RC 6: 28th June

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Capacitor & Energy

The potential energy of a group of N discrete point charges at rest is

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \quad (J)$$

$$V_k = \frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1 \\ (j\neq k)}}^{N} \frac{Q_j}{R_{jk}}$$

W_e represents only the interaction energy (mutual energy) and does not include the work required to assemble the individual point charges themselves (self-energy).

Continuous Charge Distribution

$$W_e = \frac{1}{2}\int_{V'} \rho V\, dv \quad (\mathbf{J})$$

Capacitor & Energy

$$W_e = \frac{1}{2}\int_{V'} \mathbf{D}\cdot\mathbf{E}dv \quad (\mathbf{J})$$

$$W_e = \frac{1}{2}\int_{V'} \epsilon E^2 dv$$

$$W_e = \frac{1}{2}\int_{V'} \frac{D^2}{\epsilon} dv \quad (J)$$

Electrostatic Energy Density

$$w_e = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} \quad (\mathrm{J/m^3})$$

$$w_e = \frac{1}{2}\epsilon E^2 \quad (\mathrm{J/m^3})$$

$$w_e = \frac{D^2}{2\epsilon} \quad (\mathrm{J/m^3})$$

Capacitor & Energy

Series-connected Capacitors

$$\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$$

Parallel-connected Capacitors

$$C_{\mathrm{II}} = C_1 + C_2 + \cdots + C_n$$

Two-conductor Capacitor

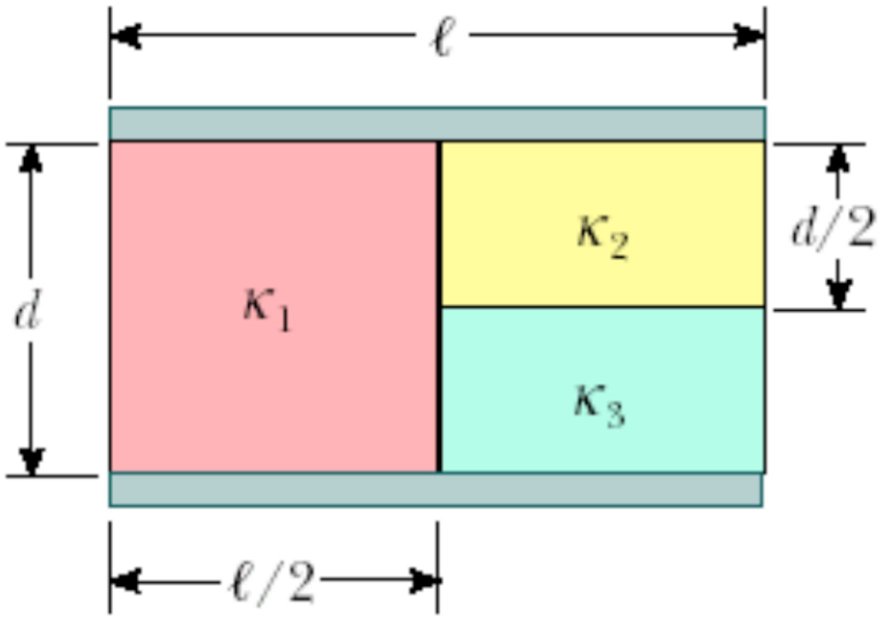
$$W_e = \frac{1}{2} C V^2 \quad (\mathrm{J})$$

