High precision study for rho-meson resonance from Lattice QCD

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Introduction

Why we study resonance from Lattice QCD?

- Lattice QCD offers us a way to study the resonances in terms of quark and gluon dynamics. It serves as a test of QCD for well determined resonance parameters.
- The techniques can be used to investigate systems where the experimental situation is less clear.
- Validate effective models used to describe hadron scattering.

How?

- We start from meson resonance because they have better signal-to-noise ratio.
- $\rho(770)$ resonance in $I = 1, J = 1$ π-π scattering channel.

Symmetries on the lattice

On the lattice, the energy eigenstates $|n>$ of the system are computed in a given irrep of the lattice symmetry group.

$$\psi_n(R^{-1}x) = \psi_n(R^{-1}(x + \mathbf{n}L)); \qquad \left\langle \hat{O}_2(t)\hat{O}_1^\dagger(0) \right\rangle = \sum_n \left\langle 0|\hat{O}_2|n \right\rangle \left\langle n|\hat{O}_1|0 \right\rangle e^{-tE_n} \quad (1)$$

Isospin, color and flavor symmetries are similar to the continuum.

Table : Irreducible representation in $SO(3), O$ and D_4

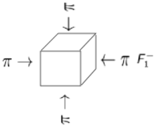
	$SO(3)$	cubic box(O_h)	elongated box(D_{4h})
irep label	$Y_{lm}; l = 0, 1...\infty$	A_1, A_2, E, F_1, F_2	A_1, A_2, E, B_1, B_2
dim	$1, 3, ..., 2l + 1, ...\infty$	$1, 1, 2, 3, 3$	$1, 1, 2, 2, 2$

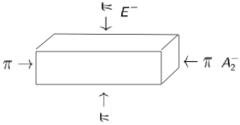
Table : Angular momentum mixing among the irreducible representations of the lattice group

O_h		D_{4h}	
irreducible representation	l	irreducible representation	l
A_1	0,4,6, ...	A_1	0,2,3, ...
A_2	3,6, ...	A_2	1,3,4, ...
F_1	1,3,4,5,6, ...	B_1	2,3,4, ...
F_2	2,3,4,5,6, ...	B_2	2,3,4, ...
E	2,4,5,6, ...	E	1,2,3,4, ...

Symmetries of the elongated box

ρ resonance is in $I = 1, J^P = 1^-$ channel for pion-pion scattering. Elongated box method tunes the momentum of the scattering particles on the lattice $\mathbf{p} \propto (\frac{2\pi}{\eta L})$.





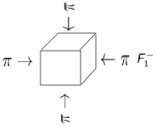
The $SO(3)$ symmetry group reduce to discrete subgroup O_h or D_{4h}

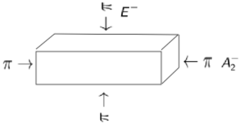
J	O_h	D_{4h}
0	A_1^+	A_1^+
1	F_1^-	$A_2^- \oplus E^-$
2	$E^+ \oplus F_2^+$	$A_1^+ \oplus B_1^+ \oplus B_2^+ \oplus E^+$
3	$A_2^- \oplus F_1^- \oplus F_2^-$	$A_2^- \oplus B_1^- \oplus B_2^- \oplus 2E^-$
4	$A_1^+ \oplus E^+ \oplus F_1^+ \oplus F_2^+$	$2A_1^+ \oplus A_2^+ \oplus B_1^+ \oplus B_2^+ \oplus 2E^+$

For the p-wave($l = 1$) scattering channel, we only need to construct the interpolating fields in F_1^- in the O_h group, A_2^- representations in D_{4h} group because the energy contribution from angular momenta $l \geq 3$ is negligible.

Lüscher's formula for elongated box [1]

Phase shift for $I = 1$, rest frame ($\mathbf{P} = 0$):





$$A_2^- : \cot\delta_1(k) \quad = \quad \mathcal{W}_{00} + \frac{2}{\sqrt{5}}\mathcal{W}_{20} \tag{2}$$

$$\tag{3}$$

$$\mathcal{W}_{lm}(1, q^2, \eta) = \frac{\mathcal{Z}_{lm}(1, q^2, \eta)}{\eta\pi^{\frac{3}{2}} q^{l+1}}; \quad q = \frac{kL}{2\pi}; \ \eta = \frac{N_{el}}{N} : \text{elongation factor} \tag{4}$$

Zeta function

$$\mathcal{Z}_{lm}(s; q^2, \eta) = \sum_{\tilde{\mathbf{n}}} \mathcal{Y}_{lm}(\tilde{\mathbf{n}})(\mathbf{n}^2 - q^2)^{-s}; \ \mathbf{n} \in \mathbf{m} \tag{5}$$

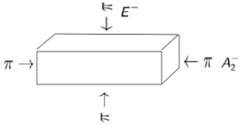
Total energy

$$E = 2\sqrt{m^2 + k^2}; \quad k = \sqrt{\left(\frac{E}{2}\right)^2 - m^2} \tag{6}$$

[1] X. Feng, X. Li, and C. Liu, Phys.Rev. D70 (2004) 014505

Lüscher's formula for boost frame

In order to obtain new kinematic region, we boost the resonance along the elongated direction.



