Problem 2

We analyze approximations to

$$\frac{d^2x}{dt^2} + 4\pi^2 x = 0, t_0 = 0, x_0 = -1, x'_0 = 0, t \in [0,1.2],$$

which has exact solution

$$x(t) = -\cos(2\pi t),$$

using Euler's Method and the fourth order Runge-Kutta method (RK4). First we show that $x(t)$ is the exact solution to the initial value problem.

1) Domain of x contains a neighborhood of t_0
The domain of $x(t)$ is $\mathbb{R}$, and therefore, contains a neighborhood of t_0.

2) $x(t_0) = x_0$

$$x(t_0) = -\cos(0) = -1$$

3) $\frac{d^2x}{dt^2} = -4\pi^2 x$

$$\frac{dx}{dt} = 2\pi \sin(2\pi t)$$

$$\frac{d^2x}{dt^2} = 4\pi^2 \cos(2\pi t) =? -4\pi^2 x$$

$$=? -4\pi^2(-\cos(2\pi t))$$

$$= 4\pi^2 \cos(2\pi t)$$

$x(t)$ satisfies all conditions necessary for it to be the unique solution to the IVP.
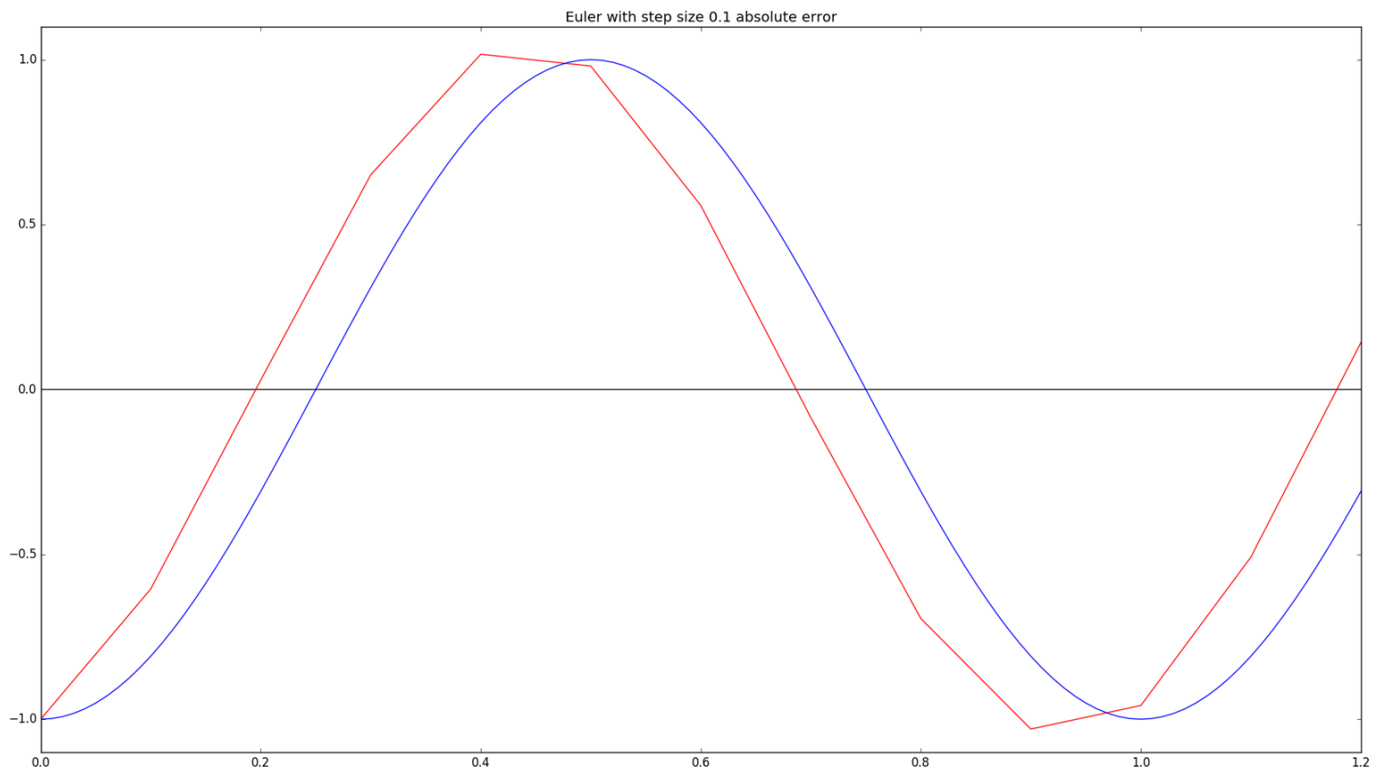
Figure 1. Graph of the exact solution (blue) and an approximation of it (red) using Euler's Method with a step size of 0.1.
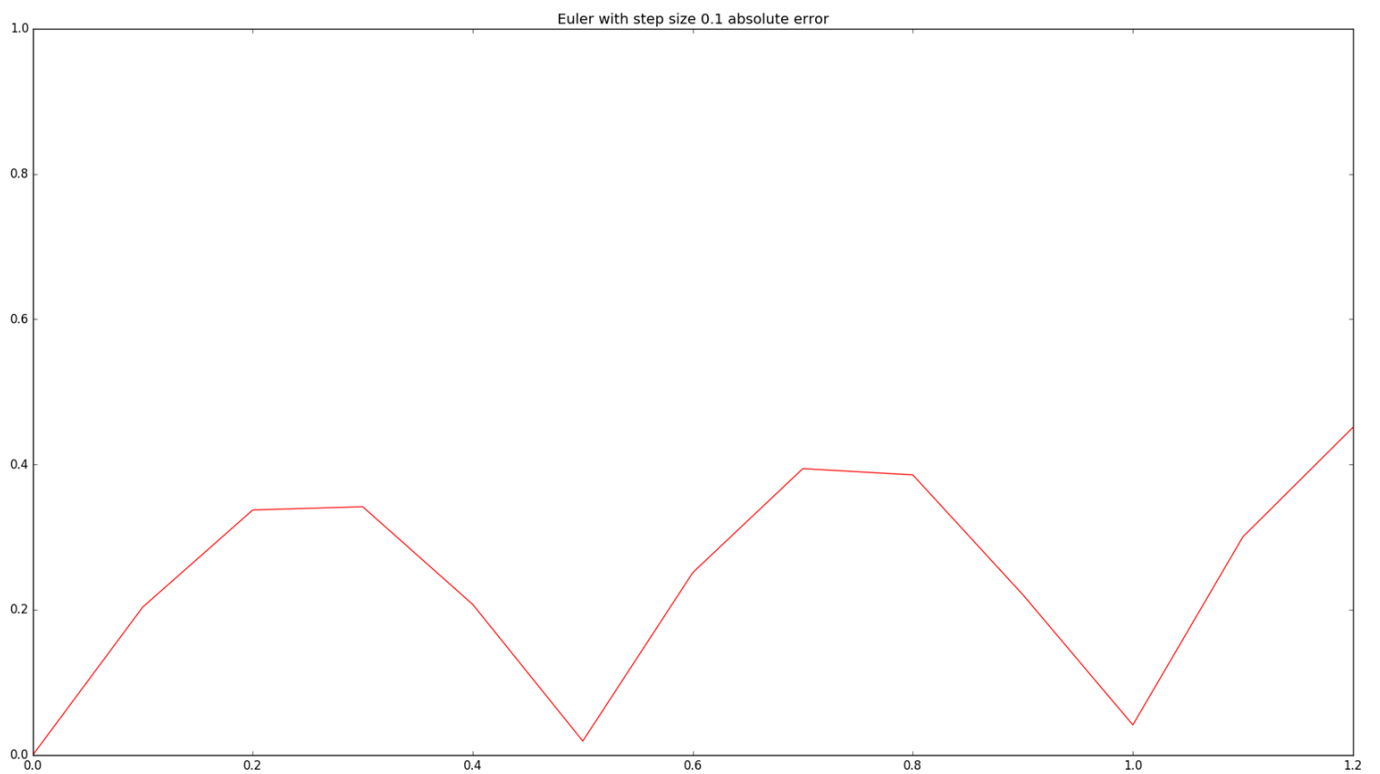

