Probability and Computing, 2nd Edition

Solutions to Chapter 3: Moments and Deviations

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July 2025

3.1

$\mathbf{E}[X^2] = \sum_{i=1}^{n} \frac{1}{n} \times i^2 = \frac{(n+1)(2n+1)}{6}$, and $\mathbf{E}[X] = \frac{n+1}{2}$.
Thus, $\mathbf{Var}[X] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{n^2-1}{12}$.

3.2

$\mathbf{E}[X] = 0$, and $\mathbf{E}[X^2] = \sum_{i=1}^{k} \frac{2}{2k+1} \times i^2 = \frac{k(k+1)}{3}$.
Thus, $\mathbf{Var}[X] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{k(k+1)}{3}$.

3.3

The variance of a single die roll is $\frac{35}{12}$ from exercise 3.1. Since all rolls are independent, $\Pr(|X - 350| \geq 50) \leq \frac{1}{50^2} \times \frac{35}{12} \times 100 = \frac{7}{60}$.

3.4

$\mathbf{Var}[cX] = \mathbf{E}[(cX - \mathbf{E}[cX])^2] = \mathbf{E}[c^2X^2 - 2cX\mathbf{E}[cX] + (\mathbf{E}[cX])^2]$
$= c^2(\mathbf{E}[X^2] - (\mathbf{E}[X])^2) = c^2\mathbf{Var}[X]$. ■

3.5

$\mathbf{Var}[X - Y] = \mathbf{E}[((X - Y) - \mathbf{E}[X - Y])^2] = \mathbf{E}[((X - \mathbf{E}[X]) - (Y - \mathbf{E}[Y]))^2]$
$= \mathbf{E}[(X - \mathbf{E}[X])^2] - 2\mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])] + \mathbf{E}[(Y - \mathbf{E}[Y])^2]$
$= \mathbf{Var}[X] - \mathbf{Cov}[X, Y] + \mathbf{Var}[Y] = \mathbf{Var}[X] + \mathbf{Var}[Y]$ $(X \perp\!\!\!\perp Y)$. ■

3.6

Let X_i $(1 \leq i \leq k)$ be the number of flips after $(i-1)$th head until ith head. Since all flips are independent, the desired variance could be computed as $\sum_{i=1}^{k} \mathbf{Var}[X_i]$. As $X_i \sim Geom(p)$, $\mathbf{Var}[X_i] = (1 - p)/p^2$ for all i. Thus, the desired variance is $k(1 - p)/p^2$.

3.7

Let X be the number of increases. Then $\Pr(X = k) = \binom{d}{k} p^k (1 - p)^{d-k}$.
Let the price of the stock after d days be V.
Then $\mathbf{E}[V] = \sum_{k=0}^{d} qr^k (\frac{1}{r})^{d-k} \binom{d}{k} p^k (1 - p)^{d-k} = \sum_{k=0}^{d} q \binom{d}{k} (pr)^k (\frac{1-p}{r})^{n-k}$.

Let $M = pr + (1-p)/r = (1-p+pr^2)/r$. Then
$\mathbf{E}[V] = M^d \sum_{k=0}^{d} q\binom{d}{k}(\frac{pr}{M})^k(\frac{1-p}{rM})^{d-k} = M^d \sum_{k=0}^{d} q\binom{d}{k}(\frac{pr^2}{rM})^k(\frac{1-p}{rM})^{d-k} = M^d q$.
Now we compute $\mathbf{E}[V^2] = \sum_{k=0}^{d} q^2 r^{2k}(\frac{1}{r})^{2d-2k}\binom{d}{k}p^k(1-p)^{d-k}$.
$\mathbf{E}[V^2] = q^2 \sum_{k=0}^{d} \binom{d}{k}(pr^2)^k(\frac{1-p}{r^2})^{d-k} = q^2\left(pr^2 + \frac{1-p}{r^2}\right)^d$ (similar to $\mathbf{E}[V]$).
Thus, $\mathbf{Var}[V] = q^2\left((pr^2+\frac{1-p}{r^2})^d - (pr+\frac{1-p}{r})^{2d}\right)$.
By plugging $q=1$ in, we get the desired result.

3.8

Let X be the running time of the given algorithm on input strings of size n. Now, let M be the longest running time of the algorithm among the input strings of size n. Then $\Pr(X \geq M) \geq 1/2^n$ by definition.
By Markov's inequality, $1/2^n \leq \Pr(X \geq M) \leq \frac{\mathbf{E}[X]}{M}$, which leads to $M \leq 2^n\mathbf{E}[X]$. Since $\mathbf{E}[X] = O(n^2)$, we get $M = O(n^2 2^n)$.

