Resonance parameters from Lattice QCD

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Overview

1. Motivation
2. Introduction to Lattice QCD
3. Hadron spectroscopy
4. Numerical Results
5. Conclusions

1
- Compute the energy spectrum in finite volume: Lattice QCD.

2
- Connect the energy spectrum in finite volume to phase shift in infinite volume: Lüscher's formula.

3
- Parameterize the phase shift and obtain the resonance parameters: Breit Wigner form, conformal mapping and UχPT.

Motivation

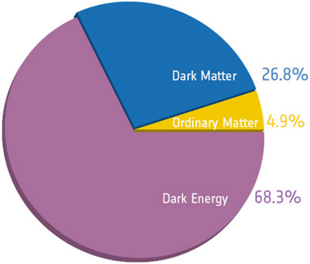
Why?

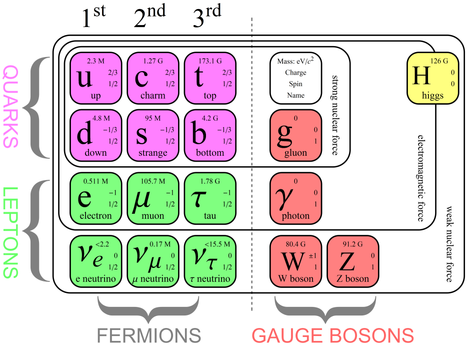
- A way to study the resonances in terms of quark and gluon dynamics; A test of QCD for well determined resonance parameters;
- The techniques can be used to investigate systems where the experimental situation is less clear.
- Validate effective models used to describe hadron scattering.

Starting point: meson resonance

- ρ in $I = 1, J^{PC} = 1^{--}$ π-π channel.
  - Better signal-noise ratio as a starting point.
  - Test the validity of lattice QCD method and its predictive power.
- σ in $I = 0, J^{PC} = 0^{++}$ π-π channel.
  - m_σ from experiment is indirectly obtained and has large systematic errors,$\Gamma_\sigma = 400 - 700\,\text{MeV} \sim m_\sigma$
  - A challenging application of lattice QCD

Quantum Chromodynamics





- The interaction between hadrons is dominated by the strong force.
- Quantum Chromodynamics (QCD) is one section of the standard model which describe the strong interaction between quarks and gluons.

$$\mathcal{L}_{QCD} = -\frac{1}{2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} - \sum_f \bar{\psi}_f \gamma^\mu \left[\partial_\mu - igA_\mu\right]\psi_f - \sum_f m_f \bar{\psi}_f \psi_f,$$

Asymptotic freedom

The coupling constant become small at high energies and vanishes in the ultraviolet limit.

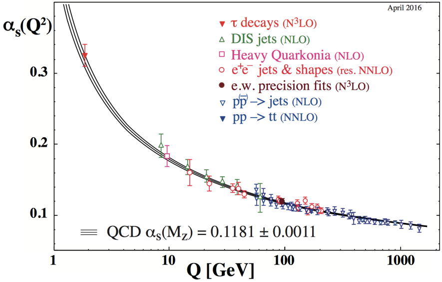


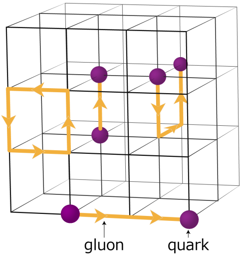
Figure: The running of coupling constant α_s as a function of momentum transfer.

$$\alpha_s(q) = \frac{c}{\log(q/\Lambda)} + \dots ; \quad \Lambda_{QCD} = 217(25)\,\text{MeV}^1.$$

In the hadronic scale, $\alpha_s \sim 1$, the non-perturbative approach is required.

[1] S. Bethke, J. Phys. G26 (2000) R27, [hep-ex/0004021].

Lattice QCD: non-perturbative approach to QCD. It formulates QCD in a discrete way.



Inputs:

- $a \leftarrow g$
- $m_{u/d} \rightarrow m_\pi$

Light hadron study:

- u and d quarks are important
- s quark introduces small correction

In this study: $N_f = 2$; $m_u = m_d > m_u^{\text{phys}} \rightarrow m_u^{\text{phys}}$
The role of LQCD in resonance study: extract energy spectrum for two-hadron states.

