Probability and Computing, 2nd Edition

Solutions to Chapter 5: Balls, Bins, and Random Graphs

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5.1

As $(1+1/n)^n$ increases, we find the smallest n to reach the threshold.
$(1+1/n)^n$ first reaches $0.99e$ at $n=50$, and $0.999999e$ at $n=499982$.
Since $(1-1/n)^n$ also increases, we solve in a similar way. $(1-1/n)^n$ first reaches $0.99/e$ at $n=51$ and $0.999999/e$ at $n=499991$.

5.2

Recall the formula used in the birthday paradox: If there are N possibilities, then we solve for the smallest n that satisfies $\prod_{i=1}^{n-1}(1-\frac{i}{N}) \approx \prod_{i=1}^{n-1} e^{-i/n} = e^{-(n-1)n/2N} < 1/2$. Note that we omitted the final approximation to derive exact numerical answers.
Regardless of whether the number of Social Security number digits is 9 or 13, using the last four digits gives $N=10000$ and this gives $n=119$.
In the case where the number of digits is 9 ($N=10^9$), we get $n=37234$.
In the case where the number of digits is 13 ($N=10^{13}$), we get $n=3723298$.

5.3

Let the number of balls thrown be m. Then the desired probability is $\prod_{i=0}^{m-1}(1-\frac{i}{n})$.
We first determine c_1. $m=c_1\sqrt{n}$ should satisfy $\prod_{i=0}^{m-1}(1-\frac{i}{n}) \le \prod_{i=0}^{m-1} e^{-i/n} = e^{-(m-1)m/2n} \le e^{-1}$. Since $(m-1)m = c_1^2 n - c_1\sqrt{n} \ge 2n$, $(c_1^2-2)\sqrt{n} \ge c_1$. Therefore, we choose c_1 that is greater than or equal to $\frac{1}{2}\left(\frac{1}{\sqrt{n}} + \sqrt{\frac{1}{n}+8}\right)$.
Now we determine c_2. To use the given hint, assume that $2m < n$.
$\prod_{i=0}^{m-1}(1-\frac{i}{n}) \ge \prod_{i=0}^{m-1}\exp(-\frac{i}{n}-\frac{i^2}{n^2}) = \exp(-\frac{m(m-1)}{2n} - \frac{(m-1)m(2m-1)}{6n^2})$
$= \exp(-\frac{m(m-1)}{2n}(1+\frac{2m-1}{3n})) \ge \exp(-\frac{m^2}{2n}(1+\frac{2m}{3n})) \ge \frac{1}{2}$ should be satisfied for $m=c_2\sqrt{n}$. This is equivalent to satisfying $\frac{c_2^2}{2}(1+\frac{2c_2}{3\sqrt{n}}) \le \ln 2$.
Since n is sufficiently large, choosing $c_2 = \sqrt{2\ln 2 - \frac{1}{\ln n}}$ yields the desired result.

5.4

Let event A indicate that there exist two or more people who share a birthday, and event B indicate that exactly two people share a birthday. Then our desired probability would be $\Pr(A-B) = \Pr(A) - \Pr(B)$ since $B \subset A$.
We first determine $\Pr(A)$, which is easy: $\Pr(A) = 1 - \prod_{i=1}^{100}\frac{366-i}{365}$.
We now determine $\Pr(B)$. If there are i shared birthdays in which each day is

shared by exactly two people, then the number of possible permutations would be the product of the following terms:
$\binom{365}{i}$ ways to choose i shared days, $\binom{100}{2i}$ ways to choose $2i$ people to share birthdays, $\prod_{j=1}^{i} \binom{2j}{2}$ ways to distribute i birthdays to $2i$ people and $\prod_{j=1}^{100-2i} (366 - i - j)$ ways to distribute unique birthdays to the rest.
Thus, $\Pr(B) = \sum_{i=0}^{50} \frac{365!100!}{i!(100-2i)!(265+i)!2^i} \times \frac{1}{365^{100}}$.
Therefore, we can determine our desired probability $\Pr(A - B) = \Pr(A) - \Pr(B)$.

5.5

Let $X \sim Poisson(\lambda)$. Then $M_X(t) = \mathbf{E}[e^{tX}] = e^{\lambda(e^t - 1)}$ holds. By computing the second derivative of $M_X(t)$ with respect to t and plugging $t = 0$ in, we get $\mathbf{E}[X^2] = \lambda + \lambda^2$. Thus, $\mathbf{Var}[X] = \lambda$ follows.

5.6

We first show that $Y \sim Poisson(\mu p)$.
$\Pr(Y = k) = \sum_{i=k}^{\infty} \Pr(X = i)\binom{i}{k} p^k (1-p)^{i-k} = \sum_{i=k}^{\infty} \frac{e^{-\mu}\mu^i}{i!} \frac{i!}{k!(i-k)!} p^k (1-p)^{i-k}$
$= \frac{e^{-\mu}(p\mu)^k}{k!} \sum_{i=k}^{\infty} \frac{(\mu(1-p))^{i-k}}{(i-k)!} = \frac{e^{-\mu}(p\mu)^k}{k!} e^{\mu(1-p)} = \frac{e^{-\mu p}(\mu p)^k}{k!}$.
We can also similarly show that $Z \sim Poisson(\mu(1-p))$.
Now we show that $\Pr(Y = i, Z = j) = \Pr(Y = i)\Pr(Z = j)$. Note that $X = Y + Z$ by definition. This allows us to write $\Pr(Y = i, Z = j)$ as $\Pr(Y = i, X = i + j) = \Pr(X = i + j)\binom{i+j}{i} p^i (1-p)^j = \frac{e^{-\mu}\mu^{i+j}}{i!j!} p^i (1-p)^j$.
Since $\Pr(Y = i)\Pr(Z = j) = \frac{e^{-p\mu}(p\mu)^i}{i!} \frac{e^{-(1-p)\mu}((1-p)\mu)^j}{j!} = \frac{e^{-\mu}\mu^{i+j} p^i (1-p)^j}{i!j!} = \Pr(Y = i, X = i + j)$, $Y \perp\!\!\!\perp Z$. ■

5.7

We first prove that $\ln(1 + x) \leq x$, which is equivalent to $1 + x \leq e^x$.
Since $\ln(1 + x) - x = -\frac{x^2}{2} + \frac{x^3}{3} - \cdots$, this can be seen as an alternating series as $\frac{x^n}{n}$ is monotonically decreasing in $|x| \leq 1$. We can apply rearrangements to the alternating series as $\ln(1 + x) - x = -\sum_{n=1}^{\infty} \left(\frac{x^{2n}}{2n} - \frac{x^{2n+1}}{2n+1} \right)$, since the Taylor expansion of $\ln(1 + x)$ is absolutely convergent (to $e^x - 1$). The rearrangement gives $\ln(1 + x) - x \leq 0$, which is the desired result.
We now prove $x + \ln(1 - x^2) \leq \ln(1 + x)$, which is equivalent to $e^x(1 - x^2) \leq 1 + x$. Since $\ln(1 - x^2) = \ln(1 + x) + \ln(1 - x)$, $x + \ln(1 - x^2) \leq \ln(1 + x)$ is reduced to $\ln(1 - x) \leq -x$. At $|x| \leq 1$, this is equivalent to $\ln(1 + x) \leq x$, which we have previously proved. ■

5.8