$\mathbf{P} \rightarrow$

$$A_2^- : \cot \delta_1(k) \quad = \quad \mathcal{W}_{00} + \frac{2}{\sqrt{5}} \mathcal{W}_{20} \tag{7}$$

$$\tag{8}$$

$$\mathcal{W}_{lm}(1, q^2, \eta) = \frac{\mathcal{Z}_{lm}^{\mathbf{P}}(1, q^2, \eta)}{\gamma \eta \pi^{\frac{3}{2}} q^{l+1}}; \quad \eta = \frac{N_{el}}{N} : \text{elongation factor}; \ \ \gamma : \text{boost factor}; \tag{9}$$

$$\mathcal{Z}_{lm}^{\hat{\mathbf{P}}}(s; q^2, \eta) = \sum_{\mathbf{n}} \mathcal{Y}_{lm}(\tilde{\mathbf{n}})(\tilde{\mathbf{n}}^2 - q^2)^{-s}; \mathbf{n} \in \frac{1}{\gamma}(\mathbf{m} + \frac{\hat{\mathbf{P}}}{2}); \tag{10}$$

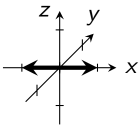
Interpolating field construction for ρ resonance

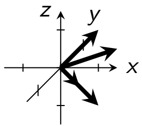
Four $q\bar{q}$ operators and two scattering operators $\pi\pi$ in A_2^- sector.

$$\rho^J(t_f) = \bar{u}(t_f)\Gamma_{t_f} A_{t_f}(\mathbf{p}) d(t_f); \quad \rho^{J\dagger}(t_i) = \bar{d}(t_i)\Gamma_{t_i}^\dagger A_{t_i}^\dagger(\mathbf{p}) u(t_i) \tag{11}$$

N	Γ_{t_f}	A_{t_f}	$\Gamma_{t_i}^\dagger$	$A_{t_i}^\dagger$
1	γ_i	$e^{i\mathbf{p}}$	$-\gamma_i$	$e^{-i\mathbf{p}}$
2	$\gamma_4\gamma_i$	$e^{i\mathbf{p}}$	$\gamma_4\gamma_i$	$e^{-i\mathbf{p}}$
3	γ_i	$\nabla_j e^{i\mathbf{p}} \nabla_j$	γ_i	$\nabla_j^\dagger e^{-i\mathbf{p}} \nabla_j^\dagger$
4	$\frac{1}{2}$	$\{e^{i\mathbf{p}}, \nabla_i\}$	$-\frac{1}{2}$	$\{e^{-i\mathbf{p}}, \nabla_i\}$

$$(\pi\pi)_{\mathbf{P},\Lambda,\mu} = \sum_{\mathbf{p_1}^*,\mathbf{p_2}^*} C(\mathbf{P},\Lambda,\mu;\mathbf{p_1};\mathbf{p_2})\pi(\mathbf{p_1})\pi(\mathbf{p_2}), \tag{12}$$





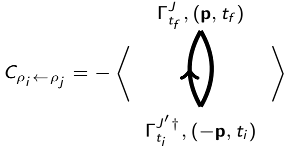
$$\pi\pi_{100}(\mathbf{p_1},\mathbf{p_2},t) = \frac{1}{\sqrt{2}}[\pi^+(\mathbf{p_1})\pi^-(\mathbf{p_2}) - \pi^+(\mathbf{p_2})\pi^-(\mathbf{p_1})]; \quad \mathbf{p_1} = (1,0,0) \quad \mathbf{p_2} = (-1,0,0)$$

$$\pi\pi_{110} = \frac{1}{2}(\pi\pi(110) + \pi\pi(101) + \pi\pi(1-10) + \pi\pi(10-1))$$