Hadron spectroscopy

STEP1

Consider the two point correlation functions for two interpolating field

$$\left\langle \hat{O}_2(t)\hat{O}_1^\dagger(0)\right\rangle = \frac{1}{Z}\int D[\psi,\bar{\psi},U]e^{-S_{QCD}[\psi,\bar{\psi},U]}O_2[\psi_t,\bar{\psi}_t,U_t]O_1^\dagger[\psi_0,\bar{\psi}_0,U_0],$$

$$Z = \int D[\psi,\bar{\psi},U]e^{-S_{QCD}[\psi,\bar{\psi},U]}.$$

In Euclidean space-time: Monte Carlo simulation

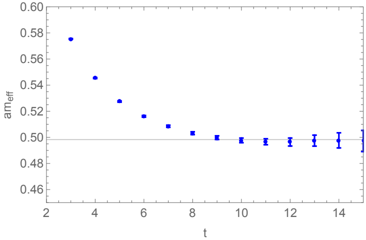
Operatorial view

$$\left\langle \hat{O}_2(t)\hat{O}_1^\dagger(0)\right\rangle = \lim_{T\to\infty}\frac{1}{Z_T}tr[e^{-(T-t)\hat{H}}\hat{O}_2 e^{-t\hat{H}}\hat{O}_1] = \sum_n \left\langle 0|\hat{O}_2|n\right\rangle \left\langle n|\hat{O}_1|0\right\rangle e^{-tE_n}$$

Choose the corresponding operators for the scattering channel we are interested in.

$$C(t)_{T\to\infty} = c_1 e^{-E_1 t}\left(1+O\left(e^{-\Delta E t}\right)\right);$$

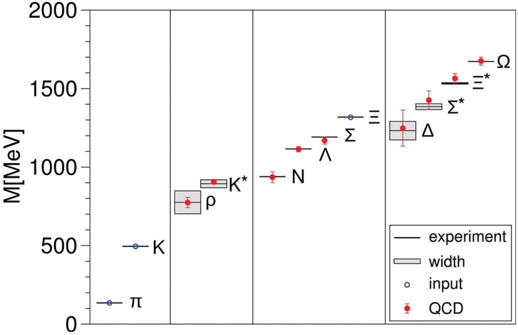
$$E(t) = -\ln\frac{C(t+1)}{C(t)}$$



Hadron spectroscopy

STEP1

- Lattice QCD has determined the single particle spectrum of hadrons [2].



$$\pi(J^P = 0^-) : \bar{d}\gamma_5 u;$$

- The resonance is unstable $\rightarrow$ quark-antiquark + hadron-hadron interpolating fields.

$$\rho(J^P = 1^-) : \pi \rightarrow\leftarrow \pi$$

[2] S. Durr et. al., Science 322 (2008) 12241227, [arXiv:0906.3599].

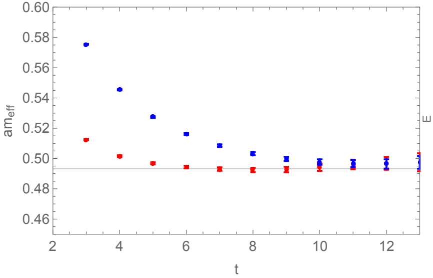
Correlation matrix:

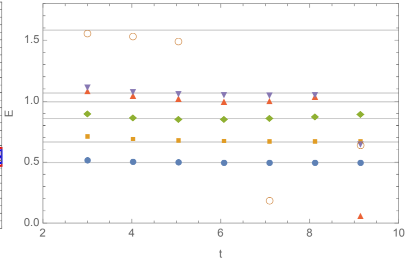
$$C(t)_{ij} =< \mathcal{O}_i(t)\mathcal{O}_j^\dagger(0) >; i,j = 1,2,...,N.$$

The eigenvalues of the correlation matrix are

$$\lambda^{(n)}(t,t_0) \propto e^{-E_n t}(1+\mathcal{O}(e^{-\Delta E_n t})), n = 1,2,...,N$$

where $\Delta E_n = E_{N+1} - E_n > E_{n+1} - E_n$.

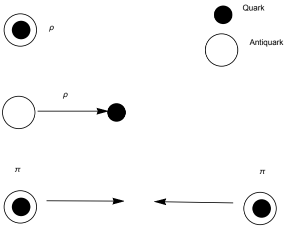




Interpolating field for ρ resonance STEP1

Four quark-antiquark operator ρ and two scattering operators $\pi\pi$ with $I = 1, J = 1$.

$$\rho^J(t_f) = \bar{u}(t_f)\Gamma_{t_f} A_{t_f}(\mathbf{p}) d(t_f).$$



N	Γ_{t_f}	A_{t_f}
1	γ_i	$e^{i\mathbf{p}}$
2	$\gamma_4\gamma_i$	$e^{i\mathbf{p}}$
3	γ_i	$\nabla_j e^{i\mathbf{p}} \nabla_j$
4	$\frac{1}{2}$	$\{e^{i\mathbf{p}}, \nabla_i\}$

$$(\pi\pi)_{\mathbf{P},\Lambda,\mu} = \sum_{\mathbf{p_1}^*,\mathbf{p_2}^*} C(\mathbf{P},\Lambda,\mu;\mathbf{p_1};\mathbf{p_2})\pi(\mathbf{p_1})\pi(\mathbf{p_2}),$$

