Geometric Variational Algorithm

定理 (Minkowsi)

Given $A_1, A_2, \cdots, A_k$ *and* $\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_k$, *such that* $\sum_{i=1}^{k} A_i \mathbf{n}_i = 0$. *There exists a convex polyhedron* P, *unique up to a translation, the area of the* i*-th face* F_i *is* A_i, *the normal to* F_i *is* $\mathbf{n}_i$.

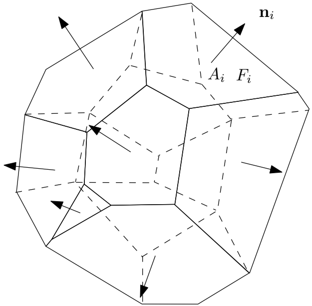


图: Minkowski problem.

定理 (Alexadnrov 1950)

Ω is a compact convex domain in $\mathbb{R}^n$, $p_1, \ldots, p_k$ are distinct vectors in $\mathbb{R}^n$, $A_1, \ldots, A_k > 0$, satisfying $\sum A_i = vol(\Omega)$, then there is a convex piecewise-linear function, unique up to a constant,

$$u(x) = \max_{i=1}^{k}\{\langle p_i, x\rangle - h_i\},$$

satisfying

$$vol(W_i) = A_i, \quad W_i = \{x | \nabla u(x) = p_i\}.$$

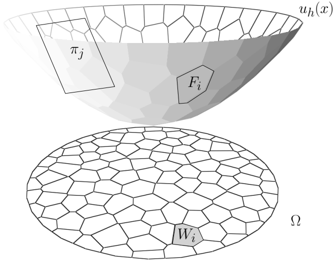


图: Alexandrov Theorem.

Alexandrov theorem is equivalent to the semi-discrete Optimal Transportation map. Let $\Omega \subset \mathbb{R}^d$ be a compact convex set, the measure μ is absolutely continuous, the target measure is the summation of Dirac measures,

$$\nu = \sum_{i=1}^{n} \nu_i \delta(y - y_i),$$

satisfying the condition $\mu(\Omega) = \sum_{i=1}^{n} \nu_i$, the transportation cost is the square of Euclidean distance $c(x, y) = \frac{1}{2}|x - y|^2$, then there is a unique Optimal Transportation map $T : (\Omega, \mu) \to (\{y_i\}_{i=1}^{n}, \nu)$, $T = \nabla u$, where the Briener potential function $u : \Omega \to \mathbb{R}$ is a PL convex function, u unique up to a constant:

$$u(x) = \max_{i=1}^{n} \{\langle x, y_i \rangle - h_i\},$$

the graph of u is the upper envelope of the supporting planes $\langle x, y_i \rangle = h_i$.

The graph of the Brenier potential induces a cell decomposition of $\mathbb{R}^d$

$$\Omega = \bigcup_{i=1}^{n} W_i(u), \qquad W_i(u) = \{x \in \Omega | \nabla u = y_i\}.$$

The Optimal Transportation map transforms each cell $W_i(u)$ to a target point y_i, $T: W_i(u) \mapsto y_i$. The Legendre dual u^* is the convex hull of the dual points $\{(y_i, h_i)\}_{i=1}^n$, each point (y_i, h_i) is dual to a supporting plane $\langle x, y_i \rangle - h_i$.

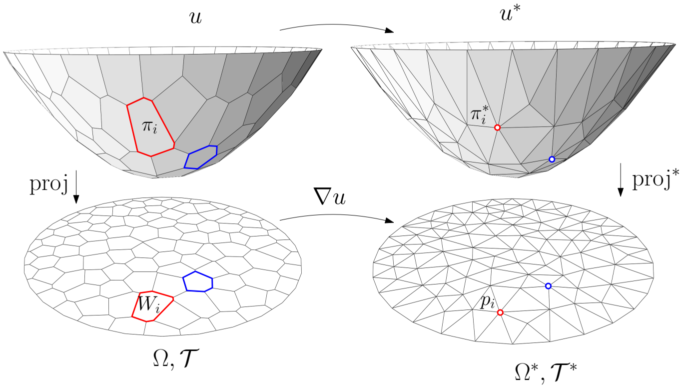


图: Semi-discrete OT map (from left to right): maps W_i to p_i. Discrete Monge-Ampère equation (from right to left): $\mu_\sigma(W_i)$ is the discrete Hessian determinant of p_i.