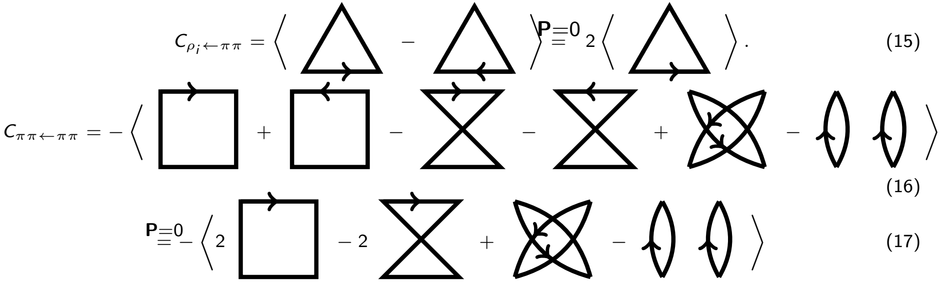
6×6 correlation matrix

$$C = \begin{pmatrix} C_{\rho^J \leftarrow \rho^{J'}} & C_{\rho^J \leftarrow \pi\pi_{100}} & C_{\rho^J \leftarrow \pi\pi_{110}} \\ C_{\pi\pi_{100} \leftarrow \rho^{J'}} & C_{\pi\pi_{100} \leftarrow \pi\pi_{100}} & C_{\pi\pi_{100} \leftarrow \pi\pi_{110}} \\ C_{\pi\pi_{110} \leftarrow \rho^{J'}} & C_{\pi\pi_{110} \leftarrow \pi\pi_{100}} & C_{\pi\pi_{110} \leftarrow \pi\pi_{110}} \end{pmatrix} . \tag{13}$$

The correlation functions: $\bar{u}(t_i) \longrightarrow u(t_f)$

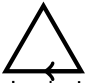


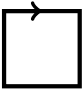
$$= - \left\langle \mathrm{Tr}[M^{-1}(t_i, t_f)\Gamma^J_{t_f} e^{i\mathbf{p}} M^{-1}(t_f, t_i)\Gamma^{J'\dagger}_{t_i} e^{-i\mathbf{p}}] \right\rangle . \tag{14}$$



Laplacian Heaviside smearing [2]

To estimate all-to-all propagators:

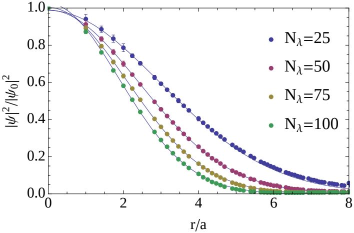


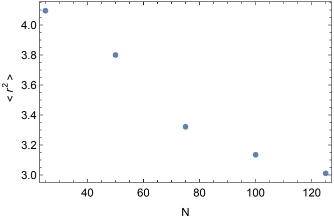


The 3-dimensional gauge-covariant Laplacian operator

$$\tilde{\Delta}^{ab}(x,y;U) = \sum_{k=1}^{3} \left\{ \tilde{U}_k^{ab}(x)\delta(y,x+\hat{k}) + \tilde{U}_k^{ba}(y)^*\delta(y,x-\hat{k}) - 2\delta(x,y)\delta^{ab} \right\}. \tag{18}$$

$$S_\Lambda(t) = \sum_{\lambda(t)}^{\Lambda} |\lambda(t)\rangle \langle\lambda(t)| \, ; \quad \tilde{u}(t) = S(t)u(t) = \sum_{\lambda_t} |\lambda_t\rangle \langle\lambda_t| \, u(t). \tag{19}$$





[2] C. Morningstar, J. Bulava, J. Foley, K. J. Juge, D. Lenkner, et al., Phys.Rev. D83 (2011) 114505

ρ energy spectrum

We implement the calculation in 3 ensembles ($\eta = 1.0, 1.25, 2.0$) at $m_\pi \approx 310\,\text{MeV}$ and 3 ensembles ($\eta = 1.0, 1.17, 1.33$) at $m_\pi \approx 227\,\text{MeV}$ with nHYP-smeared clover fermions and two mass-degenerated quark flavor.

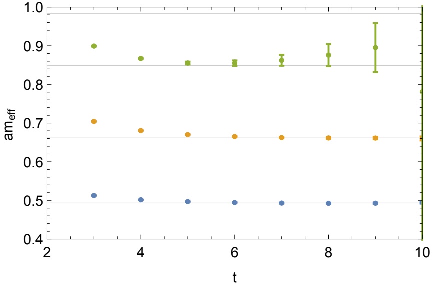
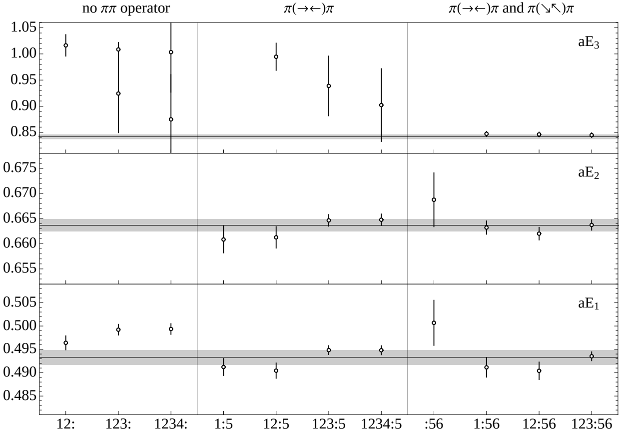


