Journal of Mathematical Physics

Four Euclidean conformal group in atomic calculations: Exact analytical expressions for the bound–bound two-photon transition matrix elements in the H atom

Cite as: J. Math. Phys. **19**, 1041 (1978); https://doi.org/10.1063/1.523766
Published Online: 11 August 2008

J. P. Gazeau

View Online Export Citation

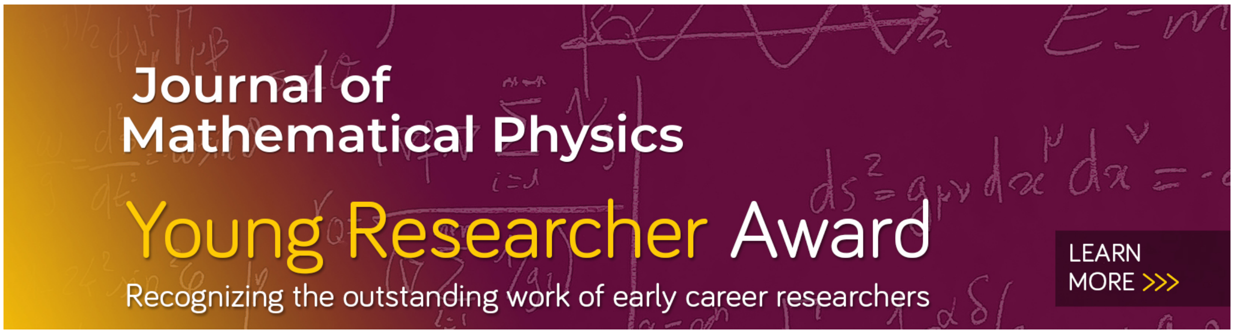
ARTICLES YOU MAY BE INTERESTED IN

Three-dimensional solitons
Journal of Mathematical Physics **19**, 1068 (1978); https://doi.org/10.1063/1.523771

Invariance and conservation laws for Lagrangian systems with one degree of freedom
Journal of Mathematical Physics **19**, 1049 (1978); https://doi.org/10.1063/1.523767

The zero-energy Coulomb problem
Journal of Mathematical Physics **19**, 1037 (1978); https://doi.org/10.1063/1.523765

AIP Publishing



© 1978 American Institute of Physics.

Four Euclidean conformal group in atomic calculations: Exact analytical expressions for the bound–bound two-photon transition matrix elements in the H atom

J. P. Gazeau

Laboratoire de Chimie Physique [a] *de l'Université Pierre et Marie Curie, 75231 Paris Cedex 05, France*
(Received 7 June 1977)

By combining Sturmian–Coulomb techniques with a local representation of the four Euclidean conformal group SU*(4) $\simeq$Spin(1,5), a compact analytical form, suitable for any analytic continuation on the energy variable, is obtained for the following bound–bound two-photon transition matrix element in the H atom: $I_{NN'}(E)=\langle N|\mathbf{p}\cdot\epsilon_1 \exp(i\mathbf{k}_1\cdot\mathbf{r})G(E)\mathbf{p}\cdot\epsilon_2\exp(i\mathbf{k}_2\cdot\mathbf{r})|N'\rangle$, where $G(E)$ is the Coulomb Green's function.

INTRODUCTION

In this paper, yet another method is presented for the calculation of matrix elements of two photon transitions between two hydrogenic bound states. Although a large amount of literature exists on this subject already,[1–12] general compact expressions valid for arbitrary states, and suitable for analytic continuation on the energy variable, are still unknown.

We propose a group theoretical technique, solving this problem completely, by using the four Euclidean conformal group, isomorphic to Spin(1, 5), the universal covering group of $SO_+(1,5)$.[13] This approach describes the transformation induced by a general boost in an "energy–momentum" space and the Fock stereographic projection[14] in terms of a conformal transformation in the quaternion field $\mathbb{H}$. Thus, this method differs from the usual ones where the boost is described by exponentiation of the infinitesimal action of the Lie algebra so(2, 4) on the Hydrogenic states.[9,15]

The organization of this paper is as follows. In Sec. 1 we formulate the Fock treatment of the H atom by introducing a "coupling constant" operator which acts on a Hilbert space, denoted by $H(p_0)$ [identical to $L^2_c(\mathrm{SU}(2))$]. Both its eigenvectors or "Sturmian functions" and its eigenvalues are dependent on the energy E which is a fixed parameter $p_0\equiv(-2mE)^{1/2}$.[16–18]

In Sec. II, we introduce the group SU*(4) $\approx$ Spin(1, 5) and define its action on $\mathbb{H}$ as a conformal transformation. Then we consider a local representation of this group on $L^2_{\mathrm{C}}(\mathrm{SU}(2))$ which is linear when it is restricted to Spin(1, 4), and define some matrix elements which are computed in Appendix A.

In Sec. III, the above elements are used to give exact analytic expressions for transition matrix elements in the H atom, and to recover the classical results for the elastic transitions in the dipole approximation and to extend it to higher orders. The formulas which are obtained are suitable for any analytic continuation.

1. A SURVEY OF THE STURMIAN PROBLEM AND THE FOCK METHOD

The Schrödinger equation for two charged particles without spin in terms of relative coordinates and momenta can be written as (in natural units)

$$(p_0^2+p^2)\psi=c^{-1}V\psi, \tag{1.1}$$

where

$$p_0=(-2mE)^{1/2},\quad c^{-1}=2m\alpha z,\quad V=\frac{1}{r}.$$

For fixed E or equivalently fixed p_0, Eq. (1.1) is also the eigenvalue equation for the so-called "Sturmian operator" or "coupling constant operator,"[16–18]

$$C\equiv(p_0^2+p^2)^{-1}V, \tag{1.2}$$

$$C\psi=c\psi. \tag{1.3}$$

When $E<0$, the spectrum of C is infinite and discrete, whereas it is continuous for $E>0$, and the algebraic relation between the eigenvalues c of C and the parameter p_0 allows one to find the energy spectrum, but the basic difference between the Hamiltonian problem and the Sturmian problem must be emphasized. For instance, C is not Hermitian in $L^2_{\mathrm{C}}(\mathbb{R}^3)$, the Hilbert space of complex square integrable functions on $\mathbb{R}^3$. It is possible to render it Hermitian in a pre-Hilbertian space in correspondence with the first by the p_0 dependent transformation: Suppose E negative, for all $\psi\in L^2_{\mathrm{C}}(\mathbb{R}^3)$ such that $|(\psi,(p_0^2+p^2)\psi)|<\infty$; we associate a weighted state

$$\psi'=(p_0^2+p^2)^{1/2}\psi, \tag{1.4}$$

where the square root makes sense since $(p_0^2+p^2)$ is diagonal in momentum space. Then

$$\begin{aligned}C'&=(p_0^2+p^2)^{1/2}C(p_0^2+p^2)^{-1/2}\\&=(p_0^2+p^2)^{-1/2}V(p_0^2+p^2)^{-1/2}\end{aligned} \tag{1.5}$$

is clearly Hermitian on the space generated by the ψ'.

A similar treatment was used by Fock.[6,14,19,20] The Fock method consists of two operations.

The change of the integration variable in the scalar product of $L^2_{\mathrm{C}}(\mathbb{R}^3)$ introduces the Hilbert space $L^2_{\mathrm{C}}(S^3)$, where S^3 is the unit sphere of $\mathbb{R}^4$. The multiplication of the states by a weight renders C Hermitian in $L^2_{\mathrm{C}}(S^3)$.

Explicitly, the Fock stereographic projection, denoted by $s(p_0)$, brings the unit sphere S^3 onto the compactified hyperplane $H(p_0)$ which is isomorphic to the momentum space,

[a]Laboratoire "Matière et Rayonnement" associé au C.N.R.S.

