Final state interactions from euclidean correlation functions ☆

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We study the three-point euclidean correlation functions and derive a theorem relating their asymptotic behaviour in euclidean time and for infinite space volume to the threshold time-like form factor and the scattering length. We comment on the relation between our approach and that in which scattering lengths are related to finite volume effects on energy levels.

The Wightman axioms allow the continuation of the correlation functions of local operators from Minkowski to euclidean four-dimensional space [1]. Conversely, the Osterwalder-Schrader theorem guarantees that euclidean correlation functions (ECF's) can be analytically continued back to Minkowski space, provided we have a local action which satisfies the so-called reflection positivity condition [2].

Of course, the possibility of analytic continuation is of little use in cases, such as the lattice simulation of QCD, where the ECF's are computed approximately and on a discrete set of points in four-space. Thus, it is of considerable interest to identify the physical quantities, if any, which can be extracted directly from the ECF's, avoiding analytic continuation.

The best example is offered by the two-point function of a local operator. Choosing one direction as "time" and making a three-dimensional Fourier transform in the other directions, one defines [#1]

$$G_q(t) \equiv \int \mathcal{D}\varphi \exp(-S) O_q(t) O(0) = \langle O_q(t) O(0) \rangle, \tag{1}$$

where

$$O_q(t) = \int d^3x \exp(-i\boldsymbol{q}\cdot\boldsymbol{x}) O(\boldsymbol{x}, t), \tag{2}$$

and $\mathcal{D}\varphi$ denotes collectively the integration over the relevant fields. The Kallën-Lehmann representation implies the asymptotic behaviour, for $t \to +\infty$:

$$G_q(t) = \frac{Z}{\sqrt{2E_q}} \exp(-E_q t) + \text{exponentially small terms}, \tag{3}$$

with

$$E_q = \sqrt{\boldsymbol{q}^2 + m^2}, \tag{4}$$

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[#1] In the following, when no ambiguity arises, 0 will denote the origin in four-dimensional space, $(\boldsymbol{0}, 0)$.

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m being the physical (i.e. Minkowski space) mass of the lowest-lying state that O can excite from the vacuum, and

$$\sqrt{Z} = \langle 0|O(0)|\boldsymbol{q}\rangle\sqrt{2E_q}. \tag{5}$$

These relations are used, in lattice QCD, to obtain e.g. the physical mass of the pion, M_π, and its coupling to the axial current, f_π.

Besides the two-point functions, very little is known on the problem we have just stated. In this note, we address ourselves to the study of the three-point functions involving a local operator and two interpolating fields of some hadron [#2]. Our work turns out to be complementary to that of ref. [5], which proposes to extract the threshold scattering amplitude from the (space) volume dependence of euclidean correlation functions, as we shall discuss later.

The basic results will be derived explicitly for the case of no internal quantum numbers, namely for the ECF of a scalar density, J, and two fields interpolating a pseudoscalar particle, which we call pion for short. We shall also consider the extension to more general kinematical situations. We restrict initially to the case where no stable state coupled to J sits below the two-particle threshold, and comment later on the effect of such a state, when present.

One result which is already known, and has been used to compute the hadronic form-factors for space-like momentum transfer [4], concerns the asymptotic behaviour of the three-point function for $t_1 \to +\infty$ and $t_2 \to -\infty$.

We define the correlation function according to

$$G(t_1 t_2; \boldsymbol{q}_1 \boldsymbol{q}_2) = \langle \varphi_{\boldsymbol{q}_1}(t_1)\varphi_{\boldsymbol{q}_2}(t_2)J(0)\rangle, \tag{6}$$

with $\varphi_{\boldsymbol{q}}(t)$ defined analogously to (2).

Saturating with the lowest-lying intermediate states and using time-translation invariance, one finds immediately

$$\lim_{\substack{t_1\to+\infty \\ t_2\to-\infty}} G = \frac{Z_\pi}{\sqrt{2E_{q_1}2E_{q_2}}}\langle \pi, \boldsymbol{q}_1|J(0)|\pi, -\boldsymbol{q}_2\rangle \exp(-E_{q_1}t_1)\exp(E_{q_2}t_2) + \text{exponentially small terms} \tag{7}$$

with

$$\sqrt{2E_{q_1}2E_{q_2}}\langle \pi, \boldsymbol{q}_1|J(0)|\pi, -\boldsymbol{q}_2\rangle = f(q^2), \quad q^2 = (E_{q_1} - E_{q_2})^2 - (\boldsymbol{q}_1 + \boldsymbol{q}_2)^2, \tag{8}$$

f being the (on-the-mass-shell, physical) form factor of J.

