A Viscous Regularization for the Seven-Equation two-phase flow Model feels like we should try to add low-Mach in the title.

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Abstract

In this paper, a viscous regularization is derived for the seven-equation two-phase flow model. The regularization ensures positivity of the entropy residual, uniqueness of the weak solution is consistent with the viscous regularization for Euler equations when one phase disappears, and does not depend on the spatial discretization scheme chosen. We also show that the viscous regularization is compatible with the generalized Harten entropies.

Key words: two-phase flow model, viscous regularization, artificial dissipative method, low-Mach regime, shocks

1. Introduction

Compressible two-phase fluid flows are found in numerous industrial applications. Their numerical solution is an ongoing area of research in modeling and simulation. A variety of two-phase models, with different levels of complexity, has been developed; for instance, the five-equation model of Kapila [1], the six-equation model [2], and more recently the Seven-Equation Model (SEM)[3]. These models are all obtained by integrating the one-phase flow balance equations weighed by a characteristic or indicator function for each phase. The resulting system of equations contains non-conservative terms and relaxation terms that describe the interaction between phases, supplemented by an equation for the void fraction. The systems of two-phase flow equations are usually solved using discontinuous discretization schemes (finite volume and discontinuous Galerkin approaches). By assuming that the system of equations is hyperbolic, a Riemann solver could be used but is often ruled out because of its complexity due to the number of equations involved. Instead, approximate

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Riemann solvers, a well-established approach for single-phase flows, are employed [**?**]add citations pertaining to discretizations of 2-phase, while ensuring the correct low-Mach asymptotic limit and deriving a consistent discretization scheme for the non-conservative terms [] add the specific papers about low-mach and non-conserv terms here [4, 5, 6, 7, 8, 9].

In this paper, we derive a viscous regularization for the Seven-Equation two-phase flow Model of [3]. The foundation for this work can be traced back to viscous regularizations for single-phase Euler and Navier-Stokes equations, notably [**?**] and the references therein. The proposed viscous regularization for the SEM is consistent with the entropy minimum principle and Harten's generalized entropies. In addition, we ensure that the regularization scales appropriately in the low-Mach regime as such situations are often encountered in practical applications; the two-phase low-Mach asymptotic study determines conditions that need to be satisfied by the artificial dissipative terms to yield a well-scaled regularization in the low-Mach case [10].

One of the key aspects of the viscous regularization derived here is that it is agnostic of the spatial discretization scheme, unlike approximate Riemann solvers. Therefore, this viscous regularization can be employed to stabilized numerical scheme both continuous and discontinuous discretizations. For examples of prior applications to the single-phase Euler equations, we refer the reader to [11, 12].

The remainder of the paper is as follows. In Section 2, the Seven-equation two-phase flow Model is recalled along with its main properties. The viscous regularization is derived in Section 3 and a low-Mach asymptotic study of the regularized equations is performed in Section 4. Finally, we conclude in Section **??** where we outline possible uses of finish.

2. The Seven-Equation two-phase flow Model

The Seven-Equation two-phase flow Model employed in this paper is obtained by assuming that each phase satisfies the single-phase Euler equations (with phase-exchange terms) and by integrating the latter over a control volume after multiplication by a phasic characteristic function. The detailed derivation can be found in [3] and we recall the SEM governing equations for phase k in interaction with phase j. In the SEM, each phase obeys the following mass, momentum and energy balance equations, supplemented by a non-conservative equation for the void fraction:

$$\frac{\partial \alpha_k A}{\partial t} + A\boldsymbol{u}_{int} \cdot \boldsymbol{\nabla} \alpha_k = A\mu_P (P_k - P_j) - \frac{\Gamma_{k\to j} A_{int} A}{\rho_{int}}\,, \tag{1a}$$

$$\frac{\partial \left(\alpha\rho\right)_k A}{\partial t} + \boldsymbol{\nabla}\cdot\left(\alpha\rho\boldsymbol{u}A\right)_k = -\Gamma_{k\to j} A_{int} A\,, \tag{1b}$$

$$\frac{\partial \left(\alpha\rho\boldsymbol{u}\right)_k A}{\partial t} + \boldsymbol{\nabla}\cdot\left[\alpha_k A \left(\rho\boldsymbol{u}\otimes\boldsymbol{u} + P\mathbb{I}\right)_k\right] = P_{int} A \boldsymbol{\nabla}\alpha_k + P_k \alpha_k \boldsymbol{\nabla} A \\ + A\lambda_u(\boldsymbol{u}_j - \boldsymbol{u}_k) - \Gamma_{k\to j} A_{int} \boldsymbol{u}_{int} A\,, \quad (1c)$$

$$\frac{\partial \left(\alpha\rho E\right)_k A}{\partial t} + \boldsymbol{\nabla}\cdot\left[\alpha_k \boldsymbol{u}_k A \left(\rho E + P\right)_k\right] = P_{int} A \boldsymbol{u}_{int}\cdot\boldsymbol{\nabla}\alpha_k - \bar{P}_{int} A \mu_P (P_k - P_j) \\ + A\lambda_u \bar{\boldsymbol{u}}_{int}\cdot(\boldsymbol{u}_j - \boldsymbol{u}_k) + \Gamma_{k\to j} A_{int}\left(\frac{P_{int}}{\rho_{int}} - H_{k,int}\right) A\,, \quad (1d)$$

where α_k, ρ_k, $\boldsymbol{u}_k$ and E_k denote the void fraction, the density, the velocity vector and the total specific energy of phase k, respectively. The phasic pressure P_k is computed from an equation of state. The cross section of the geometry is denoted by A and is only spatially dependent. A is present for completeness of the presentation and is set to 1 for most applications; however, nozzle flow problems can be solved using the one-dimensional version of the equation and setting A to the cross-sectional area of the nozzle. The interfacial pressure and velocity and their corresponding average values are denoted by P_{int}, $\boldsymbol{u}_{int}$, $\bar{P}_{int}$ and $\bar{\boldsymbol{u}}_{int}$, respectively; they are defined in Eq. (2). I do not know if we should keep the mass, momentum and energy exchange terms in the equations since they are not included in the derivations of the viscous regularization.

$$P_{int} = \bar{P}_{int} + \frac{Z_k Z_j}{Z_k + Z_j}\frac{\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||}\cdot\left(\boldsymbol{u}_j - \boldsymbol{u}_k\right), \quad (2a)$$

$$\bar{P}_{int} = \frac{Z_j P_k + Z_k P_j}{Z_k + Z_j}\,, \quad (2b)$$

$$\boldsymbol{u}_{int} = \bar{\boldsymbol{u}}_{int} + \frac{\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||}\frac{P_j - P_k}{Z_k + Z_j}\,, \quad (2c)$$

$$\bar{\boldsymbol{u}}_{int} = \frac{Z_k \boldsymbol{u}_k + Z_j \boldsymbol{u}_j}{Z_k + Z_j}\,. \quad (2d)$$

The interfacial specific total enthalpy of phase k, $H_{k,int}$, is defined as $H_{k,int} = h_{k,int} + 0.5||\boldsymbol{u}_{int}||^2$, where $h_{k,int}$ is the phasic specific enthalpy evaluated at the interface conditions (P_{int} and $T_{int} = T_{sat}(\bar{P}_{int})$). Following [3], the pressure and velocity relaxation coefficients, μ_P and λ_u respectively, are function of the acoustic impedance $Z_k = \rho_k c_k$ and the specific interfacial area A_{int} as shown in Eq. (3).

$$\mu_P = \frac{A_{int}}{Z_k + Z_j}\,, \quad (3a)$$

$$\lambda_u = \frac{1}{2}\mu_P Z_k Z_j\,. \quad (3b)$$

The specific interfacial area (i.e., the interfacial surface area per unit volume of a two-phase mixture), A_{int}, is typically dependent upon flow regime conditions and can be provided as a correlation. In [3], A_{int} is chosen to be a function of the liquid void fraction:

$$A_{int} = A_{int}^{max}\left[6.75\left(1-\alpha_{liq}\right)^2\alpha_{liq}\right], \tag{4}$$

with $A_{int}^{max} = 5100\ m^2/m^3$. With this definition, the interfacial area is zero in the limits $\alpha_k = 0$ and $\alpha_k = 1$. Lastly, $\Gamma_{k\to j}$ is the net mass transfer rate per unit interfacial area from phase k to phase j not the opposite? done. Its expression, given in Eq. (5), is obtained by considering a vaporization/condensation process that is dominated by heat diffusion at the interface [3, 13]:

$$\Gamma_{k\to j} = \frac{h_{T,k}\left(T_k - T_{int}\right) + h_{T,j}\left(T_j - T_{int}\right)}{L_v\left(T_{int}\right)}, \tag{5}$$

where $L_v\left(T_{int}\right) = h_{j,int} - h_{k,int}$ represents the latent heat of vaporization. The interface temperature is determined by the saturation constraint $T_{int} = T_{sat}(P)$ with the appropriate pressure $P = \bar{P}_{int}$ defined previously. The interfacial heat transfer coefficients for phases k and j are denoted by $h_{T,k}$ and $h_{T,j}$, respectively, and are computed from correlations [3].

The set of equations satisfied by phase j are simply obtained by substituting k by j and j by k in Eq. (1), keeping the same definition of the interfacial variables and noting that $\Gamma_{k\to j} = -\Gamma_{jk}$. The equation for the void fraction of phase j is simply replaced by the algebraic relation

$$\alpha_j = 1 - \alpha_k,$$

which reduces the number of partial differential equations from eight to seven and yields the Seven-Equation two-phase flow Model.

