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# VARIABLE STEP SIZE KALMAN FILTER USING EVENT HANDLING ALGORITHM FOR SWITCHING SYSTEMS

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## ABSTRACT

*This paper presents a novel variable step size Kalman Filter by augmenting the event handling procedure of Ordinary Differential Equation (ODE) solvers with the predictor-corrector scheme of well-known discrete Kalman Filter (KF). The main goal is to increase the estimation performance of Kalman Filter in the case of switching/stiff systems. Unlike fixed step size Kalman Filter the sample time (ST) is adapted in the proposed approach based on current estimation performance (KF innovation) of system states and can change during the estimation procedure. The proposed event handling algorithm consists of two main parts: relaxing ST and restricting ST. Relaxing procedure is used to avoid high computational time when no rapid change exists in system dynamics. Restricting procedure is considered to improve the estimation performance by decreasing the Kalman filter step size in the case of fast dynamical behavior (switching behavior). The accuracy and computational time are controlled by using design parameters. The effectiveness of the proposed approach is verified by simulation results using the bouncing ball example as a switching system.*

## INTRODUCTION

Ordinary Differential Equation (ODE) contains derivative(s) of a dependent variable (usually denoting as $y$ and corresponding derivative(s) as $\dot{y}$, $\ddot{y}$, ...) with respect to a single independent variable (usually referring to time $t$). Initial Value Problems (IVP) are based on iteratively solving an ODE by assuming an initial condition $y(t_0) = y_0$ and a period of time $t = [t_0,\ t_f]$. At each step the solver tries to use the results of previous step considering a particular algorithm (Euler's method, RungeKutta methods, etc.). The final result is given as a vector of time steps $t = [t_0,\ t_1,\ ...,\ t_f]$ and corresponding sequence of values for the dependent variable $y = [y_0,\ y_1,\ ...,\ y_f]$. Higher-order differential equations can be reformulated at each step of iterative solving procedure of IVP as a system of first-order equations

$$\begin{aligned} \dot{y}(t) &= f(t, y(t)),\ t_0 \leq t \leq t_f \\ y(t_0) &= y_0, \end{aligned} \tag{1}$$

with step size

$$h_n = t_{n+1} - t_n, \tag{2}$$

and corresponding solution

$$y(t+h) = y(t) + \int_t^{t+h} f(s, y(s))ds. \tag{3}$$

Generally even if function $f$ is a continuous function there is no guarantee that the IVP provides a unique solution. However, by considering the Picard's theorem according to [1] and considering the Lipschitz condition for function $f$, it can be stated that a unique solution exists for Eqn. (1).

In [2] a linear approximation of the general ODE Eqn. (1) is introduced as

$$\dot{y}(t) = Ay + g(t), \tag{4}$$

with $A = f_y(t_0, y_0)$ as a $m \times m$ Jacobian matrix of $f$ evaluated at $(t_0, y_0)$.

Stiff system equations contain some terms that produce a fast variation in the solution. To solve the numerical integration of stiff ordinary differential equation (1) a relative small step size $h_n$ is required in the case of fast variation in the solution and correspondingly the step size has to be relatively large (relaxed) when the solution is smooth. Consequently a perfect numerical solution should be able to solve the stiff and non-stiff ODEs. Stiff numerical methods have the ability to change the step size during solving procedure. They take small steps to obtain satisfactory results nearby solutions that vary rapidly. The main advantage of stiff solvers is the low computational time compared to non-stiff solvers. The non-stiff solutions can be used for stiff problems with a proper small step size but it takes more time to achieve the final solution because the step size is constant and can not be adapted according to actual results.

