Problem 3

Since we are given a continuous function and are tasked with finding the least squares polynomial, we can use the linear system

$$\begin{bmatrix} \int_{i=0}^{m} \phi_0\phi_0 dx & \int_{i=0}^{m} \phi_0\phi_1 dx & \cdots & \int_{i=0}^{m} \phi_0\phi_n dx \\ \vdots & & & \vdots \\ \vdots & \vdots & \int_{i=0}^{m} \phi_j\phi_k dx & \vdots \\ \vdots & & & \vdots \\ \int_{i=0}^{m} \phi_n\phi_0 dx & \int_{i=0}^{m} \phi_n\phi_1 dx & \cdots & \int_{i=0}^{m} \phi_n\phi_n dx \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_j \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \int_{i=0}^{m} f(x)\phi_0 dx \\ \vdots \\ \int_{i=0}^{m} f(x)\phi_j dx \\ \vdots \\ \int_{i=0}^{m} f(x)\phi_n\, dx \end{bmatrix}$$

to find the coefficients, where ϕ_j is the j^{th} Legendre polynomial.. Since we are using the Legendre polynomials as the set to form the least squares polynomial, the polynomials will be of the form

$$a_0\phi_0(x) + a_1\phi_1(x) + \cdots + a_n\phi_n(x)$$

Moreover, since the interval is [-1, 1], the set of Legendre polynomials becomes orthogonal, and we can find the coefficients simply by solving

$$a_k = \frac{\langle f(x), \phi_k(x)\rangle}{\langle \phi_k(x), \phi_k(x)\rangle}$$

the results of which are shown in table 1.

k	**k'th Degree Coefficient**
0	0.25
1	0.0
2	-0.46875000000
3	0.0
4	0.26367187500

Table 1. The coefficients of the n^{th} degree least squares polynomial for the given data.

As expected from the nature of polynomials, the approximating functions are only reliable on [-1, 1] (Figure 2). Outside of this interval, the polynomials venture off into infinity. This makes polynomial approximation functions a bad choice for approximating periodic functions.

