Undergraduate Research Opportunity Program
(UROP) Literature Review

Computing bounds on quantum probabilities

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Abstract

In this literature review, we introduce the problem of computing bounds on quantum probabilies and tools that researchers use to tackle the problem. We also mention briefly what we plan to do and achieve in this project.

Keywords:
semidefinite proramming, sum of squares, weighted sum of squares, polynomial optimization, 2-prover cooperative game, non-commutative variables, positivstellensatz.

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Chapter 1

Introduction

In this chapter, we introduce the problem of computing bounds on quantum probabilites in the context of 2-prover cooperative games.

1.1 Definition of a 2-prover Cooperative Game

In a 2-prover cooperative game, there are two players, namely Alice and Bob and a verifier. Let S and T be two finite sets of questions and π be a probability distribution on S $\times$ T. A verifier then chooses two questions s $\in$ S and t $\in$ T according to the distribution π. It assigns question s to Alice and question t to Bob. Let A and B be two finite sets. Upon receiving s, Alice has to reply to the verifier with an answer a $\in$ A. Similarly, Bob replies with an answer b $\in$ B. Let V: $S \times T \times A \times B \to \{0, 1\}$ be a binary function. We can view V as one of the rules of the game. After receving answers from Alice and Bob, the verifier calculates the value of V(a, b|s, t) = V(s,t,a,b). Alice and Bob wins if V(a, b|s, t) = 1 and loses otherwise. Another constraint of the game is that Alice and Bob can decide any kind of strategy beforehand; however, once the game begins, they cannot communicate with each other. We denote such a game as G = G(V, π).
The above desciption is the classical version of a cooperative game. In the quantum setting, we allow Alice and Bob to share an entangled quantum state $|\psi\rangle$ of their choice. To be precise, let $\mathcal{H}_A$ and $\mathcal{H}_B$ be the Hilbert spaces of Alice and Bob respectively. Then Alice and Bob can choose a quantum state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. In addition, Alice and Bob can decide their sets of quantum project measurements. Let $\mathcal{A} = \{A_s^a \mid a \in A, s \in S\}$ and $\mathcal{B} = \left\{B_t^b \mid b \in B, t \in S\right\}$ be the sets of project measurements of Alice and Bob respectively, where X_y^z denotes a projective measurement that results in an outcome z corresponding to an input y on a Hilbert space $\mathcal{H}$. Together with $|\psi\rangle$, $\mathcal{A}$ and $\mathcal{B}$ form Alice and Bob's strategy.

1.2 Overview of the Problem of computing bounds on quantum probabilities

Given a 2-prover cooperative game G = G(V, π) as described in the previous section, we are interested in the maximum probability that Alice and Bob win the game.
In the quantum setting, the probability that Alice and Bob give an answer (a, b) $\in$ A $\times$ B is

given by

$$\langle\psi| A_s^a \otimes B_t^b |\psi\rangle$$

Therefore, given a shared quantum state $|\psi\rangle$ and two sets of measurement $\mathcal{A}$ and $\mathcal{B}$, the probability that Alice and Bob win the game is given by

$$\sum_{a,b,s,t} \pi(s,t) V(a,b \mid s,t) \langle\psi| A_s^a \otimes B_t^b |\psi\rangle$$

Hence, the maximum probability for Alice and Bob to win the game G = G(V, π), denoted as $\omega^*(A)$, is defined as:

$$\omega^*(G) = \lim_{d\to\infty} \max_{\substack{|\psi\rangle\in\mathbb{C}\otimes\mathbb{C} \\ |||\psi\rangle||=1}} \max_{A_s^a, B_t^b} \sum_{a,b,s,t} \pi(s,t) V(a,b \mid s,t) \langle\psi| A_s^a \otimes B_t^b |\psi\rangle \tag{1.1}$$

where $A_s^a \in \mathbb{B}(\mathcal{H}_A)$ ($\mathbb{B}(\mathcal{H}_A)$ is the set of all bounded operators on the Hilbert space $\mathcal{H}_A$) and $B_t^b \in \mathbb{B}(\mathcal{H}_B)$ for some Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. In addition, the measurements A_s^a, B_t^b must satisfy $\sum_a A_s^a = \mathbb{I}_A$, $\sum_b B_t^b = \mathbb{I}_B$, $(A_s^a)^2 = A_s^a$, and $\left(B_t^b\right)^2 = B_t^b$ (since we assume A_s^a and B_t^b are projective measurements).

