Resonance parameters from Lattice QCD

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Overview

Motivation

Why we study resonance from Lattice QCD?

- Lattice QCD offers us a way to study the resonances in terms of quark and gluon dynamics. It serves as a test of QCD for well determined resonance parameters.
- The techniques can be used to investigate systems where the experimental situation is less clear.
- Validate effective models used to describe hadron scattering.

Proposed work

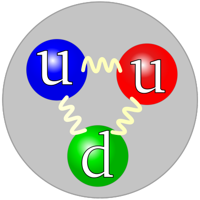
In particular

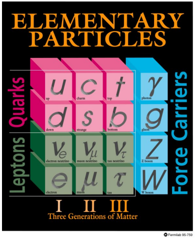
- $\rho(770)$ meson resonance in $I = 1$, $J = 1$ channel of pion-pion scattering
- $K^*(892)$ meson resonance in $I = \frac{1}{2}$, $J = 1$ channel of kaon-pion scattering
- $N(1440)$ baryon resonance

Lattice QCD meson observables have better signal-to-noise ratio than baryon's.

Quantum Chromodynamics

- Most visible matter in the universe are made up of particles called hadrons.





- The interaction between hadrons is dominated by the strong force.
- Quantum Chromodynamics (QCD) is a theory to describe the strong interaction between quarks and gluons which make up hadrons.

$$\mathcal{L}_{QCD} = -\frac{1}{2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} - \sum_f \bar{\psi}_f \gamma^\mu \left[\partial_\mu - igA_\mu\right]\psi_f - \sum_f m_f \bar{\psi}_f \psi_f\,, \qquad (1)$$

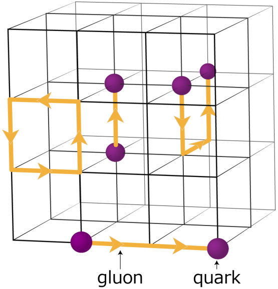
- Asymptotic freedom is an important feature of QCD. Quarks interact weakly at high energy allowing to use perturbative calculation.
- Some techniques to work with QCD: Perturbation theory, Effective field theory, Lattice QCD and so on.

Introduction to Lattice QCD

For light hadron study, non-perturbative approach is needed. Lattice QCD is a non-perturbative approach to QCD. It formulates QCD in a discrete way.

Inputs:

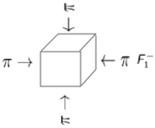
- lattice geometry N
- lattice spacing a set indirectly through the coupling constant g
- quark mass represented by pion mass m_π

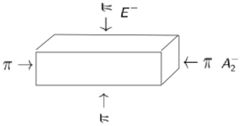


For light hadron study, only light quarks u and d are important. s quark introduces only small correction.
The role of Lattice QCD in resonance study is to extract the energy spectrum for two hadron states.

From energy spectrum to phase shift: Lüscher's formula

Phase shift for $l = 1$:





$$
\begin{aligned}
A_2^- : \cot\delta_1(k) &= \mathcal{W}_{00} + \frac{2}{\sqrt{5}}\mathcal{W}_{20} \qquad (2)\\
E^- : \cot\delta_1(k) &= \mathcal{W}_{00} - \frac{1}{\sqrt{5}}\mathcal{W}_{20} \qquad (3)
\end{aligned}
$$

$$\mathcal{W}_{lm}(1, q^2, \eta) = \frac{\mathcal{Z}_{lm}(1, q^2, \eta)}{\pi^{\frac{3}{2}}\eta q^{l+1}}; \quad q = \frac{kL}{2\pi}; \; \eta = \frac{N_{el}}{N} : \text{elongation factor} \qquad (4)$$

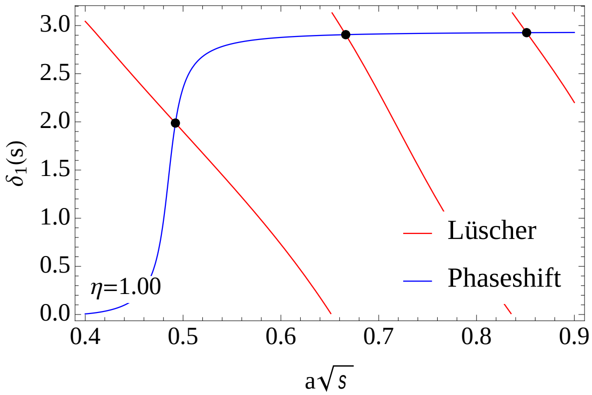
Zeta function

$$\mathcal{Z}_{lm}(s; q^2) = \sum_{\mathbf{n}\in L^3} \mathcal{Y}_{lm}(\mathbf{n})(\mathbf{n}^2 - q^2)^{-s}; \qquad (5)$$

