RC 2: 26th May

Presented by: Minjie Mao

Email: jerry-mmj@sjtu.edu.cn

JOINT INSTITUTE
交大密西根学院

Table of Content

- Quiz
- Vector Analysis
- Two fundamental postulates of electrostatics in free space

Quiz

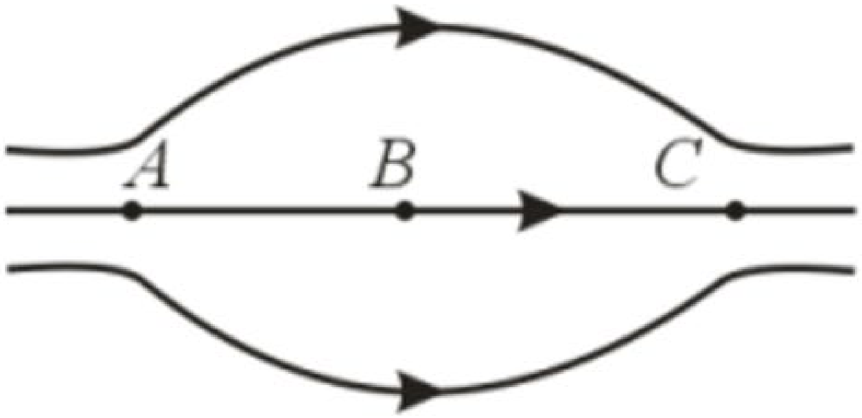
Four-point charges $-Q, -q, 2q$, and $2Q$ are placed at each corner of a square. The relation between Q and q for which the potential at the center of the square is zero is

a) $Q = -q$
b) $Q = -1/q$
c) $Q = q$
d) $Q = 1/q$

Answer a):

Quiz

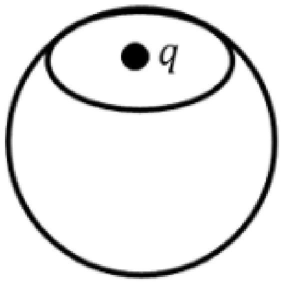
The figure below shows some of the electric field lines corresponding to an electric field. The figure suggests that,



- $E_A = E_C > E_B$
- $E_A > E_B > E_C$
- $E_A = E_C < E_B$
- $E_A = E_B = E_C$

Quiz

A point charge q is kept at the center of the circle formed by cutting a spherical shell by a plane as shown in the figure below. The electric flux ϕ linked with the remaining surface of the shell is then

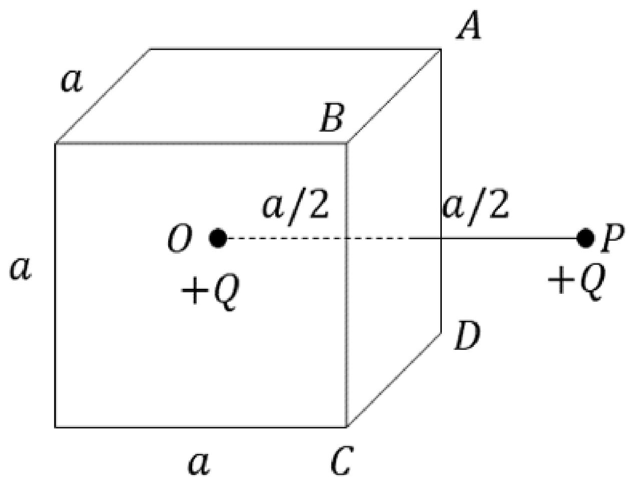


- ○ $\Phi = \frac{q}{\varepsilon_0}$
- ○ $\Phi = \frac{q}{2\varepsilon_0}$
- ○ $\Phi < \frac{q}{2\varepsilon_0}$
- ○ $\Phi > \frac{q}{2\varepsilon_0}$