Now our aim is to find bounds (particularly upperbounds) on $\omega^*(G)$, which is called the entangled value of G = G(V, π). In Chapter 3, we show that we can reduce the problem of finding bounds on $\omega^*(A)$ to finding bounds of a polynomial in non-commutative variables which are projective measurements or observables. We refer the reader to the book by Nielsen and Chuang (Nielsen & Chuang, 2010) for an excellent and detailed introduction to concepts of quantum computation and information which include quantum measurement, quantum state, projective measurement and observable.

Chapter 2

Semidefinite Programming and Finding Bounds on Polynomials in Commutative Variables

Before discussing our main problem of finding bounds on a polynomial in non-commutative variables, we go through semidenite programming, a useful tool to tackle the problem. Then we discuss the problem of finding bounds on a polynomial in commutative variables first to have a clear picture of the concepts and techniques used.

2.1 Semidefinite programming

Semidefinite programming is a kind of nonconvex programming with many applications. It can be seen as a generalization of linear programming. Here we present two equivalent definitions of a semidefinite program that are mostly used and their corresponding dual problems.

The first definition is in the thorough and excellent paper by Boyd and Vandenberghe (Vandenberghe & Boyd, 1996). Let x $\in \mathbb{R}^m$ be a variable. Let c $\in \mathbb{R}^m$ be a fixed vector. Let $F_0, F_1, ..., F_m$ be m + 1 symmetric matrices in $\mathbb{R}^{n\times n}$. Then a semidefinite program is an optimization problem:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0 \end{array} \tag{2.1}$$

where $A \succeq 0$ means that the square matrix A of order n is positive definite, i.e. $y^T A y \geq 0$ $\forall y \in \mathbb{R}^n$
The dual problem of the semidefinite program (2.1) is

$$\begin{array}{ll} \text{maximize} & -tr(F_0 Z) \\ \text{subject to} & tr\,(F_i Z) = c_i \ \forall i = 1, ..., m, \\ & Z = Z^T \succeq 0 \text{ and } Z \in \mathbb{R}^{n\times n} \end{array} \tag{2.2}$$

Another commonly used definition of a semidefinite program can be found in the paper of Parrilo (Parrilo, 2003). Its primal problem is more or less the dual problem (2.2) of the semidefinite program (2.1). That is,

$$\begin{array}{ll} \text{minimize} & tr(F_0 Z) \\ \text{subject to} & tr\,(F_i Z) = c_i \ \forall i = 1, ..., m, \\ & Z = Z^T \succeq 0 \text{ and } Z \in \mathbb{R}^{n\times n} \end{array} \tag{2.3}$$

The corresponding dual problem is

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & F(x) = F_0 - \sum_{i=1}^{m} x_i F_i \succeq 0 \end{array} \tag{2.4}$$

There are many efficient polynomial-time algorithms to solve a semidefinite program, such as primal-dual potential reduction methods (see (Vandenberghe & Boyd, 1996)). SeDuMi (Sturm,) is a popular solver for semidefinite programming.

2.2 Finding bounds on a polynomial of commutative variables

In this section, we discuss an application of semidefinite programming in computing bounds on a polynomial of commutative variables, which is quite related to our non-commutative case.

2.2.1 Global nonnegativity of multivariate polynomials

Polynomials of commutative variables

A polynomial f in n commutative variables x_1, x_2, …, x_n is a finite linear combination of monomials. It has a form:

$$f = \sum_{\alpha} c_\alpha v^\alpha = \sum_{\alpha} x_\alpha^{\alpha_1} \dots x_n^{\alpha_n} \tag{2.5}$$

where $c_\alpha \in \mathbb{R}$ and $x_\alpha^{\alpha_1} + \ldots + x_n^{\alpha_n} = \alpha$

A commutative ring of all multivariate polynomials in n commutative variables is denoted as $\mathbb{R}\left[x_1, x_2, \ldots, x_n\right]$.
The total degree of a monomial v^α is equal to $x_\alpha^{\alpha_1} + \ldots + x_n^{\alpha_n}$. The total degree of a polynomial f is the maximum total degree of all its monomials.
A homogenous polynomial (or a form) is a polymial where all its monomials have the same total degree.
A polynomial F is called to have a sum of squares decomposition if F can be expressed as:

$$F = \sum_{i} f_i^2 \tag{2.6}$$

where $f_i \in \mathbb{R}\left[x_1, x_2, \ldots, x_n\right]$

Global nonnegativity and Sum of squares relaxation

Given a polynomial $F \in \mathbb{R}\left[x_1, x_2, \ldots, x_n\right]$, the global nonnegativity problem is to check if

$$F\left(x_1, x_2, \ldots, x_n\right) \geq 0 \ \forall x_1, x_2, \ldots, x_n \in \mathbb{R} \tag{2.7}$$