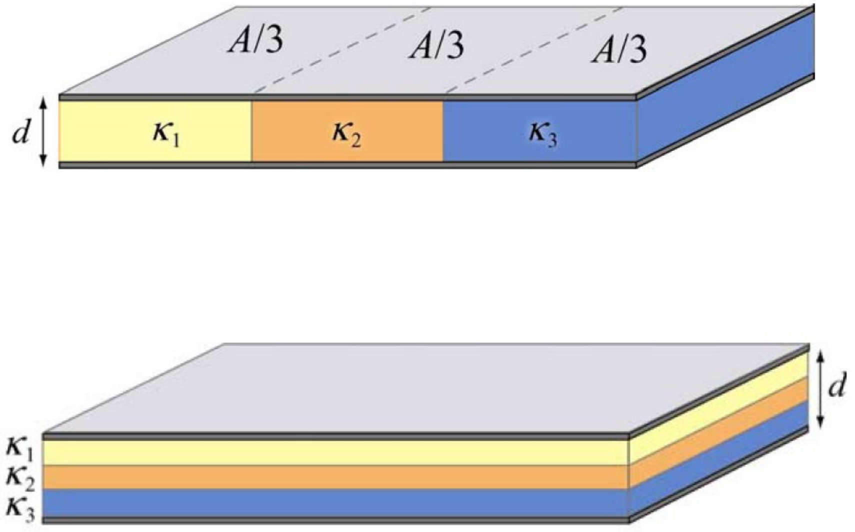
$$W_e = \frac{1}{2} Q V \quad (\mathrm{J})$$

$$W_e = \frac{Q^2}{2C} \quad (\mathrm{J})$$

Exercise 1

(b) A parallel-plate capacitor is constructed by filling the space between two square plates with blocks of three dielectric materials, as in the figure below. You may assume that $\ell >> d$. Find an expression for the capacitance of the device in terms of the plate area A and d, κ_1, κ_2, and κ_3.





Exercise 1

The capacitor can be regarded as being consisted of three capacitors, $C_1 = \frac{\kappa_1 \varepsilon_0 A/2}{d}$, $C_2 = \frac{\kappa_2 \varepsilon_0 A/2}{d/2}$ and $C_3 = \frac{\kappa_3 \varepsilon_0 A/2}{d/2}$, with C_2 and C_3 connected in series, and the combination connected in parallel with C_1. Thus, the equivalent capacitance is

$$C = C_1 + \left(\frac{1}{C_2} + \frac{1}{C_3}\right)^{-1} = C_1 + \frac{C_2 C_3}{C_2 + C_3} = \frac{\kappa_1 \varepsilon_0 A/2}{d} + \frac{\varepsilon_0 A}{d}\left(\frac{\kappa_2 \kappa_3}{\kappa_2 + \kappa_3}\right)$$

$$= \frac{\varepsilon_0 A}{d}\left(\frac{\kappa_1}{2} + \frac{\kappa_2 \kappa_3}{\kappa_2 + \kappa_3}\right)$$

Exercise 2

A parallel plate capacitor has capacitance C. It is connected to a battery of EMF ε until fully charged, and then disconnected. The plates are then pulled apart an extra distance d, during which the measured potential difference between them changed by a factor of 4. Below are a series of questions about how other quantities changed. Although they are related you do not need to rely on the answers to early questions in order to correctly answer the later ones.

a) Did the potential difference increase or decrease by a factor of 4?

INCREASE DECREASE

b) By what factor did the electric field change due to this increase in distance? Make sure that you indicate whether the field increased or decreased.

c) By what factor did the energy stored in the electric field change? Make sure that you indicate whether the energy increased or decreased.

d) A dielectric of dielectric constant κ is now inserted to completely fill the volume between the plates. Now by what factor does the energy stored in the electric field change? Does it increase or decrease?

e) What is the volume of the dielectric necessary to fill the region between the plates? (Make sure that you give your answer only in terms of variables defined in the statement of this problem, fundamental constants and numbers)

Exercise 2

(a)

$\boxed{\text{INCREASE}}$

(b)

Since the charge cannot change (the battery is disconnected) the electric field cannot change either. $\boxed{\text{No Change!}}$

(c)

The electric field is constant but the volume in which the field exists increased, so the energy must have increased. But by how much? The energy $U = \frac{1}{2}QV$. The charge doesn't change, the potential increased by a factor of 4, so the energy:

Increased by a factor of 4

(d)

Inserting a dielectric decreases the electric field by a factor of κ so it decreases the potential by a factor of κ as well. So now, by using the same energy formula $U = \frac{1}{2}QV$,

Energy decreases by a factor of κ

Exercise 2

(e)

How in the world do we know the volume? We must be able to figure out the cross-sectional area and the distance between the plates. The first relationship we have is from knowing the capacitance:

$$C = \frac{\varepsilon_0 A}{x}$$

where x is the original distance between the plates. Make sure you don't use the more typical variable d here because that is used for the distance the plates are pulled apart.

Next, the original voltage $V_0 = E\,x$, which increases by a factor of 4 when the plates are moved apart by a distance d, that is, $4\,V_0 = E\,(x{+}d)$. From these two equations we can solve for x:

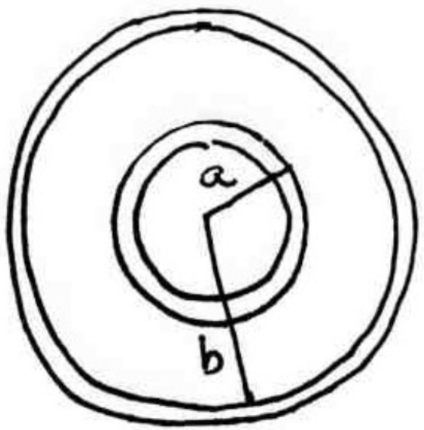
$$4V_0 = 4Ex = E(x+d) \quad \Rightarrow \quad x = d/3$$

Now, we can use the capacitance to get the area, and multiply that by the distance between the plates (now $x + d$) to get the volume:

$$\text{Volume} = A(x+d) = \frac{xC}{\varepsilon_0}(x+d) = \frac{dC}{3\varepsilon_0}\left(\frac{d}{3}+d\right) = \boxed{\frac{4d^2C}{9\varepsilon_0}}$$

Exercise 3

A capacitor consists of two concentric spherical shells. The outer radius of the inner shell is $a = 0.1\ \mathrm{m}$ and the inner radius of the outer shell is $b = 0.2\ \mathrm{m}$.



a) What is the capacitance C of this capacitor?

b) Suppose the maximum possible electric field at the outer surface of the inner shell before the air starts to ionize is $E_{\max}(a) = 3.0 \times 10^{6}\ \mathrm{V \cdot m^{-1}}$. What is the maximum possible charge on the inner capacitor?

c) What is the maximum amount of energy stored in this capacitor?

d) When $E(a) = 3.0 \times 10^{6}\ \mathrm{V \cdot m^{-1}}$ what is the potential difference between the shells?

Exercise 3

Solution:

The shells have spherical symmetry so we need to use spherical Gaussian surfaces. Space is divided into three regions (I) outside $r \geq b$, (II) in between $a < r < b$ and (III) inside $r \leq a$. In each region the electric field is purely radial (that is $\vec{\mathbf{E}} = E\hat{\mathbf{r}}$).

Region I: Outside $r \geq b$:
Region III: Inside $r \leq a$:

These Gaussian surfaces contain a total charge of 0, so the electric fields in these regions must be 0 as well.

Region II: In between $a < r < b$: Choose a Gaussian sphere of radius r. The electric flux on the surface is $\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E \cdot 4\pi r^2$

Exercise 3

The enclosed charge is $Q_{enc} = +Q$, and the electric field is everywhere perpendicular to the surface. Thus Gauss's Law becomes

$$E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

That is, the electric field is exactly the same as that for a point charge.
Summarizing:

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} & \text{for } a < r < b \\ 0 & \text{elsewhere} \end{cases}$$

We know the positively charged inner sheet is at a higher potential so we shall calculate

$$\Delta V = V(a) - V(b) = -\int_b^a \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\int_b^a \frac{Q}{4\pi\varepsilon_0 r^2} dr = \left.\frac{Q}{4\pi\varepsilon_0 r}\right|_b^a = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) > 0$$

which is positive as we expect.