Total energy

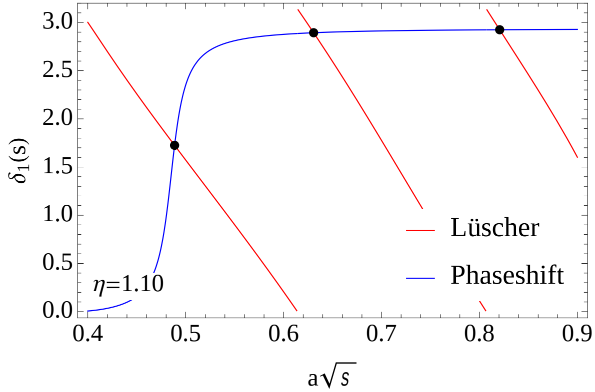
$$E = 2\sqrt{m^2 + k^2}; \quad k = \sqrt{\left(\frac{E}{2}\right)^2 - m^2} \qquad (6)$$

Lüscher's phase shift formula



$$A_2^- : \cot \delta_1(k) = \mathcal{W}_{00}(k,\eta) + \frac{2}{\sqrt{5}}\mathcal{W}_{20}(k,\eta) \tag{7}$$

Lüscher's phase shift formula



$$A_2^- : \cot \delta_1(k) = \mathcal{W}_{00}(k,\eta) + \frac{2}{\sqrt{5}} \mathcal{W}_{20}(k,\eta) \tag{8}$$

Hadron spectroscopy

Consider the two point correlation functions for two interpolating field

$$\left\langle \hat{O}_2(t)\hat{O}_1^\dagger(0)\right\rangle = \frac{1}{Z}\int D[\psi,\bar{\psi},U]e^{-S_{QCD}[\psi,\bar{\psi},U]}O_2[\psi_t,\bar{\psi}_t,U_t]O_1^\dagger[\psi_0,\bar{\psi}_0,U_0], \tag{9}$$

$$Z = \int D[\psi,\bar{\psi},U]e^{-S_{QCD}[\psi,\bar{\psi},U]}. \tag{10}$$

Operatorial view

$$\left\langle \hat{O}_2(t)\hat{O}_1^\dagger(0)\right\rangle = \lim_{T\to\infty}\frac{1}{Z_T}tr[e^{-(T-t)\hat{H}}\hat{O}_2 e^{-t\hat{H}}\hat{O}_1] = \sum_n \left\langle 0|\hat{O}_2|n\right\rangle\left\langle n|\hat{O}_1|0\right\rangle e^{-tE_n} \tag{11}$$

$$C(t)_{T\to\infty} = c_1 e^{-E_1 t}\left(1+O\left(e^{-\Delta E t}\right)\right) \tag{12}$$

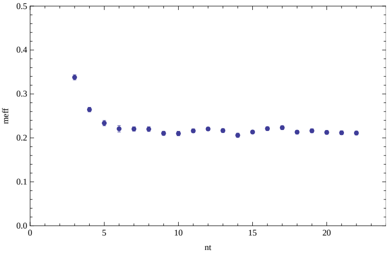
$$E(t) = \ln\frac{C(t+1)}{C(t)} \tag{13}$$

Hadronic spectroscopy

- Lattice QCD has determined the single particle spectrum of hadrons.

$$E(t) = \ln \frac{C(t+1)}{C(t)}$$

$$\pi(J^P = 0^-) : \bar{d}\gamma_5 u;$$



- The resonance is unstable from scattering experiment, we need to introduce the hadron-hadron interpolating field and include dynamic vacuum (contain sea quark creation and annihilation).

$$\rho(J^P = 1^-) : \pi \rightarrow\leftarrow \pi$$

Variational method

In principle, we can extract several energy levels from two point correlation function.

