# Study Report: Narrowing

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**Abstract**

In this report we study

## 1 Introduction

## 2 Preliminaries

## 3 Narrowing

In this section, we discuss the use of narrowing techniques on solving equations over an equational theory $E$. By solving an equation we mean finding a substitution $\sigma$ such that $s\sigma =_E t\sigma$. These solutions can be found by unification *modulo* $E$ (if such a unification algorithm exists for this theory), but here we are interested in applications of rewriting theory to solving such equations.

We start the presentation by the so-called syntactic unification, i.e. $E$-unification with empty $E$. The transformation rules given below is due to Martelli and Montanari [1], the idea is to transform sets of equations to other sets of equations until a termination state is reached; that is, a solution state or a failure one.

**Definition 1.** Let $\Sigma$ be a signature. An equational goal is a finite set of $\Sigma$-equations. Consider the special goal $\perp$ to denote a failed derivation state.

Table 1 Martelli-Montanari rules

**(1) Trivial**

$$\{x \stackrel{?}{=} x\} \cup G \implies G$$

**(2) Decompose**

$$\{f(s_1, \cdots, s_n) \stackrel{?}{=} f(t_1, \ldots, t_n)\} \cup G \implies \{s_1 \stackrel{?}{=} t_1, \ldots, s_n \stackrel{?}{=} t_n\} \cup G$$

**(3) Symbol Clash**

$$\{f(s_1, \ldots, s_n) \stackrel{?}{=} g(t_1, \ldots, t_n)\} \cup G \implies \perp \text{ if } f \neq g$$

**(4) Orient**

$$\{t \stackrel{?}{=} x\} \cup G \implies \{x \stackrel{?}{=} t\} \cup G \text{ if } t \notin V$$

**(5) Occurs Check**

$$\{x \stackrel{?}{=} t\} \cup G \implies \perp \text{ se } x \in \mathtt{vars}(t) \text{ and } x \neq t$$

**(6) Variable Elimination**

$$\{x \stackrel{?}{=} t\} \cup G \implies G\{x \mapsto t\} \text{ if } x \notin \mathtt{vars}(t)$$

The application of the above rules non-deterministically transforms goals into goals:

$$G_0 \implies \cdots \implies G_n$$

Each application of a rule will then called a *elementary derivation step*. As for the case of (Variable Elimination), we may get some substitution on the way, we make them explicit by writing:

$$G_0 \implies G_1 \implies_{\sigma_1} G_2 \implies_{\sigma_2} \cdots \implies_{\sigma_i} \cdots \implies_{\sigma_{n-1}} G_n$$

The computed solution of the derivation chain is then the composition of such substitutions in their order of appearance.

**Definition 2.** A *successful* derivation chain is a finite sequence of equational goals $G_0, G_1, \ldots, G_n$ such that the last goal is empty. We also say a derivation chain has *failed* if it last goal is the fail symbol $\perp$.

**Example 1.**

1. We want to determine an mgu of the terms $f(g(x), h(x,u))$ and $f(z, h(f(y,y), z))$. That is, solving the equational goal:

$$\begin{aligned}
&\{f(g(x), h(x,u)) = f(z, h(f(y,y), z))\}\\
&\qquad \stackrel{(2)}{\Longrightarrow} \{g(x) = z, h(x,u) = h(f(y,y), z)\}\\
&\qquad \stackrel{(4)}{\Longrightarrow} \{z = g(x), h(x,u) = h(f(y,y), z)\}\\
&\qquad \stackrel{(6)}{\Longrightarrow}_{[z/g(x)]} \{h(x,u) = h(f(y,y), g(x))\}\\
&\qquad \stackrel{(2)}{\Longrightarrow} \{x = f(y,y), u = g(x)\}\\
&\qquad \stackrel{(6)}{\Longrightarrow}_{[x/f(y,y)]} \{u = g(f(y,y))\}\\
&\qquad \stackrel{(6)}{\Longrightarrow}_{[u/f(y,y)]} \emptyset
\end{aligned}$$

   We get as the computed solution to the problem the composition $[z/g(x)][x/f(y,y)][u/g(f(y,y))]$.

2. A failing unification derivation:

$$\begin{aligned}
&\{h(x,y,x) = h(y, g(x), x)\}\\
&\qquad \stackrel{(2)}{\Longrightarrow} \{x = y, x = g(x), x = x\}\\
&\qquad \stackrel{(5)}{\Longrightarrow} \perp
\end{aligned}$$

It can be proved (see [1]) that this set of rules derive a correct and terminating procedure for syntactic unification.