Some properties of the Seven-Equation two-phase flow Model are discussed next. A set of $5+2\mathbb{D}$ (with $\mathbb{D}$ the geometry dimension) waves is present in the model: two acoustic waves per phase, one contact wave per phase per domain dimension, and one void fraction wave propagating at the interfacial velocity $\boldsymbol{u}_{int}$. These waves (eigenvalues of the Jacobian for the inviscid flux terms) and as follows for each phase k:

$$\begin{aligned}
&\lambda_{1,k} = \boldsymbol{u}_k\cdot\bar{\boldsymbol{n}} - c_k\\
&\lambda_{2,k} = \boldsymbol{u}_k\cdot\bar{\boldsymbol{n}} + c_k\\
&\lambda_{2+d,k} = \boldsymbol{u}_k\cdot\bar{\boldsymbol{n}} \text{ for } d = 1\ldots\mathbb{D}\\
&\lambda_{3+\mathbb{D}} = \boldsymbol{u}_{int}\cdot\bar{\boldsymbol{n}}\,,
\end{aligned} \tag{6}$$

where $\bar{\boldsymbol{n}}$ is an unit vector pointing to a given direction. The eigenvalues given in Eq. (6) are unconditionally real (as long as the equation of state yields a real-valued sound speed). Having real eigenvalues is an extremely valuable property for the development of numerical methods since it ensures that the system of equations is hyperbolic and well-posed.

One may relax the Seven-Equation two-phase flow Model to the ill-posed classical six-equation model, where a single pressure is used for both phases; this is accomplished by letting the pressure relaxation coefficient μ_P become very large, i.e., by letting it approach infinity. Note that as the pressure relaxation coefficient increases, so does the velocity relaxation coefficient λ_u; see Eq. (3). However, the six-equation model only relaxes the pressure parameter of the SEM and results in an ill-posed system of equations that can present unstable numerical solutions with sufficiently fine spatial resolution [3, 14]. If one lets both the pressure and the velocity relaxation parameters tend to infinity, this further relaxes the Seven-Equation two-phase flow Model to the hyperbolic and well-posed mechanical equilibrium five-equation model of Kapila [1].

Next, we consider the SEM model with pressure and velocity relaxation but omit phase exchange terms ($\Gamma_{k\to j} = 0$):

$$\frac{\partial \alpha_k A}{\partial t} + A\boldsymbol{u}_{int}\cdot\boldsymbol{\nabla}\alpha_k = A\mu_P(P_k - P_j)\,, \tag{7a}$$

$$\frac{\partial\left(\alpha\rho\right)_k A}{\partial t} + \boldsymbol{\nabla}\cdot\left(\alpha\rho\boldsymbol{u}A\right)_k = 0\,, \tag{7b}$$

$$\begin{aligned}\frac{\partial\left(\alpha\rho\boldsymbol{u}\right)_k A}{\partial t} + \boldsymbol{\nabla}\cdot\left[\alpha_k A\left(\rho\boldsymbol{u}\otimes\boldsymbol{u} + P\mathbb{I}\right)_k\right] = P_{int}A\boldsymbol{\nabla}\alpha_k + P_k\alpha_k\boldsymbol{\nabla}A \\ + A\lambda_u(\boldsymbol{u}_j - \boldsymbol{u}_k)\,,\end{aligned} \tag{7c}$$

$$\begin{aligned}\frac{\partial\left(\alpha\rho E\right)_k A}{\partial t} + \boldsymbol{\nabla}\cdot\left[\alpha_k\boldsymbol{u}_k A\left(\rho E + P\right)_k\right] = P_{int}A\boldsymbol{u}_{int}\cdot\boldsymbol{\nabla}\alpha_k - \bar{P}_{int}A\mu_P(P_k - P_j) \\ + A\lambda_u\bar{\boldsymbol{u}}_{int}\cdot(\boldsymbol{u}_j - \boldsymbol{u}_k)\,.\end{aligned} \tag{7d}$$

An entropy equation can be derived for each phase k of system Eq. (7) and the sign of the entropy material derivative can be proved positive. The entropy function for a phase k is denoted by s_k and a function of density ρ_k and internal energy e_k. The full derivation is given in Appendix A and only the final result is recalled here:

$$\begin{aligned}(s_e)_k^{-1}\alpha_k\rho_k A\frac{Ds_k}{Dt} = \mu_P\frac{Z_k}{Z_k + Z_j}(P_j - P_k)^2 + \lambda_u\frac{Z_j}{Z_k + Z_j}(\boldsymbol{u}_j - \boldsymbol{u}_k)^2 \\ \frac{Z_k}{(Z_k + Z_j)^2}\left[Z_j(\boldsymbol{u}_j - \boldsymbol{u}_k) + \frac{\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||}(P_k - P_j)\right]^2\,,\end{aligned} \tag{8}$$

where $\frac{D(\cdot)}{Dt} = \partial_t(\cdot) + \boldsymbol{u}\cdot\boldsymbol{\nabla}(\cdot)$ is the material derivative. The partial derivative of the entropy function s_k with respect to the internal energy e_k, $(s_e)_k$, is shown to be proportional to the inverse of the temperature of phase k, as in the case of the single phase Euler equations [11, 15]. The right-hand side of Eq. (8) is unconditionally positive since all terms are squared and thus, is used to demonstrate the entropy minimum principle. Furthermore, Eq. (8) is valid for both phases $\{k, j\}$ and ensures positivity of the total entropy equation that is obtained by summation over the phases:

$$\sum_k (s_e)_k^{-1} \alpha_k \rho_k A \frac{Ds_k}{Dt} = \sum_k (s_e)_k^{-1} \alpha_k \rho_k A \left(\partial_t s_k + \boldsymbol{u}_k \cdot \boldsymbol{\nabla} s_k\right) \geq 0. \tag{9}$$

Note that when one phase disappears, Eq. (9) degenerates to the single-phase entropy equation obtained from Euler equations [3, 15].

3. A viscous regularization for the Seven-Equation two-phase flow Model

We now propose to derive a viscous regularization for the Seven-Equation two-phase flow Model given in Eq. (1) by using the same methodology as for the multi-dimensional Euler equations with/without variable area [11, 10]. The method consists in adding dissipative terms to the system of equation under consideration, and re-derive the entropy equation whose sign is known to be positive to ensure uniqueness of the numerical solution [16]. Because of the addition of dissipation terms, the entropy equation is modified and contains extra terms of yet unknown sign. By carefully choosing a definition for each of the dissipation term, the sign of the entropy equation can be determined and proved positive. For the Seven-Equation two-phase flow Model, derivation of a viscous regularization can be achieved by considering either the phasic entropy equation (Eq. (8)) or the total entropy equation (Eq. (9)). In the later case, the entropy minimum principle is verified for the whole system which may not ensure positivity of the entropy equation for each phase. However, positivity of the total entropy equation can be also achieved by assuming that the entropy minimum principle holds for each phase. This stronger requirement will also ensure consistency with the single phase Euler equations when one of the phase disappears in the limit $\alpha_k \to 0$. Thus, it is chosen to work with the phasic entropy equations given in Eq. (8).

For the purpose of this section, the system of equations given in Eq. (10) is considered, which is obtained by simply omitting the mass source terms (terms proportional to $\Gamma_{k\to j}$) in Eq. (1).

$$\partial_t \left(\alpha_k A\right) + A\boldsymbol{u}_{int} \cdot \boldsymbol{\nabla} \alpha_k = A\mu_P \left(P_k - P_j\right) \tag{10a}$$

$$\partial_t \left(\alpha_k \rho_k A\right) + \boldsymbol{\nabla}\cdot\left(\alpha_k \rho_k \boldsymbol{u}_k A\right) = 0 \tag{10b}$$

$$\begin{aligned}\partial_t \left(\alpha_k \rho_k u_k A\right) + \boldsymbol{\nabla}\cdot\left[\alpha_k A \left(\rho_k \boldsymbol{u}_k \otimes \boldsymbol{u}_k + P_k \mathbb{I}\right)\right] = \\ \alpha_k P_k \boldsymbol{\nabla} A + P_{int} A \boldsymbol{\nabla} \alpha_k + A\lambda_u \left(\boldsymbol{u}_j - \boldsymbol{u}_k\right)\end{aligned} \tag{10c}$$

$$\begin{aligned}\partial_t \left(\alpha_k \rho_k E_k A\right) + \boldsymbol{\nabla}\cdot\left[\alpha_k A \boldsymbol{u}_k \left(\rho_k E_k + P_k\right)\right] = \\ A P_{int} \boldsymbol{u}_{int} \cdot \boldsymbol{\nabla} \alpha_k - \mu_P \bar{P}_{int} A \left(P_k - P_j\right) + A\lambda_u \bar{\boldsymbol{u}}_{int} \cdot \left(\boldsymbol{u}_j - \boldsymbol{u}_k\right)\end{aligned} \tag{10d}$$

