

Probability and Computing, 2nd Edition

Solutions to Chapter 5: Balls, Bins, and Random Graphs

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5.1

As $(1 + 1/n)^n$ increases, we find the smallest n to reach the threshold. $(1 + 1/n)^n$ first reaches $0.99e$ at $n = 50$, and $0.999999e$ at $n = 499982$. Since $(1 - 1/n)^n$ also increases, we solve in a similar way. $(1 - 1/n)^n$ first reaches $0.99/e$ at $n = 51$ and $0.999999/e$ at $n = 499991$.

5.2

Recall the formula used in the birthday paradox: If there are N possibilities, then we solve for the smallest n that satisfies $\prod_{i=1}^{n-1} (1 - \frac{i}{N}) \approx \prod_{i=1}^{n-1} e^{-i/n} = e^{-(n-1)n/2N} < 1/2$. Note that we omitted the final approximation to derive exact numerical answers.

Regardless of whether the number of Social Security number digits is 9 or 13, using the last four digits gives $N = 10000$ and this gives $n = 119$.

In the case where the number of digits is 9 ($N = 10^9$), we get $n = 37234$.

In the case where the number of digits is 13 ($N = 10^{13}$), we get $n = 3723298$.

5.3

Let the number of balls thrown be m . Then the desired probability is $\prod_{i=0}^{m-1} (1 - \frac{i}{n})$.

We first determine c_1 . $m = c_1\sqrt{n}$ should satisfy $\prod_{i=0}^{m-1} (1 - \frac{i}{n}) \leq \prod_{i=0}^{m-1} e^{-i/n} = e^{-(m-1)m/2n} \leq e^{-1}$. Since $(m-1)m = c_1^2 n - c_1\sqrt{n} \geq 2n$, $(c_1^2 - 2)\sqrt{n} \geq c_1$.

Therefore, we choose c_1 that is greater than or equal to $\frac{1}{2} \left(\frac{1}{\sqrt{n}} + \sqrt{\frac{1}{n} + 8} \right)$.

Now we determine c_2 . To use the given hint, assume that $2m < n$.

$\prod_{i=0}^{m-1} (1 - \frac{i}{n}) \geq \prod_{i=0}^{m-1} \exp(-\frac{i}{n} - \frac{i^2}{n^2}) = \exp(-\frac{m(m-1)}{2n} - \frac{(m-1)m(2m-1)}{6n^2})$
 $= \exp(-\frac{m(m-1)}{2n}(1 + \frac{2m-1}{3n})) \geq \exp(-\frac{m^2}{2n}(1 + \frac{2m}{3n})) \geq \frac{1}{2}$ should be satisfied for $m = c_2\sqrt{n}$. This is equivalent to satisfying $\frac{c_2^2}{2}(1 + \frac{2c_2}{3\sqrt{n}}) \leq \ln 2$.

Since n is sufficiently large, choosing $c_2 = \sqrt{2 \ln 2 - \frac{1}{\ln n}}$ yields the desired result.

5.4

Let event A indicate that there exist two or more people who share a birthday, and event B indicate that exactly two people share a birthday. Then our desired probability would be $\Pr(A - B) = \Pr(A) - \Pr(B)$ since $B \subset A$.

We first determine $\Pr(A)$, which is easy: $\Pr(A) = 1 - \prod_{i=1}^{100} \frac{366-i}{365}$.

We now determine $\Pr(B)$. If there are i shared birthdays in which each day is

shared by exactly two people, then the number of possible permutations would be the product of the following terms:

$\binom{365}{i}$ ways to choose i shared days, $\binom{100}{2i}$ ways to choose $2i$ people to share birthdays, $\prod_{j=1}^i \binom{2j}{2}$ ways to distribute i birthdays to $2i$ people and $\prod_{j=1}^{100-2i} (366 - i - j)$ ways to distribute unique birthdays to the rest.

Thus, $\Pr(B) = \sum_{i=0}^{50} \frac{365!100!}{i!(100-2i)!(265+i)!2^i} \times \frac{1}{365^{100}}$.

Therefore, we can determine our desired probability $\Pr(A-B) = \Pr(A) - \Pr(B)$.

5.5

Let $X \sim \text{Poisson}(\lambda)$. Then $M_X(t) = \mathbf{E}[e^{tX}] = e^{\lambda(e^t-1)}$ holds. By computing the second derivative of $M_X(t)$ with respect to t and plugging $t = 0$ in, we get $\mathbf{E}[X^2] = \lambda + \lambda^2$. Thus, $\mathbf{Var}[X] = \lambda$ follows.

