Topology

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Contents

1 Set theory

1.1 Inverse functions

If f : A -> B,

$$f^{-1}(B_0) = \{a|f(a) \in B_0\}$$

1.2 Relations

1.2.1 Equivalence relations

If there is a equivalence relation C on A it has the following properties,

1. Reflexivity: xCx for every x in A.
2. Symmetry: if xCy, then yCx.
3. Transitivity: if xCy and yCz, then xCz.

A **equivalence class** determined by x is given by:

$$E = \{y|y \sim x\}$$

Lemma: Two equivalence classes E and E' are either disjoint or equal.

Partition: Collection of disjoint nonempty subsets of A whose union is all of A.

Note that given any partition of A, there is exactly one equivalence relation from which it is derived.

Example: Define two points in the plane to be equivalent is they lie at the same distance from the origin. Then it is a equivalence relation and the collection of equivalence classes consists of all circles centered at the origin, along with the origin alone.

1.2.2 Order relation

1. (Comparability) For every x and y in A for which x != y, either xCy or yCx.

2. (Nonreflexivity) For no x in A does the relation xCx hold.

3. (Transitivity) If xCy and yCz, then xCz.

(a,b) is then the set {x | a < x < b}, which is an open interval. If the set is empty, a (b) is the immediate predecessor (sucessor) of b (a).

Order type: A and B have the same order type if there is a bijective correspondence between them that preserves order.

Dictionary order: A "lexicographic" order relation for cartesian products.

Least upper bound property: An ordered set A has the lub property if every nonempty subset A_0 of A that is bounded above has a least upper bound. The greatest lowerbound property is defined similarly.

1.2.3 Size of set

1. A is finite if $A \sim J_n$ for some n, where J_n is the set whose elements are the integers 1 to n.

2. A is infinite if A is not finite/ A is equivalent to one of its proper subsets.

3. A is countable if $A \sim J$

4. A is uncountable if A is neither finite nor countable

5. A is at most countable if A is finite or countable

Corollaries:

- Set of all integers is countable

Proof: The set of all integers is countable as we can set up a 1-1 correspondence: $f(n) = n/2$ when n is even, and $f(n) = -(n-1)/2$ when n is odd.

- Every infinite subset of a countable set A is countable
- The union of a sequence of countable sets is countable: Cantor diagonalisation
- If A is a countable set, and B_n is the set of all n-tuples $(a_1, \ldots, a_n)$, where $a_k \in A(k = 1, \ldots, n)$ and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_n is countable

Theorem
The set of all sequences whose elements are the digits 0 and 1 is uncountable.

1.3 Metric Spaces

A set X, whose elements we shall call points, is said to be a metric space if any two points p,q of X there is associated a real number d(p,q), such that

1. $d(p,q) > 0$, if $p \neq q$; $d(p,p) = 0$;
2. $d(p,q) = d(q,p)$
3. $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$

Any function with these properties is called a **distance function** or **metric**.

Definition
If $a_i < b_i$ for all i, then the set of points in euclidean space that satisfies the inequality $a_i \leq x_i \leq b_i$ for all i is called a **k-cell**.

If $x \in R^k$ and $r > 0$, the **open** or **closed ball** B, with center at x and radius r is defined to be the set of all $y \in R^k$ such that $|y - x| < r$ or likewise for closed balls.

A set $E \subset R^k$ is **convex** if $\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in E$, where $\boldsymbol{x}, \boldsymbol{y} \in E$ and $0 < \lambda < 1$.

Definition

1. A **neighbourhood** of p is a set $N_r(p)$ consisting of all q such that $d(p,q) < r$, for some $r > 0$. r is the **radius** of $N_r(p)$.

2. A point p is a **limit point** of the set E if *every* neighbourhood of p contains a point $q \neq p$ such that $q \in E$.

3. If $p \in E$ and p is not a limit point of E, then p is called an **isolated point** of E.

4. E is **closed** if every limit point of E is a point of E.

5. A point p is an **interior** point of E if there is a neighbourhood N of p such that $N \subset E$.

6. E is **open** if every point of E is an interior point of E.

7. The **complement** of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$

8. E is **perfect** if E is closed and every point of E is a limit point of E. i.e. a point is a limit point of E iff $p \in E$.