**Definition 3.** Say a term $s$ *narrows* to a term $t$ if there exists a non-variable position $p \in \mathtt{pos}(s)$, a variant $l \to r$ of a rewrite rule in $\mathcal{R}$, and a substitution $\sigma$ satisfying two conditions:

1. $\sigma$ is a *mgu* of $s\big|_p$ and $l$,
2. $t = (s[r]_p)\,\sigma$.

The relation $\rightsquigarrow$ is called *narrowing relation.* We write $s \rightsquigarrow_\sigma^* t$ if there exists a narrowing derivation

$$s = t_1 \rightsquigarrow_{\sigma_1} t_2 \rightsquigarrow_{\sigma_2} t_3 \rightsquigarrow \cdots \rightsquigarrow_{\sigma_{n-1}} t_n = t$$

and $\sigma$ is given by $\sigma := \sigma_1\sigma_2\cdots\sigma_{n-1}$, the computed solution of the above narrowing derivation.

*Remark* 1. Renaming of rewrite rules will be mandatory to ensure completeness of the narrowing approach. We always use a simple renaming such that $\mathtt{vars}(l) \cap \mathtt{vars}(s) = \emptyset$. This also extend to chains of narrowing derivations.

Narrowing was first introduced in the context of $E$-unification. Fay [2] and Hullot [3] shows that narrowing is a complete method for solving equations in the theory defined by a confluent and terminating term rewriting system. In fact, narrowing is a first attempt for solving equations in arbitrary equational theories $E$, with the requirement that they can be represented as a convergent rewrite system. This approach also shows an important application for the Knuth-Bendix completion procedure: it prepares the way for solving equations over $E$, by delivering a complete TRS for $E$ (if possible).

For this set of technology we give the name *narrowing*, we now present the first application (of solving equations modulo $E$) in the framework of transformation rules on sets of equational goals.

Consider an equational theory $E$ specified by a convergent rewrite system $\mathcal{R}$, which is called the equational specification of $E$. The rules below extend the Martelli-Montanari unification on syntax terms to solve equations modulo $E$.

Table 2 Narrowing rules

**(7) Left Narrowing** if $s \rightsquigarrow_\sigma s'$

$$\{s = t\} \cup G \implies {}_\sigma\{s' = t\sigma\} \cup G\sigma$$

**(8) Right Narrowing** if $t \rightsquigarrow_\sigma t'$

$$\{s = t\} \cup G \implies {}_\sigma\{s\sigma = t'\} \cup G\sigma$$

**Definition 4.** A *narrowing derivation* is a finite set of equational goals in which each step is obtained by using the rules (1)-(8). It is a successful narrowing derivation if it last goal is empty.

**Example 2.** Let $\mathcal{R} = \{g(a) \to a\}$ and consider the failing unification attempt of Example 1-(2). Observe that $h(y, g(x), x) \rightsquigarrow_{[x/a]} h(y, a, a)$. Thus by the above rules:

$$\begin{aligned}
\{h(x, y, x) = h(y, g(x), x)\} &\overset{(8)}{\Longrightarrow}{}_{[x/A]} \{h(a, y, a) = h(y, a, a)\} \\
&\overset{(2)}{\Longrightarrow} \{a = y, y = a, a = a\} \\
&\overset{(4,1)}{\Longrightarrow} \{y = a\} \\
&\overset{(8)}{\Longrightarrow}{}_{[x/A]} \{a = a\} \\
&\overset{(1)}{\Longrightarrow} \emptyset
\end{aligned}$$

The equational problem is now a successful narrowing derivation with computed answer substitution $\sigma = [x/a, y/a]$.

In order to solve an equation $s \overset{?}{=} t$ in an equational theory, corresponding to such a TRS, one can construct all possible narrowing derivations starting from the given equational goal until an equation $s' \overset{?}{=} t'$ is obtained such that $s'$ and $t'$ are indeed syntactically unifiable. Note that, if this equational goal has a solution one always get a last equation of the form $s \overset{?}{=} s$. We now investigate the semantic of solving equations using narrowing techniques.

If $(\Sigma, E)$ is an equational theory, write $[s = t]_E$ for the set of all solutions to the equation $s = t$ modulo $E$. Moreover, if $X$ is some set of substitutions, let $X\sigma$ be the set $\{\gamma\sigma \mid \gamma \in X\}$.

The narrowing relation was defined on terms rather equational goals. They act upon goals by the means of the above transformation rules.

