MATH 323 - Summer Semester

1 Session 1

1.1 Review of Set Theory

A set is a collection of object called elements (of the set)
Notation : Upper case letters are used to denote sets

Ex : A = set of all students in MATH 323.
An element of the set A is any student in MATH 323

1.1.1 Particular Set

1. Empty set
Notation : $\varnothing$
Def : Set with no element

Ex : Let A = $\{a, b, c\}$
a $\in$ A : a belongs to A
d $\notin$ A : d is not in A

1.1.2 Operations on sets

1. Union
   Let A, B be two sets.
   $$A \cup B = \{x | x \in \text{A or } x \in B\}$$
   (a) $A = \{a.b.c\}$ $\quad$ $B = \{c, d, e\}$
       $A \cup B = \{a, b, c, d, e\}$
   (b) $A = (-2, +\inf)$ $\quad$ $B = (-4, 6)$
       $A \cup B = (-4, +\inf)$
2. Intersection Let A, B be two sets
   $$A \cap B = \{x | x \in \text{A and } x \in B\}$$
   (a) $A = \{a, b, d\}$ $\quad$ $B = \{b, d, e\}$
       $A \cap B = \{b, d\}$
   (b) $A = (-2, +\inf]$ $\quad$ $B = (-6, 2)$
       $A \cap B = (-2, 2)$
   (c) $A = (2, +\inf)$ $\quad$ $B = (-6, 2)$
       $A \cap B = \{\varnothing\}$

A and B are distant

$$A \cup B = \varnothing \iff A = B = \varnothing$$

3. Complement
   Setting : Let S be a non-empty set
   Let A be a set such that $A \subseteq S$
   $A \subseteq S$ means that $\forall x \in A, x \in S$

   The complement of A WITH RESPECT TO S is the subset of S denoted A^c

   $$A^c = \{x \in \text{S and } x \notin A\}$$

   Different set, different complements (why with respect to the set is primordial)

   (a) Let S be any set
       $\varnothing \in S$
       $\varnothing^c = S$
   (b) S = {a,b,c,d,e,f,g }
       i. $A = \{a, h\}$ $\quad A \notin S$
       ii. $A = \{a, g\}$ $\quad A^c = \{b, c, d, e, f, \}$
   (c) Let S be $(-2, 4) \cup (6, +\inf)$ and A be (8,10)
       $A^c = (-2, 4) \cup (6, 8] \cup [10, +\inf]$

1.1.3 Properties of sets

1. $A \cap B = B \cap A$
   $A \cup B = B \cap A$
2. $A \cap \varnothing = \varnothing$
   $A \cup \varnothing = A$
3. $A \cup (B \cup C) = (A \cup B) \cup C$
   $A \cap (B \cap C) = (A \cap B) \cap C$
4. $(A^c)^c = A$
5. $(A \cup B)^c = A^c \cap B^c$
   $(A \cap B)^c = A^C \cup B^c$

Proof. Let's prove 5 !
E = F means

1. $E \subseteq F$
2. $F \subseteq E$

Proving $(A \cup B)^c = A^c \cap B^c$

1. $\subseteq$
   Let $x \in (A \cup B)^x$, this means that $x \notin (A \cup B)$
   $x \notin A$ and $x \notin B$
   $x \in A^c$ and $x \in B^c$
   $x \in (A^c \cap B^c)$
2. $\supseteq$
   Let $x \in (A^c \cap B^c)$
   $x \in A^c$ and $x \in B^c$
   $x \notin A$, $x \notin B$
   $x \notin (A \cup B)$
   $x \in (A \cup B)^c$

□

1.1.4 Other notation

1. $\setminus$
   $A \setminus B = \{x \in A | x \notin B\}$
   $A \setminus B = A \cap B^c$
2. $\triangle$
   $A \triangle B = A \setminus B \cup B \setminus A$

1.2 Let's start probability

Probability : studying random events as opposed to deterministic events
Random Experiments : experiments such that the outcome is unknown, but part of a set called Ω
Ex

1. Toss a coin
   $\Omega = \{H, T\}$

2. Roll a dice
   $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$
   ω_i is the number i appera on the top-face of the dice.

