A Modified Field Lagrangian for E&M

Alex Deich

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1 The New Action

The modified action used for this analysis is given by

$$S\left[A_\nu\right] = \alpha \int \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + m_0 A_\alpha A^\alpha - \kappa A_\alpha J^\alpha \right) \sqrt{-\eta}\; d^4x \tag{1}$$

with the field-strength tensor given by $F_{\alpha\beta} \equiv A_{\beta,\alpha} - A_{\alpha,\beta}$, and a conserved source J^ν. This is identical to the standard electromagnetic action but for the new coupling term $mA_\alpha A^\alpha$. Using the standard method of Euler-Lagrange analysis, and calculating the relevant terms in

$$-\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial A_{\nu,\mu}} \right) + \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \tag{2}$$

where we take the integrand of (1) as the Lagrangian $\mathcal{L}$, we find

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = m_0^2 A^\nu - \kappa J^\nu \tag{3}$$

for the full sourced field equation.

1.1 Gauge Freedom

We can immediately simplify (3) by looking at whether or not the action is insensitive to the introduction of a pure scalar field (its "gauge freedom"). Whether or not a given action is sensitive to such a perturbation is a defining quality, as it tells you about any redundant degrees of freedom in the field variables. For instance, in standard E&M (a gauge invariant theory) we can define A_ν as $A_\nu + \delta A_\nu$ with $\delta A_\nu \equiv \phi_{,\mu}$ where $\phi_{,\mu}$ is the four-gradient of some scalar field, ϕ, without changing the field equation at all. We are then free to say that under *any* circumstances, $\phi_{,\mu}$ is 0 (we made it, so we can destroy it), and this so-called Lorentz gauge allows for a wide range of simplifications and generalizations[1].

[1] There *are* other gauge choices – in the final section of this paper, we will encounter the Weyl gauge – and these are useful in different contexts. The point is that you can make it whatever you want.

So, to investigate the gauge freedom, we might perturb the action by some small scalar field gradient. Defining δA_ν as above, it is clear that the field-strength tensor does not change:

$$F_{\alpha\beta}[A_\nu + \delta A_\nu] = A_{\beta,\alpha} - A_{\alpha,\beta} + \underbrace{\phi_{,\beta\alpha} - \phi_{,\alpha\beta}}_{=0} = A_{\beta,\alpha} - A_{\alpha,\beta} \tag{4}$$

Which we already knew from the standard model of E&M. Similarly, you can show that the source term $\kappa A_\alpha J^\alpha$ remains fixed under the addition of a scalar field.

However, a problem pops up with the new term, $m_0^2 A_\alpha A^\alpha$. Starting with the field equation, (3),

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = m_0^2 A^\nu - \kappa J^\nu \tag{5}$$

$$A_\alpha \rightarrow A_\alpha + \phi_{,\alpha}$$

$$\partial_\mu \partial^\mu \left(A^\nu + \partial^\nu \phi\right) - \partial^\nu \partial_\mu \left(A^\mu + \partial^\mu \phi\right) = m_0^2 \left(A^\nu + \partial^\nu \phi\right) - \kappa J^\nu \tag{6}$$

And here, we *don't* get the nice cancellation we did in the field-strength tensor. (6) is *only* the same as (3) if we *require* that $\phi_{,\mu} = 0$. So our theory does not change under scalar addition, but only for a very specific scalar. We want this property, so this forces us into Lorentz gauge ($\partial_\nu A^\nu = 0$). Finally we have for our field equation,

$$\partial_\nu \partial^\nu A^\nu = m_0^2 A^\nu - \kappa J^\nu \tag{7}$$

2 Field Equations and Particle Interactions

So we have the field equation, (7). Given a particle with a relativistic action,

$$S = mc \int \sqrt{-\frac{dx^\mu}{dt} \eta_{\mu\nu} \frac{dx^\nu}{dt}} dt \tag{8}$$

it turns out that the simplest scalar term to couple the particle to the field is the exact same as the standard model of E&M:

$$S = mc \int \sqrt{-\frac{dx^\mu}{dt} \eta_{\mu\nu} \frac{dx^\nu}{dt}} dt + \alpha \int A_\nu \frac{dx^\mu}{dt} dt \tag{9}$$