Note that to satisfy (2.7), it is necessary that the degree of F is even. Clearly, if a polynomial F has a sum of squares decomposition, then F satisfies the condition (2.7). However, not all nonnegative polynomial has a sum of squares decomposition. In a paper by Parrilo (Parrilo, 2003), there is an example of such a polynomial, which is $M\left(x, y, z\right) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$. It is known that the global nonnegativity problem of a polymial is NP-hard. Therefore, we want to relax the condition (2.7) so that it can be solved more efficiently. A possible relaxation is to check if a polynomial F has a sum of square decomposition.

Sum of Squares and Semidefinite Programming

Consider a polynomial $F \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ of degree 2d. Let z be a row vector of all the monomials of F that have the total degree less than or equal to d, i.e.

$$z = \left[1, x_1, x_2, \ldots, x_n, x_1 x_2, \ldots, x_n^d\right]$$

Then the polynomial F can be expressed in a form:

$$F(x_1, x_2, \ldots, x_n) = z^T Q z \tag{2.8}$$

where Q is some constant matrix.
It can be shown that F has a sum of squares decomposition if and only if Q is positive semidefinite, i.e. $Q \succeq 0$. Therefore, the problem of checking if F has a sum of squares decomposition can be cast as a semidefinite program of the form (2.3) where the expression that needs minimizing is a constant. Hence, it can be determined in polynomial time in the size of the matrix Q if a polynomial F has a sum of squares decomposition.

Polynomial Optimization and Hierachy of Semidefinite Programs

Now we consider the problem of polynomial optimization in commutative variables.
Let $p, g_1, g_2, \ldots, g_m \in \mathbb{R}[x]$.

$$K = \{x \in \mathbb{R}^n | g_1(x) \geq 0, \ldots, g_m(x) \geq 0\} \tag{2.9}$$

be a semialgebraic set.
The problem of polynomial optimization can be formulated as follows:

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & p(x) - \gamma \geq 0 \ \forall x \in K \end{array} \tag{2.10}$$

In other words, the problem (2.10) finds the infimum (greatest lower bound) of the polynomial p over the semialgebraic set K. The problem of finding an upper bound of p can be formulated in a similar manner.

It can be shown that the problem (2.10) is hard. Therefore, a natural approach is to relax the problem as we do to the problem of checking global nonnegativity (2.7). In a detailed survey of the topic by Laurent (Laurent, 2000), there are many efficient relaxations which depend on additional properties of K. Here we present a relaxtion based on Putinar's Positivestellensatz.

The quadratic module generated by polynomials $g_1, \ldots, g_m$ is defined as:

$$M(g_1, \ldots, g_m) = \left\{ u_0 + \sum_{j=1}^{m} u_j g_j | u_0, u_j \in \Sigma \right\} \tag{2.11}$$

where $\Sigma \subset \mathbb{R}[x]$ is the set of all polynomials that have a sum of squares decomposition.
A quadratic module $M(g_1, \ldots, g_m)$ is Archimedean if it satisfies the following condition

$$\forall p \in \mathbb{R}[x], \exists K \in \mathbb{N}, N \pm p \in M(g_1, \ldots, g_m) \tag{2.12}$$

Now we can state Putinar's Positivstellensatz.

Theorem 1. *Let K be a semialgebraic set as in (2.9) and suppose the quadratic module $M(g_1, \ldots, g_m)$ is Archimedean. For a polynomial $F \in \mathbb{R}[x]$, if $F > 0$ on K, then $F \in M(g_1, \ldots, g_m)$.*

Note that in the problem (2.10),

$$\gamma = \sup\{\rho | p(x) - \rho \geq 0 \ \forall x \in K\} \tag{2.13}$$
$$= \sup\{\rho | p(x) - \rho > 0 \ \forall x \in K\} \tag{2.14}$$