9. E is **bounded** if there is a real number M and a point $q \in X$ such that $d(p,q) < M \forall p \in E$.

10. E is **dense** in X if every point of X is a limit point of E, or a point of E or both.

Theorem

Every neighbourhood is an open set.

Theorem

If p is a limit point of a set E, then every neighbourhood of p contains infinitely many point of E.

Corollary

A finite point set has no limit points.

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Theorem
Let E_α be a collection of sets. Then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c)$$

Theorem
A set is open iff its complement is closed
A set F is closed iff its complement is open.
Theorem

1. For any collection of open sets, $\{G_\alpha\}$, $\cup_\alpha G_\alpha$ is open.
2. For any collection of closed sets, $\{F_\alpha\}$, $\cap_\alpha G_\alpha$ is closed.
3. For any finite collection of open sets, $\cap_i G_i$ is open.

4. For any finite collection of closed sets, $\cup_i F_i$ is closed.

Definition
If X is a metric space, E is a subset of X and if E' is the set of limit points of E in X, then the **closure** of E is the set $\bar{E} = E \cup E'$. in **Theorem**

1. $\bar{E}$ is closed.

2. $E = \bar{E}$ iff E is closed.

3. $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

Theorem Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$, then $y \in \bar{E}$. Hence, $y \in E$ if E is closed.
Theorem
Suppose $Y \subset X$. A subset E of Y is open relative to Y iff $E = Y \cap G$ for some open subset G of X.
(???)

1.4 Compact sets

Definition
An **open cover** of a set E in a metric space X, we mean a collection of open subsets of X such that $E \subset \cup_\alpha G_\alpha$.
Definition
A subset K of of a metric space X is said to be **compact** if every open cover of K contains a subcover. i.e. if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \dots, \alpha_n$, such that:

$$K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$$

Theorem
Suppose $K \subset Y \subset X$. Then K is compact relative to X iff K is compact relative to Y.
Theorem
Compact subsets of metric spaces are closed.
Theorem
Closed subsets of compact sets are compact. **Corollary**: If F is closed and K is compact, then $F \cap K$ is compact.
Theorem
If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{_a lpha\}$ is non-empty, then

$\cap K_\alpha$ is nonempty. **Corollary**: If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ then $\cap_1^\infty K_n$ is not empty.

Theorem If E is an infinite subset of a compact set K, then E has a limit point in K.

Theorem If $\{I_n\}$ is a sequence of intervals in R^1, such that $I_n \supset I_{n+1}$ then $\cap_1^\infty I_n$ is not empty.

Theorem If $\{I_n\}$ is sequence of k-cells such that $I_n I_{n+1}$, then $\cap_1^\infty I_n$ is not empty.

Theorem Every k-cell is compact.

Theorem If a set in R^k has one of the following three properties, it has the other two.

1. E is closed and bounded
2. E is compact
3. Every infinite subset of E has a limit point in E.

   (b) and (c) are equivalent in any metric space but (a) does not in general imply (b) and (c).

Theorem (Weierstrass)
Every bounded infinite subset of R^k has a limit point in R^k.

1.5 Perfect sets

Theorem
A nonempty perfect set in R^k is uncountable.

1.6 Connected sets

Two subsets A,B of a metric space X are *seperated* if $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty. A subset of X is connected if it is not a union of two nonempty seperated sets.

Theorem
A subset E of the real line R^1 is connected iff it has the following property: If $x, y \in E$, $x < z < y$, then $z \in E$.

2 Topological spaces

Definition
A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties.

1. $\emptyset$ and X are in $\mathcal{T}$

2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$

3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

The collection of all subsets of X is called the **discrete topology**. The collection consisting of X and $\emptyset$ only is the **indiscrete/trivial topology**.

Let $\mathcal{T}_f$ be the collection of all subsets U of X such that X-U either is at most countable or is all of X. Then $\mathcal{T}_f$ is the **finite complement topology**.

Definition

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies. If $\mathcal{T}' \supset \mathcal{T}$, we say $\mathcal{T}'$ is **finer** than $\mathcal{T}$. If it proper contains $\mathcal{T}$, we say strictly finer than $\mathcal{T}$. The reverse is called coarser. $\mathcal{T}$ is comparable with $\mathcal{T}'$ if one is the subset of the other.

