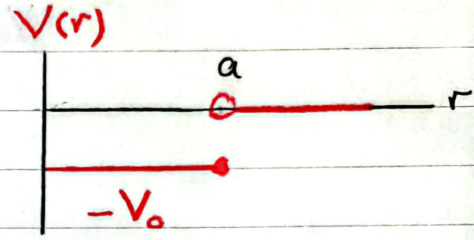
Advanced Quantum Mechanics II

Solutions to the Final Exam Spring Semester 2023



1. $H\,\psi_{E\ell m} = E\,\psi_{E\ell m}(\vec{r}),\quad \psi_{E\ell m}(\vec{r})$

$$\psi_{E\ell m}(\vec{r}) = \frac{1}{r}\,u_{E\ell}(r)\,Y_\ell^m(\theta,\varphi)$$

$$\Rightarrow \left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2}\right]u_{E\ell}(r) = \overbrace{\frac{2m}{\hbar^2}\left[E - V(r)\right]}^{\alpha^2} u_{E\ell}(r),\quad \text{let } k^2 = \frac{2m}{\hbar^2}E$$

1. Bound states $-V_0 < E \leqslant 0$ $\qquad \left(h_\ell^{(1,2)}(kr) = j_\ell(kr) \pm i\,n_\ell(kr)\right)$

$\Rightarrow$ For $0 \leqslant r \leqslant a$ $\qquad \frac{1}{r}u_{E\ell}(r) = A\,j_\ell(\alpha r),\quad \alpha = \sqrt{\frac{2m}{\hbar^2}(-|E| + V_0)} > 0$

For $a < r$ $\qquad \frac{1}{r}u_{E\ell}(r) = B\,h_\ell^{(1)}(kr),\quad k = \sqrt{-\frac{2m|E|}{\hbar^2}} = i\,K$

2. Scattering states $0 < E$

For $0 \leqslant r \leqslant a$: $\frac{1}{r}u_{E\ell}(r) = A\,j_\ell(\alpha r),\quad \alpha = \sqrt{\frac{2m}{\hbar^2}(E + V_0)} > 0$

For $a < r$: $\frac{1}{r}u_{E\ell}(r) = B\,h_\ell^{(1)}(kr) + C\,h_\ell^{(2)}(kr),\quad k^2 = \frac{2m}{\hbar^2}E > 0$

Conditions:

1. Bound states $\qquad \psi_{in}(a,\theta,\varphi) = \psi_{out}(a,\theta,\varphi) \Rightarrow A\,j_\ell(\alpha a) = B\,h_\ell^{(1)}(iKa)$

$$\frac{\partial}{\partial r}\psi_{in}(r,\theta,\varphi)\Big|_{r=a} = \frac{\partial}{\partial r}\psi_{out}(r,\theta,\varphi)\Big|_{r=a} \Rightarrow A\,j_\ell'(\alpha a) = B\,h_\ell'^{(1)}(iKa)$$

These two determine $\frac{A}{B}$ and E_n. $\int_0^\infty |\psi|^2 dV = 1 \Rightarrow$ fixes A

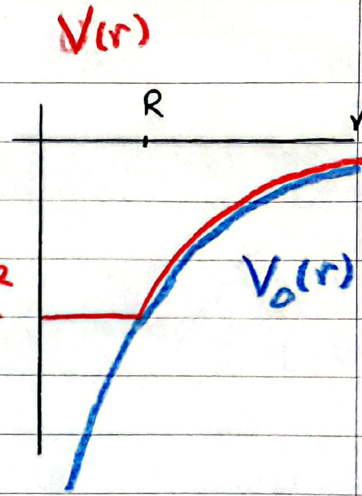
2. Scattering states $\quad A\,j_\ell(\alpha a) = B\,h_\ell^{(1)}(ka) + C\,h_\ell^{(2)}(ka)$

$$A\,j_\ell'(\alpha a) = B\,h_\ell'^{(1)}(ka) + C\,h_\ell'^{(2)}(ka)$$

+ Normalization

$\Rightarrow E > 0$ is unrestricted

2. For a spherical shell $V(r) = \begin{cases} -\dfrac{Ze^2}{R} & \text{for } r<R \\ -\dfrac{Ze^2}{r} & \text{for } R<r \end{cases}$



$$\Delta V = V(r) - V_0(r) = \begin{cases} \dfrac{Ze^2}{r} - \dfrac{Ze^2}{R} & r<R \\ 0 & R<r \end{cases}$$

$$\dfrac{Ze^2}{r} \uparrow -\dfrac{Ze^2}{r}$$

$$\Delta E^{(1)}_{1,0} = \langle 100|\Delta V|100\rangle = \int |\psi_{100}|^2 \Delta V \, dV$$

$$= \left(\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\right)^2 \int_0^R 4\pi r^2 dr\, e^{-2Zr/a_0} \left(\frac{Ze^2}{r} - \frac{Ze^2}{R}\right)$$