Quiz

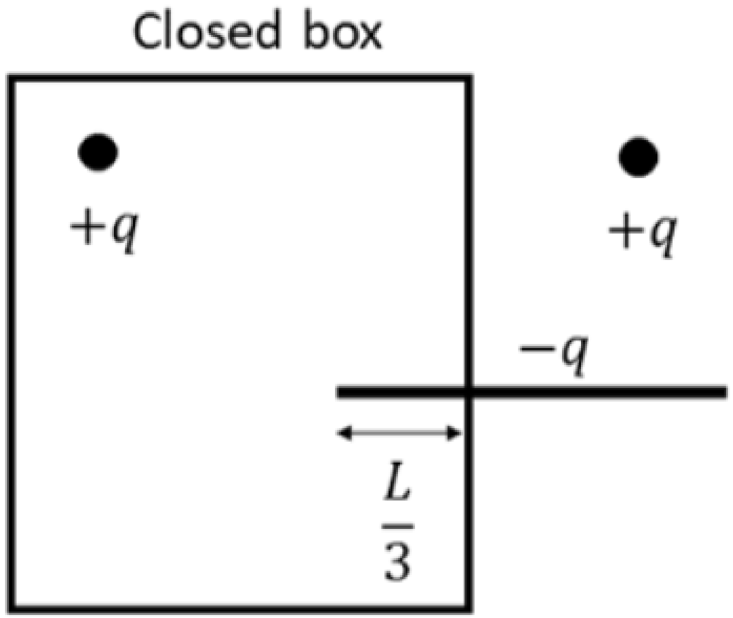
Consider a region in free space bounded by the surface of an imaginary cube having sides of length a as shown in the figure below. A charge +Q is placed at the center of the cube. P is a point outside the cube the line OP perpendicularly intersects the surface $ABCD$ at R and $OR = RP = a/2$. Charge $+Q$ is also placed at point P.

What is the flux through the five faces of the cube (taken together) other than $ABCD$?



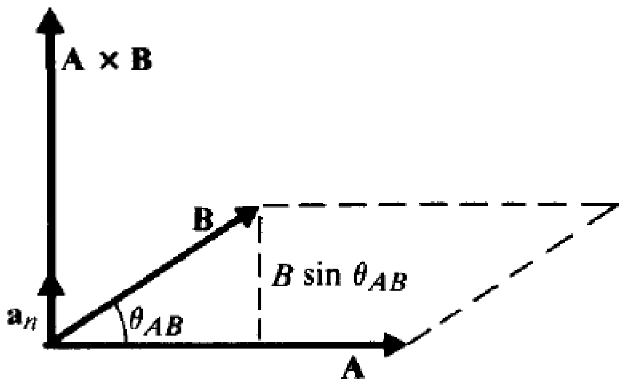
Quiz

The figure below shows two point charges $+q$ and $+q$ and a metal rod of length L carrying a uniform charge $-q$ with L/3 of its length inside the box. The electric flux through the box is