Exercise 3

We can now calculate the capacitance using the definition

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{\dfrac{Q}{4\pi\varepsilon_0}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)} = \frac{4\pi\varepsilon_0}{\left(\dfrac{1}{a} - \dfrac{1}{b}\right)} = \frac{4\pi\varepsilon_0 ab}{b-a}$$

$$C = \frac{4\pi\varepsilon_0 ab}{b-a} = \frac{(0.1\ \mathrm{m})(0.2\ \mathrm{m})}{(9\times 10^9\ \mathrm{N\cdot m^2\cdot C^2})(0.1\ \mathrm{m})} = 2.2\times 10^{-11}\ \mathrm{F}\,.$$

Exercise 3

b) Suppose the maximum possible electric field at the outer surface of the inner shell before the air starts to ionize is $E(a) = 3.0\times10^{6}\ \mathrm{V\cdot m^{-1}}$. What is the maximum possible charge on the inner capacitor?

Solution:

The electric field $E(a) = \dfrac{Q}{4\pi\varepsilon_0 a^2}$. Therefore the maximum charge is

$$Q_{\max} = 4\pi\varepsilon_0 E_{\max}(a)a^2 = \frac{(3.0\times10^{6}\ \mathrm{V\cdot m^{-1}})(0.1\ \mathrm{m})^2}{(9\times10^{9}\ \mathrm{N\cdot m^2\cdot C^2})} = 3.3\times10^{-6}\ \mathrm{C}$$

Exercise 3

c) What is the maximum amount of energy stored in this capacitor?

Solution:

The energy stored is

$$U_{\max} = \frac{Q_{\max}^2}{2C} = \frac{(3.3\times 10^{-6}\ \mathrm{C})^2}{(2)(2.2\times 10^{-11}\ \mathrm{F})} = 2.5\times 10^{-1}\ \mathrm{J}$$

d) When $E(a) = 3.0\times 10^{6}\ \mathrm{V\cdot m^{-1}}$ what is the potential difference between the shells?

Solution:

We can find the potential difference two different ways.

Using the definition of capacitance we have that

Exercise 3

$$\Delta V = \frac{Q}{C} = \frac{4\pi\varepsilon_0 E(a)a^2(b-a)}{4\pi\varepsilon_0 ab} = \frac{E(a)a(b-a)}{b}$$

$$\Delta V = \frac{(3.0\times10^6\ \mathrm{V\cdot m^{-1}})(0.1\ \mathrm{m})(0.1\ \mathrm{m})}{(0.2\ \mathrm{m})} = 1.5\times10^5\ \mathrm{V}$$

We already calculated the potential difference in part a):

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right).$$

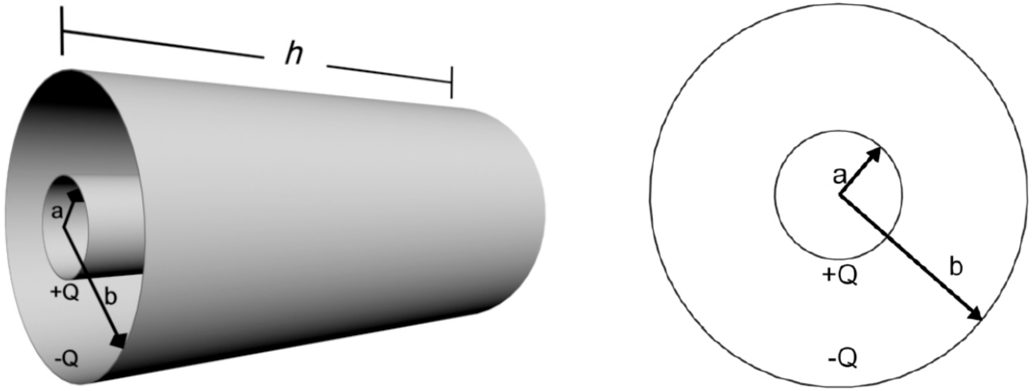
Recall that $E(a) = \dfrac{Q}{4\pi\varepsilon_0 a^2}$ or $\dfrac{Q}{4\pi\varepsilon_0^2} = E(a)a^2$. Substitute this into our expression for potential difference yielding