3.9

(a) By linearity of expectations, $\mathbf{E}[X^2] = \mathbf{E}[\sum_{i=1}^{n} X_i X] = \sum_{i=1}^{n} \mathbf{E}[X_i X]$.
Since X_i are Bernoulli random variables, $\mathbf{E}[X_i X] = \Pr(X_i = 0) \times 0 + \Pr(X_i = 1) \times \mathbf{E}[X|X_i = 1]$. $\blacksquare$
(b) Using the equation proven in (a), $\mathbf{E}[X^2] = \sum_{i=1}^{n} p \times (1+(n-1)p) = np + n(n-1)p^2$. Thus, $\mathbf{Var}[X] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = np + n(n-1)p^2 - n^2p^2 = np(1-p)$.

3.10

Let $X \sim \mathit{Geom}(p)$, and let $Y = 1$ if and only if $X = 1$ and $Y = 0$ otherwise. Then, by Lemma 2.5, $\mathbf{E}[X^3] = \Pr(Y=1)\mathbf{E}[X^3|Y=1] + \Pr(Y=0)\mathbf{E}[X^3|Y=0]$ $= p \times 1 + (1-p) \times \mathbf{E}[X^3|Y=0] = p + (1-p) \times \mathbf{E}[X^3|X>1]$.
Now, by the memoryless property of geometric distributions, $\mathbf{E}[X^3|X>1] = \mathbf{E}[(X+1)^3]$. Thus, $\mathbf{E}[X^3] = p + (1-p)(\mathbf{E}[X^3] + 3\mathbf{E}[X^2] + 3\mathbf{E}[X] + 1)$.
This leads to $\mathbf{E}[X^3] = (p^2 - 6p + 6)/p^3$.
Similarly, we can find $\mathbf{E}[X^4] = (-p^3 + 14p^2 - 36p + 24)/p^4$.

3.11

Let $X = \sum_{i<j} X_{i,j}$, where $X_{i,j}$ is an indicator variable that is 1 if a_i and a_j are inverted. Then, we compute $\mathbf{E}[X^2]$ as:

$\mathbf{E}[X^2] = \mathbf{E}[\sum_{i<j} X_{i,j}^2 + \sum_{|\{i,j,k,l\}|=4} X_{i,j}X_{k,l} + \sum_{i<j<k} 2(X_{i,j}X_{j,k} + X_{i,j}X_{i,k} + X_{i,k}X_{j,k})]$
$= \binom{n}{2} \cdot \frac{1}{2} + \binom{n}{4} \cdot \binom{4}{2} \cdot \frac{1}{4} + 2 \cdot \binom{n}{3} \cdot \left(\frac{1}{6} + \frac{1}{3} + \frac{1}{3}\right) = \frac{n(n-1)(9n^2-5n+10)}{144}$.
Since $\mathbf{E}[X] = \frac{n(n-1)}{4}$, $\mathbf{Var}[X] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{n(n-1)(2n+5)}{72}$.

3.12

Let X be a random variable such that $\Pr(X = n) = \frac{1}{\zeta(3)n^3}$.
Then $\mathbf{E}[X] = \sum_{n=1}^{\infty} n \times \frac{1}{\zeta(3)n^3} = \frac{1}{\zeta(3)} \times \frac{\pi^2}{6} < \infty$.
However, since $\mathbf{E}[X^2] = \sum_{n=1}^{\infty} n^2 \times \frac{1}{\zeta(3)n^3} = \sum_{n=1}^{\infty} \frac{1}{n} \times \frac{1}{\zeta(3)} = \infty$ (harmonic series), the variance of X is unbounded.

3.13

Let X be a random variable such that $\Pr(X = n) = \frac{1}{\zeta(k+2)n^{k+2}}$.
Then, similar to exercise 3.12, $\mathbf{E}[X^k]$ converges and $\mathbf{E}[X^{k+1}]$ diverges.