 0022-2488/78/1905-1041$1.00 © 1978 American Institute of Physics

$$x=(p_0,\mathbf{p})\in H(p_0)$$
$$\to \xi = s^{-1}(p_0)(x)\begin{cases}\xi=\dfrac{2p_0\mathbf{p}}{p_0^2+p^2},\\ \xi_0=\dfrac{p_0^2-p^2}{p_0^2+p^2}.\end{cases}\tag{1.6}$$

The relation between the Euclidean measure $d^3\mathbf{p}$ and the O(4) invariant measure on S^3 is

$$d\mu(\xi)=\left(\frac{2p_0}{p_0^2+p^2}\right)^3 d^3\mathbf{p} =\left(\frac{|\underline{1}+\xi|^2}{2p_0}\right)^3 d^3\mathbf{p},\tag{1.7}$$

where $\underline{1}\equiv(1,0)$.

The Fock correspondence $\mathcal{J}_{p_0}$ between the two Hilbert spaces $L^2_{\mathbb{C}}(\mathbb{R}^3)$ and $L^2_{\mathbb{C}}(S^3)$, the latter denoted by $\mathcal{H}(p_0)$, is

$$\psi\in L^2_{\mathbb{C}}(\mathbb{R}^3)\xrightarrow{\mathcal{J}_{p_0}}\phi =\frac{1}{\sqrt{p_0}}\left(\frac{2p_0}{|\underline{1}+\xi|^2}\right)^2\psi\circ s(p_0),\tag{1.8}$$
$$\phi\in L^2_{\mathbb{C}}(S^3)\approx\mathcal{H}(p_0),$$

and reciprocally

$$\phi\in\mathcal{H}(p_0)\xrightarrow{\mathcal{J}^{-1}_{p_0}}\psi =\sqrt{p_0}\left(\frac{2p_0}{p_0^2+p^2}\right)^2\phi\circ s^{-1}(p_0).\tag{1.8}'$$

Their respective scalar products are related by

$$(\psi_1,\psi_2)_{L^2_{\mathbb{C}}(\mathbb{R}^3)}=p_0\left(\phi_1,\left(\frac{|\underline{1}+\xi|^2}{2p_0}\right)\phi_2\right)_{\mathcal{H}(p_0)},\tag{1.9}$$

$$(\phi_1,\phi_2)_{\mathcal{H}(p_0)}=\frac{1}{p_0}\left(\psi_1,\left(\frac{2p_0}{p_0^2+p^2}\right)\psi_2\right)_{L^2_{\mathbb{C}}(\mathbb{R}^3)},\tag{1.9}'$$

and the eigenvalue equation (1.3) for the Sturmian operator is written as an integral equation,

$$\hat{C}\phi(\xi)=\frac{1}{2\pi^2}\int_{S^3}d\mu(\xi')\,|\xi-\xi'|^{-2}\phi(\xi') =\hat{c}\phi(\xi),\tag{1.10}$$

where

$$\phi=\mathcal{J}_{p_0}\psi,\quad \hat{C}=\mathcal{J}_{p_0}C\mathcal{J}^{-1}_{p_0},\quad \hat{c}=2p_0c=\frac{p_0}{m\alpha z}.$$

$\hat{C}$ is clearly Hermitian and O(4) invariant. Its eigenvalues are

$$\hat{c}_n=\frac{1}{n},\quad n\in\mathbb{N}^*.\tag{1.11}$$

A natural system of eigenvectors is the set of the spherical harmonics Y_{nlm} [20] on S^3.

Returning to the Hamiltonian problem, by solving (1.11) in p_0 and carrying out the corresponding transformation $\mathcal{J}^{-1}_{p_0}$, the well-known eigenvalues and related eigenstates in the momentum space are obtained:

$$p_0=\frac{\lambda}{n}\equiv p_n\quad \lambda\equiv m\alpha z\quad\left(\text{i.e.},\ E_n=\frac{\lambda^2}{2mn^2}\right),\tag{1.12}$$

$$\psi_{nlm}=\mathcal{J}^{-1}_{p_n}Y_{nlm},\quad \xi_n=s^{-1}(p_n)((p_n,\mathbf{p})),$$
$$\psi_{nlm}(\mathbf{p})=\frac{4p_n^{5/2}}{(p_n^2+p^2)^2}Y_{nlm}(\xi_n).\tag{1.13}$$

2. SU*(4) APPROACH TO THE STURMIAN-COULOMB PROBLEM

The dynamical symmetry O(4, 2) of the H atom has already been intensively used in atomic calculations, mainly by Barut and Kleinert[15] and by Fronsdal.[8] See also Refs. 9 and 10. The main difficulty in the generalization of these methods to many atomic calculations (transition matrix elements, etc., $\cdots$) is the translation of the action of operators on the states space in terms of "abstract rotations" deduced by exponentiation of the action of a representation of the Lie algebra o(4, 2) or of its enveloping algebra. Part of the physical meaning is lost when the matrix elements which describe the processes are re-expressed in terms of real or imaginary angles.

We exploit all the resources of the quaternionic calculus. This is an advantage in itself. Indeed, when the calculation of matrix elements describing the transition from one Coulomb state to another is required, the energy jumps and momentum transfers characterizing a "general physical boost" are described by a translation in an "energy—momentum" space. This, one naturally identifies as the quaternion field $\mathbb{H}$ when the "energy component" $p_0=(-2mE)^{1/2}$ is real. Now, the Fock projection transforms the translation group element to an $\mathrm{Sp}(1,1)\approx\mathrm{Spin}(1,4)$ element.[13] More generally if other effects are taken into account (e.g., intermediate summation in the form of a Green function, as arises in perturbation calculations) a Fock projection will give an element of the four Euclidean conformal group SU*(4)[13] which is isomorphic to Spin(1, 5). Explicitly, in this way a convenient representation of Spin(1, 5) is obtained in the form of a group of 2×2 quaternionic matrices, denoted by SU*(4), which acts on $\mathbb{H}$ as a homographic transformation. It is then a simple matter to multiply 2×2 matrices together as other processes follow. Moreover, the fact that their matrix elements, which are physically undimensioned quadrivectors, are very simply and strangely [see Eq. (2.8)] connected to the general boost parameters of the process under consideration merits deeper understanding. This will be gone into elsewhere.

Now the way in which the conformal group appears naturally in the Sturmian problem and Fock method when $p_0=(-2mE)^{1/2}$ is real, is explained.

Let us consider the quaternion field $\mathbb{H}$, the elements of which will be denoted by $x=(x_0,\mathbf{x})$, where x_0 is the scalar part and $\mathbf{x}$ the vector part[21]:

$$x=x_0\mathbf{1}+x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k},$$
$$xx'=(x_0x_0'-\mathbf{x}\cdot\mathbf{x}_0',x_0\mathbf{x}'+x_0'\mathbf{x}+\mathbf{x}\times\mathbf{x}'),\tag{2.1}$$
$$\bar{x}\equiv(x_0,-\mathbf{x}).$$

Spin(1, 5) is isomorphic to the group SU*(4) of 2×2 matrices with quaternionic entries verifying a scalar relation

$$\mathrm{SU}^*(4)=\left\{g=\begin{pmatrix}a&b\\c&d\end{pmatrix};a,b,c,d\in\mathbb{H};\ |c|\,|d|\,|ac^{-1}-bd^{-1}|=1\right\},\tag{2.2}$$

where $x \to |x|$ is the Euclidean norm in $\mathbf{H}$, also called the modulus of the quaternion x.

SU*(4) acts on $\mathbf{H}$ via conformal transformations:

$$x \in \mathbf{H}, \quad g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}^*(4), \qquad x \to g \circ x=(ax+b)(cx+d)^{-1}. \tag{2.3}$$

The following important property holds:

$$|cx+d|\,|g\cdot x-g\cdot y|\,|cy+d|=|x-y| \tag{2.4}$$

for all $x, y \in \mathbf{H}$, $g \in \mathrm{SU}^*(4)$.