$$\Delta V = E(a)a^2\left(\frac{1}{a}-\frac{1}{b}\right) = E(a)a^2\frac{(b-a)}{ab} = E(a)a\frac{(b-a)}{b}$$

in agreement with our result above.

Exercise 4

Consider two nested cylindrical conductors of height h and radii a & b respectively. A charge $+Q$ is evenly distributed on the outer surface of the pail (the inner cylinder), $-Q$ on the inner surface of the shield (the outer cylinder). You may ignore edge effects.



a) Calculate the electric field between the two cylinders ($a < r < b$).

b) Calculate the potential difference between the two cylinders:

c) Calculate the capacitance of this system, $C = Q/DV$

d) Numerically evaluate the capacitance, given: $h \cong 15$ cm, $a \cong 4.75$ cm and $b \cong 7.25$ cm.

e) Find the electric field energy density at any point between the conducting cylinders. How much energy resides in a cylindrical shell between the conductors of radius r (with $a < r < b$), height h, thickness dr, and volume $2\pi rh\, dr$? Integrate your expression to find the total energy stored in the capacitor and compare your result with that obtained using $U_E = (1/2)C(\Delta V)^2$.

Exercise 4

(a) Calculate the electric field between the two cylinders ($a < r < b$).

For this we use Gauss's Law, with a Gaussian cylinder of radius r, height l

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 2\pi r l E = \frac{Q_{inside}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{Q}{h}l \quad \Rightarrow \quad E(r)_{a<r<b} = \frac{Q}{2\pi r \varepsilon_0 h}$$

(b) Calculate the potential difference between the two cylinders:

The potential difference between the outer shell and the inner cylinder is

$$\Delta V = V(a) - V(b) = -\int_b^a \frac{Q}{2\pi r' \varepsilon_0 h} dr' = -\frac{Q}{2\pi \varepsilon_0 h} \ln r' \Big|_b^a = \frac{Q}{2\pi \varepsilon_0 h} \ln\left(\frac{b}{a}\right)$$

(c) Calculate the capacitance of this system, $C = Q/DV$

$$C = \frac{|Q|}{|\Delta V|} = \frac{|Q|}{\dfrac{|Q|}{2\pi\varepsilon_0 h}\ln\left(\dfrac{b}{a}\right)} = \frac{2\pi\varepsilon_o h}{\ln\left(\dfrac{b}{a}\right)}$$

(d) Numerically evaluate the capacitance given:
$h \cong 15$ cm, $a \cong 4.75$ cm and $b \cong 7.25$ cm

$$C = \frac{2\pi\varepsilon_o h}{\ln\left(\dfrac{b}{a}\right)} = \frac{1}{2 \cdot 9 \times 10^9 \text{ m F}^{-1}} \frac{15 \text{ cm}}{\ln\left(\dfrac{7.25 \text{ cm}}{4.75 \text{ cm}}\right)} \cong 20 \text{ pF}$$

Exercise 4

The total energy stored in the capacitor is

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\left(\frac{Q}{2\pi r\varepsilon_0 h}\right)^2$$

Then

$$dU = u_E dV = \frac{1}{2}\varepsilon_0\left(\frac{Q}{2\pi r\varepsilon_0 h}\right)^2 2\pi rh\,dr = \frac{Q^2}{4\pi\varepsilon_0 h}\frac{dr}{r}$$

Integrating we find that

$$U = \int_a^b dU = \int_a^b \frac{Q^2}{4\pi\varepsilon_0 h}\frac{dr}{r} = \frac{Q^2}{4\pi\varepsilon_0 h}\ln(b/a)\,.$$

From part d) $C = 2\pi\varepsilon_o h/\ln(b/a)$, therefore

$$U = \int_a^b dU = \int_a^b \frac{Q^2}{4\pi\varepsilon_0 h}\frac{dr}{r} = \frac{Q^2}{4\pi\varepsilon_0 h}\ln(b/a) = \frac{Q^2}{2C} = \frac{1}{2}C\Delta V^2$$

which agrees with that obtained above.