5.6

We first show that $Y \sim \text{Poisson}(\mu p)$.

$$\begin{aligned} \Pr(Y = k) &= \sum_{i=k}^{\infty} \Pr(X = i) \binom{i}{k} p^k (1-p)^{i-k} = \sum_{i=k}^{\infty} \frac{e^{-\mu} \mu^i}{i!} \frac{i!}{k!(i-k)!} p^k (1-p)^{i-k} \\ &= \frac{e^{-\mu} (\mu p)^k}{k!} \sum_{i=k}^{\infty} \frac{(\mu(1-p))^{i-k}}{(i-k)!} = \frac{e^{-\mu} (\mu p)^k}{k!} e^{\mu(1-p)} = \frac{e^{-\mu p} (\mu p)^k}{k!}. \end{aligned}$$

We can also similarly show that $Z \sim \text{Poisson}(\mu(1-p))$.

Now we show that $\Pr(Y = i, Z = j) = \Pr(Y = i) \Pr(Z = j)$. Note that $X = Y + Z$ by definition. This allows us to write $\Pr(Y = i, Z = j)$ as $\Pr(Y = i, X = i + j) = \Pr(X = i + j) \binom{i+j}{i} p^i (1-p)^j = \frac{e^{-\mu} \mu^{i+j}}{i!j!} p^i (1-p)^j$.

Since $\Pr(Y = i) \Pr(Z = j) = \frac{e^{-\mu p} (\mu p)^i}{i!} \frac{e^{-(1-p)\mu} ((1-p)\mu)^j}{j!} = \frac{e^{-\mu} \mu^{i+j} p^i (1-p)^j}{i!j!} = \Pr(Y = i, X = i + j)$, $Y \perp\!\!\!\perp Z$. ■

5.7

We first prove that $\ln(1+x) \leq x$, which is equivalent to $1+x \leq e^x$.

Since $\ln(1+x) - x = -\frac{x^2}{2} + \frac{x^3}{3} - \dots$, this can be seen as an alternating series as $\frac{x^n}{n}$ is monotonically decreasing in $|x| \leq 1$. We can apply rearrangements to the alternating series as $\ln(1+x) - x = -\sum_{n=1}^{\infty} \left(\frac{x^{2n}}{2n} - \frac{x^{2n+1}}{2n+1} \right)$, since the Taylor expansion of $\ln(1+x)$ is absolutely convergent (to $e^x - 1$). The rearrangement gives $\ln(1+x) - x \leq 0$, which is the desired result.

We now prove $x + \ln(1-x^2) \leq \ln(1+x)$, which is equivalent to $e^x(1-x^2) \leq 1+x$. Since $\ln(1-x^2) = \ln(1+x) + \ln(1-x)$, $x + \ln(1-x^2) \leq \ln(1+x)$ is reduced to $\ln(1-x) \leq -x$. At $|x| \leq 1$, this is equivalent to $\ln(1+x) \leq x$, which we have previously proved. ■

5.8

(a) Since the ball is equally likely to fall in one of the three bins, the desired probability is $1/3$.

(b) Since the bin 2 did not receive balls, we can simply think of this as throwing balls n into $n - 1$ bins. The conditional expectation would be $n/(n - 1)$.

(c) Note that the probability that bin 1 receives more balls than bin 2 is the same as that of bin 2 receiving more balls than bin 1. Thus, we first compute the probability that two bins receive the same number of balls, which is

$$P = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} \left(\frac{1}{n}\right)^{2k} \left(1 - \frac{2}{n}\right)^{n-2k}. \text{ The desired probability would be } (1 - P)/2.$$

5.9

In the given condition, the expected number of elements in a single bucket is at most a . Since $a = O(1)$, sorting all buckets can still be done in linear time.

5.10

(a) By the Poisson approximation, the probability p is bounded as $p \leq e\sqrt{n}(\frac{1}{e})^n$.

(b) $\frac{n!}{n^n}$.

(c) Since $\Pr(Z = n) = \frac{e^{-n}n^n}{n!}$ when $Z \sim \text{Poisson}(n)$, $\frac{n!}{n^n} \times \frac{e^{-n}n^n}{n!} = e^{-n}$ shows the claim. Theorem 5.6 states that the distribution (Y_1, \dots, Y_n) constrained on $\sum_{i=1}^n Y_i = n$ is equivalent to the balls and bins model. Note that each Y_i follows $\text{Poisson}(1)$ and each X_i denotes the load of the i th bin in the balls and bins model. Then using theorem 5.6, $(1/e)^n / (\frac{e^{-n}n^n}{n!}) = \frac{\Pr(\forall i, Y_i=1)}{\Pr(\sum_i Y_i=n)} = \Pr(\forall i, Y_i = 1 | \sum_i Y_i = n) = \Pr(\forall i, X_i = 1) = \frac{n!}{n^n}$.