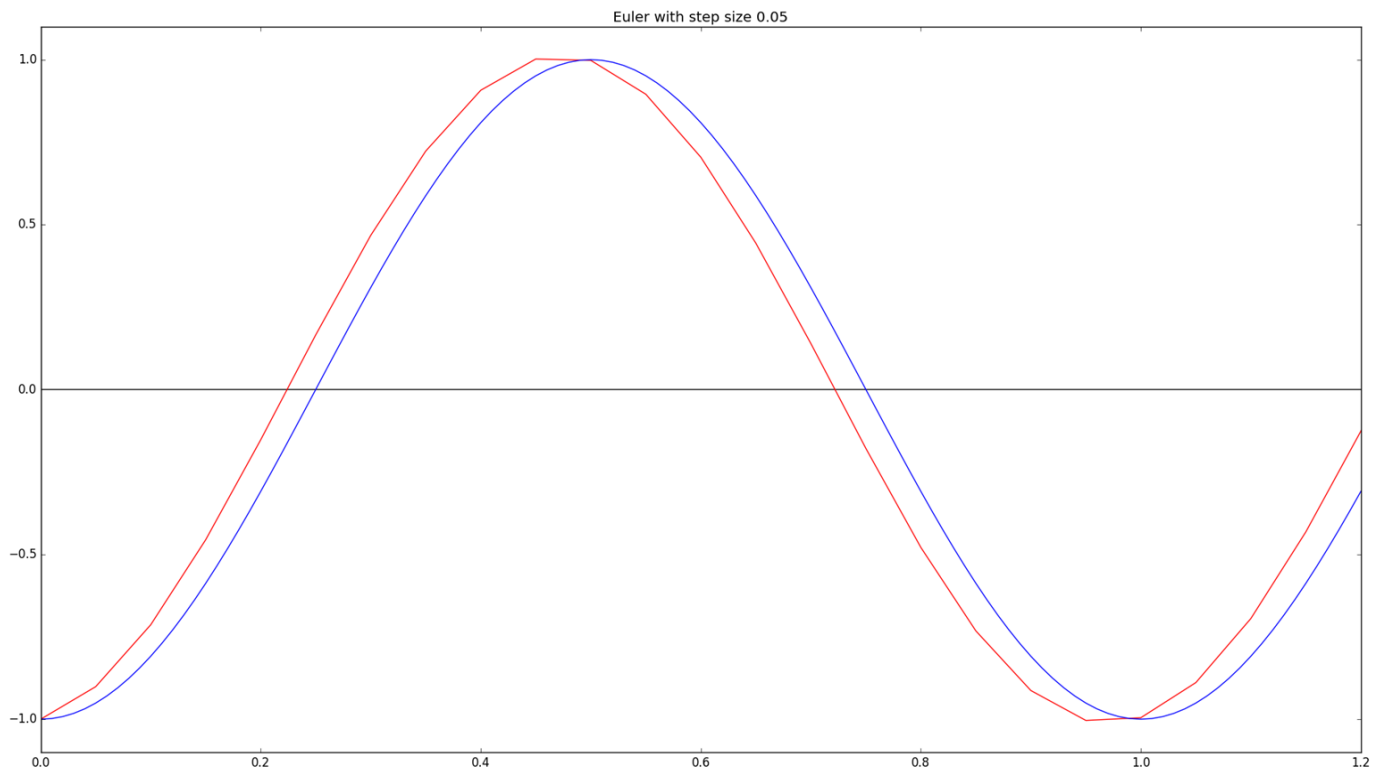Figure 2. Absolute errors between the curves in figure 1.

