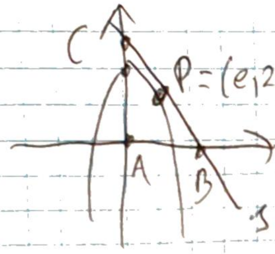
5.



$P = (e, 2-e^2), \ e>0$

$s: \ y = ax + b$

$f(x) = 2 - x^2, \ x \in R$

$f'(x) = -2x, \ x \in R$

$a = f'(e) = -2e$

$s: \ y = -2e \cdot x + b$

$2 - e^2 = -2e \cdot e + b$

$b = 2 + e^2$

$s: \ y = -2e \cdot x + 2 + e^2$

$C = (0, 2+e^2)$

$B = \left(\frac{e^2+2}{2e}, 0\right)$

$P_{ABC} = \frac{1}{2} \cdot \frac{e^2+2}{2e} \cdot e^2+2 = \frac{1}{4} \cdot \frac{(e^2+2)^2}{e}$

$g(e) = \frac{1}{4} \cdot \frac{(e^2+2)^2}{e}, \ e>0$

$g'(e) = \frac{1}{4} \cdot \frac{2(e^2+2) \cdot 2e^2 - (e^2+2)^2}{e^2} = \frac{1}{4} \cdot \frac{(e^2+2)(4e^2-e^2-2)}{e^2} =$

$= \frac{1}{4} \frac{(e^2+2)(3e^2-2)}{e^2} = \frac{3}{4} \frac{(e^2+2)(e^2-\frac{2}{3})}{e^2} =$

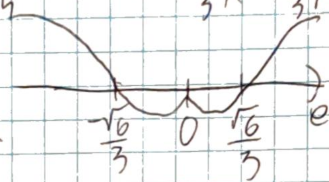
$= \frac{3}{4} \frac{(e^2+2)(e+\frac{\sqrt{6}}{3})(e-\frac{\sqrt{6}}{3})}{e^2}, \ e>0$

$g'(e) = 0$

$e = \frac{\sqrt{6}}{3}$

$g'(e) > 0$

$\frac{3}{4} e^2 (e^2+2)(e-\frac{\sqrt{6}}{3})(e+\frac{\sqrt{6}}{3}) > 0$

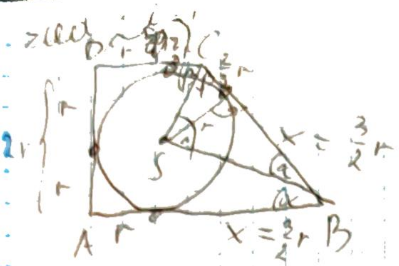


$e \in (\frac{\sqrt{6}}{3}, +\infty)$

$g'(e) < 0$

$e \in (0, \frac{\sqrt{6}}{3})$

Funkcja g rośnie w przedziale $\langle \frac{\sqrt{6}}{3}, +\infty)$ i malejąca w przedziale $(0, \frac{\sqrt{6}}{3}\rangle$ więc dla $e = \frac{\sqrt{6}}{3}$ występuje min. lokalne będące najmniejszą wartością funkcji



Δ ABC jest prostokątny, ponieważ BS oraz CS jest dwusiecznymi kątów trapezu

$$2\alpha + 2\beta = 180^\circ$$

$$\alpha + \beta = 90^\circ$$

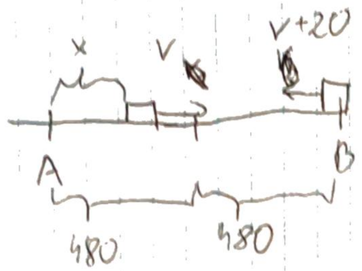
$$r^2 = \frac{2}{3}r \cdot x$$

$$\frac{3}{2}r = x$$

$$|AB| = \frac{5}{2}r$$

$$P_{ABCD} = \frac{\frac{5}{2}r + \frac{5}{3}r}{\cancel{2}} \cdot \cancel{2}r = \frac{25}{6}r^2$$

