Energy Function

Shulei ZHU

12 November 2010

Contents

1 Energy function

$$\mathcal{X}^2(\phi) = \text{argmin}_\phi \mathcal{X}^2(\phi)$$

where

$$\begin{aligned}\mathcal{X}^2(\phi) = & -2\ln\left\{\frac{1}{(2\pi)^{1/2}}\frac{1}{|\Sigma_\phi^*|}\exp\{-\frac{1}{2}(\phi - m_\phi^*)^T {\Sigma_\phi^*}^{-1}(\phi - m_\phi^*)\}\right\} \\ & -2\ln\left\{\prod_{v\in\mathcal{V}}^{N}\frac{1}{(2\pi)^{1/2}}\frac{1}{|\Sigma_v|}\exp\{-\frac{1}{2}[I_v - m_v(a_{v,1})]^T\left\{\Sigma_v(a_{v,1})\right\}^{-1}[I_v - m_v(a_{v,1})]\}\right\}\end{aligned}$$

Where we assume m_ϕ^* as $[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$ and m_ϕ^* as a 6×6 identity matrix, m_v and Σ_v represent the mean value and covariance of a given point on the curve.

$$m_v = a_{v,1}m_{v,1} + (1 - a_{v,1})m_{v,2}$$

$$\Sigma_v = a_{v,1}\Sigma_{v,1} + (1 - a_{v,1})\Sigma_{v,2}$$

$m_{v,1}(m_{v,2})$, $\Sigma_{v,1}(\Sigma_{v,2})$ are the mean values and covariance metrics in the positive(negative) normal direction near the vicinity of a given point, but the formula to calculate these four variables is very complicated, so I will not show it. $a_{v,1}$,$a_{v,2}$ are the fuzzy weights of a given point locating in side 1 and 2, N means the number of points on the curve. I_v denotes the pixel value of given point. $a_{v,1}$ and $a_{v,2}$ are given by

$$a_{v,1} = \Sigma_i^m \left\{ \frac{1}{2} \cdot erf(\frac{d_v(x)}{\sqrt{2} \cdot \sigma_v}) + \frac{1}{2} \right\}$$

$$a_{v,2} = 1 - a_{v,1}$$

here m means the number of points in the positive normal $(+\vec{n})$ direction in the vicinity of a given point, there are also m points in the negative normal direction $(-\vec{n})$, σ_v is given by

$$\sigma_v^2 = \mathbf{n}_v^T \cdot \mathbf{J}_v \cdot \mathbf{\Sigma}_\phi \cdot \mathbf{J}_v^T \cdot \mathbf{n}_v$$

where $\mathbf{n}$ is the normal vector, $\mathbf{J}_v$ denots the Jacobian of curve, i.e. the partial derivatives of c with respect to model parameters ϕ in the given point,e.g we are given a point (p_x, p_y), and model parameters are given by $[x_0, x_1, x_2, x_3, x_4, x_5]$, we can write $\mathbf{J}_v$ as

$$\mathbf{J}_v = \begin{bmatrix} \frac{\partial p_x}{\partial x_0} & \frac{\partial p_x}{\partial x_1} & \frac{\partial p_x}{\partial x_2} & \frac{\partial p_x}{\partial x_3} & \frac{\partial p_x}{\partial x_4} & \frac{\partial p_x}{\partial x_5} \\ \frac{\partial p_y}{\partial x_0} & \frac{\partial p_y}{\partial x_1} & \frac{\partial p_y}{\partial x_2} & \frac{\partial p_y}{\partial x_3} & \frac{\partial p_y}{\partial x_4} & \frac{\partial p_y}{\partial x_5} \end{bmatrix}$$

Furthermore, The energy function can be simplified as

$$\begin{aligned} \mathcal{X}^2(\phi) = &\ln(2\pi) + 2\ln|\Sigma_\phi^*| + \phi^T \Sigma_\phi^{*\,-1} \phi \\ &+ N \ln 2\pi + 2N\Sigma_{v\in\mathcal{V}}^N \ln|\Sigma_v| + \Sigma_{v\in\mathcal{V}}^N \left\{ [I_v - m_v(a_{v,1})]^T \{\Sigma_v(a_{v,1})\}^{-1} [I_v - m_v(a_{v,1})] \right\} \end{aligned}$$

2 Compute the first order derivative of energy function

$$\begin{aligned} \nabla_\phi\{\mathcal{X}^2(\phi)\} = &0 + 0 + \{\Sigma_\phi^{*\,-1}\}^T \phi + \Sigma_\phi^{*\,-1} \phi \\ &+ 0 + 0 + \Sigma_{v\in\mathcal{V}}^N \nabla_\phi \left\{ [I_v - m_v(a_{v,1})]^T \{\Sigma_v(a_{v,1})\}^{-1} [I_v - m_v(a_{v,1})] \right\} \end{aligned}$$

The question is how to calculate the last term in the above formula.

PS. You can find the expression of $a_{v,1}$ in the first part.