In reality the system dynamics is modeled by linear or nonlinear ordinary differential equation form. For example in [3] an elastic beam is modeled as linear ODE. A simple high gain Proportional-Integral-Observer (PIO) introduced in [4] is used to estimate the node displacements and disturbance estimation of the elastic beam system. The dynamics of system and observer are augmented and described in one linear ODE form. The aforementioned example requires a suitable numerical IVP. In this case the stiff ODEs are used to avoid high computational time by relaxing the integration step size when no rapid change exists in system states and disturbance dynamics, as well as to improve the estimation results by decreasing the integration step size in the case of fast dynamical behavior of disturbance (stiff behavior). In [3] simulation results are achieved using stiff ODE solver (ode15s) considering event handling and zero crossing procedures.

The paper is organized as follows: First a general definition of stiff/switching systems is provided besides the common solution. In the second part the main aspects of (Matlab-based) ODE solvers are introduced e.g. zero crossing and event handling procedures with the purpose of integration into the structure of Kalman Filter. Subsequently, the proposed variable step size Kalman Filter is detailed considering the structure and procedure of well-known discrete Kalman Filter. Furthermore, the proposed method is evaluated using simulation results of a bouncing ball system (switching system). The last section concludes the paper with a summary and conclusions.

## STIFF/SWITCHING SYSTEMS: PROBLEM DEFINITION, AND COMMON SOLUTIONS

**Definition** [5]: *The linear system of ordinary differential equations (4) is called stiff when the eigenvalues $\lambda_k$ ($k = 1, 2, \ldots, m$) of the constant coefficient matrix $A \in R^{m \times m}$ of the system have the following properties:*
*1. Re $\lambda_k < 0$ for all $k = 1, 2, \ldots, m$. This means that the system is Lyapunov stable.*
*2. The number defined as*

$$S = \frac{max_k |Re(\lambda_k)|}{min_k |Re(\lambda_k)|}, \tag{5}$$

*(so called stiffness number) is large, i.e. $S >> 1$.*
According to [6] the system is called weakly stiff in the case of $S \approx 10$ and stiff in the case of $S > 10$. In other words, the eigenvalue produces the slowest rates of change should be compared with the eigenvalue that leads to fastest rates of change.

The solution of solving stiff ODE functions is provided in different refrences considering ordinary differential problems. Using of observer equations integrating system model ODEs and solving the problem using ODE solvers is a well-known issue. On the other hand, discrete Kalman Filter can provide the same functionality as observers to estimate the system states. It contains a predictor-corrector procedure to be solved at each discrete step size. The procedure starts from an initial set of Kalman parameters and system states. The next estimation step is done based on the current step information (same concept as IVP). The sampling time is defined based on available measurements. An iterative estimation procedure is performed considering a predictor-corrector scheme.

To extend the KF procedure for nonlinear systems extended Kalman Filter (EKF) has been introduced by linearizing the nonlinear system equations around the working point at each sampling time. Different approaches have been introduced to improve the performance of KF and EKF. For example iterated extended Kalman filter (IEKF) [7] linearizes the system nonlinear equations iteratively to compensate the significant nonlinearities. In this algorithm an iterative procedure is considered for measurements update (correction part). The iterative procedure is stopped if the maximum number of iteration is reached or the difference between two iterations results is less than a pre-specified threshold. In [8] by assuming the EKF as an optimization problem, different optimization algorithms (e.g. Line Search, Quasi-Newton, and Levenberg-Marquardt) are considered to improve the performance and robustness of estimation. In [9] a variable step-length IEKF is introduced by rewriting the measurement update step and considering a variable step length in Line Search optimization procedure. In all aforementioned approaches the sampling time of estimation results is constant defined based on sampling time of measurements.

Sometimes it is necessary to estimate the switching/stiff system states in discrete time using Kalman Filter. Generally the predictor-corrector approaches are based on a pre-specified tolerance. To solve the switching/stiff problems the algorithm should be able to look back, execute the solution with smaller step size (stiff problems), execute the solution at precisely defined moments in time (switching problems), and reach the predefined tolerance. Therefore, this contribution provides a variable step size Kalman filter. The proposed variable step size Kalman Filter is able to solve the state estimation of switching/stiff problems in discrete time to avoid high computational time and to increase estimation performance. Using of the proposed variable step size Kalman Filter provides step size controlling inside the predictor-corrector procedure. The sampling time is adapted based on current estimation performance (KF innovation) of system states.