3. Tossing a coin until H appears
   $\Omega = \{"H", "TH", "TTH", "TTTH", ..., TT...TH, ...\}$

Let Ω be the set of all outcomes of a random experiment. A subset of Ω is called an event

1. Elementary event
   An event with only one outcome

2. Compounded event
   An event that has 2 or more elements

Remark : compounded events can be written as the union of elementary event

Ex :

1. Rolling a dice
   $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$
   $\{\omega_i\}$ is an elementary event (while ω_i is an outcome). There are six elementary event, namely $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$

2. Rolling a dice and getting an even number
   A is the even where the number obtained is even.
   $A = \{\omega_2, \omega_4, \omega_6\}$
   A is a compounded event therefore $A = \{\omega_2\} \cup \{\omega_4\} \cup \{\omega_6\}$

1.2.1 Power set of Ω

Let Ω be the set of outcomes for a random experiment. The power set of Ω is the set of all events (i.e. the ste of all subsets of Ω.
Notation : $\mathcal{P}(\Omega)$

Ex : Tossing a coin
$\Omega = \{H, T\}$
$\mathcal{P}(\Omega) = \{\varnothing, \{H\}, \{T\}, \Omega\}$

1.2.2 Probability

Def : Let Ω be the set of outcomes corresponding to a random experiment.
Remark : Ω is also called a sample space

A probability P is a function

$$P : \mathcal{P}(\Omega) \to [0, 1]$$

1.2.3 Properties of probabilities

1. $P(\Omega) = 1$

2. If $A_1, A_2, ..., A_n, ...$ is a sequence of pairwise, disjoint events. (i.e. $A_i \cap A_j = \varnothing$ if $i \neq j$) then

$$P\left(\bigcup_{n=1}^{\infty} A_i\right) = \sum_{n=1}^{\infty} P(A_n)$$

<u>Remark</u> : In practice, if Ω is finite, $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ where n = card(Ω). The cardinality is the number of elements in a set.

Let P be a probability on Ω. Set $P_i = P(\{\omega_i\}$. Then $\sum_{i=1}^{n} P_i = 1$

$$\Omega = \bigcup_{i=1}^{n} \{w_i\} \text{ and } \{w_i\} \cap \{w_j\} = \varnothing \text{ if } i \neq j$$

By 1 and 2

$$1 = P(\Omega) = \sum_{i=1}^{n} P(\{\omega_i\}) = \sum_{n}^{i=1} P_i$$

<u>Remark</u> : 2. holds if $A_1, A_2, ..., A_n$ is a finite sequence of pairwise, disjoint set

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$\underbrace{A_1, A_2, ..., A_n}_{\mathcal{P}(\Omega)}, \underbrace{A_{n+1}, ...}_{\varnothing}$$

$$\sum_{i=1}^{\infty} P(A_i) + P(\varnothing) = 1 + 0 = 1$$

Ex : Rolling a dice

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$$

The following defines a probability on $\mathcal{P}(\Omega)$

$$P(\{\omega_1\} = P(\{\omega_2\} = P(\{\omega_3\} = P(\{\omega_5\} = \frac{1}{6}$$

$$P(\omega_4\} = \frac{1}{4} \qquad P\{\omega_6\} = \frac{1}{12}$$

Let A be the event where the number obtained when rolling a dice is even. What is P(A).

$$A = \{\omega_2, \omega_4, \omega_6\} = \{\omega_2\} \cup \{\omega_4\} \cup \{\omega_6\}$$

$$\begin{aligned} P(A) =& P(\{\omega_2\}) + P(\{\omega_4\}) + P(\{\omega_6\}) \\ =& \frac{1}{6} + \frac{1}{4} + \frac{1}{12} \\ =& \frac{1}{2} \end{aligned}$$

2 Session 2

2.1 Properties

1. $P(\varnothing) = 0$
2. $P(A^c) = 1 - P(A)$
   This is useful when finding P(A) is difficult or complicated.
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof. 1. $P(\varnothing) = 0$

$$\begin{aligned} \Omega =& \Omega \cup \varnothing \\ \Omega \cap \varnothing =& \varnothing \end{aligned}$$

$$\begin{aligned} 1 =& P(\Omega) \\ 1 =& P(\Omega \cup \varnothing) \\ 1 =& P(\Omega) + P(\varnothing) \\ 1 =& 1 + P(\varnothing) \\ 0 =& P(\varnothing \end{aligned}$$

