

# Probability and Computing, 2nd Edition

Solutions to Chapter 5: Balls, Bins, and Random Graphs

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## 5.1

As  $(1 + 1/n)^n$  increases, we find the smallest  $n$  to reach the threshold.  $(1 + 1/n)^n$  first reaches  $0.99e$  at  $n = 50$ , and  $0.999999e$  at  $n = 499982$ . Since  $(1 - 1/n)^n$  also increases, we solve in a similar way.  $(1 - 1/n)^n$  first reaches  $0.99/e$  at  $n = 51$  and  $0.999999/e$  at  $n = 499991$ .

## 5.2

Recall the formula used in the birthday paradox: If there are  $N$  possibilities, then we solve for the smallest  $n$  that satisfies  $\prod_{i=1}^{n-1} (1 - \frac{i}{N}) \approx \prod_{i=1}^{n-1} e^{-i/n} = e^{-(n-1)n/2N} < 1/2$ . Note that we omitted the final approximation to derive exact numerical answers.

Regardless of whether the number of Social Security number digits is 9 or 13, using the last four digits gives  $N = 10000$  and this gives  $n = 119$ .

In the case where the number of digits is 9 ( $N = 10^9$ ), we get  $n = 37234$ .

In the case where the number of digits is 13 ( $N = 10^{13}$ ), we get  $n = 3723298$ .

## 5.3

Let the number of balls thrown be  $m$ . Then the desired probability is  $\prod_{i=0}^{m-1} (1 - \frac{i}{n})$ .

We first determine  $c_1$ .  $m = c_1\sqrt{n}$  should satisfy  $\prod_{i=0}^{m-1} (1 - \frac{i}{n}) \leq \prod_{i=0}^{m-1} e^{-i/n} = e^{-(m-1)m/2n} \leq e^{-1}$ . Since  $(m-1)m = c_1^2 n - c_1\sqrt{n} \geq 2n$ ,  $(c_1^2 - 2)\sqrt{n} \geq c_1$ .

Therefore, we choose  $c_1$  that is greater than or equal to  $\frac{1}{2} \left( \frac{1}{\sqrt{n}} + \sqrt{\frac{1}{n} + 8} \right)$ .

Now we determine  $c_2$ . To use the given hint, assume that  $2m < n$ .

$\prod_{i=0}^{m-1} (1 - \frac{i}{n}) \geq \prod_{i=0}^{m-1} \exp(-\frac{i}{n} - \frac{i^2}{n^2}) = \exp(-\frac{m(m-1)}{2n} - \frac{(m-1)m(2m-1)}{6n^2})$   
 $= \exp(-\frac{m(m-1)}{2n}(1 + \frac{2m-1}{3n})) \geq \exp(-\frac{m^2}{2n}(1 + \frac{2m}{3n})) \geq \frac{1}{2}$  should be satisfied for  $m = c_2\sqrt{n}$ . This is equivalent to satisfying  $\frac{c_2^2}{2}(1 + \frac{2c_2}{3\sqrt{n}}) \leq \ln 2$ .

Since  $n$  is sufficiently large, choosing  $c_2 = \sqrt{2 \ln 2 - \frac{1}{\ln n}}$  yields the desired result.

## 5.4

Let event  $A$  indicate that there exist two or more people who share a birthday, and event  $B$  indicate that exactly two people share a birthday. Then our desired probability would be  $\Pr(A - B) = \Pr(A) - \Pr(B)$  since  $B \subset A$ .

We first determine  $\Pr(A)$ , which is easy:  $\Pr(A) = 1 - \prod_{i=1}^{100} \frac{366-i}{365}$ .

We now determine  $\Pr(B)$ . If there are  $i$  shared birthdays in which each day is

shared by exactly two people, then the number of possible permutations would be the product of the following terms:

$\binom{365}{i}$  ways to choose  $i$  shared days,  $\binom{100}{2i}$  ways to choose  $2i$  people to share birthdays,  $\prod_{j=1}^i \binom{2j}{2}$  ways to distribute  $i$  birthdays to  $2i$  people and  $\prod_{j=1}^{100-2i} (366 - i - j)$  ways to distribute unique birthdays to the rest.

Thus,  $\Pr(B) = \sum_{i=0}^{50} \frac{365!100!}{i!(100-2i)!(265+i)!2^i} \times \frac{1}{365^{100}}.$

Therefore, we can determine our desired probability  $\Pr(A-B) = \Pr(A) - \Pr(B).$

## 5.5

Let  $X \sim \text{Poisson}(\lambda)$ . Then  $M_X(t) = \mathbf{E}[e^{tX}] = e^{\lambda(e^t-1)}$  holds. By computing the second derivative of  $M_X(t)$  with respect to  $t$  and plugging  $t = 0$  in, we get  $\mathbf{E}[X^2] = \lambda + \lambda^2$ . Thus,  $\mathbf{Var}[X] = \lambda$  follows.

## 5.6