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Linear response of the hydrogen atom in Stark states to a harmonic uniform electric field

Tudor A. Marian
Faculty of Physics, University of Bucharest, P.O. Box MG-11, Bucharest-Măgurele, 76900, Romania
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The influence of a weak harmonic uniform electric field, switched on adiabatically, on a nonrelativistic hydrogenlike atom is examined. Each of the ϕ- and $\mathbf{A}$-gauge first-order corrections to the wave function of a stationary state $|N\rangle$ is determined by a vector function that we denote $\mathbf{v}_N$ and $\mathbf{w}_N$, respectively. The absolute starting point of our calculations is Schwinger's formula for the Coulomb Green's function in momentum space. In the case of a bound state with definite angular momentum, we report a compact integral representation and also an explicit expression of the ϕ-gauge vector $\mathbf{v}_{nlm}$, which are analogous to those of the corresponding $\mathbf{A}$-gauge vector $\mathbf{w}_{nlm}$ studied previously. We have derived compact analytic expressions of the linear-response vectors $\mathbf{v}_{n_\xi n_\eta m}$ and $\mathbf{w}_{n_\xi n_\eta m}$ associated to an arbitrary Stark state. These are written first as contour integrals, and then explicitly in terms of a new generalized hypergeometric function with five variables, ${}_2\phi_H$, which is a finite sum of Humbert functions ϕ_1. We have calculated the static limit of the regular part of the vector $\mathbf{v}_{n_\xi n_\eta m}$. Also discussed are the Sturmian-function expansions of the linear-response vectors for angular momentum states.

I. INTRODUCTION

The problem of finding the linear response of the hydrogen atom to a weak electric field

$$\mathscr{E}(t)=\mathscr{E}_0\cos(\omega t) \tag{1}$$

consists of solving the Schrödinger equation by time-dependent perturbation theory, to first order. This problem, with an initial stationary $|nlm\rangle$ state, was introduced by Podolsky a long time ago.[1] Working in the $\mathbf{A}$ gauge, Podolsky wrote the inhomogeneous second-order differential equations for the two space-dependent factors entering into the first-order correction to the wave function, $\psi^{(1)}_{nlm}(\omega;r,\theta,\varphi,t)$. Then, he solved explicitly these equations only in the ground-state case, determining what is now usually called the Sturmian-function expansion of the correction $\psi^{(1)}_{100}(\omega;\mathbf{r},t)$. A closed form of the same correction was found much later and only for field frequencies below the photoionization threshold.[2]

Recently,[3] using the Coulomb Green's function (CGF), we have established *compact* analytic expressions of the function $\psi^{(1)}_{nlm}(\omega;\mathbf{r},t)$ for an *arbitrary* $|nlm\rangle$ state and with *no* restriction on the frequency ω. The present paper will add to our previous formulas in the $\mathbf{A}$ gauge the similar ones in the ϕ gauge. However, its main purpose is to report *complete* results concerning the linear modification of *any* stationary Stark state $|n_\xi n_\eta m\rangle$. We have obtained these principal results by two different methods. The first one is an elegant *direct* method nontrivially related to that used in the case of a $|nlm\rangle$ state: as the latter is merely mentioned in Ref. 3, both derivations will be described thoroughly in a separate paper. Here we shall briefly explain the second method, which is an *indirect* one, starting from the linear-response vectors in spherical coordinates, $\mathbf{v}_{nlm}$ and $\mathbf{w}_{nlm}$. As a rule, mathematical details are avoided throughout this paper.

In Sec. II we show that for any stationary state $|N\rangle$, the ϕ-gauge first-order correction $\widetilde{\Psi}^{(1)}_N$ to the wave function is expressed in terms of a vector function $\mathbf{v}_N$, while the corresponding $\mathbf{A}$-gauge correction $\psi^{(1)}_N$ is determined by a related vector function $\mathbf{w}_N$. We conclude this section with a discussion concerning the static limit of the linear-response problem. Section III deals with the ϕ-gauge vector $\mathbf{v}_{nlm}$, corresponding to an angular momentum state, in close analogy with our investigation, in Ref. 3, of the similar $\mathbf{A}$-gauge vector $\mathbf{w}_{nlm}$. In Sec. IV we derive compact integral representations of the linear-response vectors $\mathbf{v}_{n_\xi n_\eta m}$ and $\mathbf{w}_{n_\xi n_\eta m}$, associated to an arbitrary Stark state, while in Sec. V their explicit expressions are given. Section VI reports the static limit of the regular part of the vector $\mathbf{v}_{n_\xi n_\eta m}$. In Sec. VII we analyze first the Sturmian-function expansion of the linear-response vectors $\mathbf{v}_{nlm}$ and $\mathbf{w}_{nlm}$, complementing a very recent letter by Maquet *et al.*[4] Then we summarize the results which constitute a basis for subsequent applications. Appendix A presents certain useful expansions of the vectors $\mathbf{r}u_{nlm}$, $\mathbf{P}u_{nlm}$, $\mathbf{r}u_{n_\xi n_\eta m}$, and $\mathbf{P}u_{n_\xi n_\eta m}$, built up with the standard energy eigenfunctions $u_{nlm}(r,\theta,\varphi)$ and $u_{n_\xi n_\eta m}(\xi,\eta,\varphi)$. In Appendix B, we point out an alternative form of the linear-response vectors in spherical coordinates.

II. FIRST-ORDER PERTURBED WAVE FUNCTIONS

We consider a bound electron in the Coulomb field of a nucleus of charge Ze, fixed at the origin of the coordinate system. The external field (1) is described by potentials chosen either in the ϕ gauge,

$$\widetilde{\phi}(\mathbf{r},t)=-\mathscr{E}_0\cdot\mathbf{r}\cos(\omega t), \quad \widetilde{\mathbf{A}}=\mathbf{0}\ , \tag{2}$$

or in the **A** gauge,

$$\phi=0,\quad \mathbf{A}(t)=-\frac{c}{\omega}\mathscr{E}_0\sin(\omega t)\ . \tag{3}$$

The generating function of the gauge transformation from (2) to (3) is

$$\mathscr{f}(\mathbf{r},t)=-\frac{c}{\omega}\mathscr{E}_0\cdot\mathbf{r}\sin(\omega t)\ . \tag{4}$$

The field-atom interaction Hamiltonians in the two gauges are, respectively,

$$\tilde{H}^{(1)}=e\mathscr{E}\cdot\mathbf{r} \tag{5}$$

and

$$H^{(1)}=\frac{e}{m_e c}\mathbf{A}\cdot\mathbf{P} \tag{6}$$

(linear approximation), with m_e denoting the electron mass and $\mathbf{P}$ its momentum operator. We suppose that the electric field (1) is turned on adiabatically, with an exponential switching factor in the time interval $(-\infty,0)$. In the remote past $(t\to-\infty)$, the electron is assumed to be in a stationary state,

$$\psi_N^{(0)}(\mathbf{r},t)=\exp\left[-\frac{i}{\hbar}E_n t\right]u_N(\mathbf{r})\ , \tag{7}$$

belonging to a *complete* orthonormal set of energy eigenstates. We have denoted by N the corresponding quantum numbers: for the angular momentum states, $N=\{n,l,m\}$, while for the Stark states, $N=\{n_\xi,n_\eta,m\}$, with $n_\xi+n_\eta+|m|+1=n$.

Then, Dirac's perturbation theory gives the first-order corrections to the wave function (7), for $t\geq 0$, in the gauges (2) and (3),

$$\tilde{\psi}_N^{(1)}(\omega;\mathbf{r},t)=-\frac{e}{2}\exp\left[-\frac{i}{\hbar}E_n t\right]\times\mathscr{E}_0\cdot[\exp(-i\omega t)\mathbf{v}_N(\Omega_1;\mathbf{r})+\exp(i\omega t)\mathbf{v}_N(\Omega_2;\mathbf{r})]\ , \tag{8}$$

and

$$\psi_N^{(1)}(\omega;\mathbf{r},t)=-\frac{e}{2im_e}\exp\left[-\frac{i}{\hbar}E_n t\right]\times\mathscr{E}_0\cdot\left[\frac{1}{\omega+i0}\exp(-i\omega t)\mathbf{w}_N(\Omega_1;\mathbf{r})-\frac{1}{\omega-i0}\exp(i\omega t)\mathbf{w}_N(\Omega_2;\mathbf{r})\right]\ . \tag{9}$$