Figure 3. Graph of the exact solution (blue) and an approximation of it (red) using Euler's Method with a step size of 0.05.
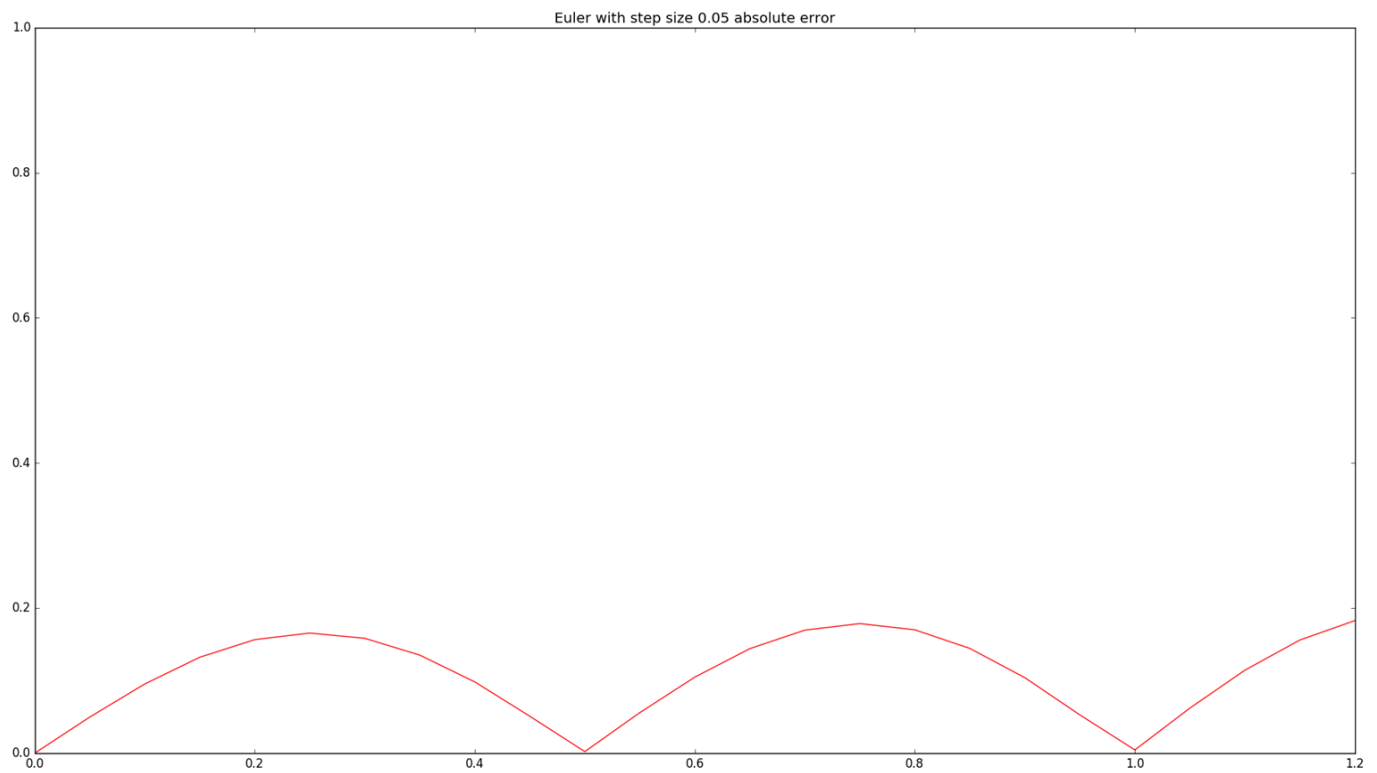

Figure 4. Absolute errors between the curves in figure 3.

Euler with step size 0.01

Figure 5. Graph of the exact solution (blue) and an approximation of it (red) using Euler's Method with a step size of 0.01.