2.1 Basis

A **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that:

1. For each $x \in X$, there is at least one basis element B containing x. (B is a cover)

2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these conditions, we define the topology generated by $\mathcal{B}$ as: A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note each element is an element of $\mathcal{T}$.

Lemma

$\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Lemma

Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X. (C make up a cover and open set of X is a superset of some C).

Proof. We will show why every element in $\mathcal{T}$ belongs in the topology generated by the basis, $\mathcal{T}'$. Since for $x \in C \subset U$, there exists a union of C which equals U. The converse follows from the previous lemma.

Lemma (Fineness)

Let $\mathcal{B}$ and $\mathcal{B}'$ be the bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on X. Then the following are equivalent.

1. $\mathcal{T}'$ is finer than $\mathcal{T}$
2. For each $x \in X$, and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.

2.1.1 Common topologies

Topology	Basis	Symbol
Standard	(a,b)	$\mathbb{R}$
Lower-limit	[a,b]	$\mathbb{R}_l$
K-topology	(a,b) and (a,b) - K	$\mathbb{R}_K$

Note: K is the set of all numbers $1/n$ for each positive integer n.

Lemma Topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

2.1.2 Subbasis

What if you extend the basis to also take finite intersections?

Definition

A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

2.2 Order topology

Definition

Let X be a set with a simple order relation. The collection of all sets of the following types:

1. All open intervals
2. All intervals of the form $[a_0, b]$ where a_0 is the smallest element if any
3. All intervals of the form $[a, b_0]$ where b_0 is the largest element if any

is the basis for the **order topology** on X.

2.3 Product topology

Definition

Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection of all sets of the form $U \times V$, where $U \in X, V \in Y$.

Theorem

If $\mathcal{B}$ is the basis for the topology of X and $\mathcal{C}$ is a basis topology of Y, then the collection:

$$\mathcal{D} = \{B \times C | B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

We are also interested in a subbasis.

Theorem

$$\mathcal{S} = \{\pi_1^{-1}(U) | U \text{ open in } X\} \cup \{\pi_2^{-1}(V) | V \text{ open in } Y\}$$

2.4 Subspace topology

Definition

Let X be a topological space with topology $\mathcal{T}$. If $Y \subset X$,

$$\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$$

is the **subspace topology** on Y. Y is then a **subspace** of X,

Lemma

A basis can be derived in a similar form. (replace Y with B).

Lemma

If U is open in Y and Y is open in X, then U is open in X.

Definition

A subset Y of X is convex in X if for each pair of points $a < b$ of Y, the entire interval of points of X lies in Y.

Theorem

Let X be an ordered set in the order topology, let Y be a subset of X that is convex in X. Then the order topology on Y, is the same as the topology Y inherits as a subspace of X.

2.5 Closed sets

Defining topological space with closed sets.

Theorem

In a topological space,

1. the empty set and the whole set are closed
2. Arbitrary intersections of closed sets are closed
3. Finite unions of closed sets are closed

2.5.1 Closures

Theorem

Let A be a subset of the topological space X.

1. Then $x \in \bar{A}$ iff every neighbourhood of x intersects A.
2. If X is given by a basis, $x \in \bar{A}$ iff every basis element containing x intersects A.

2.6 Hausdorff spaces

Usually it is nicer to have one-point sets closed like in euclidean space, as this means that sequences don't converge to multiple values for instance.

Definition

A topological space X is a **Hausdorff space**, if for each pair of distinct points in X, there exist neighborhoods of each point that are disjoint.

Theorem

Every finite point set in a hausdorff space is closed.

The Hausdorff space condition is stronger than the condition that finite point sets be closed (T_1 axiom) but that's fine. But for fun:

Theorem

Let X be a space satisfying the T_1 axiom. Let A be a subset of X, then the point x is a limit point of A iff every neighbourhood of x contains infinitely many points of A.

Back to hausdorff spaces:

Theorem

Every simply ordered set is a Hausdorff space in the order topology. Product and subspaces of hausdorff spaces are hausdorff spaces.

2.7 Continuous functions

Definition

A function $f : X \to Y$ is continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

It also suffices to show that the inverse image of each basis/subbasis element is open.

Other definitions:

Theorem

1. For every subset of X, one has $f(\bar{A}) \subset \bar{f}(A)$.
2. For every closed set of B of Y, the set $f^{-1}(B)$ is closed in X.
3. For every $x \in X$ and each neighbourhood V of f(x), there is a neighbourhood U of x such that $f(U) \subset V$.