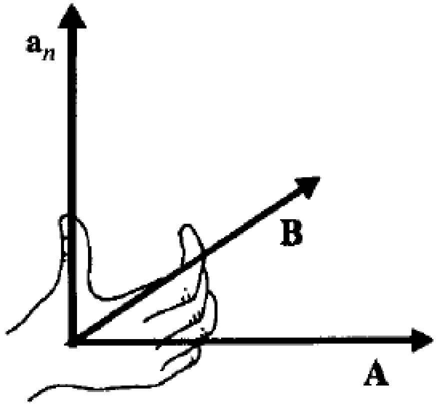


Vector Analysis

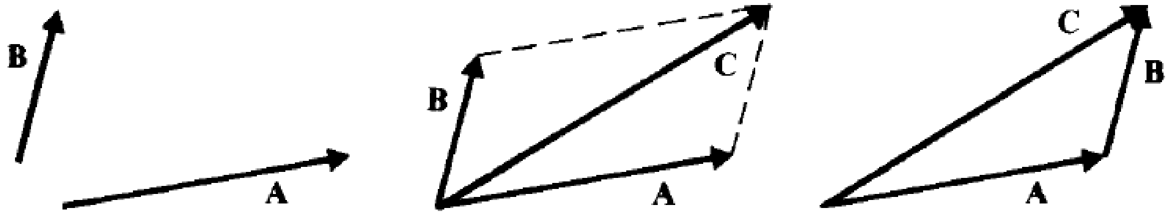
- $\vec{A} = \hat{a}_A A$
- Magnitude: $A = |\vec{A}|$
- Addition: $\vec{C} = \vec{A} + \vec{B}$
- Dot Product: $\vec{A} \cdot \vec{B} = AB \cos \theta_{AB}$
- Cross Product: $\vec{A} \times \vec{B} = \hat{a}_n |AB \sin \theta_{AB}|$



(a) $\mathbf{A} \times \mathbf{B} = \mathbf{a}_n |AB \sin \theta_{AB}|$.



(b) The right-hand rule.



(a) Two vectors, A and B. (b) Parallelogram rule. (c) Head-to-tail rule.

Cross Product (Vector Product)

$$\begin{aligned}\boldsymbol{a}\times\boldsymbol{b} &= (a_2b_3-a_3b_2)\boldsymbol{i}-(a_1b_3-a_3b_1)\boldsymbol{j}+(a_1b_2-a_2b_1)\boldsymbol{k}\\ &= \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.\end{aligned}$$

Quiz Choose the vector product of $\boldsymbol{a}=(1,2,3)$ and $\boldsymbol{b}=(3,-2,1)$.

(a) $8\boldsymbol{i}-8\boldsymbol{j}-8\boldsymbol{k}$, (b) $-4\boldsymbol{i}-10\boldsymbol{j}+4\boldsymbol{k}$,

(c) $8\boldsymbol{i}+8\boldsymbol{j}-8\boldsymbol{k}$, (d) $8\boldsymbol{i}-10\boldsymbol{j}-8\boldsymbol{k}$.

Vector Analysis

- Scalar triple product:

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\vec{B}\cdot\left(\vec{C}\times\vec{A}\right)=\vec{C}\cdot\left(\vec{A}\times\vec{B}\right)$$

- Vector triple product:

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{C}\cdot\vec{A}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)$$

Orthogonal Coordinate System

- Three surfaces are mutually perpendicular to one another

$$\mathbf{A} = \mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3},$$

Cartesian Coordinates

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$(u_1, u_2, u_3) = (x, y, z)$$

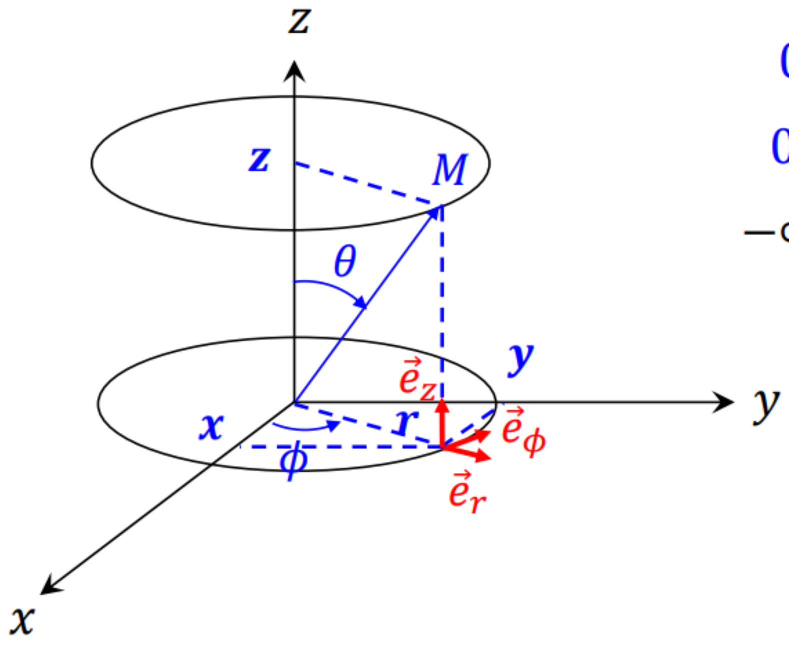
$$\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_\mathbf{y} A_y + \mathbf{a}_z A_z$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical Coordinates