$$C(t)_{T\to\infty} = \sum_n \left\langle 0|\hat{O}_2|n\right\rangle \left\langle n|\hat{O}_1^\dagger|0\right\rangle e^{-E_n t} = c_1 e^{-E_1 t} + c_2 e^{-E_2 t} + c_3 e^{-E_3 t} + \ldots \quad (14)$$

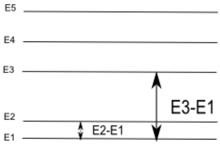
It doesn't work when two energy levels are close (eg. degenerated case)
Variational method is used to extract energy of the excited states.
Construct correlation matrix in the interpolator basis

$$C(t)_{ij} =< \mathcal{O}_i(t)\mathcal{O}_j^\dagger(0) >; i,j = 1,2,\ldots,\text{number of operators} \quad (15)$$

The eigenvalues of the correlation matrix are

$$\lambda^{(n)}(t,t_0) \propto e^{-E_n t}(1+\mathcal{O}(e^{-\Delta E_n t})), n = 1,2,\ldots,\text{number of operators} \quad (16)$$

where $\Delta E_n = E_{\text{Number of operators} + 1} - E_n$.



Larger energy gap makes the high lying energy decay faster and effective mass plateau appear in an earlier time slice.

ρ resonance study

Why do we study ρ resonance?

- Previous studies have large errorbars and discrepancies.
- Higher precision studies are needed to compute resonance parameters at different pion mass.

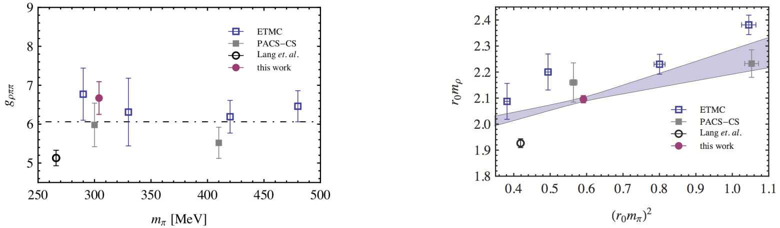


Figure: (Left) A comparison of $g_{\rho\pi\pi}$ for several previous studies [1].

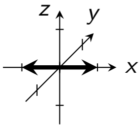
Interpolating field construction for ρ resonance

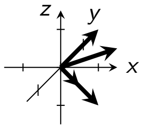
Four single-particle operator ρ and two scattering operators $\pi\pi$ in A_2^- sector.

$$\rho^J(t_f) = \bar{u}(t_f)\Gamma_{t_f} A_{t_f}(\mathbf{p})d(t_f); \quad \rho^{J\dagger}(t_i) = \bar{d}(t_i)\Gamma^\dagger_{t_i} A^\dagger_{t_i}(\mathbf{p})u(t_i) \tag{17}$$

N	Γ_{t_f}	A_{t_f}	$\Gamma^\dagger_{t_i}$	$A^\dagger_{t_i}$
1	γ_i	$e^{i\mathbf{p}}$	$-\gamma_i$	$e^{-i\mathbf{p}}$
2	$\gamma_4\gamma_i$	$e^{i\mathbf{p}}$	$\gamma_4\gamma_i$	$e^{-i\mathbf{p}}$
3	γ_i	$\nabla_j e^{i\mathbf{p}} \nabla_j$	γ_i	$\nabla_j^\dagger e^{-i\mathbf{p}} \nabla_j^\dagger$
4	$\frac{1}{2}$	$\{e^{i\mathbf{p}}, \nabla_i\}$	$-\frac{1}{2}$	$\{e^{-i\mathbf{p}}, \nabla_i\}$

$$(\pi\pi)_{\mathbf{P},\Lambda,\mu} = \sum_{\mathbf{p_1}^*,\mathbf{p_2}^*} C(\mathbf{P},\Lambda,\mu;\mathbf{p_1};\mathbf{p_2})\pi(\mathbf{p_1})\pi(\mathbf{p_2}), \tag{18}$$





$$\pi\pi_{100}(\mathbf{p_1},\mathbf{p_2},t) = \frac{1}{\sqrt{2}}[\pi^+(\mathbf{p_1})\pi^-(\mathbf{p_2}) - \pi^+(\mathbf{p_2})\pi^-(\mathbf{p_1})]; \quad \mathbf{p_1} = (1,0,0) \quad \mathbf{p_2} = (-1,0,0)$$

$$\pi\pi_{110} = \frac{1}{2}(\pi\pi(110) + \pi\pi(101) + \pi\pi(1-10) + \pi\pi(10-1))$$