In order to apply the entropy viscosity method, dissipation terms are added to each equation yielding:

$$\partial_t \left(\alpha_k A\right) + A\boldsymbol{u}_{int} \cdot \boldsymbol{\nabla}\alpha_k = A_{int} A \mu_P \left(P_k - P_j\right) + \boldsymbol{\nabla}\cdot \boldsymbol{l}_k \tag{11a}$$

$$\partial_t \left(\alpha_k \rho_k A\right) + \boldsymbol{\nabla}\cdot\left(\alpha_k \rho_k \boldsymbol{u}_k A\right) = \boldsymbol{\nabla}\cdot \boldsymbol{f}_k \tag{11b}$$

$$\begin{split}\partial_t \left(\alpha_k \rho_k \boldsymbol{u}_k A\right) + \boldsymbol{\nabla}\cdot\left[\alpha_k A \left(\rho_k \boldsymbol{u}_k \otimes \boldsymbol{u}_k + P_k \mathbb{I}\right)\right] = \\ \alpha_k P_k \boldsymbol{\nabla} A + P_{int} A \boldsymbol{\nabla}\alpha_k + A\lambda_u \left(\boldsymbol{u}_j - \boldsymbol{u}_k\right) + \boldsymbol{\nabla}\cdot \mathbb{g}_k\end{split} \tag{11c}$$

$$\begin{split}\partial_t \left(\alpha_k \rho_k E_k A\right) + \boldsymbol{\nabla}\cdot\left[\alpha_k A \boldsymbol{u}_k \left(\rho_k E_k + P_k\right)\right] = \\ P_{int} A \boldsymbol{u}_{int} \cdot \boldsymbol{\nabla}\alpha_k - \mu_P A \bar{P}_{int} \left(P_k - P_j\right) + \\ A\lambda_u \bar{\boldsymbol{u}}_{int} \cdot \left(\boldsymbol{u}_j - \boldsymbol{u}_k\right) + \boldsymbol{\nabla}\cdot\left(\boldsymbol{h}_k + \boldsymbol{u}\cdot \mathbb{g}_k\right)\end{split} \tag{11d}$$

where $\boldsymbol{f}_k$, $\mathbb{g}_k$, $\boldsymbol{h}_k$ and $\boldsymbol{l}_k$ are phasic viscous terms to be determined. The next step consists in deriving the entropy equation for the phase k, on the same model as what was done in Appendix A but with dissipative terms now present. The steps are as follows:

1. derive the phasic density and internal energy equations from Eq. (11).
2. assuming that the phasic entropy, s_k, is a function of density, ρ_k and internal energy, e_k, derive the entropy equation by using the chain rule:

$$\frac{Ds_k}{Dt} = \left(s_\rho\right)_k \frac{D\rho_k}{Dt} + \left(s_e\right)_k \frac{De_k}{Dt} \tag{12}$$

   The terms $(s_e)_k$ and $(s_\rho)_k$ denote the partial derivative of the entropy s_k with respect to e_k and ρ_k, respectively.
3. isolate the terms of interest and choose an appropriate expression for each of the dissipation terms in order to ensure positivity of the new entropy residual.

We first derive the phasic density equation for the primitive variable ρ_k by combining Eq. (11a) and Eq. (11b) to obtain:

$$\alpha_k A \left[\partial_t \rho_k + \left(\boldsymbol{u}_k - \underline{\underline{\boldsymbol{u}_{int}}}\right) \cdot \boldsymbol{\nabla}\rho_k\right] = \underline{\underline{A\rho_k \mu_P \left(P_k - P_j\right)}} + \boldsymbol{\nabla}\cdot \boldsymbol{f}_k - \rho_k \boldsymbol{\nabla}\cdot \boldsymbol{l}_k \tag{13}$$

In order to derive the phasic internal energy equation, the phasic velocity equation is obtained by subtracting the phasic density equation from the phasic momentum equation:

$$\begin{split}\alpha_k \rho_k A \left[\partial_t \boldsymbol{u}_k + \boldsymbol{u}_k \cdot \boldsymbol{\nabla}\cdot \boldsymbol{u}_k\right] + \boldsymbol{\nabla}\cdot\left(\alpha_k \rho_k A P_k \mathbb{I}\right) = \\ \alpha_k P_k \boldsymbol{\nabla} A + P_{int} A \boldsymbol{\nabla}\alpha_k + A\lambda \left(\boldsymbol{u}_j - \boldsymbol{u}_k\right) + \boldsymbol{\nabla}\cdot \mathbb{g}_k - \boldsymbol{u}_k \otimes \boldsymbol{f}_k\end{split} \tag{14}$$

After multiplying Eq. (14) by the phasic velocity vector $\boldsymbol{u}_k$, the resulting phasic kinetic energy equation is subtracted from the phasic total energy equation to obtain the internal energy equation for phase k:

$$\begin{aligned}\alpha_k\rho_k A\left[\partial_t \boldsymbol{e}_k + \boldsymbol{u}_k\cdot\boldsymbol{\nabla}\cdot\boldsymbol{e}_k\right] + \alpha_k\rho_k A P_k \boldsymbol{\nabla}\boldsymbol{u}_k = &\\ &\underline{\underline{P_{int}A\left(\boldsymbol{u}_{int}-\boldsymbol{u}_k\right)\cdot\boldsymbol{\nabla}\alpha_k}} - \underline{\underline{\alpha_k P_k \boldsymbol{u}_k\cdot\boldsymbol{\nabla}A}}\\ &\underline{\underline{-\bar{P}_{int}A\mu_P\left(P_k-P_j\right)}} + \underline{\underline{A\lambda_u\left(\boldsymbol{u}_j-\boldsymbol{u}_k\right)\cdot\left(\bar{\boldsymbol{u}}_{int}-\boldsymbol{u}_k\right)}}\\ &+\boldsymbol{\nabla}\cdot\boldsymbol{h}_k + \mathbb{g}_k:\boldsymbol{\nabla}\boldsymbol{u}_k + ||\boldsymbol{u}||_k^2\boldsymbol{f}_k\end{aligned} \tag{15}$$

The underline terms in Eq. (13) and Eq. (15) yield the positive terms in the right-hand-side of Eq. (8) and thus are ignored in the remainder of this derivation for brevity. The phasic entropy equation is now obtained by combining the phasic density equation (Eq. (13)) and the phasic internal energy equation (Eq. (15)) through the chain rule given in Eq. (12) to yield:

$$\begin{aligned}\alpha_k\rho_k A\frac{Ds_k}{Dt} = (\rho s_\rho)_k\left[\boldsymbol{\nabla}\cdot\boldsymbol{f}_k - \rho_k\boldsymbol{\nabla}\cdot\boldsymbol{l}_k\right] + &\\ (s_e)_k\left[\boldsymbol{\nabla}\cdot\boldsymbol{h}_k + \mathbb{g}_k:\boldsymbol{\nabla}\boldsymbol{u}_k + \left(||\boldsymbol{u}||_k^2 - e_k\right)\boldsymbol{\nabla}\cdot\boldsymbol{f}_k\right],&\end{aligned} \tag{16}$$

where it was assumed that the entropy of phase k satisfies the second thermodynamic law:

$$T_k\mathrm{d}s_k = \mathrm{d}e_k - P_k\frac{\mathrm{d}\rho_k}{\rho_k^2}\ , \tag{17a}$$

which implies

$$P_k(s_e)_k + \rho_k(s_\rho)_k = 0, \tag{17b}$$
$$(s_e)_k = T_k^{-1} \text{ and } (s_\rho)_k = -(s_e)_k P_k\frac{\mathrm{d}\rho_k}{\rho_k^2}.$$

Following the methodology applied in [11, 10], the right-hand side of Eq. (16) can be further simplified by using the following expression for the dissipative terms $\boldsymbol{f}_k$, $\mathbb{g}_k$ and $\boldsymbol{h}_k$:

$$\boldsymbol{f}_k = \tilde{\boldsymbol{f}}_k + \rho_k\boldsymbol{l}_k \tag{18a}$$
$$\mathbb{g}_k = \alpha_k\rho_k A\mu_k\mathbb{F}(\boldsymbol{u}_k) + \boldsymbol{f}_k\otimes\boldsymbol{u}_k \tag{18b}$$
$$\boldsymbol{h}_k = \tilde{\boldsymbol{h}}_k - \frac{||\boldsymbol{u}_k||^2}{2}\boldsymbol{f}_k + (\rho e)_k\boldsymbol{l}_k, \tag{18c}$$

where μ_k is a positive viscosity coefficient for phase k. Note the area function A in the definition of $\mathbb{g}_k$. Substituting the expression of the dissipative terms given in Eq. (18) into Eq. (16) yields:

$$
\begin{aligned}
\alpha_k\rho_k A\frac{Ds_k}{Dt} = & \underbrace{\boldsymbol{\nabla}\cdot\Big[(s_e)_k\tilde{\boldsymbol{h}}_k + \Big(e_k(s_e)_k - \rho_k(s_\rho)_k\Big)\tilde{\boldsymbol{f}}_k\Big]}_{\mathcal{R}_0} \\
& \underbrace{(s_e)_k\,\alpha_k\rho_k A\mu_k\mathbb{F}(\boldsymbol{u}_k):\boldsymbol{\nabla}\boldsymbol{u}_k}_{\mathcal{R}_1}\underbrace{-\tilde{\boldsymbol{h}}_k\cdot\boldsymbol{\nabla}(s_e)_k - \tilde{\boldsymbol{f}}_k\cdot\boldsymbol{\nabla}\left[(\,es_e\,)_k - (\rho s_\rho)_k\right]}_{\mathcal{R}_2} + \\
& \underbrace{(s_e)_k\boldsymbol{\nabla}\cdot(\rho_k e_k\boldsymbol{l}_k) - (s_e)_k e_k\boldsymbol{\nabla}\cdot(\rho_k\boldsymbol{l}_k) + \rho_k(s_\rho)_k\boldsymbol{\nabla}\cdot(\rho_k\boldsymbol{l}_k) - \rho_k^2(s_\rho)_k\boldsymbol{\nabla}\cdot\boldsymbol{l}_k}_{\mathcal{R}_3}. \qquad (19)
\end{aligned}
$$

We now split the right-hand-side of Eq. (19) into three residuals denoted by $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ and we study the sign of each of them. Since $(s_e)_k$ is defined as the inverse of the temperature and thus is positive, the sign of the first term, $\mathcal{R}_1$, is conditioned by the choice of the function $\mathbb{F}(\boldsymbol{u}_k)$ so that the product with the tensor $\boldsymbol{\nabla}\boldsymbol{u}_k$ is positive. As in [11, 10], $\mathbb{F}(\boldsymbol{u}_k)$ is chosen proportional to the symmetric gradient of the velocity vector $\boldsymbol{\nabla}^s\boldsymbol{u}_k$, whom entries are given by $\left((\boldsymbol{\nabla}^s\boldsymbol{u})_{i,j}\right)_k = \frac{1}{2}\left(\partial_{x_i}u_i + \partial_{x_j}u_j\right)_k$. With such a choice, the viscous regularization is also rotationally invariant. After a few lines of algebra, the third term $\mathcal{R}_3$ can be recast as a function of the gradient of the entropy as follows:

$$\mathcal{R}_3 = \rho_k A\boldsymbol{l}_k\cdot\boldsymbol{\nabla}s_k. \qquad (20)$$

One of the assumptions made in the entropy minimum principle is to that the entropy is at a minimum which implies that its gradient is null. Because of this, it follows that the term $\mathcal{R}_3$ is zero at the minimum and thus, the entropy minimum principle is verified independently of the definition of the dissipation term $\boldsymbol{l}_k$ used in the void fraction equation Eq. (11a). It will be explained later in this section how to obtain a definition for $\boldsymbol{l}_k$.