$$(u_1, u_2, u_3) = (r, \phi, z)$$

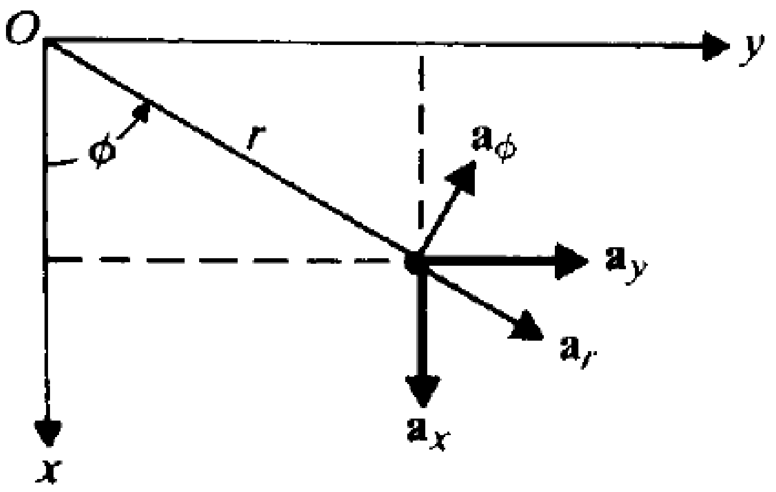
$$\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$$

$r = \sqrt{(x^2 + y^2)}$
$\varphi = \arctan(y / x)$
$z = z$

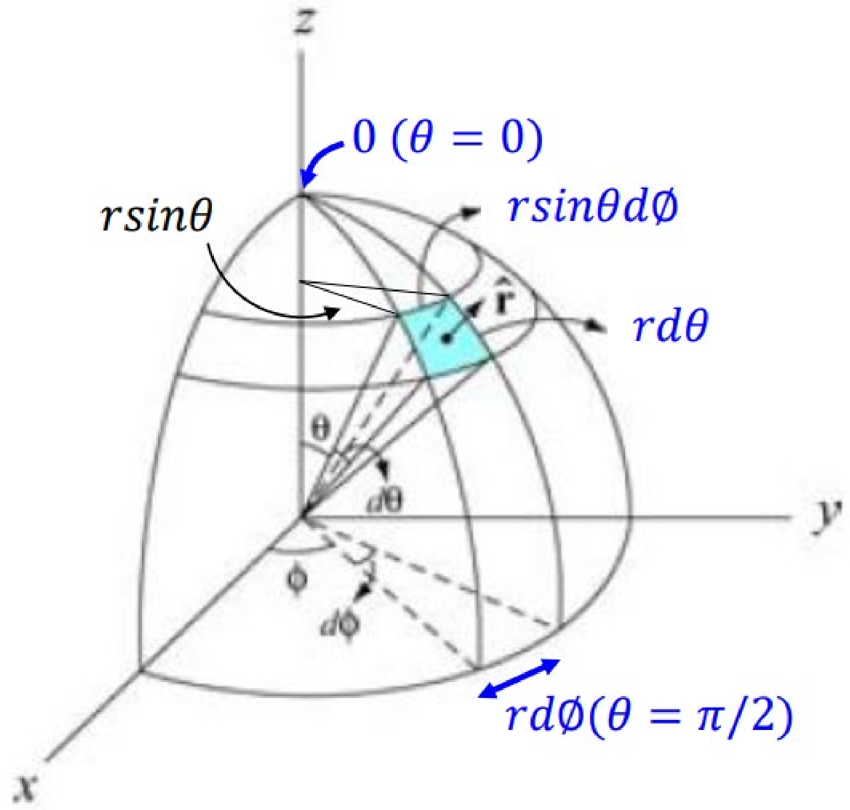
$x = r * \cos(\varphi)$
$y = r * \sin(\varphi)$
$z = z$

