AQM _ PS1

3.13)

a) problem 1.11 : $|\alpha\rangle = \cos(\beta/2)|+\rangle + e^{i\alpha}\sin(\beta/2)|-\rangle$

$$\longrightarrow \langle S_x\rangle = \frac{\hbar}{2}\langle\alpha|\big(|+\rangle\langle -| + |-\rangle\langle +|\big)|\alpha\rangle = \frac{\hbar}{2}\left(\cos\frac{\beta}{2}\sin\frac{\beta}{2}e^{-i\alpha} + \sin\frac{\beta}{2}\cos\frac{\beta}{2}e^{i\alpha}\right)$$

$$= \frac{\hbar}{2}\sin\beta\cos\alpha$$

$$\langle S_y\rangle = \frac{i\hbar}{2}\langle\alpha|\big(|-\rangle\langle +| - |+\rangle\langle -|\big)|\alpha\rangle$$

$$= -\frac{\hbar}{2}\left(\frac{e^{+i\alpha} - e^{-i\alpha}}{2}\right)\cos\beta\sin\frac{\beta}{2} = -\frac{\hbar}{2}\sin\alpha\sin\beta$$

$$\langle S_z\rangle = \frac{\hbar}{2}\langle\alpha|\big(|+\rangle\langle +| - |-\rangle\langle -|\big)|\alpha\rangle = \frac{\hbar}{2}\left[\cos^2\left(\frac{\beta}{2}\right) - \sin^2\left(\frac{\beta}{2}\right)\right] = \frac{\hbar}{2}\cos\beta$$

می‌دانیم $0 \leqslant \beta \leqslant \pi$ و $0 \leqslant \alpha \leqslant 2\pi$. با مشخص بودن $\langle S_z\rangle$، $\cos\beta$ مشخص می‌شود و با توجه به بازه‌ی β می‌توان β را به طور دقیق مشخص کرد. از $\langle S_x\rangle$ اندازه $\cos\alpha$ نیز مشخص می‌شود اما علامت α را نمی‌توان با آن مشخص کرد که در این جا علامت $\langle S_y\rangle$ راهگشا خواهد بود.

b)

اگر $\rho = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ماتریس چگالی در پایه‌ی S_z باشد خواهیم داشت:

$$[S_x] = \frac{\hbar}{2}\mathrm{Tr}\left[\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right] = \frac{\hbar}{2}(b+c) \quad (1)$$

$$[S_y] = \frac{\hbar}{2}\mathrm{Tr}\left[\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\right] = \frac{\hbar}{2}i(b-c) \quad (2)$$

$$[S_z] = \frac{\hbar}{2}\mathrm{Tr}\left[\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right] = \frac{\hbar}{2}(a-d) \quad (3)$$

هم‌چنین از آن جا که $Tr\rho = 1$، داریم $a + d = 1$ (4). بنابراین:

$$(3),(4) \longrightarrow 2a = 1 + \frac{2}{\hbar}[S_z] \quad (5)$$

$$(1),(2) \longrightarrow 2b = \frac{2}{\hbar}\big([S_x] - i[S_y]\big) \quad (6)$$

$$(4),(5) \longrightarrow d = \frac{1}{2} - \frac{[S_z]}{\hbar}$$

$$(6),(1) \longrightarrow c = \frac{1}{\hbar}\big([S_x] + i[S_y]\big)$$

$$\longrightarrow \rho = \begin{pmatrix} \frac{1}{2} + \frac{[S_z]}{\hbar} & \frac{[S_x] - i[S_y]}{\hbar} \\ \frac{[S_x] + i[S_y]}{\hbar} & \frac{1}{2} - \frac{[S_z]}{\hbar} \end{pmatrix}$$

3.14)

$\rho = \sum w_i |\alpha_i\rangle\langle\alpha_i| \; ; \; \sum w_i = 1$

We have 3 states with same w_i : $\rho = \frac{1}{3}\left[|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| + |2\rangle\langle 2| \right]$

$|\alpha\rangle\langle\alpha| = \frac{1}{2}\left(|0\rangle + |1\rangle\right)\left(\langle 0| + \langle 1|\right) = \frac{1}{2}\left[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1| \right]$

$|\beta\rangle\langle\beta| = \frac{1}{2}\left(|1\rangle + |2\rangle\right)\left(\langle 1| + \langle 2|\right) = \frac{1}{2}\left[|1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 2| \right]$

$$\rightarrow \rho = \frac{1}{6}|0\rangle\langle 0| + \frac{1}{6}|0\rangle\langle 1| + 0$$
$$+ \frac{1}{6}|1\rangle\langle 0| + \frac{2}{6}|1\rangle\langle 1| + \frac{1}{6}|1\rangle\langle 2|$$
$$+ 0 + \frac{1}{6}|2\rangle\langle 1| + \left(\frac{1}{6} + \frac{1}{3}\right)|2\rangle\langle 2|$$

$$\rightsquigarrow \rho = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{1}{2} \end{pmatrix}$$

which is normal.

$H = \hbar\omega\left(n + \frac{1}{2}\right)$

$$[H] = \mathrm{Tr}(\rho H) = \mathrm{Tr}\left[\begin{pmatrix} \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hbar\omega/2 & & 0 \\ & \frac{3}{2}\hbar\omega & \\ 0 & & \frac{5}{2}\hbar\omega \end{pmatrix} \right]$$

$$= \hbar\omega\left(\frac{1}{12} + \frac{1}{2} + \frac{5}{4}\right) = \hbar\omega\left(\frac{1 + 6 + 15}{12}\right) = \hbar\omega\,\frac{22}{12} = \boxed{\hbar\omega\,\frac{11}{6}}$$