## ZERO CROSSING AND EVENT HANDLING PROCEDURES

In this section the main ideas of zero crossing and event handling are repeated to be used later introducing the proposed algorithm. As discussed in section 1, variable step size solvers increases or decreases the step size to achieve error tolerances and required or given performance. Selection of fixed or variable step size depends on the dynamical model and implementation issues. The fixed step size solver uses a single step size for the whole simulation time and consequently the step size has to be small enough to achieve the accuracy requirements.
Implicit variable step size solvers (stiff solvers) can be used to solve stiff problems. The non-stiff solvers are ineffective on intervals where the solution changes slowly because they use time steps small enough to resolve the fastest possible change for the whole estimation time. Furthermore, the step size of non-stiff solvers has to be defined at the initialization level and can not be changed during the solving procedure.

Event handling and zero crossing procedures are usually considered in the structure of ODE solvers. Detecting the moment in time of specific events in ODEs is referred to event handling procedure. For example the exact time a ball hits the ground or the time that the ODE solutions reaches a specific value is a very specific moment in time to be defined which strongly effects the upcoming (nonlinear) system behavior. The event is detected from time step $t_n$ to $t_{n+1}$ if the conditional statement (event) becomes true. The zero crossing detection algorithm is a procedure to capture and locate events accurately. In the case of zero crossing for a predefined event, the ODE solver can be terminated or continued with different conditions according to demand. Event detection of ODE solvers contains two functions $f(t,y)$ and $g(t,y)$, and an initial condition $(t_0,y_0)$. Using event handling procedure in the structure of ODE solver means to numerically find the moment in time $t^*$ so that

$$\begin{aligned} &\dot{y}(t) = f(t,y(t)),\\ &y(t_0) = y_0,\\ &g(t^*,y(t^*)) = 0. \end{aligned} \tag{6}$$

Here $g(t,y)$ denotes the event condition which has to be defined by the programmer according to the problems requirements.

## PROPOSED APPROACH

In the following section the procedure of standard Kalman Filter and correspondingly predictor-corrector procedure is briefly explained. Afterward the algorithm of variable step size Kalman Filter is proposed and discussed in detail to enhance the performance of Kalman Filter in the case of stiff/switching problems. The proposed approach consists of event handling and zero crossing concepts according to ODE solvers with variable step size integration.

### (Standard) Kalman Filter

If the measurements contain statistical noise and other inaccuracies, Kalman Filter can be applied to estimate system states. Kalman Filter is actually a set of mathematical equations which is optimal in the sense of minimizing the estimation error covariance when some presumed conditions are met. This filter/estimator firstly proposed in [10] works in a two-step process: prediction of actual state and error covariance, correction of estimated state and error covariance.
Based on a system model a prediction is firstly performed, then the correction part is executed using a suitable designed gain known as Kalman gain. For linear well-defined system influenced by Gaussian noise (system and measurement noise) a statistically optimal solution can be reached using Kalman Filter. The algorithm is able to remove or attenuate the effects of system/measurement noise ($w_k$ and $v_k$) to reconstruct the state value $x_k$ from the real measurement $z_k$. The Kalman Filter assumes that the state at time step $k$ is calculated from the previous state step $x_{k-1}$ according to

$$\begin{aligned} &x_k = Ax_{k-1} + Bu_k + w_{k-1},\\ &z_k = Hx_k + v_k, \end{aligned} \tag{7}$$

with $A$, $B$, and $H$ matrices denoting the available model typically considered as constant. Based on (7) first of all the determination of required parameters (system noise covariance $Q$ and measurement noise covariance $R$) and initial states must be done. The system and measurement noise respectively $w_k$ and $v_k$ are assumed to be independent, white and with normal probability