The vector functions $\mathbf{v}_N$ and $\mathbf{w}_N$ are defined by equations

$$\mathbf{v}_N(\Omega;\mathbf{r})\equiv\sum_{N'}\frac{\langle u_{N'}|\mathbf{r}u_N\rangle}{E_{n'}-\Omega}u_{N'}(\mathbf{r})\ , \tag{10}$$

and

$$\mathbf{w}_N(\Omega;\mathbf{r})\equiv\sum_{N'}\frac{\langle u_{N'}|\mathbf{P}u_N\rangle}{E_{n'}-\Omega}u_{N'}(\mathbf{r})\ , \tag{11}$$

where the summations are extended over the complete set of Coulomb energy eigenfunctions in discussion. The parameters Ω_1 and Ω_2 in Eqs. (8) and (9) are

$$\Omega_1=E_n+\hbar\omega+i0,\quad \Omega_2=E_n-\hbar\omega\ . \tag{12}$$

As a consequence of the familiar identity

$$\langle u_{N'}|\mathbf{P}u_N\rangle=\frac{im_e}{\hbar}(E_{n'}-E_n)\langle u_{N'}|\mathbf{r}u_N\rangle \tag{13}$$

and of the completeness of the system $\{u_N(\mathbf{r})\}$, the vectors (10) and (11), which entirely determine the linear-response corrections (8) and (9), are connected by the simple relation

$$\frac{\hbar}{im_e}\mathbf{w}_N(\Omega;\mathbf{r})=u_N(\mathbf{r})\mathbf{r}+(\Omega-E_n)\mathbf{v}_N(\Omega;\mathbf{r})\ . \tag{14}$$

Equation (14) results in the gauge transformation of the wave function, to first order, generated by the function (4),

$$\psi_N^{(1)}(\omega;\mathbf{r},t)=\tilde{\psi}_N^{(1)}(\omega;\mathbf{r},t)+\frac{ie}{\hbar\omega}\mathscr{E}_0\cdot\mathbf{r}\sin(\omega t)\psi_N^{(0)}(\mathbf{r},t)\ . \tag{15}$$

Both the Hamiltonian (6) and the solution (9) show the **A** gauge not to be appropriate for taking the static limit $(\omega=0)$ of the linear-response problem. Even the ϕ-gauge correction (8) becomes divergent in this limit, due to the contribution to the sum (10) of the nth energy eigensubspace $\mathscr{U}_n$ of the unperturbed Coulomb Hamiltonian.

We define the vector

$$\mathbf{v}'_N(\Omega;\mathbf{r})\equiv{\sum_{N'}}'\frac{\langle u_{N'}|\mathbf{r}u_N\rangle}{E_{n'}-\Omega}u_{N'}(\mathbf{r})\ , \tag{16}$$

where the prime on the summation sign means exclusion of the eigenstates $|N'\rangle$ corresponding to the Bohr level E_n. The function (16) has a finite limit for $\Omega=E_n$, so that we can distinguish the *regular part* of the correction (8),

$$\tilde{\psi}_N^{\prime(1)}(\omega;\mathbf{r},t)=-\frac{e}{2}\exp\left[-\frac{i}{\hbar}E_n t\right]\times\mathscr{E}_0\cdot[\exp(-i\omega t)\mathbf{v}'_N(\Omega_1;\mathbf{r})+\exp(i\omega t)\mathbf{v}'_N(\Omega_2;\mathbf{r})]\ , \tag{17}$$

from its *secular terms* with $n'=n$.[5] In the case $N=\{n,l,m\}$, the function (17), with a regular behavior in the low-frequency range, has been used in evaluating the dynamic electric dipole polarizability.[6] This physical quantity should really have a well-defined static limit, but the way of simply projecting the eigensubspace $\mathscr{U}_n$ out of the solution (8) is questionable as long as one cannot explain the origin of the secular terms. It may be adopted as a conjecture which, however, is justified when directing the z axis along the external field and using accordingly the Stark states. Indeed, then the operator (5) has a diagonal matrix in the eigensubspace $\mathcal{U}_n$ and, like in Ref. 5, the secular term in Eq. (8) originates in a phase factor whose argument is proportional to the first-order dynamic level shift. Moreover, as required by the adiabatic theorem,[7] the function (17) multiplied by the abovementioned phase factor has a static limit that coincides with the first-order correction to the wave function in the static case, given by stationary perturbation theory.

III. THE VECTOR $\mathbf{v}_{nlm}(\Omega;r,\theta,\varphi)$

In Ref. 3 we have established closed-form analytic expressions of the vectors $\mathbf{w}_{nlm}(\Omega;r,\theta,\varphi)$ and $\mathbf{v}'_{nlm}(E_n;r,\theta,\varphi)$. Here we give compact formulas for the ϕ-gauge function $\mathbf{v}_{nlm}(\Omega;r,\theta,\varphi)$. We have obtained them by the method applied in the case of the $\mathbf{A}$-gauge vector $\mathbf{w}_{nlm}$, with the only difference of using Eq. (A8) instead of Eq. (A9) in the final stage of the derivation.

Substitution of Eq. (A3) into Eq. (10), written with $N=\{n,l,m\}$, yields the vector $\mathbf{v}_{nlm}$ as a linear combination of two vector spherical harmonics, for $l>0$,

$$\mathbf{v}_{nlm}(\Omega;r,\theta,\varphi)=\sum_{q=1,-1}(-q)\left[\frac{|\lambda_{l+q,l}|}{2l+1}\right]^{1/2}\mathcal{A}_{n\,l\,l+q}(\Omega;r)\mathbf{V}_{l+q\,l\,m}(\theta,\varphi)\quad(l>0)\ , \tag{18}$$

and as proportional to $\mathbf{r}$, for $l=0$,

$$\mathbf{v}_{n00}(\Omega;r,\theta,\varphi)=(4\pi)^{-1/2}\mathcal{A}_{n01}(\Omega;r)\frac{\mathbf{r}}{r}\ . \tag{19}$$

Taking into account the definition (A5) of the symbol $\lambda_{l+q,l}$, we notice that Eq. (18) reduces for $l=0$ to Eq. (19). The scalar functions $\mathcal{A}_{n\,l\,l+q}(\Omega;r)$ in Eqs. (18) and (19) ($q=\pm1$, if $l>0$, and $q=1$, if $l=0$) have the following radial eigenfunction expansion:

$$\mathcal{A}_{n\,l\,l+q}(\Omega;r)=\sum\!\!\!\!\!\!\int_{n'}\frac{R_{nl}^{n'\,l+q}}{E_{n'}-\Omega}R_{n'\,l+q}(r)\ . \tag{20}$$

The spectral sum in Eq. (20) is extended over the negative energy eigenvalues with $n'>l+q$ and over all the positive energies. $R_{nl}(r)$ are the Coulomb radial energy eigenfunctions: for $E=E_n<0$, they have the expression (A2), while for $E>0$, they are normalized in the energy scale. We also specify the usual notation

$$R_{nl}^{n'\,l+q}\equiv\int_0^\infty dr\,r^3R_{n'\,l+q}(r)R_{nl}(r)\ , \tag{21}$$

and remark that the eigenfunction expansion (20) is similar to that of the radial function $\mathcal{B}_{n\,l\,l+q}(\Omega;r)$,[8] introduced in Ref. 3.

Our first result concerning the radial function $\mathcal{A}_{n\,l\,l+q}(\Omega;r)$ is the following integral representation:

$$\begin{aligned}\mathcal{A}_{n\,l\,l+q}(\Omega;r)=&\frac{(2\kappa_n)^{1/2}}{2|E_n|}\frac{4n\tau}{[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\\&\times\sum_{s=-2}^{2}c_{n,l}^{(q,s)}\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\int_1^{(0+)}d\rho\,\rho^{-\tau}\exp\left[-\frac{\mathcal{N}_{\tau,n}}{\mathcal{N}_{n,\tau}}\frac{X}{\hbar}r\right]\mathcal{N}_{n,-\tau}^{n-1-s}\mathcal{N}_{n,\tau}^{-n-1+s}(2\kappa_{n,\tau}r)^{l+q}\\&\times{}_1F_1(l+q+1+s-n,2(l+q)+2;2\kappa_{n,\tau}r)\ ,\end{aligned} \tag{22}$$

where

$$\kappa_n\equiv\frac{Z}{na_0}\ , \tag{23}$$

with a_0 the Bohr radius, and

$$X\equiv(-2m_e\Omega)^{1/2}\quad(\mathrm{Re}X>0),\quad \tau\equiv\frac{\hbar\kappa_1}{X}\ , \tag{24}$$

$$\mathcal{N}_{\tau',\tau}\equiv\tau'+\tau+(\tau'-\tau)\rho\ , \tag{25}$$

$$\kappa_{n,\tau}\equiv\frac{4n^2\rho}{\mathcal{N}_{n,-\tau}\mathcal{N}_{n,\tau}}\kappa_n\ . \tag{26}$$

In Eq. (22), as well as in the following, the integration path in the complex ρ plane starts at $\rho=1$ (where one should take $\rho^{-\tau}=1$), encircles the origin $\rho=0$ in the counterclockwise sense and then returns to $\rho=1$. The coefficients $c_{n,l}^{(q,s)}$ are those introduced by Eq. (A6) and given in Table I. Moreover, the confluent hypergeometric functions in Eq. (22) are exactly the same as in Eq. (A6), with the only difference that they depend on the variable $2\kappa_{n,\tau}r$ instead of $2\kappa_n r$. It follows that they are proportional to Laguerre polynomials, just as in Eq. (A6).

