

Probability and Computing, 2nd Edition

Solutions to Chapter 6: The Probabilistic Method

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6.1

(a) Since all k literals must be false for a clause not to be satisfied, the probability that a clause is satisfied is $1 - 2^{-k}$. As the given SAT instance has m clauses, the expected number of satisfied clauses is $m(1 - 2^{-k})$. By the expectation argument, we know that there is an assignment that satisfies at least $m(1 - 2^{-k})$ clauses. Since counting the number of satisfied clauses in the given assignment can be done in $O(mk)$, we bound the probability p that a random assignment satisfies at least $m(1 - 2^{-k})$ clauses. Let the number of satisfied clauses be X .

$$\text{Then } m(1 - 2^{-k}) = \mathbf{E}[X] = \sum_{i < m(1 - 2^{-k})} i \Pr(X = i) + \sum_{i \geq m(1 - 2^{-k})} i \Pr(X = i) \\ \leq (m(1 - 2^{-k}) - 1)(1 - p) + mp, \text{ as } X \leq m.$$

Thus, $p \geq 1/(m2^{-k} + 1)$ holds. This indicates that the expected running time of the algorithm would be $O(mk + m^2 k 2^{-k})$.

(b) We can process each variable one by one. For each variable x , compute the conditional expectation of X while leaving all unset variables as random, for both $x = \text{True}$ or $x = \text{False}$. Then take x as the value that gives a larger conditional expectation, breaking ties arbitrarily. Since the initial expectation is $m(1 - 2^{-k})$, the assignment should satisfy at least $m(1 - 2^{-k})$ as the conditional expectation never decreases.

6.2

(a) Consider randomly assigning a color to each edge, so that each K_4 is monochromatic with a probability of $2 \times 2^{-\binom{4}{2}} = 2^{-5}$. By the linearity of expectations, the expected number of monochromatic K_4 would be $\binom{n}{4} 2^{-5}$. By the expectation argument, there exists a two-coloring of K_n where the number of monochromatic K_4 does not exceed $\binom{n}{4} 2^{-5}$.

(b) Let p be the probability that a random coloring of K_n has no more than $\binom{n}{4} 2^{-5}$ monochromatic K_4 s. Since there are $\binom{n}{4}$ K_4 s, it will take $O(n^4)$ time to check if the random coloring satisfies the statement. Let the number of monochromatic K_4 be X , and we bound p as $\binom{n}{4} 2^{-5} = \mathbf{E}[X] = \sum_{i < \binom{n}{4} 2^{-5}} i \Pr(X =$

$$i) + \sum_{i \geq \binom{n}{4} 2^{-5}} i \Pr(X = i) \leq ((\binom{n}{4} 2^{-5} - 1)p + \binom{n}{4}(1 - p)). \text{ Thus, } p \geq 1/(1 + \frac{31}{32} \binom{n}{4})$$

holds. This indicates that the expected running time of the algorithm would be $O(n^8)$.

(c) We can color each edge one by one. For each edge e_i , compute the conditional expectation of X while leaving all unset edge colors as random for both colors. Then take the color of e_i as the value that gives a smaller conditional expectation, breaking ties arbitrarily. Since the initial expectation is $\binom{n}{4} 2^{-5}$, the obtained two-coloring should have no more than $\binom{n}{4} 2^{-5}$ monochromatic K_4 s as the conditional expectation never increases.

6.3