Probability and Computing, 2nd Edition

Solutions to Chapter 5: Balls, Bins, and Random Graphs

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5.1

As $(1+1/n)^n$ increases, we find the smallest n to reach the threshold.
$(1+1/n)^n$ first reaches $0.99e$ at $n=50$, and $0.999999e$ at $n=499982$.
Since $(1-1/n)^n$ also increases, we solve in a similar way. $(1-1/n)^n$ first reaches $0.99/e$ at $n=51$ and $0.999999/e$ at $n=499991$.

5.2

Recall the formula used in the birthday paradox: If there are N possibilities, then we solve for the smallest n that satisfies $\prod_{i=1}^{n-1}(1-\frac{i}{N}) \approx \prod_{i=1}^{n-1} e^{-i/n} = e^{-(n-1)n/2N} < 1/2$. Note that we omitted the final approximation to derive exact numerical answers.
Regardless of whether the number of Social Security number digits is 9 or 13, using the last four digits gives $N=10000$ and this gives $n=119$.
In the case where the number of digits is 9 ($N=10^9$), we get $n=37234$.
In the case where the number of digits is 13 ($N=10^{13}$), we get $n=3723298$.

5.3

Let the number of balls thrown be m. Then the desired probability is $\prod_{i=0}^{m-1}(1-\frac{i}{n})$.
We first determine c_1. $m=c_1\sqrt{n}$ should satisfy $\prod_{i=0}^{m-1}(1-\frac{i}{n}) \le \prod_{i=0}^{m-1} e^{-i/n} = e^{-(m-1)m/2n} \le e^{-1}$. Since $(m-1)m = c_1^2 n - c_1\sqrt{n} \ge 2n$, $(c_1^2-2)\sqrt{n} \ge c_1$. Therefore, we choose c_1 that is greater than or equal to $\frac{1}{2}\left(\frac{1}{\sqrt{n}}+\sqrt{\frac{1}{n}+8}\right)$.
Now we determine c_2. To use the given hint, assume that $2m < n$.
$\prod_{i=0}^{m-1}(1-\frac{i}{n}) \ge \prod_{i=0}^{m-1}\exp(-\frac{i}{n}-\frac{i^2}{n^2}) = \exp(-\frac{m(m-1)}{2n} - \frac{(m-1)m(2m-1)}{6n^2})$
$= \exp(-\frac{m(m-1)}{2n}(1+\frac{2m-1}{3n})) \ge \exp(-\frac{m^2}{2n}(1+\frac{2m}{3n})) \ge \frac{1}{2}$ should be satisfied for $m=c_2\sqrt{n}$. This is equivalent to satisfying $\frac{c_2^2}{2}(1+\frac{2c_2}{3\sqrt{n}}) \le \ln 2$.
Since n is sufficiently large, choosing $c_2 = \sqrt{2\ln 2 - \frac{1}{\ln n}}$ yields the desired result.

5.4