5.11

Let X_i be the indicator variables that are 1 if there is a k -gap starting at i .

$$\text{Let } X = \sum_{i=0}^{n-k} X_i.$$

(a) The expected number of k -gaps would be $\sum_{i=0}^{n-k} \mathbf{E}[X_i] = (n - k + 1)(1 - \frac{k}{n})^m$.

(b) First, we assume that the bin loads follow the Poisson distribution to derive the Poisson-approximated Chernoff bound. We divide $\{X_i\}$ into k subsets, so that all indicator variables in the same subset are independent. First, we derive the Chernoff bound for $Y_0 = \sum_{i \geq 0} X_{ik}$ where $0 < \delta < 1$.

$\Pr(Y_0 \geq (1+\delta)\mathbf{E}[Y_0]) \leq \exp(-\frac{1}{3}\frac{\mathbf{E}[X]}{k}\delta^2)$, and $\Pr(Y_0 \leq (1-\delta)\mathbf{E}[Y_0]) \leq \exp(-\frac{1}{2}\frac{\mathbf{E}[X]}{k}\delta^2)$ holds. Therefore, $\Pr(|Y_0 - \mathbf{E}[Y_0]| \geq \delta\mathbf{E}[Y_0]) \leq 2\exp(-\frac{1}{3}\frac{\mathbf{E}[X]}{k}\delta^2)$.

With the union bound for all Y_i , we get $\Pr(|X - \mathbf{E}[X]| \geq \delta \mathbf{E}[X]) \leq 2k \exp(-\frac{1}{3} \frac{\mathbf{E}[X]}{k} \delta^2)$. Since we have used the Poisson approximation to compute the Chernoff bound, the computed upper bound should be multiplied by $e\sqrt{m}$.

5.12

Let X_i be the indicator variables that are 1 if a ball landed in bin i by itself.

(a) The expected number of balls to be *served* in this round would be $\sum_{i=1}^n \mathbf{E}[X_i] =$

$\sum_{i=1}^n b \times \frac{1}{n} (1 - \frac{1}{n})^{b-1} = b(1 - \frac{1}{n})^{b-1}$. Therefore, the expected number of balls at the start of the next round would be $b(1 - (1 - \frac{1}{n})^{b-1})$.

(b) Note that if $n = 1$, then the number of rounds required would be trivially 1. Therefore, we only consider the cases where $n \geq 2$.

Since $x_{j+1} = x_j(1 - (1 - \frac{1}{n})^{x_j-1}) \leq x_j(1 - (1 - \frac{x_j-1}{n})) = x_j \frac{x_j-1}{n} \leq \frac{x_j^2}{n}$, the inequality given in the hint is true.

With $x_1 = n(1 - (1 - \frac{1}{n})^{n-1})$, cascading the inequality yields $x_k \leq n(\frac{x_1}{n})^{2^{k-1}} = n(1 - (1 - \frac{1}{n})^{n-1})^{2^{k-1}} \leq n(1 - (1 - \frac{1}{n})^n)^{2^{k-1}}$.

Now, let k^* be the minimum k that satisfies $n(1 - (1 - \frac{1}{n})^n)^{2^{k-1}} \leq 1$, so that the operation ends after no more than $k^* + 1$ rounds. Since $1 - (1 - \frac{1}{n})^n \leq \frac{3}{4}$ at $n \geq 2$, we calculate the minimum k that satisfies $n(\frac{3}{4})^{2^{k-1}} \leq 1$.

Taking log on both sides gives $\ln n + 2^{k^*-1} \ln \frac{3}{4} \leq 0$, which is equivalent to $2^{k^*-1} \geq \frac{\ln n}{\ln \frac{4}{3}}$. Therefore, we get $k^* - 1 \geq \ln \frac{\ln n}{\ln \frac{4}{3}}$, which shows that $k^* = O(\log \log n)$.

5.13

Let the load of bin i be X_i , and let $Y_k = X_{kn/\log_2 n}$ where $k \in \mathbb{N}_0$. Then for all i , there exists $k \in \mathbb{N}$ such that $kn/\log_2 n \leq i \leq (k+1)n/\log_2 n$.

