2001 Physics GRE Solutions

1 Pendulum bob

This is *total* acceleration, so there will be two components acting on the bob: gravity and tension. Tension acts in the direction of the string, and gravity always points straight down. For this reason, we can immediately rule out (A) and (E), as those are clearly ignoring any force other than what is along the string. Furthermore, at point c, the string (and therefore the tension) is pointing in the same direction as gravity, so the arrow should be pointing totally along the string. This rules out (B) and (D).

Answer: **(C)**.

2 Friction on a turntable

There are two forces which must cancel out: Centripetal and frictional.
Therefore:

$$m\omega^2 r = mg\mu_s, \tag{1}$$

$$\Rightarrow r = \frac{g\mu_s}{\omega^2}. \tag{2}$$

However, the units need to be taken care of. We are given a frequency $\nu = 33.3$ revolutions per minute, while $[g] = [\mathrm{m}][\mathrm{s}]^{-2}$. We want to convert $[\nu]$ to an angular frequency $[\omega] = [\mathrm{rad}][\mathrm{s}]^{-1}$.

$$\omega = 2\pi \frac{\mathrm{rad}}{\mathrm{rev}} \frac{1\mathrm{min}}{60\mathrm{s}} \nu, \tag{3}$$

$$\Rightarrow r = \frac{9.8 \times 0.3}{\left(33.3 \times \frac{2\pi}{60}\right)^2} = 0.242. \tag{4}$$

Answer: **(D)**.

Alex Deich, July 2016

3 Satellite orbit

I can't think of many clever ways to narrow down the answers. It is probably just good to have Kepler's Third Law memorized:

$$T = 2\pi\sqrt{\frac{a^3}{GM}}, \tag{5}$$

where
T is the orbital period,
a is the semi-major axis, and
GM is the planet's mass times the gravitational constant.

Answer: **(D)**.

4 One-dimensional inelastic collision

As usual, start with conservation of momentum:

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_{12}. \tag{6}$$

With $m_1 = 2m$, $m_2 = m$ and $v_2 = 0$ (the problem states that the particle is at rest), we have

$$2mv_1 = 3mv_{12} \tag{7}$$

$$\Rightarrow v_{12} = \frac{2}{3} v_1. \tag{8}$$

Then, with conservation of energy,

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} (m_1 + m_2) (v_{12})^2 \tag{9}$$

$$\Rightarrow \frac{1}{2} 2mv_1^2 = \left(\frac{1}{2} 3m\right) \left(\frac{2}{3} v_1\right)^2 \tag{10}$$

$$\Rightarrow mv_1^2 = \left(\frac{3}{2} m\right) \frac{4}{9} v_1^2 \tag{11}$$

$$\Rightarrow KE_f = \frac{2}{3} KE_i \tag{12}$$

So the final energy is $\frac{2}{3}$ of the initial, meaning the system has lost a third of its kinetic energy.

Answer: **(C)**.

5 Average kinetic energy of a 3D harmonic oscillator

The equipartition theorem says that there is $\frac{1}{2}k_BT$ of energy for each degree of freedom *in the Hamiltonian*. The Hamiltonian for a harmonic oscillator is

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}k\left(q_x^2 + q_y^2 + q_z^2\right), \tag{13}$$

which has 6 degrees of freedom (3 positions and 3 momenta), so the average kinetic energy is

$$\langle KE \rangle = 6 \times \frac{1}{2}k_BT = 3k_BT. \tag{14}$$

Answer: **(D)**.

6 Work done on a gas

Adiabatic means that heat is conserved. **Adiabatic heating occurs when the pressure of a gas is increased from work done on it by its surroundings**. For an adiabatic process, $PV^\gamma = \text{const.}$, and the work done by applying pressure to a volume is $W = \int PdV$. Using these, one can derive the worthy-of-memorizing equation for the work done on a gas in an adiabatic process:

$$W_a = -\frac{1}{1-\gamma}\left(P_fV_f - P_iV_i\right). \tag{15}$$

Isothermal means that the temperature is conserved. **An isothermal process occurs slowly enough to allow the system to continually adjust to the temperature of the reservoir through heat exchange**. Therefore, $P_iV_i = nRT_i = P_fV_f = nRT_f$, and using the same definition of work above, one can derive the worthy-of-memorizing equation for the work done on a gas in an isothermal process:

$$W_i = nRT_i \ln\left(\frac{V_f}{V_i}\right) = P_iV_i \ln\left(\frac{V_f}{V_i}\right). \tag{16}$$

Right off the bat, we know that $V_f = 2V_i$, so neither W_i nor W_a will be equal to 0. This eliminates (B) and (D). Furthermore, we can see from the formulae that $W_i \neq W_a$. This makes the problem as deciding whether W_a is greater than W_i or vice-versa. One way to solve this would be by noting that for a monatomic gas, $\gamma = \frac{5}{3}$ and solving.

Another method is this: For an adiabat, $P \propto \frac{1}{V^\gamma}$, and $\gamma > 1$ for an ideal gas. For an isotherm, $P \propto \frac{1}{V}$. Therefore, for the same change in volume, the pressure of an adiabat changes more than the pressure of an isotherm. Therefore, the adiabat does more work.

Answer: **(E)**.

7 Two bar magnets

Going through the options:

(A) cannot be true because both the magnets are standing on their north ends. Matching polarities always repel.

(C) seems to imply that there are magnetic monopoles, which don't exist. In this picture, $\nabla \cdot \mathbf{B} \neq 0$.

(D) doesn't work because you'd expect cylindrical symmetry.

(E) would be good if there were longitudinal current through the cylinders, but there's not.

Answer: **(B)**.

8 Charge induced on an infinite conductor

By the conservation of charge, there should be an equal and opposite charge induced on the top face of the conductor, giving an answer of $-Q$.

Answer: **(D)**.

9 Charges on a circle

All the charges are symmetrically placed around the circle, so at the center, there is an equal contribution from every direction pointing towards the charges, giving a net magnitude of 0. One suggestion from the Internet is to take the limit as infinite charges are equally placed around the circle: This is a conducting ring, which has no field on the interior.

Answer: **(A)**.

10 Energy across a capacitor

The energy stored in a capacitor can be derived by finding the work required to establish an electric field in the capacitor:

$$W = \int_0^Q V(q)dq = \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}VQ, \tag{17}$$

where V is the voltage difference across the capacitor, C is the effective capacitance, and Q is the total charge across one of the plates.

Furthermore, it is helpful to remember the additive properties of capacitors. In parallel, n capacitors add normally:

$$C_{\text{eff}} = C_1 + C_2 + \cdots + C_n. \tag{18}$$

In series, the reciprocals add:

$$\frac{1}{C_{\text{eff}}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}. \tag{19}$$

So, for this problem we need the effective capacitance of two capacitors in series:

$$\frac{1}{C_{\text{eff}}} = \frac{1}{3\mu\text{F}} + \frac{1}{6\mu\text{F}}, \tag{20}$$

$$= \frac{9}{18\mu\text{F}}, \tag{21}$$

$$\Rightarrow C_{\text{eff}} = \frac{18}{9}\mu\text{F}. \tag{22}$$

Finally, using $\frac{1}{2}CV^2$ for the energy stored, we get

$$\frac{1}{2} \times \frac{18}{9} \times 300^2 = 300^2 = 900,000. \tag{23}$$

Not knowing the units is fine: There is only one answer which is 9 times some number of tens.

Answer: **(A)**.

11 Two-lens image

This requires two applications of the thin-lens limit of the lensmaker's formula,

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}, \tag{24}$$

where d_o is the distance from the object to the lens, d_i is the distance from the lens to the image, and f is the focal length of the lens.

For this problem, we apply the formula once for the object and the first lens, and then using that image as the "object" for the second lens. Solving for d_i for the first lens:

$$\frac{1}{d_i} = \frac{1}{20} - \frac{1}{40}, \tag{25}$$

$$= \frac{1}{40}, \tag{26}$$

$$\Rightarrow d_i = 40. \tag{27}$$

So the first image is 40 cm behind the first lens. This places the new "object" 10 cm behind the second lens. Because this is now on the opposite side of the second lens as the side which we established was positive for the first lens, we say that d_o for the second lens is -10.

Then, following through with the second lensmaker's equation,

$$\frac{1}{d_i} = \frac{1}{10} + \frac{1}{10}, \tag{28}$$

$$= \frac{1}{5}, \tag{29}$$

$$\Rightarrow d_i = 5. \tag{30}$$

So the final image is 5 cm behind the second lens.