6×6 correlation matrix

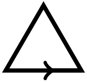
$$C = \begin{pmatrix} C_{\rho^J \leftarrow \rho^{J'}} & C_{\rho^J \leftarrow \pi\pi_{100}} & C_{\rho^J \leftarrow \pi\pi_{110}} \\ C_{\pi\pi_{100} \leftarrow \rho^{J'}} & C_{\pi\pi_{100} \leftarrow \pi\pi_{100}} & C_{\pi\pi_{100} \leftarrow \pi\pi_{110}} \\ C_{\pi\pi_{110} \leftarrow \rho^{J'}} & C_{\pi\pi_{110} \leftarrow \pi\pi_{100}} & C_{\pi\pi_{110} \leftarrow \pi\pi_{110}} \end{pmatrix} . \tag{19}$$

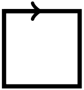
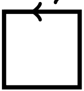
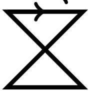
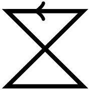
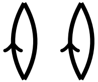
The correlation functions: $\bar{u}(t_i) \longrightarrow u(t_f)$

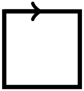
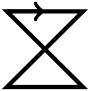
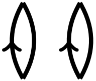
$\Gamma^J_{t_f}, (\mathbf{p}, t_f)$

$C_{\rho_i \leftarrow \rho_j} = -\Big\langle$  $\Big\rangle = -\left\langle \mathrm{Tr}[M^{-1}(t_i, t_f)\Gamma^J_{t_f} e^{i\mathbf{p}} M^{-1}(t_f, t_i)\Gamma^{J'\dagger}_{t_i} e^{-i\mathbf{p}}] \right\rangle .$ (20)

$\Gamma^{J'\dagger}_{t_i}, (-\mathbf{p}, t_i)$

$C_{\rho_i \leftarrow \pi\pi} = \Big\langle$  $-$  $\Big\rangle \overset{\mathbf{P}=0}{=} 2 \Big\langle$  $\Big\rangle .$ (21)

$C_{\pi\pi \leftarrow \pi\pi} = -\Big\langle$  $+$  $-$  $-$  $+$  $-$  $\Big\rangle$ (22)

$\overset{\mathbf{P}=0}{=} - \Big\langle 2$  -2  $+$  $-$  $\Big\rangle$ (23)

ρ energy spectrum

We implement the calculation in three ensembles ($\eta = 1.0, 1.25, 2.0$) at $m_\pi \approx 310\,\mathrm{MeV}$.

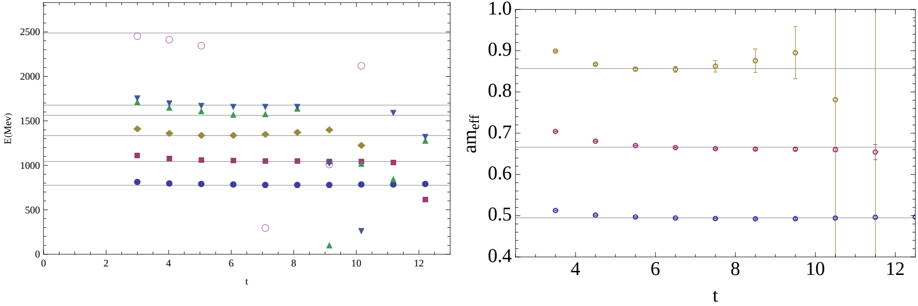
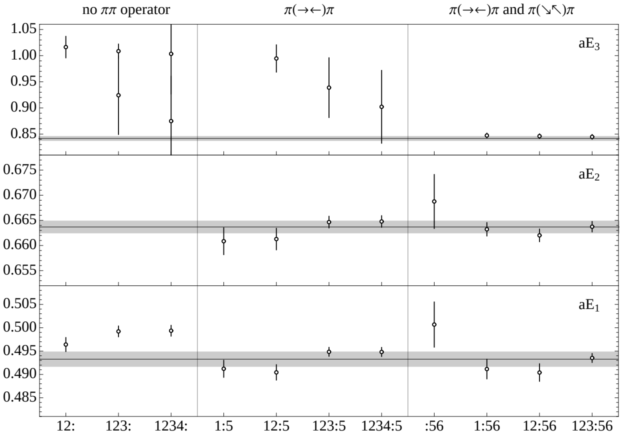