We now focus on the term denoted by $\mathcal{R}_2$, which is identical to the right-hand-side of the single phase entropy equation for Euler equations (see [11, 10]). Thus, the term $\mathcal{R}_2$ is known to be positive when (i) assumes concavity of the entropy function s_k with respect to the internal energy e_k and the specific volume $1/\rho_k$ (or convexity of $-s_k$) and (ii) chooses the following definitions for the dissipative terms $\tilde{h}_k$ and $\tilde{f}_k$:

$$\tilde{\boldsymbol{f}}_k = \alpha_k A\kappa_k\boldsymbol{\nabla}\rho_k \qquad (21a)$$

$$\tilde{\boldsymbol{h}}_k = \alpha_k A\kappa_k\boldsymbol{\nabla}\left(\rho e\right)_k, \qquad (21b)$$

where κ_k is another positive viscosity coefficient. In addition, using Eq. (21a), the term $\mathcal{R}_0$ can be recast as a function of the phasic entropy as follows:

$$\mathcal{R}_0 = \boldsymbol{\nabla}\cdot(\alpha_k A\kappa_k\rho_k\boldsymbol{\nabla}s_k) \qquad (22)$$

The entropy equation can now be written in its final form:

$$
\begin{aligned}
\alpha_k\rho_k A\frac{Ds_k}{Dt} &= \boldsymbol{f}_k\cdot\boldsymbol{\nabla}s_k + \boldsymbol{\nabla}\cdot(\alpha_k A\rho_k\kappa_k\boldsymbol{\nabla}s_k) \\
&- \alpha_k\rho_k A\kappa_k\mathbf{Q}_k + (s_e)_k\alpha_k A\rho_k\mu_k\boldsymbol{\nabla}^s\boldsymbol{u}_k:\boldsymbol{\nabla}\boldsymbol{u}_k, \qquad (23)
\end{aligned}
$$

where $\mathbf{Q}_k$ is a negative semi-definite quadratic form under the assumption of s_k being concave with respect to e_k and $1/\rho_k$, and defined as:

$$\begin{aligned} \mathbf{Q}_k &= X_k^t \Sigma_k X_k \\ \text{with } X_k &= \begin{bmatrix} \boldsymbol{\nabla}\rho_k \\ \boldsymbol{\nabla} e_k \end{bmatrix} \text{ and } \Sigma_k = \begin{bmatrix} \rho_k^{-2}\partial_{\rho_k}(\rho_k^2 \partial_{\rho_k} s_k) & \partial_{\rho_k, e_k} s_k \\ \partial_{\rho_k, e_k} s_k & \partial_{e_k, e_k} s_k \end{bmatrix}. \end{aligned}$$

Eq. (23) is used to prove the entropy minimum principle: assuming that s_k reaches its minimum value in $\boldsymbol{r}_{min}(t)$ at each time t, the gradient, $\boldsymbol{\nabla} s_k$, and Laplacian, Δs_k, of the entropy are null and positive at this particular point, respectively. Furthermore, it is recalled that the viscosity coefficients μ_k and κ_k are positive by definition. Then, because the terms in the right-hand-side of Eq. (23) are proven either positive or null when the entropy reaches a minimum value, the entropy minimum principle holds for each phase k, **independently of the definition of the dissipative term $\boldsymbol{l}_k$**, such as:

$$\alpha_k \rho_k A \partial_t s_k(\boldsymbol{r}_{min}, t)) \geq 0 \Rightarrow \partial_t s_k(\boldsymbol{r}_{min}, t)) \geq 0$$

Do we need to make the above statement a theorem or property?

It remains to obtain a definition for the dissipative term $\boldsymbol{l}_k$ used in the void fraction equation Eq. (11a). A way to achieve this is to consider the void fraction equation, by itself and notice that it is an hyperbolic equation with eigenvalue $\boldsymbol{u}_{int}$. An entropy equation can be derived and used to prove the entropy minimum principle by properly choosing the dissipative term. The objective is to ensure positivity of the void fraction and also uniqueness of the weak solution. Following the work of Guermond et al. in [17, 18], it can be shown that a dissipative term ensuring positivity and uniqueness of the weak solution for the void fraction equation, is of the form $\boldsymbol{l}_k = \beta_k A \boldsymbol{\nabla} \alpha_k$, where β_k is a positive viscosity coefficient. The dissipative term is proportional to the area A for consistency with the other terms of the void fraction equation Eq. (11a).

All of the dissipative terms are now defined and recalled here:

$$\boldsymbol{l}_k = \beta_k A \boldsymbol{\nabla} \alpha_k \tag{24a}$$

$$\boldsymbol{f}_k = \alpha_k A \kappa_k \boldsymbol{\nabla} \rho_k + \rho_k A \boldsymbol{l}_k \tag{24b}$$

$$\mathbb{g}_k = \alpha_k A \mu_k \rho \boldsymbol{\nabla}^s \boldsymbol{u}_k \tag{24c}$$

$$\boldsymbol{h}_k = \alpha_k A \kappa_k \boldsymbol{\nabla} \left(\rho e\right)_k + \boldsymbol{u}_k : \mathbb{g}_k - \frac{||\boldsymbol{u}_k||^2}{2} \boldsymbol{f}_k + (\rho e)_k \boldsymbol{l}_k \tag{24d}$$

At this point, some remarks are in order:

1. The dissipative term $\boldsymbol{l}_k$ requires the definition of a new viscosity coefficient β_k. It was shown that this viscosity coefficient is independent of the other viscosity coefficients μ_k and κ_k. Its definition should account for the eigenvalue $\boldsymbol{u}_{int}$ and the entropy equation associated with the void fraction equation.

2. The dissipative term $\boldsymbol{f}_k$ is a function of $\boldsymbol{l}_k$. Thus, all of the other dissipative terms are also functions of $\boldsymbol{l}_k$.
3. The partial derivatives $(s_e)_k$ and $(s_{\rho_k})_k$ can be computed using the definition provided in Eq. (17a) and are functions of the phasic thermodynamic variables: pressure, temperature and density.
4. All of the dissipative terms are chosen to be proportional to the void fraction α_k and the cross-sectional area A, except the one in the void fraction equation that is only proportional to A. For instance, $\alpha_k A \boldsymbol{\nabla} \rho_k$ is the flux of the dissipative term in the continuity equation through the pseudo-area, $\alpha_k A$, seen by the phase k. When one of the phases disappears, the dissipative terms must go to zero for consistency. On the other hand, when α_k goes to one, the single-phase Euler equations with variable area and with proper viscous regularization must be recovered.
5. By choosing $\beta_k = \mu_k = \kappa_k$ and $\mathbb{F}(\boldsymbol{u}_k) = \boldsymbol{\nabla} \boldsymbol{u}_k$, the dissipative terms collapse as follows:

$$\partial_t \left(\alpha_k A\right) + A \boldsymbol{u}_{int} \cdot \boldsymbol{\nabla} \alpha_k = A \mu_P \left(P_k - P_j\right) + \boldsymbol{\nabla} \cdot \left[A \kappa_x \boldsymbol{\nabla} \alpha_k\right] \tag{25a}$$

$$\partial_t \left(\alpha_k \rho_k A\right) + \boldsymbol{\nabla} \cdot \left(\alpha_k \rho_k \boldsymbol{u}_k A\right) = \boldsymbol{\nabla} \cdot \left[A \kappa_k \boldsymbol{\nabla} \left(\alpha \rho\right)_k\right] \tag{25b}$$

$$\begin{split} \partial_t \left(\alpha_k \rho_k \boldsymbol{u}_k A\right) + \boldsymbol{\nabla} \cdot \left[\alpha_k A \left(\rho_k \boldsymbol{u}_k \otimes \boldsymbol{u}_k + P_k \mathbb{I}\right)\right] = \\ \alpha_k P_k \boldsymbol{\nabla} A + P_{int} A \boldsymbol{\nabla} \alpha_k + \boldsymbol{\nabla} \cdot \left[A \kappa_k \boldsymbol{\nabla} \left(\alpha \rho \boldsymbol{u}\right)_k\right] \end{split} \tag{25c}$$

$$\begin{split} \partial_t \left(\alpha_k \rho_k E_k A\right) + \boldsymbol{\nabla} \cdot \left[\alpha_k A \boldsymbol{u}_k \left(\rho_k E_k + P_k\right)\right] = \\ P_{int} A \boldsymbol{u}_{int} \cdot \boldsymbol{\nabla} \alpha_k - \mu_P \bar{P}_{int} \left(P_k - P_j\right) + A \lambda_u \bar{\boldsymbol{u}}_{int} \cdot \left(\boldsymbol{u}_j - \boldsymbol{u}_k\right) \\ + \boldsymbol{\nabla} \cdot \left[A \kappa_k \boldsymbol{\nabla} \left(\alpha \rho E\right)_k\right] \ , \end{split} \tag{25d}$$

   to yield a viscous regularization that is analogous to the parabolic regularization for Euler equations [19]. Note that by choosing $\mathbb{F}(\boldsymbol{u}_k) = \boldsymbol{\nabla} \boldsymbol{u}_k$, the above viscous regularization is no longer rotationally invariant.
6. Compatibility of the viscous regularization proposed in Eq. (24) with the generalized entropies identified in Harten et al. [20] is demonstrated in Appendix B.