3.14

$\mathbf{Var}[\sum_{i=1}^{n} X_i] = \mathbf{E}\left[\left(\sum_{i=1}^{n}(X_i - \mathbf{E}[X_i])\right)^2\right] = \sum_{i=1}^{n} \mathbf{Var}[X_i] + \sum_{i \neq j} \mathbf{Cov}[X_i, X_j] = \sum_{i=1}^{n} \mathbf{Var}[X_i] + 2\sum_{i=1}^{n}\sum_{i<j} \mathbf{Cov}[X_i, X_j]$. ∎

3.15

$\mathbf{E}[X_iX_j] = \mathbf{E}[X_i]\mathbf{E}[X_j]$ indicates that $\mathbf{Cov}[X_i, X_j] = \mathbf{E}[X_iX_j] - \mathbf{E}[X_i]\mathbf{E}[X_j] = 0$.
Thus, $\mathbf{Var}[X] = \sum_{i=1}^{n} \mathbf{Var}[X_i]$ holds.

3.16

Suppose that we want the expectation to be μ. Then the desired X should satisfy $\Pr(X = k\mu) = 1/k$ and $\Pr(X = 0) = 1 - 1/k$.

3.17

Suppose that we want the expectation to be μ. Then in order to satisfy $\Pr(|X - \mathbf{E}[X]| \geq a) = \frac{\mathbf{Var}[X]}{a^2}$, X should satisfy:

$\Pr(X = \mu) = p$, $\Pr(X = \mu + a) = (1 - p)/2$ and $\Pr(X = \mu - a) = (1 - p)/2$.

3.18

(a) $\Pr(X - \mathbf{E}[X] \geq t\sigma[X]) = \Pr[t(X - \mathbf{E}[X]) + \sigma[X] \geq (t^2 + 1)\sigma[X]]$
$\leq \Pr[(t(X - \mathbf{E}[X]) + \sigma[X])^2 \geq (t^2 + 1)^2 \mathbf{Var}[X]]$
$\leq \mathbf{E}[(t(X - \mathbf{E}[X]) + \sigma[X])^2]/(t^2 + 1)^2 \mathbf{Var}[X]$ (Markov's inequality)
$=(t^2 \mathbf{Var}[X] + \mathbf{Var}[X])/(t^2 + 1)^2 \mathbf{Var}[X] = 1/(t^2 + 1)$. ■
(b) Since probabilities cannot be greater than 1, we only consider $t \geq 1$.
By Chebyshev's inequality, $\Pr(|X - \mathbf{E}[X]| \geq t\sigma[X]) \leq 1/t^2 \leq 2/(1 + t^2)$. ■

3.19

(i) If $\mu = m$, then the claim is trivially valid.
(ii) If $\mu < m$, then let $t = |\mu - m|/\sigma$. Then using the result of exercise 3.18(a), $\Pr(X - \mu \geq |\mu - m|) \leq \frac{\sigma^2}{(\mu - m)^2 + \sigma^2}$.
Now, from $1/2 \leq \Pr(X \geq m) \leq \Pr(X - \mu \geq |\mu - m|) \leq \frac{\sigma^2}{(\mu - m)^2 + \sigma^2}$, we conclude $(\mu - m)^2 \leq \sigma^2$, thus $|\mu - m| \leq \sigma$.
(iii) If $\mu > m$, then let $t = |\mu - m|/\sigma$. Now, substituting X into $-X$ in the result of exercise 3.18(a), we get $\Pr(-X + \mathbf{E}[X] \geq t\sigma[X]) \leq 1/(t^2 + 1)$.
This leads to $\Pr(-X + \mu \geq |\mu - m|) \leq \frac{\sigma^2}{(\mu - m)^2 + \sigma^2}$.
Similarly, from $1/2 \leq \Pr(X \leq m) \leq \Pr(-X + \mu \geq |\mu - m|) \leq \frac{\sigma^2}{(\mu - m)^2 + \sigma^2}$, we conclude $(\mu - m)^2 \leq \sigma^2$, thus $|\mu - m| \leq \sigma$. ■

3.20

We first prove $\Pr(Y \neq 0) \leq \mathbf{E}[Y]$. This is trivial since
$\mathbf{E}[Y] - \Pr(Y \neq 0) = \sum_{i=1}^{\infty} i \Pr(Y = i) - \sum_{i=1}^{\infty} \Pr(Y = i) = \sum_{i=1}^{\infty} (i - 1) \Pr(Y = i) \geq 0$.
Now we prove $\frac{\mathbf{E}[Y]^2}{\mathbf{E}[Y^2]} \leq \Pr(Y \neq 0)$. Let $X = Y | Y \neq 0$ such that $\Pr(X = x) = \Pr(Y = x | Y \neq 0)$. Since $(\mathbf{E}[X])^2 \leq \mathbf{E}[X^2]$, $\mathbf{E}[Y | Y \neq 0]^2 \leq \mathbf{E}[Y^2 | Y \neq 0]$ holds.
Now, consider that $\mathbf{E}[X] = \mathbf{E}[0] \Pr(X = 0) + \mathbf{E}[X | X \neq 0] \Pr(X \neq 0) = \mathbf{E}[X | X \neq 0] \Pr(X \neq 0)$ is valid for any random variable X.
Combined with $\mathbf{E}[Y | Y \neq 0]^2 \leq \mathbf{E}[Y^2 | Y \neq 0]$, $\left(\frac{\mathbf{E}[Y]}{\Pr(Y \neq 0)}\right)^2 \leq \frac{\mathbf{E}[Y^2]}{\Pr(Y \neq 0)}$ holds. ■