The Fock stereographic projection $s(p_0)$ is a particular case of this action. It establishes a one to one correspondence between the subgroup [isomorphic to, and briefly denoted by SU(2)] of unit modulus quaternions and the hyperplane of the quaternions having the same scalar part p_0; let us put:

$$x=(p_0, \mathbf{p}), \quad s(p_0)=\frac{1}{\sqrt{2p_0}}\begin{pmatrix} 2p_0 & 0 \\ 1 & 1 \end{pmatrix}, \quad s^{-1}(p_0)=\frac{1}{\sqrt{2p_0}}\begin{pmatrix} 1 & 0 \\ -1 & 2p_0 \end{pmatrix}, \tag{2.5}$$

where

$$\lambda\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}, \quad \lambda \in \mathbf{R},$$

and 0 and 1 henceforward denote the zero and unit elements both in $\mathbf{H}$ and in $\mathbf{R} \subset \mathbf{H}$. (1.6) is now written

$$\xi=s^{-1}(p_0)\cdot x=x\bar{x}^{-1},$$

where $\bar{x}$ is the quaternionic conjugate of x,

$$\bar{x}=(p_0, -\mathbf{p}). \tag{2.6}$$

Now, we describe in terms of the action (2.3) the general physical "boost" which also includes the Galilean boost in momentum space

$$\mathbf{p} \to \mathbf{p}'=\mathbf{p}+\mathbf{k},$$

as a scalar boost

$$p_0 \to p_0'=p_0+k_0, \qquad x=(p_0, \mathbf{p}) \to x'=(p_0', \mathbf{p}')=(p_0+k_0, \mathbf{p}+\mathbf{k})=t_K \circ x, \tag{2.7}$$

where

$$t_K=\begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix} \in T_4 \subset \mathrm{SU}^*(4), \qquad K \equiv (k_0, \mathbf{k}),$$

T_4: group of translations in $\mathbf{H}$.

The transformation induced on S^3 by the translations in $\mathbf{H}$ after the inverse stereographic projection is

$$\xi'=(s^{-1}(p_0')t_K s(p_0))\cdot\xi=h\cdot\xi, \tag{2.8}$$

where

$$\xi'=s^{-1}(p_0')\cdot x', \quad \xi=s^{-1}(p_0)\circ x,$$

$$h=\frac{1}{2(p_0p_0')^{1/2}}\begin{pmatrix} K_+ & K_- \\ \bar{K}_- & \bar{K}_+ \end{pmatrix}, \quad K_\pm=(p_0' \pm p_0, \mathbf{k}).$$

h is an element of the subgroup $\mathrm{Sp}(1,1) \approx \mathrm{Spin}(1,4)$, which leaves both SU(2) and the unit ball invariant under the conformal transformation. Spin(1,4) has been intensively studied (see for instance Fronsdal[22]), particularly under its quaternionic representation by Takahaski[23] and Strömm.[24] In atomic computations we shall use the following (nonunitary) irreducible linear representation of this group[22]:

$$f \in L^2_{\mathrm{C}}(\mathrm{SU}(2)) \approx L^2_{\mathrm{C}}(S^3),$$

$$h^{-1}=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Leftrightarrow h=\begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix},$$

$$\mathcal{T}^\rho(h)f(\xi)=|c\xi+d|^{-2\rho}f(h^{-1}\circ\xi)$$

with $\rho=1$ or 2. (2.9)

We define the matrix elements of this representation with respect to the orthonormal basis $\{Y_N\}$, $N=nlm$,

$$\mathcal{T}^\rho_{NN'}(h)=(\mathcal{T}^\rho(h)Y_{N'}, Y_N)_{L^2_{\mathrm{C}}(\mathrm{SU}(2))}. \tag{2.10}$$

Since

$$d\mu(h^{-1}\cdot\xi)=d\mu(\xi)|c\xi+d|^{-6}, \tag{2.11}$$

these matrix elements verify

$$\mathcal{T}^2_{NN'}(h)=(\mathcal{T}^1_{N'N}(h^{-1}))^*. \tag{2.12}$$

One may show that (see Appendix B):

$$\mathcal{T}^1_{NN'}(h)=\frac{n'}{n}(\mathcal{T}^1_{N'N}(h^{-1}))^*. \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\mathcal{T}^2_{NN'}(h)=\frac{n}{n'}\mathcal{T}^1_{NN'}(h). \tag{2.14}$$

Now we extend the representation $\mathcal{T}^\rho$ to SU*(4) in the following way: Let us continue an element f of $L^2_{\mathrm{C}}(\mathrm{SU}(2))$ to a function defined inside (outside) $S^3 \approx \mathrm{SU}(2)$:

$$f \to F_{\lessgtr},$$

$$F_<(x)=\frac{1}{2\pi^2}\int_{\mathrm{SU}(2)} d\mu(\xi')f(\xi')\frac{1-|x|^2}{|\xi'-x|^4}, \quad |x|<1, \tag{2.15}$$

$$F_>(x)=|x|^{-2}F_<(x^{-1}), \quad |x|>1,$$

$$F_<(x)=F_>(x)=f(x), \quad |x|=1.$$

Putting $x=|x|\xi$, $F_<$ (resp. $F_>$) as a function of the variable ξ belongs to $L^2_{\mathrm{C}}(\mathrm{SU}(2))$.

We define the local representation of SU*(4) on $L^2_{\mathrm{C}}(\mathrm{SU}(2))$ in the following way,

$$f \in L^2_{\mathrm{C}}(\mathrm{SU}(2)), \quad g^{-1}=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}^*(4),$$

$$\mathcal{T}^\rho(g)f(\xi)=|c\xi+d|^{-2\rho}F_{\lessgtr}(g^{-1}\circ\xi), \tag{2.16}$$

$\lessgtr$ according as to whether $|g^{-1}\circ\xi| \lessgtr 1$.

The representation $\mathcal{T}^\rho$ is local in the sense that it is linear for all g in some neighborhood of e, unit element of SU*(4). Now, let us consider the integral

$$S^\rho_{NN'}(g)=\int_{\mathrm{SU}(2)} d\mu(\xi)|c\xi+d|^{-2\rho}\mathcal{Y}_{N'}(g^{-1}\cdot\xi')Y^*_N(\xi), \tag{2.17}$$

where $\mathcal{Y}_N(x)$ is the harmonic polynomial ("solid harmonic") deduced from the surface harmonic by the homogeneity formula

$$\mathcal{Y}_N(x)=|x|^{n-1}Y_N\left(\frac{x}{|x|}\right).$$

It is homogeneous of degree $n-1$.

If $|g^{-1} \circ \xi| \leqslant 1$ for all $\xi \in \mathrm{SU}(2)$, $S^{\rho}_{NN'}(g)$ is the matrix element $\mathcal{T}^{\rho}_{NN'}(g)$ of the operator $\mathcal{T}^{\rho}(g)$. The same remains true if $|g^{-1} \circ \xi| \geqslant 1$ for all $\xi \in \mathrm{SU}(2)$. We give the analytic expression of the integral (2.17) in Appendix A for $\rho=1$ and $|c| < |d|$.

3. CALCULATION OF THE BOUND-BOUND TWO PHOTON TRANSITION MATRIX ELEMENTS IN THE H ATOM

In a set of particular cases, exact analytic expressions have already been available for about ten years for the following matrix elements (between bound or continuum states),

$$I^2_{NN'}(E,\epsilon_1,\mathbf{k}_1,\epsilon_2,\mathbf{k}_2) \equiv I^2_{NN'}(E) = \langle N|\mathbf{p}\cdot\epsilon_1 \exp(i\mathbf{k}_1\cdot\mathbf{r})G(E)\mathbf{p}\cdot\epsilon_2 \exp(i\mathbf{k}_2\cdot\mathbf{r})|N'\rangle, \tag{3.1}$$

where $G(E)=(H-E)^{-1}$ is the Coulomb Green function.

ϵ_i (resp. $\mathbf{k}_i$) is the polarization vector (resp. momentum) of the ith photon.