Figure : The lowest three energy states with their error bars for $\eta = 1.0$, $m_\pi = 310\,\text{MeV}$ ensemble

We extract energy E by using double exponential $f(t) = we^{-Et} + (1-w)e^{-E't}$ to do the χ^2 fitting for each eigenvalues.

Energy spectrum



$\mathcal{O}_i$	1	2	3	4	5	6
	ρ_1	ρ_2	ρ_3	ρ_4	$\pi\pi_{100}$	$\pi\pi_{110}$

(20)

Expectation for energy states

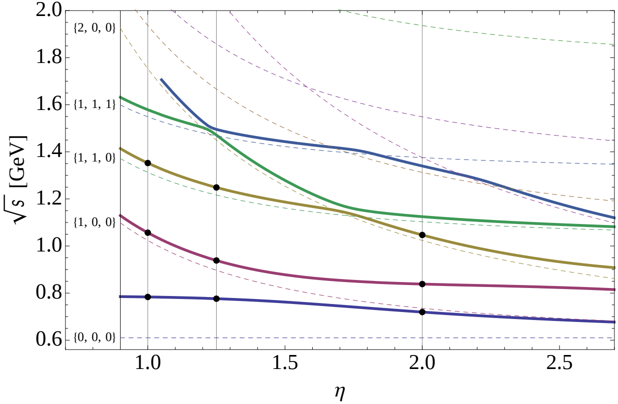


Figure : The lowest 3 energy states prediction from unitary χPT. When $\eta = 2.0$ the 3*rd* state is from operator $\pi\pi_{200}$ instead of $\pi\pi_{110}$

Phase shifts and resonance parameters

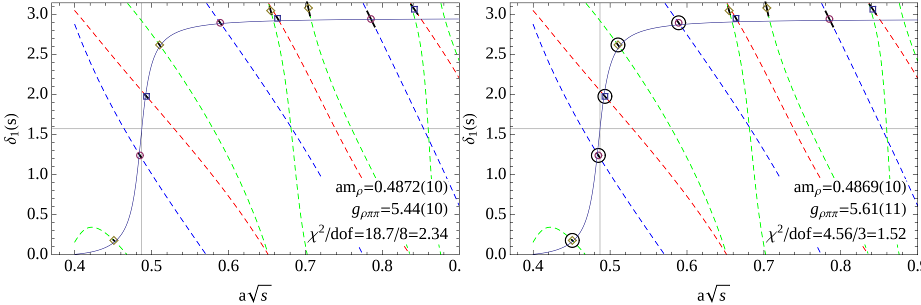


Figure : Phaseshift data from three ensembles fitted with Breit Wigner form (left) and only fit 5 data points in the resonance region .

$$\cot(\delta_1(E)) = \frac{M_R^2 - E^2}{E\Gamma_r(E)} \text{where} \quad \Gamma_r(E) \equiv \frac{g_{R12}^2}{6\pi}\frac{p^3}{E^2}. \tag{21}$$

$$\delta_1(E) = \text{arccot}\,\frac{6\pi(M_R^2 - E^2)E}{g^2p^3} \tag{22}$$

Centrifugal barrier term [3]

Based on the idea that resonance has finite spatial size, Γ_r is expected to damped faster than Breit Wigner form above the resonance region. Modify BW with a centrifugal barrier term.

$$\Gamma_r(E) = \frac{g^2}{6\pi}\frac{p^3}{E^2}\frac{1+(p_R R)^2}{1+(pR)^2}. \tag{23}$$