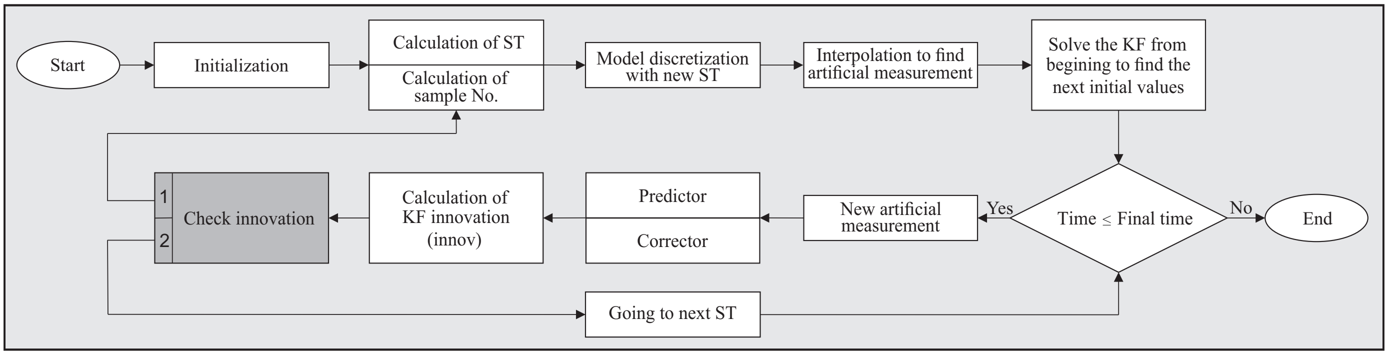


**FIGURE 1**. MAIN ALGORITHM OF VARIABLE STEP SIZE KALMAN FILTER

distributions as

$$p(w) \sim N(0,Q), \quad p(v) \sim N(0,R). \tag{8}$$

In practice, the noise covariance matrices might change during the time, however they are often assumed to be constant. There are two sets of equations for prediction and correction process. The time update projects the current state estimation ahead in time while the measurement update adjusts the projected estimation by an actual measurement at that time. The procedure of Kalman Filter can be summarized as follows:

**Prediction (time update)**

(1) Project the state ahead

$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$

(2) Project the error covariance ahead

$P_k^- = AP_{k-1}A^T + Q$

**Correction (measurement update)**

(1) Compute the Kalman gain

$K_k = P_k^- H^T (HP_k^- H^T + R)^{-1}$

(2) Update the estimation with measurement

$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$

(3) Update the error covariance

$P_k = (I - K_k H)P_k^-$

Kalman filter solution is optimal (minimizes errors in some respect) if the following conditions are satisfied: (1) the system model is precise, (2) system/measurement noises are white noise, and (3) the covariance of noise is precisely known [11]. Under these conditions, there is a unique "best" estimation $\hat{x}_k$. The process and measurement noise are assumed to be independent (of each other), white, and normal distributed.

## Variable Step Size Kalman Filter

To increase the performance of Kalman Filter in the case of switching and/or stiff systems, variable step size Kalman Filter is proposed in this contribution. It contains event handling and zero

