Problem 4: Analyzing $L_6(x)$

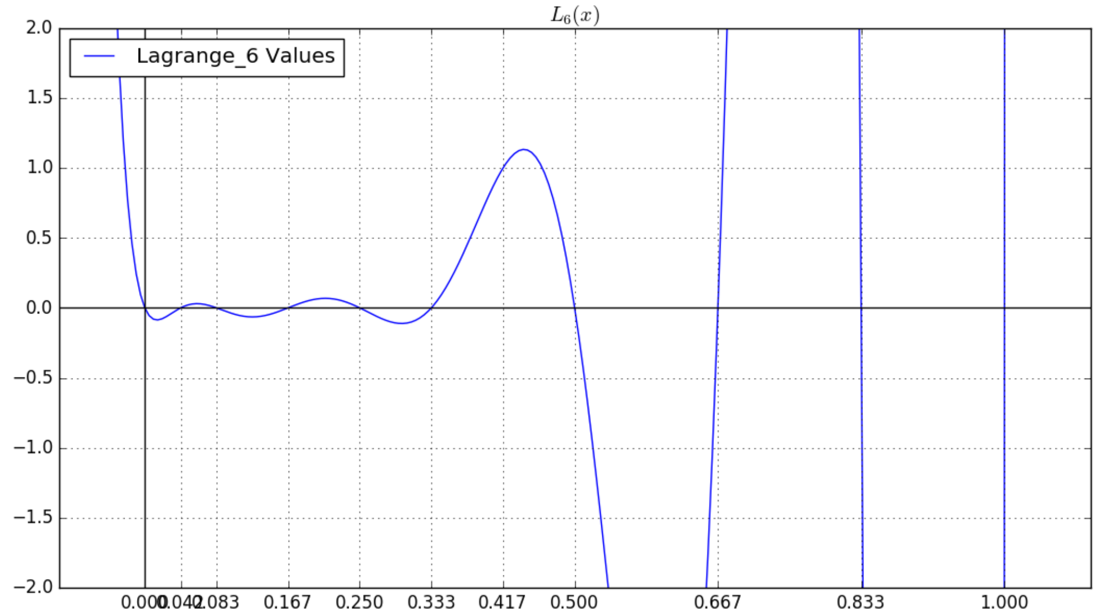

Figure 1. A plot of $L_6(x)$ that uses the nodes from problem 3. The left half is very well behaved and close to what we might intuitively picture the curve as. The right half is extremely erratic. This behavior is attributed to the spacing of the nodes. The nodes on the left half are spaced closer together, while the nodes on the right half are spaced farther apart.

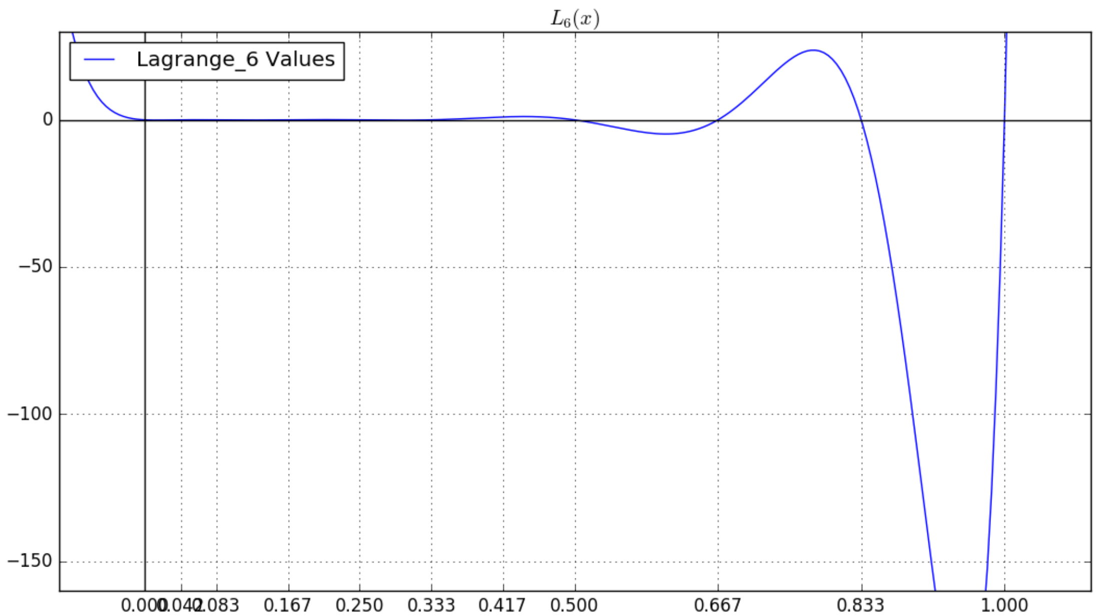

Figure 2. The same curve from figure 1 zoomed out. This shows the magnitudes of the peaks and emphasizes the scale of the deviation from zero of the right half.