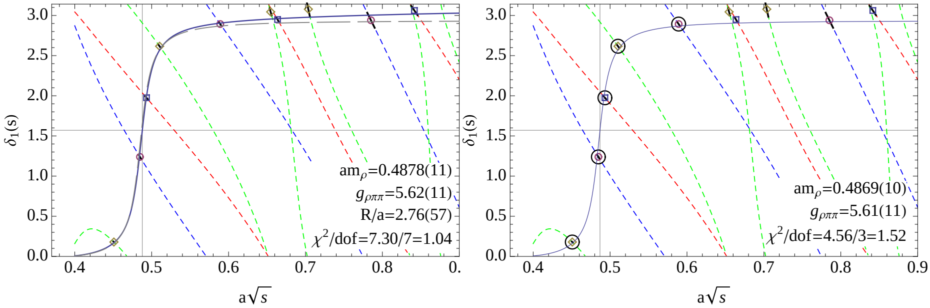
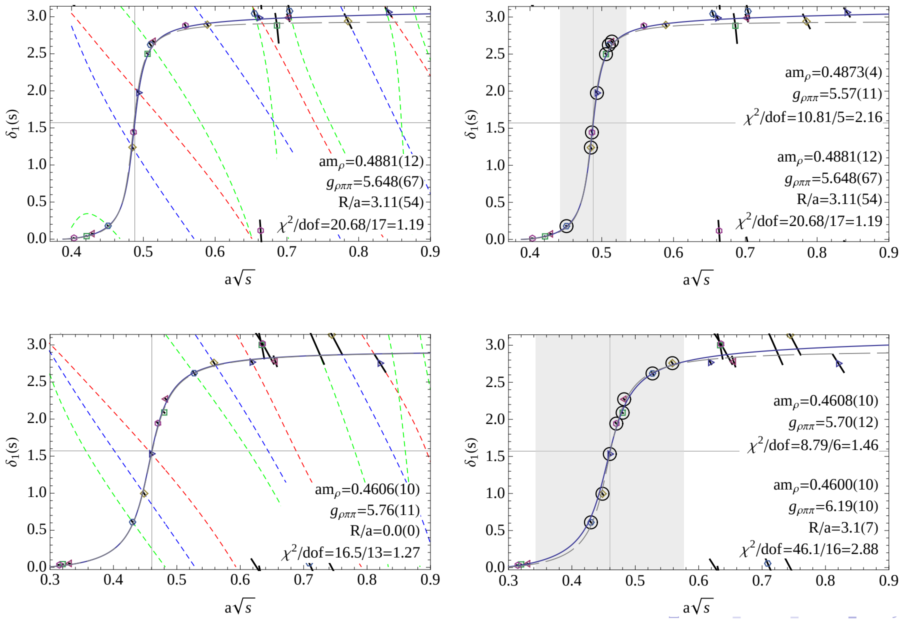


Figure : (left)Current study with LapH smearing vs (right) fitting only the resonance region

[3] F. Von Hippel and C. Quigg, Phys.Rev. D5 (1972) 624-638.

Fit in $m_\rho \pm 2\Gamma$ with BW



Fit with other models

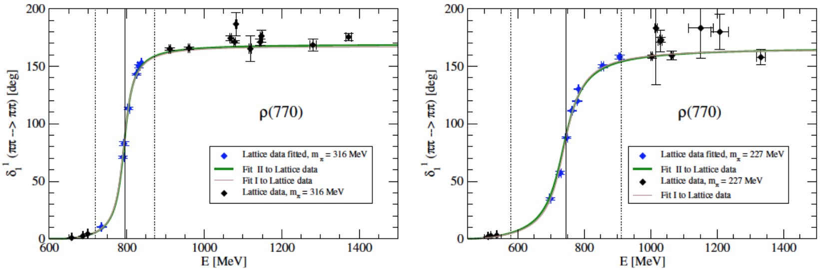
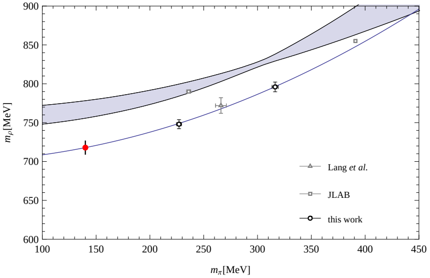


Figure : Two pion mass lattice data fitted with UχPT model in $m_\rho \pm 2\Gamma$ region.

	m_π	m_ρ	Γ_ρ	g	χ^2/dof
Breit Wigner	315	794.6(6)	37.0(2)	5.57(11)	2.16
UχPT		795.2(3)	36.1(1)		1.26
Breit Wigner	227	748.4(1.6)	71.0(8)	5.70(12)	1.46
UχPT		748.2(7)	77.0(5)		1.53

m_ρ and $g_{\rho\pi\pi}$ comparison



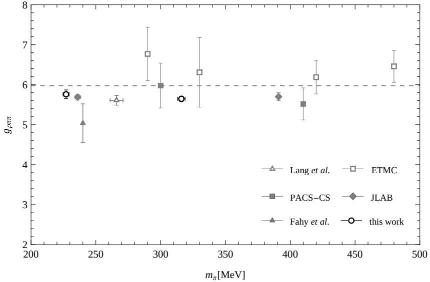


Figure : m_π^2 extrapolation to physical pion mass

The extrapolation value of m_ρ at physical pion mass is 715(7) MeV $< m_\rho = 775$ MeV.