Definition

If f is an injective continuous map, and f' is the surjective function by restricting the range of f, f is a topological imbedding if f' is a homeomorphism of X with Z.

2.7.1 Homeomorphisms

Let $f : X \rightarrow Y$ be a bijection. If both f and its inverse function are continuous, f is called a homeomorphism. Another way to define it is to say it is a bijective correspondence such that $f(U)$ is open iff U is open.

2.8 Metric topology

Definition

The collection of all \$$\epsilon$\$-balls $B_d(x, \epsilon)$ is a basis for a topology on X, called the **metric topology**, induced by d.

Definition (alt)

A set U is open in the metric topology induced by d iff for each $y \in U$, there is a $\delta > 0$ s.t. $B_d(y, \delta) \subset U$.

Definition

A topological space X is **metrizable** if there exists a metric on X that induces the topology of X. A **metric space** is a metrizable space together with a specific metric that gives the topology of X.

Lemma (Sequence lemma)

Let X be a topological space. Let $A \subset X$. If there is sequence of points of A converging to x, then $x \in \bar{A}$. the converse holds if X is metrizable.

Theorem

Let $f : X \rightarrow Y$. If the fucntion if is continuous, then for every convergent sequence $x_n \rightarrow x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

Theorem (Uniform limit theorem)

Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.

2.9 Quotient topology

Definition

Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a quotient map provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

(like a homeomorphism without being injective)

Definition

A subset C of X is saturated (with respect to the surjective map) if C contains every set $p^{-1}(y)$ it intersects. Thus C is saturated if it equals the complete inverse image of a subset of Y. To say that P is a quotient map is equivalent to saying that p is continuous and p maps saturated open sets of X of to open sets of Y.

A map is an open set if for each open set in its domain, its image is also open, and likewise for closed maps. All open/closed maps are quotient maps.

Definition

If X is a space and A is a set and if $p : X \to A$ is a surjective map, there is exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map, which is the **quotient topology** induced by p.

Definition

Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a **quotient space** of X.

What concepts do quotient maps work well with?

2.9.1 Subspaces

Theorem

Let $p : X \to Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p.; let $q : A \to p(A)$ be the map obtained by restricting p.

1. If A is either open or closed in X, then q is a quotient map

2. If p is either an open map or a closed map, then q is a quotient map.

2.9.2 Composites

Composites of quotient maps are quotient maps.

2.9.3 Products

Products of maps do not behave well, and one needs conditions such as local compactness, and that the two maps are open maps.

2.9.4 Hausdorff condition

Does not behave well.

For X^* to satisfy the T_1 axiom, one requires that each eleemtn of the partition X^* be a closed subset of X.

2.9.5 Continuous functions

Similar to how we had a criterion for determining when a map into a product space was continuous, we wish to find when a map out of a quotient space is continuous.

Theorem

Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}(y)$ for $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous iff g is continuous; f is a quotient map iff g is a quotient map.

Corollary

Let $g : X \to Z$ be a surjective continuous map. Let $X*$ be the following collection of subsets of X:

$$X^* = \{g^{-1}(\{z\})|z \in Z\}$$

Give it the quotient topology.

1. The map g induces a bijective continuous map f, which is a homeomorphism iff g is a quotient map
2. If Z is Hausdorff, so is X^*.kk

2.9.6 Topological Groups

A **topological group** G is a geoup that is also a topological space satisfying the T_1 axiom, such that the map of $G \times G$ into G sending xxy into $x\dot{y}$ and the map of G into G sending x into 1/x are continuous maps.

3 Connectedness and compactness

3.1 Connected spaces

Definition

Let X be a topological space. A **seperation** of X is a pair of disjoint nonempty open subsets of X whose union is X. The space is **connected** if there does not exist a seperation of X.

Note that connectedness is a topological property.

Another formulation of connectedness is that a space is connected iff the only subsets that are both open and closed in X are the empty set and X itself.

For a subspace of a topological space, there is another useful formulation.

Lemma

If Y is a subspace of X, a seperation of Y is a pair of disjoint nonempty sets A and B whose union is Y. neither of which contains a limit point of the other. The space Y is connected if there exists no seperation of Y.