zad. 3



$x = v_{\cancel{1}} \cdot 2 = 2v_{\cancel{1}}$

$$\begin{cases} v \cdot t = 480 - x \\ (v+20) \cdot t = 480 \end{cases}$$

$$\begin{cases} vt = 480 - 2v \\ vt + 20t = 480 \end{cases}$$

$480 - 2v + 20t = 480$

$v = 10t$

$10t^2 = 480 - 20t$

$t^2 + 2t - \cancel{24}\,48 = 0$

$\Delta = 4 + 4 \cdot 48 = 4 \cdot 49$, $\sqrt{\Delta} = 2 \cdot 7 = 14$

$t_1 = \frac{-2-14}{2} < 0$

$t_2 = \frac{-2+14}{2} = 6$

$$\begin{cases} t = 6\ [h] \\ v = 60\ [km/h] \end{cases}$$

zał.: $a \leq 0 \wedge b \leq 0$

teza: $a^5+b^5 \leq ab^4+a^4b$

d-d: Pokażę, że:

$$a^5+b^5-ab^4-a^4b \leq 0$$

$$a^4(a-b)+b^4(b-a) \leq 0$$

$$a^4(a-b)-b^4(a-b) \leq 0$$

$$(a^4-b^4)(a-b) \leq 0$$

$$(a^2-b^2)(a^2+b^2)(a+b) \leq 0$$

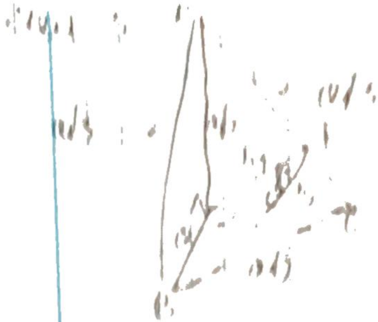
$$(a+b)(a-b)(a^2+b^2)(a-b) \leq 0$$

$$(a+b)(a-b)^2(a^2+b^2) \leq 0$$

$a+b \leq 0 \wedge (a-b)^2 \geq 0 \wedge a^2+b^2 \geq 0$, więc otrzymana nierówność jest prawdziwa, więc teza też jest prawdziwa

równość jest prawdziwa, gdy $a=b \vee a=-b$

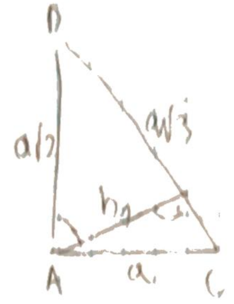
czyli $\begin{cases} a=0 \\ b=0 \end{cases}$



$x^2 = a^2 + 2a^2$

$x^2 = 3a^2, \ x>0$

$x = a\sqrt{3}$



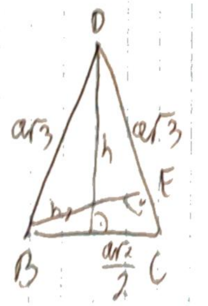
$P_{ACD} = \frac{a \cdot a\sqrt{2}}{2} = \frac{a^2\sqrt{2}}{2}$

$P_{ACD} = \frac{a\sqrt{3} \cdot h_1}{2}$

$\frac{a^2\sqrt{2}}{2} = \frac{a\sqrt{3} \cdot h_1}{2}$

$\frac{a\sqrt{2}}{\sqrt{3}} = h_1$

$h_1 = \frac{a\sqrt{6}}{3}$



$h^2 = 3a^2 - \frac{2a^2}{4}$

$h^2 = \frac{10a^2}{4}, \ h>0$

$h = \frac{a\sqrt{10}}{2}$

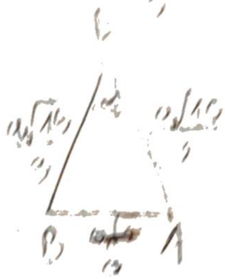
$P_{BCD} = \frac{1}{2} \cdot a\sqrt{2} \cdot \frac{a\sqrt{10}}{2} = \frac{a^2\sqrt{5}}{2}$