Exercise 5

1. The polarization in a dielectric cube of side L centered at the origin is given by $\mathbf{P} = P_0(\mathbf{a_x}x + \mathbf{a_y}y + \mathbf{a_z}z)$.

   a) Determine the surface and volume bound-charge densities.

   b) Show that the total bound charge is zero.

Exercise 5

1. The polarization in a dielectric cube of side L centered at the origin is given by $\mathbf{P} = P_0(\mathbf{a_x}x + \mathbf{a_y}y + \mathbf{a_z}z)$.

   a) Determine the surface and volume bound-charge densities.

   b) Show that the total bound charge is zero.

a) $\rho_{ps} = \bar{P}\cdot\bar{a}_n = P_0\frac{L}{2}$ on all six faces of the cube.

$\rho_p = -\bar{\nabla}\cdot\bar{P} = -3P_0$.

b) $Q_s = 6L^2\rho_{ps} = 3P_0L^3$, $Q_v = L^3\rho_p = -3P_0L^3$.

Total bound charge $= Q_s + Q_v = 0$.

Exercise 6

Assume that the $z = 0$ plane separates two lossless dielectric regions with $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 3$. If we know that $\mathbf{E_1}$ in region 1 is $\mathbf{a_x}2y - \mathbf{a_y}3x + \mathbf{a_z}(5 + z)$, what do we also know about $\mathbf{E_2}$ and $\mathbf{D_2}$ in region 2? Can we determine $\mathbf{E_2}$ and $\mathbf{D_2}$ at any point in region 2? Explain.

Exercise 6

Assume that the $z = 0$ plane separates two lossless dielectric regions with $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 3$. If we know that $\mathbf{E_1}$ in region 1 is $\mathbf{a_x}2y - \mathbf{a_y}3x + \mathbf{a_z}(5 + z)$, what do we also know about $\mathbf{E_2}$ and $\mathbf{D_2}$ in region 2? Can we determine $\mathbf{E_2}$ and $\mathbf{D_2}$ at any point in region 2? Explain.

21 At the z=0 plane : $E_1 = \bar{a}_x 2y - \bar{a}_y 3x + \bar{a}_z 5.$

$$\bar{E}_{1t}(z=0) = \bar{E}_{2t}(z=0) = \bar{a}_x 2y - \bar{a}_y 3x,$$

$$\bar{D}_{1n}(z=0) = \bar{D}_{2n}(z=0) \longrightarrow 2\bar{E}_{1n}(z=0) = 3\bar{E}_{2n}(z=0)$$

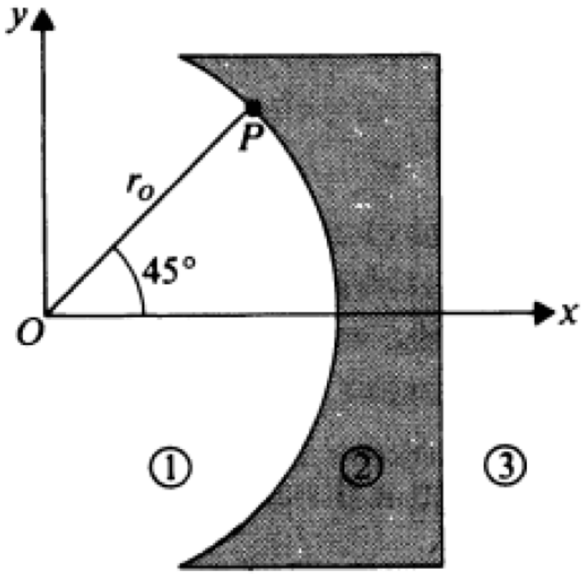
$$\longrightarrow \bar{E}_{2n}(z=0) = \frac{2}{3}(\bar{a}_z 5) = \bar{a}_z \frac{10}{3}.$$