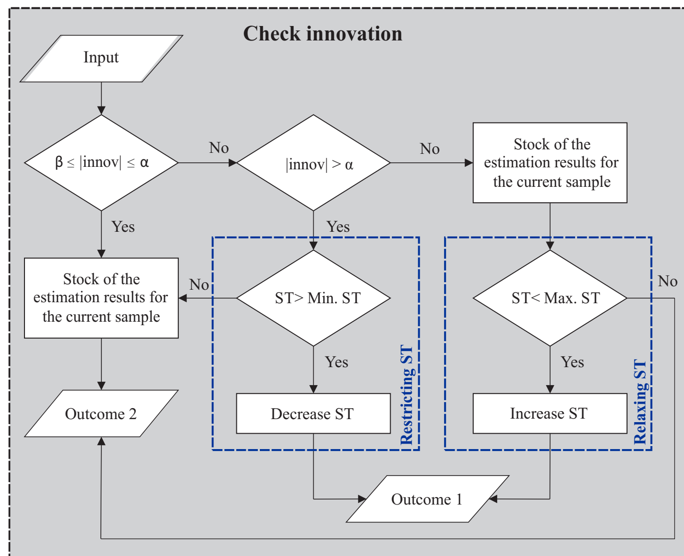


**FIGURE 2**. EVENT HANDLING PROCEDURE OF VARIABLE STEP SIZE KALMAN FILTER

crossing procedures augmented with predictor-corrector scheme. The main flowchart of variable step size Kalman Filter is illustrated in Fig. 1. The algorithm starts by initialization of Kalman Filter prerequisites e.g. initial states, covariance matrices, and initial sample time (ST). At the first step of initialization level or in the case that the event is occurred and detected during the solving procedure (ST is decreased or increased) the state space model of the system is discretized based on the new ST. Since the measurement vector is constant, interpolation is required to find the measurement vector according to new sample time. The Kalman Filter procedure is solved from the zero moment up to the current time to find the suitable initial value for the upcoming time. At each step of predictor-corrector scheme the Kalman Filter innovation is used for event handling procedure (shown in

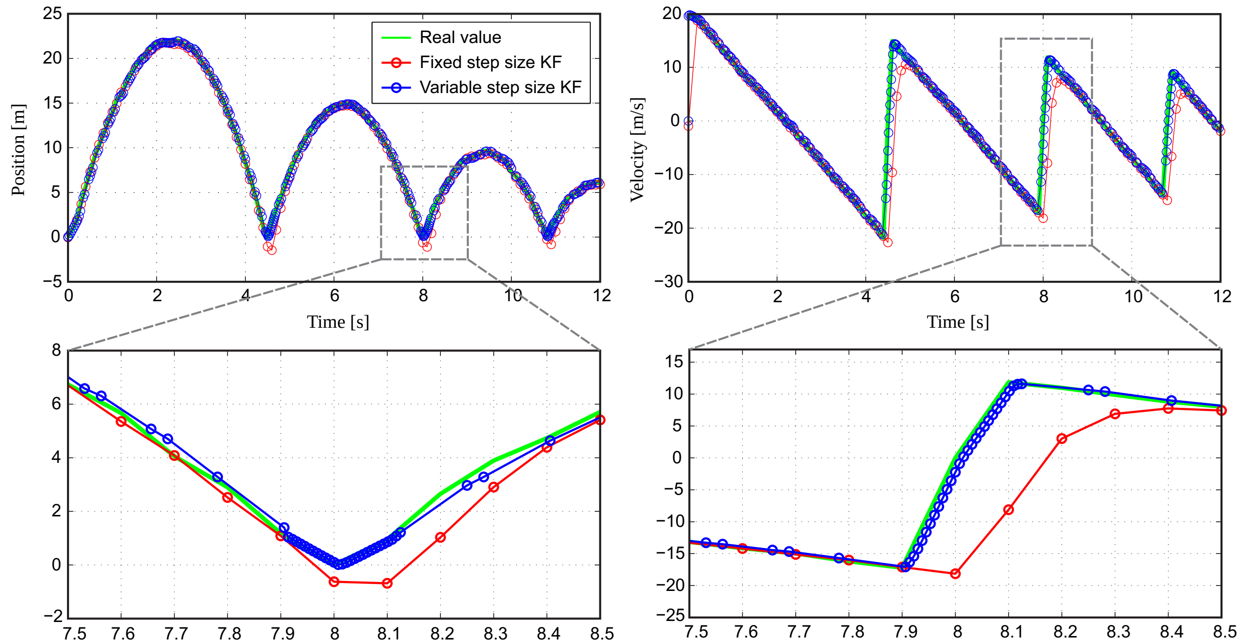


**FIGURE 3**. STATES (POSITION AND VELOCITY) ESTIMATION OF BOUNCING BALL EXAMPLE USING FIXED AND VARIABLE STEP SIZE KALMAN FILTER (KF)