7. We could add a paragraph explaining that the above viscous regularization can also be used for the five-equation model of Kapila with some very light modifications.

At this point in the paper, we have derived a viscous regularization for the Seven-Equation two-phase flow Model that ensures positivity of the entropy residual, uniqueness of the numerical solution when assuming concavity of the phasic entropy s_k, and is consistent with the viscous regularization derived for the multi-dimensional Euler equations [11, 10] in the limit $\alpha_k \to 1$. The viscous regularization involves a set of three viscosity coefficients for each phase, μ_k, κ_k and β_k, that are assumed positive. Definition of the viscosity coefficients should be devised from the scaled SEM in order to ensure well-scaled dissipative terms for a wide range of Mach numbers (subsonic, transonic and supersonic flows).

Remark. *Through the derivations of the viscous regularization, it was noted that another set of dissipative terms $\boldsymbol{f}_k$ and $\boldsymbol{l}_k$ would also ensures positivity of the entropy residual:*

$$\boldsymbol{l}_k = \beta_k T_k \left[\frac{\rho_k}{P_k + \rho_k e_k} \boldsymbol{\nabla} \left(\frac{P_k}{\rho_k e_k} \right) - \frac{1}{P_k} \boldsymbol{\nabla} \rho_k \right] \tag{26a}$$

$$\boldsymbol{f}_k = \kappa_k \boldsymbol{\nabla} \rho_k + \frac{\rho_k^2 (s_\rho)_k}{\left(\rho s_\rho - e s_e\right)_k} \boldsymbol{l}_k \tag{26b}$$

However, the definition of $\boldsymbol{l}_k$ proposed in Eq. (26a) was not considered as valid for the following reasons: positivity of the void fraction cannot be achieved and the parabolic regularization is not retrieved when assuming equal viscosity coefficients.

4. The scaled Seven-Equation two-phase flow Model with viscous regularization

When working with artificial dissipative numerical stabilization methods, great care needs to be carried to the definition of the viscosity coefficients that will determine the accuracy of the method. Generally speaking, sufficient artificial viscosity should be added into the shock and discontinuity regions to prevent spurious oscillations from forming, while little dissipation is added when the numerical solution is smooth to ensure high-order accuracy. In addition, the low-Mach asymptotic limit also has to be accounted for in the definition of the viscosity coefficients in order to recover the incompressible asymptotic equations [21, 22, 23]. The purpose of this section is to derive the scaled SEM and investigate the scaling of the dissipative terms to ensure well-scaled dissipative terms for all-Mach flows (subsonic, transonic and supersonic flows). First, the scaled SEM are derived and then, two limit cases (a) and (b) will be considered to determine appropriate scaling for the entropy viscosity coefficients so that the dissipative terms remain well-scaled for: (a) the isentropic low-Mach limit where the Seven-Equation two-phase flow Model degenerate to an incompressible system of equations in the low-Mach limit and (b) the non-isentropic limit with formation of shocks. Finally, for each case the scaling of the numerical adimensional numbers will be given. Also, because each phase can experience different flow regime e.g., supersonic gas and subsonic liquid, it is chosen to work with three distinct viscosity coefficients for each phase. The study is performed on the multi-dimensional version of the Seven-Equation two-phase flow Model with the Stiffened Gas Equation of State (SGEOS) given in Eq. (27).

$$P_k = \left(\gamma_k - 1\right) \rho_k e_k - \gamma_k P_{k,\infty} \tag{27}$$

4.1. Derivation of the scaled Seven-Equation two-phase flow Model

We consider the case where the relaxation coefficients are set to zero: the two phases do not interact and the Seven-Equation two-phase flow Model degenerates into two sets of Euler equations with a pseudo cross-section $\alpha_k A$. The first step in the study of the two limit cases (a) and (b) is to re-write each system of equations in a non-dimensional manner. To do so, the following variables are introduced for each phase k:

$$\rho_k^* = \frac{\rho_k}{\rho_{k,\infty}},\ u_k^* = \frac{\boldsymbol{u}_k}{u_{k,\infty}},\ P_k^* = \frac{P_k}{\rho_{k,\infty} c_{k,\infty}^2},\ E_k^* = \frac{E_k}{c_{k,\infty}^2},\ x^* = \frac{x}{L_\infty},$$
$$t_k^* = \frac{t_k}{L_\infty / u_{k,\infty}},\ \mu_k^* = \frac{\mu_k}{\mu_{k,\infty}},\ \kappa_k^* = \frac{\kappa_k}{\kappa_{k,\infty}},\ P_{int}^* = \frac{P_{int}}{P_{int,\infty}},$$
$$u_{int}^* = \frac{\boldsymbol{u}_{int}}{u_{int,\infty}},\ \bar{P}_{int}^* = \frac{\bar{P}_{int}}{\bar{P}_{int,\infty}},\ \bar{u}_{int}^* = \frac{\bar{u}_{int}}{\bar{u}_{int,\infty}}, \tag{28}$$

where the subscript ∞ denote the far-field or stagnation quantities and the superscript $*$ stands for the non-dimensional variables. The far-field reference quantities are chosen such that the dimensionless flow quantities are of order 1. The stagnation quantities for the pressure and velocity interfacial variables will be specified for each case. The reference phasic Mach number is given by

$$M_{k,\infty} = \frac{u_{k,\infty}}{c_{k,\infty}}. \tag{29}$$

Because we consider that phases do not interact with each other, it is assumed that the interfacial pressure and velocity scale as the phasic pressure and velocity, respectively: $P_{int,\infty} = \rho_{k,\infty} c_{k,\infty}^2$ and $u_{int,\infty} = u_{k,\infty}$. Under these assumptions, the interfacial pressure and velocity are simply replaced by P_k and $\boldsymbol{u}_k$ in the equations. Then, the system of equations with viscous regularization becomes:

$$\partial_t (\alpha_k A) + A \boldsymbol{u}_k \cdot \boldsymbol{\nabla} \cdot \alpha_k = \boldsymbol{\nabla} \cdot (A \beta_k \boldsymbol{\nabla} \alpha_k) \tag{30a}$$

$$\partial_t (\alpha_k \rho_k A) + \boldsymbol{\nabla} \cdot (\alpha_k \rho_k \boldsymbol{u}_k A) = \boldsymbol{\nabla} \cdot (A \alpha_k \kappa_k \boldsymbol{\nabla} \rho_k) + \boldsymbol{\nabla} \cdot (A \beta_k \rho_k \boldsymbol{\nabla} \alpha_k) \tag{30b}$$

$$\begin{aligned} &\partial_t (\alpha_k \rho_k u_k A) + \boldsymbol{\nabla} \cdot [\alpha_k A (\rho_k \boldsymbol{u}_k \otimes \boldsymbol{u}_k + P_k)] = \\ &\quad \alpha_k P_k \boldsymbol{\nabla} A + P_k A \boldsymbol{\nabla} \alpha_k + \boldsymbol{\nabla} \cdot (A \mu_k \alpha_k \rho_k \boldsymbol{\nabla}^s \boldsymbol{u}_k) + \\ &\quad \boldsymbol{\nabla} \cdot (A \kappa_k \alpha_k \boldsymbol{u}_k \otimes \boldsymbol{\nabla} \rho_k) + \boldsymbol{\nabla} \cdot (A \beta_k \rho_k \boldsymbol{u}_k \otimes \boldsymbol{\nabla} \alpha_k) \end{aligned} \tag{30c}$$

$$\begin{aligned} &\partial_t (\alpha_k \rho_k E_k A) + \boldsymbol{\nabla} \cdot [\alpha_k A \boldsymbol{u}_k (\rho_k E_k + P_k)] = \\ &\quad P_k A \boldsymbol{u}_k \cdot \boldsymbol{\nabla} \alpha_k + \boldsymbol{\nabla} \cdot (A \kappa_k \alpha_k \boldsymbol{\nabla} (\rho_k e_k)) + \boldsymbol{\nabla} \cdot \left(A \kappa_k \alpha_k \frac{||\boldsymbol{u}_k||^2}{2} \boldsymbol{\nabla} \rho_k \right) + \\ &\quad \boldsymbol{\nabla} \cdot (A \mu_k \alpha_k \rho_k \boldsymbol{u}_k : \boldsymbol{\nabla}^s \boldsymbol{u}_k) + \boldsymbol{\nabla} \cdot (A \beta_k \rho_k e_k \boldsymbol{\nabla} \alpha_k) \end{aligned} \tag{30d}$$