Euler with step size 0.01 absolute error

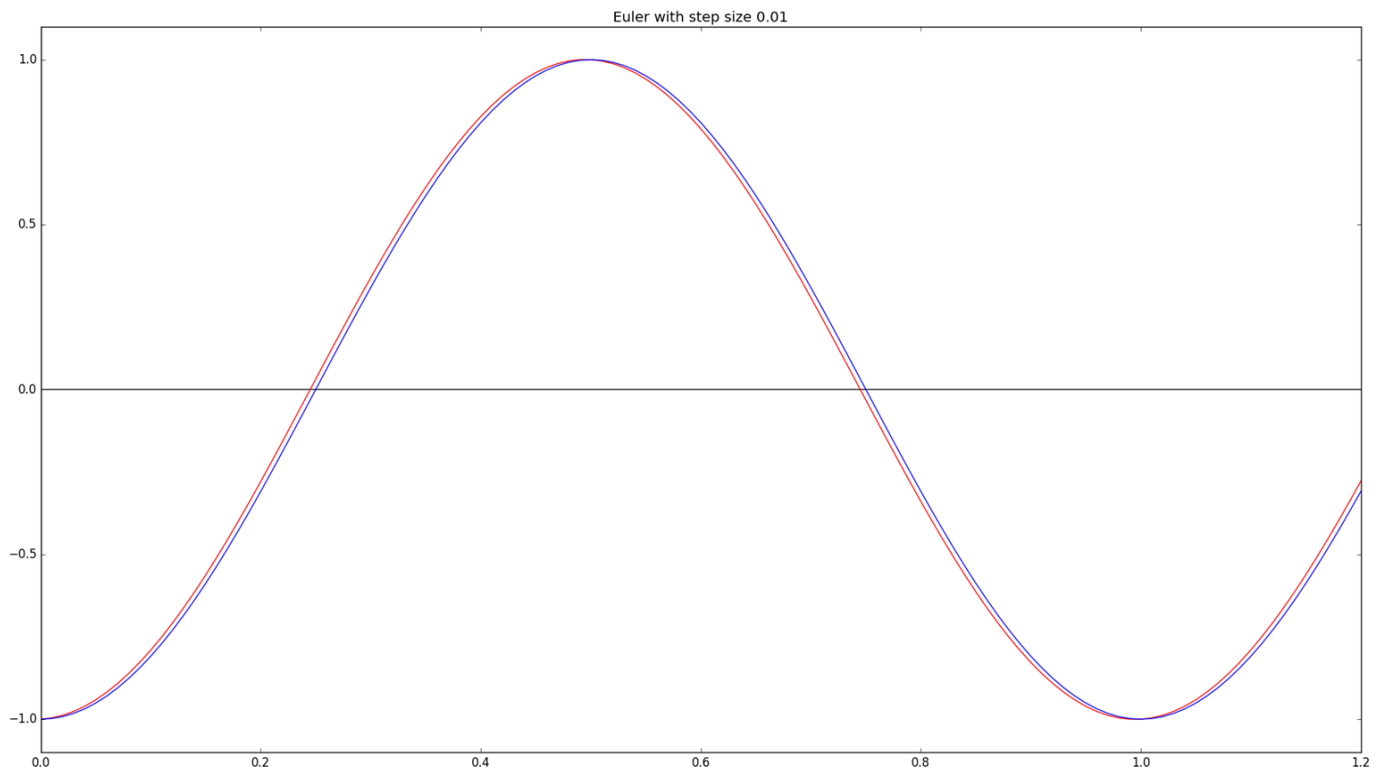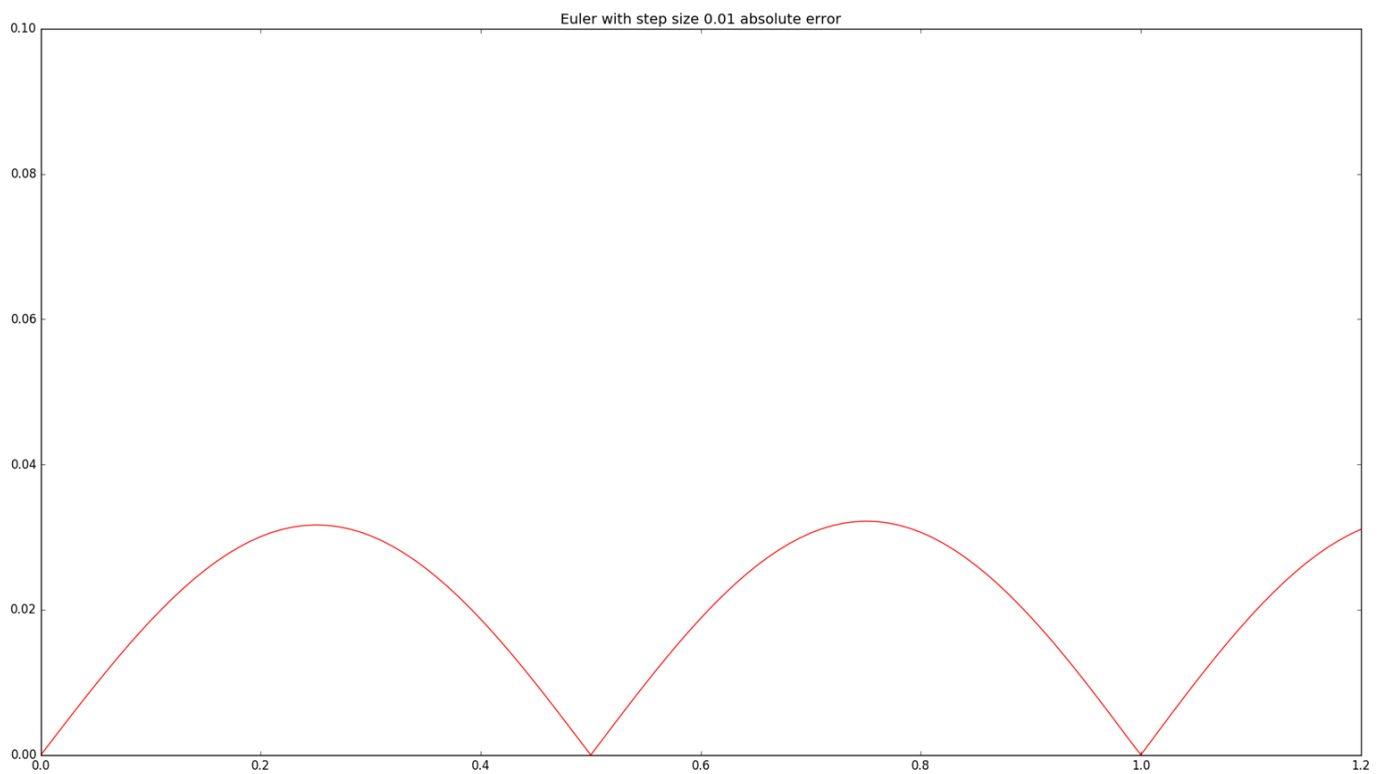