The following various techniques were used:

(i) analytic methods in configuration space, by use of the Hostler integral representation[25] or the Sturmian Coulomb Green function: in the dipole approximation[1–3] and with the retardation effects[3];

(ii) analytic methods in momentum space, by use of the Schwinger integral representation of the Coulomb Green function[26]: in the dipole approximation[4] and with retardation effects[5];

(iii) analytic methods in $L^2_{\mathbb{C}}[\mathrm{SU}(2)]$, by use of the Fock method and the Schwinger integral representation and the harmonic analysis on SU(2): in the dipole approximation[6] and with retardation effects[1,7];

(iv) algebraic techniques, by use of the dynamical group O(4, 2), with retardation effect[8–10];

(v) numerical techniques, by numerical integration of inhomogeneous differential equations.[11,12]

None of these methods is able to give a general compact analytical expression for the matrix elements $I^2_{NN'}(E)$ between any initial and final states.

It is evident that our expressions provide their own analytic continuation in contrast to the usual situation.

The detail of the method is given in Appendix C. We shall now examine the results.

Let us define the two elements of Sp(1, 1),

$$h_1=\frac{1}{2\sqrt{p_0p_n}}\begin{pmatrix} K_{1+} & -\bar{K}_{1-} \\ -K_{1-} & \bar{K}_{1+}\end{pmatrix}, \quad h_2=\frac{1}{2\sqrt{p_0p_{n'}}}\begin{pmatrix} K_{2+} & K_{2-} \\ \bar{K}_{2-} & \bar{K}_{2+}\end{pmatrix}, \quad \text{and } \exp[(x/2)\upsilon]=\begin{pmatrix} e^{x/2} & 0 \\ 0 & x^{-x/2}\end{pmatrix}, \tag{3.2}$$

where

$$p_n=(-2mE_n)^{1/2}, \quad p_{n'}=(-2mE_{n'})^{1/2}, \quad p_0=(-2mE)^{1/2}, \quad K_{1\pm}=(p_0\pm p_{n'},\mathbf{k}_1), \quad K_{2\pm}=(p_0\pm p_{n'},\mathbf{k}_2). \tag{3.3}$$

Let us put $\nu=m\alpha z/p_0$.

Then, we obtain for the matrix element (3.1) the following integral representation

$$I^2_{NN'}(E)=-m\nu(nn')^{-1/2}\sum_{N_0,N_0'} C_{N_0'N'}(\epsilon_2)C^*_{NN_0}(\epsilon_1)n_0\int_0^{+\infty}dx\, e^{\nu x}\mathcal{T}^1_{N_0N_0'}(g(x)), \tag{3.4}$$

with

$$\mathrm{SU}^*(4)\ni g^{-1}(x)=(h_1\exp[(x/2)\upsilon]h_2)^{-1}=\frac{1}{4p_0\sqrt{p_np_{n'}}}\begin{pmatrix} \bar{K}_{2+}\bar{K}_{1+}e^{-x/2}-K_{2-}K_{1-}e^{x/2} & \bar{K}_{2+}\bar{K}_{1-}e^{-x/2}-K_{2-}K_{1+}e^{x/2} \\ -\bar{K}_{2-}\bar{K}_{1+}e^{-x/2}+K_{2+}K_{1-}e^{x/2} & -\bar{K}_{2-}\bar{K}_{1-}e^{-x/2}+K_{2+}K_{1+}e^{x/2}\end{pmatrix}. \tag{3.5}$$

The matrix elements $C_{NN'}(\epsilon)$ are given by (D2). We can show that

$$|-\bar{K}_{2-}\bar{K}_{1+}e^{-x/2}+K_{2+}K_{1-}e^{x/2}| < |-\bar{K}_{2-}\bar{K}_{1-}e^{-x/2}+K_{2+}K_{1+}e^{x/2}|, \tag{3.6}$$

which corresponds to the particular case given in Appendix A. Expression (3.4) can be reduced to a finite sum of integrals $\mathcal{G}^b_a$ defined by[1,7]

$$\mathcal{G}^b_a(y,y')=\int_0^{+\infty}dx\,\exp(-bx)|y'-e^{-x}y|^{-2a}=|y'|^{2a}b^{-1}F_1(b,a,a,b+1;\rho e^{i\omega},\rho e^{-i\omega}), \tag{3.7}$$

where

$$y,y'\in\mathbf{H}, \quad |y|<|y'|, \quad \rho=|y|\,|y'|^{-1}, \quad \omega=(y,y')^{\hat{}}.$$

F_1 is an Appell function.[27] Thus

$$n_0\int_0^{+\infty}dx\,e^{\nu x}\mathcal{T}^1_{N_0N_0'}(g(x))=\sum_{q,q'} C^{qq'}_{N_0N_0'}(h_1,h_2)\mathcal{G}^{q-\nu}_{q'}(\bar{K}_{2-}\bar{K}_{1-},K_{2+}K_{1+}). \tag{3.8}$$

The expression of the coefficients $C^{qq'}_{N_0N_0'}(h_1,h_2)$ is given in Appendix E.

Let us consider explicitly two important particular cases. In the dipole approximation ($\mathbf{k}_1=\mathbf{k}_2=0$) we have

$$g^{-1}(x)=\frac{(1+\nu/n)(1+\nu/n')e^{x/2}}{4(\nu^2/nn')^{1/2}}\begin{pmatrix} e^{-x}-r_n r_{n'} & r_n e^{-x}-r_{n'} \\ r_n-r_{n'}e^{-x} & 1-r_n r_{n'}e^{-x}\end{pmatrix},\quad \text{where } r_n=\frac{1-\nu/n}{1+\nu/n},\quad r_{n'}=\frac{1-\nu/n'}{1+\nu/n'} \tag{3.9}$$

and (3.8) now becomes

$$n_0\int_0^{+\infty}dx\, e^{\nu x}\mathcal{T}^1_{N_0N_0'}(g(x))=\delta_{l_0l_0'}\delta_{m_0m_0'}[n_0n_0'(n_0-l_0-1)!\,(n_0'-l_0-1)!\,(n_0+l_0)!\,(n_0'+l_0)!]^{1/2}\sum_{q,\sigma}\left(\frac{(1-\nu^2/n^2)(1-\nu^2/n'^2)}{16\nu^2/nn'}\right)^{q-n_<}$$
$$\times\frac{(-1)^\sigma r_<^{n_<+\sigma}\,{}_2F_1(-q,-\sigma;q+n_>-n_<-\sigma+1;(r_>/r_<)^2)}{\sigma!\,q!\,(n_>-n_<+q-\sigma)!\,(n_<-l_0-1-q)!)(n_<+l_0-q)!}$$
$$\times(n_>+q-\sigma-\nu)^{-1}\,{}_2F_1(n_0+n_0',n_>+q-\sigma-\nu;n_>+q-\sigma-\nu+1;r_nr_{n'}), \tag{3.10}$$

where

$$n_{\gtrless}=\sup_{\inf}(n_0,n_0'),\quad r_<=\begin{matrix} r_{n'} & \text{if } n_0=n_< \\ r_n & \text{if } n_0'=n_<\end{matrix},\quad r_>=\begin{matrix} r_{n'} & \text{if } n_0=n_> \\ r_n & \text{if } n_0'=n_>\end{matrix}.$$

For $n=2$, $n'=1$, $l=l'=m=m'=0$, this expression is in agreement with Ref. 3. Particularly, for $N=N'$, and $\epsilon_1=\epsilon_2=\hat{z}$ (elastic scattering), we give a complete expression,

$$I^2_{NN}(E)=-\frac{m\nu}{4n}\sum_{l_0}\{A_{ll_0}S_{n-1,n-1}(l_0,\nu)+B_{ll_0}S_{n+1,n+1}(l_0,\nu)-2C_{ll_0}S_{n-1,n+1}(l_0,\nu)\}, \tag{3.11}$$

where

(1) $S_{n_0n_0'}(l_0,\nu)$

$$=[n_0n_0'(n_0-l_0-1)!\,(n_0'-l_0-1)!\,(n_0+l_0)!\,(n_0'+l_0)!]^{1/2}\left(1+\frac{\nu}{n}\right)^{-2(n_0+n_0')}$$
$$\times(-1)^{n_0-n_0'}\sum_{q=0}^{n_<-l_0-1}\frac{(1-\nu^2/n^2)^{2q+n_>-n_<}(4\nu/n)^{2n_<-2q}}{(n_<-l_0-1-q)!\,(n_<+l_0-q)!\,(n_>-n_<+q)!\,q!}$$
$$\times\frac{\Gamma(n_<-q-\nu)\Gamma(2q+n_>-n_<+1)}{\Gamma(q+n_>+1-\nu)}\,{}_2F_1(n_0+n_0',n_<-q-\nu;n_>+q+1-\nu;r_n^2),$$
$$n_{\gtrless}=\sup_{\inf}(n_0,n_0').$$

