Problem 4: Gaussian Quadrature on the interval *[0, 4]*.

To get the Gauss points for $n = 3$, we need the roots of the third degree polynomial derived by using Graham-Schmidt orthogonalization on the standard basis for the polynomials. The new basis polynomials need to yield *1* when evaluated at $x = 1$. The basis that we start with is $(1, x, x^{2,} x^3)$, which we will convert to the orthogonal basis $(P_0(x), P_1(x), P_2(x), P_3(x))$. The inner product we use is

$$\langle f(x),\ g(x)\rangle \ = \ \int_0^4 f(x)g(x)dx \quad .$$

Each vector $P_k(x)$ in the orthogonal basis will have the following form:

$$P_k(x) \ = \ \alpha(x^k - \sum_{j=0}^{k-1} \frac{\langle x^k,\ P_j(x)\rangle}{\langle P_j(x),\ P_j(x)\rangle} P_j(x)) \quad ,$$

where α is a constant that makes $P_k(1) \ = \ 1$.

We begin by finding the coefficients of each $P_k(x)$. The coefficient for $P_1(x)$ is:

$$a_0 \ = \ \frac{\langle x,\ P_0(x)\rangle}{\langle P_0(x),\ P_0(x)\rangle} \quad (1)$$

The coefficients for $P_2(x)$ are:

$$b_0 \ = \ \frac{\langle x^2,\ P_0(x)\rangle}{\langle P_0(x),\ P_0(x)\rangle} \quad (2)$$

$$b_1 \ = \ \frac{\langle x^2,\ P_1(x)\rangle}{\langle P_1(x),\ P_1(x)\rangle} \quad (3)$$

The coefficients for $P_3(x)$ are:

$$c_0 \ = \ \frac{\langle x^3,\ P_0(x)\rangle}{\langle P_0(x),\ P_0(x)\rangle} \quad (4)$$

$$c_1 \ = \ \frac{\langle x^3,\ P_1(x)\rangle}{\langle P_1(x),\ P_1(x)\rangle} \quad (5)$$

$$c_2 \ = \ \frac{\langle x^3,\ P_2(x)\rangle}{\langle P_2(x),\ P_2(x)\rangle} \quad (6)$$

In general, the coefficients for $P_n(x)$ are:

$$\omega_0 \ = \ \frac{\langle x^n,\ P_0(x)\rangle}{\langle P_0(x),\ P_0(x)\rangle}$$

...

$$\omega_j = \frac{\langle x^n,\ P_j(x)\rangle}{\langle P_j(x),\ P_j(x)\rangle}$$
$$\cdots$$
$$\omega_0 = \frac{\langle x^n,\ P_{n-1}(x)\rangle}{\langle P_{n-1}(x),\ P_{n-1}(x)\rangle} \ ,$$

therefore,

$$P_n(x) = \alpha(x^n - \omega_{n-1}P_{n-1}(x) - \ldots - \omega_0 P_0(x)) \ .$$

Now we will find the α associated with each $P_k(x)$. For $P_1(x)$,

$$\begin{aligned} P_1(x) &= \alpha(x-a_0) \\ \Rightarrow P_1(1) &= 1 = \alpha(1-a_0) \\ \Rightarrow \quad \alpha &= \frac{1}{1-a_0} \end{aligned}$$

For $P_2(x)$,

$$\begin{aligned} P_2(x) &= \alpha(x^2-b_1P_1(x)-b_0) \\ \Rightarrow P_2(1) &= 1 = \alpha(1-b_1-b_0) \\ \Rightarrow \quad \alpha &= \frac{1}{1-b_1-b_0} \end{aligned}$$

For $P_3(x)$,

$$\begin{aligned} P_3(x) &= \alpha(x^3-c_2P_2(x)-c_1P_1(x)-c_0) \\ \Rightarrow P_3(1) &= 1 = \alpha(1-c_2-c_1-c_0) \\ \Rightarrow \quad \alpha &= \frac{1}{1-c_2-c_1-c_0} \end{aligned}$$

Therefore, in general, $P_n(x) = \dfrac{x^n - \omega_{n-1}P_{n-1}(x) - \ldots - \omega_0 P_0(x)}{1-\omega_{n-1} - \ldots - \omega_0}$.

So we just need to find coefficients. We start with $P_0(x) = 1$. Plugging into equation (1), we get

$$a_0 = \frac{\int_0^4 x dx}{\int_0^4 1^2 dx} = 2 \ , \ 1-a_0=-1 \ ,$$

therefore, $P_1(x) = -x+2$.