Giving us the exact same equation of motion,

$$\frac{d}{dt} \left(\frac{m\mathbf{v}}{sqrt 1 - \frac{v^2}{c^2}} \right) = \alpha \left(\nabla A^0 c + \frac{\partial \mathbf{A}}{\partial t} \right) - \alpha \left(\mathbf{v} \times (\nabla \times \mathbf{A}) \right) \tag{10}$$

which suggests the definitions of the exact same fields:

$$\mathbf{E} \equiv -\nabla A^0 c - \frac{\partial \mathbf{A}}{\partial t} \tag{11}$$

$$\mathbf{B} \equiv \nabla \times \mathbf{A} \tag{12}$$

Of course, while this looks identical to standard E&M, remember that there is a different definition of A_ν, which will change the fields considerably.

3 Static Solutions to the Field Equations

Armed with our field equation, $\Box A^\nu = m_0^2 A^\nu - \mu_0 J^\nu$[2], we can start to think about the shapes of the fields themselves. Considering for the moment only the time-independent case, the field equation, along with the gauge condition and charge conservation give:

$$\begin{aligned} \nabla^2 V &= m_0^2 V - \mu_0 J^0 \\ \nabla^2 \mathbf{A} &= m_0^2 V - \mu_0 \mathbf{J} \\ \nabla \cdot \mathbf{A} &= 0 \\ \nabla \cdot \mathbf{J} &= 0 \end{aligned} \tag{13}$$

Where we have finally gone ahead and said that A_ν is a four-potential with 0-component V/c. We might as well at the same time define $J^0 \equiv \rho c$ for a charge density ρ.

3.1 Point Charge and the Green's Function

From this set of static field equations, we can extract the equations describing the fields themselves in cases involving high degrees of symmetry. In the case of a point source, for example, we can assume radial symmetry, $V(\mathbf{r}) = V(r)$, and solve

$$\nabla^2 V = m_0^2 V - \frac{1}{\epsilon_0}\rho \tag{14}$$

where $\rho = q\delta(r)$, a point source with charge q.

$V(r)$, then, is a Green's function, and while we can solve it informally in this case, the Green's function will appear again in more detail, so let's go through the rigamarole.

For all points $r \neq 0$, we're really solving the equation

$$\nabla^2 V = m_0^2 V. \tag{15}$$

This becomes pretty tractable when we assume spherical symmetry, because the Laplacian on the left-hand side can be replaced:

[2] I have redefined $\kappa \equiv \mu_0$, as we are more explicitly dealing with electro-esque quantities

$$
\begin{aligned}
\frac{1}{r}\frac{d^2}{dr^2}(Vr) &= m_0^2 V \\
\Rightarrow \frac{d^2}{dr^2}(Vr) &= r m_0^2 V
\end{aligned}
\tag{16}
$$

Which is straightforward to solve. The full solution is

$$V(r) = C_1 \frac{e^{-m_0 r}}{r} + C_2 \frac{e^{m_0 r}}{m_0 r} \tag{17}$$

For arbitrary C_1, C_2. Now, with a mere application of some boundary conditions, we can get our full Green's function. First of all, we know that we'd like the potential to go to 0 at spatial infinity. Of the two terms in (17), the second blows up at infinity. So $C_2 = 0$.

The second boundary condition, that $V(0) = q$, can be tackled by a direct integration of (14). Inserting what's left of (17) into (14), we have

$$\nabla^2 V = C_1 \frac{m^2}{r} e^{-m_0 r} \tag{18}$$

We can integrate this over a sphere of radius δ, and take the limit as $\delta \to 0$.