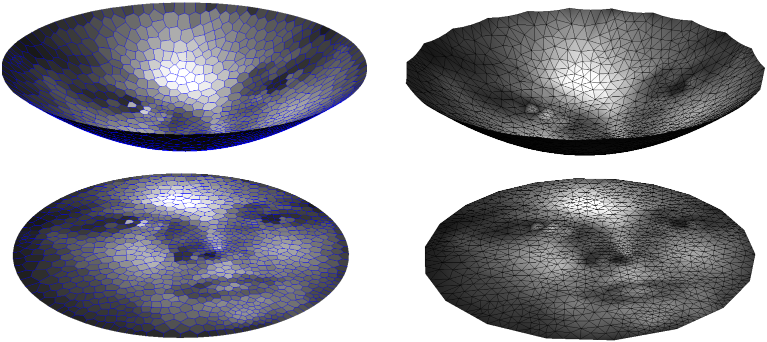


图: Semi-discrete Optimal Transportation Map

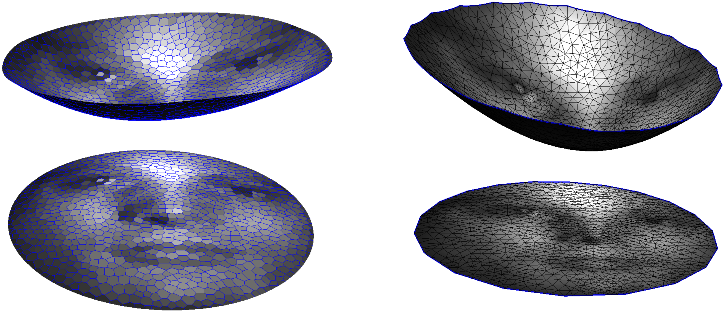


图: Semi-discrete Optimal Transportation Map

Input: Discrete point set $P = \{p_1, p_2, \cdots, p_k\}$, target measure $\{\nu_1, \nu_2, \cdots, \nu_k\}$; planar convex domain Ω, satisfying $\sum \nu_i = \text{Area}(\Omega)$;

Output: Optimal Transportation map $T : \Omega \to P$;

1. Translate, scale P, such that $P \subset \Omega$;
2. Initialize the height vector
$$\mathbf{h}^0 \leftarrow \frac{1}{2}(|p_1|^2, |p_2|^2, \cdots, |p_k|^2)^T;$$
3. Construct the supporting planes $\{\pi_i(\mathbf{h}^n)\}_{i=1}^k$
$$\pi_i(\mathbf{h}^n, x) = \langle p_i, x\rangle - h_i, \quad i = 1, 2, \ldots, k.$$

4. Construct the dual points of the supporting planes $\{\pi_i^*(\mathbf{h}^n)\}_{i=1}^k$,

$$\pi_i^*(\mathbf{h}^n) = (p_i, h_i), \quad i = 1, 2, \dots, k.$$

5. Compute the convex hull of the dual points $\text{Conv}(\{\pi_i^*(\mathbf{h}^n)\}_{i=1}^k)$, to get the Legendre dual of the potential $u^*(\mathbf{h}^n)$;
6. Compute the dual of the convex hull, get the upper envelope of the supporting planes $\text{Env}(\{\pi_i(\mathbf{h}^n)\}_{i=1}^k)$, get the Brenier potential $u(\mathbf{h}^n)$,

$$u(\mathbf{h}^n, x) = \max_{i=1}^{k} \pi_i(\mathbf{h}^n, x) = \max_{i=1}^{k}\{\langle p_i, x\rangle - h_i\}$$

7. Project the Legendre dual of the potential to get a weighted Delaunay triangulation of P, $\mathcal{T}(\mathbf{h}^n)$
8. Project the Brenier potential to get the Power diagram of Ω, $\mathcal{D}(\mathbf{h}^n)$, compute the intersection between each cell and Ω,

$$\Omega = \bigcup_{i=1}^{k} W_i(\mathbf{h}^n) = \bigcup_{i=1}^{k} \{x \in \mathbb{R}^2 | \nabla u(\mathbf{h}^n, x) = p_i\} \cap \Omega.$$

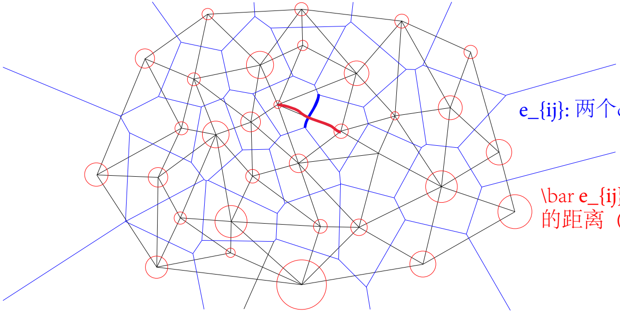
9. Compute the area of each cell, $w_i(\mathbf{h}^n)$, $i = 1, 2, \ldots, k$,

10. Compute the gradient of the energy $E(\mathbf{h})$:

$$E(\mathbf{h}) = \int^{\mathbf{h}} \sum_{i=1}^{k} w_i(\eta)\, d\eta_i - \sum_{i=1}^{k} \nu_i h_i$$

$$\nabla E(\mathbf{h}^n) = w_i(\mathbf{h}^n) - \nu_i.$$

11. If $|\nabla E(\mathbf{h}^n)|$ is less than a threshold ε, return the map $T = \nabla u(\mathbf{h}^n)$, $W_i(\mathbf{h}^n) \mapsto p_i$, $i = 1, 2, \ldots, k$.