$K\bar{K}$ channel contribution

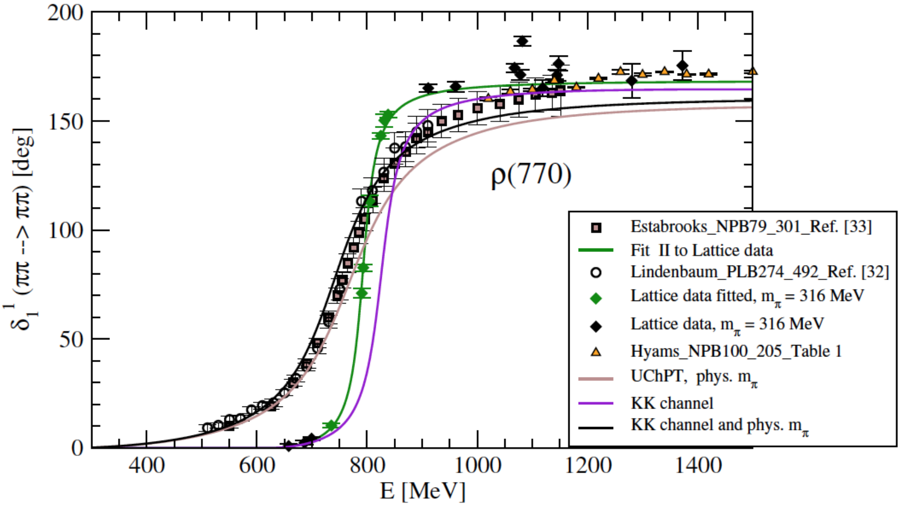


Figure : $m_\pi = 315\,\mathrm{MeV}$ data fitted with Unitary χPT model and their $K\bar{K}$ channel corrections

$K\bar{K}$ channel contribution

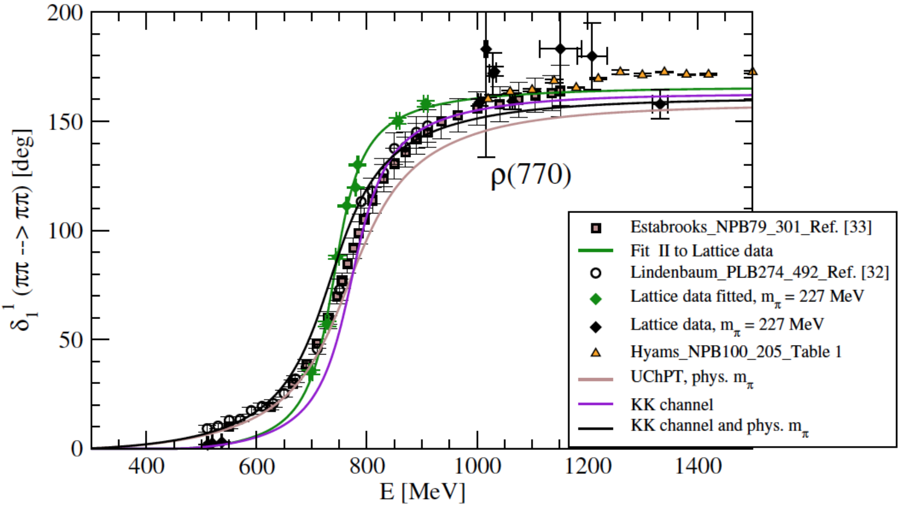


Figure : $m_\pi = 227\,\text{MeV}$ data fitted with Unitary χPT model and their $K\bar{K}$ channel corrections

$K\bar{K}$ channel contribution

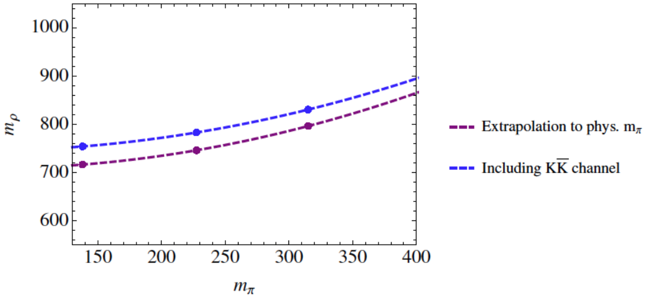


Figure : The $K\bar{K}$ channel correction vs pion mass from a global fit for two pion mass.

