Study Report: Narrowing

Deivid Vale

November 8, 2018

Abstract

Narrowing...

1 Introduction

2 Preliminaries

3 Narrowing

In this section, we discuss the use of narrowing techniques on solving equations over an equational theory E. By solving an equation we mean finding a substitution σ such that $s\sigma =_E t\sigma$. These solutions can be found by unification *modulo* E (if such a unification algorithm exists for this theory), but here we are interested in applications of rewriting theory to solving such equations.

We start the presentation by the so-called syntactic unification, i.e. E-unification with empty E. The transformation rules given below is due to Martelli and Montanari [citation], the idea is to transform sets of equations to other sets of equations until a termination state is reached; that is, a solution state or a failure one. We also extend this same set of rules for solving E-equations.

Definition 1. Let Σ be a signature. An equational goal is a finite set of Σ-equations.

Table 1 Martelli-Montanari rules

(1) Trivial

$$\{x \stackrel{?}{=} x\} \cup G \implies G$$

Delete trivial equations.

(2) Decompose

$$\{f(s_1, \cdots, s_n) \stackrel{?}{=} f(t_1, \ldots, t_n)\} \cup G \implies \{s_1 \stackrel{?}{=} t_1, \ldots, s_n \stackrel{?}{=} t_n\} \cup G$$

(3) Symbol Clash

$$\{f(s_1, \ldots, s_n) \stackrel{?}{=} g(t_1, \ldots, t_n)\} \cup G \implies \perp \text{ if } f \neq g$$

(4) Orient

$$\{t \stackrel{?}{=} x\} \cup G \implies \{x \stackrel{?}{=} t\} \cup G \text{ if } t \notin V$$

(5) Occurs Check

$$\{x \stackrel{?}{=} t\} \cup G \implies \perp \text{ se } x \in \mathtt{vars}(t) \text{ and } x \neq t$$

(6) Variable Elimination

$$\{x \stackrel{?}{=} t\} \cup G \implies G\{x \mapsto t\} \text{ if } x \notin \mathtt{vars}(t)$$

The application of the above rules non-deterministically transforms goals into goals:

$$G_0 \implies \cdots \implies G_n$$

Each application of a rule will then called a *elementary derivation step*. As for the case of (Variable Elimination), we may get some substitution on the way, we make them explicit by writing:

$$G_0 \implies G_1 \implies_{\sigma_1} G_2 \implies_{\sigma_2} \cdots \implies_{\sigma_i} \cdots \implies_{\sigma_{n-1}} G_n$$

The computed solution of the derivation chain is then the composition of such substitutions in their order of appearance.

Definition 2. A *successful* derivation chain is a finite sequence of equational goals $G_0, G_1, \ldots, G_n$ such that the last goal is empty. We also say a derivation chain has *failed* if it end is the fail symbol $\perp$.

Example 1. 1. We want to determine an mgu of the terms $f(g(x), h(x, u))$ and $f(z, h(f(y, y), z))$. That is, solving the equational goal:

$$
\begin{aligned}
&\{f(g(x), h(x, u)) = f(z, h(f(y, y), z))\} \\
&\quad \stackrel{(2)}{\Longrightarrow} \{g(x) = z, h(x, u) = h(f(y, y), z)\} \\
&\quad \stackrel{(4)}{\Longrightarrow} \{z = g(x), h(x, u) = h(f(y, y), z)\} \\
&\quad \stackrel{(6)}{\Longrightarrow}_{[z/g(x)]} \{h(x, u) = h(f(y, y), g(x))\} \\
&\quad \stackrel{(2)}{\Longrightarrow} \{x = f(y, y), u = g(x)\} \\
&\quad \stackrel{(6)}{\Longrightarrow}_{[x/f(y,y)]} \{u = g(f(y, y))\} \\
&\quad \stackrel{(6)}{\Longrightarrow}_{[u/f(y,y)]} \emptyset
\end{aligned}
$$

We get as the computed solution to the problem the composition $[z/g(x)][x/f(y, y)][u/g(f(y, y))]$.

2. A failing unification derivation:

$$
\begin{aligned}
&\{h(x, y, x) = h(y, g(x), x)\} \\
&\quad \stackrel{(2)}{\Longrightarrow} \{x = y, x = g(x), x = x\} \\
&\quad \stackrel{(5)}{\Longrightarrow} \perp
\end{aligned}
$$

It can be proved that this set of rules derive a correct and terminating procedure for syntactic unification. (add citation here).

Definition 3. Say a term s *narrows* to a term t if there exists a non-variable position $p \in \texttt{pos}(s)$, a variant $l \to r$ of a rewrite rule in $\mathcal{R}$, and a substitution σ satisfying two conditions:

1. σ is a *mgu* of $s|_p$ and l,
2. $t = (s[r]_p)\,\sigma$.