12. Compute the Hessian matrix of the energy $E(\mathbf{h}^n)$

$$\frac{\partial^2 E(\mathbf{h}^n)}{\partial h_i \partial h_j} = \frac{\partial w_i(\mathbf{h}^n)}{\partial h_j} = -\frac{|e_{ij}|}{|\bar{e}_{ij}|},$$

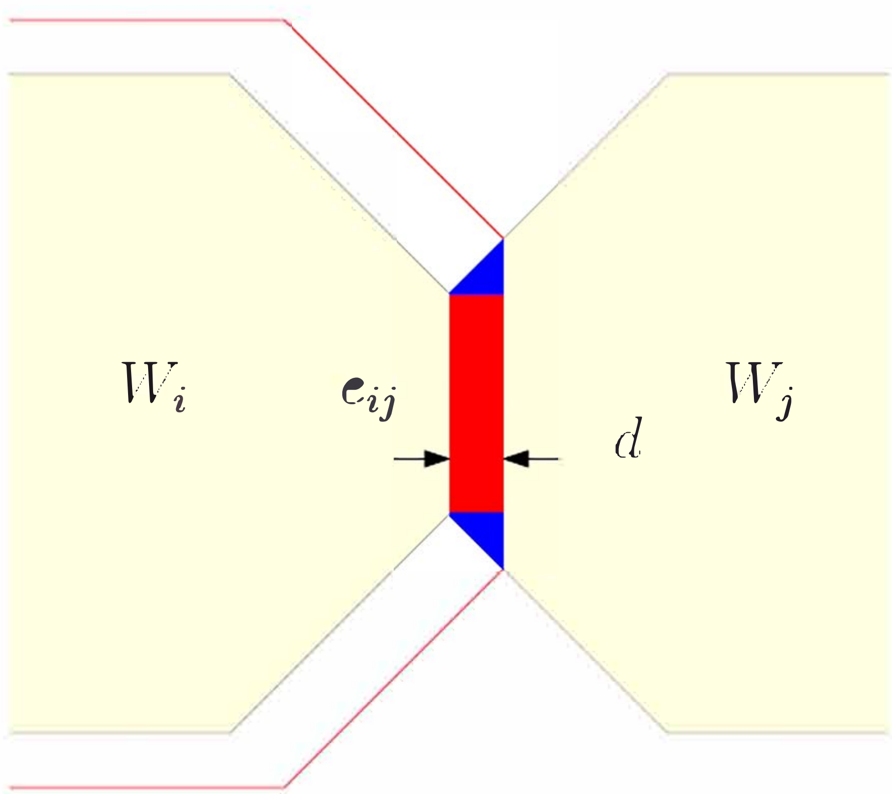
$$\frac{\partial^2 E(\mathbf{h}^n)}{\partial h_i^2} = \frac{\partial w_i(\mathbf{h}^n)}{\partial h_i} = -\sum_{i \neq j} \frac{\partial w_i(\mathbf{h}^n)}{\partial h_j},$$

Variational Proof

Lemma

The following symmetric relation holds, $w_i(\mathbf{h})$ is the area of face F_i:

$$\frac{\partial w_i(\mathbf{h})}{\partial h_j} = \frac{\partial w_j(\mathbf{h})}{\partial h_i} = -\frac{|e_{ij}|}{|\bar{e}_{ij}|} \leq 0.$$



Proof.

$\forall x \in e_{ij}$, $\langle p_i, x\rangle - h_i = \langle p_j, x\rangle - h_j$, hence $\langle p_i - p_j, x\rangle = h_i - h_j$. Change $h_i \to h_i + \delta h_i$, then $x \to x + d$, $|d| = \frac{\delta h_i}{|p_i - p_j|}$,

$$\delta w_j = -|e_{ij}||d| + o(\delta h_i^2) = -\frac{|e_{ij}|}{|p_i - p_j|}\delta h_i$$

$\bar{e}_{ij} = |p_i - p_j|$. □

13. Solve the linear system

$$\mathrm{Hess}(\mathbf{h}^n)\mathbf{d} = \nabla E(\mathbf{h}^n),$$

with linear constraint

$$d_1 + d_2 + \cdots + d_k = 0.$$

14. Set the initial step length $\lambda \leftarrow 1$,
15. Construct supporting planes $\{\pi_i(\mathbf{h}^n + \lambda\mathbf{d})\}_{i=1}^k$, dual points $\{\pi_i(\mathbf{h}^n + \lambda\mathbf{d})^*\}_{i=1}^k$
16. Construct the convex hull $\mathrm{Conv}\left(\{\pi_i(\mathbf{h}^n + \lambda\mathbf{d})^*\}_{i=1}^k\right)$
17. If there is a dual point $\pi_i(\mathbf{h}^n + \lambda\mathbf{d})^*$ which is not on the convex hull, let $\lambda \leftarrow \lambda/2$, repeat steps 15, 16, until all the dual points are on the convex hull;

18. Update the height vector

$$\mathbf{h}^{n+1} \leftarrow \mathbf{h}^{n} + \lambda \mathbf{d};$$

19. Repeat step 3 to step 18.