**Example 3.** Consider the TRS

$$\mathcal{R} = \begin{cases} \rho_1 : 0 + x \to x \\ \rho_2 : s(x) + y \to s(x + y) \end{cases}$$

The propositions above will be useful to prove the various versions of “Lifting Lemmas” of this study.

**Proposition 1.** If $t$ is a term and $\gamma$ a substitution then $\mathtt{vars}(t\gamma) = (\mathtt{vars}(t) \setminus \mathtt{dom}(\gamma)) \cup \mathtt{vran}(\gamma|_{\mathtt{vars}(t)})$.

**Proposition 2.** Suppose we have substitutions $\gamma, \theta, \theta'$ and sets $A, B$ of variables such that $(B \setminus \mathtt{dom}(\gamma)) \cup \mathtt{vran}(\gamma) \subseteq A$. If $\theta =^A \theta'$ then $\gamma\theta =^B \gamma\theta'$.

**Proposition 3.** Let $\mathcal{R}$ be a TRS and suppose we have sets $A, B$ of variables and substituions $\gamma, \theta, \theta'$ such that the following consditions are satisfied:

1. $\theta|_A$ is $\mathcal{R}$-normalized,
2. $\theta =^A \gamma\theta'$,
3. $B \subseteq (A \setminus \mathtt{dom}(\gamma)) \cup \mathtt{vran}(\gamma|_A)$

Then $\theta'|_B$ is also normalized.

*Proof.* Let $x \in B$. We have to show that $x\theta'$ is an $\mathcal{R}$-normal form. If $x \in A \setminus \mathtt{dom}(\gamma)$ then $x\theta' = x(\gamma\theta') = x\theta$ which is an $\mathcal{R}$-normal form by the first condition. If $x \in \mathtt{vran}(\gamma|_A)$ then there exists a variable $y \in A$ such that $x \in \mathtt{vars}(y\gamma)$. Also, by condition (2), we have $x\theta' \leq (y\gamma)\theta' = y\theta$. By condition (1) $y\theta$ is an $\mathcal{R}$-normal form and hence its subterm $x\theta'$ is also an $\mathcal{R}$-normal form. $\square$

**Lemma 1** (Lifting Lemma)**.** Let $\mathcal{R}$ be a TRS. Suppose we have terms $s$ and $t$, a normalized substitution $\theta$ and a finite set of variables $V$ such that $\mathtt{vars}(s) \cup \mathtt{dom}(\theta) \subseteq V$ and $t = s\theta$. If $t \twoheadrightarrow t'$ then there exist a term $s'$ and substitutions $\theta', \gamma$ such that:

1. $s \rightsquigarrow^*_\gamma s'$,
2. $t' = s'\theta'$,
3. $\theta\big|_V = \gamma\theta'$,
4. $\theta'$ is $\mathcal{R}$-normalized.

Futhermore, we may assume that the narrowing derivation $s \rightsquigarrow^*_\sigma s'$ and the rewrite sequence $t \twoheadrightarrow t'$ employ the same rules at the same positions.

*Proof.* The proof is by induction on the length of the reduction sequence from $t$ to $t'$. If $t = t'$, a reduction of length zero, then the result crearly follows. Suppose $t \to t_1 \twoheadrightarrow t'$ is a reduction sequence of length $n+1$.

$$\begin{tikzcd}
s \arrow[r, rightsquigarrow, "\gamma_1"'] & s_1 \arrow[r, rightsquigarrow, "*", "\gamma'"'] & s' \\
t = s\theta \arrow[u, dashed] \arrow[r, twoheadrightarrow, "\mathcal{R}"'] & t_1 \arrow[u, dashed] \arrow[r, twoheadrightarrow, "*", "\mathcal{R}"'] \arrow[ur, phantom, "I.H"] & t' = s'\theta' \arrow[u, dashed]
\end{tikzcd}$$