$P_{BCD} = \frac{a\sqrt{3} \cdot h_2}{2}$

$\frac{a\sqrt{3} \cdot h_2}{2} = \frac{a^2\sqrt{5}}{2}$

$h_2 = \frac{a^2\sqrt{5}}{a\sqrt{3}}$

$h_2 = \frac{a\sqrt{15}}{3}$

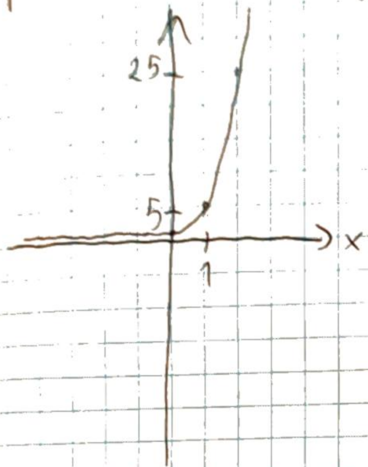


cos = 0, ponieważ kąt to α = 90°, więc cos 90° = 0 [illegible] f

$f(x) = p \cdot 5^{x} + (p+3) \cdot 5^{-x} - 4, \ x \in R$

$p \cdot 5^{x} + (p+3) \cdot 5^{-x} - 4 = 0$

$p \cdot 5^{x} + (p+3) \cdot \frac{1}{5^{x}} - 4 = 0, \quad 5^{x} = t$



Jedno rozwiązanie f generuje:
- 0 rozw. dla zmiennej x gdy $t \leq 0$
- 1 rozw. $t > 0$

$p \cdot t + (p+3) \cdot \frac{1}{t} - 4 = 0, \ t \neq 0$

$p t^{2} - 4 t + (p+3) = 0$

I $p = 0$

$-4t = 0$

$t = 0$

sprzeczność

$p = 0$ nie spełnia war. zadania

II $p \neq 0$

1° 1. $\Delta = 0$

2. $t_0 > 0$

ad. 1

$\Delta = 16 - 4(p+3) = -4p + 4$

$-4p + 4 = 0$

$p = 1 \wedge p \neq 0$

$p = 1$

ad. 2

$t_0 = \frac{4}{2} = 2 > 0$

$p \in R - \{0\}$

ad. 1 ∧ ad. 2

$p = 1$

2° 1. $\Delta > 0$

jedno ujemne, drugie zero

2. $(t_1 + t_2 < 0 \wedge t_1 t_2 = 0)$

$\vee \ t_1 t_2 < 0$

jedno dodatnie, drugie ujemne

ad. 1

$-4p + 4 > 0$

$p < 1 \wedge p \neq 0$

$p \in (-\infty, 0) \cup (0, 1)$

ad. 2

$t_1 + t_2 = 4$

$t_1 t_2 = p+3$

$(4<0 \wedge p+3=0) \vee p+3<0$

sprzeczność

$p<-3 \wedge p \neq 0$

$p \in (-\infty, -3)$

ad. 1 $\wedge$ ad. 2

$p \in (-\infty, -3)$

$1^0 \vee 2^0$

$p \in (-\infty, -3) \cup \{1\}$

I $\vee$ II

$p \in (-\infty, -3) \cup \{1\}$

zad. 7

$$\lim_{n \to \infty} \frac{\frac{1}{25} \cdot \left(\frac{1}{5}\right)^n + 2 \cdot \left(\frac{1}{4}\right)^n}{25 \cdot \left(\frac{1}{5}\right)^n + \frac{1}{2} \cdot \left(\frac{1}{4}\right)^n} = \lim_{n \to \infty} \frac{\left(\frac{1}{4}\right)^n \left(\frac{1}{25} \cdot \overbrace{\left(\frac{4}{5}\right)^n}^{\to 0} + 2\right)}{\left(\frac{1}{4}\right)^n \left(25 \cdot \underbrace{\left(\frac{4}{5}\right)^n}_{\to 0} + \frac{1}{2}\right)} = \frac{2}{\frac{1}{2}} = 4$$

$W(1) = S_p + S_{np} = 4$

$W(-1) = S_p - S_{np} = 2$

$$\begin{cases} S_p = 3S_{np} \\ S_p + S_{np} = 4 \end{cases}$$

$3S_{np} + S_{np} = 4$

$$\begin{cases} S_{np} = 1 \\ S_p = 3 \end{cases}$$

$W(-1) = 3 - 1 = 2$

a) z tw. o reszcie

$R(x) = W(1) = 4$

b) $R(x) = W(-1) = 2$

c) $W(x) = (x^2-1) \cdot Q(x) + ax + b$

$W(1) = (1-1) \cdot Q(1) + a + b = a + b$

$W(-1) = (1-1) \cdot Q(-1) - a + b = -a + b$

$$\begin{cases} a+b=4 \\ -a+b=2 \end{cases}$$

$$2b=6$$

$$\begin{cases} b=3 \\ a=1 \end{cases}$$

$h(x)=x+3$

zad. 3

$$f(x)=\begin{cases} \frac{x^m-1}{x-1}, & x\neq 1 \\ a_m, & x=1 \end{cases}$$

$$\lim_{x\to 1^-} \frac{x^m-1}{x-1} = \lim_{x\to 1^-} \frac{\cancel{(x-1)}(x^{m-1}+x^{m-2}+\ldots+1)}{\cancel{x-1}} = \underbrace{1+1+\ldots+1}_{m \text{ razy}} = m$$