Secondly, the function $\mathcal{A}_{n\,l\,l+q}(\Omega;r)$ may be expressed in terms of the generalized hypergeometric function $\phi_H\equiv{}_1\phi_H$, defined in Ref. 3, by its series expansion.[9] It is important to mention here an alternative definition of ${}_1\phi_H$:

$$
{}_1\phi_H(a;b,a_1,c_1;c;x,x',y,z) \equiv (1-x)^{-a} \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \frac{-ie^{-i\pi a}}{2\sin(\pi a)}
\times \int_1^{(0+)} d\rho\, \rho^{a-1}(1-\rho)^{c-a-1} \exp\left[\frac{y\rho/(1-x)}{1+x\rho/(1-x)}\right]\left[1+\frac{x}{1-x}\rho\right]^{b-c}
\times \left[1-\frac{x'}{1-x}\rho\right]^{-a_1} {}_1F_1\left[a_1,c_1;\frac{z\rho/(1-x)}{[1+x\rho/(1-x)][1-x'\rho/(1-x)]}\right], \tag{27}
$$

where $a \neq 1,2,3,\ldots$, a_1 is a nonpositive integer, $\mathrm{Re}(c-a)>0$, and $x \neq 1$. In virtue of Eq. (27), the function (22) may be written as

$$
\mathcal{A}_{n\,l\,l+q}(\Omega;r) = \frac{(2\kappa_n)^{1/2}}{2|E_n|} \frac{1}{[2(l+q)+1]!} \left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2} \frac{\tau}{n}(2\kappa_n r)^{l+q} \exp\left[-\frac{X}{\hbar}r\right]
\times \sum_{s=-2}^{2} c_{n,l}^{(q,s)} \left[\frac{n-\tau}{2n}\right]^{n-(l+q)-1-s} \left[\frac{n+\tau}{2n}\right]^{-n+s-\tau} \frac{1}{l+q+1-\tau}
\times {}_1\phi_H\left[l+q+1-\tau; -n-\tau+1+s, l+q+1+s-n, 2(l+q)+2; l+q+2-\tau; \frac{n-\tau}{2n}, -\frac{(n+\tau)^2}{2n(n-\tau)}, \frac{n-\tau}{2\tau}2\kappa_n r, \frac{2n}{n-\tau}2\kappa_n r\right]. \tag{28}
$$

Equation (28) is similar to the explicit expression of the radial function $\mathcal{B}_{n\,l\,l+q}(\Omega;r)$.[10]

If in Eq. (14) we set $N=\{n,l,m\}$ and then decompose the occurring vectors in terms of vector spherical harmonics, we get the identity[11]

$$
\mathcal{B}_{n\,l\,l+q}(\Omega;r) = rR_{nl}(r) - (E_n - \Omega)\mathcal{A}_{n\,l\,l+q}(\Omega;r). \tag{29}
$$

Starting from the integral representation (22) and the similar one of the function $\mathcal{B}_{n\,l\,l+q}(\Omega;r)$,[12] we have *checked* Eq. (29) after a lengthy calculation.

We denote by $\mathcal{A}'_{n\,l\,l+q}(\Omega;r)$ the regular part of the function $\mathcal{A}_{n\,l\,l+q}(\Omega;r)$. From its eigenfunction expansion[13] it follows that the secular term of the radial function (20) is

$$
\mathcal{A}_{n\,l\,l+q}(\Omega;r) - \mathcal{A}'_{n\,l\,l+q}(\Omega;r)
= \begin{cases} \dfrac{3}{2\kappa_n} \dfrac{(n^2-\lambda_{l+q,l}^2)^{1/2}}{\Omega - E_n} R_{n\,l+q}(r), & l+q<n, \\ 0, & l+q=n. \end{cases} \tag{30}
$$

TABLE I. The coefficients $c_{n,l}^{(q,s)}$ in the expansion (A8) of the vector $\mathbf{r}u_{nlm}(r,\theta,\varphi)$.

q	s	$c_{n,l}^{(q,s)}$
1	-2	$-\frac{1}{2}(l+1+n)(l+2+n)(l+3+n)$
1	-1	$-(l+1+n)(l+2+n)(l-2n)$
1	0	$3n(l+1+n)(l+1-n)$
1	1	$(l+1-n)(l+2-n)(l+2n)$
1	2	$\frac{1}{2}(l+1-n)(l+2-n)(l+3-n)$
-1	-2	$-\frac{1}{2}(l+1+n)$
-1	-1	$l+1+2n$
-1	0	$-3n$
-1	1	$-(l+1-2n)$
-1	2	$\frac{1}{2}(l+1-n)$

It is also easy to recover Eq. (30) from Eq. (28).

IV. THE VECTORS $\mathbf{v}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)$ AND $\mathbf{w}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)$ AS CONTOUR INTEGRALS

We have used two distinct methods in deriving compact integral representations of the linear-response vectors (10) and (11) corresponding to the Stark states.

(i) A direct method which, thanks to adequate changes, parallels the procedure developed previously for the angular momentum states. The starting point in both cases is an integral representation of a certain generating function.[14] In turn, this has been derived formerly,[15] making use of Schwinger's formula for the CGF in momentum space.[16] We stress the usefulness of Eqs. (A12) and, respectively, (A13) in obtaining our final expressions of the vectors $\mathbf{v}_{n_\xi n_\eta m}$ and $\mathbf{w}_{n_\xi n_\eta m}$.

(ii) An indirect method, exploiting the linearity of Park's relation between the energy eigenfunctions in parabolic and spherical coordinates,[17]

$$
u_{n_\xi n_\eta m}(\xi,\eta,\varphi) = (-1)^{n_\xi + (1/2)(m+|m|)} \times \sum_{l=|m|}^{n-1} \langle Jm_1, Jm_2 | JJ, lm \rangle \times u_{nlm}(r,\theta,\varphi). \tag{31}
$$

In Eq. (31), the energy eigenfunctions have the usual expressions (A1), (A2) and (A10), (A11), while the quantum numbers J, m_1, and m_2 occurring in the Clebsch-Gordan coefficients are defined as

$$
J = \tfrac{1}{2}(n-1), \quad m_{1,2} = \tfrac{1}{2}[m \mp (n_\xi - n_\eta)].
$$

Taking into account the definitions (10) and (11), Eq. (31) yields the vector identity

$$\mathbf{v}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)=(-1)^{n_\xi+(1/2)(m+|m|)}\times\sum_{l=|m|}^{n-1}\langle Jm_1,Jm_2|JJ,lm\rangle\times\mathbf{v}_{nlm}(\Omega;r,\theta,\varphi)\ ,\qquad(32)$$

and a similar one connecting the vectors $\mathbf{w}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)$ and $\mathbf{w}_{nlm}(\Omega;r,\theta,\varphi)$. Now, substitution of Eq. (B1) into Eq. (32) allows us to apply once more the linear transformation (31) *without* making explicit use of its coefficients. In this way, we get a closed form of the function $\mathbf{v}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)$ analogous to (B1),

$$\mathbf{v}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)=\frac{1}{\kappa_n|E_n|}n\tau\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\int_1^{(0+)}d\rho\,\rho^{-\tau}\mathcal{N}_{n,-\tau}^{n-3}\mathcal{N}_{n,\tau}^{-n-3}\exp\left[-\frac{\mathcal{N}_{\tau,n}}{\mathcal{N}_{n,\tau}}\frac{X}{2\hbar}(\xi+\eta)\right]\times\mathcal{R}_{n,\tau}\left[\exp[\tfrac{1}{2}\kappa_{n,\tau}(\xi+\eta)]u_{n_\xi n_\eta m}\left[\frac{\kappa_{n,\tau}}{\kappa_n}\xi,\frac{\kappa_{n,\tau}}{\kappa_n}\eta,\varphi\right]\right]\ .\qquad(33)$$

Similar operations with Eq. (B3) lead to a parallel expression of the function $\mathbf{w}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)$,

$$\mathbf{w}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)=i\frac{\hbar\kappa_n}{|E_n|}n\tau\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\int_1^{(0+)}d\rho\,\rho^{-\tau}\mathcal{N}_{n,-\tau}^{n-3}\mathcal{N}_{n,\tau}^{-n-3}\exp\left[-\frac{\mathcal{N}_{\tau,n}}{\mathcal{N}_{n,\tau}}\frac{X}{2\hbar}(\xi+\eta)\right]\times\mathcal{P}_{n,\tau}\left[\exp[\tfrac{1}{2}\kappa_{n,\tau}(\xi+\eta)]u_{n_\xi n_\eta m}\left[\frac{\kappa_{n,\tau}}{\kappa_n}\xi,\frac{\kappa_{n,\tau}}{\kappa_n}\eta,\varphi\right]\right]\ .\qquad(34)$$