Integrating and employing the divergence theorem to the left hand side gives

$$C_1 e^{m_0 \delta}\left(-4\pi + 4\pi\delta - m_0^2 \delta\right) = -\frac{q}{\epsilon_0} \tag{19}$$

And, taking the limit $\delta \to 0$ results in

$$C_1 = \frac{q}{4\pi\epsilon_0}. \tag{20}$$

Giving us the full point source potential (which is also the Green's function),

$$V(r) = \frac{q}{4\pi\epsilon_0 r} e^{-m_0 r} \tag{21}$$

This means that the general potential for any given charge distribution can be determined from integrating the Green's function

$$V(\mathbf{r}) = \int \rho \mathbf{r}' G(\mathbf{r}, \mathbf{r}') d\tau' = \int \frac{\rho(\mathbf{r}') e^{-m_0 r}}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} d\tau' \tag{22}$$

(The assumption that the integration works is legitimate– The field equations are linear, so a sum of solutions is also a solution.)

3.2 Static Wire

We can also get the potential around an infinitely long charged wire. Performing the exact same Green's function strategy in cylindrical coördinates (assuming axial symmetry) with $\rho = \lambda\delta(s)$, results in a different form of (15):

$$\frac{\partial^2 V}{\partial s^2}s + \frac{\partial V}{\partial s} = s m_0^2 V \tag{23}$$

It should be self-evident (downright obvious, actually), that the solution to (23) is given by a sum of Bessel functions:

$$V(s) = C_1 J_0(i m_0 s) + C_2 Y_0(-i m_0 s) \tag{24}$$

Applying the same boundary conditions, $C_1 = 0$, and integrating in the same way as before, $C_2 = \lambda/4\epsilon_0$. Finally, noting that $Y_0(-ims) = K_0(ms)$, we get

$$V(s) = \frac{\lambda}{4\epsilon_0} K_0(ms). \tag{25}$$

If you don't believe the differential equation magic here, a nice visual confirmation of this appears in a paper in progress by my colleague Charlie McIntyre. Instead of solving the Green's function for a wire, he numerically integrates (21) which, when plotted, looks identical to K_0.

As you can see in figures 1 and 2, both of the potentials so far derived look superficially like the standard $V \sim 1/r$ and $V \sim \ln(1/s)$ potentials from standard E&M. It's important to keep in mind that although the forms of the equations are quite different, it's (in this analysis, so far) possible that this action is the "real" one, and that it is only due to a sufficiently small value of m_0 that we have gone for 150 years with the standard Maxwellian story. (After all, going back to the action at the beginning – the limit as $m_0 \to 0$ leaves us with regular E&M.)

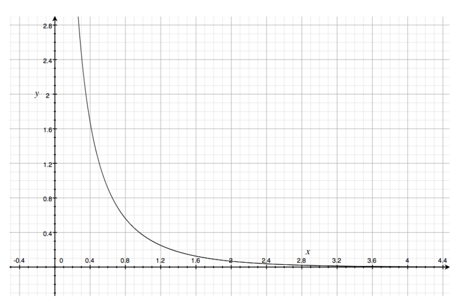

Figure 1: $V(r) \sim \frac{1}{r}e^{-r}$

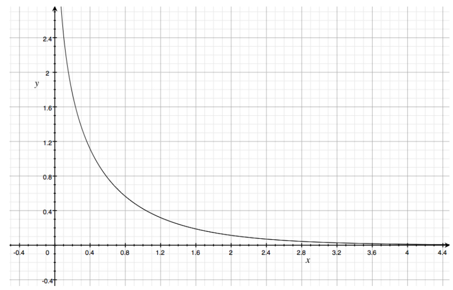

Figure 2: $V(s) \sim K_0(s)$

3.3 Current-carrying wire

Finally, we can get the vector potential for a current-carrying wire pretty easily because this theory was developed to as a relativistic one from the start. The potential is just

$$\mathbf{A} = \begin{pmatrix} c \\ \mathbf{v} \end{pmatrix} \frac{\lambda}{4\epsilon_0} K_0(ms). \tag{26}$$