$$\lim_{x\to 1^+} \frac{x^m-1}{x-1} = m$$

$f(1)=a_m$

f jest ciągła w punkcie $x=1$, więc

$$a_m=m$$

$a_2=2$

$a_6=6$

zad. 1

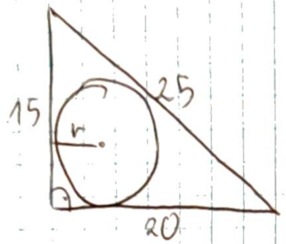
$x, y \in \mathbb{Z}$

$(x-2y-1)(x+2y+1) = 3$

$\begin{cases} x-2y-1=3 \\ x+2y+1=1 \end{cases} \vee \begin{cases} x-2y-1=-3 \\ x+2y+1=-1 \end{cases} \vee \begin{cases} x-2y-1=1 \\ x+2y+1=3 \end{cases} \vee \begin{cases} x-2y-1=-1 \\ x+2y+1=-3 \end{cases}$

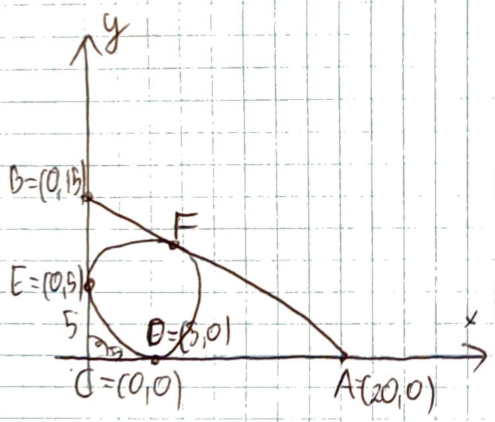
...

zad. 7



$r = \frac{20+15-25}{2} = 5\,[cm]$

$225+400$
625



$|AE| = \sqrt{20^2+5^2} = \ldots$

$|DB| = \sqrt{5^2+15^2} = \ldots$

$AB:\ y = ax+b$

$b = 15$

$y = ax+15$

$20a+15=0$

$a = -\frac{15}{20}$

$a = -\frac{3}{4}$

$y = -\frac{3}{4}x+15$

$S=(5,5),\ r=5$

$o: (x-5)^2+(y-5)^2=25$

$\begin{cases} (x-5)^2+(y-5)^2=25 \\ y=-\frac{3}{4}x+15 \end{cases}$

$(x-5)^2+(-\frac{3}{4}x+10)^2=25$

$x^2-10x+\cancel{25}+\frac{9}{16}x^2-15x+100=\cancel{25}$

$\frac{25}{16}x^2-25x+100=0$

$\frac{1}{16}x^2-x+4=0$

$x^2-16x+64=0$

$(x-8)^2=0$

$\begin{cases} x=8 \\ y=9 \end{cases}$

$F=(8,9)$

$|CF| = \sqrt{8^2+9^2}$

zad 6

$\left(\operatorname{tg}x, 1, \frac{\cos x}{1+\sin x}\right)$ - c. ar. , $x \neq \frac{\pi}{2}+k\pi \wedge x \neq \frac{3\pi}{2}+2k\pi \wedge x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$k \in \mathbb{Z}$

$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$1 = \frac{\operatorname{tg}x + \frac{\cos x}{1+\sin x}}{2}$$

$$2 = \operatorname{tg}x + \frac{\cos x}{1+\sin x}$$

$$\cancel{2+2\sin x = \frac{\sin x + \sin^2 x}{\cos x}}$$

$$2 = \frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x}$$

$$2(1+\sin x)\cos x = \sin x(1+\sin x) + \cos^2 x$$

$$2\cos x + 2\sin x\cos x = \sin x + \sin^2 x + \cos^2 x$$

$$2\cos x(1+\sin x) = \sin x + 1$$

$$2\cos x(1+\sin x) - (\sin x + 1) = 0$$

$$(\sin x + 1)(2\cos x - 1) = 0$$

$\sin x = -1 \quad \vee \quad \cos x = \frac{1}{2}$

sprzeczność

$$\left(x = \frac{\pi}{3}+2k\pi \vee x = -\frac{\pi}{3}+2k\pi\right) \wedge x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$x \in \left\{-\frac{\pi}{3}, \frac{\pi}{3}\right\}$$