Figure 1 Lifting Lemma

Let $t$ contract to $t_1$ using a position $p \in \mathtt{pos}(t)$ and a variant $l \to r$ of a rule from $\mathcal{R}$ such that $\mathtt{vars}(l) \cap V = \emptyset$. We use the fact that $t = s\theta$ to write $(s\theta)\big|_p = l\tau$ for some substitution $\tau$ with $\mathtt{dom}(\tau) \subseteq \mathtt{vars}(l)$. Since $\theta$ is normalized we have $p$ is a non-variable position and hence $(s\theta)\big|_p = (s\big|_p)\theta$. Let $\gamma = \tau \cup \theta$ so $(s\big|_p)\gamma = (s\big|_p)\theta = l\tau$ then $s\big|_p$ and $l$ are unifiable. Consider $\gamma_1$ as an idempotent mgu of $s\big|_p$ and $l$. By Lemma [1] $\mathtt{dom}(\gamma_1) \cup \mathtt{vran}(\gamma_1) = \mathtt{vars}(s\big|_p) \cup \mathtt{vars}(l)$. Let $s_1 = (s[r]_p)\gamma_1$. By definition 3,

$$s \rightsquigarrow_{\gamma_1} s_1 \tag{1}$$

Since $\gamma_1 \leq \gamma$, there exists a substitution $\rho$ such that $\gamma = \gamma_1\rho$. Let $V_1 = (V \setminus \mathtt{dom}(\gamma_1)) \cup \mathtt{vran}(\gamma_1)$ and define $\theta_1 = \rho\big|_{V_1}$. Clearly $\mathtt{dom}(\theta_1) \subseteq V_1$. Also,

$$\begin{aligned}
\mathtt{vars}(s_1) &= \mathtt{vars}(s[r]_p\gamma_1) \\
&\subseteq \mathtt{vars}(s[l]_p\gamma_1) \\
&= \mathtt{vars}(s\gamma_1), \text{ by Proposition 2} \\
&\subseteq V_1.
\end{aligned}$$

Therefore, $\mathtt{vars}(s_1) \cup \mathtt{dom}(\theta_1) \subseteq V_1$.

Using the equality $\theta_1 =^{V_1} \rho$ we obtain

$$\begin{aligned}
s_1\theta_1 = s_1\rho &= ((s[r]_p)\gamma_1)\rho \\
&= (s[r]_p)\gamma \\
&= (s\gamma)[r\gamma]_p
\end{aligned}$$

and since $V \cap \mathtt{dom}(\tau) = \emptyset$ (remember that $\mathtt{vars}(l) \cap V = \emptyset$ and $\mathtt{dom}(\tau) \subseteq \mathtt{vars}(l)$), one have $\gamma =^V \theta_1$. Likewise $\gamma =^{\mathtt{vars}(r)} \tau$. Hence the term $(s\gamma)[r\gamma]_p$ is equal to $(s\theta)[r\theta]_p = t_1$. Thus

$$t_1 = s_1\theta_1 \tag{2}$$

Next we show that $\gamma_1\theta_1 =^V \theta$. Proposition 2 yields $\gamma_1\theta_1 =^V \gamma_1\rho$. Since $\gamma =^V \theta$ and using the equality $\gamma = \gamma_1\rho$ we have

$$\begin{aligned}
\gamma_1\theta_1 &=^V \gamma_1\rho \\
&=^V \gamma \\
&=^V \theta
\end{aligned} \tag{3}$$

Finally we show that $\theta_1$ is normalized. Since $\mathtt{dom}(\theta_1) \subseteq V_1$ if suffices to show that $\theta_1\big|_{V_1}$ is normalized. Let $B = (V \setminus \mathtt{dom}(\gamma_1)) \cup \mathtt{vars}(\gamma_1\big|_V)$. Proposition 3 (with $A = V$) yields the normalization of $\theta_1\big|_B$. Remember that $\mathtt{vran}(\gamma_1) \subseteq \mathtt{vars}(s\big|_p) \cup \mathtt{vars}(l)$. Let $x \in \mathtt{vran}(\gamma_1)$. Idempotence of $\gamma_1$ yields $x \notin \mathtt{dom}(\gamma_1)$. If $x \in \mathtt{vars}(s\big|_p) \subseteq V$ then $x \in V \setminus \mathtt{dom}(\gamma_1)$. If $x \in \mathtt{vars}(l)$ then $x \in \mathtt{vars}(l\gamma_1) = \mathtt{vars}((s\big|_p)\gamma_1)$ and thus $x \in \mathtt{vran}(\gamma_1\big|_V)$. So $\mathtt{vran}(\gamma_1) \subseteq B$, and hence $B = V_1$. So $\theta_1$ is normalized.

