Problem 1: Prove that there exists a polynomial $q_k(x)$ such that $L_k(x) = q_k(x)(x - x_k) + 1$

To start, we let *N* be the total number of nodes, and *n* be *N – 1*.

We start with $L_k(x) = \dfrac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}$

and multiply by $\dfrac{(x-x_k)}{(x-x_k)}$

to get

$$L_k(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x-x_k)} \frac{(x-x_k)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}$$

We can write the products as expanded polynomials.

$$\prod_{j \in X}(x_k - x_j) = x_k^n + a_{n-1}x_k^{n-1} + \ldots + a_1 x_k + a_0 = x_k^n + \sum_{i=0}^{n-1} a_i x_k^i \qquad (1)$$

where *X* is the set of nodes without x_k .

To reduce the left fraction, we use polynomial long division.

$$x^{n-1} + b_1 x^{n-2} + b_2 x^{n-3} + b_3 x^{n-4} + b_4 x^{n-5} + \ldots + b_{n-2} x + b_{n-1}$$

$$x - x_k \;\big|\; x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + a_{n-4}x^{n-4} + \ldots + a_1 x + a_0$$

$$\underline{-x^n + x_k x^{n-1}}$$

$$b_1 x^{n-1}$$

$$\underline{-b_1 x^{n-1} + x_k b_1 x^{n-2}}$$

$$b_2 x^{n-2}$$

$$\underline{-b_2 x^{n-2} + x_k b_2 x^{n-3}}$$

$$b_3 x^{n-3}$$

$$\underline{-b_3 x^{n-3} + x_k b_3 x^{n-4}}$$

$$b_4 x$$

$$\ldots$$

$$b_{n-1} x$$

$$\underline{-b_{n-1} x + x_k b_{n-1}}$$

$$b_n$$

Now let's write down each b_j .

$$b_1 = a_{n-1} + x_k$$
$$b_2 = a_{n-2} + b_1 x_k$$
$$b_3 = a_{n-3} + b_2 x_k$$
$$b_4 = a_{n-4} + b_3 x_k$$

We can see that, for the j'th level of the division, b_j will be the sum of a_{n-j} and $b_{j-1} x_k$. This is apparent by looking at each step of the division process. Thus, we can write a recurrence relation:

$$b_j = a_{n-j} + b_{j-1} x_k, \quad b_0 = 1$$

Finding a closed form is not hard. By substitution, we get

$$b_0 = 1$$
$$b_1 = a_{n-1} + b_0 x_k = a_{n-1} + x_k$$
$$b_2 = a_{n-2} + b_1 x_k = a_{n-2} + a_{n-1} x_k + x_k^2$$
$$b_3 = a_{n-3} + b_2 x_k = a_{n-3} + a_{n-2} x_k + a_{n-1} x_k^2 + x_k^3$$

...

$b_j = x_k^j + \sum_{i=0}^{j-1} a_{n-j+i} x_k^i$, which can be proven by using induction.

When $j = n$, we get

$$b_n = x_k^n + \sum_{i=0}^{n-1} a_i x_k^i \qquad (2)$$

Gathering the results from the division, we get the following;

$$\frac{(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x-x_k)} = \sum_{j=0}^{n-1} b_j x^{n-1-j} + \frac{b_n}{(x-x_k)} \; .$$

Thus,

$L_k(x) = \left(\sum_{j=0}^{n-1} b_j x^{n-1-j} + \frac{b_n}{(x-x_k)}\right) \frac{(x-x_k)}{\prod_{j\in X}(x_k - x_j)}$. Distributing, we get

$= \frac{(x-x_k)}{\prod_{j\in X}(x_k-x_j)} \sum_{j=0}^{n-1} b_j x^{n-1-j} + \frac{b_n}{\prod_{j\in X}(x_k-x_j)}$. From equations 1 and 2, we get

$$= \frac{(x-x_k)}{\prod_{j\in X}(x_k-x_j)} \sum_{j=0}^{n-1} b_j x^{n-1-j} + \frac{x_k^n + \sum_{i=0}^{n-1} a_i x_k^i}{x_k^n + \sum_{i=0}^{n-1} a_i x_k^i}$$

$$= \frac{(x-x_k)}{\prod_{j\in X}(x_k-x_j)} \sum_{j=0}^{n-1} b_j x^{n-1-j} + 1$$

Thus, $q_k(x) = \frac{\sum_{j=0}^{n-1} b_j x^{n-1-j}}{\prod_{j\in X}(x_k-x_j)}$, where $b_j = x_k^j + \sum_{i=0}^{j-1} a_{n-j+i} x_k^i$ and X is the set of nodes without x_k .

Therefore, $L_k(x) = q_k(x)(x-x_k)+1$, which completes the proof.

$q_k(x)$ is a polynomial of degree *n-1*, which, since *n = N – 1*, means that it is of degree *N – 2*.

Since $L_k(x)=0$ if *x* is a node not equal to x_k, we can immediately note $q_k(x_j)=\frac{-1}{x_j-x_k}, j \neq k$

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