Figure 1 shows $L_6(x)$ for the nodes in problem 3. With the knowledge gained from that problem, this graph is not unexpected. The interval that contains closer-spaced nodes is close to 0. The curve varies wildly for the intervals that contain nodes that are more spread apart. These are the nodes that are greater than 0.5. We might be able to correlate behavior between the graphs of $L_6(x)$ and the interpolant. We know from problem 3 that successive nodes that are farther apart introduce large errors. The graph of $L_6(x)$ shows explosive behavior for these nodes. However, for the nodes that are close, the curve is like a calm ripple. Larger spaced nodes seem to make the graph look like monsterous waves. Combining these two observations, we may be able to conclude that the more an interval deviates from 0, the greater the error in the approximation for values of x in that interval.

Figure 2 shows a zoomed out version of the graph in figure 1. We can appreciate the magnitudes of the peaks of the curve when $x > 0.5$. Observing that larger local extrema may produce larger approximation errors is clearer from this point of view.

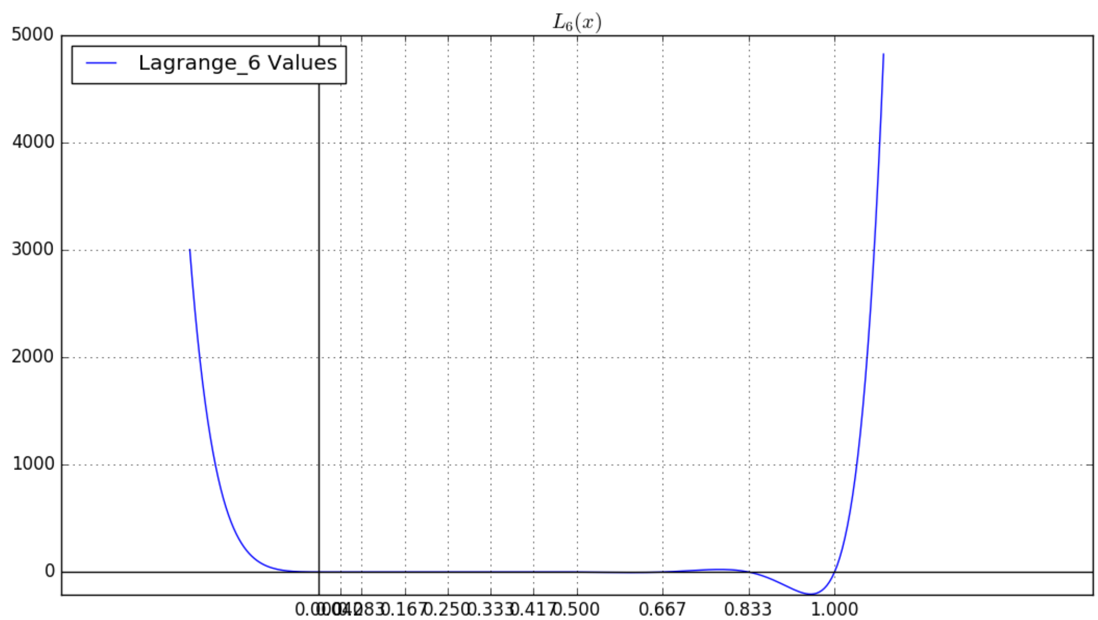

Figure 3. The same curve from figure 1 zoomed out to show the behavior of the curve when x is beyond the interval bounded by the nodes. Both sides approach infinity.

Figure 3 shows what happens when we try to evaluate a value of x that is outside the interval bounded by the first and last nodes. The curve explodes and increases upwards to infinity. Saying that approximations for x in this range would be unreliable is a tremendous understatement.

