State Estimation and Localization

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Outline

State Estimation

Crucial capability of autonomous vehicles to determine their global and local positions within their surrounding environment

Components of the vehicle localization

- **Multiple sensors**
  - Global Navigation Satellite System (GNSS) receivers
  - Inertial Measurement Units (IMU) such as gyroscopes and accelerometers
  - Additional dead reckoning sensors inside the vehicle such as: wheel speeds, transmission gear and speeds, throttle, brake, and steering wheel position
- **Validation system** to mitigate sensor errors

Limited precision of the sensors

- Associated uncertainty (error) in measurements from real sensors
- Combination of various sensor measurements into a state estimate to improve the overall state estimation
- Track of the uncertainties

State Estimation

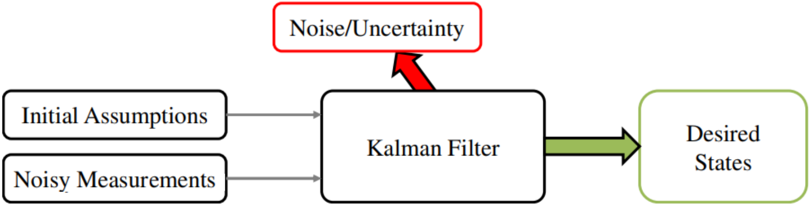
Definition of the basic concepts of state estimation via the development of the **Kalman filter** in 1960

- **Notion of observability**
  - State inferred from a set of measurements in a dynamic system
- **Estimation of the system state in the presence of measurement noise**

Classical Kalman filter for **linear** systems with measurements corrupted by **Gaussian noise**

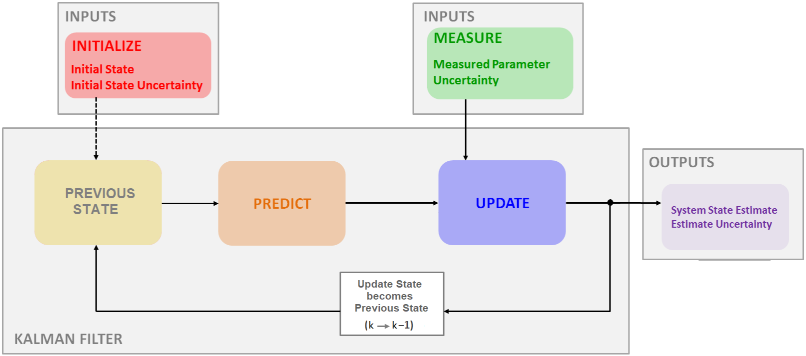
Kalman Filter

- Input to the Kalman filter: initial state and system model
  - Required model assumptions
- Consideration of the sensor errors, noise, and the different update rates of each sensor
- Goal: estimation of the states via the measurements minimizing the uncertainty



Kalman Filter

Kalman Filter: recursive approach for state estimation of a dynamic system in the presence of noise



Kalman Filter

- Kalman Filter: operation on linear state space
- Application of motion and observation models

$$\mathbf{x}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1} + \mathbf{w}_{k-1}, \quad k = 1, ..., K \tag{1}$$

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k, \quad k = 0, ..., K \tag{2}$$

where k is the discrete-time index and K its maximum.

Kalman Filter

Variable	Description	Dimension
$\mathbf{x}$	State Vector	$n_x \times 1$
$\mathbf{y}$	Output Vector	$n_y \times 1$
$\mathbf{u}$	Input Vector	$n_u \times 1$
$\mathbf{w}$	Process Noise Vector	$n_x \times 1$
$\mathbf{v}$	Measurement Noise Vector	$n_y \times 1$
$\mathbf{F}$	System Matrix – State	$n_x \times n_x$
$\mathbf{G}$	System Matrix – Input	$n_x \times n_u$
$\mathbf{H}$	Observation Matrix	$n_y \times n_x$

Note that the number of states, outputs, and inputs are independent, and therefore the matrices **G** and **H** can be non-square, but **F** is always a square matrix. The state dynamics are described by (1), while the output equations are given by (2). Vectors are denoted in bold font, so (1) and (2) are vector equations.