Answer: **(A)**.

12 Spherical concave mirror

We can start with the lensmaker's equation as used above,

$$\frac{1}{d_i} = \frac{1}{f} - \frac{1}{d_o}. \tag{31}$$

Looking at the picture in the problem, we can see that $d_o < f$. Therefore, $\frac{1}{d_o} > \frac{1}{f}$, and so d_i will be negative, placing the image behind the mirror.

Answer: **(E)**.

13 Resolving two stars

I can't think of any way to do this problem but to memorize the Rayleigh criterion:

$$\theta = 1.22\frac{\lambda}{D}, \tag{32}$$

which describes the requirement on the diameter of a lens, D, in order to resolve two objects separated by an angle θ, at some wavelength of light, λ.

In this particular problem, we have $\theta = 3 \times 10^{-5}$ rad, and $\lambda = 600$ nm. Plugging it all in, we have

$$D = \frac{1.22 \times 600 \times 10^{-9} \text{ m}}{3 \times 10^{-5} \text{ rad}} = 2.4 \text{ cm} \tag{33}$$

Answer: **(B)**.

14 Cylindrical gamma-ray detector

The "8-centimeter-long" fact is not useful. All that matters is the flux through the face of the detector. When the detector is *right at* the source, it's collecting all of the radiation that leaves from that half of the source – hence it's collecting 50% of emitted radiation. At a distance of 1 meter, the radiation is now spread out over a sphere of surface area $4\pi r^2 = 4\pi$ m^2. The face of the detector is only occupying $16\pi \times 10^{-4}$ m^2. Therefore, the fraction of radiation which is detected is a ratio of these two areas:

$$\frac{16\pi \times 10^{-4} \text{ m}^2}{4\pi \text{ m}^2} = 4 \times 10^{-4}. \tag{34}$$

Answer: **(C)**.

15 Measurement precision

Precision is a measure of *random* error. If the precision is high, the there is low random error. With low random error, the statistical variability is low, so the measurements are clustered closely regardless of how well they agree with the actual value.

Answer: **(A)**.

16 Uncertainty

In addition to generally knowing about random error estimation, you need to know two things specifically for this problem: 1) that radioactive decay follows a Poisson distribution, and 2) that the standard deviation of a Poisson distribution is the square root of the average.

Then, you can estimate the expected uncertainty (usually taken as the error of the standard deviation, σ_x) from the fact that the error in the standard deviation, $\Delta\sigma_x$ goes as

$$\Delta\sigma_x = \frac{\sigma_x}{\sqrt{N}}, \tag{35}$$

for N trials.

So, calculating the average, we have

$$\langle x \rangle = \frac{3+0+2+1+2+4+0+1+2+5}{10} = 2, \tag{36}$$

so the standard deviation is

$$\sigma_x = \sqrt{\langle x \rangle} = \sqrt{2}. \tag{37}$$

Assuming that by "uncertainty of 1 percent" they mean that the error should be the average plus or minus 1%, we are looking for an uncertainty of $\Delta\sigma_x = 0.002$. So, using the fact mentioned above, we can solve for the number of trials required for this uncertainty.

$$0.02 = \frac{\sqrt{2}}{\sqrt{N}}, \tag{38}$$

$$\Rightarrow N = \frac{2}{0.0004} = 5,000. \tag{39}$$

Answer: **(D)**.

17 Filling electron shells

For problems like these, I think it's best to just memorize how electron shells are filled up. The rules I keep in mind are these:

1. s-orbitals can take 2 electrons, p-orbitals can take 6 and d-orbitals can take 10.
2. The pattern is s-s-p-s-p-s-d.
3. If the orbitals are in the right order, and they each have as many electrons as they can take (except, perhaps the last one), then that's the right answer.

Answer: **(B)**.

18 Energy to remove a helium electron

The Bohr model is applicable here, because helium isn't too different from hydrogen. The energy for an electron in the n^{th} state of a hydrogenic atom with Z protons is given by

$$E_n = Z^2 \frac{R_E}{n^2}, \tag{40}$$

where R_E is a Rydberg, the energy of the ground state of hydrogen, 13.6 eV.

Because Helium has two protons ($Z = 2$), it can accept two electrons into its ground state, so $n = 1$ for this problem. If the helium were singly ionized and only had a single electron, as He+, then the energy to remove that electron would be, from the Bohr model above,

$$E_1 = 4 \times 13.6 \text{ eV} = 54.4. \text{ eV}. \tag{41}$$

It's tempting to say that the energy required to remove the two electrons of neutral helium would just be 2×54.4 eV $= 108.8$ eV, but this is not the case. Considering one electron at a time, one can see that the first one is not *just* experiencing the tug of the nucleus, but is *also* being pushed away by a Coulomb repulsion from the *other electron*. Therefore, we expect it to be slightly easier to remove the first of two electrons than to remove one solitary electron.

In lieu of perturbation theory, we can subtract the theoretical value of 54.4 eV for a single ground state energy from the experimental value of 79 eV for complete ionization. What's left over will be the energy that was required to remove an electron whose exit was being pushed along by a Coulomb repulsion. So,

$$79 \text{ eV} - 54.4 \text{ eV} = 24.6 \text{ eV}. \tag{42}$$

Answer: **(A)**.

19 Source of the Sun's energy

It is not necessary to know the whole proton-proton chain to answer this problem. Just note that a helium has 4 things of roughly proton mass (2 protons and 2 neutrons), and that $\mathrm{n} + \mathrm{e}^- \to \mathrm{p}$. So in order to make helium, you need 4 hydrogens and enough energy to collide half of the electrons and protons.

Answer: **(B)**.

20 Definition of bremsstrahlung

The way I remember this is that bremsstrahlung is German for "breaking radiation" which reminds me that it's analogous to breaking the sound barrier (it doesn't matter that they're different meanings of "break"). The only answer that sounds like it has anything to do with breaking a sound barrier is the one that mentions rapid deceleration in a medium.

Answer: **(E)**.

21 Comparing Lyman-α and Balmer-α wavelengths

This requires knowing the Rydberg formula for the energy, E, of a photon released in the transition from one energy level, n_i to another, n_f:

$$E = E_f - E_i = R_E \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right), \tag{43}$$

where R_E is the Rydberg energy, 13.6 eV. The energy of a photon emitted by a Lyman-α transition (from $n = 2$ to $n = 1$) is then

$$E_{L\alpha} = R_E \left(1 - \frac{1}{4} \right) = \frac{3}{4} R_E. \tag{44}$$

While Balmer-α transitions (from $n = 3$ to $n = 2$) release photons with energies of

$$E_{B\alpha} = R_E \left(\frac{1}{4} - \frac{1}{9} \right) = \frac{5}{36} R_E, \tag{45}$$

and, taking the ratio,

$$\frac{E_{L\alpha}}{E_{B\alpha}} = \frac{3 \times 36}{4 \times 5}, \tag{46}$$

$$= \frac{27}{5}. \tag{47}$$

Finally, the problem is asking for the wavelength of the light. $E = hc/\lambda$, and taking the ratio has dealt with the implicit hc, so we only need to take the reciprocal of 27/5.

Answer: **(B)**.

22 Orbital system parameters

The hint is in the phrase "very small moon". The question is implying that the mass of the moon is negligible compared to the mass of the planet. This means that the center of gravity will be very close to the center of the planet, and so you can treat the planet as stationary. Then, think about the equations of orbital motion. Kepler's Third Law (problem 3) describes the period and semi major axes of the orbit, but makes no mention of mass. If you're solving for a circular orbit with forces (or any orbit, but circular is easy), the smaller mass will cancel:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \tag{48}$$

There are probably other examples, but these should be sufficient to convince you the answer is the moon's mass.

Answer: **(A)**.

23 Particle accelerating around a circle

I think this problem is worded confusingly. There's a particle going around a circle at a rate of 10 ms^{-1}, and so it's experiencing the standard centripetal acceleration of

$$\mathbf{a}_{\mathrm{c}} = -\frac{v^2}{r}\hat{\mathbf{r}} = -\frac{100}{10}\hat{\mathbf{r}} = -10\hat{\mathbf{r}}. \tag{49}$$

But this particle is *also* undergoing a tangential acceleration of 10 ms^{-2}, in the direction of the particle's velocity. That means that the particle is experiencing two orthogonal accelerations of equal magnitude. The only way that these can combine is to form a vector at 45°to both. To answer the question, note that one of the accelerations is in the same direction as the velocity, so the angle is the same.

Answer: **(C)**.