Therefore, problem (2.10) can be relaxed as

$$\begin{array}{ll} \text{maximize} & \gamma_t^{SOS} \\ \text{subject to} & p(x) - \gamma_t^{SOS} = s_0 + \sum_{j=1}^m s_j g_j \\ & s_0, s_j \in \Sigma \\ & \deg(s_0), \deg(s_j g_j) \leq 2t \end{array} \tag{2.15}$$

where $\deg(f)$ is the degree of a polynomial f.
Equivalently, the problem (2.15) can be reformulated as

$$\begin{array}{ll} \text{maximize} & \gamma_t^{SOS} \\ \text{subject to} & p(x) - \gamma_t^{SOS} - \sum_{j=1}^m s_j g_j \in \Sigma \\ & s_j \in \Sigma \\ & \deg(s_j g_j) \leq 2t \end{array} \tag{2.16}$$

Note that in the formulation of the problems (2.15) and (2.16), in fact we construct a hierachy of semidefinite programs in t. Clearly, $\gamma_t^{SOS} \leq \gamma$ (γ is the value we try to maximize in the problem (2.10)). According to the survey by Laurent, under certain conditions on the semialgebraic set K, there is a finite convergence of $\{\gamma_t^{SOS}\}$ to γ.

Chapter 3

Computing Bounds on Quantum Probabilities

In this chapter, we show that the problem of finding bounds on quantum probablities, particularly finding bounds on $\omega^*(G)$ in (1.1), can be relaxed to the problem of polynomial optimization in non-commutative variables.

3.1 Field-theoretic value of $\mathbf{G} = \mathbf{G}(\pi, V)$

The field-theoretic value, denoted as $\omega^f(G)$, of a 2-prover game with classical verifier G = G(π, V) is defined as

$$\omega^f(G) = \sup_{A_s'^a, B_t'^b} \left\| \sum_{a,b,s,t} \pi(s,t) V(a,b|s,t) A_s'^a B_t'^b \right\| \tag{3.1}$$

where $\|X\| = \max\left\{\lambda | \lambda \text{ is an eignvalue of } X^\dagger X\right\}$ ($X^\dagger$ is the Hermitian conjugate of the operator X), $A_s'^a \in \mathbb{B}(\mathcal{H})$ and $B_t'^b \in \mathbb{B}(\mathcal{H})$ for the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ ($\mathcal{H}_A$ and $\mathcal{H}_B$ are Hilbert spaces of Alice and Bob respectively) satisfying $A_s'^a, B_t'^b \succeq 0$, $\sum_a A_s'^a = \sum_b B_t'^b = \mathbb{I}$ for all s, t and $\left[A_s'^a, B_t'^b\right] = 0$ ($[X,Y] = XY - YX$ is the commutator of X and Y) for all $s \in S$, $t \in T$, $a \in A$, $b \in B$ (The sets S, T, A, B are defined in Section 1.1).
The theoretical foundation to introduce $\omega^f(G)$ is a well-known mathematical result which is that if a Hilbert space is finite-dimensional, imposing the commutativity constraints is equivalent to demanding a tensor product structure. We will state this theorem in a rigorous form in the context of a 2-prover cooperative game as follows:

Theorem 2. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space, and let $\left\{A_s'^a \in \mathbb{B}(\mathcal{H}) | s \in S\right\}$ and $\left\{B_t'^b \in \mathbb{B}(\mathcal{H}) | t \in T\right\}$. Then the following statements are equivalent:*

1. *For all $s \in S$, $t \in T$, $a \in A$ and $b \in B$, it holds that $\left[A_s'^a, B_t'^b\right] = 0$.*
2. *There exist Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$ such that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and operators $A_s^a \in \mathcal{H}_A$, $B_t^b \in \mathcal{H}_B$ such that $A_s'^a = A_s^a \otimes \mathbb{I}_B$ and $B_t'^b = \mathbb{I}_A \otimes B_t^b$.*

The proof of the direction (2) $\Rightarrow$ (1) is simple because by a property of tensor product $A_s'^a B_t'^b = [A_s^a \otimes \mathbb{I}_B]\left[\mathbb{I}_A \otimes B_t^b\right] = (A_s^a \mathbb{I}_A) \otimes \left(\mathbb{I}_B B_t^b\right) = A_s^a \otimes B_t^b = B_t'^b A_s'^a$. The proof of the other direction of Theorem 2 and a general version of Theorem 2 are presented in the paper by Wehner et al. (Wehner, Toner, Liang, & Doherty, 2008). Also, in that paper, it is proved that

$$\omega^* (G) \leq \omega^f (G) \tag{3.2}$$

3.2 Finding Bounds on Quantum Probabilities and Polynomial Optimization in Non-Commutative Variables

In this section, we will draw a connection between the problem of finding bounds on quantum probabilities and the problem of polynomial optimization in non-commutative variables.

