
If vector E = (1, 2) and vector F = (-2, 3), what is the sum E + F?
Find the sum of vectors G = (4, -3) and H = (-1, 6).
Vector I = (2, 5) is added to vector J = (-4, 2). Determine the resulting vector.

Subtraction of Vectors:

6. Given vectors K = (7, 5) and L = (3, 2), find the difference K - L.
Vector M = (6, -4) is subtracted from vector N = (-2, 3). What is the resulting vector?
If vector O = (2, 7) and vector P = (1, 1), what is the difference O - P?
Determine the result of subtracting vector Q = (-3, 5) from vector R = (4, 1).
Vector S = (-2, 3) is subtracted from vector T = (1, -4). Find the resulting vector.

Scaling of Vectors:

11. Given vector V = (3, 6), scale it by a factor of 2.
Vector W = (-2, 4) is scaled by a factor of 3. What is the resulting vector?
If vector X = (1, -3) is scaled by a factor of 0.5, what is the resultant vector?
Scale vector Y = (5, -2) by a factor of 4.

Vector Z = (-4, 8) is scaled by a factor of 0.25. Determine the resulting vector.

Mixed Operations:

1-. Find the resultant of adding vector A = (3, 4) to twice vector B = (-2, 1).

2-Vector C = (5, -2) is added to three times vector D = (-3, 7) and then scaled by a factor of 0.5. What is the resulting vector? (c + 3D)x 0.5=

(-2, 9.5)

3-Determine the resultant of subtracting twice vector E = (1, 2) from vector F = (-2, 3) and then scaling the result by a factor of 3.

(F - 2E) x 3

4-Vector G = (4, -3) is subtracted from three times vector H = (-1, 6). What is the resulting vector?

3H- G

(-7,21)

5-Find the sum of vector I = (2, 5) and vector J = (-4, 2), then scale the result by a factor of 0.75.

(I+ J) x 0.75

(-1.5, 5.25)

(2+(-4), 5+2)= (-2, 7)x 0.75

Addition of Vectors:

Given vectors A = (-1, 3) and B = (2, -5), find the sum A + B.

(-1+2, 3+(-5))= (-1, -2)

Vector C = (-4, 7) is added to vector D = (3, -2). What is the resultant vector?

(-4+3, 7-2)= (-1,5)

If vector E = (6, -9) and vector F = (-7, 4), what is the sum E + F?

Find the sum of vectors G = (5, 2) and H = (-3, 6).

Vector I = (2, -8) is added to vector J = (-1, 3). Determine the resulting vector.

Subtraction of Vectors:

6. Given vectors K = (4, -2) and L = (-3, 5), find the difference K - L.

Vector M = (-6, 9) is subtracted from vector N = (7, -4). What is the resulting vector?

N-M

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If vector O = (-1, 6) and vector P = (3, 2), what is the difference O - P?
Determine the result of subtracting vector Q = (2, -3) from vector R = (-5, 1).
R-Q
(-5-(2),1-(-3))
(-7,4)
Vector S = (-8, 5) is subtracted from vector T = (1, -7). Find the resulting vector.
T-S
(1-(-8), -7-5)
(9,-12)
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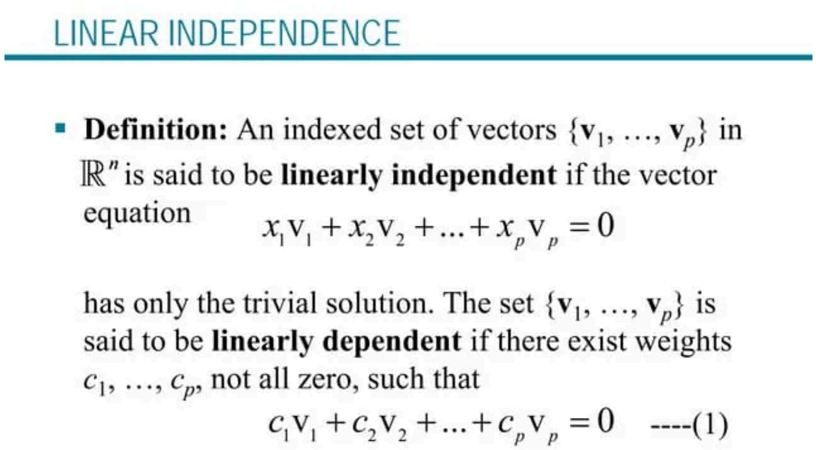
Independent Vectors:



LINEAR INDEPENDENCE

- **Definition:** An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1 \mathrm{v}_1 + x_2 \mathrm{v}_2 + \ldots + x_p \mathrm{v}_p = 0$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1 \mathrm{v}_1 + c_2 \mathrm{v}_2 + \ldots + c_p \mathrm{v}_p = 0 \quad \text{----(1)}$$

Description: Vectors that aren't linear combinations of each other.

Example: [1,0] and [0,1] are independent.

Importance: Ensures diverse and non-redundant information in data sets.

Dependent Vectors:

Description: Vectors that can be expressed as a linear combination of others.

Example: [2,2] can be written as 2 x [1,1].

Importance: Identifying dependent vectors helps in reducing dimensionality and redundancy.

Linear Combination:

Description: Constructing a vector by a combination of scaling and adding other vectors.

Example: Combining 2 x [1,2] and 3 x [0,1] yields [2,5].

Importance: A foundational concept in understanding vector spaces and relationships.

Span of Vectors:

Span of set of vectors

Definition:

If $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S.

$$span(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \Lambda + c_k\mathbf{v}_k \mid \forall c_i \in R\}$$

(the set of all linear combinations of vectors in S)

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

Description: The set of all vectors achievable by linear combinations of a given set.

Example: The span of {[1,0], [0,1]} includes every point in the 2D plane.

Importance: Indicates the "coverage" or "reach" of our vectors in the space.

Dot Product of Two Vectors:

Description: The sum of products of the respective components of two vectors.

Example: The dot product of [1,2] and [3,4] is (13) + (24) = 11.

Importance: Central in measuring vector similarity and projection.

Norms of Vectors:

Vector Norms

DEF: A norm is a function $\|\cdot\|: R^n \to R$ that satisfies

$$
\begin{aligned}
&(1)\ \ \|x\| \geq 0, \text{ and } \|x\| = 0 \text{ only if } x = 0,\\
&(2)\ \ \|x + y\| \leq \|x\| + \|y\|,\\
&(3)\ \ \|\alpha x\| = |\alpha|\,\|x\|.
\end{aligned}
$$

p-norms: The most important class of vector norms

$$
\begin{aligned}
\|x\|_1 &= \sum_{i=1}^{m} |x_i|,\\
\|x\|_2 &= \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} = \sqrt{x^* x},\\
\|x\|_\infty &= \max_{1\leq i\leq m} |x_i|,\\
\|x\|_p &= \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p} \quad (1 \leq p < \infty).
\end{aligned}
$$

Example: $x = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$

$$
\begin{aligned}
\|x\|_1 &= 10\\
\|x\|_2 &= \sqrt{4+25+9} \approx 6.1644\\
\|x\|_\infty &= 5\\
\|x\|_p &= \sqrt[p]{2^p + 5^p + 3^p}
\end{aligned}
$$

Description: The length or magnitude of a vector.

Example: The L2 (Euclidean) norm of [3,4] is 5.

Importance: Provides magnitude which is essential in numerous algorithms, especially in normalization.

Distance Between Vectors:

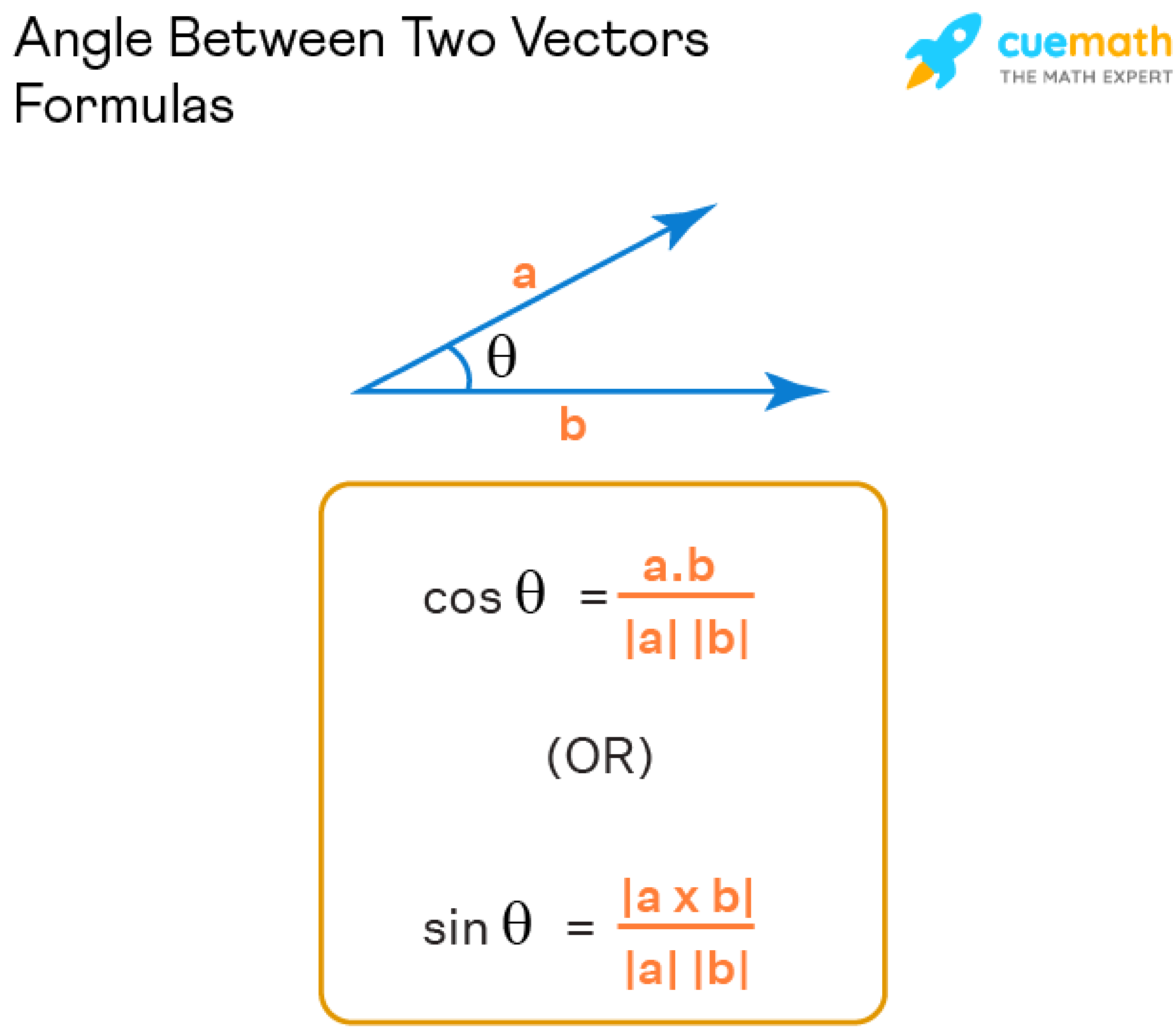
$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$$

Description: The length of the vector resulting from the subtraction of two vectors, typically measured in L2 norm.

Example: Distance between [1,2] and [3,4] is $\sqrt{((3-1)^2 + (4-2)^2)} = \sqrt{8}$.

Importance: Core in clustering and similarity checking algorithms.

Angle Between Vectors:

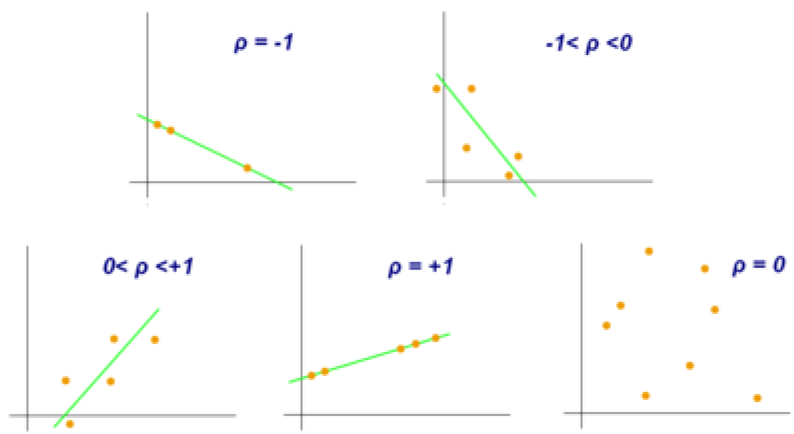


Description: Measure of orientation between two vectors.