Kalman Filter

- State vector $\mathbf{x}$ estimated by the Kalman filter
- Output vector, $\mathbf{y}$, measured by sensors
- Input vector, $\mathbf{u}$, with information coming into the filter in order to define the system dynamics

Terms $\mathbf{w}$ and $\mathbf{v}$

- random variables representing the process and measurement noise, respectively
- assumed to be independent, white, and with normal probability distributions
- used to determine information about the process and measurement noise covariance matrices $\mathbf{Q}$ and $\mathbf{R}$

$$p(\mathbf{w}) \sim \mathcal{N}(0, \mathbf{Q}) \tag{3}$$

$$p(\mathbf{v}) \sim \mathcal{N}(0, \mathbf{R}) \tag{4}$$

Kalman Filter

F matrix

- state at the previous time step *k-1* to the state at current step *k* in the state dynamics (1)

G matrix

- optional control input **u** to the state **x** in the state dynamics (1)

H matrix

- relation between the state and the measurement in the output equations (2)

In practice, these matrices can vary with time, but cannot change with respect to the states or inputs. For many problems, these matrices are constant.

Kalman Filter

Predicted state vector from the state dynamic equation (1) by:

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}\hat{\mathbf{x}}_{k-1} + \mathbf{G}\mathbf{u}_{k-1} + \mathbf{w}_{k-1} \tag{5}$$

where

- $\hat{\mathbf{x}}_{k|k-1}$: predicted state vector (before measurements)
- $\hat{\mathbf{x}}_{k-1}$: previous estimated state vector
- $\mathbf{u}$ is the input vector (omitted for systems with no external influences)
- $\mathbf{F}$ (prediction) and $\mathbf{G}$ (control) defined by the system dynamics.

The subscript *k/k-1* is a shorthand notation for the state at discrete time *k* given its previous state at discrete time *k-1*.

Kalman Filter

Assumptions:

- $\mathbf{x}_k$ and $\mathbf{w}_k$ are zero mean and uncorrelated.
- $\mathbf{u}_k = 0$

Predicted state covariance matrix:

$$\mathbf{P}_{k|k-1} = E\{\hat{\mathbf{x}}_{k|k-1}\hat{\mathbf{x}}_{k|k-1}^{\mathrm{T}}\} \tag{6}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}\mathbf{P}_{k-1}\mathbf{F}^{T} + \mathbf{Q} \tag{7}$$

where

- $\mathbf{P}_{k|k-1}$: predicted state covariance matrix
- $\mathbf{P}_{k-1}$: previous estimated state covariance matrix $E\{\hat{\mathbf{x}}_{k-1}\hat{\mathbf{x}}_{k-1}^{\mathrm{T}}\}$
- $\mathbf{Q}$: process noise covariance matrix given by $E\{\mathbf{w}_{k-1}\mathbf{w}_{k-1}^{T}\}$

Kalman Filter

Both equations (5) e (7) are responsible for projecting forward (in time) the current state and error covariance estimates to obtain the *a posteriori* estimates for the next time step.

Kalman Filter

Kalman gain matrix $\mathbf{K}_k$: determination of how much of the new measurements $\mathbf{y}_k$ are necessary to update the state estimate $\mathbf{x}_k$
Kalman gain matrix calculated by:

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}^T(\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^T + \mathbf{R})^{-1} \tag{8}$$

where

- $\mathbf{H}$: matrix relating state and measurements
- $\mathbf{R}$: measurement noise covariance matrix given by $E\{\mathbf{v}_{k-1}\mathbf{v}_{k-1}^T\}$

Interpretation of the Kalman gain:

$$\mathbf{K}_k = \frac{\text{Uncertainty in Estimate}}{\text{Uncertainty in Estimate} + \text{Uncertainty in Measurement}}$$

Kalman Filter

State vector $\mathbf{x}_k$

- updated by scaling the **innovation** defined as the discrepancy between the actual measurement, $\mathbf{y}_k$, and the predicted measurement $\mathbf{H}\hat{\mathbf{x}}_{k|k-1}$
- innovation given by the Kalman gain matrix in order to correct the prediction by the appropriate amount:

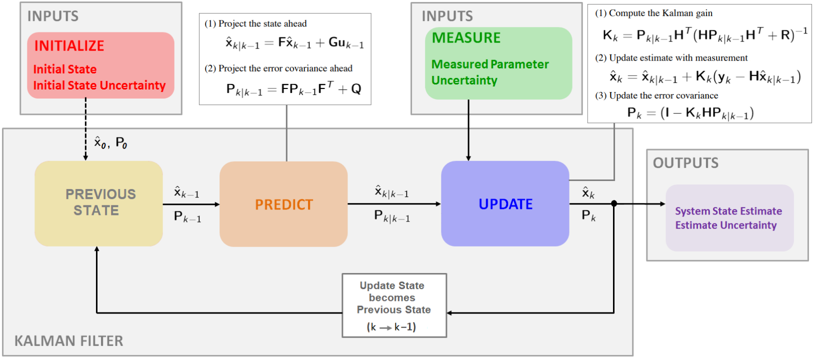
$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{y}_k - \mathbf{H}\hat{\mathbf{x}}_{k|k-1}) \tag{9}$$

Similarly, the state covariance updated by

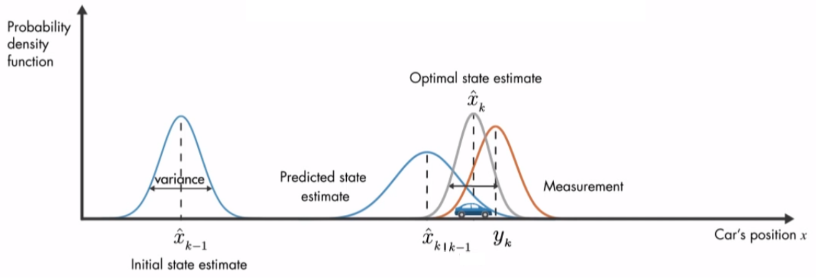
$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{P}_{k|k-1} \tag{10}$$

where $\mathbf{I}$ is an identity matrix.

Kalman Filter



Kalman Filter



Linear Kalman Filter Example

Let assume a vehicle moving in one dimension.

The system dynamic model described by kinematic equations:

$$x_k = x_{k-1} + \dot{x}_{k-1}\Delta t + \frac{1}{2}\ddot{x}\Delta t^2 \tag{11}$$

$$\dot{x}_k = \dot{x}_{k-1} + \ddot{x}\Delta t \tag{12}$$

Linear Kalman Filter Example

Let the object simple state vector having position and velocity.

$$\mathbf{x}_k = \begin{bmatrix} x_k \\ \dot{x}_k \end{bmatrix} \tag{13}$$

We assume:

a. both variables are random and Gaussian distributed
b. each variable has a **mean** value μ
c. each variable has a **variance** σ^2

Linear Kalman Filter Example

Input vector defined as:

$$\mathbf{u}_k = \ddot{x}$$

Acceleration proportional to the force applied over the object. In this sense, $\ddot{x}$ cannot be a state in the filter.

Linear Kalman Filter Example

Using basic kinematic formula in (13), we have:

$$\mathbf{x}_k = \begin{bmatrix} x_{k-1} + \dot{x}_{k-1}\Delta t + \frac{1}{2}\ddot{x}\Delta t^2 \\ \dot{x}_{k-1} + \ddot{x}\Delta t \end{bmatrix} \tag{14}$$

Rewriting in terms of the state vector (5):

$$\hat{\mathbf{x}}_{k|k-1} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \hat{\mathbf{x}}_{k-1} + \begin{bmatrix} \frac{\Delta t^2}{2} \\ \Delta t \end{bmatrix} \mathbf{u}_k$$

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}\hat{\mathbf{x}}_{k-1} + \mathbf{G}\mathbf{u}_{k-1}$$

$$\mathbf{F} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} \frac{\Delta t^2}{2} \\ \Delta t \end{bmatrix}, \mathbf{u}_{k-1} = \ddot{x}$$

Linear Kalman Filter Example

For this problem, we are considering constant velocity, i.e. $\ddot{x} = 0$.

Assumed Initial State Vector

$$\hat{\mathbf{x}}_0 = \begin{bmatrix} 10 \\ 90 \end{bmatrix}$$

Assumed Initial State Error Covariance Matrix

$$\mathbf{P}_0 = \begin{bmatrix} 30 & 0 \\ 0 & 10 \end{bmatrix}$$

Time Increment

$$\Delta t = 1s$$

We are assuming the process errors are small enough, so we can ignore them and do not need to model the process noise. Thus, the process noise covariance matrix, $\mathbf{Q}$, can be set to zero:

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Linear Kalman Filter Example

Now, we need to consider the measurement part of the system.