(2)
$$A_{ll_0}=\frac{(n+l)(n+l-1)(l+m)(l-m)}{n(n-1)(2l+1)(2l-1)}\delta_{l-1\,l_0}+\frac{(n-l-1)(n-l-2)(l+1+m)(l+1-m)}{n(n-1)(2l+1)(2l+3)}\delta_{l+1\,l_0},$$
$$B_{ll_0}=\frac{(n-l)(n-l+1)(l+m)(l-m)}{n(n+1)(2l+1)(2l-1)}\delta_{l-1\,l_0}+\frac{(n+l+1)(n+l+2)(l+1+m)(l+1-m)}{n(n+1)(2l+1)(2l+3)}\delta_{l+1\,l_0},$$
$$C_{ll_0}=\left(\frac{(n+l)(n+l-1)(n-l)(n-l+1)}{n^2(n-1)(n+1)}\right)^{1/2}\frac{(l+m)(l-m)}{(2l+1)(2l-1)}\delta_{l-1\,l_0}$$
$$+\left(\frac{(n-l-1)(n-l-2)(n+l+1)(n+l+2)}{n^2(n-1)(n+1)}\right)^{1/2}\frac{(l+1+m)(l+1-m)}{(2l+1)(2l+3)}\delta_{l+1\,l_0}.$$

For $n=1$ and 2, $l=m=0$, this expression is in agreement with the results of the previous works.[1-4]

CONCLUSION

All these results can be analytically continued to positive energies $E=-p_0^2/2m>0$ or equivalently to purely imaginary values of ν. Moreover, the method can be easily generalized to the higher order processes, where the following integrals must appear at the end of the calculus,

$$\int_0^{+\infty}dx_1\cdots\int_0^{+\infty}dx_n\exp\left(\sum_{i=1}^n\nu_ix_i\right)\mathcal{T}^1_{N_0N_0'}\left(\left(\prod_{i=1}^n h_ie^{(x_i/2)\nu}\right)h_{n+1}\right),$$
$$h_i\in \mathrm{Sp}(1,1). \tag{3.12}$$

Then one has to express (3.12) in terms of known special functions. Finally, similar group theoretical methods can be used to calculate the matrix elements $I_{NN'}(E)$ between arbitrary continuum states. These different cases will be dealt with in another work.

ACKNOWLEDGMENTS

The author expresses his gratitude to Professor Y. Heno, Professor M. Gavrila, Dr. S. Klarsfeld, and Dr. A. Maquet for useful discussions.

APPENDIX A

Our aim is to calculate the integral

$$S^1_{NN'}(g)=\int_{SU(2)}d\mu(\xi)\,|c\xi+d|^{-2}\mathcal{Y}_{N'}(g^{-1}\cdot\xi)Y^*_N(\xi), \tag{A1}$$

for

$$g^{-1}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SU^*(4),\quad |c|<|d|.$$

It is easier to evaluate the integral

$$\mathcal{S}^1_{N,N'}(g)\equiv\int_{SU(2)}d\mu(\xi)\,|c\xi+d|^{-2}D^{j'}_{m_1'm_2'}(g^{-1}\cdot\xi)D^{j*}_{m_1m_2}(\xi), \tag{A2}$$

$\mathcal{N} \equiv j m_1 m_2$,

where the $D^j_{m_1 m_2}(x)$ are the homogeneous harmonic polynomials on $\mathbb{H}$ deduced from the usual matrix elements of the unitary irreducible representations of SU(2) by the homogeneity formula[28]

$$D^j_{m_1 m_2}(x) = |x|^{2j} D^j_{m_1 m_2}\left(\frac{x}{|x|}\right).$$

In another paper[23] we have established three fundamental properties verified by these polynomials: A finite difference equation, an addition formula, and an expansion formula.

Putting

$$\sigma^j_m = [(j-m)!\,(j+m)!]^{-1/2}, \qquad \sigma^j_{m_1 m_2} = \sigma^j_{m_1}\sigma^j_{m_2}, \tag{A3}$$

we have successively:

a finite difference equation:

$$(\sigma^j_{m_1 m_2}) D^j_{m_1 m_2}(x) = \binom{2j}{2j'} \sum_{m_1', m_2'} (\sigma^{j-j'}_{m_1-m_1' m_2-m_2'}) D^{j-j'}_{m_1-m_1' m_2-m_2'}(x) \times (\sigma^{j'}_{m_1' m_2'}) D^{j'}_{m_1' m_2'}(x); \tag{A4}$$

an addition theorem,

$x, x' \in \mathbf{H}$,

$$\sigma^j_{m_1 m_2} D^j_{m_1 m_2}(x+x') = \sum_{j', m_1', m_2'} \sigma^{j-j'}_{m_1-m_1' m_2-m_2'} D^{j-j'}_{m_1-m_1' m_2-m_2'}(x)\, \sigma^{j'}_{m_1' m_2'} D^{j'}_{m_1' m_2'}(x'); \tag{A5}$$

an expansion theorem:

$x, x' \in \mathbf{H}, \quad |x| < |x'|$:

$$(\sigma^j_{m_1 m_2})^{-1} |x+x'|^{-2} D^j_{m_1 m_2}((x+x')^{-1}) = \sum_{j', m_1', m_2'} (-1)^{2j'} \sigma^{j'}_{m_1' m_2'} D^{j'}_{m_1' m_2'}(x) (\sigma^{j+j'}_{m_1+m_2' m_2+m_1'})^{-1} \times |x'|^{-2} D^{j+j'}_{m_1+m_2' m_2+m_1'}(x'^{-1}). \tag{A6}$$

Let us add the fundamental group representation property, $x, x' \in \mathbb{H}$:

$$D^j_{m_1 m_2}(xx') = \sum_{m'} D^j_{m_1 m'}(x) D^j_{m' m_2}(x'). \tag{A7}$$

Combining these four formulas, we easily obtain the "four Euclidean conformal transformation" formula, $|cx| < |d|$,

$$|cx+d|^{-2} D^{j'}_{m_1' m_2'}((ax+b)(cx+d)^{-1}) = \sum_{\mu} D^{j'}_{m_1' \mu}((ax+b))\,|cx+d|^{-1} D^{j'}_{\mu m_2'}((cx+d)^{-1}) = \sum_{j, m_1, m_2} F(jm_1m_2; j'm_1'm_2'; g) \frac{\sigma^{j'}_{m_2}\sigma^j_{m_1}}{\sigma^{j'}_{m_1}\sigma^j_{m_1}} D^j_{m_1 m_2}(x),$$

where

$$F(jm_1m_2; j'm_1'm_2'; g) = \sum_{\substack{j_i, m_{i1}, m_{i2} \\ 1\le i\le 4}} \delta_{j', j_1+j_2}\delta_{j_4, j'+j_3}\delta_{j, j_1+j_3}\delta_{m_1', m_2+m_{21}} \times \delta_{m_{42}, m_2'+m_{31}}\delta_{m_{41}, m_2+m_{22}}\delta_{m_1, m_{12}+m_{32}}(-1)^{2j_3} (\sigma^{j_4}_{m_{41} m_{42}})^{-2} |d^2|^{-(2j_4+1)} \prod_{i=1}^{4} \sigma^{j_i}_{m_{i1} m_{i2}} D^{j_i}_{m_{i1} m_{i2}}(g_i) \tag{A8}$$

with $g_1 = a,\ g_2 = b,\ g_3 = c,\ g_4 = \bar{d}$.