1° $x = -\frac{\pi}{3}$

$$\left(\operatorname{tg}\left(\frac{\pi}{3}\right), 1, \frac{\cos\left(\frac{\pi}{3}\right)}{1+\sin\left(-\frac{\pi}{3}\right)}\right) = \left(-\sqrt{3}, 1, \frac{\frac{1}{2}}{1-\frac{\sqrt{3}}{2}}\right) = \left(-\sqrt{3}, 1, \frac{1}{2-\sqrt{3}}\right) =$$

$= \left(-\sqrt{3}, 1, 2+\sqrt{3}\right)$, $r > 0$

2° $x = \frac{\pi}{3}$

$\left(\operatorname{tg}\frac{\pi}{3}, 1, \frac{\cos\frac{\pi}{3}}{1+\sin\frac{\pi}{3}}\right) = \left(\sqrt{3}, 1, 2-\sqrt{3}\right)$, $r < 0$

sprzeczność

$\Rightarrow x = \frac{\pi}{3}$

zad. 5

zał.: $a = \underbrace{11\ldots1}_{n}\underbrace{22\ldots2}_{n+1}5,\ n \in N$ rozpisując powstają wyrazy c. geo.

teza: $a = k^2,\ k \in N$

d-d: $a = \cancel{5+20+200+\ldots 20\cdot 10^{n+1}} +$

$\cancel{1\cdot 10^{n+1} + 1\cdot 10^{n+1}\cdot 10 + \ldots + 1\cdot 10^{n+1}\cdot 10^{n-1}} =$

$\cancel{= 5 + 20\cdot\frac{10^{n+1}-1}{10-1} + 1\cdot 10^{n+1}\cdot\frac{10^n-1}{10-1}} =$

$\cancel{= 5 + 20\cdot\frac{10^{n+1}-1}{9} + 10^{n+1}\cdot\frac{10^n-1}{9}} =$

$\cancel{= \frac{45 + 20\cdot 10^{n+1} - 20 + 10^{n+1}\cdot 10^n - 10^{n+1}}{9}} =$

$\cancel{= 25 + 19\cdot 10^{n+1} + 10^{2n+1}}$

$$= 5 + 2\cdot 10 + 2\cdot 10^2 + \ldots + 2\cdot 10^{n+1} + 1\cdot 10^{n+2} + 1\cdot 10^{n+2}\cdot 10 + \ldots + 1\cdot 10^{n+2}\cdot 10^{n-1} =$$

$$= 5 + 20\cdot\frac{10^{n+1}-1}{9} + 1\cdot 10^{n+2}\cdot\frac{10^n-1}{9} =$$

$$= \frac{45 + 20\cdot 10^{n+1} - 20 + 10^{2n+2} - 10^{n+2}}{9} =$$

$$= \frac{25 + 20\cdot 10^{n+1} + (10^{n+1})^2 - 10\cdot 10^{n+1}}{9} =$$

$$= \frac{25 + 10\cdot 10^{n+1} + (10^{n+1})^2}{9} = \frac{5^2 + 2\cdot 5\cdot 10^{n+1} + (10^{n+1})^2}{9} =$$

$$= \frac{(5+10^{n+1})^2}{9} = \left(\frac{10^{n+1}+5}{3}\right)^2$$

liczba $10^{n+1}+5$ jest podzielna przez 3, ponieważ suma jej cyfr $=6$

więc $\frac{10^{n+1}+5}{3}$ jest liczbą naturalną c.n.d.

zad. 7

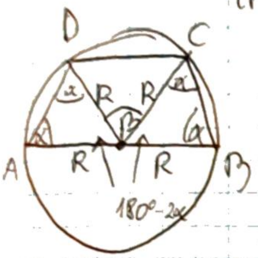
$P(A) = \binom{n}{k} \cdot \left(\frac{1}{2}\right)^k \cdot \left(\frac{1}{2}\right)^{n-k}$

~~P(B)~~

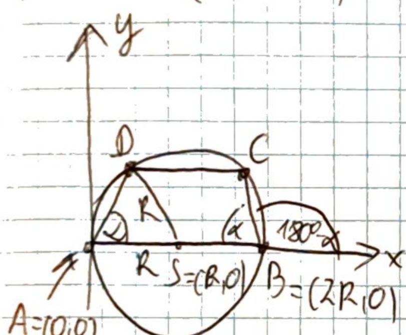
~~P(C)~~ =

zad. 3

trapez jest równoramienny.