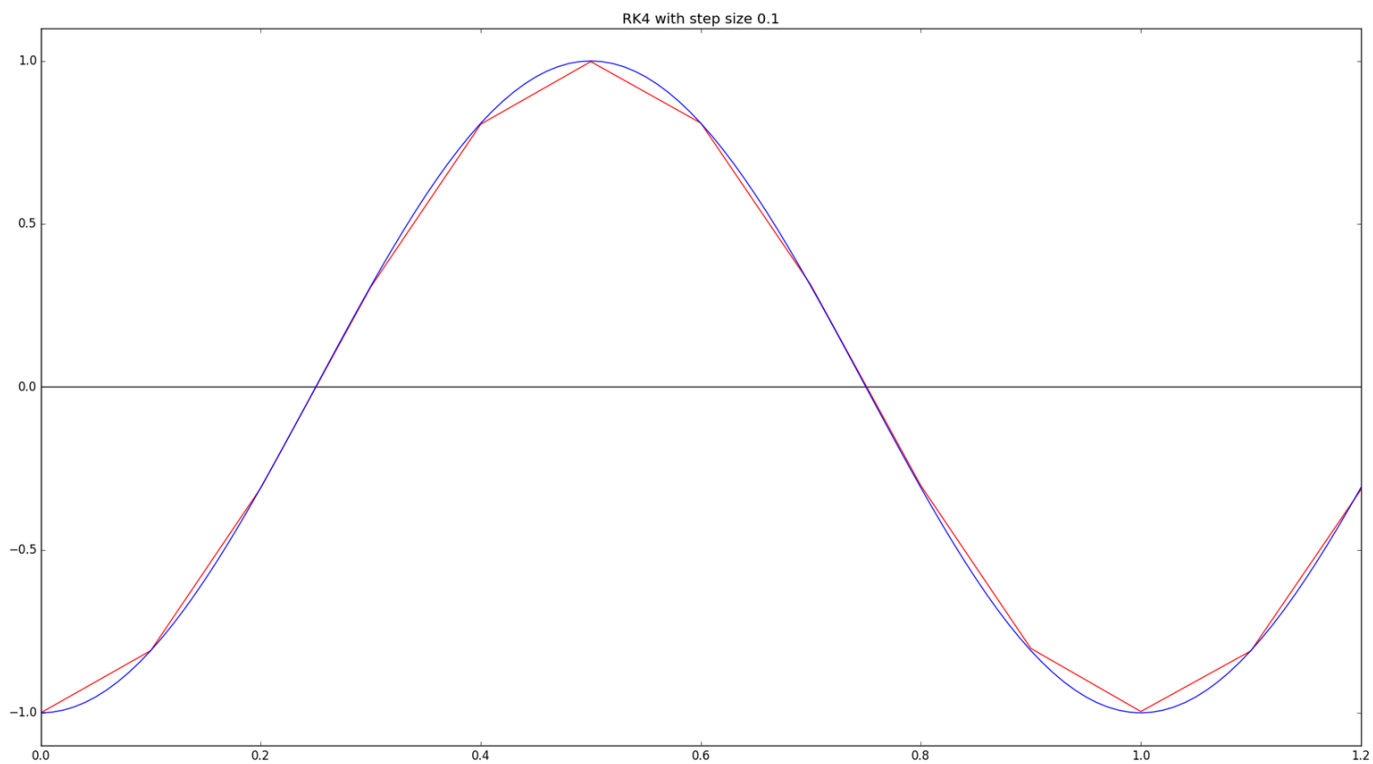Figure 6. Absolute errors between the curves in figure 5.

