Probability and Computing, 2nd Edition

Solutions to Chapter 6: The Probabilistic Method

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6.1

(a) Since all k literals must be false for a clause not to be satisfied, the probability that a clause is satisfied is $1-2^{-k}$. As the given SAT instance has m clauses, the expected number of satisfied clauses is $m(1-2^{-k})$. By the expectation argument, we know that there is an assignment that satisfies at least $m(1-2^{-k})$ clauses. Since counting the number of satisfied clauses in the given assignment can be done in $O(mk)$, we bound the probability p that a random assignment satisfies at least $m(1-2^{-k})$ clauses. Let the number of satisfied clauses be X.
Then $m(1-2^{-k}) = \mathbf{E}[X] = \sum_{i<m(1-2^{-k})} i \Pr(X=i) + \sum_{i \geq m(1-2^{-k})} i \Pr(X=i)$
$\leq \left(m(1-2^{-k})-1\right)(1-p) + mp$, as $X \leq m$.
Thus, $p \geq 1/(m2^{-k}+1)$ holds. This indicates that the expected running time of the algorithm would be $O(mk + m^2k2^{-k})$.
(b) We can process each variable one by one. For each variable x, compute the conditional expectation of X while leaving all unset variables as random, for both $x = True$ or $x = False$. Then take x as the value that gives a larger conditional expectation, breaking ties arbitrarily. Since the initial expectation is $m(1-2^{-k})$, the assignment should satisfy at least $m(1-2^{-k})$ as the conditional expectation never decreases.

6.2

(a) Consider randomly assigning a color to each edge, so that each K_4 is monochromatic with a probability of $2 \times 2^{-\binom{4}{2}} = 2^{-5}$. By the linearity of expectations, the expected number of monochromatic K_4 would be $\binom{n}{4}2^{-5}$. By the expectation argument, there exists a two-coloring of K_n where the number of monochromatic K_4 does not exceed $\binom{n}{4}2^{-5}$.
(b) Let p be the probability that a random coloring of K_n has no more than $\binom{n}{4}2^{-5}$ monochromatic K_4s. Since there are $\binom{n}{4}$ K_4s, it will take $O(n^4)$ time to check if the random coloring satisfies the statement. Let the number of monochromatic K_4 be X, and we bound p as $\binom{n}{4}2^{-5} = \mathbf{E}[X] = \sum_{i<\binom{n}{4}2^{-5}} i \Pr(X = i) + \sum_{i \geq \binom{n}{4}2^{-5}} i \Pr(X=i) \leq \left(\binom{n}{4}2^{-5}-1\right)p + \binom{n}{4}(1-p)$. Thus, $p \geq 1/(1+\frac{31}{32}\binom{n}{4})$ holds. This indicates that the expected running time of the algorithm would be $O(n^8)$.
(c) We can color each edge one by one. For each edge e_i, compute the conditional expectation of X while leaving all unset edge colors as random for both colors. Then take the color of e_i as the value that gives a smaller conditional expectation, breaking ties arbitrarily. Computing conditional expectations at each step would take $O(n^2)$, as the number of K_4 containing a particular edge is $\binom{n-2}{2}$. As there are $\binom{n}{2}$ edges in K_n, the running time would be $O(n^4)$. Since the initial expectation is $\binom{n}{4}2^{-5}$, the obtained two-coloring should have no more than $\binom{n}{4}2^{-5}$ monochromatic K_4s as the conditional expectation never increases.

6.3