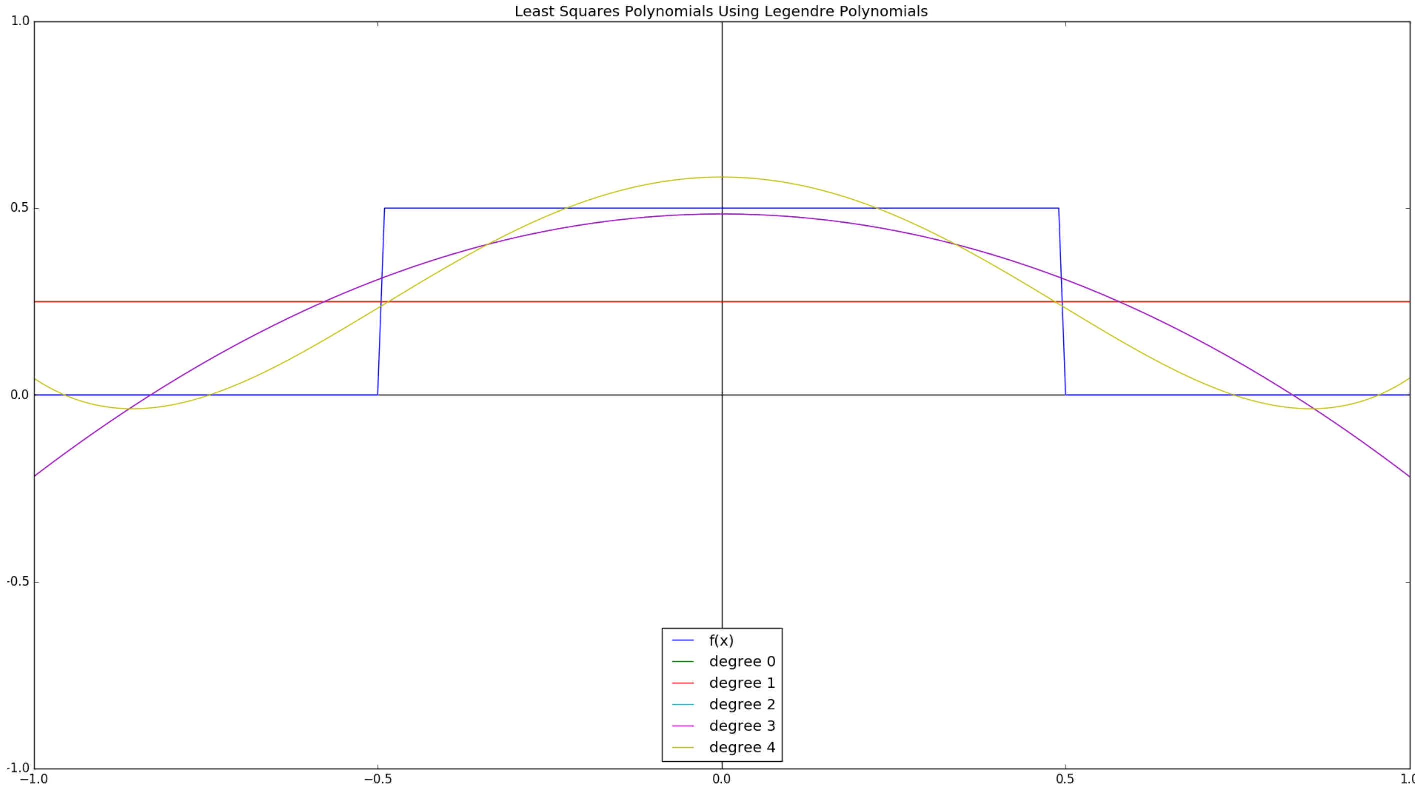

Figure 1. Least squares polynomials in [-1, 1]. The polynomials of degree 0 and 1 are the same, as are the polynomials of degree 2 and 3 (since a_1 and a_3 are zero).

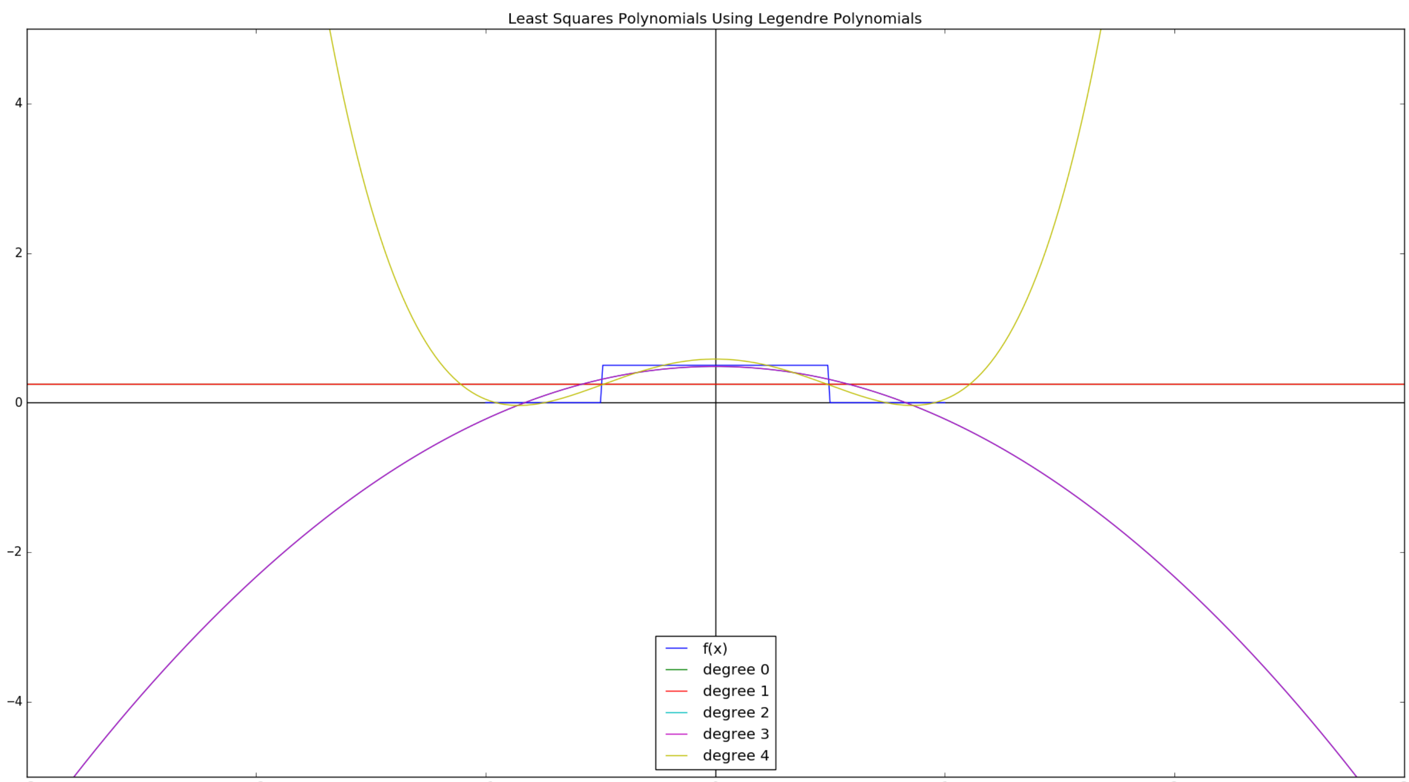

Figure 2. Least squares polynomials outside of [-1, 1].