Figure 7. Graph of the exact solution (blue) and an approximation of it (red) using the RK4 method with a step size of 0.1.
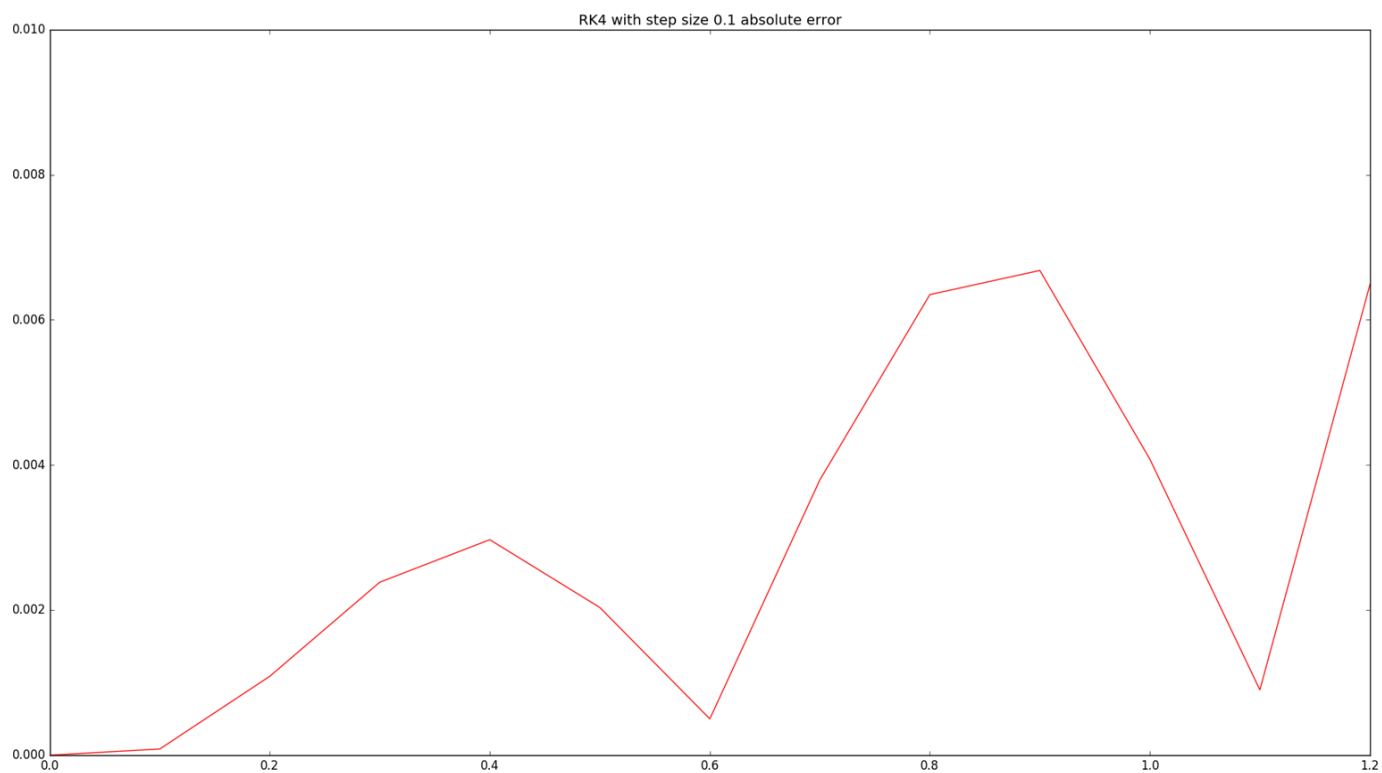

