Problem 2

Since we are given a discrete set of data points and are tasked with finding the least squares exponential, we use the nonlinear system (letting *a* be positive to simplify some calculations)

$$f_1(x) = \frac{\partial E}{\partial a} = 2\sum_{k=0}^{5}(y_k - be^{ax_k})(-bx_k e^{ax_k}) = 0$$

$$f_2(x) = \frac{\partial E}{\partial b} = 2\sum_{k=0}^{5}(y_k - be^{ax_k})(-e^{ax_k}) = 0$$

,where $\mathrm{E}(a, b) = \sum_{k=0}^{5}(y_k - be^{ax_k})$, to find the coefficients a and b. Both equations are set to zero to find their minimum. To solve this, we use Newton's Method, which requires the following equations:

$$\frac{\partial f_1}{\partial a} = 2\sum_{0}^{5}(b^2 x_k^2 e^{2ax_k} - (y_k - be^{ax_k})bx_k^2 e^{ax_k})$$

$$\frac{\partial f_1}{\partial b} = 2\sum_{0}^{5}(bx_k e^{2ax_k} - (y_k - be^{ax_k})x_k e^{ax_k})$$

$$\frac{\partial f_2}{\partial a} = \frac{\partial f_1}{\partial b}, by\ Clairaut's\ Theorem$$

$$\frac{\partial f_2}{\partial b} = 2\sum_{0}^{5} e^{2ax_k}$$

Then we apply Newton's method to solve:

$$\begin{bmatrix} \frac{\partial f_1(a,b)}{\partial a} & \frac{\partial f_1(a,b)}{\partial b} & -f_1(a,b) \\ \frac{\partial f_2(a,b)}{\partial a} & \frac{\partial f_2(a,b)}{\partial b} & -f_2(a,b) \end{bmatrix}$$

until necessary. We choose the initial guess to be *a = 0.72005416, b = 2.1450464*. We got these values by solving the system

$$1.0442 = be^{-a0.9998}$$
$$0.4660 = be^{-a2.1203}$$

by first substituting for *b*. Table 1 shows the values of *a* and *b* when applying Newton's Method.

a	b	$f_1(a, b)$	$f_2(a, b)$
-0.698308785055	1.9524134382	0.00426806347043	-0.00491427954298
-0.699939809444	1.95535180212	1.55988397904e-05	2.83425536939e-06
-0.699942841078	1.95535266987	1.44420163328e-10	3.36833824721e-11
-0.699942841105	1.95535266987	-3.05311331772e-16	-5.13478148889e-16
-0.699942841105	1.95535266987	6.24500451352e-17	7.329206686e-17

Table 1. Successive iterations of Newton's Method.

After five iterations, we get $a = -0.69994284110$, $b = 1.95535266987$. accurate to the decimal places shown. Thus, the least squares exponential is:

$$1.95535266987e^{-0.69994284110x}.$$

The sum of the squared errors is 0.01376077449.