detail in Fig. 2). The proposed event handling algorithm consists of two main parts: restricting ST and relaxing ST. Four parameters are considered as design parameters (Min. ST, Max. ST, $\alpha$, and $\beta$). When ST changes the predictor-corrector scheme is stopped. According to Fig. 1 the variable step size Kalman Filter procedure is continued by applying a new discretization of system model and new initialization using interpolated measurement vector.

## SIMULATION RESULTS USING BOUNCING BALL SYSTEM

To verify the proposed approach namely variable step size Kalman Filter the bouncing ball example is considered. The dynamics of system show a stiff system structure. This system is used in literature as a switching example to show the advantages of event handling and variable step size ODE solvers. Therefore, it is used here to illustrate the advantages of proposed variable step size Kalman filter including event handling procedure. The dynamics of the bouncing ball describe the motion of ball when it falls, hits the ground, and bounces back. The second order differential equation

$$\ddot{x} = -g, \tag{9}$$

is used to model the system behavior where $x$ denotes the ball position and $g = 9.81\ m/s^2$ denotes the acceleration due to gravity. The dynamics of free ball are described in terms of position and velocity as a first order differential equation

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -9.81, \end{aligned} \tag{10}$$

with position $x_1$ and velocity $x_2$. The bounce occurs at the moment when the ball hits the ground. At this moment the behavior of system changes and the system dynamics has stiff/switching behavior. By assuming an idealized model, the ball collides with a fixed surface and bounces according to

$$x_2^+ = -kx_2^-, \tag{11}$$

with velocity before the collision $x_2^-$, velocity after the collision $x_2^+$, and a related coefficient of restitution $k$. Therefore, after collision the velocity reverses and decreases by coefficient of restitution. In this work the collision is defined at the moment in time for $x_1 = 0$. The value of restitution is considered as $k = 0.7$. Using stiff ODE solvers provides suitable step size control nearby the rapid behavior. Accordingly in this contribution variable step size Kalman Filter is considered to overcome the problems regarding fixed step size solution to increase the performance of system state estimation.

The results are illustrated in Fig. 3 for fixed and variable step size Kalman Filter. From the results it can be concluded that