4 Point Source Solution

It would be nice to have an integral from of the full field equation, (7)

$$\partial_\nu \partial^\nu A^\nu = m_0^2 A^\nu - \kappa J^\nu$$

To do this, we can just do a four-dimensional Green's function solution, where the Green's function we're finding satisfies

$$\left[-\frac{1}{2} \frac{\partial^2}{\partial t^2} + \nabla^2 - m_0^2 \right] G(\mathbf{r}, \mathbf{r}', t, t') = -\delta^4(x) \tag{27}$$

Now, defining

$$\tilde{G}(\mathbf{r}, t) = \int_\infty^\infty G(\mathbf{r}, t) e^{i2\pi f t} dt \tag{28}$$

as the temporal Fourier transform of G, we can convert this into a spatial problem,

$$\left[\left(\frac{2\pi f}{c} \right)^2 + \nabla^2 - m_0^2 \right] \tilde{G} = -\delta^3(x) \tag{29}$$

And, assuming spherical symmetry, for $r \neq 0$,

$$\tilde{G} = C_1 \frac{e^{r\left(\sqrt{-\left(\frac{2\pi f}{c}\right)^2 - m_0^2}\right)}}{r} + C_2 \frac{e^{r\left(\sqrt{\left(\frac{2\pi f}{c}\right)^2 - m_0^2}\right)}}{r\sqrt{\left(\frac{2\pi f}{c}\right)^2 - m_0^2}} \tag{30}$$

And, to recover the Green's function from the field equation, we want to set constants C_1, C_2 to $a/4\pi$ and $(1-a)/4\pi$ respectively.

5 Stress Tensor

Starting with the definition of the stress tensor,

$$\mathcal{T}^{\mu\nu} = -\left(2\frac{\partial \mathcal{L}}{\partial \eta_{\mu\nu}} + \eta_{\mu\nu}\mathcal{L}\right) \tag{31}$$

and calculating the relevant quantities from (1), we find

$$\mathcal{T}^{\mu\nu} = F^{\mu\nu}F^{\nu}_{\sigma} - m_0 A^{\mu}A^{\nu} + \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}m_0^2 A_{\alpha}A^{\alpha}\eta_{\mu\nu} \tag{32}$$

Which is to say, identical to the standard E&M stress tensor but with two new potential terms.

Considering the definition of $F^{\alpha\beta}$, we can write the components in terms of the fields $\mathbf{E}$ and $\mathbf{B}$:

$$\begin{aligned}
\mathcal{T}^{00} &= \frac{1}{2\mu_0}\left(\frac{E^2}{c^2} + B^2\right) - \frac{2m_0}{c^2}V^2 \\
\mathcal{T}^{0j} &= \frac{1}{2m_0 c}(\mathbf{E}\times\mathbf{B})^j - \frac{m_0 V}{c}A^j - \frac{1}{2}m_0^2 A^2 \\
\mathcal{T}^{ij} &= -\frac{1}{\mu_0}\left[\frac{1}{c^2}E^iE^j + B^iB^j - \frac{1}{2}g^{ij}\left(\frac{E^2}{c^2} + B^2\right) - A^iA^j - \frac{1}{2}m_0^2 A^2\right]
\end{aligned} \tag{33}$$

Reading the $\mathcal{T}^{00}$ component as the energy density, it becomes clear that the total energy density of stored in the field of a point source diverges:

$$\int \frac{1}{2\mu_0}\left(\frac{E^2}{c^2} + B^2\right) - \frac{2m_0}{c^2}V^2 = \infty \tag{34}$$

The trace of the tensor comes from $\mathbf{tr}\left(\mathcal{T}^{\mu\nu}\right) = \mathcal{T}^{\mu\nu}\eta_{\mu\nu}$:

$$\mathbf{tr}\left(\mathcal{T}^{\mu\nu}\right) = -m_0 A^{\nu}A_{\nu} \tag{35}$$

5.1 Vacuum Solutions

In a vacuum, the wave equation reads

$$\Box A^{\mu} = m_0^2 A^{\mu} \tag{36}$$

The solution we're interested in at the moment is the wave equation, which in the language of four-vectors reads

$$A^{\mu} = P^{\mu} e^{i\kappa_\alpha x^\alpha} \tag{37}$$

With constants P^{μ} and κ_α.