24 Throwing a stone

There is no air resistance, so v_x should be constant, so we should only consider solutions with plot II in the first column. The ball starts by going up, so v_y is at positive at first, and should undergo constant deceleration. As the ball begins to fall, v_y is negative. The only plot which fits this description is III.

Answer: **(C)**.

25 Moment of inertia of packed pennies

To do this problem, refer to the front of the test for the moment of inertia of a solid disk about an axis going through its center: $I = \frac{1}{2}mr^2$. A very quick way I did this was to say that the whole collection of pennies was a solid disk of mass $7m$ and radius $3r$. Then,

$$I = \frac{1}{2}mr^2, \tag{50}$$

$$m \to 7m, r \to 3r, \tag{51}$$

$$= \frac{1}{2} \times 7m \times (3r)^2, \tag{52}$$

$$= \frac{63}{2}mr^2. \tag{53}$$

The available answer this is closest to is 55/2, which is the correct answer. However, if the correct answer weren't also the largest, this estimate is not nearly close enough to be sure. A rigorous way to do this is to use the parallel axis theorem.

If you know the moment of inertia of an object around one axis, the parallel axis theorem provides a way to calculate the moment of inertia around any other axis which is parallel to the original. In this case, we know the moment of inertia of one penny around the axis that goes through its center of mass,

$$I_{\text{CM}} = \frac{1}{2}mr^2 \tag{54}$$

If we now rotate that penny around an axis a distance d from its center, the parallel axis theorem says that the new moment of inertia is given by

$$I = I_{\text{CM}} + md^2 \tag{55}$$

In calculating the contribution to the total angular momentum of each extremal penny in the arrangement given in the problem, we are rotating each of the exterior pennies around an axis $2r$ from their centers. Then, we can add these moments to the moment of the middle penny being rotated around *its* center to get the total moment of inertia. For the moment of inertia of each exterior penny,

$$I_{\text{ext.}} = I_{\text{CM}} + m\left(2r\right)^2, \tag{56}$$
$$= \frac{1}{2}mr^2 + m \times 4r^2, \tag{57}$$
$$= \frac{9}{2}mr^2. \tag{58}$$

So each exterior penny contributes $(9/2)\,mr^2$ to the total moment of inertia which we can now calculate by summing the moments of all the constituents.

$$I_{\text{tot.}} = I_{\text{CM}} + 6 \times \frac{9}{2}mr^2, \tag{59}$$
$$= \frac{1}{2}mr^2 + \frac{54}{2}mr^2, \tag{60}$$
$$= \frac{55}{2}mr^2. \tag{61}$$

Answer: **(E)**.

26 Speed of the end of a falling rod

This problem also requires looking at the front of the test for (or memorizing) the moment of inertia of a rod of length L and mass M about its center of mass (so that it's spinning like a baton): $I = \frac{1}{12}ML^2$. However, the rod in the problem is not rotating around its center of mass. It's pivoting on one end, a distance $L/2$ from its center of mass. We can get the moment of inertia of this rotation with the parallel axis theorem.

$$I = I_{\text{CM}} + Md^2, \tag{62}$$
$$d \to \frac{L}{2}, \tag{63}$$
$$I = M\frac{L^2}{12} + M\frac{L^2}{4}, \tag{64}$$
$$= \frac{1}{3}ML^2. \tag{65}$$

Now we can use conservation of energy to solve for the speed of the end of the rod. Setting the rotational energy equal to the gravitational potential energy from the rod's center of mass (which is located at $L/2$),

$$\frac{1}{2}I\omega^2 = Mg\frac{L}{2}, \tag{66}$$

$$\Rightarrow \omega^2 = \frac{MgL}{I}, \tag{67}$$

$$= \frac{3g}{L}. \tag{68}$$

To convert the radial speed, ω to the linear speed of the end of the rod, we note that $\omega = v/r$, and in the case of the end of the rod, $r = L$. So,

$$\frac{v^2}{L^2} = \frac{3g}{L}, \tag{69}$$

$$\Rightarrow v = \sqrt{3gL}. \tag{70}$$

Answer: **(C)**.

27 Eigenvalues of the Hamiltonian operator

An axiom of quantum mechanics is that the Hamiltonian operator is hermitian, and a requirement of hermitian operators is that their eigenvalues are always real. If the Hamiltonian operator weren't hermitian, then observations of a particle's energy could give imaginary results, which would have no physical meaning.

Answer: **(A)**.

28 Orthogonal states

That two vectors are orthogonal means their inner product is equal to zero. Taking the inner product of ψ_1 and ψ_2,

$$\langle\psi_1|\psi_2\rangle = (5 \times 1) + ((-3) \times (-5)) + (2 \times x) = 20 + 2x \tag{71}$$

The problem asks for the value of x which ensures ψ_1 and ψ_2 are orthogonal, which, from demanding the above inner product be equal to zero, is -10.

Answer: **(E)**.

29 Expectation value of an operator

The whole point of eigenvalues is that you can replace an operator acting on an eigenvector with the corresponding eigenvalue, a scalar. In this case, we take the inner product of the wavefunction with the wavefunction after it has been acted upon by the operator $\hat{O}$,

$$\left\langle \psi \middle| \hat{O}\psi \right\rangle. \tag{72}$$

To calculate $\left| \hat{O}\psi \right\rangle$, we can just push the operator through ψ,

$$\begin{aligned}
\left| \hat{O}\psi \right\rangle &= \frac{1}{\sqrt{6}}\hat{O}\psi_{-1} + \frac{1}{\sqrt{2}}\hat{O}\psi_1 + \frac{1}{\sqrt{3}}\hat{O}\psi_2, && (73)\\
&\text{replacing the operator with the eigenvalues,} && (74)\\
&= \frac{1}{\sqrt{6}}(-1)\psi_{-1} + \frac{1}{\sqrt{2}}(1)\psi_1 + \frac{1}{\sqrt{3}}(2)\psi_2, && (75)\\
&= -\frac{1}{\sqrt{6}}\psi_{-1} + \frac{1}{\sqrt{2}}\psi_1 + \frac{2}{\sqrt{3}}\psi_2. && (76)
\end{aligned}$$

Then, calculating the inner product of this acted-upon wavefunction with the original wavefunction,

$$\begin{aligned}
\left\langle \psi \middle| \hat{O}\psi \right\rangle &= \left(-\frac{1}{\sqrt{6}} \times \frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} \times \frac{2}{\sqrt{3}}\right), && (77)\\
&= -\frac{1}{6} + \frac{1}{2} + \frac{2}{3} = 1. && (78)
\end{aligned}$$

Answer: **(C)**.

30 Possible Hydrogen wavefunctions

Any wavefunction must be normalizable, which is to say

$$\int_{-\infty}^{\infty} \psi^* \psi dr = 1, \tag{79}$$

because between the limits of $\pm\infty$, the particle must be *somewhere* (that is, if you're in position space; the analogous requirement in momentum space is that the particle must have *some* finite momentum). The only function which is normalizable is option I.

Answer: **(A)**.

31 Energy levels in positronium

To derive the ground state energy of positronium, we can treat it as hydrogenic and use the Bohr model. The Bohr model predicts that the energy of different levels are proportional to the reduced mass, $\mu = \frac{m_1 m_2}{m_1 + m_2}$, of the system,

$$E_n = -\frac{\mu q_e^4}{8h^2\epsilon_0^2}\frac{1}{n^2}. \tag{80}$$

In the case of hydrogen, m_1 and m_2 are the masses of the proton and electron, respectively. In that case, $m_p \gg m_{e^-}$, and so $\mu \approx m_{e^-}$. In this case, the positron and electron are the same mass, so $\mu = m_e/2$, roughly half the mass of hydrogen. Everything else in the equation for E_n stays the same: they're all constants (besides n, the energy level). This means that we can just divide in half any hydrogen energy to get the corresponding positronium energy. So, the ground state of positronium should be

$$\frac{13.6}{2} = 6.8. \tag{81}$$

Now we can use the Rydberg equation for energy of a photon emitted from an electron transitioning from state n_i to n_f,

$$E = E_1 \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right), \tag{82}$$

where E_1, rather than the Rydberg from the last time we applied this equation to hydrogen, is now the ground state energy of positronium, 6.8 eV. Plugging everything in, an electron transitioning from the $n = 3$ state to the $n = 1$ state of positronium will emit a photon with energy

$$E = 6.8 \times \left(1 - \frac{1}{9} \right) = 6.8 \times \frac{8}{9} = 6.0 \text{ eV}. \tag{83}$$

Answer: **(A)**.