Example: For vectors A and B, angle θ is found by: $\cos(\theta) = (A.B) / (||A|| ||B||)$.

Importance: Understanding vector orientation aids in many analytical tasks.

Correlation Coefficient:

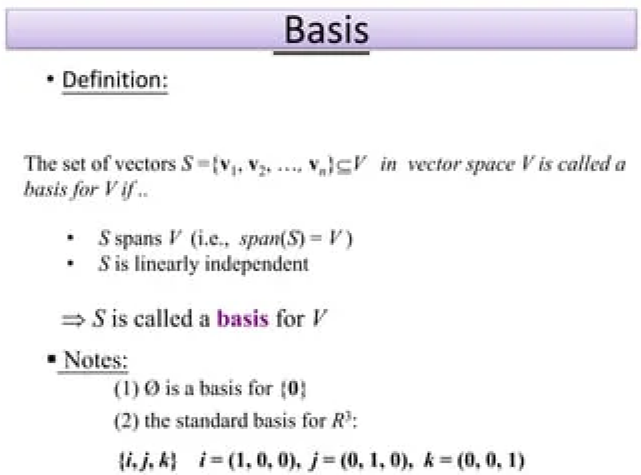


Description: Quantifies the linear relationship between two sets of data.

Example: Age and bone density might have a correlation coefficient of -0.7.

Importance: Key statistics in data analysis to understand relationships.

Basis in Vectors:

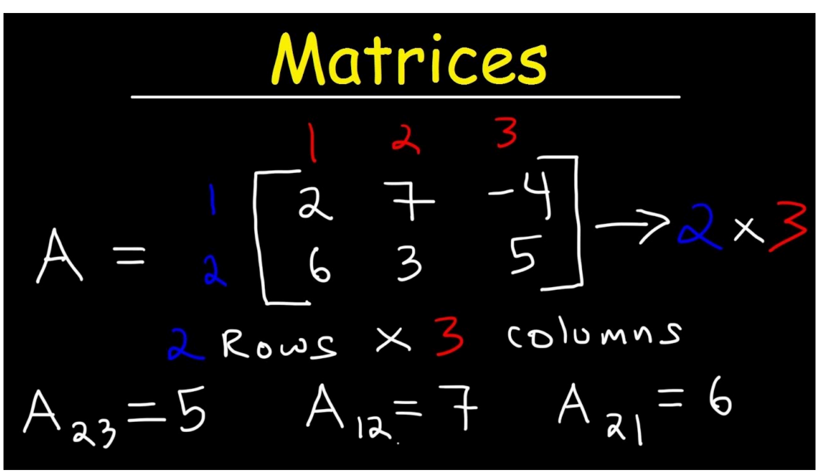


Description: A set of vectors that are linearly independent and span the space.

Example: In 2D, vectors [1,0] and [0,1] can serve as a basis.

Importance: Basis vectors offer a simplified perspective and are pivotal in vector space transformations.

Matrices:



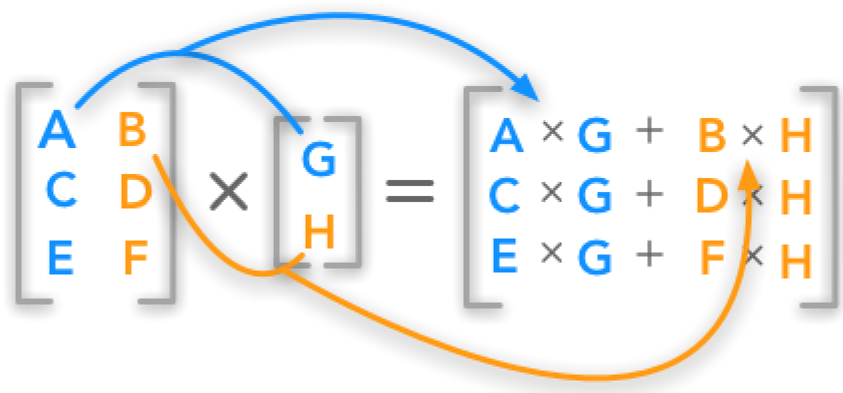
Description: A rectangular array of numbers, symbols, or expressions arranged in rows and columns.