Then using the scaling introduced in Eq. (28), the scaled equations for the phase k with viscous regularization are:

$$\partial_{t^*} (\alpha_k A)^* + A^* \boldsymbol{u}_k^* \cdot \boldsymbol{\nabla}^* \alpha_k^* = \frac{1}{\text{Pé}_{k,\infty}^\beta} \boldsymbol{\nabla} \cdot^* (A \beta_k \boldsymbol{\nabla}^* \alpha_k)^* \tag{31a}$$

$$\partial_{t^*}\left(\alpha_k\rho_k A\right)^* + \boldsymbol{\nabla}\cdot^*\left(\alpha_k\rho_k\boldsymbol{u}_k A\right)^* = \\ \frac{1}{\text{Pé}^{\kappa}_{k,\infty}}\boldsymbol{\nabla}\cdot^*\left(A\kappa_k\boldsymbol{\nabla}^*\rho_k\right)^* + \frac{1}{\text{Pé}^{\beta}_{k,\infty}}\boldsymbol{\nabla}\cdot^*\left(A\beta_k\rho_k\boldsymbol{\nabla}^*\alpha_k\right)^* \tag{31b}$$

$$\partial_{t^*}\left(\alpha_k\rho_k u_k A\right)^* + \boldsymbol{\nabla}\cdot^*\left[\alpha_k A\left(\rho_k\boldsymbol{u}_k\otimes\boldsymbol{u}_k\right)\right]^* + \frac{A\alpha_k^*}{M^2_{k,\infty}}\boldsymbol{\nabla}^* P_k^* = \\ \frac{1}{M^2_{k,\infty}}\alpha_k^* P_k^*\boldsymbol{\nabla}^* A^* + \frac{1}{M^2_{k,\infty}}P_k^* A^*\boldsymbol{\nabla}^*\alpha_k^* + \frac{1}{\text{Re}_{k,\infty}}\boldsymbol{\nabla}\cdot^*\left(A\alpha_k\mu_k\rho_k\boldsymbol{\nabla}^s\boldsymbol{u}_k\right)^* \\ + \frac{1}{\text{Pé}^{\kappa}_{k,\infty}}\boldsymbol{\nabla}\cdot^*\left(A\alpha_k\kappa_k\boldsymbol{u}_k\otimes\boldsymbol{\nabla}^*\rho_k\right)^* + \frac{1}{\text{Pé}^{\beta}_{k,\infty}}\boldsymbol{\nabla}\cdot^*\left(A\beta_k\rho_k\boldsymbol{u}_k\otimes\boldsymbol{\nabla}\alpha_k\right)^* \tag{31c}$$

$$\alpha_k^* A^*\left[\partial_t\left(\rho_k E_k\right) + \boldsymbol{u}_k\cdot\boldsymbol{\nabla}^*\left(\rho_k E_k\right)\right]^* + \alpha_k\boldsymbol{\nabla}\cdot^*\left(A\boldsymbol{u}_k P_k\right) + \rho_k^* E_k^*\alpha_k^*\boldsymbol{\nabla}\cdot^*\left(\boldsymbol{u}A\right)_k^* = \\ \frac{1}{\text{Pé}^{\kappa}_{k,\infty}}\boldsymbol{\nabla}\cdot^*\left(A\alpha_k\kappa_k\boldsymbol{\nabla}\left(\rho_k e_k\right)\right)^* + \frac{M^2_{k,\infty}}{\text{Pé}^{\kappa}_{k,\infty}}\boldsymbol{\nabla}\cdot^*\left(A\alpha_k\kappa_k\frac{||\boldsymbol{u}_k||^2}{2}\boldsymbol{\nabla}\rho\right)^* + \\ \frac{M^2_{k,\infty}}{\text{Re}_{k,\infty}}\boldsymbol{\nabla}\cdot^*\left(A\alpha_k\mu_k\rho_k\boldsymbol{u}_k:\boldsymbol{\nabla}^s\boldsymbol{u}_k\right)^* + \frac{1}{\text{Pé}^{\beta}_{k,\infty}}\boldsymbol{\nabla}\left(\rho_k e_k\right)^*\cdot\left(A\beta_k\boldsymbol{\nabla}\alpha_k\right)^* \\ - \frac{M^2_{k,\infty}}{\text{Pé}^{\beta}_{k,\infty}}\rho_k\frac{||\boldsymbol{u}_k^2||}{2}\boldsymbol{\nabla}\cdot\left(\beta_k A\boldsymbol{\nabla}\alpha_k\right) \tag{31d}$$

where the phasic numerical Reynolds ($\text{Re}_{k,\infty}$) and Péclet ($\text{Pé}^{\kappa}_{k,\infty}$ and $\text{Pé}^{\beta}_{k,\infty}$) numbers are defined as:

$$\text{Re}_{k,\infty} = \frac{u_{k,\infty}L_\infty}{\mu_{k,\infty}}\ ,\text{Pé}^{\kappa}_{k,\infty} = \frac{u_{k,\infty}L_\infty}{\kappa_{k,\infty}} \text{ and } \text{Pé}^{\beta}_{k,\infty} = \frac{u_{k,\infty}L_\infty}{\beta_{k,\infty}}\ . \tag{32}$$

Note that the phasic energy equation was recast under a non-conservative form by using the void fraction equation (Eq. (31a)) to facilitate the derivations when trying to recover the divergence constraint onto the velocity in the low-Mach asymptotic regime. The numerical Reynolds and Péclet numbers defined in Eq. (32) are related to the phasic entropy viscosity coefficients $\mu_{k,\infty}$, $\kappa_{k,\infty}$ and $\beta_{k,\infty}$. Thus, once a scaling (in powers of $M_{k,\infty}$) is obtained for $\text{Re}_{k,\infty}$, $\text{Pé}^{\kappa}_{k,\infty}$ and $\text{Pé}^{\beta}_{k,\infty}$ in the two limit cases (a) and (b), it will impose a condition onto the definition of the phasic viscosity coefficients μ_k, κ_k and β_k. For brevity, the superscripts * are omitted in the remainder of this section.

4.2. Scaling of $Re_{k,\infty}$, $Pé^{\kappa}_{k,\infty}$ and $Pé^{\beta}_{k,\infty}$ in the low-Mach asymptotic regime (case (a))

In the low-Mach isentropic limit, the Seven-Equation two-phase flow Model converges to an incompressible system of equations, that is characterized for each phase with pressure fluctuations of order $M^2_{k,\infty}$ and the divergent constraint on the velocity: $\boldsymbol{\nabla}\cdot(A\boldsymbol{u}_k) = 0$. When adding dissipative terms, as is the case with the entropy viscosity method, the main properties of the low-Mach asymptotic limit must be preserved. We begin by expanding each variable in powers of the Mach number. As an example, the expansion for the pressure is given by:

$$P_k(\boldsymbol{r},t) = P_{k,0}(\boldsymbol{r},t) + P_{k,1}(\boldsymbol{r},t)M_{k,\infty} + P_{k,2}(\boldsymbol{r},t)M_{k,\infty}^2 + \dots \tag{33}$$

By studying the resulting momentum equations for various powers of M_∞, it is observed that the leading- and first-order pressure terms, $P_{k,0}$ and $P_{k,1}$, are spatially constant if and only if $\mathrm{Re}_{k,\infty} = \mathrm{Pé}^{\kappa}_{k,\infty} = \mathrm{Pé}^{\beta}_{k,\infty} = 1$. In this case, we have at order $M_{k,\infty}^{-2}$:

$$\boldsymbol{\nabla} P_{k,0} = 0 \tag{34a}$$

and at order $M_{k,\infty}^{-1}$

$$\boldsymbol{\nabla} P_{k,1} = 0\,. \tag{34b}$$

From Eq. (34) we infer that the leading- and first-order pressure terms are spatially independent which ensures pressure fluctuations of order Mach number square, as expected in the low-Mach asymptotic limit. Using the scaling $\mathrm{Re}_{k,\infty} = \mathrm{Pé}^{\kappa}_{k,\infty} = \mathrm{Pé}^{\beta}_{k,\infty} = 1$, the second-order momentum equations and the leading-order expressions for the void fraction, continuity and energy equations are:

$$\partial_t \left(A\alpha_k\right)_0 + \boldsymbol{u}_{k,0}\cdot\boldsymbol{\nabla}\alpha_{k,0} = \boldsymbol{\nabla}\cdot\left(A\beta_k\boldsymbol{\nabla}\alpha_k\right)_0 \tag{35a}$$

$$\partial_t \left(A\alpha_k\rho_k\right)_0 + \boldsymbol{\nabla}\cdot\left(A\alpha_k\rho_k\boldsymbol{u}_k\right)_0 = \boldsymbol{\nabla}\cdot\left(A\alpha_k\kappa_k\boldsymbol{\nabla}\rho_k\right)_0 + \boldsymbol{\nabla}\cdot\left(A\beta_k\boldsymbol{\nabla}\alpha_k\right)_0 \tag{35b}$$

$$\begin{aligned}\partial_t(\alpha_k A\rho_k\boldsymbol{u}_k)_0 + \boldsymbol{\nabla}\cdot(A\alpha_k\rho_k\boldsymbol{u}_k\otimes\boldsymbol{u}_k)_0 + A\alpha_k\boldsymbol{\nabla}P_{k,2} = \\ \boldsymbol{\nabla}\cdot\left[A\alpha_k\left(\mu_k\rho_k\boldsymbol{\nabla}^s\boldsymbol{u}_k + \kappa_k\boldsymbol{u}_k\otimes\boldsymbol{\nabla}\rho_k\right)\right]_0 + \boldsymbol{\nabla}\cdot\left(A\beta_k\rho\boldsymbol{u}\boldsymbol{\nabla}\alpha_k\right)_0\end{aligned} \tag{35c}$$

$$\begin{aligned}\alpha_{k,0}A\left[\partial_t(\rho_k E_k) + \boldsymbol{u}_k\cdot\boldsymbol{\nabla}\left(\rho_k E_k\right)\right]_0 + \alpha_{k,0}\boldsymbol{\nabla}\cdot\left[A\boldsymbol{u}_k P_k\right]_0 + \\ \alpha_{k,0}\rho_{k,0}E_{k,0}\boldsymbol{\nabla}\cdot\left(\boldsymbol{u}_k A\right)_0 = \boldsymbol{\nabla}\cdot\left[A\alpha_k\kappa_k\boldsymbol{\nabla}\left(\rho_k e_k\right)\right] \\ + A\beta_{k,0}\boldsymbol{\nabla}\left(\rho_k e_k\right)_0\cdot\boldsymbol{\nabla}\alpha_{k,0}\end{aligned} \tag{35d}$$

where the notation $(fg)_0$ means that we only keep the 0^{th}-order terms in the product fg. The set of equations given in Eq. (35) are similar to the multi-dimensional single-phase Euler equations with variable area when seeing $A\alpha_k$ as a pseudo-area [10]. The leading-order of the Stiffened Gas Equation of State (Eq. (27)) is given by

$$P_{k,0} = (\gamma_k - 1)\rho_{k,0}E_{k,0} - \gamma P_{k,\infty} = (\gamma_k - 1)\rho_0 e_{k,0} - \gamma_k P_{k,\infty}\,. \tag{36}$$