The relation $\rightsquigarrow$ is called *narrowing relation.* We write $s \rightsquigarrow^*_\sigma t$ if there exists a narrowing derivation

$$s = t_1 \rightsquigarrow_{\sigma_1} t_2 \rightsquigarrow_{\sigma_2} t_3 \rightsquigarrow \cdots \rightsquigarrow_{\sigma_{n-1}} t_n = t$$

and σ is given by $\sigma := \sigma_1\sigma_2\cdots\sigma_{n-1}$. We consider σ as the computed solution to the above narrowing derivation.

Remark 1. Renaming of rewrite rules will be mandatory to ensure completeness of the narrowing approach. We always use a simple renaming such that $\texttt{vars}(l) \cap \texttt{vars}(s) = \emptyset$. This also extend to chains of narrowing derivations.

Narrowing was first introduced in the context of E-unification. Fay [cite] and Hullot [1] shows that narrowing is a complete method for solving equations in the theory defined by a confluent and terminating term rewriting system. In fact, narrowing is a first attempt for solving equations in arbitrary equational theories E, with the requirement that they can be represented as a convergent rewrite system. This approach also shows an important application for the Knuth-Bendix completion procedure: it prepares the way for solving equations over E, by delivering a complete TRS for E (if possible).

For this set of technology we give the name narrowing, we now present the first application (of solving equations modulo E) in the framework of transformation rules on sets of equational goals.

Consider an equational theory E specified by a convergent rewrite system $\mathcal{R}$, which is called the equational specification of E.

Table 2 Narrowing rules

(7) Left Narrowing if $s \rightsquigarrow_\sigma s'$

$$\{s = t\} \cup G \implies {}_\sigma\{s' = t\sigma\} \cup G\sigma$$

(8) Right Narrowing if $t \rightsquigarrow_\sigma t'$

$$\{s = t\} \cup G \implies {}_\sigma\{s\sigma = t'\} \cup G\sigma$$

Example 2. Let $\mathcal{R} = \{g(a) \to a\}$ and consider the failing unification attempt of Example 1-(2). Observe that $h(y, g(x), x) \rightsquigarrow_{[x/a]} h(y, a, a)$. Thus by the above rules:

$$\begin{aligned}
\{h(x, y, x) = h(y, g(x), x)\} &\overset{(8)}{\Longrightarrow}{}_{[x/A]} \{h(a, y, a) = h(y, a, a)\} \\
&\overset{(2)}{\Longrightarrow} \{a = y, y = a, a = a\} \\
&\overset{(4,1)}{\Longrightarrow} \{y = a\} \\
&\overset{(8)}{\Longrightarrow}{}_{[x/A]} \{a = a\} \\
&\overset{(1)}{\Longrightarrow} \emptyset
\end{aligned}$$

The equational problem is now a successful narrowing derivation with computed answer substitution $\sigma = [x/a, y/a]$.

In order to solve an equation $s = t$ in an equational theory, corresponding to such a TRS, one can construct all possible narrowing derivations starting from the given equational goal until an equation $s' = t'$ is obtained such that s' and t' are indeed syntactically unifiable. Note that, if this equational goal has a solution we always will get a last equation of the form $s = s$. We now investigate the semantic of solving equations using narrowing techniques.

If (Σ, E) is an equational theory, write $[s = t]_E$ for the set of all solutions to the equation $s = t$ modulo E. Moreover, if X is some set of substitutions, let $X\sigma$ be the set $\{\gamma\sigma \mid \gamma \in X\}$.

The narrowing relation was defined on terms rather equational goals. They act upon goals by the means of the above transformation rules.

Example 3. Consider the TRS

$$\mathcal{R} = \begin{cases} \rho_1 : 0 + x \to x \\ \rho_2 : s(x) + y \to s(x + y) \end{cases}$$

The propositions above will be useful to prove the various versions of "Lifting Lemmas" of this study.

Proposition 1. If t is a term and γ a substitution then $\mathtt{vars}(t\gamma) = (\mathtt{vars}(t) \setminus \mathtt{dom}(\gamma)) \cup \mathtt{vran}(\gamma|_{\mathtt{vars}(t)})$.

Proposition 2. Suppose we have substitutions γ, θ, θ' and sets A, B of variables such that $(B \setminus \mathtt{dom}(\gamma)) \cup \mathtt{vran}(\gamma) \subseteq A$. If $\theta =^A \theta'$ then $\gamma\theta =^B \gamma\theta'$.

Proposition 3. Let $\mathcal{R}$ be a TRS and suppose we have sets A, B of variables and substituions γ, θ, θ' such that the following consditions are satisfied:

1. $\theta|_A$ is $\mathcal{R}$-normalized,
2. $\theta =^A \gamma\theta'$,
3. $B \subseteq (A \setminus \mathtt{dom}(\gamma)) \cup \mathtt{vran}(\gamma|_A)$

Then $\theta'|_B$ is also normalized.

