Answers to Reviewer #1

Ms. Ref. No.: JOMP-D-15-00204
Title: "Viscous Regularization for the Non-equilibrium Seven-Equation Two-Phase Flow Model',
Springer Journal of Scientific Computing

In blue are reviewer's comments, in black are our answers, in red are Jean's comments, and in green are Marco's comments.

> Line numbers: we will need to convert them the the Springer format before we resubmit. The current numbers are obtained using another format!

Viscous regularization of hyperbolic conservation laws is a meaningful approach to stabilize the numerical discretization of such problems. In the present manuscript the authors derive a viscous regularization for a two-phase model following previous work by Guermond and Popov for a single-phase fluid. The basic idea is to design regularization terms such that the entropy residuals (corresponding to the physical entropy and Harten's generalized entropies) are positive. For scalar problems in conservative form it can be proven that this ensures uniqueness of the weak solution. Therefore, positivity of the entropy residuals is considered a useful property also for systems.

The subject can be considered very interesting and relevant. The presentation is clear and the analysis is sound. However, some issues are not properly addressed and the presentation of the numerical results needs to be improved before I can recommend this paper for publication.

Thank you for your very thorough review.

1. The hyperbolic system (1) corresponding to the seven-equation two-phase ow model is not in conservative form because of the gradients $\nabla\alpha_k$ in (1a), (1c) and (1d). Therefore classical results from the theory of conservation laws cannot be directly applied to the present system. Based on the DLM theory [2] the classical definitions of a weak solution and the entropy solution, respectively, can be extended for quasi- conservative problems introducing a path, see [3, 4, 2]. Existence of a solution to a problem in non-conservative form has been investigated in [5]. In [6] existence of a solution to the Riemann problem has been verified. In particular, shock waves are influenced by the path since it enters the generalized Rankine-Hugoniot jump conditions, see [4]. Note that rarefaction waves and contact discontinuities are not affected by the choice of the path. It would be interesting for the reader to know what notion of weak solution and entropy solution is applied here.

The problem under consideration is indeed a nonlinear hyperbolic system in non-conservative form. We have added a paragraph reviewing previous work dealing with weak solution and entropy condition for non-conservative system of equations in the introduction (see lines 12-56). We also modified the section entitled *Methodology* accordingly (see lines 193-228).

We thank you for your insightful review. The following references have been added: [1], [2], [3], [4] and [5]. Marco, can you add that? done

Marco and I think we have a pretty good handle on this. It will include citing the proper references and the relevant theorems. We must remember to thank the reviewer for pointing this out to us. This was invaluable and improved the quality of the manuscript. We believe that the reviewer is actually giving us the answer to his question. In a very polite way, he is guiding us towards the answer. References [3] and the paper by S. Bianchini et al. entitled "Vanishing viscosity Solutions of Nonlinear Hyperbolic Systems" contain the theorems related to weak and entropy solutions when stabilizing non-conservative hyperbolic system of equations with a vanishing viscosity numerical method.

2. In the abstract, see page 1, line 20, the authors claim that regularization ensures uniqueness of the weak solution. No evidence is given in the manuscript, i.e., neither a reference nor a proof, that justifies this statement. For scalar problems in conservative form it is well-known that uniqueness is ensured if an entropy inequality holds for all entropy-entropy flux pairs. However, the problem at hand is not in conservative form. See also the first remark.

Our wording was incorrect and we have corrected it: we meant to state that the viscous regularization ensures convergence of the numerical solution to a weak (minimum-entropy satisfying) solution and not uniqueness of the weak solution. See also our answer to your first remark.

Here too we will need to cite the proper theorems from the list of references giving to us. Also we should be more careful about the wording and write 'entropy solution' instead of 'uniqueness of the weak solution'.

3. In (3) the authors give a particular choice for the interfacial pressure and velocity. It should be mentioned that other choices are considered in the literature for which the second law of thermodynamics can be verified, see e.g. [7].

The definitions of the interfacial pressure and velocity are not unique, as you rightfully mentioned. We added References 8 and 30 with further examples of the interfacial variables. We now emphasize that the theoretical results devised hold as long as the choice of the interfacial variables preserve the entropy inequality (see paragraph between equations (3d) and (4a)).

We should also emphasize that the results of the paper are independent of the choice of the definition for the interfacial variables as long as the entropy inequality is preserved.

4. On page 5, l. 34-37, it should be mentioned that hyperbolicity also requires a full set of linearly independent (left/right) eigenvectors. For the

considered system this only holds true if the non-resonance condition is satisfied, see [1].