Figure: (Left) All 6 energy states (plateaus) vs t in $\eta = 1.0$ ensemble. (Right) A closer look into the first three energy states with their error bars

$$\lambda^{(n)}(t, t_0) \propto e^{-E_n t}(1 + \mathcal{O}(e^{-\Delta E_n t})); \qquad E_n = ln\frac{\lambda(t+1)}{\lambda(t)} \tag{24}$$

We extract energy E by using double exponential $f(t) = we^{-Et} + (1-w)e^{-E't}$ to do the χ^2 fitting for each eigenvalues.

Energy spectrum stability for interpolating field basis



$\mathcal{O}_i$	1	2	3	4	5	6
	ρ_1	ρ_2	ρ_3	ρ_4	$\pi\pi_{100}$	$\pi\pi_{110}$

(25)

Expectation for energy states

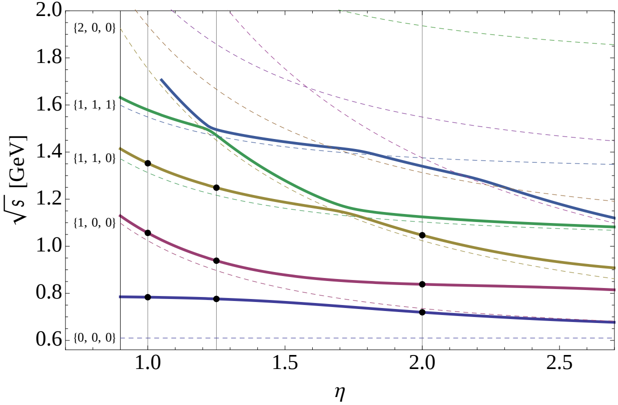
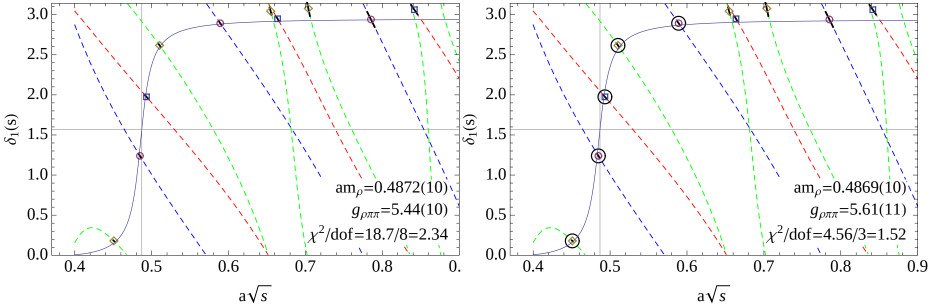


Figure: The lowest 3 energy states prediction from unitary χPT. When $\eta = 2.0$ the 3*rd* state is from operator $\pi\pi_{200}$ instead of $\pi\pi_{110}$

Extract parameters from phase shift

Figure: Phaseshift data from three ensembles fitted with Breit Wigner form (left) and only fit 5 data points in the resonance region .



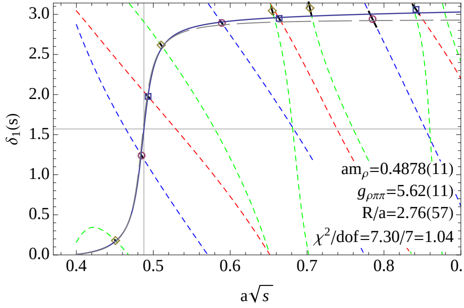
$$\cot(\delta_1(E)) = \frac{M_R^2 - E^2}{E\Gamma_r(E)} \text{where} \quad \Gamma_r(E) \equiv \frac{g_{R12}^2}{6\pi}\frac{p^3}{E^2}. \tag{26}$$

$$\delta_1(E) = \text{arccot}\,\frac{6\pi(M_R^2 - E^2)E}{g^2 p^3} \tag{27}$$

Centrifugal barrier term [2]

