EXERCISE 2 - GLOBAL OPTIMIZATION

A. Linear Programming Reformulation

Derivation of the problem formulation in the canonical form of Linear Programming

- Let the set S of the input data be partitioned into an inlier-set $S_I \subseteq S$ and an outlier-set $S_O = S \backslash SI$. The model $\Theta = T = (T_x, T_y) \in \mathbb{R}^2$ where T_x and T_y represent the translation along the x and y axis. The i-th input correspondence (p_i, p'_i) where p_i and p'_i represent the points in the left and right images. Their x and y coordinates are written $p_i = (x_i, y_i)$ and $p'_i = (x'_i, y'_i)$. We have n input correspondences, i.e. $i = 1, \ldots, n$. The maximization problem is define as:

$$\begin{aligned} \max_{\Theta, S_I} \quad & card(S_I) \\ s.t. \, & |x_i + T_x - x'_i| \leq \delta, \forall i \in S_I \subseteq S \\ & |y_i + T_y - y'_i| \leq \delta, \forall i \in S_I \subseteq S \end{aligned} \tag{1}$$

- By introducing binary variable $z_i \in \{0, 1\}, i = 1, \ldots, n$, we let $z_i = 1$ represents the i-th correspondence is an inlier, and introducing the upperbound and lowerbound for translation, we have the following reformulation:

$$\begin{aligned} \max_{\Theta, \mathbf{z}} \quad & \sum_{i=1}^{N} z_i \\ s.t. \; & z_i |x_i + T_x - x'_i| \leq z_i \delta, \; \forall i \in S_I \subseteq S \\ & z_i |y_i + T_y - y'_i| \leq z_i \delta, \; \forall i \in S_I \subseteq S \\ & z_i \in [0, 1], \; \forall i \in S_I \subseteq S \\ & \underline{T_x} \leq T_x \leq \bar{T}_x, \; \underline{T_y} \leq T_y \leq \bar{T}_y \end{aligned} \tag{2}$$

  - Note that we relax the binary bariables z_i to a continuous interval. Because if z_i is fractional, maximizing the objective would force $z_i = 1$. Since the binary variables appear on both sides of the inequalities the constraints become active as soon as $z_i > 0$.
- By removing the absolute sign, we have:

$$\begin{aligned} z_i(x_i + T_x - x'_i) \geq -z_i \delta, \\ z_i(x_i + T_x - x'_i) \leq z_i \delta, \\ z_i(y_i + T_y - y'_i) \geq -z_i \delta, \\ z_i(y_i + T_y - y'_i) \leq z_i \delta, \end{aligned} \tag{3}$$

- By converting the bilinear term to the concave and convex term, and introducing auxiliary variables $w_{ix} = z_i T_x, w_{iy} = z_i T_y$, we have (only x is shown):

$$\begin{aligned} w_{ix} = z_i T_x \geq max(\underline{z_i} T_x + \underline{T_x} z_i - \underline{z_i} \underline{T_x}, \bar{z}_i T_x + \bar{T}_x z_i - \bar{z}_i \bar{T}_x) \\ w_{ix} = z_i T_x \leq min(\bar{z}_i T_x + \underline{T_x} z_i - \bar{z}_i \underline{T_x}, \underline{z_i} T_x + \bar{T}_x z_i - \underline{z_i} \bar{T}_x) \end{aligned} \tag{4}$$

- By removing the max and min, and replacing the parameters in (3), we have the final result:

$$
\begin{aligned}
&\max_{\Theta,\mathbf{z}} \sum_{i=1}^{N} z_i \\
&s.t. \forall i \in S_I \subseteq S \\
&z_i \in [0,1] \\
&\underline{z_i} T_x + \underline{T_x} z_i - w_{ix} \leq \underline{z_i T_x} \\
&\bar{z}_i T_x + \bar{T}_x z_i - w_{ix} \leq \bar{z}_i \bar{T}_x \\
&-\bar{z}_i T_x - \underline{T_x} z_i + w_{ix} \leq -\bar{z}_i \underline{T_x} \\
&-\underline{z_i} T_x - \bar{T}_x z_i + w_{ix} \leq -\underline{z_i} \bar{T}_x \\
&\underline{z_i} T_y + \underline{T_y} z_i - w_{iy} \leq \underline{z_i T_y} \\
&\bar{z}_i T_y + \bar{T}_y z_i - w_{iy} \leq \bar{z}_i \bar{T}_y \\
&-\bar{z}_i T_y - \underline{T_y} z_i + w_{iy} \leq -\bar{z}_i \underline{T_y} \\
&-\underline{z_i} T_y - \bar{T}_y z_i + w_{iy} \leq -\underline{z_i} \bar{T}_y \\
&z_i(x_i - x_i' - \delta) + w_{ix} \leq 0, \\
&z_i(-x_i + x_i' - \delta) - w_{ix} \leq 0, \\
&z_i(y_i - y_i' - \delta) + w_{iy} \leq 0, \\
&z_i(-y_i + y_i' - \delta) - w_{iy} \leq 0, \\
&\underline{T_x} \leq T_x \leq \bar{T}_x,\ \underline{T_y} \leq T_y \leq \bar{T}_y
\end{aligned} \tag{5}
$$

- To build the canonical form, we have:

$$
\begin{aligned}
&\min_{x} c^T \mathbf{x} \\
&s.t. \mathbf{A}\mathbf{x} \leq b \\
&and\ l_b \leq \mathbf{x} \leq u_b
\end{aligned} \tag{6}
$$

  - $\mathbf{x} = [z_1, \ldots, z_n, w_{1x}, \ldots, w_{nx}, w_{1y}, \ldots, w_{ny}, T_x, T_y]^T$
  - $l_b = [\underline{z_1}, \ldots, \underline{z_n}, -\infty, \ldots, -\infty, \underline{T_x}, \underline{T_y}]$ ($\underline{z_i} = 0$)
  - $u_b = [\bar{z_1}, \ldots, \bar{z_n}, \infty, \ldots, \infty, \bar{T}_x, \bar{T}_y]$ ($\bar{z}_i = 1$)
  - $c = [-1, \ldots, -1, 0, \ldots, 0, 0, 0]$

$$
A_i = \begin{bmatrix}
0 & \dots & \underline{T_x} & \dots & 0, & 0 & \dots & -1 & \dots & 0, & 0 & \dots & 0 & \dots & 0, & \underline{z_i} & 0 \\
0 & \dots & \bar{T}_x & \dots & 0, & 0 & \dots & -1 & \dots & 0, & 0 & \dots & 0 & \dots & 0, & \bar{z}_i & 0 \\
0 & \dots & -\underline{T_x} & \dots & 0, & 0 & \dots & 1 & \dots & 0, & 0 & \dots & 0 & \dots & 0, & -\bar{z}_i & 0 \\
0 & \dots & -\bar{T}_x & \dots & 0, & 0 & \dots & 1 & \dots & 0, & 0 & \dots & 0 & \dots & 0, & -\underline{z_i} & 0 \\
0 & \dots & \underline{T_y} & \dots & 0, & 0 & \dots & 0 & \dots & 0, & 0 & \dots & -1 & \dots & 0, & 0 & \underline{z_i} \\
0 & \dots & \bar{T}_y & \dots & 0, & 0 & \dots & 0 & \dots & 0, & 0 & \dots & -1 & \dots & 0, & 0 & \bar{z}_i \\
0 & \dots & -\underline{T_y} & \dots & 0, & 0 & \dots & 0 & \dots & 0, & 0 & \dots & 1 & \dots & 0, & 0 & -\bar{z}_i \\
0 & \dots & -\bar{T}_y & \dots & 0, & 0 & \dots & 0 & \dots & 0, & 0 & \dots & 1 & \dots & 0, & 0 & -\underline{z_i} \\
0 & \dots & (x_i - x_i' - \delta) & \dots & 0, & 0 & \dots & 1 & \dots & 0, & 0 & \dots & 0 & \dots & 0, & 0 & 0 \\
0 & \dots & (-x_i + x_i' - \delta) & \dots & 0, & 0 & \dots & -1 & \dots & 0, & 0 & \dots & 0 & \dots & 0, & 0 & 0 \\
0 & \dots & (y_i - y_i' - \delta) & \dots & 0, & 0 & \dots & 0 & \dots & 0, & 0 & \dots & 1 & \dots & 0, & 0 & 0 \\
0 & \dots & (-y_i + y_i' - \delta) & \dots & 0, & 0 & \dots & 0 & \dots & 0, & 0 & \dots & -1 & \dots & 0, & 0 & 0
\end{bmatrix} \tag{7}
$$