32 Relativistic momentum

The total energy of a particle is given by

$$E = \sqrt{m^2c^4 + p^2c^2}, \tag{84}$$

and we are told that the total energy is equal to twice its rest energy, or

$$E = 2mc^2. \tag{85}$$

Plugging in for the total energy and solving, we have

$$2mc^2 = \sqrt{m^2c^4 + p^2c^2}, \tag{86}$$

$$\Rightarrow 4m^2c^4 = m^2c^4 + p^2c^2, \tag{87}$$

$$\Rightarrow p^2c^2 = 3m^2c^4, \tag{88}$$

$$\Rightarrow p = \sqrt{3}mc. \tag{89}$$

Answer: **(D)**.

33 Decaying pion

This problem is really about being comfortable with changing between relativistic reference frames. Is proper time longer or shorter than a time at rest? What about proper distance? This gets really confusing, and I have a few ways of remembering how it goes. First, I think about proper time. I know that any spacetime interval is conserved between reference frames. If you have an interval, $\Delta s^2 = -c\Delta t^2 + \Delta x^2$ in one reference frame, you know that it's the same if you measure it in coordinates which are moving (I'll denote moving coordinates – and therefore proper length and time – with a bar):

$$\Delta s^2 = -c\Delta t^2 + \Delta x^2 = \Delta \bar{s}^2 = -c\Delta \bar{t}^2 + \Delta \bar{x}^2 \tag{90}$$

Then, I think about my own reference frame. When I'm stopped, all of my motion is through time, which I am accomplishing at c. If I speed up, my Δs must stay the same, and so I must be accomplishing less of my travel through time and more through space. This helps me recall that **moving clocks run slow**, so a moving clock will measure *less* time than a clock at rest. Then, I know that lengths do the opposite thing, so I know that **moving rulers shrink** and that a proper length is *longer* than a length at rest.

Now, if I'm given a moving time, $\Delta\bar{t}$, I know that to get the time at rest, Δt, I need to change it by a factor of γ. But γ is a somewhat messy fraction. Do I multiply or divide $\Delta\bar{t}$ by γ in order to make it smaller? I have no mnemonic for this; I just remember that γ works like a regular number (for $v < c$) and that multiplying will make things bigger and dividing makes things smaller. Now I have the equations $\Delta t = \gamma\Delta\bar{t}$ and $\Delta x = \frac{1}{\gamma}\Delta\bar{x}$.

So, this question gives a time interval of the pion when it's moving, $\Delta\bar{t} = 10^{-8}$ s, a length interval at rest, $\Delta x = 30$ m, and asks for the speed which reconciles these two numbers. A trick which eluded me for a surprisingly long time was to rewrite the length interval as a time interval. This allows us to use $\Delta t = \gamma\Delta\bar{t}$ (remember, the time at rest must be bigger than the moving time, so we multiply by γ).

$$\Delta t = \gamma\Delta\bar{t}, \tag{91}$$

$$\Delta t \to \frac{\Delta x}{v}, \tag{92}$$

$$\Rightarrow \frac{\Delta x}{v} = \gamma\Delta\bar{t} = \sqrt{\frac{1}{1 - v^2/c^2}}\Delta\bar{t}, \tag{93}$$

$$\Rightarrow v = \sqrt{\frac{1}{c^2\Delta\bar{t}^2 + \Delta x^2}}c\Delta x. \tag{94}$$

First, do these units work out? Yes: The denominator of the fraction has $[\mathrm{m}]^2/[\mathrm{s}]^2 \times [\mathrm{s}]^2$, which gives $[\mathrm{m}]^2$. This is added to another $[\mathrm{m}]^2$, and it's in the denominator of a square root, ultimately giving a factor of 1/[m]. That leaves us with [m]/[s] $\times$ [m]/[m] = [m]/[s], which is a velocity. Good.

Now we are given $\Delta x = 30$ m and $\Delta\bar{t} = 10^{-8}$ s. Furthermore, a few of the answers (except the first one) are with 10% of the approximation $c = 3 \times 10^8$ ms^{-1} so that tells us we should use $c = 2.98 \times 10^8$ ms^{-1} instead. So,

$$v = \sqrt{\frac{1}{2.98 \times 10^{16} \times 10^{-16} + 900}} 30c, \tag{95}$$

$$= \sqrt{\frac{1}{902.98}} 30c, \tag{96}$$

$$= .998c = 2.98 \times 10^8 \text{ ms}^{-1}. \tag{97}$$

Answer: **(D)**.

34 Making events simultaneous

In one frame, the two events are separated by a spacetime interval $\Delta s^2 = -c\Delta t^2 + \Delta x^2$. This same interval is observed by a moving frame (with coordinates $\bar{t}$ and $\bar{x}$). If the events are simultaneous, then we know that $\Delta\bar{t} = 0$, and so

$$\Delta s^2 = -c\Delta t^2 + \Delta x^2 = \Delta\bar{x}^2. \tag{98}$$

Furthermore, we know that any separation in space must be real, so $\Delta\bar{x}^2 > 0$. Therefore,

$$-c\Delta t^2 + \Delta x^2 > 0, \tag{99}$$

$$\Rightarrow \Delta x^2 > c\Delta t^2, \tag{100}$$

$$\Rightarrow \frac{\Delta x^2}{\Delta t^2} > c. \tag{101}$$

Alternately, you could note that the requirement that $\Delta\bar{x} > 0$ requires also that $\Delta s > 0$, which, with this metric signature, is a space-like interval. Such intervals require faster-than-light travel.

Answer: **(C)**.

35 Temperature dependence of a blackbody

This is just about having memorized the blackbody relationship, which is that the energy per time (power, P) per area, A, is proportional to the fourth power of temperature, T by $\sigma \approx 5.67 \times 10^{-8}\text{Wm}^{-2}\text{K}^{-4}$, the Stefan-Boltzmann constant:

$$\frac{P}{A} = \sigma T^4. \tag{102}$$

Therefore, if an object triples its temperature, its energy per time per area will increase by a factor of $3^4 = 81$.

Answer: **(E)**.

36 An expanding adiabatic gas

Refer to the answer to question 6 and consider each of the answers.
(A) is true because an adiabatic process is defined as one which conserves heat.
(B) is true because a change in entropy can be written $\Delta S = \Delta Q/T$ for a change in heat ΔQ and temperature T. For an adiabatic process, $\Delta Q = 0$.
(C) is true because the change of internal energy is the negative of the work done on or by the gas, which is defined as $W = \int PdV$.
(D) is true for the same reason as (C).
(E) is *not* true because this process is *adiabatic*, which makes no promises about temperature, unlike an isothermal process.

Answer: **(E)**.

37 A PV Diagram

First I will go over the rigorous way to calculate the answer, then I will go over some quick tricks for PV diagrams.

Rigorous solution

Rigorously, you have to remember the equations relating work, pressure and volume, and then calculate the work for each segment. The solution will be the sum of each. The trick comes in not knowing the exact pressure at B. There are a couple of ways to deal with this.

The relevant equations are

$$W = \int_{V_1}^{V_2} PdV, \tag{103}$$

and, for an isothermal process,

$$W = P_i V_i \ln\left(\frac{V_f}{V_i}\right). \tag{104}$$

Now, going counter-clockwise starting at A, we compute the work done in segment AB. Normally, we would use the integral $W = \int PdV$, but this is just a flat line with constant pressure, so we can write it as a simple difference:

$$W_{AB} = P\Delta V, \tag{105}$$

$$= 200 \times (V_b - 2)\,. \tag{106}$$

Then, for the segment BC, we can't really use the integral formulation for work, because it's some curve we don't know. Instead, we'll use the logarithm for isothermal work:

$$W_{BC} = P_f V_f \ln\left(\frac{V_f}{V_i}\right), \tag{107}$$

$$= 500 \times 2 \times \ln\left(\frac{2}{V_b}\right). \tag{108}$$

Finally, for CA, there is no change in volume, so the work done is 0. So the total work is just $W = W_{AB} + W_{BC}$. However, there's still the unknown quantity V_b. We can solve for this by noting that the fact that the path from B to C is isothermal, the quantity PV will be conserved along it:

$$P_B V_B = P_C V_C, \tag{109}$$

$$\Rightarrow 200 \times V_B = 500 \times 2, \tag{110}$$

$$\Rightarrow V_B = 5. \tag{111}$$

Now, plugging V_B into the equation for total work,

$$W = 200 \times (5 - 2) + 500 \times 2 \times \ln\left(\frac{2}{5}\right), \tag{112}$$

$$= 600 + 400 \ln\left(\frac{2}{5}\right). \tag{113}$$

To quickly estimate $\ln(2/5)$, note that $e \approx 2.7 \approx 5/2 = (2/5)^{-1}$. So, $\ln(2/5) \approx -1$. Then,

$$W = 600 \times 1000 \times -1 = -400. \tag{114}$$

Qualitative solution

Now, there are some ways to really speed up the calculation of this problem. First, if a PV cycle is counter-clockwise, then that means that there is work being done on the system and the total work of the system will be negative. This immediately knocks out the first two solutions for this problem. Furthermore, you know that the answer is not 0 because there is only one segment which can contribute 0 work, and the other paths must be contributing non-zero work to the process. This leaves (D) and (E) as the possible answers. Then, to calculate the area, you can approximate the cycle as a triangle and wind up with a sufficiently accurate answer.