When a ball is thrown into the bin i , only one of the two bins (bin $kn/\log_2 n$ and bin $(k+1)n/\log_2 n$) must be chosen together. This means that $X_i \leq Y_k + Y_{k+1}$. Since only one of the bins in $S = \{i | i = kn/\log_2 n\}$ gets a ball within a player's round, we can see Y_k as each bin in the model of balls and bins with $\log_2 n$ bins and $\log_2 n$ balls.

Recall that the probability that the maximum load is more than $3 \ln n / \ln \ln n$ is at most $1/n$ in the model of n balls and n bins (Lemma 5.1). Therefore, we bound the probability that the maximum load is greater than $6M = 6 \ln \log_2 n / \ln \ln \log_2 n$, considering $X_i \leq Y_k + Y_{k+1}$. Thus, we can derive the

upper bound as $\Pr(\sum_{i=0}^{n-1} \mathbf{1}_{X_i \geq 6M} > 0) \leq \Pr(\sum_{i=0}^{\log_2 n-1} \mathbf{1}_{Y_i \geq 3M} > 0) \leq \frac{1}{\log_2 n}$. Thus,

the maximum load is less than $6 \ln \log_2 n / \ln \ln \log_2 n = O(\log \log n / \log \log \log n)$ with probability $1 - \frac{1}{\log_2 n}$ which approaches 1 as $n \rightarrow \infty$.

5.14

- (a) $\Pr(Z = \mu + h) - \Pr(Z = \mu - h - 1) = \frac{e^{-\mu} \mu^{\mu+h}}{(\mu+h)!} - \frac{e^{-\mu} \mu^{\mu-h-1}}{(\mu-h-1)!} = \frac{\mu^{\mu-h}}{(\mu+h)!} (\mu^{2h} - \sum_{i=1}^h (\mu^2 - i^2))$. Since $\mu^2 > \mu^2 - i^2$, $\Pr(Z = \mu + h) - \Pr(Z = \mu - h - 1) \geq 0$ holds and the claim is proved.
- (b) $\Pr(Z \geq \mu) \geq \sum_{h=0}^{\mu-1} \Pr(Z = \mu + h) \geq \sum_{h=0}^{\mu-1} \Pr(Z = \mu - h - 1) = \Pr(Z < \mu) = 1 - \Pr(Z \geq \mu)$ shows that $2\Pr(Z \geq \mu) \geq 1$, proving the claim.
- (c) Numerical validation can show that $\Pr(Z \geq \mu) \leq 1/2$ for all integers μ from 1 to 10.

5.15

- (a) $\mathbf{E}[f(Y_1^{(m)}, \dots, Y_n^{(m)})] = \sum_{k=0}^{\infty} \mathbf{E}[f(X_1^{(k)}, \dots, X_n^{(k)})] \Pr(\sum_{i=1}^n Y_i^{(m)} = k)$ holds (recall the proof of Theorem 5.7).

If $\mu(m) = \mathbf{E}[f(X_1^{(m)}, \dots, X_n^{(m)})]$ is monotonically increasing in m , then we have $\mu(k) \geq \mu(m)$ for $k \geq m$ for some m . Thus, $\mathbf{E}[f(Y_1^{(m)}, \dots, Y_n^{(m)})] = \sum_{k < m} \mu(k) \Pr(\sum_{i=1}^n Y_i^{(m)} = k) + \sum_{k \geq m} \mu(k) \Pr(\sum_{i=1}^n Y_i^{(m)} = k)$ ($f \geq 0$)

$$\geq \sum_{k \geq m} \mu(m) \Pr(\sum_{i=1}^n Y_i^{(m)} = k) = \mathbf{E}[f(X_1^{(m)}, \dots, X_n^{(m)})] \Pr(\sum Y_i^{(m)} \geq m).$$

Similarly, if $\mu(m)$ is monotonically decreasing in m , then we have $\mu(k) \geq \mu(m)$ for $k \leq m$ for some m . Thus,

$$\begin{aligned} \mathbf{E}[f(Y_1^{(m)}, \dots, Y_n^{(m)})] &= \sum_{k \leq m} \mu(k) \Pr(\sum_{i=1}^n Y_i^{(m)} = k) + \sum_{k > m} \mu(k) \Pr(\sum_{i=1}^n Y_i^{(m)} = k) \\ &\geq \sum_{k \leq m} \mu(m) \Pr(\sum_{i=1}^n Y_i^{(m)} = k) = \mathbf{E}[f(X_1^{(m)}, \dots, X_n^{(m)})] \Pr(\sum Y_i^{(m)} \leq m). \end{aligned}$$

- (b) Since the sum of independent Poisson random variables is also a Poisson random variable, $\sum Y_i^{(m)} \sim \text{Poisson}(m)$. Using the result of exercise 5.14.(b) and 5.15.(a), $\mathbf{E}[f(Y_1^{(m)}, \dots, Y_n^{(m)})] \geq \mathbf{E}[f(X_1^{(m)}, \dots, X_n^{(m)})] \Pr(\sum Y_i^{(m)} \geq m) \geq \mathbf{E}[f(X_1^{(m)}, \dots, X_n^{(m)})] \times \frac{1}{2}$. Thus, Theorem 5.10 is proved for the monotonically increasing case.