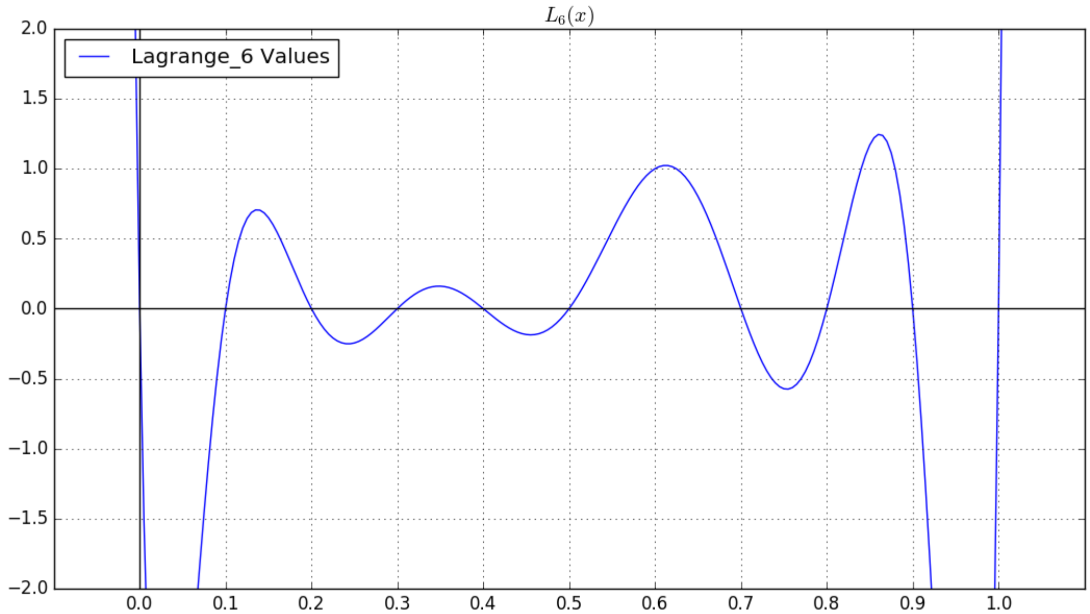

Figure 4. A plot of $L_6(x)$ that uses nodes from *0.0* to *1.0*, at intervals of *0.1*. The center region of the curve is what we expect to see: close to zero except at x_6. The curve noticably deviates from zero near the edges. It also deviates from zero on the left edge, a region where the curve in figure 1 stayed close to zero.

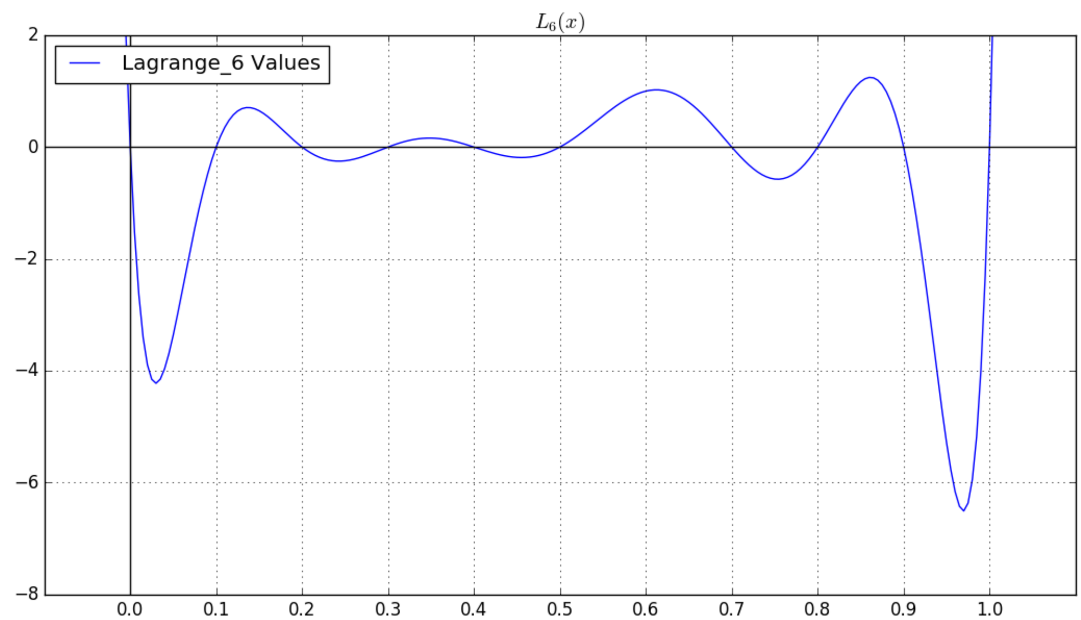

Figure 5. The same curve as in figure 4 zoomed out. The behavior at the edges, although not as bad as in figure 1, is significant.