3.1.1 Forming connected spaces from given ones

Lemma

If the sets C and D form a seperation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.

Properties:

1. The union of a collection of connected subspaces of X that have a point in common is connected.

2. Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.

3. The image of a connected space under a continuous map is connected.

4. A finite cartesian product of connected spaces is connected.

3.2 Connected subspaces of the real line

A simply ordered set L having more than one element is called a **linear continuum** if the following hold:

1. L has the least upper bound property

2. If $x < y$, there exists z such that $x < z < y$

Theorem

If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L.

Corollary

The real line is connected and so are intervals and rays

Theorem (Intermediate value theorem)

Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between f(a) and f(b), then there exists a point c of X such that f(c) = r.

3.3 Path connectedness

Definition

Given points x and y of the space X, a **path** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that f(a) = x and f(b) = y. A space X is said to be path connected if every pair of points of X can be joined by a path in X.

Although a path-connected space is connected, the converse may not hold (Such as the ordered square).

3.4 Components and local connectedness

Definition

Given X, define an equivalence relation on X by setting $x\tilde{y}$ if there is a connected subspace of X containing both x and y. The equivalence classes are the components of X.

Theorem

The components of X are connected disjoint subspaces of X whose union is X. such that nonempty connected subspace of X intersects only one of them.

Theorem

the path components are defined similarly, with an equivalence relation when there is a path from x to y in X.

Theorem

The path components of X are path-connected disjoint subpaces of X whose union is X, such that each nonempty path-connected subspace of X intersects only one of them.

Definition

A space X is said to be **locally connected at x**, if for every neighbourhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is said simply to be locally connected. Similarly, a space is **locally path connected at x** if for every neighbourhood U of X, there is a path-connected neighborhood V of x contained in U.

Theorem

A space X is locally connected iff for every open set U of X, each component of U is open in X.

Theorem

A space X is locally path connected iff for every open set U of X, each path component of U is open in X.

Theorem

If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

3.5 Compact spaces

Definition

A space X is **compact** if every open covering A of X contains a finite subcollection that also covers X.

Theorem

Every closed subspace of a compact space is compact.

Theorem

Every compact subspace of a Hausdorff space is closed

Lemma

If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.

Theorem

The image of a compact space under a continuous map is compact.

Theorem

Let $f : X \to Y$ is a bijective continuous function. If X is a compact and Y is Hausdorff, then f is a homeomorphism

Useful for proving a map is a homeomorphism.

Theorem

The product of finitely many compact spaces is compact.

Lemma (The tube lemma)

Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighbourhood of x_0 in X.

For infinite products, we require the Tychanoff theorem.

3.5.1 Finite intersection

Following is another formulation of compactness.

Definition

A collection $\mathcal{C}$ of subsets of X is said to have the **finite intersection property** if for every finite subcollection, its intersection is nonempty.

Theorem

Let X be a topological space. Then X is compact iff for every collection of closed sets in X having the finite intersection property, the intersection of all its elements is nonempty.

3.6 Compact subspaces of the real line

Theorem

Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.

Corollary

Every closed interval in $\mathbb{R}$ is compact

Theorem

A subspace A of R^n is compact iff it is closed and bounded in the euclidean or square metric.

Theorem (EVT)

Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

3.6.1 Uniform continuity theorem

Definition

Let (X,d) be a metric space. let A be a nonempty subst of X. For each $x \in X$, we define the distance from x to A by the equation:

$$d(x, A) = \inf\{d(x, a) | a \in A\}$$

Lemma (The Lebesgue number lemma)

Let $\mathcal{A}$ be an open covering of the metric space (X,d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less that δ, there exists an element of $\mathcal{A}$ containing it.

δ is known as the **Lebesgue number**.

Definition

A function f fomr the metric space (X, d_X) to the metric space (Y, d_Y) is said to be **uniformly continuous** if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X.

Theorem (Uniform continuity theorem)

Let $f : X \to Y$ be a continuous map of the compact metric space (X,dx) to the metric space (Y,dy). Then f is uniformly continuous.

3.6.2 Uncountability of real numbers

Definition

A point x of a space X is said to be an **isolated point** of X if the one-point set $\{x\}$ is open in X.

Theorem

Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.

Corollary

Every closed interval in $\mathbb{R}$ is uncountable.