3.21

(a) Let $Y = |X - \mathbf{E}[X]|$. Then by Markov's inequality,
$\Pr(Y > t \sqrt[k]{\mathbf{E}[Y^k]}) = \Pr(Y^k > t^k \mathbf{E}[Y^k]) \leq \Pr(Y^k \geq t^k \mathbf{E}[Y^k]) \leq 1/t^k$.
(b) If k is odd, then $Y^k \neq (X - \mathbf{E}[X])^k$ and $(X - \mathbf{E}[X])^k$ may not always be positive. Therefore, we cannot apply Markov's inequality in this case.

3.22

Let X_i be indicator variables that are 1 if $\pi(i) = i$. Then $X_i \sim Bernoulli(1/n)$. Then $\mathbf{Var}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbf{Var}[X_i] + 2\sum_{i=1}^{n}\sum_{i<j} \mathbf{Cov}[X_i, X_j]$.
Now, since $X_i \sim Bernoulli(1/n)$, $\mathbf{Var}[X_i] = \frac{1}{n}\left(1 - \frac{1}{n}\right)$.
$\mathbf{Cov}[X_i, X_j] = \mathbf{E}[X_i X_j] - \mathbf{E}[X_i]\mathbf{E}[X_j]$. Since $\mathbf{E}[X_i X_j] = \frac{1}{n(n-1)}$, $\mathbf{Cov}[X_i, X_j] = \frac{1}{n(n-1)} - \frac{1}{n^2}$.
Thus, $\mathbf{Var}[\sum_{i=1}^{n} X_i] = 1 - \frac{1}{n} + n(n-1)\left(\frac{1}{n(n-1)} - \frac{1}{n^2}\right) = 1$.

3.23

(a) Since each coin is fair, the pair of coins to decide the value of Y_i can be one of $(H, T), (T, H), (H, H), (T, T)$ with equal probability. Thus, $\Pr(Y_i = 0) = \Pr(Y_i = 1) = 1/2$.
(b) Let the ith coin be denoted as C_i. Then if the first pair is C_1, C_2, the second pair is C_1, C_3 and the third pair is C_2, C_3,
$\Pr(Y_1 = Y_2 = Y_3 = 0 \neq 1/8 = \Pr(Y_1 = 1)\Pr(Y_2 = 1)\Pr(Y_3 = 1)$.
(c) Let ith pair be C_a, C_b and jth pair be C_c, C_d.
If $|\{a, b, c, d\}| = 4$, then Y_i and Y_j are independent. The claim trivially holds.
If $|\{a, b, c, d\}| = 3$, then $\mathbf{E}[Y_i Y_j] = \Pr(Y_i = Y_j = 1) = 1/4 = \mathbf{E}[X_i]\mathbf{E}[X_j]$.
(d) $\mathbf{Var}[Y] = \mathbf{Var}[\sum_{i=1}^{m} Y_i] = \sum_{i=1}^{m} \mathbf{Var}[Y_i]$ holds from the result of exercise 3.15.
Since $\mathbf{Var}[Y_i] = 1/4$, $\mathbf{Var}[Y] = \frac{n(n-1)}{8}$.
(e) By Chebyshev's inequality, $\Pr(|Y - \mathbf{E}[Y]| \geq n) \leq \frac{\mathbf{Var}[Y]}{n^2} = \frac{n-1}{8n} \leq \frac{1}{8}$.

3.24

3.25

3.26

By Chebyshev's inequality, $\Pr\left(\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right| > \epsilon\right) \leq \frac{\mathbf{Var}[\sum_{i=1}^{n} X_i/n]}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$.
Since $0 \leq \lim_{n\to\infty} \Pr\left(\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right| > \epsilon\right) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2} = 0$, the desired result is obtained by the squeeze theorem.