Aphysical Extensions of the Coin Flipping Problem

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Abstract

Consider a flipping coin with no thickness and which lands on a perfectly damped surface (so it doesn't bounce). How can you predict what side it will land on based on the initial conditions? We consider this problem in 3 contexts: 1) a simple vacuum, 2) a linear drag force, and 3) the high-speed relativistic limit.

1 Simple case

This problem was first explored by Keller (1986)[1]. In that paper, the author considers a 2-dimensional model of a coin flipping about its diameter in a vacuum. In this paper, we consider two extensions of this so-called "Keller model". Later, Diaconis et al.(2007)[2] extended this model to 3 dimensions, and found, surprisingly, that the coin's precession can induce a preference for the face it started on 51% of the time.

In this section, we will rederive the result that Keller found. The math is straightforward, but it has a rewarding little graphical representation at the end.

If the coin starts and ends at the coordinate's origin, the only two initial conditions which matter are its translational and rotational velocities. Therefore, we're looking for a statement of $\theta_F(\omega, v)$ which gives the final angle as a function of these quantities.

In order to find this relationship, consider the requirements for the coin to land on the same side it was flipped from. It would have to be in the air long enough to make an integer number of rotations given angular velocity ω. We know already that $\theta(t) = \omega t$, and thankfully, the amount of time the coin is in the air, t, is dependent only on its velocity. From kinematics, we can figure out that the time the coin spends in the air is $t = 2v/g$, meaning that the final angle as a function of its velocities is

$$\theta_F = 2\frac{\omega v}{g}. \tag{1}$$

So, in order for the coin to land on the same face as it started, θ_F needs to be less than half-integer multiples of $\pi/2$. To determine which side the coin will land on then, we're interested in lines of constant θ_F. Given the form of equation 1, these are hyperbolas:

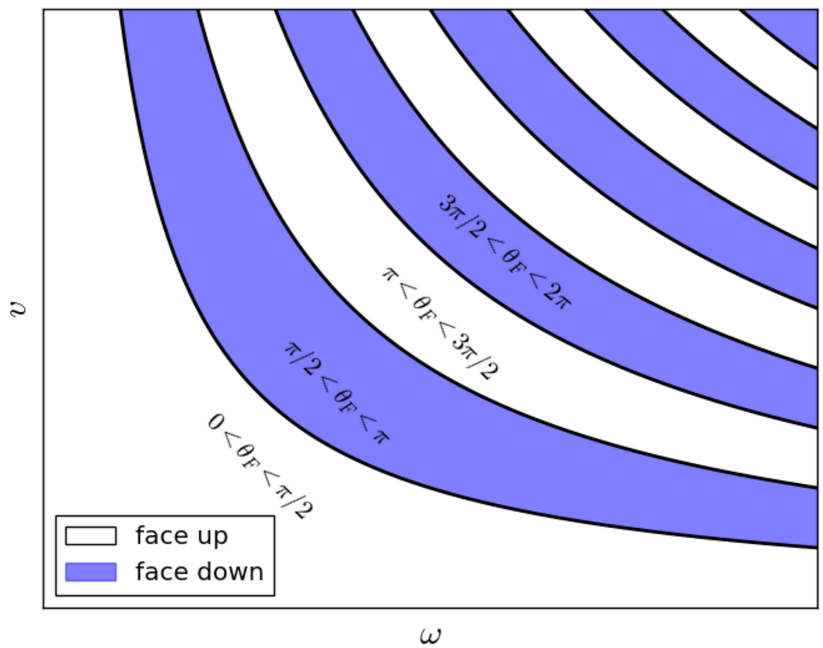


Fig. 1: The values of θ_F in $\omega - v$ space

I think this is pretty neat. The math is simple, but it makes a model that holds up to intuition: At small ω or v, the regions of face-up and face-down are large, and this makes sense: if the coin is flipping very slowly and doesn't go very high, obviously it's very likely that it will land on the same side as it started. But at high ω and v, the coin spends a lot longer in the air, flipping many times; of course it's more difficult to predict what side it will land on.

