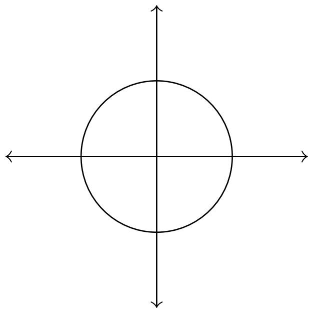
Groups 4 Subgroups

Contents

1 intro

$$\text{Consider } U = \{ z \in \mathbb{C} \mid |z| = 1 \} \subseteq \mathbb{C}.$$



note:
each elem of U is defined by $\theta \in [0, 2\pi) = \mathbb{R}_{2\pi}$

$$\text{Every angle } \theta \in \mathbb{R}_{2\pi} \text{ given by } f : U \to R_{2\pi},\ f(z) = \theta \text{ for } z = e^{i\theta}$$

$$+$$

$$f^{-1} : \mathbb{R}_{2\pi} \to U,\ f^{-1}(\theta) = e^{i\theta}$$

Consider U with multiplication:

Let $z_1 = e^{i\theta_1},\ z_2 = e^{i\theta_2} \in U$

then $z_1 \cdot z_2 = e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$ where $\theta_1,\ \theta_2 \in \mathbb{R}_{2\pi},\ \theta_1 + \theta_2 \in \mathbb{R}_{2\pi}$

Set U is <u>closed</u> under multiplication.

2 definition

def **algebraic structure**

Let S be a set.
Let $\star$ be a binary operation on S
$\quad \star : S \times S \to S$
$(S, \star)$ is an **algebraic structure**

So $(U, \cdot)$ and $(\mathbb{R}_{2\pi}, +_{2\pi})$ are algebraic structures such that there is a "1-1" relation between U and $\mathbb{R}_{2\pi}$ and operations are preserved.

$$(U, \cdot) \simeq (\mathbb{R}_{2\pi}, +_{2\pi})$$

or

$$U \simeq \mathbb{R}_{2\pi}$$

3 isomorphisms

To show two algebraic structures are isomorphic, find a bijective function that preserves the operations. Isomorphic structures share the same algebraic properties (associativity, commutative, distributive...).

To show two algebraic structures are <u>not</u> isomorphic, find a property that they don't share (the easiest to check is cardinality).

3.1 Examples

$$(\mathbb{N}, +) \not\cong (\mathbb{Z}, +)$$

because $\mathbb{N}$ has no additive identity

$$(\mathbb{Z}^*, \cdot) \not\cong (\mathbb{Q}^*, \cdot)$$

because not all elements of $\mathbb{Z}^*$ have inverses $\left(\frac{n}{m}\right)^{-1} = \frac{m}{n}$

$$(\mathbb{R}, +) \not\cong (\mathbb{R}, \cdot)$$

because $x \star x + 1 = 0$ is solveable in $(\mathbb{R}, +) : 2x + 1 = 0$ but unsolveable in $(\mathbb{R}, \cdot) : x^2 + 1 = 0$