It is useful to recall at this moment the contravariant spherical components of an arbitrary vector $\mathbf{v}$,

$$v^{-1}=\frac{1}{\sqrt{2}}(v_x+iv_y),\quad v^0=v_z,\quad v^1=-\frac{1}{\sqrt{2}}(v_x-iv_y)\ ,\qquad(35a)$$

as well as the covariant ones,[18]

$$v_\mu=(-1)^\mu v^{-\mu}\quad(\mu=-1,0,1)\ .\qquad(35b)$$

Indeed, taking into account Eqs. (B2) and (B4), we have found it convenient to write explicitly, in parabolic coordinates, the components (35a) of the vectors (33) and (34). Then, we have carried out an appropriate algebra involving the confluent hypergeometric function.[19] Here we write down the final formulas, given also, in a more straightforward way, by our method (i),

$$\begin{aligned}v_{n_\xi n_\eta m;}{}^{\mu}(\Omega;\xi,\eta,\varphi)=&(-1)^{(1/2)(\mu+|\mu|)}2^{-|\mu|/2}\frac{(2\kappa_n)^{1/2}}{2|E_n|}\frac{4n\tau}{[(|m-\mu|)!]^2}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n2\pi}\right]^{1/2}e^{i(m-\mu)\varphi}\\&\times\sum_{\tilde M=0,M}\sum_{s=-2}^{2}(1-\tfrac{1}{2}\delta_{M-\tilde M+s,\tilde M})\\&\times\Bigg[c_{n_\xi,n_\eta,|m|}^{(M,\tilde M,s)}\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\\&\times\int_1^{(0+)}d\rho\,\rho^{-\tau}\exp\left[-\frac{\mathcal{N}_{\tau,n}}{\mathcal{N}_{n,\tau}}\frac{X}{2\hbar}(\xi+\eta)\right]\mathcal{N}_{n,-\tau}^{n-1-s}\mathcal{N}_{n,\tau}^{-n-1+s}(\kappa_{n,\tau}^2\xi\eta)^{|m-\mu|/2}\\&\times{}_1F_1(-n_\xi+M-\tilde M+s,|m-\mu|+1;\kappa_{n,\tau}\xi)\\&\times{}_1F_1(-n_\eta+\tilde M,|m-\mu|+1;\kappa_{n,\tau}\eta)-(-1)^\mu(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)\Bigg]\ ,\end{aligned}\qquad(36)$$

and

$$w_{n_\xi n_\eta m;}{}^{\mu}(\Omega;\xi,\eta,\varphi)=(-1)^{(1/2)(\mu+|\mu|)}2^{-|\mu|/2}\frac{im_e}{\hbar}(2\kappa_n)^{1/2}\frac{4n\tau}{[(|m-\mu|)!]^2}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n2\pi}\right]^{1/2}e^{i(m-\mu)\varphi}$$
$$\times\sum_{\tilde{M}=0,M}\sum_{s=-1,1}(1-\tfrac{1}{2}\delta_{M-\tilde{M}+s,\tilde{M}})$$
$$\times\left[d^{(M,\tilde{M},s)}_{n_\xi,n_\eta,|m|}\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\right.$$
$$\times\int_1^{(0+)}d\rho\,\rho^{-\tau}\exp\left[-\frac{\mathcal{N}_{\tau,n}}{\mathcal{N}_{n,\tau}}\frac{X}{2\hbar}(\xi+\eta)\right]\mathcal{N}^{n-1-s}_{n,-\tau}\mathcal{N}^{-n-1+s}_{n,\tau}(\kappa^2_{n,\tau}\xi\eta)^{|m-\mu|/2}$$
$$\times{}_1F_1(-n_\xi+M-\tilde{M}+s,|m-\mu|+1;\kappa_{n,\tau}\xi)$$
$$\left.\times{}_1F_1(-n_\eta+\tilde{M},|m-\mu|+1;\kappa_{n,\tau}\eta)-(-1)^{\mu}(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)\right]. \quad (37)$$

In Eqs. (36) and (37), besides the notations (23)–(26), we have used the parameter (A14) and the symbol $(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)$, explained at the end of Appendix A. The coefficients $c^{(M,\tilde{M},s)}_{n_\xi,n_\eta,|m|}$ and $d^{(M,\tilde{M},s)}_{n_\xi,n_\eta,|m|}$ are precisely those occurring in Eqs. (A12) and (A13): they are listed in Tables II and III. Furthermore, also the functions ${}_1F_1$ entering Eqs. (36) and (37) are those from Eqs. (A12) and (A13) with the variables changed by the substitution $\kappa_n\to\kappa_{n,\tau}$. Consequently, all of them are proportional to Laguerre polynomials.

V. EXPLICIT FORM OF THE VECTORS $\mathbf{v}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)$ AND $\mathbf{w}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)$

In order to write compactly, in explicit form, the linear-response vectors (36) and (37), we introduce a generalized hypergeometric function denoted ${}_2\phi_H$ and given by the following contour integral:

$${}_2\phi_H(a;b,a_1,a_2,c_1,c_2;c,x,x',y,z_1,z_2)\equiv(1-x)^{-a}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\frac{-ie^{-i\pi a}}{2\sin(\pi a)}$$
$$\times\int_1^{(0+)}d\rho\,\rho^{a-1}(1-\rho)^{c-a-1}\exp\left[\frac{y\rho/(1-x)}{1+x\rho/(1-x)}\right]$$
$$\times\left[1+\frac{x}{1-x}\rho\right]^{b-c}\left[1-\frac{x'}{1-x}\rho\right]^{-(a_1+a_2)}$$
$$\times{}_1F_1\left[a_1,c_1;\frac{z_1\rho/(1-x)}{[1+x\rho/(1-x)][1-x'\rho/(1-x)]}\right]$$
$$\times{}_1F_1\left[a_2,c_2;\frac{z_2\rho/(1-x)}{[1+x\rho/(1-x)][1-x'\rho/(1-x)]}\right], \quad (38)$$

where $a\neq 1,2,3,\ldots$, a_1 and a_2 are nonpositive integers, $\mathrm{Re}(c-a)>0$, and $x\neq 1$. The hypergeometric function (38), with seven parameters and five variables, has a structure similar to that of ${}_1\phi_H$, Eq. (27). In explicit form it is a finite sum of Humbert functions ϕ_1:[20]

$${}_2\phi_H(a;b,a_1,a_2,c_1,c_2;c;x,x',y,z_1,z_2)=\sum_{\nu'=0}^{\infty}\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}\frac{(a)_{\nu'+\nu_1+\nu_2}(a_1)_{\nu_1}(a_2)_{\nu_2}(a_1+a_2+\nu_1+\nu_2)_{\nu'}}{(c)_{\nu'+\nu_1+\nu_2}(c_1)_{\nu_1}(c_2)_{\nu_2}\nu'!\nu_1!\nu_2!}$$
$$\times x'^{\nu'}z_1^{\nu_1}z_2^{\nu_2}\phi_1(a+\nu'+\nu_1+\nu_2,b+\nu',c+\nu'+\nu_1+\nu_2;x,y), \quad (39)$$

with $(a)_\nu$ standing for Pochhammer's symbol.

Now, using Eq. (38), the ϕ-gauge vector (36) may be written explicitly as

$$
\begin{aligned}
v_{n_\xi n_\eta m;}{}^{\mu}(\Omega;\xi,\eta,\varphi) ={}& (-1)^{(1/2)(\mu+|\mu|)}2^{-|\mu|/2}\frac{(2\kappa_n)^{1/2}}{2|E_n|}\frac{\tau}{n}\frac{1}{[(|m-\mu|)!]^2} \\
&\times\left[\frac{(n_\xi+|m|!(n_\eta+|m|)!}{n_\xi!n_\eta!4n2\pi}\right]^{1/2} e^{i(m-\mu)\varphi}\exp\left[-\frac{X}{2\hbar}(\xi+\eta)\right](\kappa_n^2\xi\eta)^{|m-\mu|/2} \\
&\times\sum_{\tilde{M}=0,M}\sum_{s=-2}^{2}(1-\tfrac{1}{2}\delta_{M-\tilde{M}+s,\tilde{M}})\left[\frac{n-\tau}{2n}\right]^{n-|m-\mu|-1-s}\left[\frac{n+\tau}{2n}\right]^{-n+s-\tau}\frac{1}{|m-\mu|+1-\tau} \\
&\times\Bigg[c^{(M,\tilde{M},s)}_{n_\xi,n_\eta,|m|}\;{}_2\phi_H\Bigg[|m-\mu|+1-\tau;-n-\tau+1+s,-n_\xi+M-\tilde{M}+s\;, \\
&\qquad -n_\eta+\tilde{M},|m-\mu|+1,|m-\mu|+1; \\
&\qquad |m-\mu|+2-\tau;\frac{n-\tau}{2n},-\frac{(n+\tau)^2}{2n(n-\tau)}\;, \\
&\qquad \frac{n-\tau}{2\tau}\kappa_n(\xi+\eta),\frac{2n}{n-\tau}\kappa_n\xi,\frac{2n}{n-\tau}\kappa_n\eta\Bigg] \\
&-(-1)^{\mu}(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)\Bigg]\;. \qquad (40)
\end{aligned}
$$

We notice that each function ${}_2\phi_H$ in Eq. (40) has only four independent parameters. A similar formula holds for the **A**-gauge vector (37).