Example: A 2x2 matrix might look like:

[1 2]
[3 4]

Importance: Matrices are fundamental in data representation, transformations, and are the core structures in multiple data science algorithms.

Matrix-Vector Multiplication:



$$\begin{bmatrix} A_x & A_y & A_z \end{bmatrix}\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = A_xB_x + A_yB_y + A_zB_z = \vec{A}\cdot\vec{B}$$

Description: This operation involves taking a matrix and a vector, and producing another vector. Each entry of the resulting vector is computed as a dot product between the corresponding row of the matrix and the original vector.

Example:
Let's consider a 2x2 matrix A and a 2x1 vector v:

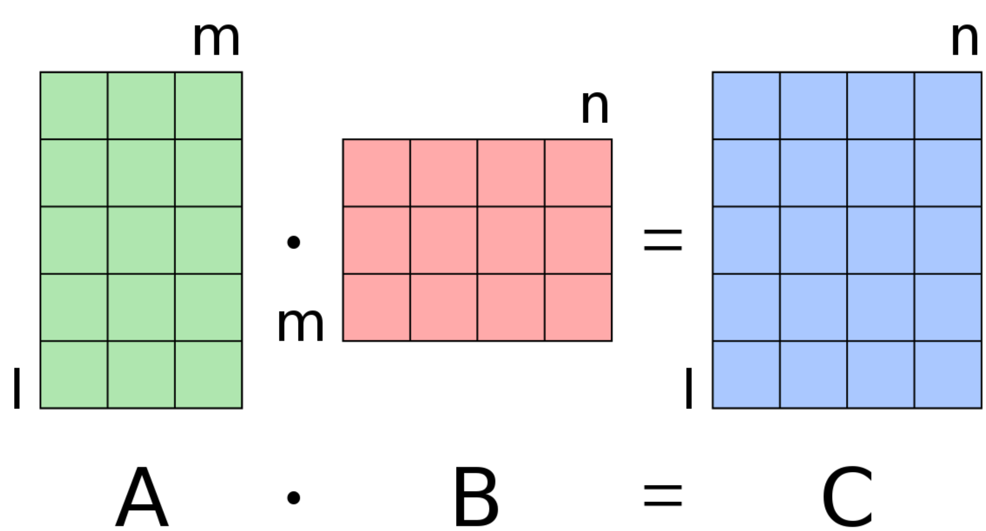
$$A = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} \qquad v = \begin{vmatrix} 1 \\ 2 \end{vmatrix}$$

Resulting vector Av will be:

$$\begin{vmatrix} 2*1 + 3*2 \\ 4*1 + 5*2 \end{vmatrix} = \begin{vmatrix} 8 \\ 14 \end{vmatrix}$$

Importance for Data Analysis: Matrix-vector multiplication is fundamental in many algorithms, especially in systems of linear equations. For example, in linear regression, it's used to compute predictions from coefficients and input features.

Matrix-Matrix Multiplication:



Description: When you multiply two matrices, the resulting matrix has entries determined by taking the dot product of the corresponding row of the first matrix with the corresponding column of the second matrix.

Example:

Consider two 2x2 matrices A and B:

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```
A = | 1  2 |      B = | 2  0 |
    | 0  1 |          | 1  3 |
```
The resulting matrix AB will be:

```
| 1*2 + 2*1  1*0 + 2*3 |   =   | 4  6 |
| 0*2 + 1*1  0*0 + 1*3 |       | 1  3 |
```

Importance for Data Analysis: Matrix-matrix multiplication is foundational in numerous areas of data science, especially in transformations, deep learning (neural networks), and anywhere where systems of linear transformations are utilized.

By understanding these linear algebra concepts, one can grasp the underlying mechanics of many data science algorithms and methods. These foundations enable precise data manipulation, transformation, and analysis.