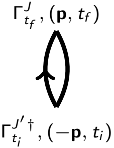
$$\pi\pi_{100}(\mathbf{p_1},\mathbf{p_2},t) = \frac{1}{\sqrt{2}}[\pi^+(\mathbf{p_1})\pi^-(\mathbf{p_2}) - \pi^+(\mathbf{p_2})\pi^-(\mathbf{p_1})]; \quad \mathbf{p}_1 = (1,0,0) \quad \mathbf{p}_2 = (-1,0,0)$$

$$\pi\pi_{110} = \frac{1}{2}(\pi\pi(110) + \pi\pi(101) + \pi\pi(1-10) + \pi\pi(10-1))$$

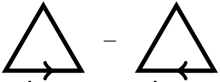
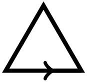
6×6 correlation matrix

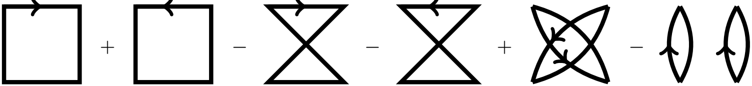
$$C = \begin{pmatrix} C_{\rho^J \leftarrow \rho^{J'}} & C_{\rho^J \leftarrow \pi\pi 100} & C_{\rho^J \leftarrow \pi\pi 110} \\ C_{\pi\pi 100 \leftarrow \rho^{J'}} & C_{\pi\pi 100 \leftarrow \pi\pi 100} & C_{\pi\pi 100 \leftarrow \pi\pi 110} \\ C_{\pi\pi 110 \leftarrow \rho^{J'}} & C_{\pi\pi 110 \leftarrow \pi\pi 100} & C_{\pi\pi 110 \leftarrow \pi\pi 110} \end{pmatrix}.$$

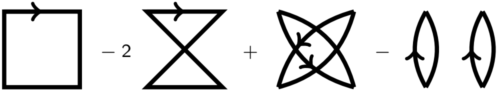
The correlation functions: $\bar{u}(t_i) \longrightarrow u(t_f)$

$C_{\rho_i \leftarrow \rho_j} = -\Big\langle$ 

$\Big\rangle = -\left\langle \mathrm{Tr}[M^{-1}(t_i, t_f)\Gamma^J_{t_f} e^{i\mathbf{p}} M^{-1}(t_f, t_i)\Gamma^{J'\dagger}_{t_i} e^{-i\mathbf{p}}] \right\rangle.$

$C_{\rho_i \leftarrow \pi\pi} = \Big\langle$  $\Big\rangle \overset{\mathbf{P}=0}{\equiv} 2 \Big\langle$  $\Big\rangle.$

$C_{\pi\pi \leftarrow \pi\pi} = -\Big\langle$  $\Big\rangle$

$\overset{\mathbf{P}=0}{\equiv} -\Big\langle 2$  $\Big\rangle$

ρ energy spectrum

STEP1

- $m_\pi \approx 315$ MeV: $\eta = 1.0, 1.25, 2.0$; $P = (0,0,0), (1,0,0)$.
- $m_\pi \approx 227$ MeV: $\eta = 1.0, 1.17, 1.33$; $P = (0,0,0), (1,0,0)$.

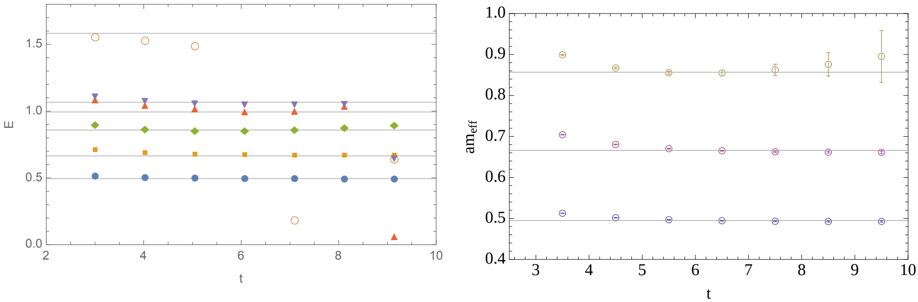


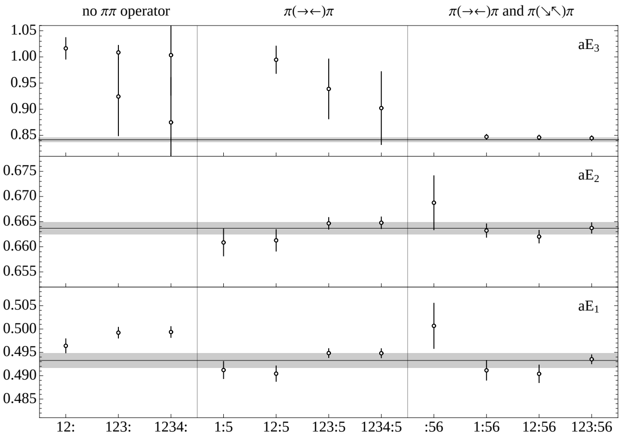
Figure: (Left) All 6 energy states (plateaus) vs t in $\eta = 1.0$ ensemble. (Right) A closer look into the first three energy states with their error bars

$$\lambda^{(n)}(t, t_0) \propto e^{-E_n t}(1 + \mathcal{O}(e^{-\Delta E_n t}));$$