The correlation function can also be computed for different time ordering

$$0 \ll t_2 \ll t_1. \tag{9}$$

Since the times are both large, one would expect naively the external pions to go on mass-shell again, and the correlation function to yield the form factor in the time-like region. However, the relation cannot be that simple. The form factor in the time-like region is complex (it has the phase of the scattering amplitude [6]) while G is the Fourier transform of a real function and it has, therefore, only trivial phases. In fact, as we shall see the asymptotic behaviour of G in the region (9) is *not* related to anything of immediate physical significance. In particular, G involves off-mass-shell amplitudes, so that, among other things, it will depend from the particular definition of the interpolating field.

[#2] Three-point correlations functions have been considered in the past years by several groups, in connection with the calculation of the weak and electromagnetic couplings of light or charmed hadrons, and of the weak non-leptonic kaon amplitudes [3,4].

A simple and elegant result obtains, however, when the external particles are at rest, namely for $\boldsymbol{q}_1=\boldsymbol{q}_2=0$. We find [#3]

$$G(t_1,t_2,\mathbf{0},\mathbf{0}) \underset{(t_1\gg t_2\gg 0)}{=} \frac{Z_\pi}{(2M_\pi)^2}\exp(-M_\pi t_1)\exp(-M_\pi t_2)f(4M_\pi^2)\left(1-a\sqrt{\frac{M_\pi}{4\pi t_2}}+\cdots\right), \tag{10}$$

where a is the physical π-π S-wave scattering length and dots denote terms suppressed by at least one further power of t_2. The form factor at threshold and the scattering length can both be obtained directly from the asymptotic behaviour of the euclidean three-point function [#4].

We illustrate now the derivation of eq. (10). We first let $t_1\to+\infty$, and specialize to the center of mass frame $(\boldsymbol{q}_1=-\boldsymbol{q}_2=\boldsymbol{q})$.

We get

$$G(t_1,t_2;\boldsymbol{q},-\boldsymbol{q}) \xrightarrow[\substack{t_1\to+\infty\\ t_2>0}]{} \exp(-E_q t_1)\frac{\sqrt{Z_\pi}}{\sqrt{2E_q}}F_{\boldsymbol{q}}(t_2), \tag{11}$$

where

$$F_{\boldsymbol{q}}(t_2)\equiv\langle\pi,\boldsymbol{q}|\varphi_{-\boldsymbol{q}}(t_2)J(0)|0\rangle\equiv\langle\pi,\boldsymbol{q}|\exp(Ht_2)\varphi_{-\boldsymbol{q}}(0)\exp(-Ht_2)J(0)|0\rangle. \tag{12}$$

H is the physical, Minkowski space, hamiltonian and $\varphi_{-\boldsymbol{q}}(0)$ the Heisenberg field at zero time. We insert now a complete set of eigenstates of H which, for convenience, are choosen to be "out" states. We obtain

$$F_q(t_2)=\sum_n(2\pi)^3\delta^{(3)}(\boldsymbol{P}_n)\exp[-(E_n-E_q)t_2]\langle\pi,\boldsymbol{q}|\varphi(0)|n,\text{out}\rangle\langle n,\text{out}|J(0)|0\rangle. \tag{13}$$

Separating the disconnected terms:

$$F_{\boldsymbol{q}}(t_2)=\frac{\sqrt{Z_\pi}}{\sqrt{2E_q}}\langle\pi,\boldsymbol{q};\pi,-\boldsymbol{q}\ \text{out}|J(0)|0\rangle+F_{\boldsymbol{q}}^{\text{conn}}(t_2), \tag{14}$$

with F^{conn} defined as in eq. (13), with the matrix element of the pion field restricted to the connected contributions.

Next, we define

$$\langle\pi,\boldsymbol{q}|\varphi(0)|n,\text{out}\rangle_{\text{conn}}=\frac{[\mathcal{M}(\boldsymbol{q},-\boldsymbol{q};n)]^*}{-p^2+M_\pi^2+\mathrm{i}\epsilon}\frac{\sqrt{Z_\pi}}{\sqrt{2E_q}}N_n, \tag{15}$$

where

$$p\equiv(E_n-E_q,-\boldsymbol{q}) \tag{16}$$

and N_n contains a factor $(\sqrt{2E})^{-1}$ for each particle in the state $|n,\text{out}\rangle$. The reason for this definition is of course that the LSZ-reduction formula shows that, when the four-momentum associated to the field $\varphi(0)$ goes on the mass-shell $(p^2=M_\pi^2)$, $\mathcal{M}$ reduces to the invariant scattering amplitude of the process