$$
b_i = [\underline{z_i T_x}, \bar{z}_i \bar{T}_x, -\bar{z}_i \underline{T_x}, -\underline{z_i} \bar{T}_x, \underline{z_i T_y}, \bar{z}_i \bar{T}_y, -\bar{z}_i \underline{T_y}, -\underline{z_i} \bar{T}_y, 0, 0, 0, 0]^T
$$

$$
A = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}
$$

B. Code for Branch and Bound

The code implementing branch and bound for consensus set maximization with a 2D translation model: See the jupyter notebook file in the code folder

C. Results

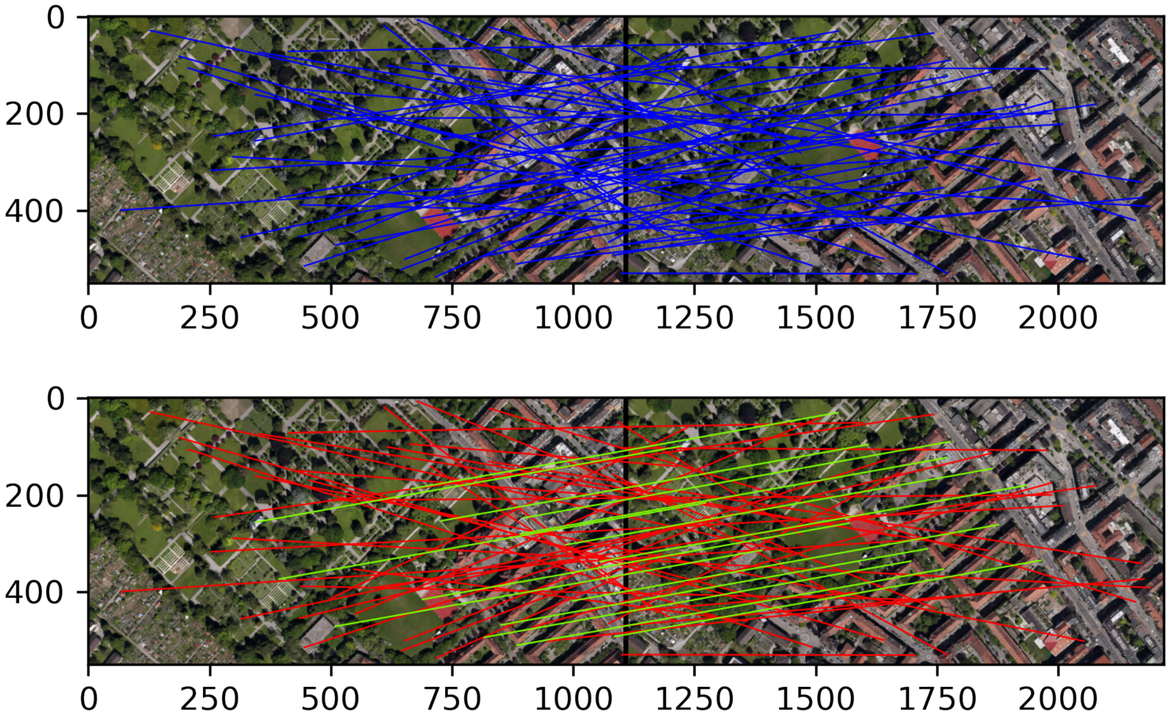
The results of the translation model, and the indices of the inliers and outliers obtained by branch and bound

```
ThetaLowerBound: (-274, -276)
ThetaUpperBound: (0, 0)
ObjLowerBound: 15
ObjUpperBound: 15.866963107682938
ThetaOptimizer: (-232.0, -154.0)

Inlier indices:
[2, 7, 8, 14, 15, 19, 25, 30, 31, 33, 34, 39, 41, 44, 50]
Outlier indices:
[0, 1, 3, 4, 5, 6, 9, 10, 11, 12, 13, 16, 17, 18, 20, 21, 22, 23, 24, 26, 27, 28, 29, 32, 35, 36, 37, 38, 40, 42, 43, 45, 46, 47, 48, 49, 51, 52, 53, 54]
```