Proof. Let $x \in B$. We have to show that $x\theta'$ is an $\mathcal{R}$-normal form. If $x \in A\setminus\mathtt{dom}(\gamma)$ then $x\theta' = x(\gamma\theta') = x\theta$ which is an $\mathcal{R}$-normal form by the first condition. If $x \in \mathtt{vran}(\gamma|_A)$ then there exists a variable $y \in A$ such that $x \in \mathtt{vars}(y\gamma)$. Also, by condition (2), we have $x\theta' \leq (y\gamma)\theta' = y\theta$. By condition (1) $y\theta$ is an $\mathcal{R}$-normal form and hence its subterm $x\theta'$ is also an $\mathcal{R}$-normal form. $\square$

Lemma 1 (Lifting Lemma)**.** Let $\mathcal{R}$ be a TRS. Suppose we have terms s and t, a normalized substitution θ and a finite set of variables V such that $\mathtt{vars}(s) \cup \mathtt{dom}(\theta) \subseteq V$ and $t = s\theta$. If $t \twoheadrightarrow_{\mathcal{R}} t'$ then there exist a term s' and substitutions θ', γ such that:

1. $s \rightsquigarrow_{\sigma}^{*} s'$,
2. $t' = s'\theta'$,
3. $\theta\big|_V = \gamma\theta'$,
4. θ' is $\mathcal{R}$-normalized.

Futhermore, we may assume that the narrowing derivation $s \rightsquigarrow_{\sigma}^{*} s'$ and the rewrite sequence $t \twoheadrightarrow_{\mathcal{R}} t'$ employ the same rules at the same positions.

Proof. The proof is by induction on the length of the reduction sequence from t to t'. If $t = t'$, a reduction of length zero, then the result crearly follows. Suppose $t \rightarrow_{\mathcal{R}} t_1 \twoheadrightarrow_{\mathcal{R}} t'$ is a reduction sequence of length $n + 1$.

$$\begin{tikzcd}
s \arrow[r, rightsquigarrow, "\gamma_1"'] & s_1 \arrow[r, rightsquigarrow, "*", "\gamma'"'] & s' \\
t = s\theta \arrow[u, dashed] \arrow[r, twoheadrightarrow, "\mathcal{R}"'] & t_1 \arrow[u, dashed] \arrow[r, twoheadrightarrow, "*", "\mathcal{R}"'] \arrow[ur, phantom, "I.H"] & t' = s'\theta' \arrow[u, dashed]
\end{tikzcd}$$

Figure 1 Lifting Lemma

Let t contract to t_1 using a position $p \in \mathtt{pos}(t)$ and a variant $l \rightarrow r$ of a rule from $\mathcal{R}$ such that $\mathtt{vars}(l) \cap V = \emptyset$. We use the fact that $t = s\theta$ to write $(s\theta)\big|_p = l\tau$ for some substitution τ with $\mathtt{dom}(\tau) \subseteq \mathtt{vars}(l)$. Since θ is normalized we have p is a non-variable position and hence $(s\theta)\big|_p = (s\big|_p)\theta$. Let $\gamma = \tau \cup \theta$ so $(s\big|_p)\gamma = (s\big|_p)\theta = l\tau$ then $s\big|_p$ and l are unifiable. Consider γ_1 as an idempotent mgu of $s\big|_p$ and l. By Lemma [1] $\mathtt{dom}(\gamma_1) \cup \mathtt{vran}(\gamma_1) = \mathtt{vars}(s\big|_p) \cup \mathtt{vars}(l)$. Let $s_1 = (s[r]_p)\gamma_1$. By definition 3,

$$s \rightsquigarrow_{\gamma_1} s_1 \tag{1}$$

Since $\gamma_1 \leq \gamma$, there exists a substitution ρ such that $\gamma = \gamma_1\rho$. Let $V_1 = (V \setminus \mathtt{dom}(\gamma_1)) \cup \mathtt{vran}(\gamma_1)$ and define $\theta_1 = \rho\big|_{V_1}$. Clearly $\mathtt{dom}(\theta_1) \subseteq V_1$. Also,

$$\begin{aligned}
\mathtt{vars}(s_1) &= \mathtt{vars}(s[r]_p\gamma_1) \\
&\subseteq \mathtt{vars}(s[l]_p\gamma_1) \\
&= \mathtt{vars}(s\gamma_1), \text{ by Proposition 2} \\
&\subseteq V_1.
\end{aligned}$$

Therefore, $\mathtt{vars}(s_1) \cup \mathtt{dom}(\theta_1) \subseteq V_1$.