$$\pi(\boldsymbol{q})+\pi(-\boldsymbol{q})\to n, \tag{17}$$

i.e.:

$$(2\pi)^4\delta^3(\boldsymbol{P}_n)\delta(E_n-2E_q)[\mathrm{i}\mathcal{M}(\boldsymbol{q},-\boldsymbol{q};n)]N_n\frac{1}{2E_q}=\langle n,\text{out}|\pi,\boldsymbol{q};\pi,-\boldsymbol{q}\ \text{in}\rangle-\langle n,\text{out}|\pi,\boldsymbol{q};\pi,-\boldsymbol{q}\ \text{out}\rangle. \tag{18}$$

[#3] The form-factor at threshold is real, since the S-wave π-π scattering phase vanishes with the center-of-mass momentum.

[#4] In the general case, one can still obtain some information by making a model of the correlation function in Minkowski space, e.g. with a phenomenological lagrangian, and determine the parameters of the model from a fit of the numerical data to the model-amplitude, continued to euclidean space. It is quite obvious that the test of the fundamental theory obtained in this way can be considerably obscured by the possible inadequacies of the model itself.

This happens for the unique value $E_n = 2E_q$, for which the matrix element of the pion field in eq. (13) has a simple pole. Separating explicitly the absorptive part of the pole, we obtain

$$F_q^{\rm conn}(t_2) = \frac{\sqrt{Z_\pi}}{\sqrt{2E_q}} \exp(-E_q t_2)$$

$$\times \left(\tfrac{1}{2} \sum_n (2\pi)^4 \delta^3(\boldsymbol{P}_n) \delta(E_n - 2E_q) [\mathrm{i}\mathcal{M}(\boldsymbol{q}, -\boldsymbol{q};\, n)]^* \langle n, \mathrm{out} | J(0) | 0 \rangle + P_q(t_2) \right), \tag{19}$$

$$P_q(t_2) = -\wp \sum_n \exp[-(E_n - 2E_q) t_2] (2\pi)^3 \delta^3(\boldsymbol{P}_n) N_n \frac{[\mathcal{M}(\boldsymbol{q}, -\boldsymbol{q};\, n)]^* \langle n;\, \mathrm{out} | J(0) | 0 \rangle}{E_n (E_n - 2E_q)}, \tag{20}$$

where $\wp$ denotes the principal value.

Using eq. (18), the first term in eq. (19) reduces to a combination of form factors for "in" and "out" states so that, in conclusion, we obtain

$$G(t_1, t_2;\, \boldsymbol{q}, -\boldsymbol{q}) \xrightarrow[\binom{t_1 \to +\infty}{t_2 > 0}]{} \frac{Z_\pi}{2E_q} \exp(-E_q t_1) \exp(-E_q t_2)$$

$$\times [\tfrac{1}{2}(\langle \pi, \boldsymbol{q};\, \pi, -\boldsymbol{q}\ \mathrm{out} | J(0) | 0 \rangle + \langle \pi, \boldsymbol{q};\, \pi, -\boldsymbol{q}\ \mathrm{in} | J(0) | 0 \rangle) + P_q(t_2)]. \tag{21}$$

The first term in the square bracket corresponds to the naive expectation, except for the appearance of an average between in and out states which makes it real. Indeed it contains the time-like form factor of J. However, for $\boldsymbol{q} \neq 0$, the leading behaviour for $t_2 \to +\infty$ is determined by the lower tip of the integration range in P_q. Since this corresponds to $E_n = 2M_\pi$, we expect:

$$P_q(t_2) \underset{t_2 \to +\infty}{\sim} \exp[2(E_q - M_\pi) t_2] \tag{22}$$

up to negative powers of t_2, with a coefficient proportional to an off-shell amplitude, with no direct meaning in terms of observable quantities. The exception is at $\boldsymbol{q} = 0$, when $E_n = 2M_\pi$ corresponds to the on-shell amplitude at threshold.

To analyze the leading behaviour in t_2 for this case, it is sufficient to restrict to a two-pion intermediate state. We find

$$P_0(t_2) = \lim_{\delta \to 0^+} \left[-\tfrac{1}{4} \int_{M_\pi}^{\infty} \frac{k}{4k_0} \frac{\mathrm{d}k_0}{(2\pi)^3} \mathrm{d}\Omega \exp[-(2k_0 - M_\pi) t_2] [\mathcal{M}(k_0, \Omega)]^* \langle \pi, \boldsymbol{k};\, \pi, -\boldsymbol{k}\ \mathrm{out} | J(0) | 0 \rangle \wp \left(\frac{1}{k_0 - M_\pi - \delta} \right) \right]$$

$$+\text{non leading terms}, \tag{23}$$

with

$k = \sqrt{k_0^2 - M_\pi^2}$.

