Sets 4 Relations

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1 notation

$\mathbb{R}$	the set of real numbers	
$\mathbb{C}$	the set of complex numbers	
$\mathbb{Q}$	the set of rational numbers	$\{ \frac{n}{m} \mid n, m \in \mathbb{Z} \wedge m \neq 0 \wedge \gcd(n, m) = 1 \}$
$\mathbb{Z}$	the set of integers	
$\mathbb{N}$	the set of natural numbers	$\{ 1,\ 2,\ 3,\ \ldots \}$
A^*	the non-zero elements of A	$\{ a \in A \mid a \neq 0 \}$
A^+	the positive elements of A	$\{ a \in A \mid a > 0 \}$
A^-	the negative elements of A	$\{ a \in A \mid a < 0 \}$
$\mathbb{R} \setminus \mathbb{Q}$	the set of irrational numbers	$\{ a \in \mathbb{R} \mid a \notin \mathbb{Q} \}$ note: $(\mathbb{R} \setminus \mathbb{Q}) \cap \mathbb{Q} = \varnothing$
$\exists x$	there exists at least one x	
$\exists' x$	there exists only one x	

2 set definitions

def **subset**

A is a **subset** of B $A \subseteq B$ iff $\forall a \in A,\ a \in B$

def **proper subset**

A is a **proper subset** of B iff $\forall a \in A, a \in B \ \wedge\ \exists b \in B, b \notin A$ (denoted $A \subset B$)

def **relation**

a **relation** $\mathcal{R}$ on a set A is a subset of $A \times A$ $\qquad \mathcal{R} \subseteq A \times A$

def **equivalence relation**

$\mathcal{R}$ is an **equivalence relation** iff

1) it is **reflexive**

   def **reflexive**

   $\forall a \in A,\ a\mathcal{R}a \quad ((a, a) \in \mathcal{R})$

2) it is **symmetric**

   def **symmetric**

   $\forall\, a,\, b \in A,\ a\mathcal{R}b \Rightarrow b\mathcal{R}a$

3) it is **transitive**

   def **transitive**

   $\forall\, a,\, b,\, c \in A,\ a\mathcal{R}b \wedge b\mathcal{R}c \Rightarrow a\mathcal{R}c$

3 set operations

$A \cup B$	union	$\{\, x \mid x \in A \vee x \in B \,\}$
$A \cap B$	intersection	$\{\, x \mid x \in A \wedge x \in B \,\}$
$A \setminus B$	complement of B in A	$\{\, x \mid x \in A \wedge x \notin B \,\}$
$A \times B$	cartesian product of A and B	$\{\, (x, y) \mid x \in A \wedge x \in B \,\}$

note:

$$\left.\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \end{aligned}\right\} \text{commutative}$$

$$\left. A \times B \neq B \times A \right\} \text{not commutative}$$

4 properties of relations

def **injective**

a function $f : A \to B$ is **injective** (one-to-one) iff $\forall\, a_1, a_2 \in A,\ f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

def **surjective**

$f : A \to B$ is **surjective** (onto) iff $\forall b \in B\ \exists a \in A$ where $f(a) = b$

def **bijective**

a function $f : A \to B$ is **bijective** iff it is injective (one-to-one) and surjective (onto B).

note:

a bijective function has an inverse

5 cardinality

def **equinumerous**

having the same cardinality (size)

let S be a collection of sets and $\mathcal{R}$ a relation on S.

let A, B be sets and $A, B \in S$. $A\mathcal{R}B$ iff $\exists f$ where $f : A \to B$ and f is bijective. if $A\mathcal{R}B$, A and B are equinumerous because they have a one-to-one correspondence

consider:

1) $\forall A \in S$, $A\mathcal{R}A$ because $\exists f = \mathrm{id}_A : A \to A$ as f, bijective, and so $\mathcal{R}$ is reflexive
2) $\forall A, B \in S$, $A\mathcal{R}B \Rightarrow B\mathcal{R}A$ because $\exists f : A \to B$ and as f is bijective $\exists g = f^{-1} : B \to A$, so $\mathcal{R}$ is symmetric
3) $\forall A, B, C \in S$, $A\mathcal{R}B \wedge B\mathcal{R}C \Rightarrow A\mathcal{R}C$ as $\exists f : A \to B$, $\exists g : B \to C$ and f, g are bijective $\Rightarrow \exists h : A \to C$ so $\mathcal{R}$ is transitive

thus $\mathcal{R}$ is an equivalence relation on S and partitions S $\square$

note:

$E_A = \{ \mathcal{R} \in S \mid A\mathcal{R}B \}$ contains all the sets of size $|A|$

def **denumerable**

a set A is **denumerable** iff A is equinumerous to $\mathbb{N}$

def **countable**

a set A is **countable** iff A is finite or denumerable