Based on the idea that resonance has finite spatial size, Γ_r is expected to damped faster than Breit Wigner form above the resonance region. Modify BW with a centrifugal barrier term.

$$\Gamma_r(E) = \frac{g^2}{6\pi}\frac{p^3}{E^2}\frac{1+(p_R R)^2}{1+(pR)^2}. \tag{28}$$



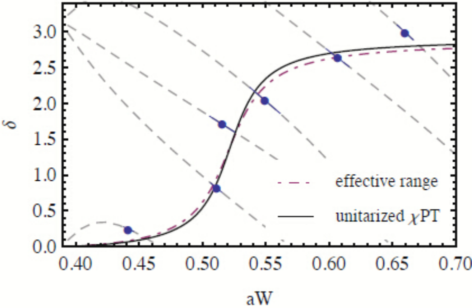
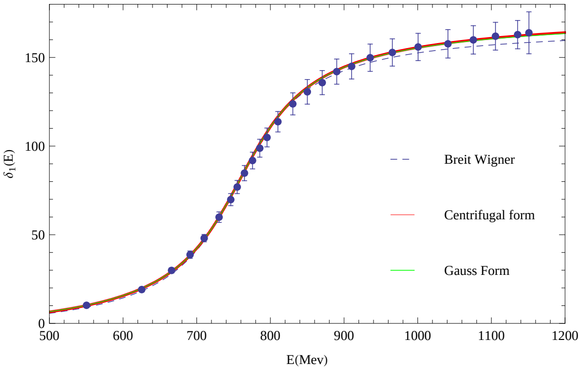


Figure: (left)Current study with LapH smearing vs (right) previous study with stochastic method [1]

Experiment data [3]



$$\Gamma_{BW}(E) = \frac{g^2}{6\pi}\frac{p^3}{E^2}$$

$$\Gamma_{CF}(E) = \frac{g^2}{6\pi}\frac{p^3}{E^2}\frac{1+(p_R R)^2}{1+(pR)^2}$$

$$\Gamma_{GA}(E) = \frac{g^2}{6\pi}\frac{p^3}{E^2}\frac{e^{-p^2/6\beta^2}}{e^{-p_R^2/6\beta^2}}$$

Figure: $\pi\pi$ phase shift below $K\bar{K}$ threshold in experiment

[3] Estabrooks, P. and Martin, Alan D. Nucl.Phys. B79 (1974) 301

m_ρ and $g_{\rho\pi\pi}$ comparison

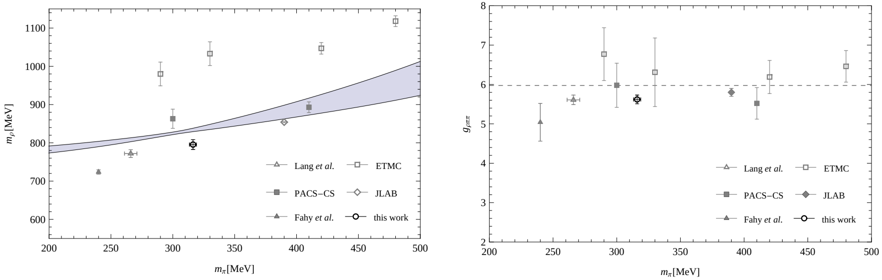
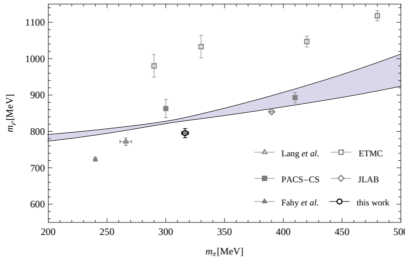
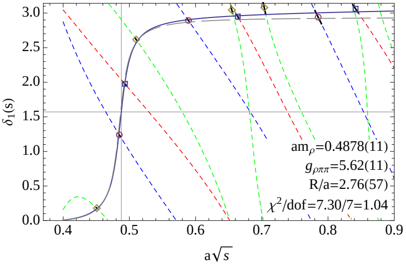


Figure: (left)ρ resonance mass and (right) $g_{\rho\pi\pi}$

Future plan

- Apply similar ρ resonance analysis (rest frame $\mathbf{P} = (0,0,0)$ to three lower pion mass $m_\pi \approx 220\,\text{MeV}$ ensembles with $nx = (24, 28, 32)$, $nt = 64$.