Figure 8. Absolute errors between the curves in figure 7.

t	Exact	Approximate	Absolute Error
0.1	-0.809016994375	-1.0	0.190983005625
0.2	-0.309016994375	-0.605215823956	0.296198829581
0.4	0.809016994375	0.650961236378	0.158055757997
0.5	1.0	1.01643529494	0.0164352949403
0.8	-0.309016994375	-0.0854105075599	0.223606486815
0.9	-0.809016994375	-0.694800685093	0.114216309282
1.0	-1.0	-1.02989454665	0.0298945466474
1.1	-0.809016994375	-0.958402338192	0.149385343817

Table 1. The approximate solution using Euler's Method with step size 0.1

t	Exact	Approximate	Absolute Error
0.1	-0.809016994375	-0.901303955989	0.0922869616142
0.2	-0.309016994375	-0.455566892258	0.146549897883
0.4	0.809016994375	0.722976492587	0.0860405017884
0.5	1.0	1.00199261857	0.00199261856887
0.8	-0.309016994375	-0.178619651431	0.130397342944
0.9	-0.809016994375	-0.732176804373	0.0768401900017
1.0	-1.0	-1.00381420858	0.00381420858143
1.1	-0.809016994375	-0.888939710516	0.0799227161415

Table 2. The approximate solution using Euler's Method with step size 0.05

t	Exact	Approximate	Absolute Error
0.1	-0.809016994375	-0.827433779738	0.018416785363
0.2	-0.309016994375	-0.338715393841	0.029698399466
0.4	0.809016994375	0.790795614871	0.0182213795043
0.5	1.0	1.00001611644	1.61164385375e-05
0.8	-0.309016994375	-0.279918571702	0.0290984226734
0.9	-0.809016994375	-0.791112372215	0.0179046221595
1.0	-1.0	-1.00003196558	3.19655828138e-05
1.1	-0.809016994375	-0.82685177119	0.0178347768154

Table 3. The approximate solution using Euler's Method with step size 0.01

t	Exact	Approximate	Absolute Error
0.1	-0.809016994375	-0.80910185138	8.48570055323e-05
0.2	-0.309016994375	-0.310104008738	0.0010870143628
0.4	0.809016994375	0.806046873672	0.00297012070284
0.5	1.0	0.997964059004	0.00203594099616
0.8	-0.309016994375	-0.302668755017	0.00634823935767
0.9	-0.809016994375	-0.802335813378	0.00668118099676
1.0	-1.0	-0.995919916214	0.00408008378567
1.1	-0.809016994375	-0.809917297839	0.000900303464367

Table 4. The approximate solution using the RK4 method with step size 0.1

      The graphs show increasing errors as values between the extrema are approximated, and decreasing errors as values near the extrema are approximated. This is counter-intuitive, and against our findings from problem 1, in which a changing slope caused worse approximations. However, this is due to the errors that manifest in the nonlinear regions, which propagate to the linear regions. The linear regions actually have the lowest change in their error values. Compared to the errors in the nonlinear regions, they are relatively constant. This shows that the approximations are behaving normally, despite the seemingly odd behavior of the errors (which is likely due to the periodic nature of the exact solution).