By Equation (3.2), if we find an upperbound ϑ of $\omega^f (G)$, then ϑ is an upperbound of $\omega^* (G)$. Recall that in a 2-prover cooperative game, we consider only projective measurements. Therefore, we can assume the operators $A_s^{\prime a}, B_t^{\prime b}$ in Equation (3.1) are orthogonal projectors. Note that an operator X is orthogonal projectors if and only if $X^\dagger = X$ and $X^2 = X$. Hence, we have:

$$\left\| \sum_{a,b,s,t} \pi (s,t) V (a,b|s,t) A_s^{\prime a} B_t^{\prime b} \right\| = \max \left\{ \lambda | \lambda \text{ is an eigenvalue of } \sum_{a,b,s,t} \pi (s,t) V (a,b|s,t) A_s^{\prime a} B_t^{\prime b} \right\} \tag{3.3}$$

Let ν be a real number. Consider the expression:

$$q_\nu = \nu \mathbb{I} - \sum_{a,b,s,t} \pi (s,t) V (a,b|s,t) A_s^{\prime a} B_t^{\prime b} \tag{3.4}$$

If $q_\nu \succeq 0$, then clearly for all $A_s^{\prime a}, B_t^{\prime b}$, we have

$$\nu \geq \max \left\{ \lambda | \lambda \text{ is an eigenvalue of } \sum_{a,b,s,t} \pi (s,t) V (a,b|s,t) A_s^{\prime a} B_t^{\prime b} \right\} \tag{3.5}$$

Hence, $\nu \geq \omega^f (G) \geq \omega^* (G)$. Thus, ν is an upperbound of $\omega^* (G)$.

Therefore, the problem of finding a smallest upperbound (supremum) of $\omega^f (G)$ as well as $\omega^* (G)$ can be expressed as:

$$\begin{array}{ll} \text{minimize} & \nu \\ \text{subject to} & q_\nu = \nu \mathbb{I} - \sum_{a,b,s,t} \pi (s,t) V (a,b|s,t) A_s^{\prime a} B_t^{\prime b} \succeq 0 \\ & \sum_{a \in A} A_s^{\prime a} - \mathbb{I} = 0 \text{ and } \sum_{b \in B} B_t^{\prime b} - \mathbb{I} = 0 \\ & \left(A_s^{\prime a} \right)^2 = A_s^{\prime a} \succeq 0 \text{ and } \left(B_t^{\prime b} \right)^2 = B_t^{\prime b} \succeq 0 \ \forall a \in A, b \in B, s \in S, t \in T \\ & \left[A_s^{\prime a}, B_t^{\prime b} \right] = 0 \ \forall a \in A, b \in B, s \in S, t \in T \\ & \left(A_s^{\prime a} \right)^\dagger = A_s^{\prime a} \text{ and } \left(B_t^{\prime b} \right)^\dagger = B_t^{\prime b} \ \forall a \in A, b \in B, s \in S, t \in T \end{array} \tag{3.6}$$

Clearly, the optimization problem (3.6) is a polynomial optimization in non-commutable and Hermitian variables $A_s^{\prime a}, B_t^{\prime b}$ (An operator X is Hermitian if $X^\dagger = X$).

3.3 Helton and McCullough's Positivstellensatz and Semidefinite Programming Hierachy

In this section, we introduce the Helton and McCullough's Positivstellensatz, which is used to relax Problem (3.6). We also discuss a hierachy of semidefinite program which is used to solve a relaxation of Problem (3.6).

Let $\mathbb{C}\left[x, x^{\dagger}\right]$ be the set of polynomials in the 2n noncommutative variables $x = (x_1, \ldots, x_n)$ and $x^{\dagger} = \left(x_1^{\dagger}, \ldots, x_n^{\dagger}\right)$ with coefficients in $\mathbb{C}$. Let $\mathcal{P} \subset \mathbb{C}\left[x, x^{\dagger}\right]$ be a collection of Hermitian polynomials. The convex cone $\mathcal{C}_{\mathcal{P}}$ generated by $\mathcal{P}$ consists of polynomials of the form

$$q = \sum_{i=1}^{M} r_i^{\dagger} r_i + \sum_{j=1}^{N} \sum_{k=1}^{L} s_{jk}^{\dagger} p_j s_{jk} \tag{3.7}$$

where $p_i \in \mathcal{P}$, M, N, L are finite, and $r_i, s_{jk} \in \mathbb{C}\left[x, x^{\dagger}\right]$.
Equation (3.7) is called a weighted sum of squares (WSOS) representation of a polynomial q.