Use $f(t) = we^{-Et} + (1-w)e^{-E' t}$ to fit each eigenvalues $\rightarrow E_n$

Energy spectrum stability

STEP1



$\mathcal{O}_i$	1	2	3	4	5	6
	ρ_1	ρ_2	ρ_3	ρ_4	$\pi\pi_{100}$	$\pi\pi_{110}$

Expectation for energy states

STEP1

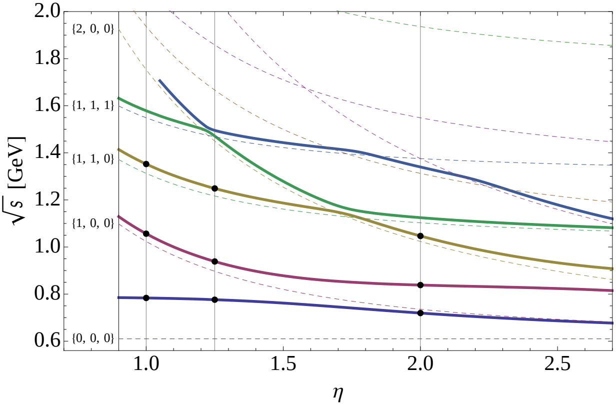
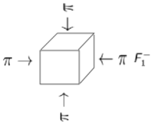
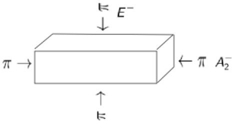


Figure: The lowest 3 energy states prediction from a plausible parameterization of data.

Lüscher's formula [3] STEP2

Two particle scattering in the elastic scattering region: $J = 1$, rest frame ($\mathbf{P} = 0$)





$$A_2^- : \cot \delta_1(k) = \mathcal{W}_{00} + \frac{2}{\sqrt{5}} \mathcal{W}_{20} \tag{1}$$

$$\tag{2}$$

$$\mathcal{W}_{lm}(1, q^2, \eta) = \frac{\mathcal{Z}_{lm}(1, q^2, \eta)}{\eta \pi^{\frac{3}{2}} q^{l+1}}; \quad q = \frac{kL}{2\pi}; \ \eta = \frac{N_{el}}{N} : \text{elongation factor}$$

Zeta function

$$\mathcal{Z}_{lm}(s; q^2, \eta) = \sum_{\tilde{\mathbf{n}}} \mathcal{Y}_{lm}(\tilde{\mathbf{n}})(\mathbf{n}^2 - q^2)^{-s}; \ \mathbf{n} \in \mathbf{m}$$

Total energy

$$E = 2\sqrt{m^2 + k^2}; \quad k = \sqrt{\left(\frac{E}{2}\right)^2 - m^2}$$

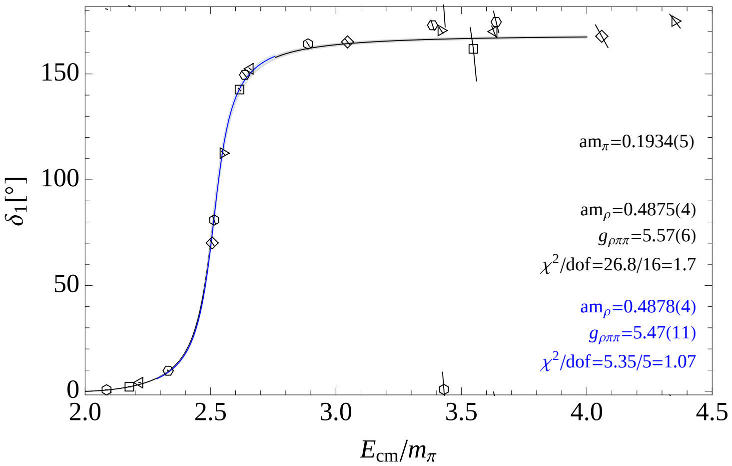
[3] X. Feng, X. Li, and C. Liu, Phys.Rev. D70 (2004) 014505, [hep-lat0404001].

Parametrization

STEP3

- Breit Wigner:
  - model independent
  - narrow resonance (like ρ)
- Conformal mapping:
  - model independent
  - broad resonance (like σ)
- Unitarized χPT:
  - inspired by χ PT
  - combined fit simultaneously for multiple channels (ρ, σ) and multiple quark masses
  - extrapolate the physical observables to physical point
  - model dependent