We fully agree with you and modified the text accordingly (see lines 151-158).

Another great remark. We will add this

5. In view of my first remark it would be interesting to know whether the field associated to the eigenvalue λ_{3+dim} in (6) is linearly degenerated. In this case the jump in α corresponds to a contact discontinuity. still noting here ...

We should verify this. We need to ask Ray for the eigenvectors of the SEM if he has them and them check that we have a contact discontinuity I have not added anything about that yet, but I believe it is done in a reference I have to find.

6. For the numerical computations in Section 5.2 the stiffened gas law (32) has to be modified to account for the heat of formation q_k.

Thank you for pointing out this. It was a typo. We fully agree with you and have corrected it (the mistake also occurred at another location).

7. For both computations presented in Section 5 the authors should provide more information on the discretization, e.g., the type of scheme (FVM/DG, explicit/implicit) and its main ingredients. In particular, the discretization of the non-conservative terms in (1) is of interest. Furthermore, the spatial/temporal dicretization and the CFL number should be mentioned.

We had forgotten to give a short paragraph explaining our space/time discretization. Actually, it was present in a previous version of the manuscript and we removed it by mistake! It has been added now (see paragraph at the beginning of Section 5).

8. For both test cases it would be interesting to investigate grid convergence because the chosen regularization terms are grid-dependent and vanish with increasing resolution, i.e., stabilization by viscous regularization is successively reduced. In particular, the grid convergence study is recommended for the shock tube problem because the numerical results show a severe smearing for both the contact wave and the shock wave.

I hesitate a bit now that I re-read the comments. Are they asking for convergence order? We may not ha

We performed a grid convergence for the first test and plotted the numerical solutions of the phasic density (figure 2a) and the phasic volume fraction (figure 2b) for comparisons. It is then observed that the contact and shock waves are better resolved as the mesh is refined (see lines 572-576).

Marco will run finer grid simulations for both problems and plot all of the numerical solutions on a same figure in order to eyeball the increasing resolution effects.

9. Since the authors only give the initial data for velocity, pressure and temperature, the authors should also provide a caloric equation of state to compute the density, i.e., T_k.

Thank you for pointing out this. We have added it.

10. The authors verified positivity of the entropy residual. Therefore it would be interesting to see whether the numerical scheme maintains this property. Therefore I would be interested to see plots of the numerical entropy residual to check whether it remains positive.

Positivity of the entropy residual is indeed ensured for a theoretical point of view. However, when numerically evaluating the entropy residual, negative values can be obtained in the vicinity of the shock. This phenomenon has been previously discussed in [38]. We added a remark explaining such a behavior in Remark-2, lines 399-406.

11. Besides the two computations presented here I would like to recommend further numerical tests exhibiting stronger variations in the volume fractions and the density, e.g., a shock tube problem for a water-gas configuration with $(\rho, uP, \alpha)_{water} = (1000kg.m^3, 0m/s, 10^5 Pa, 0:99)$ and $(\rho, uP, \alpha)_{gas} = (0.2kg.m^3, 0m/s, 2000Pa, 0:99)$, i.e., the initial states are almost "pure" states of either water or gas, with material parameters as given in [8].

Marco will run such a case. The reference given by the reviewer does not seem to be correct. But I think this test is very common.

12. In the conclusion, see page 23, line 41/42 the authors claim that the regularization ensures uniqueness of the numerical solution. Since the authors do not give a uniqueness proof this statement seems to be too strong and should be rephrased. See also my first and second remark.

We corrected this statement that is incorrect and rephrased it in lines 701-708.

This will be corrected once we answer the reviewer's remarks #1+2.

References

[1] F. Alleges, B. Merlet, Approximate shock curbes for non-conservative hyperbolic systems in one space dimension, J. Hyperbolic Differ. Equ. 1 (4) (2004) 769–788.

[2] G. L. Floch, Shock waves for nonlinear hyperbolic systems in nonconservative forms, Tech. rep., Institute for Mathematics and its Applications, Minneapolis, MN (1989).

[3] G. L. Floch, Entropy weak solutions to nonlinear hyperbolic systems in nonconservative form, Comm. in Partial Differential Equations 13 (6) (1988) 669–727.

[4] D. Maso, G. L. Floch, P. Murat, Definition and weak stability of a nonconservative product, Journal de Mathematiques Pures et Appliquees 74 (6) (1995) 483–548.

[5] S. Bianchini, A. Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems, Annals of Mathematics 161 (1) (2005) 223–342.