One can extract the secular terms of the function $v_{n_\xi n_\eta m;}{}^{\mu}$ either directly from the eigenfunction expansion (10) or from the power series expansion of Eq. (40). Both procedures yield the formula

$$
\begin{aligned}
v_{n_\xi n_\eta m;}{}^{\mu}(\Omega;\xi,\eta,\varphi)-v'_{n_\xi n_\eta m;}{}^{\mu}(\Omega;\xi,\eta,\varphi) ={}& (-1)^{(1/2)(\mu+|\mu|)}2^{-|\mu|/2}\frac{(2\kappa_n)^{1/2}}{\Omega-E_n}\frac{1}{[(|m-\mu|)!]^2} \\
&\times\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n2\pi}\right]^{1/2} e^{i(m-\mu)\varphi}e^{-(1/2)\kappa_n(\xi+\eta)}(\kappa_n^2\xi\eta)^{|m-\mu|/2}(1-\tfrac{1}{2}\delta_{M,0}) \\
&\times[C^{(-1;M)}_{n_\xi,n_\eta,|m|}\;{}_1F_1(-n_\xi+M,|m-\mu|+1;\kappa_n\xi) \\
&\times{}_1F_1(-n_\eta,|m-\mu|+1;\kappa_n\eta)-(-1)^{\mu}(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)]\;, \qquad (41)
\end{aligned}
$$

with the coefficients $C^{(-1;M)}_{n_\xi,n_\eta,|m|}$ given in Table IV.

VI. THE VECTOR $\mathbf{v}'_{\eta_\xi n_\eta m}(E_n;\xi,\eta,\varphi)$

As shown by Eq. (17) written in the *static* limit $\omega=0$, the vector $\mathbf{v}'_{n_\xi n_\eta m}(E_n;\xi,\eta,\varphi)$ determines the regular part of the ϕ-gauge first-order correction $\tilde{\psi}^{(1)}_{n_\xi n_\eta m}(0;\xi,\eta,\varphi,t)$ to a Stark-state wave function. We have derived its explicit expression in three independent ways.

(1) Starting from Eqs. (B5) and (B6), and then handling Eq. (31) as pointed out in Sec. IV, when we have presented our indirect method (ii) of getting the linear-response vectors in parabolic coordinates.

(2) Collecting from the power-series expansion of the function (40) all the terms that neither vanish nor become infinite as $\tau\rightarrow n$.

(3) On account of the relation

$$
\frac{\hbar}{im_e}[\mathbf{w}_N(E_n+\delta\Omega;\mathbf{r})-\mathbf{w}_N(E_n;\mathbf{r})]=\delta\Omega\mathbf{v}'_N(E_n;\mathbf{r})+O[(\delta\Omega)^2]\;, \qquad (42)
$$

which is a consequence of Eqs. (10) and (14), collecting from the power-series expansion of the function $w_{n_\xi n_\eta m;}{}^{\mu}(\Omega;\xi,\eta,\varphi)$ all the terms proportional to $\delta\tau\equiv\tau-n$.

In each of these three methods, the operations mentioned above are followed eventually by adequate algebraic transformations involving the function ${}_1F_1$.[19] Our final result is

$$v'_{n_\xi n_\eta m;}{}^{\mu}(E_n;\xi,\eta,\varphi)=(-1)^{(1/2)(\mu+|\mu|)}2^{-|\mu|/2}\frac{(2\kappa_n)^{1/2}}{4|E_n|}\frac{1}{[(|m-\mu|)!]^2}$$
$$\times\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi!n_\eta!4n2\pi}\right]^{1/2}e^{i(m-\mu)\varphi}e^{-(1/2)\kappa_n(\xi+\eta)}(\kappa_n^2\xi\eta)^{|m-\mu|/2}$$
$$\times\Bigg[\Bigg[-2\delta_{\mu,0}(|m|+1+n_\xi)n_\eta\,{}_1F_1(-n_\xi-1,|m|+1;\kappa_n\xi)\,{}_1F_1(-n_\eta+1,|m|+1;\kappa_n\eta)$$
$$+\sum_{\tilde M=0,M}\sum_{s=-2}^{2}(1-\tfrac{1}{2}\delta_{M-\tilde M+s,\tilde M})C^{(0;M,\tilde M,s)}_{n_\xi,n_\eta,|m|}\,{}_1F_1(-n_\xi+M-\tilde M+s,|m-\mu|+1;\kappa_n\xi)$$
$$\times\,{}_1F_1(-n_\eta+\tilde M,|m-\mu|+1;\kappa_n\eta)\Bigg]-(-1)^\mu(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)\Bigg]\,, \qquad (43)$$

where the coefficients $C^{(0;M,\tilde M,s)}_{n_\xi,n_\eta,|m|}$ are the integer numbers listed in Table V. Accordingly, each function ${}_1F_1$ in Eq. (43) is proportional to a Laguerre polynomial.

VII. DISCUSSION AND SUMMARY

It is instructive to derive the Sturmian-function expansions of the scalar radial functions $\mathcal{A}_{n\,l\,l+q}(\Omega;r)$, Eq. (20), and $\mathcal{B}_{n\,l\,l+q}(\Omega;r)$.[8] First we recall that the radial CGF,

$$g_l(\Omega;r,r')=-\sum_{n''}\!\!\!\!\!\!\int\;\frac{R_{n''l}(r)R_{n''l}(r')}{E_{n''}-\Omega}\,, \qquad (44)$$

may be expanded in terms of the radial Coulomb Sturmian functions,[21,22]

$$S_{n_rl}(\Omega;r)=\frac{2Z^{1/2}}{ea_0}\frac{1}{\tau}\frac{1}{(2l+1)!}\left[\frac{(n_r+2l+1)!}{n_r!}\right]^{1/2}$$
$$\times e^{-(X/\hbar)r}\left[2\frac{X}{\hbar}r\right]^l{}_1F_1\left[-n_r,2l+2;2\frac{X}{\hbar}r\right]\,, \qquad (45)$$

according to Hostler's formula,[23]

$$g_l(\Omega;r,r')=\sum_{\nu=0}^{\infty}\frac{-\tau}{\nu+l+1-\tau}S_{\nu l}(\Omega;r)S_{\nu l}(\Omega;r')\,. \qquad (46)$$

By virtue of Eqs. (21) and (44), the ϕ-gauge radial function (20) may be written alternatively

$$\mathcal{A}_{n\,l\,l+q}(\Omega;r)=-\int_0^\infty dr'\,r'^3g_{l+q}(\Omega;r,r')R_{nl}(r')\,. \qquad (47)$$

From Eqs. (46) and (47) we get the Sturmian-function expansion

$$\mathcal{A}_{n\,l\,l+q}(\Omega;r)=\sum_{\nu=0}^{\infty}\frac{\tau}{\nu+l+q+1-\tau}$$
$$\times\left[\int_0^\infty dr'\,r'^3S_{\nu\,l+q}(\Omega;r')\right.$$
$$\left.\times R_{nl}(r')\right]S_{\nu\,l+q}(\Omega;r)\,, \qquad (48)$$

TABLE II. The coefficients $c^{(M,\tilde M,s)}_{n_\xi,n_\eta,|m|}$ in the expansion (A12) of the vector $\mathbf{r}u_{n_\xi n_\eta m}(\xi,\eta,\varphi)$.

M	$\tilde M$	s	$c^{(M,\tilde M,s)}_{n_\xi,n_\eta,\lvert m\rvert}$
0	0	-2	$\frac{1}{2}(\lvert m\rvert+1+n_\xi)(\lvert m\rvert+2+n_\xi)$
0	0	-1	$-(\lvert m\rvert+1+n_\xi)(\lvert m\rvert+2+2n_\xi)$
0	0	0	$3n(n_\xi-n_\eta)$
0	0	1	$-n_\xi(\lvert m\rvert+2n_\xi)$
0	0	2	$\frac{1}{2}n_\xi(n_\xi-1)$
1	0	-2	$-(\lvert m\rvert+1+n_\xi)(\lvert m\rvert+2+n_\xi)(\lvert m\rvert+1+n_\eta)$
1	0	-1	$(3n-\lvert m\rvert+1)(\lvert m\rvert+1+n_\xi)(\lvert m\rvert+1+n_\eta)$
1	0	0	$-3nn_\xi(\lvert m\rvert+1+n_\eta)$
1	0	1	$n_\xi(n_\xi-1)(\lvert m\rvert+1+n_\eta)$
1	0	2	0
1	1	-2	0
1	1	-1	$(\lvert m\rvert+1+n_\xi)(\lvert m\rvert+2+n_\xi)n_\eta$
1	1	0	0
1	1	1	$(3n+\lvert m\rvert-1)n_\xi n_\eta$
1	1	2	$-n_\xi(n_\xi-1)n_\eta$
-1	0	-2	0
-1	0	-1	$\lvert m\rvert+1+n_\xi$
-1	0	0	$-3n$
-1	0	1	$3n-\lvert m\rvert-1$
-1	0	2	$-n_\xi$
-1	-1	-2	$-(\lvert m\rvert+1+n_\xi)$
-1	-1	-1	$3n+\lvert m\rvert+1$
-1	-1	0	0
-1	-1	1	n_ξ
-1	-1	2	0

TABLE III. The coefficients $d^{(M,\tilde M,s)}_{n_\xi,n_\eta,|m|}$ in the expansion (A13) of the vector $\mathbf{P}u_{n_\xi n_\eta m}(\xi,\eta,\varphi)$.