$$\therefore \bar{E}_2(z=0) = \bar{a}_x 2y - \bar{a}_y 3x + \bar{a}_z \frac{10}{3},$$

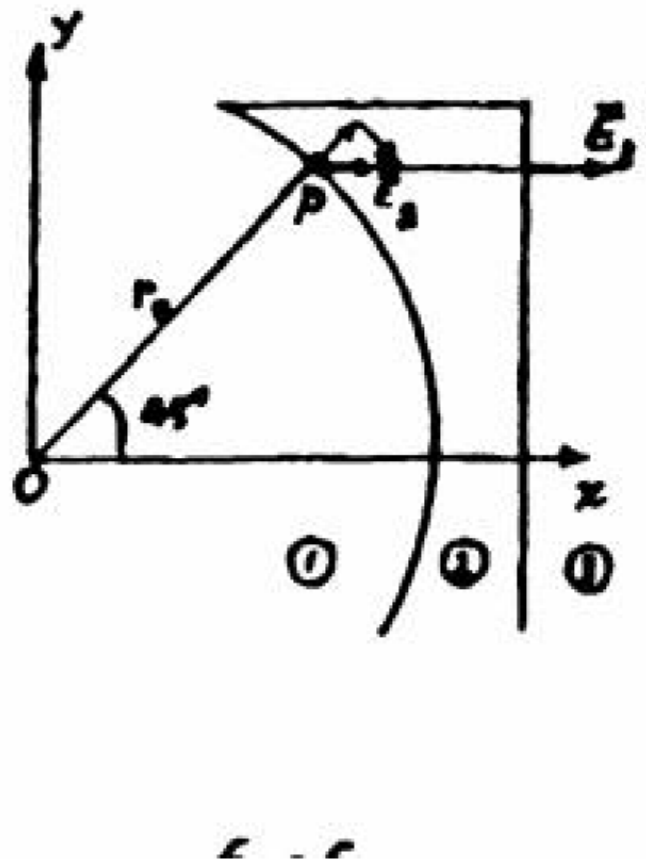
$$\bar{D}_2(z=0) = (\bar{a}_x 6y - \bar{a}_y 9x + \bar{a}_z 10)\epsilon_0.$$

Exercise 7

Dielectric lenses can be used to collimate electromagnetic fields. As shown in the figure below, the left surface of the lens is that of a circular cylinder, and the right surface is a plane. If $\mathbf{E_1}$ at point $P(r_0, 45°, z)$ in region 1 is $\mathbf{a_r}5 - \mathbf{a}_\phi 3$, what must be the dielectric constant of the lens in order that $\mathbf{E_3}$ in region 3 is parallel to the x-axis?



Exercise 7



Assume $\bar{E}_2 = \bar{a}_r E_{2r} + \bar{a}_\phi E_{2\phi}$

B.C.: $\bar{a}_n \times \bar{E}_1 = \bar{a}_n \times \bar{E}_2 \longrightarrow E_{2\phi} = -$

For $\bar{E}_3$, and hence $\bar{E}_2$, to be parallel to the x-axis,

$$E_{2\phi} = -E_{2r} \longrightarrow E_{2r} = 3.$$

B.C.: $\bar{a}_n \cdot \bar{D}_1 = \bar{a}_n \cdot \bar{D}_2 \longrightarrow 5 = 3$

$\therefore \; \epsilon_{r2} = 5/3.$

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Thank You

Presented by Minjie