Playing with the E&M Toy Model: A Charged Spinning Sphere

1 Comparing the E&M Analogy with Kerr

I argued in [insert section number here] that E&M is a strong analog to gravity. That was a generic field theoretic argument; let's take advantage of this analogy to use electromagnetism to get some concrete predictions about gravity. Taking the analogy as valid, we can construct a toy model to compare with Kerr.

To review, mapping gravity to electromagnetism, we can say:

$$m \to q \text{ (mass becomes charge)} \tag{1}$$

$$G \to \frac{1}{4\pi\epsilon_0} \text{ (constants switch)} \tag{2}$$

$$+ \to - \text{ (gravity is attractive; electricity, repulsive)} \tag{3}$$

Then, making special relativistic considerations, we can derive necessary magnetic fields as corrections to both electricity and gravity.

So, we're studying orbits in the Kerr spacetime, which is the consequence of a spinning mass. Accounting for the above mapping then, we should expect the analogous E&M system to be represented by Fig.1: A charged spinning sphere. Specifically, we want to study the orbit of a charged particle near the sphere. More to the point of this thesis, we seek how the orbits of the particles change when the spin of the central sphere changes.

2 A Description of the Relevant Magnetic Vector Potential

All quantities refer to Fig.1. Our system is a solid sphere of radius R with constant surface charge density σ, rotating about the z axis with some angular frequency ω.

The magnetic vector potential at an arbitrary point (r, θ, ϕ) outside the sphere is given by $\mathbf{A} = \frac{\mu_0 R^4 \sigma}{3} \frac{\sin\theta}{r^2} \hat{\phi}$. The derivation for this is reproduced in Appendix [insert number]. For our purposes, it is helpful to reparameterize the vector potential with the angular

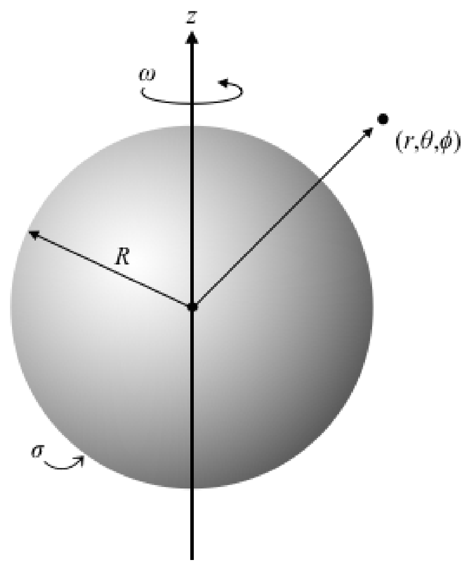

Figure 1

momentum, l and total charge Q instead[1]:

$$\mathbf{A} = \frac{1}{2}\frac{\mu_0 Q l}{4\pi}\frac{\sin\theta}{r^2}\hat{\phi} \tag{4}$$

The electric potential is unchanged from a static sphere: $V = \frac{Q}{4\pi\epsilon_0 r}$.

Then, to probe the trajectories of a particle of charge q near the spinning sphere, we need to look at the Lagrangian $L = \frac{1}{2}m\mathbf{v}^2 - qV + q\mathbf{v}\cdot\mathbf{A}$:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r\cos\dot{\theta} + r\sin^2\theta\dot{\phi}\right) - \frac{qQ}{4\pi\epsilon_0 r} + \frac{\mu_0 qQl}{8\pi}\frac{sin^2\theta}{r}\dot{\theta}. \tag{5}$$

For this analysis, we restrict ourselves to the $\theta = \pi/2$ plane. Having done this, the Lagrangian reveals two constants: Angular momentum about the z axis, $J_z \equiv \frac{\partial L}{\partial \dot{\phi}}$, and the total energy, E. Rewriting the resultant equations of motion in terms of these constants is helpful in setting initial conditions.

The relevant equations of motion are

$$\ddot{r} = \frac{J_z}{4\pi m^2 r^3} + \frac{qQ}{4\pi\epsilon_0 m r^2} - \frac{\mu_0 qQl}{8\pi m^2}\left(\frac{J_z}{3r^4} - \frac{\mu_0 qQl}{16\pi r^5}\right) \tag{6}$$

$$\dot{\phi} = \frac{J_z}{r^2 m} - \frac{\mu_0 qQl}{8\pi m r^3} \tag{7}$$

[1]This change is made using the angular momentum per unit mass, $I\omega/M$, with the moment of inertia of a sphere, $I = 2MR^2/5$.