M	$\tilde M$	s	$d^{(M,\tilde M,s)}_{n_\xi,n_\eta,\lvert m\rvert}$
0	0	-1	$-(\lvert m\rvert+1+n_\xi)$
0	0	1	n_ξ
1	0	-1	$2(\lvert m\rvert+1+n_\xi)(\lvert m\rvert+1+n_\eta)$
1	0	1	0
1	1	-1	0
1	1	1	$-2n_\xi n_\eta$
-1	0	-1	0
-1	0	1	-2
-1	-1	-1	2
-1	-1	1	0

which involves a discrete sum only, unlike the eigenfunction expansion (20). Now, the remaining radial integration in Eq. (48) can be readily performed by using Eq. (A6) and a known integral,[24]

$$\mathcal{A}_{n\,l\,l+q}(\Omega;r)=\frac{ea_0}{2Z^{1/2}}\tau\frac{(2\kappa_n)^{1/2}}{2|E_n|}\frac{1}{[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\left[\frac{4n\tau}{(n-\tau)(n+\tau)}\right]^{l+q+1}$$
$$\times\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu+l+q+1-\tau}\left[\frac{[\nu+2(l+q)+1]!}{\nu!}\right]^{1/2}$$
$$\times\left[\sum_{s=-2}^{2}c_{n,l}^{(q,s)}\left[\frac{n-\tau}{n+\tau}\right]^{\nu+n-s}{}_2F_1\left[-\nu,l+q+1+s-n;2(l+q)+2;-\frac{4n\tau}{(n-\tau)^2}\right]\right]S_{\nu\,l+q}(\Omega;r)\;. \tag{49}$$

A similar reasoning leads to the following expansion of the **A**-gauge radial function $\mathcal{B}_{n\,l\,l+q}(\Omega;r)$:

$$\mathcal{B}_{n\,l\,l+q}(\Omega;r)=\frac{ea_0}{2Z^{1/2}}\tau(2\kappa_n)^{1/2}\frac{1}{[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\left[\frac{4n\tau}{(n-\tau)(n+\tau)}\right]^{l+q+1}$$
$$\times\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu+l+q+1-\tau}\left[\frac{[\nu+2(l+q)+1]!}{\nu!}\right]^{1/2}$$
$$\times\left[\sum_{s=-1,1}d_{n,l}^{(q,s)}\left[\frac{n-\tau}{n+\tau}\right]^{\nu+n-s}{}_2F_1\left[-\nu,l+q+1+s-n;2(l+q)+2;-\frac{4n\tau}{(n-\tau)^2}\right]\right]S_{\nu\,l+q}(\Omega;r)\;, \tag{50}$$

where the coefficients $d_{n,l}^{(q,s)}$ are introduced by Eq. (A7) and listed in Table VI. Note that all the Gauss hypergeometric functions ${}_2F_1$ entering the expansions (49) and (50) are polynomials of bounded degree.

First, we remark that Eq. (49) is equivalent to the Sturmian-expansion formula derived by Maquet *et al.*,[4] but, in our opinion, is better suited for a systematic study. Second, in the important particular case of the ground state, Eq. (50) becomes

$$\mathcal{B}_{101}(\Omega;r)=\frac{2^4}{3}\kappa_1^{3/2}\frac{\tau}{(1+\tau)^4}e^{-(X/\hbar)r}r\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu+2-\tau}(\nu+1)(\nu+2)(\nu+3)\left[\frac{1-\tau}{1+\tau}\right]^\nu{}_1F_1\left[-\nu,4;2\frac{X}{\hbar}r\right]\;. \tag{51}$$

Substitution of Eq. (51) via

$$\mathbf{w}_{100}(\Omega;\mathbf{r})=\frac{im_e}{\hbar}(4\pi)^{-1/2}\mathcal{B}_{101}(\Omega;r)\frac{\mathbf{r}}{r}$$

into Eq. (9) written for the ground-state case, with $\mathcal{E}_0=\mathcal{E}_0\mathbf{e}_z$, yields Podolsky's classic result.[25]

Now, it is worth stressing the following point. Starting from the series expansion of the function ${}_1\phi_H$,[9] we have succeeded (after performing carefully a nontrivial hypergeometric-function algebra) to cast it in the form

$$\frac{1}{l+q+1-\tau}\,{}_1\phi_H\left[l+q+1-\tau;-n-\tau+1+s,l+q+1+s-n,2(l+q)+2;l+q+2-\tau;\frac{n-\tau}{2n}\,,\right.$$
$$\left.-\frac{(n+\tau)^2}{2n(n-\tau)},\frac{n-\tau}{2\tau}2\kappa_n r,\frac{2n}{n-\tau}2\kappa_n r\right]$$
$$=\frac{1}{[2(l+q)+1]!}\left[\frac{n+\tau}{2n}\right]^{\tau-(l+q)-1}$$
$$\times\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu+l+q+1-\tau}\frac{[\nu+2(l+q)+1]!}{\nu!}\left[\frac{n-\tau}{n+\tau}\right]^\nu$$
$$\times\,{}_2F_1\left[-\nu,l+q+1+s-n;2(l+q)+2;-\frac{4n\tau}{(n-\tau)^2}\right]{}_1F_1\left[-\nu,2(l+q)+2;2\frac{X}{\hbar}r\right]\;. \tag{52}$$

TABLE IV. The coefficients $C^{(-1;M)}_{n_\xi,n_\eta,|m|}$ in the explicit expression of the secular terms of the vector $\mathbf{v}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)$, Eq. (41).

M	$C^{(-1;M)}_{n_\xi,n_\eta,\lvert m\rvert}$
0	$-3(n_\xi-n_\eta)$
1	$3n_\xi(\lvert m\rvert+1+n_\eta)$
-1	3

TABLE VI. The coefficients $d^{(q,s)}_{n,l}$ in the expansion (A9) of the vector $\mathbf{P}u_{nlm}(r,\theta,\varphi)$.

q	s	$d^{(q,s)}_{n,l}$
1	-1	$(l+1+n)(l+2+n)$
1	1	$-(l+1-n)(l+2-n)$
-1	-1	1
-1	1	-1

By introducing Eq. (52) into our compact formulas, Eq. (28) of the present work and Eq. (28) of Ref. 3, we recover the Sturmian-function expansions (49) and (50), respectively.

To sum up, we have pointed out that the linear modification of an arbitrary energy eigenstate $|N\rangle$ of a hydrogenic atom due to a weak harmonic electric field is determined in the ϕ and $\mathbf{A}$ gauges by the vectors $\mathbf{v}_N(\Omega;\mathbf{r})$, Eq. (10), and, respectively, $\mathbf{w}_N(\Omega;\mathbf{r})$, Eq. (11). The function $\mathbf{v}_{nlm}(\Omega;r,\theta,\varphi)$ is a linear combination of two vector spherical harmonics, Eq. (18), whose coefficients are proportional to the scalar functions $\mathcal{A}_{n\,l\,l+q}(\Omega;r)$ $(q=\pm1)$, Eq. (20). We have written the radial function $\mathcal{A}_{n\,l\,l+q}(\Omega;r)$ in closed form, first as a contour integral, Eq. (22), and then explicitly, Eq. (28); its Sturmian-function expansion, Eq. (49), is finally established. Our main results are the integral representations (36) and (37) of the linear-response vectors $\mathbf{v}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)$ and $\mathbf{w}_{n_\xi n_\eta m}(\Omega;\xi,\eta,\varphi)$ for an arbitrary Stark state. We have also derived their explicit expressions [Eq. (40) and a similar formula for $\mathbf{w}_{n_\xi n_\eta m}$], as well as that of the vector $\mathbf{v}'_{n_\xi n_\eta m}(E_n;\xi,\eta,\varphi)$, connected with the static-field case [Eq. (43)].

In a subsequent paper,[24] we apply our compact integral representations of the linear-response vectors in evaluating the Kramers-Heisenberg matrix elements for bound-bound transitions of hydrogenlike atoms. Closely related quantities, such as dynamic electric dipole polarizabilities and bound-free two-photon transition amplitudes are under investigation and their expressions will be reported and analyzed elsewhere.

TABLE V. The coefficients $C^{(0;M,\tilde{M},s)}_{n_\xi,n_\eta,|m|}$ in the explicit expression of the vector $\mathbf{v}'_{n_\xi n_\eta m}(E_n;\xi,\eta,\varphi)$, Eq. (43).