RESULT

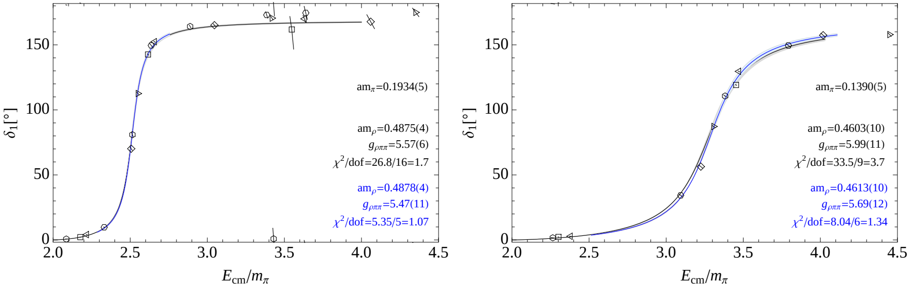


Breit Wigner form: $\cot(\delta_1(E)) = \dfrac{M_R^2 - E^2}{E\Gamma_r(E)}, \quad \Gamma_r(E) \equiv \dfrac{g_{R12}^2}{6\pi}\dfrac{p^3}{E^2}.$

$$\delta_1(E) = \operatorname{arccot} \frac{6\pi(M_R^2 - E^2)E}{g^2 p^3}$$

ρ parameterization

RESULT



	m_π	m_ρ(MeV)	Γ_ρ(MeV)
Breit Wigner	315	795.5(0.7)	35.7(1.4)
UχPT		796(1)	35(1)
Breit Wigner	227	749.2(1.6)	81.7(3.3)
UχPT		748(1)	77(1)

$K\bar{K}$ channel contribution

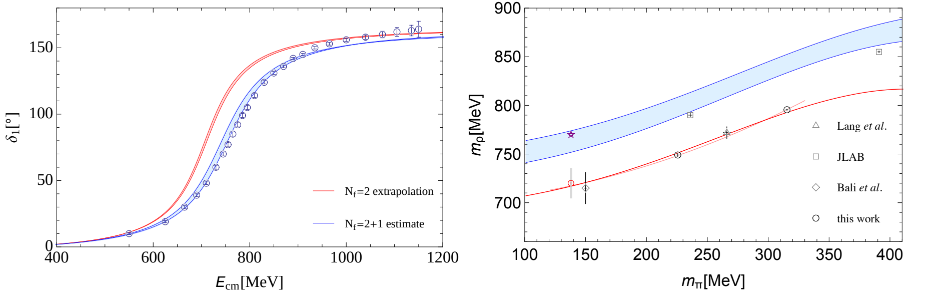


Figure: (Left) Chiral extrapolation of the phase shift to the physical pion mass (red band), obtained from the simultaneous fit to 315 MeV and 227 MeV pion masses. The blue band: phaseshift with $K\bar{K}$. Open circles: experiment data [5]. $m_{\rho\text{extr}}$ is 50 MeV(8%) lower than experiment value 775 MeV.

- statistically error, models, extrapolation, finite volume effect and lattice spacing(2%);
- s quark correction ($K\bar{K}$ channel)

~~We study the $K\bar{K}$ effect using UχPT~~model with input parameters m_π, f_π, f_K and $\hat{l}_{1,2}$.

[5] S. D. Protopopescu, et. al, Phys. Rev. D7 (1973) 1279.

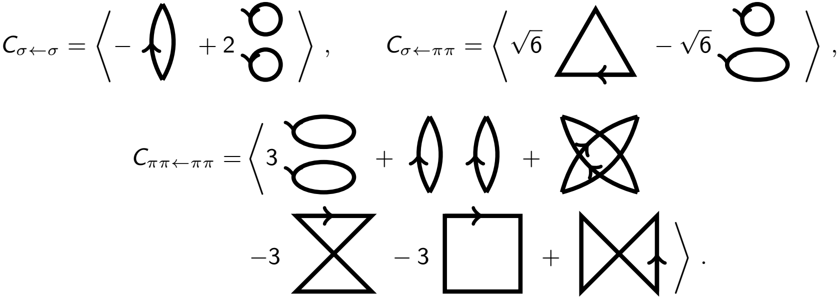
Four quark-antiquark operator σ and two scattering operators with $I = 0, J^{PC} = 0^{++}$.

$$\sigma(\Gamma_i A(\mathbf{p}), t) = \frac{1}{\sqrt{2}}[\bar{u}(t)\Gamma_i A(\mathbf{p})u(t) + \bar{d}(t)\Gamma_i A(\mathbf{p})d(t)].$$

i	Γ_i	$A(\mathbf{p})$
1	$\mathbf{1}$	$e^{i\mathbf{p}}$
2	γ_i	$e^{i\mathbf{p}}\nabla_i$
3	$\mathbf{1}$	$\nabla_i e^{i\mathbf{p}}\nabla_i$
4	$\mathbf{1}$	$\nabla_i^4 e^{i\mathbf{p}}\nabla_i^4$

$$\pi\pi(\boldsymbol{p}_1, \boldsymbol{p}_2) = \frac{1}{\sqrt{3}}\{\pi^+(\boldsymbol{p}_1)\pi^-(\boldsymbol{p}_2) + \pi^-(\boldsymbol{p}_1)\pi^+(\boldsymbol{p}_2) + \pi^0(\boldsymbol{p}_1)\pi^0(\boldsymbol{p}_2)\}.$$

$$\boldsymbol{p}_1 = -\boldsymbol{p}_2 = \{(0,0,0), (1,0,0)\}.$$



$\boldsymbol{P} = (0,0,0)$ system couples to vacuum, need vacuum subtraction.