Cartesian and Cylindrical Coordinates

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$



Spherical Coordinates



$$(u_1, u_2, u_3) = (R, \theta, \phi)$$

$$\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi$$

$R = \sqrt{(x^2 + y^2 + z^2)}$ $\quad x = R * \sin(\theta) * \cos(\varphi)$

$\varphi = \arctan(y / x)$ $\quad y = R * \sin(\theta) * \sin(\varphi)$

$\theta = \arccos(z / R)$ $\quad z = R * \cos(\theta)$

Summary

- Matric Coefficients:

	h_1	h_2	h_3
Cartesian Coordinates	1	1	1
Cylindrical Coordinates	1	r	1
Spherical Coordinates	1	R	$R\sin\theta$

- Differential Elements:

$$d\vec{\ell} = \mathbf{a}_{u_1}\,(h_1 du_1) + \mathbf{a}_{u_2}\,(h_2 du_2) + \mathbf{a}_{u3}\,(h_3 du_3)$$

$$ds_1 = h_2 h_3 du_2 du_3$$

$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

Gradient of a Scalar Field

- Definition: The vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar.

$$\nabla V \triangleq \mathbf{a}_n \frac{dV}{dn}$$

$$\nabla V = \mathbf{a}_{u_1} \frac{\partial V}{h_1 \partial u_1} + \mathbf{a}_{u_2} \frac{\partial V}{h_2 \partial u_2} + \mathbf{a}_{u_3} \frac{\partial V}{h_3 \partial u_3}$$

Divergence of a Vector Field

- Definition: The divergence of a vector field A at a point is the net outward flux of A per unit volume as the volume about the point tends to zero

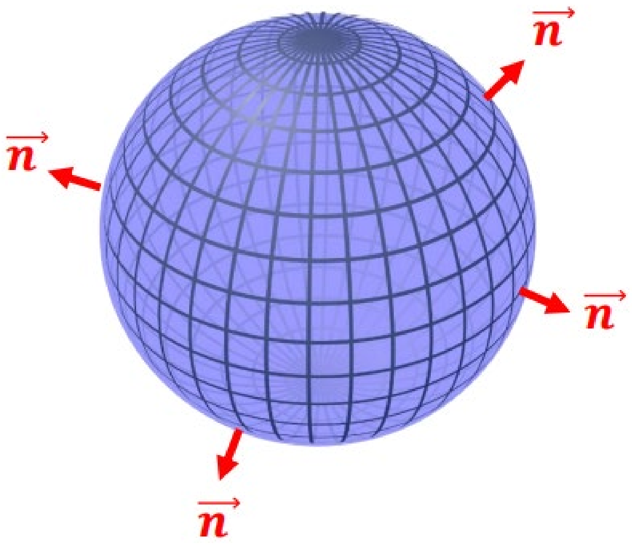
$$\operatorname{div}\mathbf{A} \triangleq \lim_{\Delta v \to 0} \frac{\oint_s \mathbf{A} \cdot d\mathbf{s}}{\Delta v}$$

$$\boldsymbol{\nabla} \cdot \mathbf{A} \equiv \operatorname{div}\mathbf{A}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

Divergence Theorem

- Interpretation: The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.



$$\int_V \nabla \cdot \mathbf{A} dv = \oint_S \mathbf{A} \cdot d\mathbf{s}$$

Curl of a Vector Field

- Definition:
- The curl of a vector field is a vector whose magnitude is the maximum net circulation of A per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum.

