**Problem 4**

To find all solutions to the following nonlinear system of equations:

$$2x^2 - y^2 = 0$$
$$2xy^2 - x^2 = 1\text{'}$$

we note that any solution will be the intersection points of $y = \pm x\sqrt{2}$and $y = \pm\sqrt{\frac{1+x^2}{2x}}$. To get the intersection points, we equate the two functions:

$$\pm x\sqrt{2} = \pm\sqrt{\frac{1+x^2}{2x}}$$
$$\Rightarrow 2x^2 = \frac{1+x^2}{2x} \quad .$$
$$\Rightarrow 4x^3 - x^2 - 1 = 0$$

Any solution to the nonlinear system will have values of *x* that are roots to the cubic polynomial $4x^3 - x^2 - 1$. Graphing this polynomial reveals only one real-valued root which is near *x* = 0.7. Using Newton's Method with *f(x)* = $4x^3 - x^2 - 1$, *f'(x)* = $12x^2 - 2x$, and an initial value of 0.5, the one and only root is 0.725270085, correct to 9 decimal places. Plugging this into either of the equations of *y* above, we get $y = 1.025686790$ and $y = -1.025686790$. These are the only intersection points, and thus, the only solutions to the nonlinear system.

Alternatively, we could have proceeded by writing *y* in terms of *x*, as follows:

$$y^2 = 2x^2$$
$$\Rightarrow 2x2x^2 - x^2 = 1 \cdot$$
$$\Rightarrow 4x^3 - x^2 - 1 = 0$$

This cubic, as shown above, has *x* = 0.725270085 as its only solution. Plugging this into either equation in the system yields $y = \pm 1.025686790$, exactly as before.

Yet another method for finding a solution to the nonlinear system is to use the two-dimensional version of Newton's Method with

$$\vec{f}(x,y) = \begin{bmatrix} 2x^2 - y^2 \\ 2xy^2 - x^2 - 1 \end{bmatrix}$$

and

$$J(x,y) = \begin{bmatrix} 4x & -2y \\ 2y^2 - 2\text{x} & 4xy \end{bmatrix}$$

Starting with an initial guess of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we get the following sequence of approximations.

$$\vec{f}(1,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\vec{f}(0.75,\ 1) = \begin{bmatrix} 0.125 \\ -0.0625 \end{bmatrix},$$
$$\vec{f}(0.725, 1.025) = \begin{bmatrix} 6.25e-4 \\ .00221875 \end{bmatrix},$$

$$\vec{f}(0.7252702703,\ 1.025687212) = \begin{bmatrix} -3.261677e-7 \\ 1.3734906e-6 \end{bmatrix},$$
$$\vec{f}(0.7252700851, 1.025686791) = \begin{bmatrix} 0.0000000000 \\ 0.0000000000 \end{bmatrix}.$$

Thus, the solution to the system when using the two-dimensional Newton's Method with an initial guess of $x = 1$ and $y = 1$ is $x = 0.725270085$ and $y = 1.025686791$.