2. $P(A^c) = 1 - P(A)$

$$\begin{aligned} \Omega =& A \cup A^c \\ A \cap A^C =& \varnothing \end{aligned}$$

$$\begin{aligned} 1 =& P(\Omega) \\ 1 =& P(A \cup A^c) \\ 1 =& P(A) + P(A^c) \\ P(A) =& 1 - P(A^c) \end{aligned}$$

3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Remark : Another way to express $A \cup B$ is $(A \setminus B) \cup (B \setminus A) \cup (A \cap B)$
We can therefore say that $P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A)$

Remark : Another way to express $A = (A \setminus B) \cup (A \cap B)$
Remark : $(A \setminus B) \cap (A \cap B) = \varnothing$

$$\begin{aligned} P(A) =& P(A \setminus B) + P(A \cap B) \\ P(B) =& P(B \setminus A) + P(A \cap B) \\ P(A) + P(B) =& P(A \setminus B) + 2 \times P(A \cap B) + P(B \setminus A) \\ P(A) + P(B) =& P(A \cup B) + P(A \cap B) \\ P(A \cup B) =& P(A) + P(B) - P(A \cap B) \end{aligned}$$

$\square$

2.2 Discrete Probability Space

Let Ω be a sample space. Ω is said to be discrte if Ω is either finite or infinite but countable.

Exemple of a finite sample space
Tossing a coin $\Omega = \{"H", "T", \}$
Example of an infinite but countale sample space

Tossing a coin until you get heads : $\Omega = \{"H", "TH", "TTH", ..., \underbrace{TT...T}_{n-1}H, ...\}$

Remark : If Ω is discrete and infinite it will have the following form

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n, \omega_{n+1}, ...\}$$

To define a probability on $\mathcal{P}(\Omega)$ we need to choose $P_n = P(\{\omega_n\}), P_n \in (0, 1)$ such that $\sum_{n=1}^{\infty} P_N = 1$

Ex : Toss a coin unitl you get H

$$\begin{aligned}\Omega =& \{\omega_1, \omega_2, ..., \omega_n, ...\}\\ \omega_n =& "\underbrace{TT...T}_{n-1}H" \qquad \text{H appears on the nth term}\end{aligned}$$

Suppose that $P(\{\omega_n\}) = C3^{-n}$

$$\begin{aligned}\sum_{n=1}^{\infty} C3^{-n} =& 1\\ C\sum_{n=1}^{\infty} 3^{-n} =& 1\\ C =& \frac{1}{\sum_{n=1}^{\infty} 3^{-n}}\\ C =& \frac{1}{\frac{\frac{1}{3}}{1-\frac{1}{3}}}\\ C =& \frac{\frac{2}{3}}{\frac{1}{3}}\\ C =& 2\end{aligned}$$

Let A be the event that H appears on an even number of tries

$$\begin{aligned}A =& \{\omega_2, \omega_4, \omega_6, ..., \omega_{2n}, ...\}\\ P(A) =& P(2) + P(4) + P(6) + ... + P(2n) + ...\\ P(A) =& \sum_{n=1}^{\infty} P_{2n}\\ P(A) =& \sum_{n=1}^{\infty} 2 \times 3^{-2n}\\ P(A) =& 2\sum_{n=1}^{\infty} 3^{-2n}\\ P(A) =& 2\sum_{n=1}^{\infty} (\frac{1}{9})^n\\ P(A) =& \frac{2 \times \frac{1}{9}}{1 - \frac{8}{9}}\\ P(A) =& \frac{\frac{2}{9}}{\frac{8}{9}} = \frac{1}{4}\end{aligned}$$

2.3 Equiprobability

Setting : Ω is a finite sample space

$$\Omega = \{\omega_1, ..., \omega_n\} \qquad N = \text{card}(\Omega)$$

Equiprobability assumption is that every elementary event has the same probability namely $\frac{1}{N}$ or $\frac{1}{\text{card}(\Omega)}$

Let A be an event

$$\begin{aligned} P(A) = & \sum_{\omega_i \in A} P(\omega_i) \\ = & \frac{1}{N}\text{card}(A) \\ = & \frac{\text{card}(A)}{\text{card}(\Omega)} \end{aligned}$$