M	$\tilde{M}$	s	$C^{(0;M,\tilde{M},s)}_{n_\xi,n_\eta,\lvert m\rvert}$
0	0	-2	$\frac{1}{2}(\lvert m\rvert+1+n_\xi)(\lvert m\rvert+2+n_\xi)$
0	0	-1	$-(\lvert m\rvert+1+n_\xi)(3n-2n_\xi-\lvert m\rvert+1)$
0	0	0	$-3(n_\xi-n_\eta)$
0	0	1	$n_\xi(3n-2n_\xi-\lvert m\rvert-3)$
0	0	2	$-\frac{1}{2}n_\xi(n_\xi-1)$
1	0	-2	$-(\lvert m\rvert+1+n_\xi)(\lvert m\rvert+2+n_\xi)(\lvert m\rvert+1+n_\eta)$
1	0	-1	$(3n+\lvert m\rvert+5)(\lvert m\rvert+1+n_\xi)(\lvert m\rvert+1+n_\eta)$
1	0	0	$n_\xi(\lvert m\rvert+1+n_\eta)(2n_\xi-2n_\eta+2\lvert m\rvert+3)$
1	0	1	$n_\xi(n_\xi-1)(\lvert m\rvert+1+n_\eta)$
1	0	2	0
1	1	-2	0
1	1	-1	$-(\lvert m\rvert+1+n_\xi)(\lvert m\rvert+2+n_\xi)n_\eta$
1	1	0	0
1	1	1	$-(3n-\lvert m\rvert-5)n_\xi n_\eta$
1	1	2	$n_\xi(n_\xi-1)n_\eta$
-1	0	-2	0
-1	0	-1	$-(\lvert m\rvert+1+n_\xi)$
-1	0	0	$-(2n_\xi-2n_\eta+2\lvert m\rvert-3)$
-1	0	1	$-(3n+\lvert m\rvert-5)$
-1	0	2	n_ξ
-1	-1	-2	$-(\lvert m\rvert+1+n_\xi)$
-1	-1	-1	$3n-\lvert m\rvert+5$
-1	-1	0	0
-1	-1	1	n_ξ
-1	-1	2	0

APPENDIX A: FINITE EXPANSIONS OF THE VECTORS $\mathbf{r}u_{nlm}$, $\mathbf{P}u_{nlm}$, $\mathbf{r}u_{n_\xi n_\eta m}$, AND $\mathbf{P}u_{n_\xi n_\eta m}$

We treat first the case of the eigenfunctions in spherical coordinates

$$u_{nlm}(r,\theta,\varphi)=R_{nl}(r)Y_{lm}(\theta,\varphi)\ , \tag{A1}$$

with

$$R_{nl}(r)=\frac{1}{(2l+1)!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}(2\kappa_n)^{3/2}\times e^{-\kappa_n r}(2\kappa_n r)^l\,{}_1F_1(l+1-n,2l+2;2\kappa_n r)\ , \tag{A2}$$

and $Y_{lm}(\theta,\varphi)$ denoting a usual spherical harmonic.[26] We start from the well-known formulas[27]

$$\frac{\mathbf{r}}{r}Y_{lm}(\theta,\varphi)=\sum_{q=1,-1}(-q)\left[\frac{|\lambda_{l+q,l}|}{2l+1}\right]^{1/2}\mathbf{V}_{l+q\,l\,m}(\theta,\varphi) \tag{A3}$$

and

$$\nabla[R_{nl}(r)Y_{lm}(\theta,\varphi)]=\sum_{q=1,-1}(-q)\left[\frac{|\lambda_{l+q,l}|}{2l+1}\right]^{1/2}\times\left[\left[\frac{d}{dr}+\frac{1+\lambda_{l+q,l}}{r}\right]R_{nl}(r)\right]\mathbf{V}_{l+q\,l\,m}(\theta,\varphi)\ , \tag{A4}$$

where $V_{ljm}(\theta,\varphi)$ are the vector spherical harmonics[28] and the symbol $\lambda_{l+q,l}$ has the values

$$\lambda_{l+q,l}=\begin{cases}-(l+1), & q=1\\ l, & q=-1\ .\end{cases} \tag{A5}$$

Combining some recursion relations for the confluent hypergeometric function ${}_1F_1$,[19] we get the following expansion formulas:

$$rR_{nl}(r)=\frac{1}{[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\frac{(2\kappa_n)^{1/2}}{\kappa_n r}e^{-\kappa_n r}(2\kappa_n r)^{l+q}$$
$$\times\sum_{s=-2}^{2}c_{n,l}^{(q,s)}\,{}_1F_1(l+q+1+s-n,2(l+q)+2;2\kappa_n r),\quad q=\begin{cases}\pm1 & \text{for } l>0\\ 1 & \text{for } l=0\end{cases} \tag{A6}$$

$$\left[\frac{d}{dr}+\frac{1+\lambda_{l+q,l}}{r}\right]R_{nl}(r)=-\frac{1}{[2(l+q)+1]!}\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}\frac{(2\kappa_n)^{1/2}\kappa_n}{r}e^{-\kappa_n r}(2\kappa_n r)^{l+q}$$
$$\times\sum_{s=-1,1}d_{n,l}^{(q,s)}\,{}_1F_1(l+q+1+s-n,2(l+q)+2;2\kappa_n r),\quad q=\begin{cases}\pm1 & \text{for } l>0\\ 1 & \text{for } l=0\ .\end{cases} \tag{A7}$$

The coefficients $c_{n,l}^{(q,s)}$ in Eq. (A6) and $d_{n,l}^{(q,s)}$ in Eq. (A7) are listed in Tables I and VI, respectively. An inspection of Table I shows that while $c_{n,l}^{(-1,-2)}$ and $c_{n,l}^{(-1,2)}$ are either integers or half-odd integers, the other eight coefficients $c_{n,l}^{(q,s)}$ have only integer values. We mention that, as a consequence of Eq. (A7), the coefficients $d_{n,l}^{(q,s)}$, all of them integers, occur also in Eq. (20) of Ref. 3, where they are written differently, as $d_{n,l}^{(l+q,-s)}$. Now, by substituting Eq. (A6) into Eq. (A3) and Eq. (A7) into Eq. (A4), we get the contravariant spherical components (35a) of the vectors $\mathbf{r}u_{nlm}$ and $\mathbf{P}u_{nlm}$,

$$x^{\mu}u_{nlm}(r,\theta,\varphi)=\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}(2\kappa_n)^{1/2}\frac{1}{\kappa_n r}e^{-\kappa_n r}$$
$$\times\sum_{q=1,-1}(-q)\left[\frac{|\lambda_{l+q,l}|}{2l+1}\right]^{1/2}\frac{\langle l+q\ m-\mu,1\mu|l+q\ 1,lm\rangle}{[2(l+q)+1]!}(2\kappa_n r)^{l+q}Y_{l+q\ m-\mu}(\theta,\varphi)$$
$$\times\sum_{s=-2}^{2}c_{n,l}^{(q,s)}\,{}_1F_1(l+q+1+s-n,2(l+q)+2;2\kappa_n r)\quad(\mu=-1,0,1) \tag{A8}$$

and

$$P^{\mu}u_{nlm}(r,\theta,\varphi)=\left[\frac{(n+l)!}{(n-l-1)!2n}\right]^{1/2}(2\kappa_n)^{1/2}\frac{i\hbar\kappa_n}{r}e^{-\kappa_n r}$$
$$\times\sum_{q=1,-1}(-q)\left[\frac{|\lambda_{l+q,l}|}{2l+1}\right]^{1/2}\frac{\langle l+q\ m-\mu,1\mu|l+q\ 1,lm\rangle}{[2(l+q)+1]!}(2\kappa_n r)^{l+q}Y_{l+q\ m-\mu}(\theta,\varphi)$$
$$\times\sum_{s=-1,1}d_{n,l}^{(q,s)}\,{}_1F_1(l+q+1+s-n,2(l+q)+2;2\kappa_n r)\quad(\mu=-1,0,1)\ . \tag{A9}$$

The second case we are dealing with is that of the eigenfunctions in parabolic coordinates

$$u_{n_\xi n_\eta m}(\xi,\eta,\varphi)=\frac{(2\kappa_n)^{3/2}}{2n^{1/2}}f_{n_\xi|m|}(\kappa_n\xi)f_{n_\eta|m|}(\kappa_n\eta)(2\pi)^{-1/2}e^{im\varphi}\ , \tag{A10}$$

where

$$f_{\nu|m|}(\zeta)\equiv\frac{1}{(|m|)!}\left[\frac{(\nu+|m|)!}{\nu!}\right]^{1/2}e^{(-1/2)\zeta}\zeta^{|m|/2}\,{}_1F_1(-\nu,|m|+1;\zeta)\ . \tag{A11}$$