2 Including drag

Now assume there's a linear wind resistance on the coin. This will affect not only how long the coin is in the air, but how many turns it makes.

2.1 Translational drag

The the distance the coin travels is now not the standard $\frac{1}{2}at^2$ from kinematics. Instead, it's the solution to the second-order differential equation

$$\ddot{x} + \beta\dot{x} + g = 0, \tag{2}$$

where β is the damping parameter due to the drag of the ambient medium on the spinning coin. This equation of motion is solved with an exponential ansatz,

$$\dot{x}(t) = (v_i + \zeta)e^{-\beta t} - \zeta, \tag{3}$$

for initial velocity v_i and where I have defined $\zeta = g/b$, a natural speed of the coin.

Setting equation 3 to zero, we can solve for the amount of time the coin spends on the way up. Then, the total time the coin spends in the air will be not more than double that time. I'll approximate it for now and say that the total time in the air is really twice the time spent going up:

$$\dot{x}(t) = (v_i + \zeta)e^{-\beta t} - \zeta = 0, \tag{4}$$

$$\Rightarrow t_{\frac{1}{2}} = -\frac{1}{\beta} \ln \left[\zeta \left(v_i + \zeta\right)^{-1}\right]. \tag{5}$$

2.2 Rotational drag

The slowing down of the coin's rotation has a very similar derivation to the derivation above, but with two notes: 1) There is no gravity term, and 2) the damping parameter β is being approximated as identical to the translational drag coefficient. This approximation is reasonable: In either case, it's a flat disk hitting the ambient gas perpendicularly. Whether there are two betas or one doesn't make a huge difference mathematically– in the case of two, they would multiply to give an effective β.

The equation of motion for the rotation is

$$\ddot{\theta} + \beta\dot{\theta} = 0. \tag{6}$$

Another exponential ansatz gets the job done,

$$\theta(t) = -\eta e^{-\beta t} + \eta, \tag{7}$$

where I have defined $\eta \equiv \omega_i/\beta$, another potentially helpful quantity. Unlike ζ, this one is dimensionless.

2.3 The new θ_F

Now combining the two new equations 5 and 7, we can get the new $\theta_F(\omega, v)$.

$$\theta_F = -\eta \exp\left[-\beta\left(-\frac{2}{\beta}\ln\left(\zeta\left(v_i + \zeta\right)^{-1}\right)\right)\right] + \eta \tag{8}$$

$$= -\eta\zeta^2\left(v_i + \zeta\right)^{-2} + \eta. \tag{9}$$

Expanding equation 9 about small β to first order gets us precisely $2\omega v/g$ from before, which is provides convincing reassurance.

The form of equation 9 has a direct effect on the shape of the face up regions of figure 1. As you increase β (corresponding to flipping the coin in thicker and thicker environments), the regions get more vertical (figure 2). As the atmosphere gets thicker, the coin stops spinning sooner and sooner. For this reason, it doesn't matter how high it goes because it spends most of the journey not spinning.

3 The relativistic limit

In the limit as v approaches c, the constant force of gravity will induce hyperbolic motion in the coin, rather than parabolic. This will change the amount of time the coin spends in the air at speeds close to c.

Velocity as a function of time with a constant force has the form, as given in [3]:

$$v(t) = \frac{Ft}{m\sqrt{1 + \frac{F^2 t^2}{m^2 c^2}}} + v_i, \tag{10}$$

for force F and initial velocity v_i. Setting this to 0 and solving for t gives

$$t = \frac{m v_i}{F\sqrt{1 - v_i^2/c^2}} = \frac{m v_i}{F}\gamma_i. \tag{11}$$

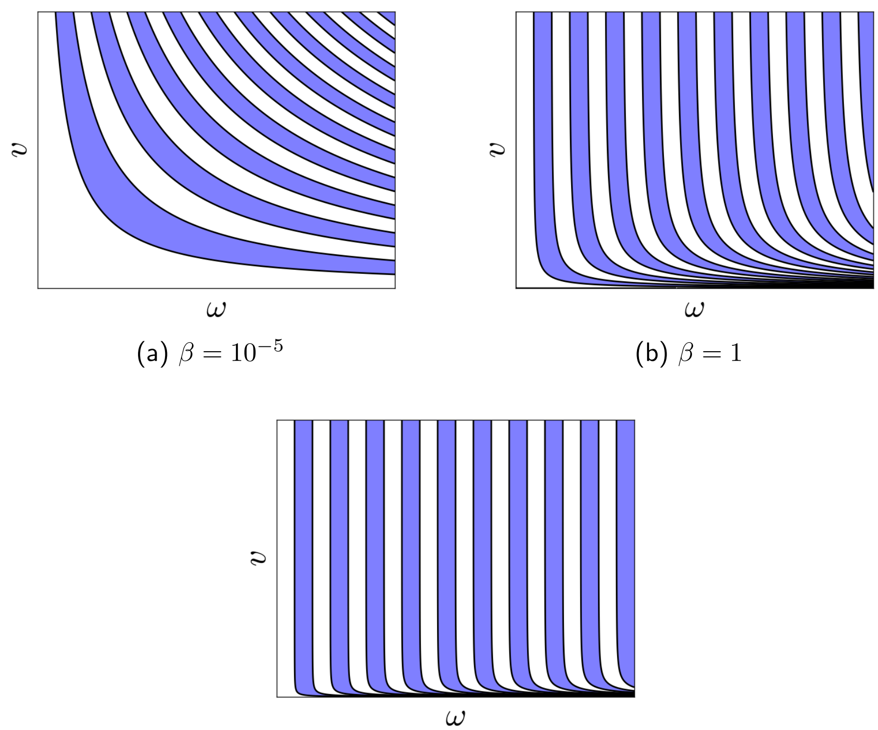
With $F = g$ and $m = 1$, we recover the relativistic θ_F equation, which is, rather pleasingly, just a Lorentz factor away from equation 1:

$$\theta_F = 2\gamma\frac{\omega v}{g}. \tag{12}$$

This Lorentz factor has the effect of degenerating the faced-ness at speeds close to c (figure 3).

References

[1] J. Keller, On the Probability of Heads, Am. Math. Mon. 93 (1986) 191-197
[2] P. Diaconis, S. Holmes, and R. Montgomery, Dynamical Bias in the Coin Toss, SIAM Review, 49, (2007) 211-235
[3] D. Griffiths, *Introduction to Electrodynamics*, Prentice Hall, 3ed. (1999) 509-511



(a) $\beta = 10^{-5}$

(b) $\beta = 1$

(c) $\beta = 10$

Fig. 2: The values of θ_F as β increases. The colors are the same here as in figure 1, with white corresponding to face up and blue to face down.

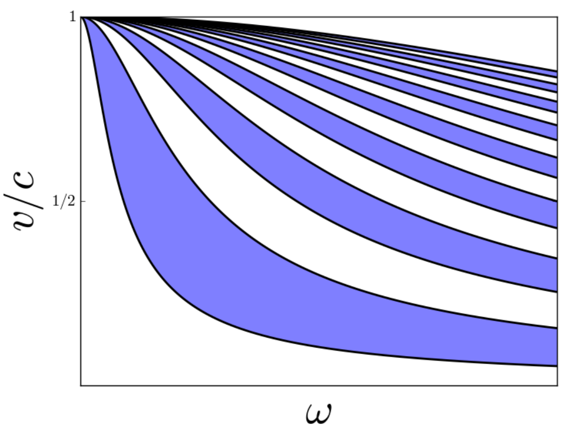


Fig. 3: θ_F as v approaches c. The faced-ness degenerates as the speed of the coin approaches c.