problem 1 simplification

In[327]:=
$$\text{rules} = \left\{A0 \to 2\,\frac{a^2}{e}\,\sqrt{1-e^2}\,\text{ArcSin}[e],\ A1 \to \frac{\sqrt{1-e^2}}{e^3}\,\text{ArcSin}[e] - \frac{1-e^2}{e^2},\ A2 \to \frac{\sqrt{1-e^2}}{e^3}\,\text{ArcSin}[e] - \frac{1-e^2}{e^2},\ A3 \to -2\,\frac{\sqrt{1-e^2}}{e^3}\,\text{ArcSin}[e] + \frac{2}{e^2}\right\};$$

$$\text{a3rule} = a3 \to a\,\sqrt{1-e^2};$$

$$\text{Collect}\left[\text{Simplify}\left[\frac{-M}{4\,a^2\,a3}\left(A0 + \frac{1}{5}\,a^2\,(A1+A2) + \frac{1}{5}\,a3^2\,A3\right) /.\ \text{rules} /.\ \text{a3rule}\right],\ \text{ArcSin}[e],\ \text{Simplify}\right]$$

Out[329]=
$$-\frac{3\,M\,\text{ArcSin}[e]}{5\,a\,e}$$

problem 2 integral

In[345]:=
$$J[l\_] :=$$
$$\left(\text{int} = \frac{-2\,\pi\,\left(M / \left(4\,\pi\,a^2\,a3/3\right)\right)}{(l+3)\,M\,a^l}\,\text{Integrate}\left[\left(\frac{1}{a^2} - e^2\,\frac{u^2}{a^2\,(e^2-1)}\right)^{-(l+3)/2}\,\text{LegendreP}[l, u],\ u\right];\right.$$
$$\text{Simplify}\left[(\text{int} /.\ u \to 1) - (\text{int} /.\ u \to -1) /.\ a3 \to a\,\sqrt{1-e^2},\right.$$
$$\left.\left. e > 0\ \&\&\ 1 > e\ \&\&\ a > a3\ \&\&\ a3 > 0\ \&\&\ \{a, a3, e\} \in \text{Reals}\right]\right)$$

In[360]:=
$$\text{For}\left[ii = 2,\ ii \le 10,\ ii{+}{+},\right.$$
$$\text{Print}\left[\text{Subscript}[\text{"J"}, ii],\ \text{"="},\ J[ii],\ \text{", and "},\right.$$
$$\left.\text{HoldForm}\left[\frac{3\,(-1)^{\ell/2}+1}{(\ell+1)\,(\ell+3)}\,e^{\ell}\right] /.\ \ell \to ii,\ \text{"="},\ \frac{3\,(-1)^{\ell/2}+1}{(\ell+1)\,(\ell+3)}\,e^{\ell} /.\ \ell \to ii\right]$$
$$\left.\right]$$

$$J_2=\frac{e^2}{5},\ \text{and}\ \frac{\left(3\ (-1)^{2/2}+1\right)\ e^2}{(2+1)\ (2+3)}=-\frac{2\ e^2}{15}$$

$$J_3=0,\ \text{and}\ \frac{\left(3\ (-1)^{3/2}+1\right)\ e^3}{(3+1)\ (3+3)}=\left(\frac{1}{24}-\frac{i}{8}\right)\ e^3$$

$$J_4=-\frac{3\ e^4}{35},\ \text{and}\ \frac{\left(3\ (-1)^{4/2}+1\right)\ e^4}{(4+1)\ (4+3)}=\frac{4\ e^4}{35}$$

$$J_5=0,\ \text{and}\ \frac{\left(3\ (-1)^{5/2}+1\right)\ e^5}{(5+1)\ (5+3)}=\left(\frac{1}{48}+\frac{i}{16}\right)\ e^5$$

$$J_6=\frac{e^6}{21},\ \text{and}\ \frac{\left(3\ (-1)^{6/2}+1\right)\ e^6}{(6+1)\ (6+3)}=-\frac{2\ e^6}{63}$$

$$J_7=0,\ \text{and}\ \frac{\left(3\ (-1)^{7/2}+1\right)\ e^7}{(7+1)\ (7+3)}=\left(\frac{1}{80}-\frac{3\ i}{80}\right)\ e^7$$

$$J_8=-\frac{e^8}{33},\ \text{and}\ \frac{\left(3\ (-1)^{8/2}+1\right)\ e^8}{(8+1)\ (8+3)}=\frac{4\ e^8}{99}$$

$$J_9=0,\ \text{and}\ \frac{\left(3\ (-1)^{9/2}+1\right)\ e^9}{(9+1)\ (9+3)}=\left(\frac{1}{120}+\frac{i}{40}\right)\ e^9$$

$$J_{10}=\frac{3\ e^{10}}{143},\ \text{and}\ \frac{\left(3\ (-1)^{10/2}+1\right)\ e^{10}}{(10+1)\ (10+3)}=-\frac{2\ e^{10}}{143}$$