$\beta = 180° - 2(180° - 2\alpha) = 180° - 360° + 4\alpha = 4\alpha = 180°$



AD: $y = ax + b$

$a = tg\alpha,\ b = 0$

$y = tg\alpha \cdot x$

$D = (d,\ d\,tg\alpha)$

o: $(x-R)^2 + y^2 = R^2$

$(d-R)^2 + d^2 tg^2\alpha = R^2$

$d^2 - 2Rd + \cancel{R^2} + d^2 tg^2\alpha = \cancel{R^2}$

$d(d - 2R + d\,tg^2\alpha) = 0$

$d = 0$ sprzeczność

~~$d = 2R\ tg^2\alpha$~~

$d(1 + tg^2\alpha) = 2R$

$d\left(1 + \frac{\sin^2\alpha}{\cos^2\alpha}\right) = 2R$

$d\,\frac{\sin^2\alpha + \cos^2\alpha}{\cos^2\alpha} = 2R$

$\frac{d}{\cos^2\alpha} = 2R$

$d = 2R\cos^2\alpha$

$D = (\cancel{d,\ 2R\cos^2\alpha} \cdot 2R\cos^2\alpha,\ 2R\cos^2\alpha \cdot \frac{\sin\alpha}{\cos\alpha}) =$

$= (2R\cos^2\alpha,\ 2R\sin\alpha\cos\alpha)$

$BC: \ y = ax + b$

$a = tg(180° - \alpha) = -tg\,\alpha$

$y = -tg\,\alpha \cdot x + b$

$-tg\,\alpha \cdot 2R + b = 0$

$b = 2R\,tg\,\alpha$

$BC: \ y = -tg\,\alpha \cdot x + 2R\,tg\,\alpha = tg\,\alpha\,(2R - x)$

$C = (e,\ tg\,\alpha\,(2R - e))$

$\cancel{c^2 = (e-R)^2 + tg^2\alpha\,(2R-e)}$

$tg\,\alpha\,(2R - e) = 2R \sin\alpha\cos\alpha$

$\frac{\sin\alpha}{\cos\alpha}(2R - e) = 2R\sin\alpha\cos\alpha \ \Big| : \sin\alpha$, bo $\sin\alpha \neq 0$

$\frac{2R - e}{\cos\alpha} = 2R\cos\alpha$

$2R - e = 2R\cos^2\alpha$

$e = 2R(1 - \cos^2\alpha)$

$e = 2R\sin^2\alpha$

$C = (2R\sin^2\alpha,\ tg\,\alpha \cdot 2R\cos^2\alpha) = (2R\sin^2\alpha,\ 2R\sin\alpha\cos\alpha)$

$|CD| = 2R\sin^2\alpha - 2R\cos^2\alpha = 2R(\sin^2\alpha - \cos^2\alpha) = -2R(\cos^2\alpha - \sin^2\alpha) =$
$= -2R\cos 2\alpha$

$h = 2R\sin\alpha\cos\alpha = R \cdot 2\sin\alpha\cos\alpha = R\sin 2\alpha$

$P_{ABCD} = \frac{2R + (-2R\cos 2\alpha)}{2} \cdot R\sin 2\alpha = \frac{2R(1 - \cos 2\alpha)}{2} \cdot R\sin 2\alpha =$

$= R(1 - \cos 2\alpha) \cdot R\sin 2\alpha = R^2 \sin 2\alpha\,(1 - \cos 2\alpha)$

zad. 4

w_1, w_2 - wydajności automatów

$$\begin{cases}(w_1+w_2)\cdot 6 = P \\ w_1\cdot 2 + w_2\cdot 6 = \frac{1}{2}P\end{cases}$$

$$\begin{cases}6w_1 + 6w_2 = P \\ 4w_1 + 12w_2 = P\end{cases}$$

$$6w_1 + 6w_2 = 4w_1 + 12w_2$$

$$2w_1 = 6w_2$$

$$w_1 = 3w_2$$

$$18w_2 + 6w_2 = P$$

$$\begin{cases}w_2 = \frac{P}{24} \\ w_1 = \frac{P}{8}\end{cases}$$

Odp. 1szy w ciągu 8h, drugi w ciągu 24h.

$s\,x)$

$\sin^2 x)$