σ energy spectrum

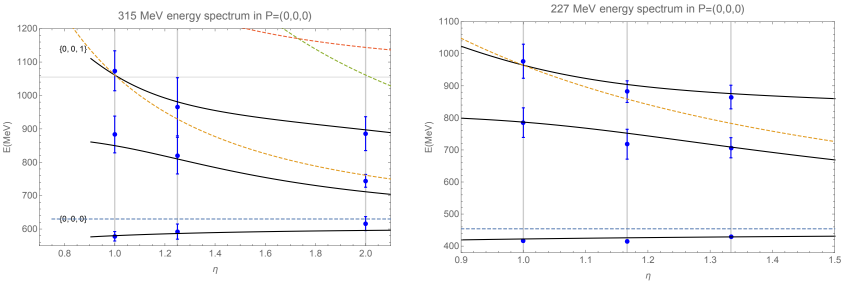


Figure: σ energy spectrum at two pion mass ensembles $m_\pi \approx 315\,\text{MeV}$ (left) and $m_\pi \approx 227\,\text{MeV}$ (right). The curves are the prediction from a parameterization with the parameters fitted from the ρ data.

Lüscher's formula for $J = 0$:

$$\cot \delta_0 = \mathcal{W}_{00} = \frac{\mathcal{Z}_{00}(1, q^2; \eta)}{\pi^{3/2}\eta q} .$$

σ parametrization

RESULT

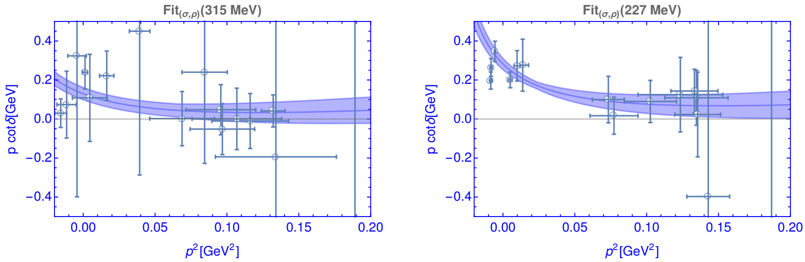
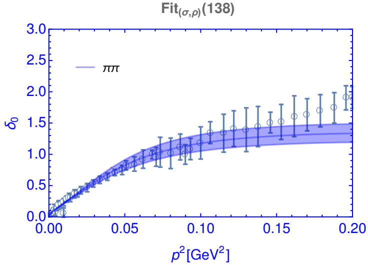


Figure: $p \cot \delta_0$ as a function of scattering momentum in the center of mass frame. The curve is the UχPT model with parameters fixed to the combined σ and ρ channel fit.(the highest energy data point in $m_\pi = 315\,\text{MeV}$ and the two highest energy data points in $m_\pi = 227\,\text{MeV}$ are excluded from the fit)

Approach	138 MeV			227 MeV			315 MeV		
	$\text{Re}\sqrt{s_0}$	$\text{Im}\sqrt{s_0}$	$\lvert g\rvert$[GeV]	$\text{Re}\sqrt{s_0}$	$\text{Im}\sqrt{s_0}$	$\lvert g\rvert$[GeV]	$\text{Re}\sqrt{s_0}$	$\text{Im}\sqrt{s_0}$	$\lvert g\rvert$[GeV]
Conformal	-	-	-	460^{+30}_{-60}	-180^{+30}_{-30}	$3.16^{+0.1}_{-0.1}$	660^{+50}_{-70}	-150^{+50}_{-40}	$4.0^{+0.2}_{-0.2}$
UχPT($\sigma+\rho$)	440^{+10}_{-16}	-240^{+20}_{-20}	$2.97^{+0.02}_{-0.02}$	500^{+20}_{-20}	-160^{+15}_{-15}	$3.02^{+0.014}_{-0.04}$	600^{+30}_{-40}	-80^{+80}_{-20}	$3.9^{+0.5}_{-0.2}$
PDG[6]	$400-550$	$200-350$							

Table: σ resonance pole mass and pole residues comparison.

[10](Particle Data Group), Chin. Phys. C, 40, 100001 (2016) and 2017 update.



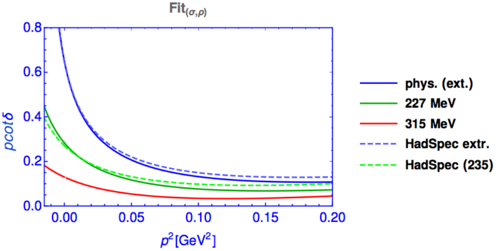


Figure: σ phase shift exptrapolation to physical point based on σ and ρ data. Comparison of the present result of the fit to the data with the fit from HadronSpectrum collaboration [9] [10] and their extrapolation to the physical pion masses.