3.7 Limit point compactness

Also known as **Frechet compactness**, or **Bolzano-Weierstrass property**, and was the former definition of compactness whereas the covering formulation was called "bicompactness".

Theorem

Compactness implies limit point compactness, but not conversely.

Definition

Let X be a topological space. It is **sequentially compact** if every sequence of points of X has a convergent subsequence.

But metrizable spaces are very nice so:

Theorem

Let X be a metrizable space. Then the three defintions of compactness are equivalent.

3.8 Local compactness

We wish to prove the basic theorem that any locally compact Hausdorff space can be imbedded in a certain compact Hausdorff space that is called its **one-point compactification.**

Definition

A space X is said to be **locally compact at x** if there is some compact subspace C of X that contains a neighbourhood of x.

Metrizable spaces and compact Hausdorff spaces are very well behaved. A subspace of a metrizable space is also metrizable but the subspace of a compact Hausdorff space need not be compact.

Theorem

Let X be a space. Then X is locally compact hausdorff iff there exists a space Y satisfying the following conditions:

1. X is a subspace of Y.
2. The set Y - X consists of a single point.
3. Y is a compact Hausdorff space.

If there are two spaces satisfying these conditions, then there is a homeomorphism between them that equals the identity map on X.

Definition

If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be a **compactification** of X. If Y-X equals a single point, then Y is called the **one-point compactification** of X.

But our definition of local compactness does not involve arbitrarily small neighbourhoods like the other definitions. Thus, here is a definition which is equivalent when X is Hausdorff.

Theorem

Let X be a Hausdorff space. Then X is locally compact iff given x in X, and given a neighborhood U of x, there is a neighbourhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

Corollary

Let X be locally compact Hausdorff; Let A be a subspace of X. If A is closed/open in X, then A is locally compact.

If A is closed, we don't need the Hausdorff condition.

Corollary

A space X is homeomorphic to an open subspace of a compact Hausdorff space iff X is locally compact Hausdorff.

This follows from the previous corollary and the second last theorem.

4 Countability and Seperation axioms

We wish to prove the Urysohn metrization theorem, which says that if a topological space satisfies a certain countability axiom (the second) and a certain seperation axiom (the regularity axiom), then X can be imbedded in a metric space and is thus metrizable.

Another imbedding theorem useful in geometry is that given a space that is a compact manifold, we show that it can be imbedded in some finite dimensional euclidean space.

4.1 Countability Axioms

Definition

A space X is said to have a **countable basis at x** if there is a countable collection $\mathcal{B}$ of neighbourhoods of x such that each neighbourhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the **first countability axiom**, or be **first-countable**.

Theorem

Let X be a topological space.

1. Let A be a subset of X. If there is a sequence of points A converging to x, then $x \in \bar{A}$. The converse holds if X is first-countable.

2. Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $\{x_n\}$ converges to $f(x)$. The converse holds if X is first-countable.

Definition

If a space X has a countable basis for its topology, then X is said to satisfy the **second countability axiom**, or to be **second-countable**.

This axiom implies the first, and is usually satisfied by familiar spaces.

Theorem

A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces if first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.

Definition

A subset A of X is said to be **dense** in X if $\bar{A} = X$.

Theorem

Suppose X has a countable basis. Then:

1. Every open covering of X contains a countable subcollection covering X. (**Lindelof space**)
2. There exists a countable subset of X that is dense in X. (**seperable**)

4.2 Seperation axioms

We will introduce seperation axioms stronger than Hausdorff.

Definition

Suppose that one-point sets are closed in X. Then X is said to be **regular** if for each pair consisting of a point x and a closed set B disjoint from X, there exist disjoint open sets containing x and B, respectively. The space X is said to be **normal** if for each pair A,B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

A regualar space is Hausdorff and a normal space is regular (Though we need to include the condition that one-point sets be closed. See the two-point space in the indiscrete topology satisfies the other parts of the definitions of regularity and normality without being Hausdorff).

Lemma

Let X be a topological space. Let one-point sets in X be closed.

1. X is regular iff given a point x of X and a neighbourhood U of x, there is a neighbourhood V of x such that $\bar{V} \subset U$.
2. X is normal iff given a closed se A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$.

Theorem

1. A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.
2. A subspace of a regular space is regular; a product of regular spaces is regular

But there is no analagous theorem for normal spaces