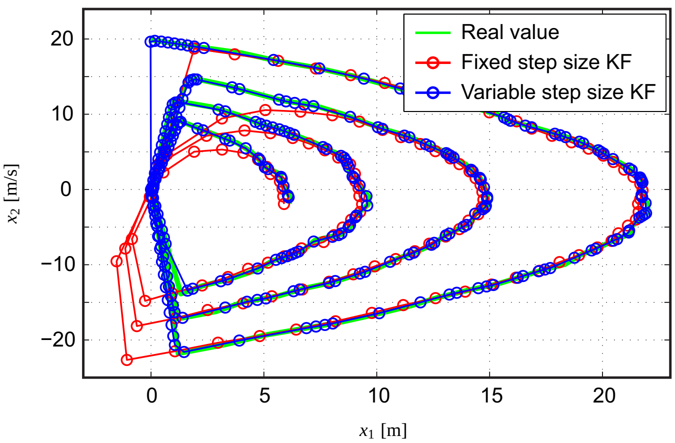


**FIGURE 4**. PHASE PORTRAIT OF BOUNCING BALL EXAMPLE CONSIDERING FIXED AND VARIABLE STEP SIZE KALMAN FILTER (KF)

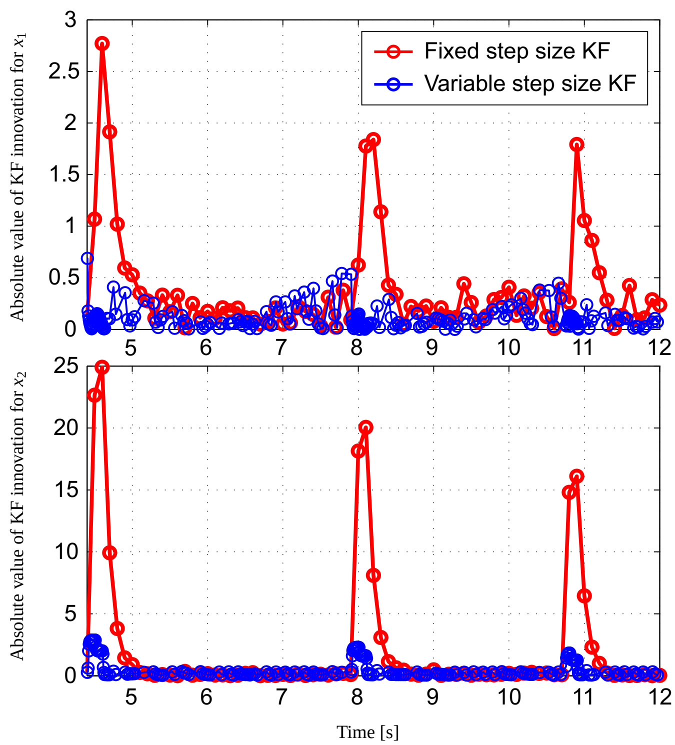


**FIGURE 5**. KALMAN FILTER INNOVATION CONSIDERING FIXED AND VARIABLE STEP SIZE APPROACHES

using variable step size Kalman Filter introduced in this work provides more sufficient estimation results compare to fixed step Kalman Filter especially nearby rapid behavior of the system (stiff problems). In other words, the proposed solution improves the estimation results by decreasing the step size in the procedure of predictor-corrector scheme and in the case of fast dynamical behavior. On the other side by relaxing the step size when no rapid change exists, the proposed approach decreases the high computational time of predictor-corrector scheme.

According to Fig. 3 the time step in the variable step size Kalman Filter solution appears to oscillate in the nonbounce region, even thought the system is smooth during this phase. As explained previously changing of sample time is based on Kalman filter innovation value. Due to the desired accuracy of Kalman filter estimation in each region, the sample time can be changed when smaller innovation is required. This oscillating behavior of sample time can be controlled by changing the value of the design parameters Min. ST, Max. ST, $\alpha$, and $\beta$.

Phase portrait of bouncing ball example considering the results of fixed and variable step size Kalman Filter is illustrated in Fig. 4. From the results it can be concluded that at the point when the ball hits the ground ($x_1 = 0$) the proposed variable step Kalman Filter provides more accurate estimation of system position and velocity compare to the standard fixed step Kalman Filter. In Fig. 5 the absolute value of Kalman Filter innovation is illustrated for system position and velocity estimation results. From the figure it can be concluded that using of proposed approach leads to less innovation value nearby the rapid changes of the system dynamics. The mean square error (MSE) of Kalman Filter innovation is achieved as 12.71 and 1.85 for fixed step and variable step KF respectively. In Fig. 6 the mean square error of Kalman Filter innovation is illustrated regarding increasing of measurement and system noise covariance. It is evident from this figure that using of proposed approach leads to less MSE considering different level of measurement and system noise.

In Fig. 7 MSE of Kalman Filter innovation is shown for different measurement sample time. As illustrated the fixed step KF accuracy decreases by increasing the sample time of system measurement because less information is available about the system behavior during the time. On the other side, using of the proposed variable step KF leads to higher accuracy of estimation also in the case of using larger sample time and the MSE value remains in a proper lower area compare to fixed step KF.

