Problem 4

Since we are given a continuous function and are tasked with finding the least squares trigonometric function, we can use the same linear system as in problem 3 to find the coefficients, where

$$\phi_0(x) = \frac{1}{2}$$
$$\phi_j(x) = \cos(j\pi x), j > 0$$

Next, we show that these functions form a set that is orthogonal on *[-1, 1]*.

$$\int_{-1}^{1} \cos^2(k\pi x) dx$$
$$= \int_{-1}^{1} \frac{1}{2}(1 + \cos(2k\pi x)) dx$$
$$= \frac{1}{2}\left(x + \frac{\sin(2k\pi x)}{2k\pi}\right) \Big|_{x=-1}^{x=1}$$
$$= 1, \quad k = j$$

$$\int_{-1}^{1} \cos(k\pi x)\cos(j\pi x)\, dx$$

$$= \int_{-1}^{1} \frac{1}{2}(\cos((k+j)\pi x) + \cos((k-j)\pi x)) dx$$

$$= \frac{1}{2}\left(\frac{\sin((k+j)\pi x)}{(k+j)\pi} + \frac{\sin((k-j)\pi x)}{(k-j)\pi}\right) \Big|_{x=-1}^{x=1}$$

$$= 0, k \neq j$$

$$\int_{-1}^{1} 1 dx = 2, k = j = 0$$

Since $\{\phi_j\}$ is an orthogonal set on *[-1, 1]*, we can find the coefficients simply by solving

$$a_k = \frac{\langle f(x), \phi_k(x)\rangle}{\langle \phi_k(x), \phi_k(x)\rangle}$$

the results of which are shown in table 1.

k	k'th Degree Coefficient
0	0.5
1	0.31828282031
2	0.00000000353
3	-0.07551094820
4	0.00003409977
5	0.04268419894

Table 1. The coefficients of the k'th degree least squares approximation for the given data.

As can be seen by the graphs, the approximating functions are periodic with a period of 2. If *f*(*x*) is itself periodic, then these approximating functions would be a good choice to approximate it.