So, we can measure the length of AB with our pencil, and see that it's approximately as long as the distance from the origin to the "2" on the x-axis. That means the area of the triangle ABC is approximately $1/2 \times (500 - 200) \times 2 = 300$. So, the work calculated this way is -300, which agrees even better with the provided solution than the rigorous method above.

Answer: **(D)**.

38 RLC circuit component values

This problem requires you to know that the current is maximized at the resonant frequency. The resonant frequency, ω, occurs when the complex impedance of the circuit vanishes. That is to say,

$$X_L = X_C, \tag{115}$$

$$\Rightarrow \omega L = \frac{1}{\omega C}, \tag{116}$$

$$\Rightarrow C = \frac{1}{\omega^2 L}, \tag{117}$$

for a circuit with a capacitor of capacitance C and an inductor of inductance L. Plugging in the values provided, we have

$$C = \frac{1}{(10^3 \text{ rad/s})^2 \times 25 \times 10^{-3} \text{ H}}, \tag{118}$$

$$= \frac{1}{25} \times 10^{-3} \text{ F}, \tag{119}$$

$$= 40 \times 10^{-6} \text{ F}. \tag{120}$$

Answer: **(D)**.

39 High-pass filters

First I'll give a qualitative solution requiring little written math, and then I'll go through the rigorous method. The qualitative solution is better for the test itself, as it takes very little time, but I wouldn't understand it if I didn't know the mathematical solution.

Qualitative solution

Qualitatively, recall what high- and low-pass filters *do*: A high-pass filter takes in an AC signal, and if the frequency is sufficiently high produces an output signal such that $V_{\text{in}} \approx V_{\text{out}}$. Otherwise, if the frequency is too low, no signal is passed to the output wire.

Another way to say this is that the voltage drop measured across the whole circuit should be approximately the same as the drop measured across the input. Low-pass filters are reversed; their goal is to measure a *lower* voltage drop, and allow lower frequencies through.

Now recall the equations for the complex impedance of capacitors and inductors:

$$Z_C = \frac{1}{i\omega C}, \tag{121}$$

$$Z_L = i\omega L. \tag{122}$$

So as the frequency increases, it is Z_L that gets large, while Z_C gets small. Now, which of the circuits will measure a higher voltage drop across the element labeled "Out"? Going through each option:

I. The inductor's complex impedance is proportional to frequency. At high frequencies (for a high-pass filter), the inductor will have a high voltage drop, leaving the resistor with a relatively small one. This is *not* a high-pass filter.

II. The resistor's impedance does not depend on frequency, but as said above, the inductor's impedance rises linearly. Therefore, at high frequencies, the inductor will have a higher drop than the resistor. This *is* a high-pass filter.

III. The capacitor's inductance is inversely proportional to frequency. By the reverse logic of circuit I, this *is* a high-pass filter.

IV. By the reasoning for the above circuits, this is *not* a high-pass filter.

Rigorous solution

Now, to solve this rigorously but slowly, you need to know that the total impedance is the sum of all complex impedances,

$$Z_{\text{tot.}} = R + i\left(\omega L + \frac{1}{\omega C}\right), \tag{123}$$

for a circuit with a resistor, inductor and capacitor.

* It is worth memorizing, though unnecessary for this specific problem, that total impedance is a complex number. This allows you to talk about the angle ϕ between the real and the complex components. This produces the handy reformulating of the

total impedance as

$$Z_{\text{tot.}} = |Z|\, e^{i\phi}, \tag{124}$$

$$|Z| = \sqrt{R_{\text{eff}}^2 + Z_C^2 + Z_L^2}, \tag{125}$$

$$\phi = \arctan\left(\frac{Z_{\text{eff.}}}{R_{\text{eff.}}}\right), \tag{126}$$

where $Z_{\text{eff.}} = Z_C + Z_L$ and $R_{\text{eff.}}$ are the effective complex impedances and effective resistances, respectively.

So, to do this problem, you must find the output voltage by using Ohm's law, $V = IZ$ (for complex impedance Z), and look at the behavior of the circuit as $\omega \to \infty$. You can solve for I with $I = V_{\text{in.}}/Z$, which is to say dividing the *input* voltage by whatever impedance it encounters. This is also helpful because it will give an output voltage in terms of the input voltage, and comparing those two is exactly what we need to do to determine if something is a high- or low-pass filter. Going through each circuit, we have:

I. In this case, $Z = R + i\omega L$. Therefore, $I = V_{\text{in.}}/\left(R + i\omega L\right)$, and so the output voltage is this current multiplied by the resistor at the end,

$$V_{\text{out.}} = R\frac{V_{\text{in.}}}{R + i\omega L}. \tag{127}$$

Now, look at the behavior at high frequency. As $\omega \to \infty$, $V_{\text{out.}} \to 0$. Therefore, at high frequency, the output turns off. This is therefore *not* a high-pass filter.

II. Here, $Z = R + i\omega L$ as before, but now the current must pass by the inductor instead of the resistor. Therefore, the input voltage is multiplied by the inductor's complex impedance instead,

$$V_{\text{out.}} = i\omega L\frac{V_{\text{in.}}}{R + i\omega L}. \tag{128}$$

Whether or not this converges provides an opportunity to review L'Hôpital's rule. If the limits of the numerator and the denominator exist, and they both converge to either 0 or $\pm\infty$, then L'Hôpital's rule applies, which says that not only does the limit of the fraction exist, but that it is equal to the limit of the derivative of the numerator divided by the derivative of the denominator. In math,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}, \tag{129}$$

$$\text{provided that } \lim_{x\to c} f(x), \lim_{x\to c} g(x) = 0 \text{ or } \pm\infty. \tag{130}$$

Here, we can quickly check that this condition is satisfied, and so we need only take the derivative of the numerator and the denominator with respect to ω and examine the limit of that.

$$\lim_{\omega\to\infty} \frac{V_{\text{in.}} iL}{iL} = V_{\text{in.}}. \tag{131}$$

So we have $V_{\text{out.}} \approx V_{\text{in.}}$ and this is a high-pass filter.

III. Here, $Z = R + 1/i\omega C$. The circuit is analogous to the first one in setting up the equation for the output voltage: you just swap out $i\omega L$ for $1/i\omega C$. So,

$$V_{\text{out.}} = R\frac{V_{\text{in.}}}{R + 1/i\omega C}. \tag{132}$$

In the limit as ω gets large, then, the denominator approaches R. This cancels with the R in the numerator, leaving an output voltage equal to the input. Therefore, this is a high-pass filter.

IV. As before, $Z = R + 1/i\omega C$, but the final circuit element is the capacitor. So the the output voltage will be

$$V_{\text{out.}} = (1/i\omega C)\,\frac{V_{\text{in.}}}{R + 1/i\omega C}, \tag{133}$$

which goes to 0 as $\omega \to \infty$. This is a low-pass filter.

Answer: **(D)**.

40 An RL circuit

As soon as the switch is closed, the battery is going to be pumping the inductor full of current, eventually leaving it impenetrable. Right away, this removes all the possible solutions which are not decaying, leaving only (D) and (E). Then you must have memorized the *time constant*, τ. You can derive τ by examining the response of the differential equation governing the system to a step function (provided here by flipping the switch). The time constant represents the amount of time for the system to decay if it continued at the original rate. In the case of the RL circuit, the differential equation is $V = \ddot{Q}L + \dot{Q}R$, and the time constant is $\tau = L/R$. For an RC circuit, by the way, $V = -RC\dot{V}$, with $\tau = RC$.