Using Eq. (36), the energy equation can be recast as a function of the leading-order pressure, P_0, as follows:

$$\begin{aligned}A\alpha_{k,0}\left[\partial_t\left(P_k\right) + (\gamma_k - 1)\boldsymbol{u}_k\cdot\boldsymbol{\nabla}P_k\right]_0 + \\ (\gamma_k - 1)\alpha_{k,0}\boldsymbol{\nabla}\cdot\left[A\boldsymbol{u}_k P_k\right]_0 + \alpha_{k,0}\left(P_{k,0} + \gamma_k P_{k\infty}\right)\boldsymbol{\nabla}\cdot\left(\boldsymbol{u}_k A\right)_0 = \\ \left[\boldsymbol{\nabla}\cdot\left(A\alpha_k\kappa_k\boldsymbol{\nabla}(P_k)\right)_0 + A\beta_{k,0}\boldsymbol{\nabla}P_{k,0}\cdot\boldsymbol{\nabla}\alpha_{k,)}\right]\,.\end{aligned} \tag{37}$$

From Eq. (34a), we infer that P_0 is spatially constant. Thus, Eq. (37) becomes

$$\frac{A}{\gamma\left(P_{k,0}+P_{k,\infty}\right)}\frac{dP_0}{dt} = -\boldsymbol{\nabla}\cdot\left(\boldsymbol{u}_k A\right)_0 \tag{38}$$

and, at steady state, we have

$$\boldsymbol{\nabla}\cdot\left(\boldsymbol{u}_k A\right)_0 = 0\,. \tag{39}$$

That is, the leading-order of the product of velocity and cross section is divergence-free which corresponds to what is obtained when dealing with the multi-dimensional Euler equations with variable area. Note that when assuming a constant cross section A, the usual divergence constraint, $\boldsymbol{\nabla}\cdot\boldsymbol{u}_{k,0}$ is recovered. Also, Eq. (38) is slightly modified due to the use of the Stiffened Gas Equation of State in the asymptotic limit. However, the Ideal Gas Equation of State degenerates from the Stiffened Gas Equation of State by simply setting $P_{k,\infty}=0$ which yields the usual leading-order single-phase energy equation with constant cross section:

$$\frac{1}{\gamma P_{k,0}}\frac{dP_0}{dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{u}_{k,0} \tag{40}$$

The same reasoning can be applied to the leading-order of the continuity equation (Eq. (35b)) to show that the material derivative of the density variable is stabilized by well-scaled dissipative terms:

$$\left.\frac{\mathrm{D}\alpha_k\rho_k}{\mathrm{D}t}\right|_0 := \partial_t\left(\alpha_k\rho\right)_0 + \boldsymbol{u}_{k,0}\cdot\boldsymbol{\nabla}\cdot\left(\alpha_k\rho_k\right)_0 = \frac{1}{A}\boldsymbol{\nabla}\cdot\left[\alpha_k A\kappa_k\boldsymbol{\nabla}\rho + A\beta_k\rho_k\boldsymbol{\nabla}\alpha_k\right]_0\,. \tag{41}$$

Therefore, we conclude that by setting the Reynolds and Péclet numbers to one, the incompressible fluid results are retrieved in the low-Mach limit when employing the compressible Seven-Equation two-phase flow Model with viscous regularization and without relaxation terms.

4.3. Scaling of $Re_{k,\infty}$, $P\acute{e}^{\kappa}_{k,\infty}$ and $P\acute{e}^{\beta}_{k,\infty}$ for non-isentropic flows (case (b))

Next, we consider the non-isentropic case. Recall that even subsonic flows can present shocks (for instance, a step initial condition in the pressure will trigger shock formation, independently of the Mach number). The non-dimensional form of the Seven-Equation two-phase flow Model given in Eq. (31) provides some insight on the dominant terms as a function of the Mach number. This is particular obvious in the momentum equation, Eq. (31c), where the gradient of pressure is scaled by $1/M^2_{k,\infty}$. In the non-isentropic case, we no longer have $\frac{\boldsymbol{\nabla}P_k}{M^2_{k,\infty}} = \boldsymbol{\nabla}P_{k,2}$ and therefore the pressure gradient term may need to be stabilized by some dissipative terms of the same scaling so as to prevent spurious oscillations from forming. By inspecting the dissipative terms presents in the momentum equation, having a dissipative term that scales as $1/M^2_{k,\infty}$ leads to a total of eight different options. Only three of them are investigated for brevity (note that the five other options can be ruled out by following the same reasoning as what is done next):

$$(i)\ \mathrm{Re}_{k,\infty} = 1\ , \mathrm{Pé}^{\kappa}_{k,\infty} = M^2_{k,\infty} \text{ and } \mathrm{Pé}^{\beta}_{k,\infty} = 1\ ,$$
$$(ii)\ \mathrm{Re}_{k,\infty} = 1\ , \mathrm{Pé}^{\kappa}_{k,\infty} = 1 \text{ and } \mathrm{Pé}^{\beta}_{k,\infty} = M^2_{k,\infty} \text{ or}$$
$$(iii)\ \mathrm{Re}_{k,\infty} = M^2_{k,\infty}\ , \mathrm{Pé}^{\kappa}_{k,\infty} = 1 \text{ and } \mathrm{Pé}^{\beta}_{k,\infty} = 1\ .$$

Any of these choices will also affect the stabilization of the void fraction, continuity and energy equations. For instance, using Péclet numbers equal to $M^2_{k,\infty}$ may effectively stabilize the void fraction and continuity equations in the shock region but this may also add an excessive amount of dissipation for subsonic flows at the location of the contact wave. Such a behavior may not be suitable for accuracy purpose, making options (i) and (ii) inappropriate. The same reasoning, left to the reader, can be carried out for the energy equation (Eq. (31d)) and results in the same conclusion. The remaining choice, option (iii), has the proper scaling: in this case, only the dissipation terms involving $\boldsymbol{\nabla}^{s,*}\boldsymbol{u}^*_k$ scale as $1/M^2_{k,\infty}$ since $\mathrm{Re}_{k,\infty} = M^2_{k,\infty}$, leaving the regularization of the void fraction and continuity equations unaffected because $\mathrm{Pé}^{\beta}_{k,\infty} = \mathrm{Pé}^{\kappa}_{k,\infty} = 1$. I feel we need another short section to explain how the two above limit cases ca be merge into one

5. Conclusions

We derived a viscous regularization for the well-posed Seven-Equation two-phase flow Model that ensures positivity of the entropy residual, uniqueness of the numerical solution when assuming concavity of the phasic entropy s_k, is consistent with the viscous regularization derived for the multi-dimensional Euler equations in the limit $\alpha_k \to 1$ and does not depend on the scheme discretization. It was also shown that the viscous regularization is compatible with the generalized Harten entropies that were initially derived for Euler equations. The viscous regularization involves a set of three positive viscosity coefficients for each phase, β_k, μ_k and κ_k that are defined from the scaled SEM to ensure well-scaled dissipative terms. We introduced three numerical non-dimensionalized numbers for each phase, Re_k, $\mathrm{Pé}^{\mu}_k$ and $\mathrm{Pé}^{\kappa}_k$ and devised their scaling in two cases: the low-Mach asymptotic limit and for non-isentropic flows. In the later case, it was demonstrated that the incompressible system of equations is recovered when assuming that all of the non-dimensionalized numbers scale as one. The study of the former case showed that the scaling of the Péclet numbers remain the same whereas the scaling of the Reynolds number Re_k has to be modified and set to M^2_k to ensure well-scaled dissipative terms in the phasic momentum equations. Because the numerical non-dimensionalized numbers are related to the scaling of the phasic viscosity coefficients, the above scaling should be used either to assess the accuracy of the viscosity coefficient definitions or derive definition for the viscosity coefficients.

Deriving a definition for the phasic viscosity coefficients should rely on existing numerical methods for scalar and system of hyperbolic equations. For instance, it is known that artificial dissipative methods are used to solved for Euler equations: Lapidus [24, 25], pressure-based [26] and entropy-based [17, 12] numerical methods. Once a definition for the viscosity coefficients is derived and found consistent with the scaling of the numerical non-dimensionalized numbers, the numerical methods can be tested by solving two-phase shock tubes using various discretization methods. Note that the viscous regularization proposed in this paper is discretization independent.