[8] M. Döring, B. Hu, and M. Mai, [arXiv:1610.1007].

[9] R. A. Briceno, J. J. Dudek, R. G. Edwards, and D. J. Wilson, Phys. Rev. Lett. 118 (2017), no. 2022002, [arXiv:1607.0590].

Conclusions

- Determine ρ and σ resonance parameters at $m_\pi = 315\,\text{MeV}$ and $227\,\text{MeV}$ and extrapolate them to physical point using UχPT model.
- For the ρ resonance, the gap between the experimental value of the ρ mass and the extrapolated ρ mass to the physical point can be explained by the Kaon effects.
- The pole for the σ resonance extrapolated to physical point agrees well with estimates in the Particle Data Book. Our results are compatible with other lattice QCD calculations.

Outlook

- Explore new channels:
  - a_1: includes ρ-π and σ-π scattering
  - Δ: N-π scattering
- More sophisticated analysis for σ meson:
  - detect $f_0(980)$ without the strange quark?
  - improve the statistical errors to become competitive with other method!

Acknowledgement

- My advisor Prof. Alexandru for instructing my thesis.
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- Prof. Doering, Raquel, Maxim for their contribution to different parameterizations in this research and their suggestions and feedbacks for my thesis.
- The numerical work was carried out on the GWU Colonial One computer cluster and GWU IMPACT collaboration clusters.

References

- D. Guo, A. Alexandru, R. Molina and M. Doering, Rho resonance parameters from lattice QCD, Phys. Rev. D 94, 034501 (2016)
- D. Guo, A. Alexandru, R. Molina and M. Doering and M. Mai, Extraction of isoscalar π-π phase-shifts from lattice QCD. (in preparation)

Appendix: Symmetries on the lattice

On the lattice, the energy eigenstates $|n>$ of the system are computed in a given irrep of the lattice symmetry group.

$$\psi_n(R^{-1}x) = \psi_n(R^{-1}(x + \mathbf{n}L)); \qquad \left\langle \hat{O}_2(t)\hat{O}_1^{\dagger}(0) \right\rangle = \sum_n \left\langle 0|\hat{O}_2|n \right\rangle \left\langle n|\hat{O}_1|0 \right\rangle e^{-tE_n}$$

Isospin, color and flavor symmetries are similar to the continuum.

Table: Irreducible representation in $SO(3)$, O and D_4

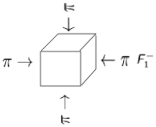
	$SO(3)$	cubic box(O_h)	elongated box(D_{4h})
irep label	$Y_{lm}; l = 0, 1...\infty$	A_1, A_2, E, F_1, F_2	A_1, A_2, E, B_1, B_2
dim	$1, 3, ..., 2l + 1, ...\infty$	$1, 1, 2, 3, 3$	$1, 1, 2, 2, 2$

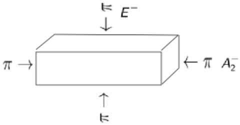
Table: Angular momentum mixing among the irreducible representations of the lattice group

O_h		D_{4h}	
irreducible representation	l	irreducible representation	l
A_1	0,4,6, ...	A_1	0,2,3, ...
A_2	3,6, ...	A_2	1,3,4, ...
F_1	1,3,4,5,6, ...	B_1	2,3,4, ...
F_2	2,3,4,5,6, ...	B_2	2,3,4, ...
E	2,4,5,6, ...	E	1,2,3,4, ...

Appendix: Symmetries of the elongated box

ρ resonance is in $I = 1, J^P = 1^-$ channel for pion-pion scattering. Elongated box method tunes the momentum of the scattering particles on the lattice $\mathbf{p} \propto (\frac{2\pi}{\eta L})$.



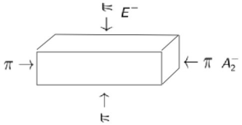


The $SO(3)$ symmetry group reduce to discrete subgroup O_h or D_{4h}

J	O_h	D_{4h}
0	A_1^+	A_1^+
1	F_1^-	$A_2^- \oplus E^-$
2	$E^+ \oplus F_2^+$	$A_1^+ \oplus B_1^+ \oplus B_2^+ \oplus E^+$
3	$A_2^- \oplus F_1^- \oplus F_2^-$	$A_2^- \oplus B_1^- \oplus B_2^- \oplus 2E^-$
4	$A_1^+ \oplus E^+ \oplus F_1^+ \oplus F_2^+$	$2A_1^+ \oplus A_2^+ \oplus B_1^+ \oplus B_2^+ \oplus 2E^+$

$I = 1, J^P = 1^-$ channel: F_1^- for O_h and A_2^- for D_{4h}. The energy contribution from angular momenta $l \geq 3$ is negligible.
$I = 0, J^P = 0^+$ channel: A_1^+. The energy contribution from angular momenta $l \geq 2$ is negligible.