To estimate the leading behaviour, we can set $k_0 = M_\pi$ everywhere inside the integral, except in the exponent, which provides the convergence factor, and in k, which vanishes there. $\mathcal{M}$ becomes the on-shell amplitude at threshold, related to the scattering length by

$$\mathcal{M}|_{k_0 = M_\pi} = 32\pi a M_\pi, \tag{24}$$

and one gets the result given in eq. (10). We can add the following remarks.

(i) The physically interesting case of a vector current and an isovector pion, is obtained in a straightforward way. One defines:

$$\langle \varphi_q^+(t_1) \varphi_{-q}(t_2) J_i(0) \rangle \equiv 2q_i G(t_1, t_2;\, \boldsymbol{q}, -\boldsymbol{q}), \tag{25}$$

and obtained, for $t_1 \to +\infty$ and $0 \ll t_2 \ll t_1$,

$$G(t_1, t_2;\, \boldsymbol{0}, \boldsymbol{0}) \sim \frac{Z_\pi}{(2M_\pi)^2} \exp(-M_\pi t_1) \exp(-M_\pi t_2) f(4M_\pi^2) \left(1 - \frac{1}{\sqrt{2}} \frac{vM}{6t_2} \sqrt{\frac{M}{4\pi t_2}} + \cdots \right), \tag{26}$$

where f is the vector form factor:

$$2E_q\langle\pi^+, \boldsymbol{q}; \pi^-, -\boldsymbol{q}\ \mathrm{out}|J_i|0\rangle = 2q_i f, \tag{27}$$

and υ, the scattering volume, is related to the $J=I=1$ phase shift near threshold according to

$$\delta_{1,1} = \tfrac{1}{3}\upsilon q^3 + \cdots . \tag{28}$$

(ii) When a stable state σ, of mass μ, coupled to J, is present below threshold, we have an additional contribution to eq. (10), of the form

$$G^J_{\mathrm{pole}} = \frac{\sqrt{Z_\pi}}{\sqrt{2M_\pi}} \exp(-M_\pi t_1) \exp[-(\mu - M_\pi) t_2] \langle\pi, \boldsymbol{q}=0|\varphi(0)|\sigma, \boldsymbol{q}=0\rangle\langle\sigma, \boldsymbol{q}=0|J(0)|0\rangle, \tag{29}$$

which, however, involves an off-mass-shell matrix element. In principle, one could isolate the uninteresting pole contribution although in practice this can be challenging.

(iii) The worst situation applies when there is a continuum below the two-particle threshold, e.g. in the case of the nucleon form-factor in the time-like region. In this case no physical quantity can be obtained directly from the correlation function in the region (9).

In conclusion, we have presented a theorem which allows to obtain information about low-energy scattering amplitudes and form factors in the time-like region from ECF's, avoiding problems related to analytic continuation. The result in eq. (10) is exact in the infinite (space) volume limit. For a finite volume, L^3, it holds, provided

$$LM_\pi \gg 1, \tag{30}$$

$$t\Delta E \ll 1, \tag{31}$$

where ΔE is the level spacing due to the discretization of momentum spectrum. The second condition guarantees that the sum over discrete levels is well approximated by the infinite volume, continuum, distribution. According to ref. [5], an alternative approach is possible where the first inequality is kept, but the second is reversed (recent developments of this idea can be found in ref. [7]). In this case, the large-time limit is dominated by the lowest, (discrete) energy eigenvalue, slightly different from $2M_\pi$ due to computable finite-volume corrections, proportional to the scattering length.

To implement our method in practical lattice calculations one must isolate terms which drop with the square root of euclidean time, a task of considerable difficulty. If this can be done, however, one has access to several low-energy hadron parameters, such as the π–π, π–N, or N–N scattering lengths and the threshold π–π form factors. Selecting exotic quantum numbers in the S-channel, one can avoid problems related to the presence of stable poles below threshold in π–N and N–N. As for π–π, there is no stable state in the physical world. However this may be the case in present lattice QCD calculations, usually done with quark masses large enough for the ρ-meson to lie below the two-pion threshold.

From a more general point of view, our analysis gives precise limitations to the physical conclusions one can draw from numerical simulations, in cases where two or more particles are present on the lattice in the initial or final state.

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