While the prescription is to find some restriction on P^{μ}, we are forced to more or less leave it in the equation as is. This is ultimately another consequence of the non-gauge invariance of this action. For when we compute (36) with (37), we wind up with

$$\begin{aligned} &-P^{\mu}\kappa^{\nu}\kappa_{\nu} e^{i\kappa_\alpha x^\alpha} = P^{\mu} e^{i\kappa_\alpha x^\alpha} \\ &\Rightarrow \kappa^{\nu}\kappa_{\nu} = -m_0^2 \end{aligned} \tag{38}$$

But when try to gauge fix again – by the same procedure, $A_\nu \to A_\nu + \phi_{,\nu}$ for some $\phi = Qe^{i\kappa_\alpha x^\alpha}$– and insist that $\Box\phi = 0$, we get that

$$\kappa^{\nu}\kappa_{\nu} = 0 \tag{39}$$

which is in conflict with (38).

So we're left just carrying around the P^{μ} in the vacuum solution. The fields go like this:

$$\mathbf{E} = -ice^{i\kappa_\alpha x^\alpha}\left(\kappa_0 \mathbf{P} - P^0 \kappa\right) \tag{40}$$

$$\mathbf{B} = i\kappa \times \mathbf{P} e^{i\kappa_\alpha x^\alpha} \tag{41}$$

The energy density of the fields is given by $\mathcal{T}^{00}$:

$$\begin{aligned} u &= \frac{1}{2\mu_0}\left(\frac{E^2}{c^2} + B^2\right) - \frac{2m_0}{c^2}V^2 \\ &= \frac{1}{2\mu_0}\left(c^2(\kappa_0^2 P^2 + P_0^2\kappa^2) - 2\mathbf{P}\cdot\kappa\kappa_0 P^0\right) e^{i\kappa_\alpha x^\alpha} - \frac{2m_0}{c^2}V^2 \end{aligned} \tag{42}$$

The momentum density can be read from $\mathcal{T}^{0j}$:

$$\mathcal{T}^{0j} = \frac{1}{2m_0 c}\left(\mathbf{E}\times\mathbf{B}\right)^j - \frac{m_0 V}{c}A^j - \frac{1}{2}m_0^2 A^2 \tag{43}$$

$$= -\frac{(-\kappa_0 c)}{\mu_0} P^2 e^{i\kappa_\alpha x^\alpha}\mathbf{k} - \frac{m_0 V}{c}A^j - \frac{1}{2}m_0^2 A^2 \tag{44}$$

Magnetohydrodynamics of the Modified Field Lagrangian

1 Magnetohydrodynamics

Magnetohydrodynamics (MHD) describes the motion of a charged fluid by combining the equations of E&M with those of Navier-Stokes fluid analysis. In this section I will outline the derivation of the MHD equations for the modified Lagrangian of this project.

1.1 What We're Looking For

To get a sense of what we're trying to derive in a set of MHD equations, let's refine our question. What do we want to know about a charged fluid? A reasonable answer is that we would like to describe the forces acting on the fluid at any given time, given some initial conditions. Well, if we have some notion of how its magnetic field (or vector potential) evolves in time, then we've answered this to a pretty reasonable degree. So that's what we're looking for: some kind of $d\mathbf{B}/dt$ statement.

Presumably, a magnetic fluid will be self-attracting. Particularly, every "little bit" of fluid will be exerting some kind of Lorentz force on every other "little bit". This suggests that we shouldn't be thinking of the fluid as a collection of discrete charges (as it surely is in reality), but rather as a continuum. In this scenario, everything becomes a density:

$$\begin{aligned} \mathbf{J} &\to \mathbf{j} \\ \mathbf{F} &\to \mathbf{f} \\ q &\to \rho_c \end{aligned}$$

(Which is to say, current becomes a current density per volume, force becomes a force density per volume, and a charge is a charge density ρ_c.)

