General Physics II

Solutions to the Final Exam   Spring 2023

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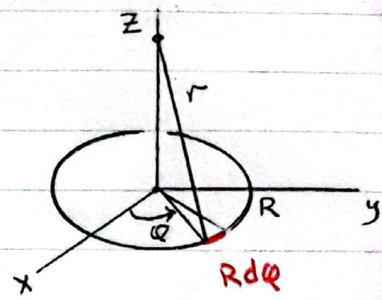
1. a) $V(z) = k\int \frac{dq}{r} = k\int_0^{2\pi} \frac{R\lambda\, d\varphi}{\sqrt{z^2+R^2}} = \frac{kR\lambda}{\sqrt{z^2+R^2}}\int_0^{2\pi} d\varphi$

$$= \frac{k\,\lambda R 2\pi}{\sqrt{z^2+R^2}} = \frac{kq}{\sqrt{z^2+R^2}}$$



$\lambda = \frac{q}{2\pi R}$

b) $\vec{E} = -\vec{\nabla} V = -(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z})\, V(z)$

$$= -\hat{z}\frac{d}{dz}V(z) = -\hat{z}\frac{d}{dz}\frac{kq}{\sqrt{z^2+R^2}} = +\frac{kqz}{(z^2+R^2)^{3/2}}\hat{z}$$

c) For $z \ll R$: $V(z) = \frac{kq}{R\sqrt{1+z^2/R^2}} \xrightarrow{z\ll R} \frac{kq}{R}\left(1 - \frac{z^2}{2R^2}\right) \xrightarrow{z=0} \frac{kq}{R}$

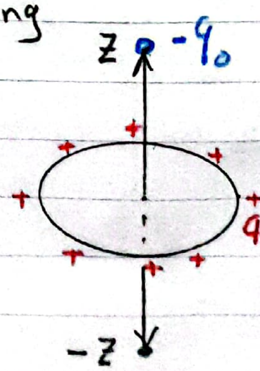
$$\vec{E}(z) = \frac{kqz\hat{z}}{R^3(1+z^2/R^2)^{3/2}} \xrightarrow{z\ll R} \frac{kqz}{R^3}\left(1 - \frac{3}{2}\frac{z^2}{R^2}\right) \xrightarrow{z=0} 0$$

For $z \gg R$: $V(z) = \frac{kq}{z\sqrt{1+R^2/z^2}} \xrightarrow{z\gg R} \frac{kq}{z}\left(1 - \frac{R^2}{2z^2}\right) \xrightarrow{z\ggg R} \frac{kq}{z}$

$$\vec{E}(z) = \frac{kqz\hat{z}}{z^3(1+R^2/z^2)^{3/2}} \xrightarrow{z\gg R} \frac{kq}{z^2}\hat{z}\left(1 - \frac{3}{2}\frac{R^2}{z^2}\right) \xrightarrow{z\ggg R} \frac{kq}{z^2}\hat{z}$$

$\vec{E}(0) = 0$ by symmetry. For $z \gg R$ the results are those of a point charge.

d) the negative test charge is attracted to the ring and executes periodic motion with amplitude z about zero.



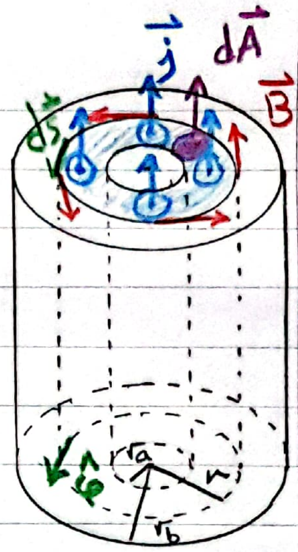
For $z \ll R$ the potential energy is

$$U(z) = -q_0 V(z) = \underbrace{-\frac{kqq_0}{R}}_{\text{constant}} + \underbrace{\frac{kqq_0}{2R^3}}_{\sim\frac{1}{2}K}z^2$$

quadratic in z $\Rightarrow$ simple harmonic motion

$\sim \frac{1}{2}Kz^2 \Rightarrow \omega_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{kqq_0}{R^3 m}}$

2. $\int \vec{B}\cdot d\vec{s} = \mu_0 I_{in} = \mu_0 \int_{r_a}^{r} \vec{j}\cdot d\vec{A}$



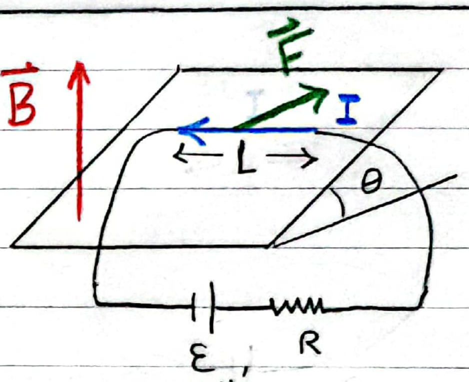
$$\Rightarrow B\int ds = \mu_0 \int_{r_a}^{r} \frac{j_0}{a} r' (r' dr' d\varphi)$$

$$\Rightarrow 2\pi r B = \frac{\mu_0 j_0}{a} 2\pi \frac{1}{3}(r^3 - r_a^3)$$

$$I = \int_{r_a}^{r_b} \vec{j}\cdot d\vec{A} = \frac{j_0}{a} 2\pi \frac{1}{3}(r_b^3 - r_a^3)$$