2.4 Counting tools

2.4.1 Cartesian product

Let Ω_1 and Ω_2 be 2 sample spaces. The cartesian product of Ω_1 and Ω_2 is

$$\Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) | \omega_1 \in \Omega_1 \text{ and } \omega_2 \in \Omega_2\}$$

Ω_1 and Ω_2 are finite

$$\text{card}(\Omega_1 \times \Omega_2) = \text{card}(\Omega_1) \times \text{card}(\Omega_2)$$

Ex : Rolling a fair dice twice such that the sum of the two dices is 10. Fair means the experiment is equiprobable.

$$\begin{aligned} \Omega_1 = & \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\} \\ \Omega_2 = & \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\} \end{aligned}$$

$$\begin{aligned} \Omega = & \Omega_1 \times \Omega_2 \\ \Omega = & 36 \text{ combinations} \end{aligned}$$

$$\begin{aligned} A = & \{(\omega_4, \omega_6), (\omega_5, \omega_5), (\omega_6, \omega_6)\} \\ P(A) = & \frac{\text{card}(A)}{\text{card}(\Omega} \\ P(A) = & \frac{3}{36} = \frac{1}{12} \end{aligned}$$

2.4.2 Permutation

Def : A permutation of r elements chosen from n elements is equivalent to drawing successively without replacement (W.O.R) r elements from a set of n elements. The number of permutation is

$$P_r^n = n \times (n-1) \times (n-2) \times ... \times (n-r+1)$$
$$P_r^n = \frac{n!}{(n-r)!}$$

Ex : An urn contains 4 balls (1 red, 1 green, 1 yellow and 1 blue). Draw successively WOR 2 balls from the urn. What's the probability that one of the balls drawn is green.

$$\Omega = \text{set of permutations of two balls chosen out of 4 balls}$$
$$\text{card}(\Omega) = P_2^4 = \frac{4!}{2!} = 12$$
$$A = \text{one of the balls is green}$$
$$A = \{\text{"}GX\text{"}, \text{"}XG\text{"}\} \qquad \text{X = 3 since there are three other colors to choose from (aside from green)}$$
$$\text{card}(A) = 6$$
$$P(A) = \frac{6}{12} = \frac{1}{2}$$

Ex : A monkey is playing with the letters of the word "ALLEE". What is the probability that the two E's will come together.

The number of distinc permutations of the letters in ALLEE is $\frac{5!}{2!2!}$. Let A be the event that the two E s are together.
Possibilites

$$A = \left\{ \begin{array}{lcl} EEXXX & \frac{3!}{2!} = & 3 \\ XEEXX & \frac{3!}{2!} = & 3 \\ XXEEX & \frac{3!}{2!} = & 3 \\ XXXEE & \frac{3!}{2!} = & 3 \end{array} \right\} = 12$$

$$P(A) = \frac{12}{\frac{5!}{2!2!}}$$
$$P(A) = \frac{2 \times 2 \times 2 \times 2 \times 3}{2 \times 2 \times 3 \times 2 \times 5}$$
$$P(A) = \frac{2}{5}$$

2.4.3 Combination

Def : A combination of r elements chosen from n elements is equivalent to drawing simultaneously r elements from a set of n elements. Note that each combination of r element selected from n element yields r! permutation of r elements selected from n elements.

Total number of combination of r elements selected from n

$$C_r^n = \frac{P_r^n}{r!} = \frac{n!}{r!(n-r)!}$$

Remark : C_r^n is also the number of subsets with r elements of a set with n elements

Ex : A hand of 5 cards is selected from a deck of 52 cards. What is the probability it will contain at least one jack.

Remark : For problems involving cards, it is assumed there are 52 cards divided in 4 suites. 13 cards per suite.

Let's use A^c and A. Here A is a hand with 1, 2, 3 or 4 jacks. A^c is a hand with no jacks. A^c is easier to determine.

$$\begin{aligned} P(A) =& 1 - P(A^c) \\ P(A) =& 1 - \frac{C^4 8_5}{C^5 2_5} \end{aligned}$$

Properties of combination

1. $C_0^n = 1 = C_n^n$
2. $C_r^n = C_{n-r}^n$
3. $C_r^n = C_r^{n-1} + C_{r-1}^{n-1}$

Proof. Omit for now as it involves graph and I suck at graphs right now lol □