If one can derive a formal proof on the statement of exercise 5.14.(c), Theorem 5.10 can also be proved for the monotonically decreasing case.

5.16

- (a) Expectations are computed as $\mathbf{E}[X_1 X_2 \cdots X_k] = \Pr(X_1 X_2 \cdots X_k = 1) = (1 - \frac{k}{n})^n$ and $\mathbf{E}[Y_1 Y_2 \cdots Y_k] = \Pr(Y_1 Y_2 \cdots Y_k = 1) = p^k = (1 - \frac{1}{n})^{k \times n}$. Since $1 - \frac{k}{n} \leq (1 - \frac{1}{n})^k$ by the Bernoulli inequality, $\mathbf{E}[X_1 X_2 \cdots X_k] \leq \mathbf{E}[Y_1 Y_2 \cdots Y_k]$.

(b) $\mathbf{E}[e^{tX}] = \mathbf{E}\left[\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right] = \sum_{k=0}^{\infty} \mathbf{E}\left[\frac{(tX)^k}{k!}\right]$ holds since \mathbb{N} is countable, and the same applies to $\mathbf{E}[e^{tY}]$. Therefore, we show $\mathbf{E}[X^k] \leq \mathbf{E}[Y^k]$ for $\forall k \in \mathbb{N}$. $\mathbf{E}[X^k] = \mathbf{E}[(X_1 + X_2 + \dots + X_n)^k]$ holds, and $(X_1 + X_2 + \dots + X_n)^k$ is the sum of products where each product is in the form of $X_{i_1}X_{i_2} \dots X_{i_k}$ ($|\{i_1, i_2, \dots, i_k\}| \leq k$). Since X_i are indicator variables, $X_i^n = X_i$ holds for $\forall n \in \mathbb{N}$, so the repeats can be ignored from the product. Suppose that $|\{i_1, i_2, \dots, i_k\}| = m$. Then the result of 5.16.(a) applies with $k = m$. Thus, $\mathbf{E}[X^k] \leq \mathbf{E}[Y^k]$ holds and, therefore, $\mathbf{E}[e^{tX}] \leq \mathbf{E}[e^{tY}]$ holds.

(c) $\Pr(X \geq (1 + \delta)\mu) \leq \frac{\mathbf{E}[e^{tX}]}{e^{t(1+\delta)\mu}} \leq \frac{\mathbf{E}[e^{tY}]}{e^{t(1+\delta)\mu}}$ holds. Since $\mathbf{E}[e^{tY}] = (1 - p + pe^t)^n$, we can choose $t = \ln(1 + \delta)$ to derive $\Pr(X \geq (1 + \delta)\mu) \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu$.

5.17

(a) Since there are $\binom{n}{5}$ possible 5-cliques and each 5-clique has $\binom{5}{2} = 10$ edges, the expected number of 5-cliques is $\binom{n}{5} \times p^{10}$. Solving $\binom{n}{5} \times p^{10} = 1$ for p gives $p = \binom{n}{5}^{-1/10}$.

(b) Since there are $\frac{1}{2} \binom{n}{6} \binom{6}{3}$ possible $K_{3,3}$ subgraphs and each $K_{3,3}$ subgraph has nine edges, the expected number of $K_{3,3}$ subgraphs is $\frac{1}{2} \binom{n}{6} \binom{6}{3} \times p^9$. Solving $\frac{1}{2} \binom{n}{6} \binom{6}{3} \times p^9 = 1$ for p gives $p = \left\{\frac{1}{2} \binom{n}{6} \binom{6}{3}\right\}^{-1/9}$.

(c) Since there are $(n-1)!/2$ possible Hamiltonian cycles and each Hamiltonian cycle has n edges, the expected number of Hamiltonian cycles is $\frac{1}{2}(n-1)! \times p^n$. Solving $\frac{1}{2}(n-1)! \times p^n = 1$ for p gives $p = \left\{\frac{1}{2}(n-1)!\right\}^{-1/n}$.

5.18