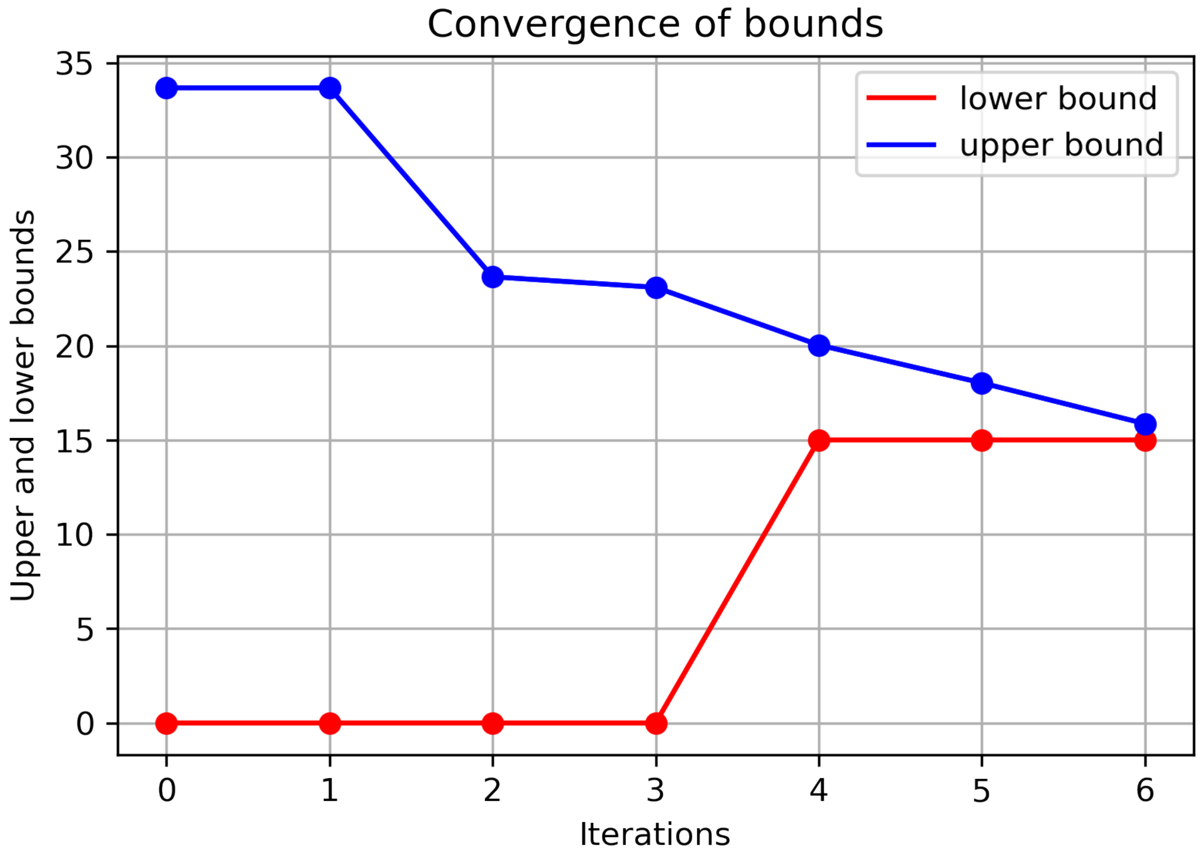
D. Correspondences Figure

A figure showing the identified inlier and outlier correspondences



E. Cardinality Bounds Figure

A figure showing the convergence of the cardinality bounds i.e. the highest lower bound obtained so far, and the highest upper bound still in the list.


Convergence of bounds
lower bound
upper bound
Upper and lower bounds
Iterations
0
5
10
15
20
25
30
35
0
1
2
3
4
5
6