Figure 4 shows $L_6(x)$ using nodes that are equally spaced by *0.1* units, from *0.0* to *1.0*. There are several noticable differences from the curve in figure 1. First is that the wild variations when $x > 0.5$ are gone. This is presumably due to using nodes that are closer to eachother. However, a surprising difference is the behavior when $x < 0.2$. In this interval, the curve in figure 2 is very close to zero, but in figure 5, the curve deviates considerably from it. This is strange because the nodes are all separated by *0.1*, so we would expect to see uniform behavior throughout the curve. One reason for the difference might be that the nodes used for the curve in figure 1 were separated by an even smaller distance than *0.1*. Moreover, the first and second nodes had the smallest magnitude of separation. These two factors might be what cause figure 1's curve to be so well-bahaved in this interval.

We see this deviation from zero again in figure 4 when $x > 0.8$. Although the deviation is much more tame than in figure 1, it still exists.

These irregularites in the edges highlight the behavior near the center: the interval $[0.2, 0.8]$ is close to zero (except at $x_6 = 0.6$ which is where it should increase to *1)*. A hypothesis for this change in behavior again features the spacing of the nodes. Perhaps separating all nodes throughout the entire interval by the same distance is not such a good idea after all. Although this strategy seems to work fine for the nodes in the center, it seems to fail for the nodes on the edge of the interval. More care might be needed in choosing the distance of separation for these nodes. The curve in figure 1 placed its first three nodes *1/24* units apart from eachother, and its next three nodes *1/12* units. If we employ a similar tactic to the nodes for this curve, that is, gradually increasing the spacing of the nodes as they approach the center, we get the curve in figure 6. Notice that the large oscillations at the edges are now gone. This result supports the hypothesis that equally spacing the nodes throughout the entire interval is not a good strategy for getting optimal approximations with Lagrange interpolation.

A great deal of insight to the behavior of Lagrange polynomials has been gained from these results. Mainly, that the placement of the nodes is extremely important. Although we cannot say with certainty yet, a good strategy for choosing nodes seems to be equally spacing them near the center, and gradually decreasing the spacing near the edges.

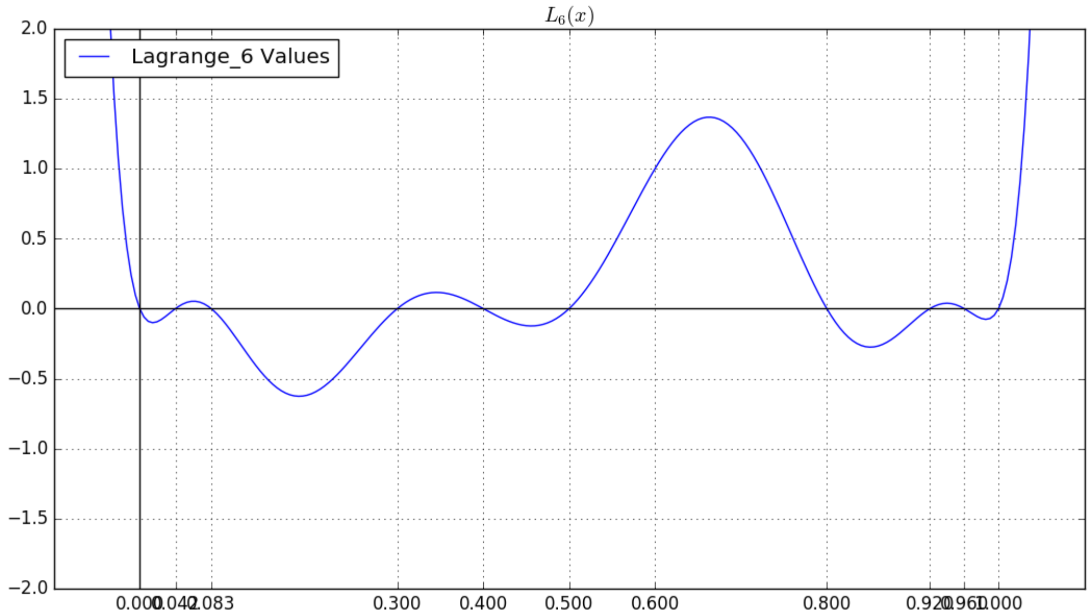

Figure 6. A plot of $L_6(x)$ that uses a combination of the nodes in the curves of figures 1 and 4: *0, 1/24, 2/24, 3/10, 4/10, 5/10, 6/10, 8/10, 9,2/10, 9.6/10, 1*. The edges no longer oscillate at high magnitudes. The curve is close to zero throughout the entire interval, except near x_6 where it should reach *1*.