By use of the unitary transformation for connecting a given $j = (n-1)/2$, the set of the $\mathcal{Y}_{nlm}$, and that of the $D^j_{m_1 m_2}$,

$$\mathcal{Y}_{nlm}(x) = \left(\frac{n}{2\pi^2}\right)^{1/2} i^l \sum_{m_1, m_2} (2l+1)^{1/2}(-1)^{j-m_2} \times \begin{pmatrix} j & j & l \\ m_1 & -m_2 & m \end{pmatrix} D^j_{m_1 m_2}(x), \tag{A9}$$

we finally obtain ($n = 2j+1,\ n' = 2j'+1$):

$$S^1_{NN'}(g) = i^{l'-l}[(2l+1)(2l'+1)]^{1/2}\left(\frac{n'}{n}\right)^{1/2} \times \sum_{\substack{m_1, m_2 \\ m_1', m_2'}} (-1)^{j-m_2+j'-m_2'-1} \begin{pmatrix} j & j & l \\ m_1 & -m_2 & m \end{pmatrix} \begin{pmatrix} j' & j' & l' \\ m_1' & -m_2' & m' \end{pmatrix} \times \frac{\sigma^j_{m_2}\sigma^{j'}_{m_2'}}{\sigma^j_{m_1}\sigma^{j'}_{m_1'}} F(jm_1m_2; j'm_1'm_2'; g). \tag{A10}$$

For $g \in \mathrm{SL}(2, \mathbb{R}) \subset \mathrm{SU}^*(4)$, this expression is reduced to a hypergeometric polymial,

$$S^1_{NN'}(g) = \delta_{ll'}\delta_{mm'}\left(\frac{n'}{n}\right)^{1/2}\left(\frac{n_> - l - 1)!\,(n_> + l)!}{n_< - l - 1)!\,(n_< + l)!}\right)^{1/2} \times d^{-(n_>+l+1)} a^{n_<-l-1} \frac{(\gamma(b,c))^{n_>-n_<}}{(n_> - n_<)!} \times {}_2F_1\left(l+1-n_<, n_>+l+1; n_>-n_<+1; \frac{bc}{ad}\right) \tag{A11}$$

$$n_{\gtrless} = \begin{matrix}\sup \\ \inf\end{matrix}(n, n'), \quad \gamma(b,c) = \begin{cases} b, & n_> = n', \\ -c & n_> = n. \end{cases}$$

In the latter case, another expression is deduced from the following formula,[27]

$${}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{-\alpha}\,{}_2F_1\left(\alpha, \gamma-\beta; \gamma; \frac{z}{z-1}\right)$$

$$S^1_{NN'}(g) = \delta_{ll'}\delta_{mm'}\left(\frac{n'(n_> - l - 1)!\,(n_> + l)!}{n(n_< - l - 1)!\,(n_< + l)!}\right)^{1/2} \times d^{-(n+n')} \frac{(\gamma(b,c))^{n_>-n_<}}{(n_> - n_<)!} \times {}_2F_1(l+1-n_<, -(n_<+l), n_>-n_<+1; -bc). \tag{A12}$$

APPENDIX B

We state here the formula (2.13)

$$\mathcal{T}^1_{NN'}(h) = \frac{n'}{n}\mathcal{T}^{1*}_{N'N}(h^{-1}), \quad \text{for all } h \in \mathrm{Sp}(1,1). \tag{2.13}$$

For all $h \in \mathrm{Sp}(1,1)$, we have the following factorization[23]

$$h = k\alpha(t)k', \tag{B1}$$

where $k, k' \in \mathrm{Spin}(4)$, maximal compact subgroup of $\mathrm{Sp}(1,1)$, $\alpha(t) \in A$, one parameter subgroup of the matrices:

$$\alpha(t) = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix}, \quad \alpha^{-1}(t) = \alpha(-t).$$

Thus

$$\mathcal{T}^1_{nlm, n'l'm'}(k\alpha(t)k') = \sum_{\lambda\mu} \mathcal{T}^1_{nlm, n\lambda\mu}(k)\mathcal{T}^1_{n\lambda\mu, n'\lambda\mu}(\alpha(t))\mathcal{T}^1_{n'\lambda\mu, n'l'm'}(k'). \tag{B2}$$

$\mathcal{T}^1$ is unitary when restricted to Spin(4). Thus

$$\mathcal{T}^1_{nlm,n\lambda\mu}(k)=\mathcal{T}^{1*}_{n\lambda\mu,nlm}(k^{-1}),$$
$$\mathcal{T}^1_{n'\lambda\mu,n'l'm'}(k')=\mathcal{T}^{1*}_{n'l'm',n'\lambda\mu}(k'^{-1}). \tag{B3}$$

On the other hand, we can state from (A12) that $\mathcal{T}^1$ satisfies

$$\mathcal{T}^1_{n\lambda\mu,n'\lambda\mu}(\alpha(t))=\frac{n'}{n}\mathcal{T}^1_{n'\lambda\mu,n\lambda\mu}(\alpha(-t)),\quad \text{for all } \alpha(t)\in A. \tag{B4}$$

Let us also remark that this matrix element is real. Inserting (B3) and (B4) in (B2), we finally obtain:

$$\begin{aligned}\mathcal{T}^1_{nlm,n'l'm'}&(k\alpha(t)k')\\&=\frac{n'}{n}\sum_{\lambda,\mu}\mathcal{T}^{1*}_{n'l'm',n'\lambda\mu}(k'^{-1})\mathcal{T}^1_{n'\lambda\mu,n\lambda\mu}(\alpha^{-1}(t))\mathcal{T}^{1*}_{n\lambda\mu,nlm}(k^{-1})\\&=\frac{n'}{n}\mathcal{T}^{1*}_{n'l'm',nlm}(k'^{-1}\alpha^{-1}(t)k^{-1})\\&=\frac{n'}{n}\mathcal{T}^{1*}_{n'l'm',nlm}(h^{-1}).\end{aligned} \tag{B5}$$

APPENDIX C

In this appendix, we shall examine the details of the calculation of the matrix elements $I_{NN'}(E)$.

In a first step, we suppose E negative and we reexpress the calculus on $L^2_{\mathbb{C}}(\mathrm{SU}(2))$ as in Refs. 7 and 26 by means of:

(i) the inverse stereographic projection (1.6) or (2.6),

$$\xi_n=s^{-1}(p_n)\cdot(p_n,\mathbf{p}),\quad \xi'_{n'}=s^{-1}(p_{n'})\cdot(p_{n'},\mathbf{p}'),$$
$$\xi=s^{-1}(p_0)\cdot(p_0,\mathbf{p}), \tag{C1}$$

where

$$p_n=(-2mE_n)^{1/2},\quad p_{n'}=(-2mE_{n'})^{1/2},\quad p_0=(-2mE)^{1/2},$$

and consequently

$$\mathbf{p}\cdot\boldsymbol{\epsilon}_1=\left(\frac{p_n^2+p^2}{2p_n}\right)\xi_n\cdot\epsilon_1,$$
$$\mathbf{p}'\cdot\boldsymbol{\epsilon}_2=\left(\frac{p_{n'}^2+p'^2}{2p_{n'}}\right)\xi'_{n'}\cdot\epsilon_2; \tag{C2}$$

(ii) the Fock transformations (1.13):

$$Y_N\in\mathcal{H}(p_n)\to\mathcal{F}_n^{-1}Y_N=\psi_N,$$
$$Y_{N'}\in\mathcal{H}(p_{n'})\to\mathcal{F}_{n'}^{-1}Y_{N'}=\psi_{N'}; \tag{C3}$$