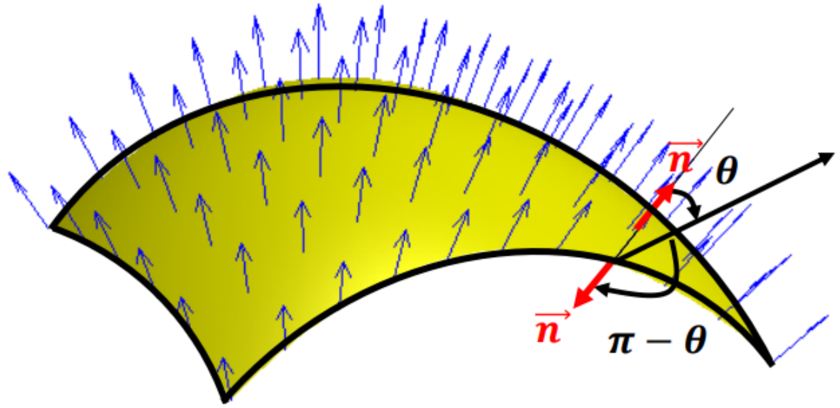
$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \mathbf{a}_{u_1} h_1 & \mathbf{a}_{u_2} h_2 & \mathbf{a}_{u_3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Exercise 1

The curl of $\boldsymbol{F}(x, y, z) = 3x^2\boldsymbol{i} + 2z\boldsymbol{j} - x\boldsymbol{k}$

Stoke's Theorem

- Interpretation: Surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.



$$\int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\ell$$

Exercise 2

For constants a, b, c, m, consider the vector field

$$\vec{F} = (ax + by + 5z)\vec{i} + (x + cz)\vec{j} + (3y + mx)\vec{k}.$$

(a) Suppose that the flux of $\vec{F}$ through any closed surface is 0. What does this tell you about the value of the constants a, b, c and m?

(b) Suppose instead that the line integral of $\vec{F}$ around any closed curve is 0. What does this tell you about the values of the constants a, b, c and m?

Exercise 3

Exercise 3. Let f be a scalar field and $\boldsymbol{F}(x, y, z)$ and $\boldsymbol{G}(x, y, z)$ be vector fields. What, if anything, is wrong with each of the following expressions

(a) $\boldsymbol{\nabla} f = x^3 - 4y$,

(b) $\boldsymbol{\nabla} \cdot \boldsymbol{F} = \boldsymbol{i} - x^2 y \boldsymbol{j} - z \boldsymbol{k}$,

(c) $\boldsymbol{\nabla} \times \boldsymbol{G} = \boldsymbol{\nabla} \cdot \boldsymbol{F}$.

Exercise 4

Find the flux of $\vec{F} = (x^2 + y^2)\vec{k}$ through the disk of radius 3 centred at the origin in the xy plane and oriented upward.

Exercise 5

Find the gradient of $V = e^{-(x^2+y^2)}$ in cylindrical (polar) coordinates.

Exercise 6

Given a vector function $\boldsymbol{F} = \boldsymbol{a}_x(x + c_1 z) + \boldsymbol{a}_y(c_2 x - 3z) + \boldsymbol{a}_z(x + c_3 y + c_4 z)$.

a) Determine the constants c_1, c_2, and c_3 if $\boldsymbol{F}$ is irrotational.

b) Determine the constant c_4 if $\boldsymbol{F}$ is also solenoidal.

c) Determine the scalar potential function V whose negative gradient equals $\boldsymbol{F}$.

Exercise 7

For vector function $\boldsymbol{A} = \boldsymbol{a}_r r^2 + \boldsymbol{a}_z 2z$, verify the divergence theorem for the circular cylindrical region enclosed by $r = 5$, $z = 0$ and $z = 4$.

Two fundamental postulates of electrostatics in free space

Postulates of Electrostatics in Free Space	
Differential Form	Integral Form
$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$
$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\ell = 0$

Thank You

Presented by Minjie