$$= \frac{4\pi}{\pi}\left(\frac{Z}{a_0}\right)^3 \left(\frac{a_0}{2Z}\right)^2 Ze^2 \int_0^{x_0} x^2 dx\, e^{-bx}\left(\frac{1}{x} - \frac{1}{x_0}\right), \quad x_0 = \frac{2ZR}{a_0} \ll 1,\ b=1$$

$$= \frac{Z^2e^2}{a_0}\left(-\frac{d}{db} - \frac{1}{x_0}\frac{d^2}{db^2}\right)\frac{e^{-bx_0} - 1}{-b} \longrightarrow \approx \frac{1}{\cancel{b}}\left(\cancel{b}x_0 - \frac{1}{2}b^{\cancel{2}}x_0^2 + \frac{1}{6}b^{\cancel{3}\,2}x_0^3\right)$$

$$\approx \frac{Z^2e^2}{a_0}\left(+\frac{1}{2}x_0^2 - \frac{1}{3}x_0^2\right) = \frac{Z^2e^2}{a_0}\frac{1}{6}\frac{4Z^2R^2}{a_0^2} = \frac{2}{3}\frac{Z^4e^2R^2}{a_0^3} > 0$$

Since the electron 'feels' a less negative potenial for $r<R$, the energy shipt is positive.

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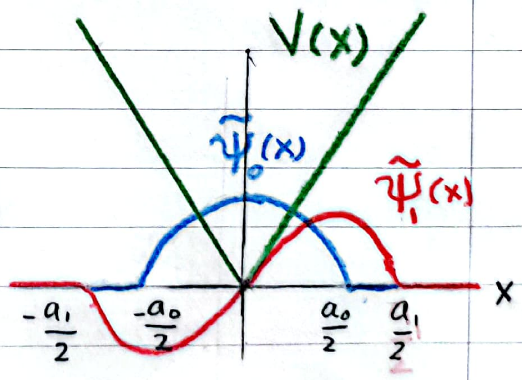
3. $$\tilde{E}(a_1) = \int \tilde{\psi}_1^*(x)\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\tilde{\psi}_1(x)\,dx$$

$$= 2\,\frac{2}{a_1}\int_0^{a_1/2}\left[\frac{\hbar^2}{2m}\left(\frac{2\pi}{a_1}\right)^2 + Ax\right]\sin^2\left(\frac{2\pi x}{a_1}\right)dx$$

$$= \frac{4}{a_1}\left[\frac{\hbar^2}{2m}\left(\frac{2\pi}{a_1}\right)^2\frac{a_1}{4} + A\frac{a_1^2}{16}\right]$$



$$\tilde{\psi}_1(x) = \sqrt{\frac{2}{a_1}}\sin\left(\frac{2x}{a_1}\pi\right),\ -\frac{a_1}{2}<x<\frac{a_1}{2}$$

$$0 = \frac{d\tilde{E}}{da_1} = -\frac{\hbar^2}{2m}\frac{4\pi^2}{a_1^3}2 + \frac{A}{4} \Rightarrow a_1 = \left(\frac{16\pi^2\hbar^2}{mA}\right)^{1/3} \Longrightarrow E = \left[\frac{(\pi\hbar A)^2}{4m}\right]^{1/3}\left(\underset{\text{kinetic}}{\frac{1}{2}} + \underset{\text{potential}}{1}\right)$$

4. $$C_n^{(1)}(t) = -\frac{i}{\hbar}\int_0^t e^{i\omega_{n0}t'}\langle n|V(t')|0\rangle dt' = -\frac{i}{\hbar}A\int_0^t e^{i\omega_{n0}t' - \frac{t'}{\tau}}\langle n|x|0\rangle dt'$$

$$\langle n|x|0\rangle = \langle n|\sqrt{\frac{\hbar}{2m\omega_0}}(a^{+} + \overset{0}{a})|0\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\langle n|1\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\delta_{n1}$$

Therefore, only transition to $|1\rangle$ is allowed (to first order)

$$\Rightarrow C_1^{(1)}(t) = -\frac{i}{\hbar}A\,\frac{e^{(i\omega_{10} - \frac{1}{\tau})t} - 1}{i\omega_{10} - \frac{1}{\tau}}\sqrt{\frac{\hbar}{2m\omega_0}}$$

$$\Rightarrow \text{Probability} \approx |C_1^{(1)}(t)|^2 = \frac{|A|^2}{\hbar^2}\,\frac{\left[e^{(i\omega_{10} - \frac{1}{\tau})t} - 1\right]\left[e^{(-i\omega_{10} - \frac{1}{\tau})t} - 1\right]}{\omega_{10}^2 + \frac{1}{\tau^2}}\,\frac{\hbar}{2m\omega_0}$$

$$= \frac{|A|^2}{\hbar^2}\,\frac{2}{\omega_{10}^2 + \frac{1}{\tau^2}}\left[1 - e^{-\frac{t}{\tau}}\cos(\omega_{10}t)\right]\frac{\hbar}{2m\omega_0}$$

where $E_n = (n + \frac{1}{2})\hbar\omega_0$, $\hbar\omega_{nm} = E_n - E_m = (n - m)\hbar\omega_0$