(iii) the Schwinger–Sturmian expansion[26] of the Coulomb Green function, taking into account the retardation effect:

$$\begin{aligned}(\exp(i\mathbf{k}_1\cdot\mathbf{r})&G(E)\exp(i\mathbf{k}\cdot\mathbf{r}))(\mathbf{p},\mathbf{p}')\\&=G(E)(\mathbf{p}-\mathbf{k}_1,\mathbf{p}'+\mathbf{k}_2)\\&=-\frac{m}{p_0}\left(\frac{p_0^2+|\mathbf{p}-\mathbf{k}_1|^2}{2p_0}\right)^{-1}\left(\frac{p_0^2+|\mathbf{p}'+\mathbf{k}_2|^2}{2p_0}\right)^{-2}\\&\quad\times\sum_{N_1}\left(1-\frac{\nu}{n_1}\right)^{-1}Y_{N_1}(h_1^{-1}\cdot\xi_n)Y^*_{N_1}(h_2\cdot\xi_{n'}),\end{aligned} \tag{C4}$$

where $\nu=m\alpha z/p_0$, and h_1 and h_2 are elements of Sp(1,1) induced by the general boosts in $\mathbf{H}$,

$$\bar{K}_1=(p_0-p_n,-\mathbf{k}_1),\quad K_2=(p_0-p_{n'},\mathbf{k}_2),$$

and the inverse stereographic projections

$$h_1^{-1}=s^{-1}(p_0)t_{\bar{K}_1}s(p_n),$$
$$h_2=s^{-1}(p_0)t_{K_2}s(p_{n'}).$$

By introducing

$$K_{1\pm}=(p_0\pm p_n,\mathbf{k}_1),$$
$$K_{2\pm}=(p_0\pm p_{n'},\mathbf{k}_2), \tag{C5}$$

we obtain

$$h_1=\frac{1}{2\sqrt{p_0p_n}}\begin{pmatrix}K_{1+}&-\bar{K}_{1-}\\-K_{1-}&\bar{K}_{1+}\end{pmatrix},$$
$$h_2=\frac{1}{2\sqrt{p_0p_{n'}}}\begin{pmatrix}K_{2+}&K_{2-}\\\bar{K}_{2-}&K_{2+}\end{pmatrix}; \tag{C6}$$

(iv) the relations (1.9) between the scalar products of the two Hilbert spaces $L^2_{\mathbb{C}}(\mathbb{R}^3)$ and $L^2_{\mathbb{C}}(\mathrm{SU}(2))$.

Finally, we have to compute

$$\begin{aligned}I^2_{NN'}(E)=-\frac{m}{p_0}(p_np_{n'})^{1/2}&\sum_{N_1}\left(1-\frac{\nu}{n_1}\right)^{-1}\\&\times(\xi'_{n'}\cdot\epsilon_2Y_{N'},\mathcal{T}^2(h_2^{-1})Y_{N_1})_{\mathcal{H}(p_{n'})}\\&\times(\mathcal{T}^2(h_1)Y_{N_1},\xi_n\cdot\epsilon_1Y_N)_{\mathcal{H}(p_n)},\end{aligned}$$

where $\mathcal{T}^2$ is defined by (2.9).

Reducing the product $\xi\cdot\epsilon Y_N(\xi)$ is easy. One may define an operator $C(\epsilon)$ which is the image of an element of the SU*(4) enveloping algebra under its representation in $\mathcal{L}(L^2_{\mathbb{C}}(\mathrm{SU}(2)))$.

The matrix elements $C_{N'N}(\epsilon)$ are given explicitly in Appendix D:

$$\xi\cdot\epsilon Y_N(\xi)\equiv C(\epsilon)Y_N(\xi)$$
$$=\sum_{N'}C_{N'N}(\epsilon)Y_{N'}(\xi). \tag{C8}$$

Thus, in terms of the matrix elements of the representation $\mathcal{T}^2$ of Sp(1,1) and of the operators $C(\epsilon_1)$ and $C(\epsilon_2)$, Eq. (C7) is written:

$$\begin{aligned}I^2_{NN'}(E)=-\frac{m}{p_0}(p_np_{n'})^{1/2}&\sum_{N_0,N_0'}C_{N_0'N'}(\epsilon_2)C^*_{NN_0}(\epsilon_1)\\&\times\left[\sum_{N_1}\left(1-\frac{\nu}{n_1}\right)^{-1}\mathcal{T}^{2*}_{N_0'N_1}(h_2^{-1})\mathcal{T}^2_{N_0N_1}(h_1)\right].\end{aligned} \tag{C9}$$

By use of (2.12) and (2.14), the expansion between the brackets is equal to

$$\sum_{N_1}(n_1-\nu)^{-1}\mathcal{T}^1_{N_0N_1}(h_1)\mathcal{T}^1_{N_1N_0'}(h_2). \tag{C10}$$

Now, as[9,26]

$$(n_1-\nu)^{-1}=\int_0^{+\infty}dx\exp[(\nu-n_1)x], \tag{C11}$$

let us put

$$\upsilon\equiv\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \exp[(x/2)\upsilon]=\begin{pmatrix}\exp(x/2)&0\\0&\exp(-x/2)\end{pmatrix}, \tag{C12}$$

and realize the following homogeneity property, deduced from the integral (2.17)

$$\exp(-n_1x)\mathcal{T}^1_{N_0N_1}(h_1)=S^1_{N_0N_1}(h_1\exp[(x/2)\upsilon]). \tag{C13}$$

Since $|(\exp[-(x/2)\upsilon]h)\cdot\xi|<1$ for all $x>0$, $\xi\in\mathrm{SU}(2)$, $h\in\mathrm{Sp}(1,1)$, $S^1_{N_0N_1}(h_1\exp[(x/2)\upsilon])$ is merely the matrix element of the local representation (2.16) of SU*(4), and we can apply the fundamental property of the group representation to sum the expansion (C10) and to obtain Eq. (3.4),

$$\sum_{N_1} T^1_{N_0 N_1}(h_1 \exp[(x/2)\upsilon]) T^1_{N_1 N_0}(h_2) = T^1_{N_0 N_0'}(h_1 \exp[(x/2)\upsilon] h_2). \tag{C14}$$

APPENDIX D

We give in this appendix the expression of the matrix elements $C_{N'N}(\epsilon)$ defined by (C8),

$$\boldsymbol{\xi}\cdot\boldsymbol{\epsilon} Y_N(\xi) \equiv C(\boldsymbol{\epsilon}) Y_N(\xi) = \sum_{N'} C_{N'N}(\boldsymbol{\epsilon}) Y_{N'}(\xi). \tag{C8}$$

For all $N=(n,l,m)$, let us define the numbers:

$$a(n,l)=\frac{1}{2}\left(\frac{(n+l)(n+l-1)}{n(n-1)}\right)^{1/2}, \quad a(1,0)=0,$$

$$b(l,m)=\left(\frac{(l+m)(l-m)}{(2l+1)(2l-1)}\right)^{1/2}, \qquad b(0,0)=0, \tag{D1}$$

$$c(l,m)=\left(\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}\right)^{1/2}, \quad c(0,0)=0.$$

Then

$$C_{N'N}(\epsilon)=\epsilon_z\delta_{mm'}T^m_{nl,n'l'}+\tfrac{1}{2}(\epsilon_x+i\epsilon_y)\delta_{m-1,m'}U^m_{nl,n'l'} - \tfrac{1}{2}(\epsilon_x-i\epsilon_y)\delta_{m+1,m'}U^{-m}_{nl,n'l'}, \tag{D2}$$

where

$$T^m_{nl,n'l'}=\delta_{n-1,n'}(\delta_{l-1,l'}a(n,l)b(l,m)-\delta_{l+1,l'}a(n,-l-1) \times b(l+1,m)) - \delta_{n+1,n'}(\delta_{l-1,l'}a(n+1,-l)b(l,m) - \delta_{l+1,l'}a(n+1,l+1)b(l+1,m)), \tag{D3}$$

$$U^m_{nl,n'l'}=\delta_{n-1,n'}(\delta_{l-1,l'}a(n,l)c(l,m) + \delta_{l+1,l'}a(n,-l-1)c(l+1,-m)) - \delta_{n+1,n'}(\delta_{l-1,l'}a(n+1,-l)c(l,m) + \delta_{l+1,l'}a(n+1,l+1)c(l+1,-m)). \tag{D4}$$