m_π	m_ρ	Γ_ρ	m'_ρ	Γ'_ρ	χ^2/dof
315	795.5(9)	36.5(3)	830.6(9)	51.4(3)	1.26
227	747.5(9)	77.4(6)	780.2(5)	97.8(3)	
138	719.7(1.6)	120.4(1.0)	754.6(0.5)	150.7(3)	

Global fit with UχPT for both pion mass. The definition of m_ρ is the value energy when the phase shift has $\pi/2$. The PDG value of ρ meson resonance parameter are $m_\rho = 775.49$ MeV and $\Gamma_\rho = 149.1$ MeV.

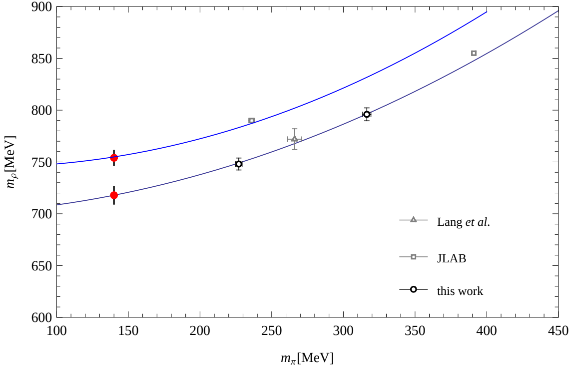


Figure : Result with $K\bar{K}$ channel

Conclusions

- We complete a precision study of ρ resonance with LapH smearing method and obtain the resonance parameters at $m_\pi \approx 310\,\text{MeV}$ and $m_\pi \approx 227\,\text{MeV}$.
- For precise energy results, the Breit Wigner form only works in the resonance region. Modification to the BW is needed when applied to a wider energy region.
- The extrapolation of m_ρ to physical pion mass is smaller than $m_\rho^{\text{phy}} = 775\,\text{MeV}$ in a $N_f = 2$ situation, we believe that this comes from the absence of strange quark and the $K\bar{K}$ channel which is supported by the unitary χPT study.

- X. Feng, X. Li, and C. Liu, *Two particle states in an asymmetric box and the elastic scattering phases*, *Phys.Rev.* **D70** (2004) 014505, [hep-lat/0404001].
- C. Morningstar, J. Bulava, J. Foley, K. J. Juge, D. Lenkner, et al., *Improved stochastic estimation of quark propagation with Laplacian Heaviside smearing in lattice QCD*, *Phys.Rev.* **D83** (2011) 114505, [arXiv:1104.3870].
- F. Von Hippel and C. Quigg, *Centrifugal-barrier effects in resonance partial decay widths, shapes, and production amplitudes*, *Phys.Rev.* **D5** (1972) 624–638.
- M. Luscher and U. Wolff, *How to calculate the elastic scattering matrix in two-dimensional quantum field theories by numerical simulation*, *Nucl.Phys.* **B339** (1990) 222–252.
- P. Estabrooks and A. D. Martin, *pi pi Phase Shift Analysis Below the K anti-K Threshold*, *Nucl.Phys.* **B79** (1974) 301.

Symmetries on the lattice

The $SO(3)$ symmetry group reduce to discrete subgroup O_h or D_{4h}

Table : Resolution of $2J+1$ spherical harmonics into the irreducible representations of O_h and D_{4h}

J	O_h	D_{4h}
0	A_1^+	A_1^+
1	F_1^-	$A_2^- \oplus E^-$
2	$E^+ \oplus F_2^+$	$A_1^+ \oplus B_1^+ \oplus B_2^+ \oplus E^+$
3	$A_2^- \oplus F_1^- \oplus F_2^-$	$A_2^- \oplus B_1^- \oplus B_2^- \oplus 2E^-$
4	$A_1^+ \oplus E^+ \oplus F_1^+ \oplus F_2^+$	$2A_1^+ \oplus A_2^+ \oplus B_1^+ \oplus B_2^+ \oplus 2E^+$

Assume that the energy contribution from angular momenta $l \geq 3$ is negligible. For example, if we study the p-wave($l=1$) scattering channel, we should construct the interpolating field in F_1^- in the O_h group, A_2^- and E^- representations in D_{4h} group.