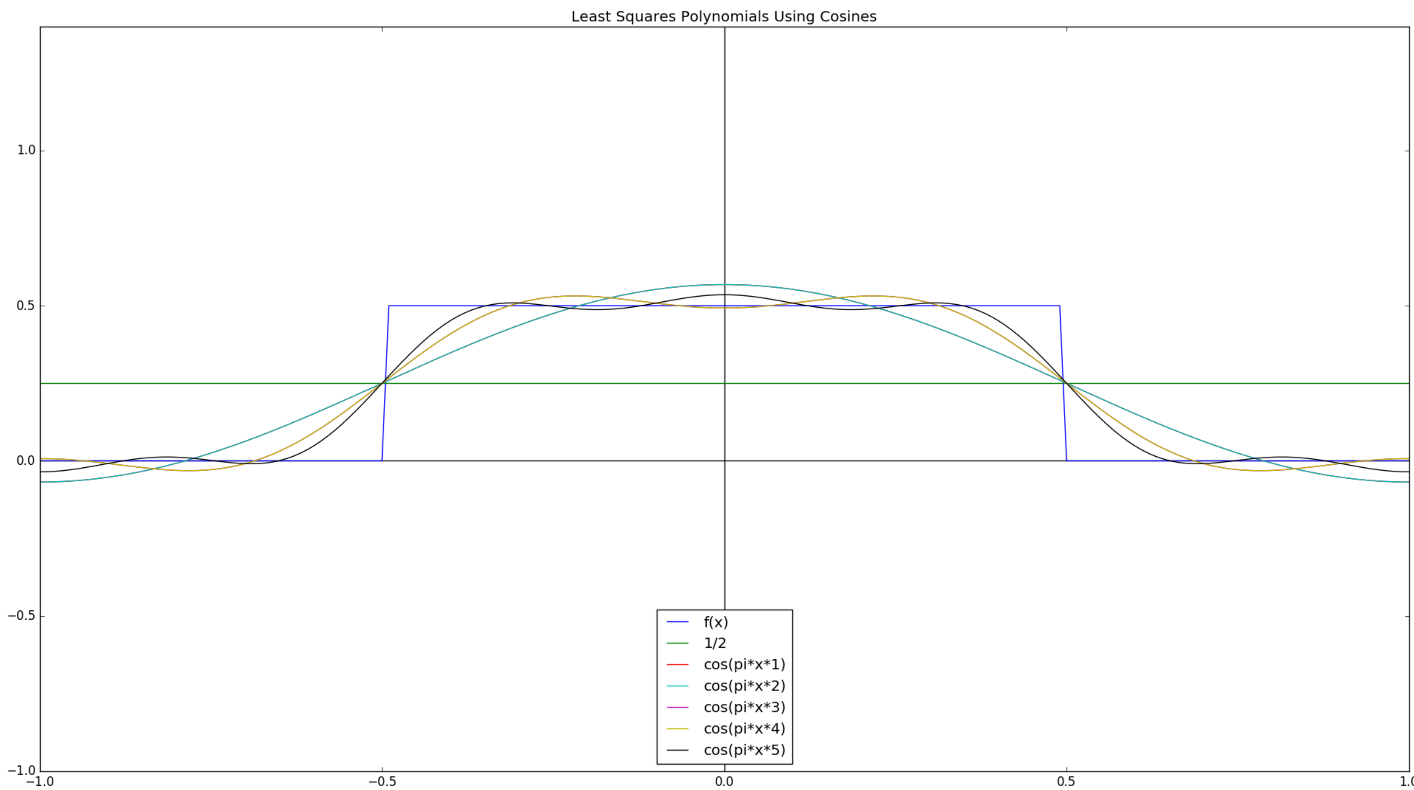

Figure 1. Least squares trigonometric approximations to *f*(*x*) on [-1, 1]. Where the legend reads $\cos(k\pi x)$, it means that the associated curve is defined by $\frac{1}{2}a_0 + a_1\cos(1\pi x) + a_2\cos(2\pi x) + \cdots + a_k\cos(k\pi x)$.

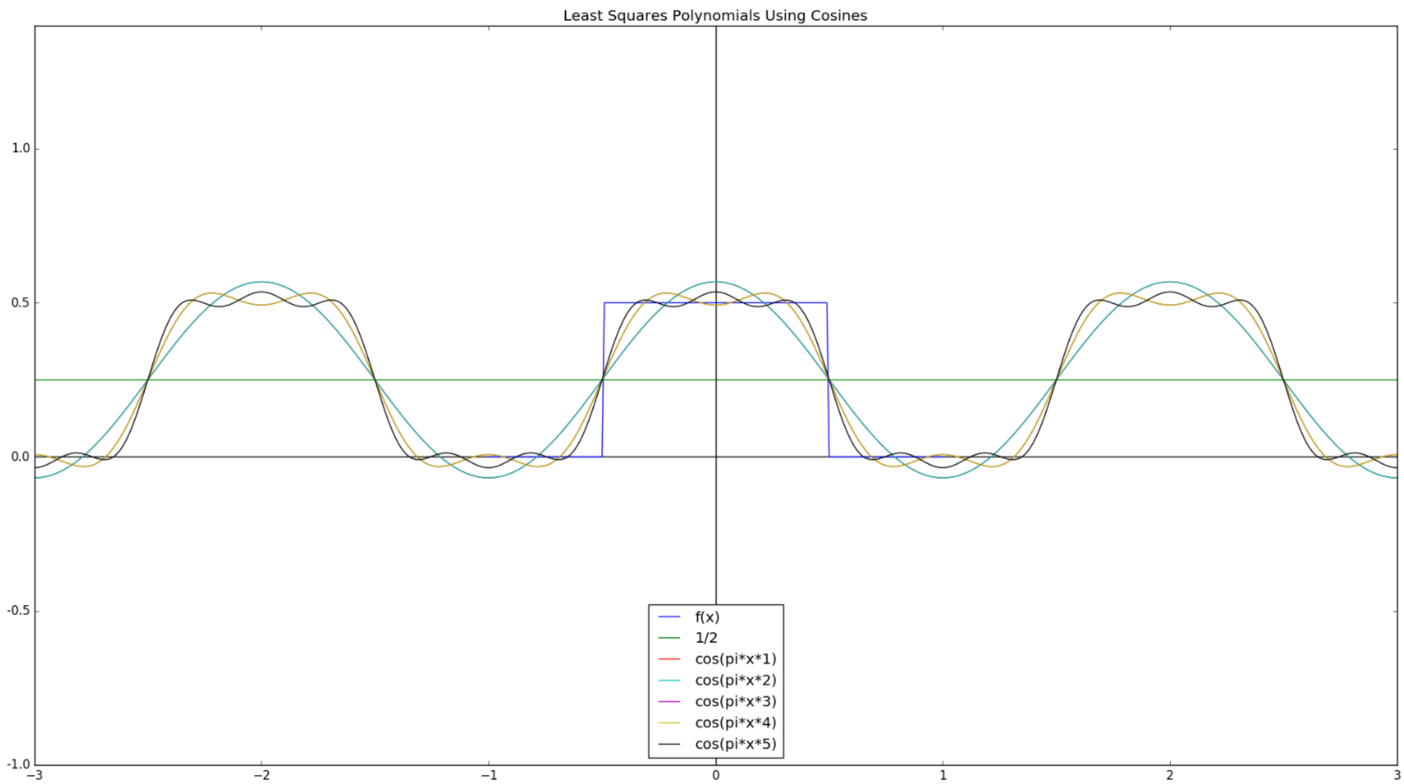

Figure 2. Least squares trigonometric approximations to $f(x)$ outside of [-1, 1].