APPENDIX E

In this appendix, we make explicit the coefficient $C^{qq'}_{N_0N_0'}(h_1,h_2)$ appearing in Eq. (3.8). If we consider the expression

$$T^1_{N_0N_0'}(g(x)) \equiv S^1_{N_0N_0'}(g(x))$$

which is given by Eq. (A10), we note that it is a question of expanding, in powers of e^{-x} and $|K_{2+}K_{1+} - e^{-x}\bar{K}_{2-}\bar{K}_{1-}|$, the function

$$F\left(\frac{n_0-1}{2}m_1m_2;\frac{n_0'-1}{2}m_1'm_2';g(x)\right).$$

Let us note that $g^{-1}(x)$, given by Eq. (3.5), is the sum of two matrices,

$$g^{-1}(x)=\frac{e^{x/2}}{4p_0\sqrt{p_np_{n'}}}\left[\begin{pmatrix} -K_{2-}K_{1-} & -K_{2-}K_{1+}\\ K_{2+}K_{1-} & K_{2+}K_{1+}\end{pmatrix} + e^{-x}\begin{pmatrix} \bar{K}_{2+}\bar{K}_{1+} & \bar{K}_{2+}\bar{K}_{1-}\\ -\bar{K}_{2-}\bar{K}_{1+} & -\bar{K}_{2-}\bar{K}_{1-}\end{pmatrix}\right]. \tag{E1}$$

Thus, by use of the homogeneity properties of the polynomials $D^j_{m_1m_2}$ and of the addition theorem (A5), we obtain:

$$j_0=\frac{n_0-1}{2}, \quad j_0'=\frac{n_0'-1}{2},$$

$$C^{qq'}_{N_0N_0'}(h_1,h_2)=16p_0^2p_np_{n'}[n_0n_0'(2l_0+1)(2l_0'+1)]^{1/2}i^{-(l_0+l_0')} \times \sum_{\substack{j_i,j_i',m_{i\frac{1}{2}},m'_{i\frac{1}{2}}\\ m_1,m_2,m_1,m_2'}} \delta_{2j_4+1,q'}\delta_{2\Sigma_i j_i'+1,q}\delta_{j_0',j_1+j_2}$$
$$\times\delta_{j_4,j_0'+j_3}\delta_{j_0,j_1+j_3}\delta_{m_1',m_{11}+m_{21}}\delta_{m_{42},m_2'+m_{31}} \times\delta_{m_{41},m_2+m_{22}}\delta_{m_1,m_{12}+m_{32}}(-1)^{j_0+j_0'+2j_3-m_2-m_2'}$$
$$\times\begin{pmatrix} j_0 & j_0 & l_0\\ m_1 & -m_2 & m_0\end{pmatrix}\begin{pmatrix} j_0' & j_0' & l_0'\\ m_1' & -m_2' & m_0'\end{pmatrix} \times(\sigma^{j_4}_{m_{41}m_{42}})^{-2}\frac{\sigma^{j_0}_{m_2}\sigma^{j_0'}_{m_2'}}{\sigma^{j_0}_{m_1}\sigma^{j_0'}_{m_1'}}\prod_{i=1}^{4}\sigma^{j_i-j_i'}_{m_{i1}-m_{i1}m_{i2}-m_{i2}} \times\sigma^{j_i'}_{m_{i1}'m_{i2}'}D^{j_i-j_i'}_{m_{i1}-m_{i1}'m_{i2}-m_{i2}'}(A_i)D^{j_i'}_{m_{i1}'m_{i2}'}(B_i),$$

$$A_1=-K_{2-}K_{1-}, \quad A_2=-K_{2-}K_{1+},$$
$$A_3=K_{2+}K_{1-}, \quad A_4=\bar{K}_{1+}\bar{K}_{2+},$$
$$B_1=\bar{K}_{2+}\bar{K}_{1+}, \quad B_2=\bar{K}_{2+}\bar{K}_{1-},$$
$$B_3=-\bar{K}_{2-}\bar{K}_{1+}, \quad B_4=-K_{1-}K_{2-}. \tag{E2}$$

[1]A. Maquet, Phys. Rev. A 15, 1088 (1977).
[2]Ya.I. Granovskii, Zh. Eksp. Teor. Fiz. 56, 605 (1969) [Sov. Phys. JETP 29, 333 (1969)].
[3]S. Klarsfeld, Lett. Nuovo Cimento, Ser. I, 1, 682 (1969); 2, 548 (1969); 3, 395 (1970).
[4]M. Gavrila, Phys. Rev. 163, 147 (1967).
[5]M. Gavrila and A. Costescu, Phys. Rev. A 2, 1752 (1970); 4, 1688 (1971).
[6]M. Lieber, Phys. Rev. 174, 2037 (1968).
[7]J.P. Gazeau, C.R. Acad. Sci. Paris B 272, 501 (1971); J.P. Gazeau and A. Maquet, C.R. Acad. Sci. Paris B 272, 1097 (1971).
[8]C. Fronsdal, Phys. Rev. 179, 1513 (1969).
[9]R.W. Huff, Phys. Rev. 186, 1367 (1969).
[10]G. Bisiacchi and G. Calucci, Phys. Rev. 181, 185 (1969).
[11]A. Dalgarno and J.T. Lewis, Proc. Phys. Soc. A 233, 70 (1969).
[12]Y. Gontier and M. Trahin, Phys. Rev. 172, 83 (1968); Phys. Rev. A 4, 1896 (1971).
[13]S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic, New York, 1962); G. Grensing, J. Math. Phys. 16, 312 (1975).
[14]W. Fock, Z. Phys. 98, 145 (1935).
[15]A.O. Barut and H. Kleinert, Phys. Rev. 156, 1541 (1967); C. Fronsdal, Phys. Rev. 156, 1665 (1967); A.O. Barut and H. Kleinert, Phys. Rev. 157, 1180 (1967); A.O. Barut in *Lie Algebras, Applications and Computational Methods*, edited by B. Kolman (Soc. Ind. Appl. Math. Philadelphia, 1973), p. 79.
[16]M. Rotenberg, Ann. Phys. (N.Y.) 19, 262 (1962).
[17]M. Rotenberg, *Advances in Atomic and Molecular Physics* edited by R.D. Bates and I. Estermann (Academic, New York, 1970), Vol. 6.
[18]A. Joseph, Int. J. Quant. Chem. 1, 535 (1967).
[19]A. Decoster, thesis, Rapport C.E.A.-R-4112 (1970); Nuovo Cimento A 68, 105 (1970).
[20]M. Bander and C. Itzyckson, Rev. Mod. Phys. 38, 330 (1966); R.J. Finkelstein, J. Math. Phys. 8, 443 (1967); R. Finkelstein and D. Levy, J. Math. Phys. 8, 2147 (1967).
[21]N. Bourbaki, *Topologie générale* (Hermann, Paris, 1974), Chap. 8.
[22]C. Fronsdal, "Lectures on Bound Electron Problems," International Centre for Theoretical Physics, Miramare Trieste (1970).
[23]R. Takahashi, "Sur les représentations unitaires des groupes de Lorentz généralisés," Bull. Soc. Math. France 91, 289 (1963).
[24]S. Ström, Ark. Fys. 33, 465 (1967); Ann. Inst. H. Poincaré 13, 77 (1970).
[25]L. Hostler, J. Math. Phys. 5, 1235 (1964).
[26]J. Schwinger, J. Math. Phys. 5, 1606 (1964).
[27]P. Appell and J. Kampe de Feriet, *Fonctions Hypergéométriques and Hypersphériques. Polynômes d'Hermite* (Gauthiers-Villars, Paris, 1926).
[28]J.D. Talman, *Special functions. A Group Theoretic Approach* (Based on lectures by Eugène P. Wigner) (Benjamin, New York, 1968).
[29]J.P. Gazeau, "Fifth International Colloquium in Group Theoretical Methods in Physics," Montreal, July 1976.