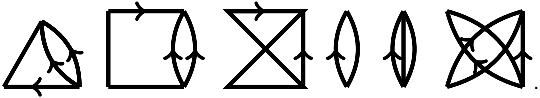




- Apply boost frame method $\mathbf{P} \neq 0$ for ρ resonance to all the ensembles to get more phase shift data below the resonance region.

Future plan

- To study K^* resonance with a narrow decay width 46 MeV, we need to apply the boost frame method to given lattice ensembles. Compute the strange quark propagator $\bar{s}_{(t_i)} \longrightarrow s_{(t_f)}$ and additional quark diagrams due to symmetry reduced.
- Study the N resonance which has more complexity in computing the quark diagrams in the correlation function such as



Timeline

- Complete the computation of LapH quark propagators for three lower pion mass ($m_\pi \approx 230\,\mathrm{MeV}$) ensembles ($n_x = 24, 28, 32$; $n_y = n_z = 24$; $n_t = 64$) and finish the rest frame ($\mathbf{P} = 0$) analysis by June 2015.
- Extend Lüscher's phase shift formula to boosted frame and finish computing the boosted ($\mathbf{P} \neq \mathbf{0}$) energy spectrum for all ensembles (310 MeV and 230 MeV) by September 2015.
- Write a paper summarizing the study of ρ resonance by September 2015.
- Compute LapH quark propagators for the strange valence quark for all ensembles by November 2015.
- Compute the correlation functions of the nucleon-pion system for all ensembles and analyze the energy spectrum by Spring 2016.
- Finish computing the correlation functions of K^* resonance and obtain the phase-shift pattern and its resonance parameters by December 2016.
- Write thesis and defend by Summer 2017.

Appendix-A:Symmetry on the lattice

The eigenstates $|n>$ are computed in a given irrep of the lattice symmetry group.

$$\psi_n(R^{-1}x) = \psi_n(R^{-1}(x + \mathbf{n}L)); \qquad \left\langle \hat{O}_2(t)\hat{O}_1^\dagger(0) \right\rangle = \sum_n \left\langle 0|\hat{O}_2|n \right\rangle \left\langle n|\hat{O}_1|0 \right\rangle e^{-tE_n} \quad (29)$$

Symmetries: Isospin, flavor, translation, rotation, inversion, etc.

Table: Irreducible representation in $SO(3), O$ and D_4

	$SO(3)$	cubic box(O_h)	elongated box(D_{4h})
irep label	$Y_{lm}; l = 0, 1 ... \infty$	A_1, A_2, E, F_1, F_2	A_1, A_2, E, B_1, B_2
dim	$1, 3, ..., 2l + 1, ...\infty$	$1, 1, 2, 3, 3$	$1, 1, 2, 2, 2$

Table: Angular momentum mixing among the irreducible representations of the lattice group

O_h		D_{4h}	
irreducible representation	l	irreducible representation	l
A_1	0,4,6, ...	A_1	0,2,3, ...
A_2	3,6, ...	A_2	1,3,4, ...
F_1	1,3,4,5,6, ...	B_1	2,3,4, ...
F_2	2,3,4,5,6, ...	B_2	2,3,4, ...
E	2,4,5,6, ...	E	1,2,3,4, ...

Appendix-A:Symmetry on the lattice

The $SO(3)$ symmetry group reduce to discrete subgroup O_h or D_{4h}

Table: Resolution of $2J+1$ spherical harmonics into the irreducible representations of O_h and D_{4h}

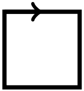
J	O_h	D_{4h}
0	A_1^+	A_1^+
1	F_1^-	$A_2^- \oplus E^-$
2	$E^+ \oplus F_2^+$	$A_1^+ \oplus B_1^+ \oplus B_2^+ \oplus E^+$
3	$A_2^- \oplus F_1^- \oplus F_2^-$	$A_2^- \oplus B_1^- \oplus B_2^- \oplus 2E^-$
4	$A_1^+ \oplus E^+ \oplus F_1^+ \oplus F_2^+$	$2A_1^+ \oplus A_2^+ \oplus B_1^+ \oplus B_2^+ \oplus 2E^+$