So, we know that the Lorentz force (which was derived earlier) will be relevant:

$$\mathbf{f} = \rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B} \tag{1}$$

(This, incidentally, gives us a heck of an indication about where to look for our $\mathbf{B}$-field time evolution – it's hidden in the curl of $\mathbf{E}$.)

But this isn't the whole story. We also have to consider the last two letters of MHD and look at what Navier-Stokes gives us. First, the divergence theorem applied to an arbitrary boundary will give you mass conservation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{u} = 0 \tag{2}$$

(with fluid density ρ and velocity $\mathbf{u}$) but this isn't too exciting because there's nowhere to insert E&M– it just tells us about the fluid's density and mass, neither of which source the fields we're interested in.
More interesting to us is the formulation of Newton's Second Law for fluids:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u}\right) = -\nabla p + \rho\nu\nabla^2\mathbf{u} + \sum \mathbf{f} \tag{3}$$

(with pressure p and viscosity ν) Where the last term is the sum of all other forces, and where E&M will make things interesting.

So that's how the story's going to go, and why we want it to go there.

1.2 Magnetic Induction Equation

Starting with our equations of $\mathbf{E}$ and $\mathbf{B}$[1],

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} - \mu^2 V \tag{4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} - \mu^2\mathbf{A} \tag{7}$$

we can make some extremely helpful simplifications. In the MHD that we're considering, the motions of the fluids will be much slower than the speed of light. If we define L, T, and $U = L/T$ as the characteristic scales of length, time, and speed, respectively, and E and B the strengths of the fields, we can say the following:

$$\frac{E}{L} \sim \frac{B}{T} \text{ (from (5))} \tag{8}$$

$$\Rightarrow \frac{E}{B} \sim \frac{L}{T} = U \tag{9}$$

[1]The derivation of these follows directly from their definitions. My reproducing the arithmetic would not be helpful.

Then, from (7), we know

$$1 = \frac{1}{\nabla \times \mathbf{B}} \left(\mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \mu^2 \mathbf{A} \right) \tag{10}$$

Let's look at the second term from (10):

$$\mu_0 \epsilon_0 \frac{\partial \mathbf{E}/\partial t}{\nabla \times \mathbf{B}} \approx \mu_0 \epsilon_0 \frac{E/T}{B/L} \sim \mu_0 \epsilon_0 \frac{E}{B} \frac{L}{T} = \frac{1}{c^2} U^2 \tag{11}$$

So we have that the second term goes as the square of the characteristic speed divided by c. But in this model, we set $U \ll c$, so the whole $\frac{\partial \mathbf{E}}{\partial t}$ term can be ignored.

So now, we are free to say that

$$\begin{aligned} \nabla \times \mathbf{B} &\approx \mu_0 \mathbf{j} - m_0 \mathbf{A} && (12) \\ &= \mu_0 \left(\mathbf{E} + \mathbf{u} \times \mathbf{B} \right) - m_0 \mathbf{A} && (13) \\ \Rightarrow \nabla \times \left(\nabla \times \mathbf{B} \right) &= \nabla \times \left(\mu_0 \left(\mathbf{E} + \mathbf{u} \times \mathbf{B} \right) \right) - m_0 \mathbf{B} && (14) \end{aligned}$$

And, after some rearranging, we get

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{\mu_0} \nabla^2 \mathbf{B} - \frac{1}{\mu_0} m_0^2 \mathbf{B} \tag{15}$$

This is the induction equation, one of the most important equations of MHD.