## CONCLUSION AND FURTHER DISCUSSIONS

This paper proposes a variable step size Kalman Filter for stiff problems. Event handling and zero crossing algorithms are used in the structure of the proposed approach. At each step of predictor-corrector scheme the Kalman Filter innovation is used for event handling procedure. In the case that an event is occurred and detected, the sample time is increased (relaxing procedure) or decreased (restricting procedure), the predictor-corrector scheme is stopped, and the procedure is continued by new discretization of system model and new initialization using interpolated measurement vector. In the case that no event is

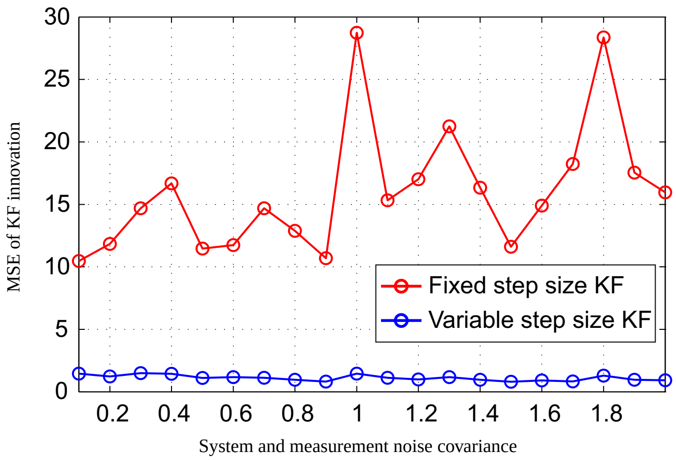


**FIGURE 6**. MEAN SQUARE ERROR (MSE) OF KALMAN FILTER INNOVATION CONSIDERING FIXED AND VARIABLE STEP SIZE APPROACHES

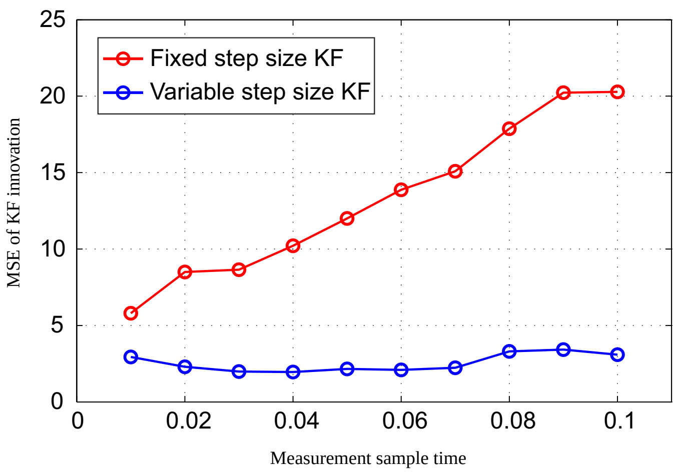


**FIGURE 7**. MEAN SQUARE ERROR (MSE) OF KALMAN FILTER INNOVATION CONSIDERING DIFFERENT SAMPLE TIME FOR SYSTEM MEASUREMENT

detected, the predictor-corrector scheme can be continued with the current considered ST. Four parameters are considered as design parameters for user. The Min. ST and Max. ST illustrate the minimum and maximum accepted ST to control the computational time when $\alpha$ and $\beta$ ($\beta < \alpha$) parameters are considered to control and influence the performance of Kalman Filter. The bouncing ball example is used as a stiff system to illustrate the advantages of the proposed approach compared to standard discrete Kalman Filter. Simulation results clearly demonstrate the effectiveness of the proposed estimation approach especially close to rapid behavioral changes of the system dynamics (stiff behavior). In other words, the proposed solution improves the estimation performance by decreasing the step size in the procedure of predictor-corrector scheme according to current estimation accuracy and in the case of fast dynamical behavior.

In the future, some issues still need to be improved regarding the implementation of proposed approach in different simulation and experimental platforms. More case studies have to be provided to compare the results and to prove the capability of the proposed approach. Furthermore, a numerical example can be defined to analyze the algorithm steps for one of the time steps with more details.

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