Assume that the energy contribution from angular momenta $l \geq 3$ is negligible. For example, if we study the p-wave($l=1$) scattering channel, we should construct the interpolating field in F_1^- in the O_h group, A_2^- and E^- representations in D_{4h} group.

correlation function

The number of propagators need to be computed

N^4



N^8

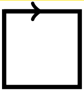
Method for evaluating all-to-all propagators:
stochastic method

- introduce stochastic noise
- repeat the propagator calculation

LapH smearing method

- compute propagators in momentum space with certain momentum cut-off
- no need to repeat propagator calculation

Appendix-B:Laplacian Heaviside smearing [4]

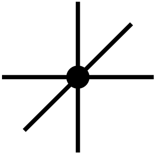



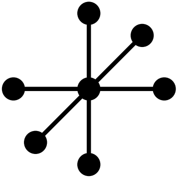
To estimate multi-hadron needs all-to-all propagator:
stochastic method, LapH smearing method
The 3-dimensional gauge-covariant Laplacian operator

$$\tilde{\Delta}^{ab}(x,y;U)=\sum_{k=1}^{3}\left\{\tilde{U}_k^{ab}(x)\delta(y,x+\hat{k})+\tilde{U}_k^{ba}(y)^*\delta(y,x-\hat{k})-2\delta(x,y)\delta^{ab}\right\}. \quad (30)$$

Definition of quark smearing operator (only acts spatial and color components)

$$S_\Lambda(t)=\sum_{\lambda(t)}^{\Lambda}|\lambda(t)\rangle\langle\lambda(t)|; \quad \tilde{u}(t)=S(t)u(t)=\sum_{\lambda_t}|\lambda_t\rangle\langle\lambda_t|\,u(t). \quad (31)$$


$u(t)$


$\tilde{u}(t)$

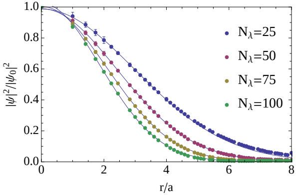


Figure: point source smeared with energy cutoff

LapH smearing

Benefit from LapH smearing:

- Keep low frequency mode up to Λ cutoff to compute the all to all propagators, $u(x) \longrightarrow u(y)$. The number of propagators $M^{-1}(t_f, t_i)$ need to be computed reduce from 6.34×10^{13} in position space to 3.7×10^{8} in momentum space for the $24^3 48$ ensemble.
- The effective mass reach a plateau in an earlier time slice.

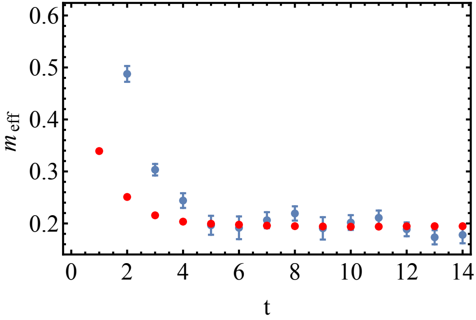


Figure: pion effective mass with (red) and without LapH smearing (blue)

Appendix-c: Fitting phase-shift

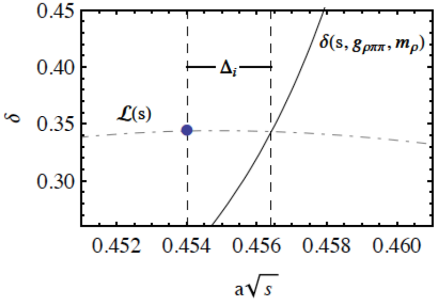


Figure: χ^2 fitting for the phase shift data to Breit Wigner form

$$\chi^2 = \Delta^T COV^{-1} \Delta \tag{32}$$

where

$$\Delta_i = \sqrt{s_i^{\text{curve}}} - \sqrt{s_i^{\text{data}}} \tag{33}$$

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C. Morningstar, J. Bulava, J. Foley, K. J. Juge, D. Lenkner, et al., *Improved stochastic estimation of quark propagation with Laplacian Heaviside smearing in lattice QCD*, *Phys.Rev.* **D83** (2011) 114505, [arXiv:1104.3870].