In the same way that we eliminated any mention of $\mathbf{E}$ from the induction equation, we can do the same to the momentum conservation equation (3), and after applying the same steps, we get to the final three equations of MHD, for the modified E&M we're working with:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} &= 0 && (16) \\ \rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \right) &= -\nabla p + \rho \nu \nabla^2 \mathbf{u} + \frac{1}{\mu_0} (\nabla \times \mathbf{B} + m_0 \mathbf{A}) \times \mathbf{B} && (17) \\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{\mu_0} \nabla^2 \mathbf{B} - \frac{1}{\mu_0} m_0^2 \mathbf{B} && (18) \end{aligned}$$

2 Simulation

For simulating the modified MHD equations, I modified the open-source PLUTO code (plutocode.ph.unito.it). PLUTO's numerical methods are written in Fortran, while it uses Python's excellent matplotlib library for plotting purposes.

PLUTO uses an HLLC (Harten-Lax-van Leer-Contact) Riemann solver for solving the MHD system of equations. For our case, this system is as above. Implementing the HLLC

Riemann solver, then, is a matter of finding the eigenvalues, eigenvectors, and introducing the new $\mathbf{B}$ terms to the induction and momentum equations.

The initial conditions for the system were really tricky to nail down. What worked for the standard MHD model didn't now, and they had to be tinkered with significantly. Having finished it though, I have run three simulations which are plotted below They are density plots of a bubble of magnetic fluid expanding, all stopped at the same timestep.

As you can see, for a very small value of m_0, there is almost no effect, but when m_0 is of order unity, the whole face of the bubble density gets smoothed over. I cannot think of an intuitive explanation of this effect given the equations.

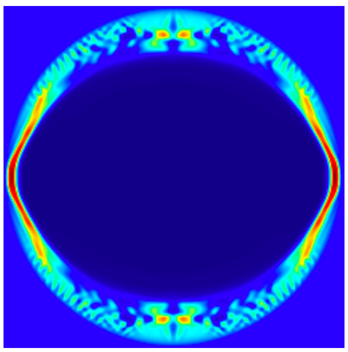

Figure 1: $m_0 = 0$

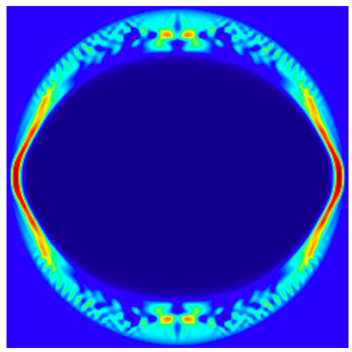

Figure 2: $m_0 = 0.001$

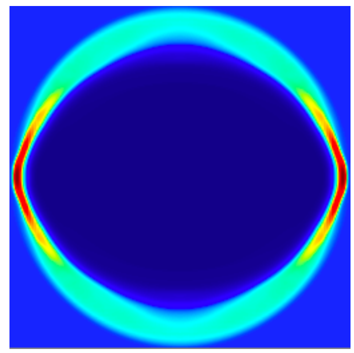

Figure 3: $m_0 = 1$

3 Future Work

There are two things I'd like to do in the future:

1) More computational simulation. I'd like to pin down the nature of the changes to the MHD equations with more simulation. I also admit that, given the strange outcome of my simulation above, I may not have implemented the changes correctly.

2) Pen-and-paper analysis. There are a lot of things to study with just the raw equations. One item of interest would be to calculate the direction of propagation of Alfvén waves. Alfén waves are transverse waves in a magnetic fluid whose restoring force is provided by the magnetic field. In standard MHD, they propagate parallel to the magnetic field, but that might not be the case here. (The analysis is simple enough– perturb (17) and linearize; a wave equation for $\mathbf{j}$ is bound to pop out– I just didn't get a chance to go through the derivation.)

4 Sources

- Sidney Moreland, in private communication
- Katherine Newton, private communication
- Joel Franklin, private communication
- Will Deich, private communication
- The PLUTO User's group at groups.google.com/plutousers
- David Griffiths' *Introdcution to Electrodynamics*
- David Fearn's MHD lecture notes at http://www.maths.gla.ac.uk/ drf/