**The algorithm**

Given a distance matrix *D* *ij*  , define

*M* *ij* =(*D* 2 1*j* +*D* 2 *i*1 −*D* 2 *ij)*  / 2 .

One thing that is good to know in case the dimensionality of the data that generated the distance matrix is not known is that the smallest (Euclidean) dimension in which the points can be embedded is given by the rank *k* of the matrix *M*  . No embedding is possible if *M* is not positive semi-definite.

The coordinates of the points can now be obtained by eigenvalue decomposition: if we write *M*=*USU* *T*  , then the matrix *X*=*U* √ S

(you can take the square root element by element) gives the positions of the points (eachrow corresponding to one point). Note that, if the data points can be embedded in *k*  dimensional space, only *k* columns of *X* will be non-zero (corresponding to *k* non-zero eigenvalues of *M* ).

**Why does this work?**

If *D*  comes from distances between points, then there are **x** *i* ∈R *m*   such that

*D* 2 *ij* =(**x** *i* −**x** *j* ) 2 =**x** 2 *i* +**x** 2 *j* −2**x** *i* ⋅**x** *j* .

Then the matrix *M* defined above takes on a particularly nice form:

*M* *ij* =(**x** *i* −**x** 1 )⋅(**x** *j* −**x** 1 )≡∑ *a*=1 *m* *x* ~  *ia* *x* ~  *ja* ,

where the elements *x* ~  *ia* =*x* *ia* −*x* 1*a*   can be assembled into an *n*×*m*  matrix *X* ~  . In matrix form,

*M*=*X* ~ *X* ~  *T* .

Such a matrix is called a [Gram matrix](http://en.wikipedia.org/wiki/Gramian_matrix). Since the original vectors were given in *m*  dimensions, the rank of *M*  is at most *m*  (assuming *m*≤*n* ).

The points we get by the eigenvalue decomposition described above need not exactly matchthe points that were put into the calculation of the distance matrix. However, they However, they can be obtained from them by a rotation and a translation. This can be proved for example by doing a singular value decomposition of *X* ~   , and showing that if *X* ~ *X* ~  *T* =*XX* *T*   (where *X*  can be obtained from the eigenvalue decomposition, as above, *X*=*US*  √  ), then *X*  must be the same as *X* ~   up to an orthogonal transformation.cobtaine from them by a rotation and a translation.

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