The induction hypothesis give us a term $s'$ and substitutions $\theta'$, $\gamma'$ such that

$$s_1 \rightsquigarrow^*_{\gamma'} s' \tag{4}$$
$$t' = s'\theta' \tag{5}$$
$$\theta_1 =^{V_1} \gamma'\theta' \tag{6}$$
$$\theta' \text{ is normalized} \tag{7}$$

Moreover, we can assume that $s_1 \rightsquigarrow^*_{\gamma'} s'$ and $t_1 \twoheadrightarrow t'$ using the same rules at the same positions. Let $\gamma = \gamma_1\gamma'$. By joining (1) and (4) we get $s \rightsquigarrow^*_{\gamma} s'$. By construction this narrowing derivation employs the same positions as the rewrite sequence $t \twoheadrightarrow t'$. It remains to show that $\gamma\theta' =^V \theta$. Proposition 2 applied to (6) give $\gamma_1\gamma'\theta' =^V \gamma_1\theta_1$ and hence

$$\begin{aligned}\gamma\theta' &=^V \gamma_1\theta_1\\ &= \theta, \text{ by equation (3).}\end{aligned}$$

□

Now we have the following theorem which express the completeness of narrowing on solving equations modulo $E$.

**Theorem 1** (Narrowing Completeness)**.** Let $\mathcal{R}$ be a complete TRS for the equational theory $E$. Let, moreover, $s, t$ be terms and $\sigma \in [s = t]_E$. Then there is a successful derivation starting with $P_0 = \{s = t\}$, using the rules (1)-(8), such that the computed solution $\tau$ is a solution for $s = t$ with $\tau \leq \sigma$.

*Proof.* Let $\sigma\downarrow$ be the $\mathcal{R}$-normal form of $\sigma$. Notice that $\sigma =_E \sigma\downarrow$, hence $s\sigma\downarrow =_E t\sigma\downarrow$. Confluence of $\mathcal{R}$ yields the existence of a common reduct $r$, that is, $s\sigma\downarrow \twoheadrightarrow r \twoheadleftarrow t\sigma\downarrow$. Now using the Lifting Lemma with $V_1 = \mathtt{vars}(s) \cup \mathtt{vars}(t)$ we get a term $s'$ and substitutions $\rho, \tau$ such that

1. $s \rightsquigarrow^*_\tau s'$
2. $r = s'\rho$,
3. $\tau\rho =^{V_1} \sigma\downarrow$

Hence we have a derivation $\{s = t\} \implies \cdots \implies \{s' = t\tau\}$ consisting of left narrowing steps with computed answer substitution $\tau$.

In a similar way, using (3) above, it follows that $t\tau\rho = t\sigma\downarrow\sigma \twoheadrightarrow r$. We again apply the Lifting Lemma (with $V_2 = \mathtt{vars}(s', s, t)$) which give us a term $t'$ and substitutions $\gamma, \eta$ such that

4. $t\tau \rightsquigarrow^*_\eta t'$,
5. $r = t'\gamma$,
6. $\eta\gamma =^{V_2} \rho$.

Again we can prolong the above narrowing derivation with a sequence of right narrowing steps

$$\{s' = t\} \implies \cdots\{s'\gamma = t'\}$$

and obtain as compute answer $\tau\eta$.

Using (6), (5) and (2) above one get

$$s'\eta\gamma = s'\rho = r = t'\gamma$$

So $s'\eta$ and $t'$ are unifiable. The Martelli-Montanari rules yields an mgu $\gamma'$ of $s'\eta$ and $t'$. So one have a successful derivation with computed solution $\tau\eta\gamma'$. This is indeed a solution: since $s \rightsquigarrow^*_\tau s'$ and $t\tau \rightsquigarrow^*_\eta t'$, we have $s\tau =_E s'$ and

$$t\tau\eta\gamma' =_E s'\eta\gamma' = t'\gamma' =_E t\tau\eta\gamma'.$$

It remains to show that $\tau\eta\gamma' \precsim_E^{V_1}$. Since $\gamma'$ is most general, there exists a $\rho'$ such that

$$\tau\eta\gamma'\rho' = \tau\eta\gamma =_E \tau\rho =_E \sigma\downarrow =_E \sigma.$$

Hence, $\tau\eta\gamma' \precsim_E^{V_1} \sigma$. As required. $\square$

From the above proof one can see that we can indeed drop the strong normalization condition if only normalized substitution is considered. Strong normalization of $\mathcal{R}$ is only used in the normalization of $\sigma$ to $\sigma\downarrow$, hence we can strengthen this result by dropping the strong normalization requirement and restricting ourselves to normalizable substitutions. Since in a weakly normalizing TRS every substitution is normalizable, we obtain the following result.

**Corollary 1.** Narrowing is complete for weakly normalizing and confluent Term Rewriting Systems.

# References


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