When applying the recursion relations mentioned before, we finally obtain the following decompositions of the contravariant spherical components (35a) of the vectors $\mathbf{r}u_{n_\xi n_\eta m}$ and $\mathbf{P}u_{n_\xi n_\eta m}$:

$$x^{\mu}u_{n_\xi n_\eta m}(\xi,\eta,\varphi)=(-1)^{(1/2)(\mu+|\mu|)}2^{-|\mu|/2}\frac{1}{[(|m-\mu|)!]^2}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi! n_\eta! 4n\,2\pi}\right]^{1/2}$$
$$\times(2\kappa_n)^{1/2}\frac{2}{\kappa_n(\xi+\eta)}e^{-(1/2)\kappa_n(\xi+\eta)}(\kappa_n^2\xi\eta)^{|m-\mu|/2}e^{i(m-\mu)\varphi}$$
$$\times\sum_{\tilde{M}=0,M}\sum_{s=-2}^{2}(1-\tfrac{1}{2}\delta_{M-\tilde{M}+s,\tilde{M}})[c^{(M,\tilde{M},s)}_{n_\xi,n_\eta,|m|}\ {}_1F_1(-n_\xi+M-\tilde{M}+s,|m-\mu|+1;\kappa_n\xi)$$
$$\times\ {}_1F_1(-n_\eta+\tilde{M},|m-\mu|+1;\kappa_n\eta)$$
$$-(-1)^{\mu}(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)]\quad(\mu=-1,0,1)\ ,\tag{A12}$$

and

$$P^{\mu}u_{n_\xi n_\eta m}(\xi,\eta,\varphi)=(-1)^{(1/2)(\mu+|\mu|)}2^{-|\mu|/2}\frac{1}{[(|m-\mu|)!]^2}\left[\frac{(n_\xi+|m|)!(n_\eta+|m|)!}{n_\xi! n_\eta! 4n\,2\pi}\right]^{1/2}$$
$$\times(2\kappa_n)^{3/2}\frac{i\hbar}{\xi+\eta}e^{-(1/2)\kappa_n(\xi+\eta)}(\kappa_n^2\xi\eta)^{|m-\mu|/2}e^{i(m-\mu)\varphi}$$
$$\times\sum_{\tilde{M}=0,M}\sum_{s=-1,1}(1-\tfrac{1}{2}\delta_{M-\tilde{M}+s,\tilde{M}})[d^{(M,\tilde{M},s)}_{n_\xi,n_\eta,|m|}\ {}_1F_1(-n_\xi+M-\tilde{M}+s,|m-\mu+1;\kappa_n\xi)$$
$$\times\ {}_1F_1(-n_\eta+\tilde{M},|m-\mu|+1;\kappa_n\eta)$$
$$-(-1)^{\mu}(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)]\quad(\mu=-1,0,1)\ .\tag{A13}$$

The coefficients $c^{(M,\tilde{M},s)}_{n_\xi,n_\eta,|m|}$ in Eq. (A12) and $d^{(M,\tilde{M},s)}_{n_\xi,n_\eta,|m|}$ in Eq. (A13) are integer numbers listed in Tables II and III, respectively. In Eqs. (A12) and (A13) we have used the parameter

$$M\equiv|m-\mu|-|m|\ ,\tag{A14}$$

which can take on the values $0,\pm1$. The symbol $(\xi\leftrightarrow\eta,n_\xi\leftrightarrow n_\eta)$ means the preceding expression inside the same parentheses with the quoted quantities interchanged.

We conclude with the remark that, according to Tables I–III and VI all the functions ${}_1F_1$ occurring in Eqs. (A8), (A9), (A12), and (A13) are in fact proportional to Laguerre polynomials.

APPENDIX B: ALTERNATIVE EXPRESSIONS OF THE LINEAR-RESPONSE VECTORS IN SPHERICAL COORDINATES

After an adequate algebra, our result concerning the ϕ-gauge vector $\mathbf{v}_{nlm}(\Omega;r,\theta,\varphi)$, which is given by Eqs. (18) and (22), can be cast in the form

$$\mathbf{v}_{nlm}(\Omega;r,\theta,\varphi)=\frac{1}{\kappa_n|E_n|}n\tau\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\int_1^{(0+)}d\rho\,\rho^{-\tau}\mathcal{N}^{n-3}_{n,-\tau}\mathcal{N}^{-n-3}_{n,\tau}\exp\left[-\frac{\mathcal{N}_{\tau,n}}{\mathcal{N}_{n,\tau}}\frac{X}{\hbar}r\right]\mathcal{R}_{n,\tau}\left[\exp(\kappa_{n,\tau}r)u_{nlm}\left[\frac{\kappa_{n,\tau}}{\kappa_n}r,\theta,\varphi\right]\right]\ ,\tag{B1}$$

where $\mathcal{R}_{n,\tau}$ is the following dimensionless linear vector operator:

$$\mathcal{R}_{n,\tau}=2\mathcal{N}^4_{n,-\tau}\kappa^2_{n,\tau}r\mathbf{r}-2\mathcal{N}^3_{n,-\tau}(\mathcal{N}_{n,-\tau}-\mathcal{N}_{n,\tau})\kappa_{n,\tau}[2\mathbf{r}+\mathbf{r}(\mathbf{r}\cdot\nabla)+r^2\nabla]$$
$$+\mathcal{N}_{n,-\tau}(\mathcal{N}_{n,-\tau}-\mathcal{N}_{n,\tau})^2r\nabla[n(\mathcal{N}_{n,-\tau}-\mathcal{N}_{n,\tau})+(2\mathcal{N}_{n,-\tau}+\mathcal{N}_{n,\tau})(1+\mathbf{r}\cdot\nabla)]$$
$$-\tfrac{1}{2}(\mathcal{N}_{n,-\tau}-\mathcal{N}_{n,\tau})^3[n(\mathcal{N}_{n,-\tau}-\mathcal{N}_{n,\tau})+(\mathcal{N}_{n,-\tau}+\mathcal{N}_{n,\tau})(2+\mathbf{r}\cdot\nabla)]\left[\frac{1}{\kappa_{n,\tau}}\nabla(\mathbf{r}\cdot\nabla)-\frac{\mathbf{r}}{r}[\mathbf{r}\cdot\nabla-(n-1)]\right]\ .\tag{B2}$$

In a similar manner, the expression of the $\mathbf{A}$-gauge vector $\mathbf{w}_{nlm}(\Omega;r,\theta,\varphi)$ as a contour integral[29] leads to the formula

$$\mathbf{w}_{nlm}(\Omega;r,\theta,\varphi)=i\frac{\hbar\kappa_n}{|E_n|}n\tau\frac{ie^{i\pi\tau}}{2\sin(\pi\tau)}\int_1^{(0+)}d\rho\,\rho^{-\tau}\mathcal{N}^{n-3}_{n,-\tau}\mathcal{N}^{-n-3}_{n,\tau}\exp\left[-\frac{\mathcal{N}_{\tau,n}}{\mathcal{N}_{n,\tau}}\frac{X}{\hbar}r\right]\mathcal{P}_{n,\tau}\left[\exp(\kappa_{n,\tau}r)u_{nlm}\left[\frac{\kappa_{n,\tau}}{\kappa_n}r,\theta,\varphi\right]\right]\ ,\tag{B3}$$

where $\mathcal{P}_{n,\tau}$ is a linear vector operator simpler than $\mathcal{R}_{n,\tau}$,

$$\mathcal{P}_{n,\tau}=\mathcal{N}_{n,-\tau}\mathcal{N}_{n,\tau}\left[2\mathcal{N}^2_{n,-\tau}(\kappa_{n,\tau}\mathbf{r}-r\nabla)+(\mathcal{N}^2_{n,-\tau}-\mathcal{N}^2_{n,\tau})\left[\frac{1}{\kappa_{n,\tau}}\nabla(\mathbf{r}\cdot\nabla)-\frac{\mathbf{r}}{r}[\mathbf{r}\cdot\nabla-(n-1)]\right]\right] . \tag{B4}$$

Finally, the vector $\mathbf{v}'_{nlm}(E_n;r,\theta,\varphi)$, given in Ref. 3,[30] can be written as

$$\mathbf{v}'_{nlm}(E_n;r,\theta,\varphi)=\frac{1}{8\kappa_n|E_n|}e^{-\kappa_n r}\mathcal{R}'_n[e^{\kappa_n r}u_{nlm}(r,\theta,\varphi)] , \tag{B5}$$

with $\mathcal{R}'_n$ denoting the linear vector operator

$$\mathcal{R}'_n=2\kappa_n^2 r\mathbf{r}+2(5n-3)\kappa_n\mathbf{r}-4\kappa_n[2\mathbf{r}(\mathbf{r}\cdot\nabla)+r^2\nabla]+r\nabla[10(\mathbf{r}\cdot\nabla)-(5n+1)]$$
$$-[5(\mathbf{r}\cdot\nabla)-1]\left[\frac{1}{\kappa_n}\nabla(\mathbf{r}\cdot\nabla)-\frac{\mathbf{r}}{r}[\mathbf{r}\cdot\nabla-(n-1)]\right] . \tag{B6}$$

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[9]Reference 3, Eqs. (29) and (30).

[10]Reference 3, Eq. (28).

[11]This identity coincides with Eq. (14) of Ref. 3.

[12]See Ref. 3, Eqs. (17) and (20).

[13]Reference 3, Eq. (40).

[14]Reference 3, Eq. (16).

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[26]Reference 18, p. 21, Eq. (2.5.5).

[27]Reference 18, p. 84, Eqs. (5.9.16) and (5.9.17).

[28]Reference 18, p. 83, Eq. (5.9.10). Edmonds denotes a vector spherical harmonic by $\mathbf{Y}_{jlm}$, while our notation is $\mathbf{V}_{ljm}$.

[29]Such an expression is given by Eqs. (9), (17), and (20) of Ref.3.

[30]See Ref. 3, Eqs. (37) and (40) written with $\Omega=E_n$, Eqs. (43) and (44).