Using the equality $\theta_1 =^{V_1} \rho$ we obtain

$$\begin{aligned}
s_1\theta_1 = s_1\rho &= ((s[r]_p)\gamma_1)\rho \\
&= (s[r]_p)\gamma \\
&= (s\gamma)[r\gamma]_p
\end{aligned}$$

and since $V \cap \mathtt{dom}(\tau) = \emptyset$ (remember that $\mathtt{vars}(l) \cap V = \emptyset$ and $\mathtt{dom}(\tau) \subseteq \mathtt{vars}(l)$), one have $\gamma =^V \theta_1$. Likewise $\gamma =^{\mathtt{vars}(r)} \tau$. Hence the term $(s\gamma)[r\gamma]_p$ is equal to $(s\theta)[r\theta]_p = t_1$. Thus

$$t_1 = s_1\theta_1 \tag{2}$$

Next we show that $\gamma_1\theta_1 =^V \theta$. Proposition 2 yields $\gamma_1\theta_1 =^V \gamma_1\rho$. Since $\gamma =^V \theta$ and using the equality $\gamma = \gamma_1\rho$ we have

$$\begin{aligned}
\gamma_1\theta_1 &=^V \gamma_1\rho \\
&=^V \gamma \\
&=^V \theta
\end{aligned} \tag{3}$$

Finally we show that θ_1 is normalized. Since $\mathtt{dom}(\theta_1) \subseteq V_1$ if suffices to show that $\theta_1|_{V_1}$ is normalized. Let $B = (V \setminus \mathtt{dom}(\gamma_1)) \cup \mathtt{vars}(\gamma_1|_V)$. Proposition 3 (with $A = V$) yields the normalization of $\theta_1|_B$. Remember that $\mathtt{vran}(\gamma_1) \subseteq \mathtt{vars}(s|_p) \cup \mathtt{vars}(l)$. Let $x \in \mathtt{vran}(\gamma_1)$. Idempotence of γ_1 yields $x \notin \mathtt{dom}(\gamma_1)$. If $x \in \mathtt{vars}(s|_p) \subseteq V$ then $x \in V \setminus \mathtt{dom}(\gamma_1)$. If $x \in \mathtt{vars}(l)$ then $x \in \mathtt{vars}(l\gamma_1) = \mathtt{vars}((s|_p)\gamma_1)$ and thus $x \in \mathtt{vran}(\gamma_1|_V)$. So $\mathtt{vran}(\gamma_1) \subseteq B$, and hence $B = V_1$. So θ_1 is normalized.

The induction hypothesis give us a term s' and substitutions θ', γ' such that

$$s_1 \rightsquigarrow^*_{\gamma'} s' \tag{4}$$

$$t' = s'\theta' \tag{5}$$

$$\theta_1 =^{V_1} \gamma'\theta' \tag{6}$$

$$\theta' \text{ is normalized} \tag{7}$$

Moreover, we can assume that $s_1 \rightsquigarrow^*_{\gamma'} s'$ and $t_1 \twoheadrightarrow_{\mathcal{R}} t'$ using the same rules at the same positions. Let $\gamma = \gamma_1\gamma'$. By joining (1) and (4) we get $s \rightsquigarrow^*_{\gamma} s'$. By construction this narrowing derivation employs the same positions as the rewrite sequence $t \twoheadrightarrow_{\mathcal{R}} t'$. It remains to show that $\gamma\theta' =^V \theta$. Proposition 2 applied to (6) give $\gamma_1\gamma'\theta' =^V \gamma_1\theta_1$ and hence

$$\begin{aligned} \gamma\theta' &=^V \gamma_1\theta_1 \\ &= \theta, \text{ by equation (3).} \end{aligned}$$

□

Now we have the following theorem which express the completeness of narrowing on solving equations modulo E.

Theorem 1 (Narrowing Completeness)**.** Let $\mathcal{R}$ be a complete TRS for the equational theory E. Let, moreover, s, t be terms and $\sigma \in [s = t]_E$. Then there is a successful derivation starting with $P_0 = \{s = t\}$, using the rules (1)-(8), such that the computed solution τ is a solution for $s = t$ with $\tau \leq \sigma$.

Proof. Let $\sigma\downarrow$ be the $\mathcal{R}$-normal form of σ. Notice that $\sigma =_E \sigma\downarrow$, hence $s\sigma\downarrow =_E t\sigma\downarrow$. Confluence of $\mathcal{R}$ yields the existence of a common reduct r, that is, $s\sigma\downarrow \twoheadrightarrow_{\mathcal{R}} t \twoheadleftarrow$ □

References

[1] J.-M. Hullot. Canonical forms and unification. In W. Bibel and R. Kowalski, editors, *5th Conference on Automated Deduction Les Arcs, France, July 8–11, 1980*, pages 318–334, Berlin, Heidelberg, 1980. Springer Berlin Heidelberg.