Consider G = G(π, V). Denote $\mathcal{A}' = \left\{A_s'^a \mid a \in A, s \in S\right\}$ and $\mathcal{B}' = \left\{B_t'^b \mid b \in B, t \in T\right\}$ where $A_s'^a \in \mathbb{B}(\mathcal{H})$ and $B_t'^b \in \mathbb{B}(\mathcal{H})$ for the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ ($\mathcal{H}_A$ and $\mathcal{H}_B$ are Hilbert spaces of Alice and Bob respectively). Let us choose

$$\mathcal{P} = \left\{i\left[A_s'^a, B_t'^b\right]\right\} \cup \left\{\sum_{a \in A} A_s'^a - \mathbb{I}\right\} \cup \left\{\sum_{b \in B} B_t'^b - \mathbb{I}\right\} \cup \left\{\left(A_s'^a\right)^2 - A_s'^a\right\} \cup \left\{\left(B_t'^b\right)^2 - B_t'^b\right\} \tag{3.8}$$

Now we will state the Helton and McCullough's Positivstellensatz in the context of a 2-prover game.

Theorem 3. *Let $G = G(\pi, V)$ be a 2-prover cooperative game, and let $\mathcal{C}_{\mathcal{P}}$ be the cone generated by the set $\mathcal{P}$ defined in (3.8). Set*

$$q_\nu = \nu\mathbb{I} - \textstyle\sum_{a,b,s,t} \pi(s,t) V(a,b|s,t) A_s'^a B_t'^b.$$

If $q_\nu \succ 0$, then $q_\nu \in \mathcal{C}_{\mathcal{P}}$.

We refer the reader to the paper by Wehner et al. (Wehner et al., 2008) for a more general version of Theorem 3 and its proof.
By Theorem 3, we have a relaxation version of Problem (3.6) as follows:

$$\begin{array}{ll} \text{minimize} & \nu \\ \text{subject to} & \nu\mathbb{I} - \sum_{a,b,s,t} \pi(s,t) V(a,b|s,t) A_s'^a B_t'^b \succeq 0 = \sum_{i=1}^{M} r_i^{\dagger} r_i + \sum_{j=1}^{N} \sum_{k=1}^{L} s_{jk}^{\dagger} p_j s_{jk} \\ & \max\left(\deg(r_i), \deg\left(s_{jk}^{\dagger} p_j s_{jk}\right)\right) \leq 2t \end{array} \tag{3.9}$$

where $p_j \in \mathcal{P}$, and r_i, s_{jk} are some arbitrary polynomial in variables $A_s'^a, B_t'^b$.
Note that for each fixed value of t, Problem (3.9) is a semidefinite program. Hence, we can form a hierachy of semidefinite programs based on the value of t. Let $\omega_t^{sdp}(G)$ denote the solution to the semidefinite program (3.9) corresponding to t. Then Wehner et al. proves that $\lim_{n\to\infty} \omega_t^{sdp}(G) = \omega^f(G)$.
Another relaxtion of Problem (3.6) is discussed in the paper by Pironio et al. (Pironio, Navascués, & Acín, 2010) which involves moment matrices and localizing matrices. This relaxation version turns out to be the primal program of (3.9).

Chapter 4

Conclusion

In conclusion, the problem of finding upperbounds on quantum probabilities can be relaxed to the problem of polynomial optimization in non-commutative variables. Moreover, the latter problem can be solved by a hierachy of semidefinite programs. We know that in many cases, a sum-of-squares relaxation of the polynomial optimization problem in commutative variables can have a hierachy of semidefinite programs which converges to the extreme values in finite steps. In addition, there are many solvers for the sum-of-squares relaxation problems in the commutative case, such as YALMIP (Ĺofberg, 2004), or SOSTOOLS (Parrilo, Prajna, Papachrisodoulou, & Seiler,). Unlike the commutative case, we do not know much about the rate of convergence of a hierachy of semidefinite programs in the non-commutative case. Neither do we know much about the optimal upperbounds of many N-prover cooperative games. In this project, we plan to design a MATLAB package to solve the sum-of-squares relaxation of the problem of polynomial optimization in non-commutative variables. It can be viewed as a pre-processor which generates the constraints and the minimization objective for the semidefinite program (3.9) or its primal program. Then it invokes SeDuMi solver to solve the semidefinite program. Then we use the package to research some open problems in finding bounds on quantum probabilities in the context of N-prover cooperative games.

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