Plugging into equation (2) and (3) results in,

$$b_0 = \frac{\int_0^4 x^2 dx}{\int_0^4 1^2 dx} = \frac{16}{3}$$

$$b_1 = \frac{\int_0^4 x^2(-x+2) dx}{\int_0^4 (-x+2)^2 dx} = -4 \quad , \quad 1-b_1-b_0=\frac{-1}{3} \quad ,$$

therefore, $P_2(x)=-3(x^2+4(-x+2)-\frac{16}{3})$.

Finally, plugging into equations (4), (5), and (6) gives us

$$c_0 = \frac{\int_0^4 x^3 dx}{\int_0^4 1^2 dx} = 16$$

$$c_1 = \frac{\int_0^4 x^3(-x+2) dx}{\int_0^4 (-x+2)^2 dx} = \frac{-72}{5}$$

$$c_2 = \frac{\int_0^4 x^3(-3)(x^2+4(-x+2)-\frac{16}{3}) dx}{\int_0^4 (-3)(x^2+4(-x+2)-\frac{16}{3})^2 dx}$$

$$= \frac{-3\int_0^4 (x^5-4x^4+\frac{8}{3}x^3) dx}{9\int_0^4 (x^4-8x^3+\frac{64}{3}x^2-\frac{64}{3}x+\frac{64}{9}) dx} \quad , \quad 1-c_2-c_1-c_0=\frac{7}{5} \quad ,$$

$$= -2$$

therefore, $P_3(x)=\frac{5}{7}(x^3-6x^2+\frac{48}{5}x-\frac{16}{5})$.

We will rewrite $P_3(x)$ as

$$P_3(x)=\frac{1}{7}(5x^3-30x^2+48x-16) \quad (7)$$

so that the Rational Zeros Theorem can be applied to it. If $P_3(x)$ has rational roots, then they will be of the form $\frac{\textit{factor of } 16}{\textit{factor of } 5}$. We can ignore values that are over 4 or under 0, since the roots should be in *[0, 4]*. This leaves *1/1, 1/5, 2/1, 2/5, 4/1, 4/5, 8/5,* and *16/5*. Plugging each into $P_3(x)$, we find that the only rational root is *2*. Thus,

$$P_3(x) = (x-2)(a_2x^2+a_1x+a_0) = a_2x^3+(a_1-2a_2)x^2+(a_0-2a_1)x-2a_0 \quad (8).$$

Equating (7) and (8) gives

$$\begin{matrix} a_0=8 \\ a_1=-20 \\ a_2=5 \end{matrix} \quad ,$$

therefore,

$$P_3(x) = (x-2)(5x^2-20x+8) \quad .$$

The roots of the quadratic factor are

$$\frac{20\pm\sqrt{(-20)^2-4(5)(8)}}{2(5)} = 2\pm\frac{2}{5}\sqrt{15} \quad .$$

Thus, the roots x_1, x_2, and x_3 of $P_3(x)$ are

$$x_1 = 2-\frac{2}{5}\sqrt{15}, \quad x_2 = 2, \quad x_3 = 2+\frac{2}{5}\sqrt{15} \quad .$$

The roots are between *0* and *4*, which also makes them the Gauss points for *n = 3* that we are searching for.

Now we can get the Gauss weights w_1, w_2, and w_3 by integrating the k'th Lagrange polynomial using the Gauss points as the nodes.

$$\begin{aligned} w_k &= \int_0^4 L_k(x)\,dx \\ &= c_k\int_0^4 (x-x_i)(x-x_j)\,dx \\ &= c_k\left(\frac{4^3}{3} - \frac{(x_i+x_j)}{2}4^2 + x_ix_j4\right) \end{aligned}$$

$$\text{where} \quad \begin{matrix} c_k=\dfrac{1}{(x_k-x_i)(x_k-x_j)}, \\ 1\le k\le 3, \\ j\ne i\ne k \end{matrix} \quad ,$$

so that

$$w_1 = \frac{10}{9},$$
$$w_2 = \frac{16}{9}, \quad .$$
$$w_3 = \frac{10}{9}$$

Thus, $\int_0^4 f(x)\,dx \approx \sum_{k=1}^{3} w_k f(x_k)$. $\begin{array}{c} x_1 = 2-\frac{2}{5}\sqrt{15}, \;\; x_2 = 2, \;\; x_3 = 2+\frac{2}{5}\sqrt{15} \\ w_1 = \frac{10}{9}, \;\; w_2 = \frac{16}{9}, \;\; w_3 = \frac{10}{9} \end{array}$.