Finally, we have $\tau = L/R$, and $L = 10$ mH, and $R = 2\Omega$. So,

$$\tau = 10 \times 10^{-3}\ \text{H}/2\Omega = 5 \times 10^{-3}\ \text{s} = 5\ \text{ms}. \tag{134}$$

Answer: **(D)**.

41 Magnetic charge in Maxwell

The logic for this question is largely similar to a question about magnetic monopoles. You make analogous equations for **B** that you already have for **E**, and Maxwell's equations become symmetrical for **B** and **E**.

I. Presumably, if magnetic charge exists, then it only sources magnetic fields when it's not moving, and would leave the electric field sourced by electric charge untouched. This equation describes the electric field due to electric charge.

II. For the same reason as above, we would expect the magnetic charge to source a diverging field, if it behaves like electric charge.

III. We know that moving electric charge, electric current, does induce a curling magnetic field. Therefore, by analogy, moving magnetic current would induce a curling electric field.

IV. The equation for the curl of the electric field now contains only contributions from the magnetic field and the magnetic current (if we follow the prescription we just established above). Therefore, by analogy, the equation for the curl of the magnetic field should contain only contributions from the electric field and current, as it already does.

Answer: **(E)**.

42 Current through loops

It is handy here to recall the rule of thumb that systems will induce current in such a way that the resulting magnetic field "fights" the change in magnetic flux. (Some texts say "nature abhors a change in flux", but I've always personally found that statement difficult to understand.) In the system provided, the center ring is moving towards A and away from B. Taking one ring at a time:

- B: The center ring is moving *away* from ring B, so the flux is decreasing. In order to try and maintain the magnetic field B was experiencing, it will want a field pointing the same direction as the center ring, and so it will induce a current moving in the same direction, counterclockwise.
- A: The center ring is moving *towards* ring A, so its flux in increasing. Therefore, A will want a field pointing in the opposite direction of the center ring's, so its current will be the opposite direction, clockwise

Answer: **(C)**.

43 Angular momentum commutation relations

For this problem, one can either recall the identity for combining commutators, or do the brute-force method and derive the general rule. I'll do the latter.

The given commutations, written out, are

$$[L_x, L_y] = L_xL_y - L_yL_x = i\hbar L_z, \tag{135}$$
$$[L_y, L_z] = L_yL_z - L_zL_y = i\hbar L_x, \text{and} \tag{136}$$
$$[L_z, L_x] = L_zL_x - L_xL_z = i\hbar L_y. \tag{137}$$

So when we write out the desired commutation relation, it is clear that certain bits can be replaced by the relations above:

$$[L_xL_y, L_z] = L_xL_yL_z - L_zL_xL_y, \tag{138}$$
$$= L_x\left(i\hbar L_x + L_zL_y\right) - (i\hbar L_y + L_xL_z)L_y, \tag{139}$$
$$= i\hbar\left(L_x^2 - L_y^2\right). \tag{140}$$

In the lines above, I have inadvertently derived the identity which you might memorize,

$$[AB, C] = [A, B]C + B[A, C]. \tag{141}$$

Answer: **(D)**.

44 Energy of a particle in a box

This problem has a lot of red herrings. The only thing you need to know are the energies of the energy eigenstates. The measurement of the energy of a particle with a ensemble wavefunction will only give you the energy of one of the constituent wavefunctions.

Therefore, the only equation we need to concern ourselves with is

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = n^2E_1. \tag{142}$$

So, we're looking for answers which are both square multiples of E_1 and appear in the ensemble wavefunction. Incidentally, there's only one answer which is even a square, so we go with that one, but what if there were two which were perfect squares, say $9E_1$ and $25E_1$? We would still go with $9E_1$ because we see that the wavefunction has an eigenstate in the state $n = \sqrt{9} = 3$ but not in $n = \sqrt{25} = 5$.

Answer: **(D)**.

45 Harmonic oscillator eigenstates

This problem is largely similar to problem 29. We're given the eigenvalues of an operator and asked to find the expectation value of some ensemble state. The prescription is exactly the same: Apply the operator to each constituent eigenfunction, and take the inner product of the ensemble state to determine the expectation value.

The operator of interest is

$$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle\,, \tag{143}$$

which we will apply to the wavefunction

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{14}}|1\rangle - \frac{2}{\sqrt{14}}|2\rangle + \frac{3}{\sqrt{14}}|3\rangle\,, && (144)\\ &= \frac{1}{\sqrt{14}}\left(|1\rangle - 2|2\rangle + 3|3\rangle\right) && (145)\end{aligned}$$

Then we will take the inner product $\langle\psi|H|\psi\rangle$ to find the expectation value. Applying the operator goes like this:

$$\begin{aligned} H|\psi\rangle &= \frac{1}{\sqrt{14}}\left(H|1\rangle - 2H|2\rangle + 3H|3\rangle\right), && (146)\\ &= \frac{1}{\sqrt{14}}\left(\hbar\omega\left(1+\frac{1}{2}\right)|1\rangle - 2\hbar\omega\left(2+\frac{1}{2}\right)|2\rangle + 3\hbar\omega\left(3+\frac{1}{2}\right)|3\rangle\right), && (147)\\ &= \frac{\hbar\omega}{\sqrt{14}}\left(\frac{3}{2}|1\rangle - 2\frac{5}{2}|2\rangle + 3\frac{7}{2}|3\rangle\right). && (148)\end{aligned}$$

Then, to take the inner product of $H|\psi\rangle$ with $\langle\psi|$, multiply the coefficients of matching eigenfunctions (remember, the inner product is ultimately a dot product):

$$\begin{aligned} \langle\psi|H|\psi\rangle &= \frac{1}{\sqrt{14}} \times \frac{\hbar\omega}{\sqrt{14}}\left(1\times\frac{3}{2} + 2\times 2\frac{5}{2} + 3\times 3\frac{7}{2}\right), && (149)\\ &= \frac{43}{14}\hbar\omega. && (150)\end{aligned}$$

Answer: **(B)**.

46 Dependence of the de Broglie wavelength on a potential

This problem is asking us to manipulate two equations which depend on momentum (de Broglie wavelength and classical kinetic energy) and asks us to find the wavelength when the energy changes. The actual calculations are relatively straightforward; I think most of the trick to this problem is in remembering the de Broglie wavelength and that classically, kinetic energy is related to momentum.

So, before the particle enters the potential (while it's still free), it has a wavelength and energy

$$\lambda = \frac{h}{p}, \text{ and} \tag{151}$$

$$E = \frac{p^2}{2m}, \tag{152}$$

respectively. Afterward, the particle will have some wavelength, λ', and energy, E',

$$\lambda' = \frac{h}{p'}, \tag{153}$$

$$E' = \frac{p^2}{2m} - V = E - V. \tag{154}$$

And, because we want an equation for λ', I'll combine the two equations above like so:

$$\lambda' = \frac{h}{\sqrt{2m(E-V)}} \tag{155}$$

It is difficult at first glance to see what the best way is to relate the equations describing the particle before and after entering the potential. I think it is easiest is to see that Plank's constant is going to be the same in both, and to write out Plank's constant in terms of the relevant quantities. (This is sort of using a "conservation of Plank's constant", but that's also sort of dumb[1].).

So, we can write Plank's constant in terms of the particle's energy and wavelength before it hits the potential:

$$\lambda = \frac{h}{p}, \tag{156}$$

$$p = \sqrt{2mE}, \tag{157}$$

$$\Rightarrow \lambda = \frac{h}{\sqrt{2mE}}, \tag{158}$$

$$\Rightarrow h = \lambda\sqrt{2mE}. \tag{159}$$

[1] h is a fundamental constant, so it's not really conserved, it's an invariant. Thanks to my colleague Mike Sommer for pointing out this little piece of pedantry.

Then, we can plug this in for h in the equation for λ':

$$\lambda' = \lambda \frac{\sqrt{2mE}}{\sqrt{2m(E-V)}}, \tag{160}$$

$$= \lambda \sqrt{\frac{E}{E-V}}, \tag{161}$$

$$= \lambda \left(\frac{E-V}{E}\right)^{-1/2}, \tag{162}$$

$$= \lambda \left(1 - \frac{V}{E}\right)^{-1/2}. \tag{163}$$

Answer: **(E)**.

47 The change in entropy of an expanding container

I'm aware of two solutions to this which are both pretty quick; which one you should use depends on whether you're more comfortable with the equations of statistical mechanics or thermal physics. I'll go over both here.