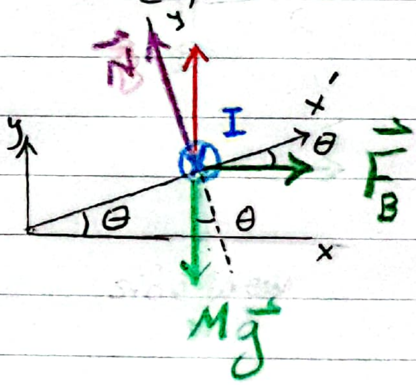
$$\vec{B}(r) = \frac{\mu_0}{2\pi r}\left(I \frac{r^3 - r_a^3}{r_b^3 - r_a^3}\right)\hat{\varphi}$$

3. $\vec{F}_B = I\vec{L}\times\vec{B} = ILB\,\hat{x}$



Equilibrium $\Rightarrow$ $\hat{x}'$: $F_B\cos\theta = Mg\sin\theta$

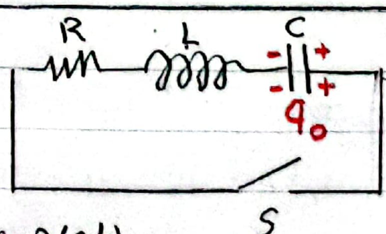
$\hat{y}'$: $N = Mg\cos\theta$



$$\Rightarrow F_B = ILB = Mg\tan\theta$$

$$\Rightarrow I = \frac{Mg}{LB}\tan\theta$$

4. $L\frac{dI}{dt} + IR + \frac{q}{C} = 0$



$$\Rightarrow L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \quad \text{Try } q(t) = q_0\exp(at)$$

$$\Rightarrow q_0\left(La^2 + Ra + \frac{a}{C}\right) = 0 \Rightarrow a = \frac{1}{2L}\left(-R \pm \sqrt{R^2 - 4\frac{L}{C}}\right) = -\frac{R}{2L} \pm i\overbrace{\sqrt{\underbrace{\frac{1}{LC}}_{\omega_0^2} - \left(\frac{R}{2L}\right)^2}}^{\omega'}$$

$$q(t) = A e^{(-\frac{R}{2L} + i\omega')t} + B e^{(-\frac{R}{2L} - i\omega')t} \in \mathbb{R} \Rightarrow B = A^*$$

$$A = |A|e^{i\phi} \Rightarrow B = |A|e^{-i\phi}$$

$$\Rightarrow q(t) = |A|\left\{ e^{\left[\left(-\frac{R}{2L} + i\omega'\right)t + i\phi\right]} + e^{\left[\left(-\frac{R}{2L} - i\omega'\right)t - i\phi\right]} \right\}$$

$$= 2|A|\, e^{-\frac{R}{2L}t} \cos(\omega' t + \phi)$$

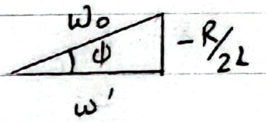
$$I(t) = \frac{dq}{dt} = 2|A| e^{-\frac{R}{2L}t}\left[-\omega' \sin(\omega' t + \phi) - \frac{R}{2L}\cos(\omega' t + \phi)\right]$$

Initial conditions

$$q(0) = q_0 = 2|A|\cos(\phi)$$

$$I(0) = 0 = 2|A|\left[-\omega' \sin(\phi) - \frac{R}{2L}\cos(\phi)\right] \Rightarrow \tan\phi = -\frac{R}{2L\omega'}$$

$$\sqrt{\omega'^2 + \left(\frac{R}{2L}\right)^2} = \sqrt{\omega_0^2 - \cancel{\left(\frac{R}{2L}\right)^2} + \cancel{\left(\frac{R}{2L}\right)^2}} = \omega_0 \qquad \cos\phi = \frac{\omega'}{\omega_0}$$



$$V_C = \frac{1}{C} q(t) = \frac{1}{C} q_0 \frac{\omega_0}{\omega'} e^{-\frac{R}{2L}t} \cos(\omega' t + \phi)$$

$$V_R = R I(t) = -R q_0 \frac{\omega_0^2}{\omega'} e^{-\frac{R}{2L}t}\left[\overbrace{\frac{\omega'}{\omega_0}}^{\cos\phi} \sin(\omega' t + \phi) + \overbrace{\frac{R/2L}{\omega_0}}^{-\sin\phi} \cos(\omega' t + \phi)\right]$$

$$= -R q_0 \frac{\omega_0^2}{\omega'} e^{-R/2L\, t} \sin(\omega' t + \cancel{\phi} - \cancel{\phi})$$

$$V_L = L\frac{dI}{dt} = -L q_0 \frac{\omega_0^3}{\omega'} e^{-\frac{R}{2L}t}\left[\overbrace{\frac{\omega'}{\omega_0}}^{\cos\phi} \cos\omega' t - \overbrace{\frac{R}{2L\omega_0}}^{+\sin\phi} \sin\omega' t\right]$$

$$= -L q_0 \frac{\omega_0^3}{\omega'} e^{-\frac{R}{2L}t} \cos(\omega' t - \phi)$$

You can check that

$$V_C(t) + V_R(t) + V_L(t) = 0$$