1. For the function $g(x)=e^{\frac{-x^2}{2}}$ use the centered difference formula with h = $10^{-1}, 10^{-2}$ $\cdots 10^{-20}$ to create a table of estimates of $g^{'}(1.4)$. Include a column for the relative error of each estimate.

Using the central difference formula, we were able to create the table of estimates of $g^{'}(1.4)$ below. We can see that an error occurs when h is less than or equal to 10^{-16} because it is causing the central difference formula to divide by a very small number. We can also see that the relative error reaches its lowest point around the fifth iteration when h is 10^{-5} but then the relative error begins to increase again after that.

$g^{'}(1.4)$ = -0.52543553839195933364

h-value	Estimate	Relative error
10^{-1}	-0.52452445426194760358	0.00173395985509470487
10^{-2}	-0.52542643080942530442	0.00001733339652262963
10^{-3}	-0.52543544731642466417	0.00000017333341202652
10^{-4}	-0.52543553748118920765	0.00000000173336224796
10^{-5}	-0.52543553838713119575	0.00000000000918883010
10^{-6}	-0.52543553832884448695	0.00000000012011910514
10^{-7}	-0.52543553841211121380	0.00000000003835271635
10^{-8}	-0.52543553397121911530	0.00000000841347776335
10^{-9}	-0.52543558393125522343	0.00000008666961513331
10^{-10}	-0.52543552842010399218	0.00000001897826586298
10^{-11}	-0.52543525086434783589	0.00000054721767084446
10^{-12}	-0.52546855755508659058	0.00006284151092693245
10^{-13}	-0.52513549064769904362	0.00057104577505083661
10^{-14}	-0.52458037913538646535	0.00162752458501378501
10^{-15}	-0.58286708792820718372	0.10930275046109579062
10^{-16}	0.0	1.0
10^{-17}	0.0	1.0
10^{-18}	0.0	1.0
10^{-19}	0.0	1.0
10^{20}	0.0	1.0

Table 1. Approximations to g'(1.4) using the Central Difference formula with several step sizes. There is a certain step size that yields the best approximation. Smaller step sizes cause the approximations to diverge. Relative errors between the exact and approximated values increase past a vertain step size due to errors in finite-precision arithmetic.

We expect that the approximations will get better as h approaches 0, as this means that the approximated slope is closer to that of the line tangent to *g(1.4)*. However, due to the limitations in precision when using finite precision floating point arithmetic, we know that there must be a value of h for which approximations are no longer accurate.

The approximations become better as h decreases from its initial value of *10e-1* to *10e-5* (table 1). The relative errors become smaller and smaller. However, any lower of a step size, and the relative errors actually begin to increase. In addition to dividing by smaller numbers, we associate this behavior to the subtraction of nearly equal numbers in the central difference formula. As h decreases, it becomes so small that both g*(x + h)* and *g(x – h)* return the same value. This happens when *h = 10e-16*. The difference between the two evaluations is 0. This is due to the way floating point numbers are stored. Some large numbers have the same floating point representation. The same applies for small numbers. In evaluating $g(x + h)$ and $g(x - h)$, h is small enough so that $x + h$ is equivalent to $x - h$. This is exactly what can be seen when $h = 10e\text{-}17$. In the computer, $x + 10e\text{-}17 = x - 10e\text{-}17$. We simply cannot make an approximation that requires arithmetic with very small numbers; the hardware is not good enough.

This relates to the increase in relative error that we see because with each evaluation of the central difference formula, $x – h$ and $x + h$ become less precise, thus reducing the accuracy of subsequent approximations.