For elongated box and boosted case along elongation direction:



$$\mathbf{P} \rightarrow$$

$$A_2^- : \cot\delta_1(k) \quad = \quad \mathcal{W}_{00} + \frac{2}{\sqrt{5}}\mathcal{W}_{20}$$

$$\mathcal{W}_{lm}(1, q^2, \eta) = \frac{\mathcal{Z}_{lm}^{\mathbf{P}}(1, q^2, \eta)}{\gamma\eta\pi^{\frac{3}{2}} q^{l+1}}; \quad \eta = \frac{N_{el}}{N} \text{ : elongation factor;} \quad \gamma \text{ : boost factor;}$$

$$\mathcal{Z}_{lm}^{\hat{\mathbf{P}}}(s; q^2, \eta) = \sum_{\mathbf{n}} \mathcal{Y}_{lm}(\tilde{\mathbf{n}})(\tilde{\mathbf{n}}^2 - q^2)^{-s}; \mathbf{n} \in \frac{1}{\gamma}(\mathbf{m} + \frac{\hat{\mathbf{P}}}{2});$$

Appnedix: Laplacian Heaviside smearing [11]

To estimate all-to-all propagators: 
The 3-dimensional gauge-covariant Laplacian operator

$$\tilde{\Delta}^{ab}(x,y;U) = \sum_{k=1}^{3} \left\{ \tilde{U}_k^{ab}(x)\delta(y,x+\hat{k}) + \tilde{U}_k^{ba}(y)^*\delta(y,x-\hat{k}) - 2\delta(x,y)\delta^{ab} \right\}.$$

$$S_\Lambda(t) = \sum_{\lambda(t)}^{\Lambda} |\lambda(t)\rangle \langle\lambda(t)|\,; \quad \tilde{u}(t) = S(t)u(t) = \sum_{\lambda_t} |\lambda_t\rangle \langle\lambda_t|\, u(t).$$

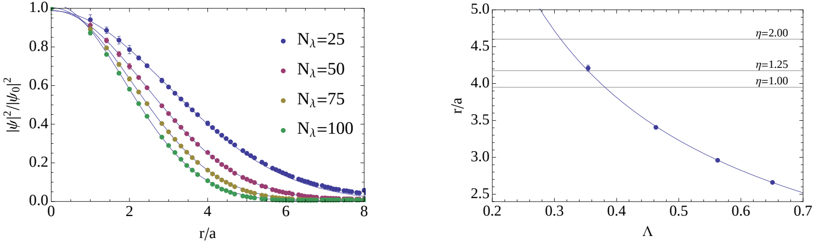


Figure: Smearing radius for pion mass $m_\pi = 315\,\mathrm{MeV}$ and lattice spacing $a \approx 0.121\,\mathrm{fm}$. We choose $N = 100$ which corresponds to a 0.5 fm smearing radius that is comparable to the radius of ρ and σ. $4 \times N_v \times N_t \ll 12 \times N_x \times N_y \times N_z$.

Appendix:conformal mappings

Free parameters: α, B_0, B_1 and B_2.

$$\tan\delta_{IL} = -\rho(s) K_{IL}(s); \quad \rho(s) = \sqrt{1 - 4M_\pi^2/s}\,,$$

$$K_{00}^{-1}(s) = \frac{M_\pi^2}{s - s_A}\left(\frac{2s_A}{M_\pi\sqrt{s}} + \sum_{i=1}^{3} B_i \omega^i(s)\right),$$

where

$$\omega(s) = \frac{\sqrt{s} - \alpha\sqrt{4M_\circ - s}}{\sqrt{s} + \alpha\sqrt{4M_\circ - s}}\,.$$

Appendix:UχPT model

We fix the $\pi\pi \to K\bar{K} and K\bar{K} \to K\bar{K}$transitions from a fit to the physical data, while keeping $l_{1,2}$ for the $\pi\pi$ transition at the values we got from fitting our data.

$$T_{IL}(s) = \frac{1}{V_{IL}^{-1}(s) - G(s)}\,, \tag{3}$$

where $V(s)$ is the chiral potential and $G(s)$ is the two-pion loop function defined via

$$G(s) = i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 - m_\pi^2 + i\epsilon} \frac{1}{(P-q)^2 - m_\pi^2 + i\epsilon}\,, \tag{4}$$

where P is the total four-momentum of the system. Note that this equation corresponds exactly to the parametrization Eq. 33 after re-shuffling the real part of the loop-function G into the K-Matrix and making the appropriate normalization. The chiral potential including contract terms up to next-to-leading order (NLO) reads for the two channels

$$V_{11}(s) = \frac{4m_\pi^2 - s}{6(f_\pi^2 - 8\,\hat{l}_1 m_\pi^2 + 4\,\hat{l}_2 s)}\,, \tag{5}$$

$$V_{00}(s) = \frac{3(m_\pi^2 - 2s)^2}{6f_\pi^2(m_\pi^2 - 2s) + 8(L_a m_\pi^4 + s(L_b m_\pi^2 + L_c s))}\,. \tag{6}$$