Variational method [4]

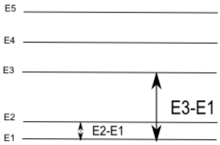
Variational method is used to extract energy of the excited states.
Construct correlation matrix in the interpolator basis

$$C(t)_{ij} = < \mathcal{O}_i(t)\mathcal{O}_j^\dagger(0) >; i,j = 1,2,...,\text{number of operators} \tag{24}$$

The eigenvalues of the correlation matrix are

$$\lambda^{(n)}(t,t_0) \propto e^{-E_n t}(1 + \mathcal{O}(e^{-\Delta E_n t})), n = 1,2,...,\text{number of operators} \tag{25}$$

where $\Delta E_n = E_{\text{Number of operators}+1} - E_n$.



Larger energy gap makes the high lying energy decay faster and effective mass plateau appear in an earlier time slice.

Appendix-B: LapH smearing

Benefit from LapH smearing:

- Keep low frequency mode up to Λ cutoff to compute the all to all propagators, $u(x) \longrightarrow u(y)$. The number of propagators $M^{-1}(t_f, t_i)$ need to be computed reduce from 6.34×10^{13} in position space to 3.7×10^{8} in momentum space for the $24^3 48$ ensemble.
- The effective mass reach a plateau in an earlier time slice.

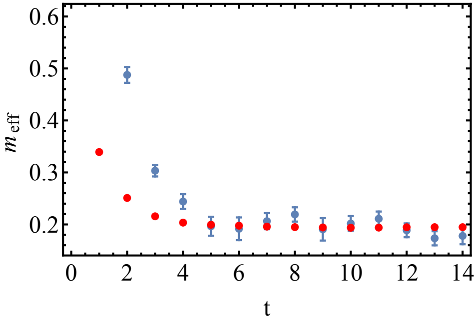


Figure : pion effective mass with (red) and without LapH smearing (blue)

Appendix-C: Fitting phase-shift

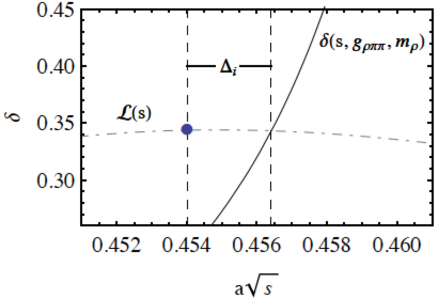


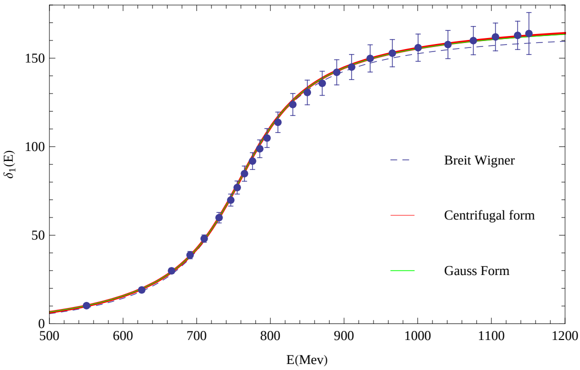
Figure : χ^2 fitting for the phase shift data to Breit Wigner form

$$\chi^2 = \Delta^T COV^{-1} \Delta \tag{26}$$

where

$$\Delta_i = \sqrt{s_i^{\text{curve}}} - \sqrt{s_i^{\text{data}}} \tag{27}$$

Appendix-D: Experiment data [5]



$$\Gamma_{BW}(E) = \frac{g^2}{6\pi}\frac{p^3}{E^2}$$

$$\Gamma_{CF}(E) = \frac{g^2}{6\pi}\frac{p^3}{E^2}\frac{1+(p_R R)^2}{1+(pR)^2}$$

$$\Gamma_{GA}(E) = \frac{g^2}{6\pi}\frac{p^3}{E^2}\frac{e^{-p^2/6\beta^2}}{e^{-p_R^2/6\beta^2}}$$

Figure : $\pi\pi$ phase shift below $K\bar{K}$ threshold in experiment

[5] Estabrooks, P. and Martin, Alan D. Nucl.Phys. B79 (1974) 301

K-matrix method

$$T^{-1} = V^{-1} - G = \frac{-3(f^2 - 8l_1 m_\pi^2 + 4l_2 W^2)}{2p^2} - \mathrm{Re}G(W) + \frac{ip}{8\pi W} \tag{28}$$

For K-matrix method the $\mathrm{Re}G(W) = 0$.

$$T = \frac{-8\pi W}{p\cot\delta p - ip} \tag{29}$$

$$l_1 = \frac{1}{8\pi^2}\left(-\frac{1}{2}\frac{m_\rho^2}{g_{\rho\pi\pi}^2} + f^2\right) \tag{30}$$

$$l_2 = -\frac{1}{8g_{\rho\pi\pi}^2} \tag{31}$$