Thermal physics solution

For this solution, you must know the second law of thermal dynamics (which you should absolutely have memorized in general anyway),

$$\Delta S = \int \frac{1}{T} dQ. \tag{164}$$

Then, the solution requires recognizing that the expansion of the container from $V \to 2V$ is isothermal ("A sealed and *thermally insulated* container..."). That means that the total heat change will be equal to the work done on the system, $dQ = dW$, and we already know from problem 6 that $W = \int P dV$ and, for an isotherm, $W = nRT \ln(V_f/V_i)$. Therefore,

$$dQ = PdV, \tag{165}$$

$$\Rightarrow \Delta S = \int \frac{P}{T} dV = nRT \ln(2V/V), \tag{166}$$

$$= nRT \ln(2). \tag{167}$$

Statistical mechanics solution

In stat mech, the relevant definition of entropy is

$$\Delta S = k_B \ln \Omega, \tag{168}$$

where Ω is the number of microstates of the system. Every additional particle will contribute one microstate for every particle already in the system. That is to say, n particles will have 2^n microstates. Therefore,

$$\Delta S = k_B \ln\left(2^n\right), \tag{169}$$

$$= k_B n \ln(2). \tag{170}$$

Then, we know that $k_B = R/N_A$, where N_A is Avogadro's number. So,

$$\Delta S = nR\ln(2) \tag{171}$$

Answer: **(B)**.

48 Estimating the root-mean-square velocity of a gas

This problem is kind of scary, but there's a nice approximation. If you remember that root-mean-square is kind of an average, then you can treat the gasses classically, and compare the velocities derived from their kinetic energies. Furthermore, because we know that the gasses are kept at the same constant temperature, we know from the equipartition theorem that their kinetic energies are both equal to $3/2k_BT$. This means that their kinetic energies are equal, so,

$$KE_{O_2} = \frac{1}{2}m_{O_2}v_{O_2}^2, \tag{172}$$

$$KE_{N_2} = \frac{1}{2}m_{N_2}v_{N_2}^2, \tag{173}$$

$$\Rightarrow \frac{v_{O_2}}{v_{N_2}} = \sqrt{\frac{m_{O_2}}{m_{N_2}}}, \tag{174}$$

$$= \sqrt{\frac{32}{28}} = \sqrt{\frac{8}{7}} \tag{175}$$

Answer: **(C)**.

49 Partition function and degeneracy

I can think of no clever way to do this other than to have the Maxwell-Boltzmann partition theorem memorized. This says,

$$Z = \sum_i g_i e^{-\epsilon_i/k_BT}, \tag{176}$$

where g and ϵ are the degeneracy and energy of a given state, respectively. Now, we are told that there are two states, each with a degeneracy of 2, and one has energy equal to ϵ

while the other has 2ϵ. So,

$$Z = 2e^{\epsilon/k_BT} + 2e^{2\epsilon/k_BT}, \quad (177)$$

$$= 2\left[e^{\epsilon/k_BT} + e^{2\epsilon/k_BT}\right]. \quad (178)$$

Answer: **(E)**.

50 Speed of sound in a cold flute

It is sufficient for this problem to know that the speed of sound goes as $\sqrt{T}$, so for temperatures greater than 0, speed of sound in a gas goes linearly to first order, $v \approx c + kT$, for some gas-specific constants c and k. Therefore, for small changes in temperature, the change in sound speed is proportional to the change in temperature. That means that if the temperature drops 3%, so does the sound speed. 3% of 440 is $3/100 \times 440 = 3 \times 4.4 = 13.2$, so the new speed of sound is $440 - 13.2 = 426.8$

Answer: **(B)**.

51 Light loss through polarizers

Every time light passes through a polarizing filter, the only thing that passes is the component of the electric field (and associated magnetic field) which is parallel to the polarization. This means that final filter will not block all the light, even though it is perpendicular to the first filter: The light coming through the second filter has a component of electric field pointing in the direction of the last polarizer, and so some of it will pass through.

For an incident beam of light with intensity I_0, the intensity, I, after passing through a filter of angle θ will be $I = I_0 \cos\theta$. Each filter is rotated 45°, with respect to the one before it, so the beam of light incident on each filter will be reduced by a factor of $\cos 45° = 1/2$. As a polarizer will cut in half the intensity of unpolarized light, the sequence of filters will render the final intensity $I_0 \to I_0/2 \to I_0/4 \to I_0/8$.

Answer: **(B)**.

52 Volume of a primitive unit cell

A primitive unit cell is a cell for which every lattice point is connected by a vertex. While the Simple Cubic (SC) lattice is its own primitive cell, the Body-Centered Cubic (BCC) and Face-Centered Cubic (FCC) have lattice points hanging out in between vertices. A handy rule is that the volume of a primitive cell is a^3/N where a is the side length of the conventional cell and N is the number of unique lattice points required to define the primitive cell. SC lattices require only 1 point to uniquely define the cell, so that confirms that it is its own primitive cell with volume a^3. BCC and FCC require 2 and 4 unique points, respectively, so they have volumes of $a^3/2$ and $a^3/4$.

Answer: **(C)**.

53 Temperature dependence of a semiconductor's resistivity

While I don't know of any clever way to rigorously do this problem, and it ultimately needs you to just know that $\rho \propto 1/T$, there is an argument to make the answer guessable. Semiconductors rely on the constituent atoms having some electrons in the conduction band of the p-type material available to make the jump to holes in the n-type. If the temperature is sufficiently low, however, these atoms will hunker down in their potential wells and refuse to jump. Therefore, you expect it to have very high resistivity at low temperatures, and, thankfully, there is only one plot which looks like that.

Answer: **(B)**.

54 Estimating impulse

Even if you didn't know that impulse is the integral of force, the problem gives you a well-defined and easy to calculate geometrical shape. It is my experience that such problems usually want something having to do with the area of the geometrical shape. Here, the area is given by

$$A = J = \frac{1}{2} \times 2 \times 2 = 2. \tag{179}$$

Answer: **(C)**.

55 Conservation of momentum at an angle

When the smaller particle has come to a stop, it has deposited momentum mv into the particle with mass $2m$, which breaks up into m-sized particles. We know exactly how much momentum these particles have in the x-direction: $mv/2$. That they now have a component of momentum in y means that their total speed must be greater than $v/2$.

Answer: **(E)**.

56 Volume of gas required to float a mass

When a body is in a fluid, the fluid will push on the body, causing a buoyant force. The buoyant force can be calculated by integrating the pressure due to the surrounding fluid over the surface area of the body. It works out to

$$\mathbf{f} = \rho_f g V_b \hat{\mathbf{r}}, \tag{180}$$

for a body of volume V_b in a fluid with density ρ_f. g is the force due to gravity, and the force points in the opposite direction that gravity is acting. Based on this equation, we can see that the density of helium provided in the question is not useful. You need only the density of the surrounding fluid, air.

So, this question is asking us to solve for the volume of helium, V_b, required for the buoyant force to impart $10\text{m/s}^2 \times 300$ kg $= 3000$ N. Rearranging the equation above and plugging in, we have

$$\begin{aligned} V_b &= \frac{\mathbf{f}}{\rho_f g}, && (181)\\ &= \frac{3000}{1.29 \times 10}, && (182)\\ &\approx \frac{3000}{13} \approx 230. && (183)\\ & && (184) \end{aligned}$$

Answer: **(D)**.

57 Force due to a jet of water

This is a perfect opportunity for unit analysis. The only solution with units of force is the first one:

$$\begin{aligned} \left[\rho v^2 A\right] &= \frac{[\text{kg}]}{[\text{m}]^3} \frac{[\text{m}]^2}{[\text{s}]^2} [\text{m}]^2, && (185)\\ &= \frac{[\text{kg}]\,[\text{m}]}{[\text{s}]^2}. && (186) \end{aligned}$$

Answer: **(A)**.

58 Deflection of a proton in electric and magnetic fields

This problem is all about manipulating the equation for Lorentz force, which, for a particle of charge q traveling with a velocity $\mathbf{v}$, and interacting with electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, reads

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \tag{187}$$

Then, there are two tricks which together make up the bulk of the solution.

For one, we know how fast the proton is going because it has been accelerated through a potential V. The potential will accelerate the proton according to $F = q\mathbf{E} = -q\nabla V$. Note that this is *not* the same $\mathbf{E}$ as the electric field pointing in the x-direction as described in the problem. This is merely the initial electric field used to get the proton up to speed; it is forgotten thereafter.

Two, that the proton does not deflect from its trajectory means that it is not experiencing a net force. That means that

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} = 0, \tag{188}$$

$$\Rightarrow \mathbf{E} = -\mathbf{v} \times \mathbf{B}. \tag{189}$$

So the force due to the electric field is equal to force due to the magnetic field when the proton is traveling with velocity $\mathbf{v}$.