- Suppose position measured using a radar
- There is some uncertainty in the measurement, $\mathbf{v}$

Since the position can be written in terms of the state vector, we have:

$$\mathbf{y}_k = \mathbf{H}\mathbf{x}_k + \mathbf{v}_k$$

$$\mathbf{y}_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_k + \mathbf{v}_k$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Linear Kalman Filter Example

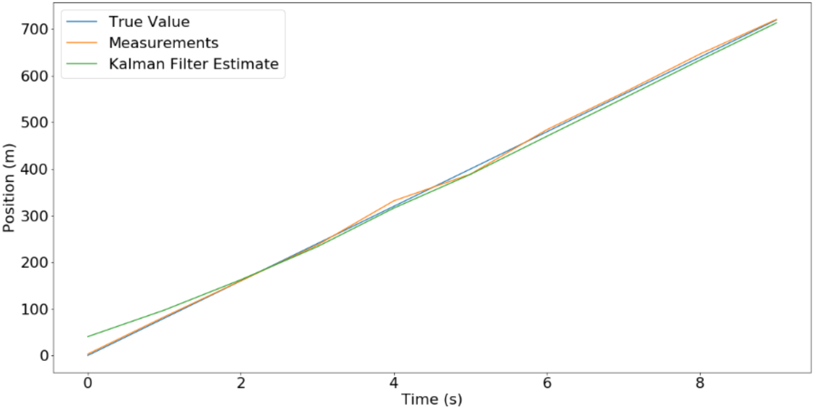
Consider the measurement system has a standard deviation of error of 5 m, which is a variance of 25 m^2.

Because the fact that there is only one term in the output vector, the resulting measurement noise covariance matrix reduces to a scalar value:

$$\mathbf{R} = 25 \ m^2$$

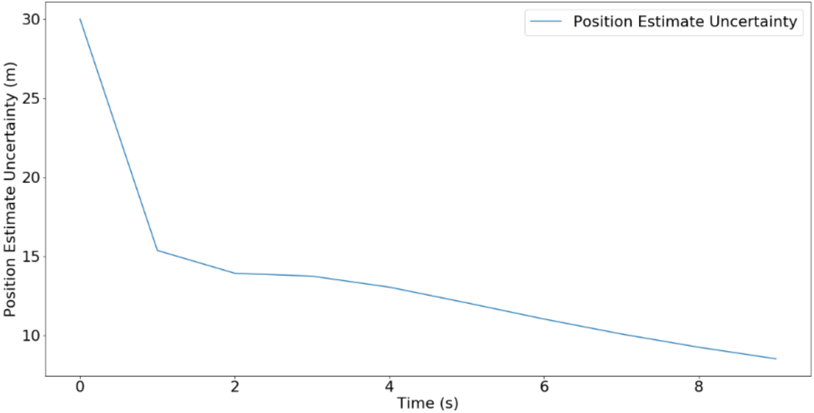
Linear Kalman Filter Example

This Kalman filtering example was executed over 10 time steps, and the measurements and estimates are plotted on the chart below.



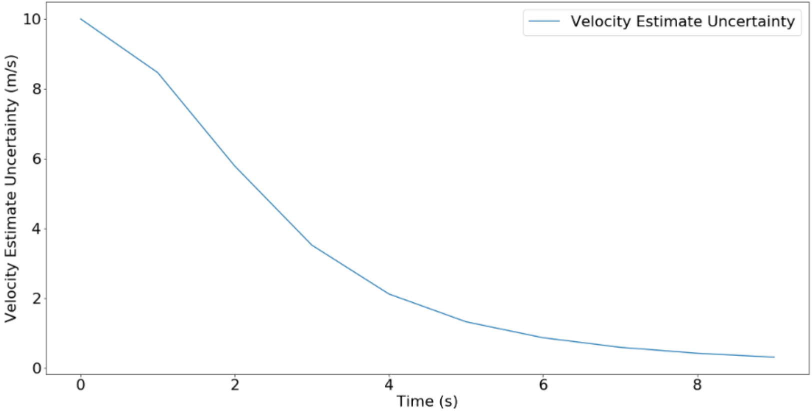
Linear Kalman Filter Example

The chart below shows the position estimate uncertainty. The initial value is 30 because our assumed initial covariance matrix $\mathbf{P}_0$.



Linear Kalman Filter Example

The chart below shows the velocity estimate uncertainty. The initial value is 10 because our assumed initial covariance matrix $\mathbf{P}_0$.



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[3] Timothy D. Barfoot. *State Estimates for Robotics*. Cambridge University Press, 2019.

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