Now, when the proton accelerates through a potential with double the magnitude of the original, the new velocity, $\mathbf{v}'$ will also be double:

$$\mathbf{F} = -q\nabla V, \tag{190}$$

$$\Rightarrow \mathbf{v} = \frac{1}{m_p}\int \nabla V\,dt, \text{ (for proton mass } m_p) \tag{191}$$

$$V \to 2V, \tag{192}$$

$$\Rightarrow \mathbf{v}' = \frac{2}{m_p}\int \nabla V\,dt = 2\mathbf{v}. \tag{193}$$

So now the total force will be

$$\mathbf{F} = q\left(-\mathbf{v} \times \mathbf{B}\right) + q\left(2\mathbf{v} \times \mathbf{B}\right). \tag{194}$$

The magnetic contribution is now greater than the electric contribution, so the proton will deflect in the direction that the magnetic field wills it. By the right-hand rule, that is the $-x$-direction.

Answer: **(B)**.

59 LC circuit-SHO equivalence

The equation of motion for a simple harmonic oscillator is

$$m\frac{d^2x}{dt^2} + kx = 0, \tag{195}$$

for a particle of mass m on a spring with spring constant k. Comparing with the equation given for the LC circuit, we can draw analogies between the coefficients of like terms:

$$m \to L, \tag{196}$$

$$k \to 1/C. \tag{197}$$

Finally, because Q is the quantity whose time-evolution you care about (it's being differentiated), it is playing the role of x.

Answer: **(B)**.

60 Flux of a sheet through a sphere

Electric flux is defined

$$\Phi = \int \mathbf{E} \cdot d\mathbf{a}, \tag{198}$$

which is one of the first applications of Gauss' law one sees in E&M. Because it's integrating the electric field over a surface area, it is equal to the charge enclosed in that surface divided by ϵ_0:

$$\Phi = \int \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc.}}}{\epsilon_0}. \tag{199}$$

So that's what we want to find for this problem: The charge enclosed by the sphere. This means finding the area of the circle of sheet which intersects the sphere.

I found the radius of this circle by drawing the radius of the sphere, R, to where the sheet meets the edge of the sphere. This makes a right triangle, with sides R, x, and $\sqrt{R^2 - x^2}$. The area of the circle with radius $\sqrt{R^2 - x^2}$ is $\pi\left(R^2 - x^2\right)$. Then, the total charge contained on this circle is this area times the charge density. Divide this by ϵ_0 to get the electric flux through the sphere,

$$\Phi = \frac{\pi\left(R^2 - x^2\right)\sigma}{\epsilon_0}. \tag{200}$$

This agrees with some limits, as well: When $x = R$, the sheet is outside of the sphere, and so the total flux through the sphere should be 0. Furthermore, when $x = 0$, the greatest amount of the sheet is contained in the sphere, and so the flux should be maximized.

Answer: **(D)**.

61 Electromagnetic wave hitting conductor

This question requires you to memorize the boundary conditions for electromagnetic waves which are changing media. I'll spend this problem talking about what these boundary conditions mean because they're a point of personal confusion, then I'll apply them to the problem of waves hitting a conductor.

Boundary conditions in E&M

The boundary conditions for electric and magnetic fields are

$$\text{(i) } \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}, \quad \text{(ii) } \epsilon_1 \mathbf{E}_1^{\perp} = \epsilon_2 \mathbf{E}_2^{\perp} + \sigma_f, \tag{201}$$

$$\text{(ii) } \mathbf{B}_1^{\perp} = \mathbf{B}_2^{\perp}, \text{ and (iv) } \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} + \left(\mathbf{K}_f \times \hat{\mathbf{n}}\right). \tag{202}$$

These describe how the electromagnetic field in one medium ($\mathbf{E}_1$ and $\mathbf{B}_1$) is related to the electromagnetic field in another medium ($\mathbf{E}_2$ and $\mathbf{B}_2$) *right at the boundary.* ϵ_1 and ϵ_2 are the permittivities of the two media, and μ_1 and μ_2 are the permeabilities. σ_f is the free surface charge and $\mathbf{K}_f$ is the free surface current. Finally, $\hat{\mathbf{n}}$ is a unit vector pointing perpendicularly from the surface.

These equations describe how the components of $\mathbf{E}$ and $\mathbf{B}$ which are parallel ($\|$) and perpendicular ($\perp$) change on either side of a boundary between media. What I consider the important lesson from these boundary conditions is that the parallel components of $\mathbf{E}$ are *continuous*, while the perpendicular components are *dis*continuous. The story for the magnetic field is swapped. Knowing these boundary conditions goes a long way to helping us solve problems of reflection and transmission of electromagnetic waves.

Reflection and transmission of electromagnetic waves on perfect conductors

Applying the above to reflection and transmission necessarily makes the assumption that when an electromagnetic wave comes in contact with a conductor, some of the wave will be transmitted and some of the wave will be reflected.

Thinking only about the electric field for a moment, this means that the total field on the left side will be composed of the incident field, E_I and the reflected field, E_R. The right side will only be composed of the transmitted field, E_T. In this case, there is no component of the electric field perpendicular to the conductor: The wave is traveling perpendicular to the conductor, so $\mathbf{E}$ and $\mathbf{B}$ must both necessarily be parallel (If it helps, recall the Poynting vector, $\mathbf{S} = \frac{1}{\mu}\left(\mathbf{E} \times \mathbf{B}\right)$, points in the direction of travel, so both the fields must be perpendicular to the direction of travel. In this case, that means they're parallel to the conductor.).

So right away, we can apply two of the boundary conditions. Condition (i) implies that

$$E_I + E_R = E_T, \tag{203}$$

and we know there is no perpendicular component, so $\sigma_f = 0$ by (ii). The question asks about the total fields on the left side of the conductor, so it would be really great if we knew something about E_T. Unfortunately, I don't know of a quick way to derive ${E_T}^2$, but when confronted with a problem like this, I think about how if you get inside a metal cage, your phone won't have a signal. That must mean that the transmitted electric field is 0. Therefore, $E_I = -E_R$, and so the total field on the left side of the conductor is 0. Because of the way the answers are provided, this is sufficient to give you the correct answer, but I'll provide an argument for why the magnetic field should be double the incident field.

The reflected electric wave is traveling the exact opposite direction of the incident wave, and so the Poynting vector of the reflected wave will read

$$\mathbf{S}_R = -\mathbf{S}_I = \mu\left(-\mathbf{E}_I \times \mathbf{B}_R\right), \tag{204}$$

which, when you go through the right-hand rule, reveals that B_R must be pointing in the same direction as B_I. Thus $\mathbf{B}$ will constructively interfere, and, as it's in phase, it will double its magnitude.

This is a long way of saying a general truth about waves incident on conductors: **Conductors kill the electric field and double the magnetic field.**

Answer: **(C)**.

62 Deriving mass from cyclotron frequency

"Cyclotron frequency" just means that a magnetic field is making a particle going in a circle. Therefore, to find its mass, we can set centrifugal force equal to Lorentz force and solve for mass:

$$m\omega^2 r = 2qvB, \tag{205}$$

$$\Rightarrow m = \frac{2qvB}{\omega^2 r}, \tag{206}$$

$$= \frac{2q\omega B}{\omega^2}, \tag{207}$$

$$= \frac{2qB}{\omega}. \tag{208}$$

Where above I have used the fact that $\omega = v/r$ to cancel one of the ω's in the denominator. Then, before plugging in, make sure to notice that you need to convert the normal frequency, given in Hertz, to an angular frequency. This is especially treacherous here, as one of the incorrect answers is equal to what you would get if you forget to do this.

$$\omega = 2\pi\nu = 2\pi \times 1600 \text{ Hz}. \tag{209}$$

Then we can plug in. In the interest of speed, notice that none of the answers have the same coefficient being multiplied by some number of tens. This means we can drop

[2] For the full thing, see p.396-7 of Griffiths, 3ed.

all orders of magnitude from the calculation, and only worry about finding the number in front.

$$
\begin{aligned}
m &= \frac{2qB}{\omega}, && (210)\\
&\text{(get the electron charge provided from the front of the test)} && (211)\\
&= \frac{2 \times 16 \times \pi}{2\pi \times 16 \times 4}, && (212)\\
&= \frac{1}{4}, && (213)\\
&= .25. && (214)
\end{aligned}
$$

And there is only answer which is .25 times some number of tens.

Answer: **(A)**.