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Appendix A Entropy equation for the multi-dimensional seven equation model without viscous regularization

This appendix provides the steps that lead to the derivation of the phasic entropy equation of the Seven-Equation two-phase flow Model [3]. For the purpose of this appendix, two phases are considered and denoted by the indexes j and k. In the Seven-Equation two-phase flow Model, each phase obeys to the following set of equations ($Eq.$ (42)):

$$\partial_t (\alpha_k A) + A\boldsymbol{u}_{int} \cdot \boldsymbol{\nabla}\alpha_k = A\mu_P (P_k - P_j) \tag{42a}$$

$$\partial_t (\alpha_k \rho_k A) + \boldsymbol{\nabla}\cdot(\alpha_k \rho_k \boldsymbol{u}_k A) = 0 \tag{42b}$$

$$\partial_t (\alpha_k \rho_k \boldsymbol{u}_k A) + \boldsymbol{\nabla}\cdot[\alpha_k A (\rho_k \boldsymbol{u}_k \otimes \boldsymbol{u}_k + P_k \mathbb{I})] = \\ \alpha_k P_k \boldsymbol{\nabla} A + P_{int} A \boldsymbol{\nabla}\alpha_k + A\lambda_u (\boldsymbol{u}_j - \boldsymbol{u}_k) \tag{42c}$$

$$\partial_t (\alpha_k \rho_k E_k A) + \boldsymbol{\nabla}\cdot[\alpha_k A \boldsymbol{u}_k (\rho_k E_k + P_k)] = \\ P_{int} A \boldsymbol{u}_{int} \cdot \boldsymbol{\nabla}\alpha_k - \mu_P \bar{P}_{int} (P_k - P_j) + \bar{\boldsymbol{u}}_{int} A \lambda_u (\boldsymbol{u}_j - \boldsymbol{u}_k) \tag{42d}$$

where ρ_k, $\boldsymbol{u}_k$, E_k and P_k are the density, the velocity, the specific total energy and the pressure of phase k, respectively. The pressure and velocity relaxation parameters are denoted by μ_P and λ_u, respectively. The variables with subscript $_{int}$ correspond to the interfacial variables and a definition is given in Eq. (43). The cross section A is only function of space: $\partial_t A = 0$.

$$\begin{cases} P_{int} = \bar{P}_{int} - \frac{\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||} \frac{Z_k Z_j}{Z_k + Z_j} (\boldsymbol{u}_k - \boldsymbol{u}_j) \\ \bar{P}_{int} = \frac{Z_k P_j + Z_j P_k}{Z_k + Z_j} \\ \boldsymbol{u}_{int} = \bar{\boldsymbol{u}}_{int} - \frac{\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||} \frac{P_k - P_j}{Z_k + Z_j} \\ \bar{\boldsymbol{u}}_{int} = \frac{Z_k \boldsymbol{u}_k + Z_j \boldsymbol{u}_j}{Z_k + Z_j} \end{cases} \tag{43}$$

where $Z_k = \rho_k c_k$ and $Z_j = \rho_j c_j$ are the impedance of phases k and j, respectively. The speed of sound is denoted by the symbol c. The function $sgn(x)$ returns the sign of the variable x.
The first step consists of rearranging the equations given in Eq. (43) using the primitive variables $(\alpha_k, \rho_k, \boldsymbol{u}_k, e_k)$, where e_k is the specific internal energy of k^{th} phase. We introduce the material derivative $\frac{D(\cdot)}{Dt} = \partial_t(\cdot) + \boldsymbol{u}_k \cdot \boldsymbol{\nabla}(\cdot)$ for simplicity.
The continuity equation is modified as follows:

$$\alpha_k A \frac{D\rho_k}{Dt} + \rho_k A \mu_P (P_k - P_j) + \rho_k A (\boldsymbol{u}_k - \boldsymbol{u}_{int}) \cdot \boldsymbol{\nabla}\alpha_k + \rho_k \alpha_k \boldsymbol{\nabla}\cdot(A\boldsymbol{u}_k) = 0 \tag{44}$$

The momentum and continuity equations are combined to yield the velocity equation:

$$\alpha_k \rho_k A \frac{D\boldsymbol{u}_k}{Dt} + \boldsymbol{\nabla}\left(\alpha_k A P_k\right) = \alpha_k P_k \boldsymbol{\nabla} A + P_{int} A \boldsymbol{\nabla} \alpha_k + A\lambda_u \left(\boldsymbol{u}_j - \boldsymbol{u}_k\right) \tag{45}$$

The internal energy is obtained by subtracting the total energy from the kinetic equation defined as $\boldsymbol{u}_k$·Eq. (45):

$$\begin{aligned}\alpha_k \rho_k A \frac{De_k}{Dt} + \boldsymbol{\nabla}\cdot\left(\alpha_k \boldsymbol{u}_k A P_k\right) - \boldsymbol{u}_k \cdot \boldsymbol{\nabla}\left(\alpha_k A P_k\right) = P_{int} A \left(\boldsymbol{u}_{int} - \boldsymbol{u}_k\right) \cdot \boldsymbol{\nabla} \alpha_k \\ - \alpha_k P_k \boldsymbol{u}_k \cdot \boldsymbol{\nabla} A - \bar{P}_{int} A \mu_P \left(P_k - P_j\right) + A\lambda_u \left(\boldsymbol{u}_j - \boldsymbol{u}_k\right) \cdot \left(\bar{\boldsymbol{u}}_{int} - \boldsymbol{u}_k\right)\end{aligned} \tag{46}$$

In the next step, we assume the existence of a phase wise entropy s_k function of density ρ_k and internal energy e_k. Using the chain rule,

$$\frac{Ds_k}{Dt} = (s_\rho)_k \frac{D\rho_k}{Dt} + (s_e)_k \frac{De_k}{Dt}, \tag{47}$$

along with the internal energy (Eq. (46)) and the continuity equations (Eq. (44)), the following entropy equation is obtained:

$$\begin{aligned}\alpha_k \rho_k A \frac{Ds_k}{Dt} + \underbrace{A\left(P_k (s_e)_k + \rho_k^2 (s_\rho)_k\right) \boldsymbol{u}_k \cdot \boldsymbol{\nabla} \alpha_k + \alpha_k \left(P_k (s_e)_k + \rho_k^2 (s_\rho)_k\right) \boldsymbol{u}_k \cdot \boldsymbol{\nabla} A}_{\text{(a)}} = \\ (s_e)_k P_{int} A \left[(\boldsymbol{u}_{int} - \boldsymbol{u}_k) \cdot \boldsymbol{\nabla}\alpha_k - \bar{P}_{int} A \mu_P (P_k - P_j) + A\lambda_u (\bar{\boldsymbol{u}}_{int} - \boldsymbol{u}_k) \cdot (\boldsymbol{u}_j - \boldsymbol{u}_k)\right] - \\ \rho^2 (s_\rho)_k \left[\mu_P A (P_k - P_j) + A(\boldsymbol{u}_k - \boldsymbol{u}_{int}) \cdot \boldsymbol{\nabla}\alpha_k\right]\end{aligned} \tag{48}$$

where $(s_e)_k$ and $(s_\rho)_k$ denote the partial derivatives of the entropy s_k with respect to the internal energy e_k and the density ρ_k, respectively. The second term, (a), in the left hand side of Eq. (48) can be set to zero by assuming the following relation between the partial derivatives of the entropy s_k:

$$P_k (s_e)_k + \rho_k^2 (s_\rho)_k = 0. \tag{49}$$

The above equation is equivalent to the application of the second thermodynamic law when assuming reversibility:

$$T_k ds_k = de_k - \frac{P_k}{\rho_k^2} d\rho_k \text{ with } (s_e)_k = \frac{1}{T_k} \text{ and } (s_\rho)_k = -\frac{P_k}{\rho_k^2}(s_e)_k \tag{50}$$

Thus, equation Eq. (48) can be rearranged using the relation $(s_\rho)_k = -\frac{P_k}{\rho_k^2}(s_e)_k$:

$$\begin{aligned}((s_e)_k)^{-1} \alpha_k \rho_k \frac{Ds}{Dt} = \underbrace{\left[P_{int}(\boldsymbol{u}_{int} - \boldsymbol{u}_k) + P_k(\boldsymbol{u}_k - \boldsymbol{u}_{int})\right] \cdot \boldsymbol{\nabla}\alpha_k}_{\text{(b)}} + \\ \underbrace{\mu_P (P_k - P_j)(P_k - \bar{P}_{int})}_{\text{(c)}} + \underbrace{\lambda_u (\boldsymbol{u}_j - \boldsymbol{u}_k) \cdot (\bar{\boldsymbol{u}}_{int} - \boldsymbol{u}_k)}_{\text{(d)}}\end{aligned} \tag{51}$$

The right hand side of equation Eq. (51) is split into three terms (b), (c) and (d) that will be dealt with separately. The terms (c) and (d) can be easily recast by using the definitions of $\bar{\boldsymbol{u}}_{int}$ and $\bar{P}_{int}$ given in equation Eq. (43):

$$\begin{aligned}\mu_P(P_k - P_j)(P_k - \bar{P}_{int}) &= \mu_P \frac{Z_k}{Z_k + Z_j}(P_j - P_k)^2 \\ \lambda_u(\boldsymbol{u}_j - \boldsymbol{u}_k)\cdot(\bar{\boldsymbol{u}}_{int} - \boldsymbol{u}_k) &= \lambda_u \frac{Z_j}{Z_k + Z_j}(\boldsymbol{u}_j - \boldsymbol{u}_k)^2\end{aligned} \tag{52}$$

By definition, μ_P, λ_u and Z_k are all positive. Thus, the above terms (c) and (d) are unconditionally positive.

It remains to look at the last term (b). Once again, by using the definition of P_{int} and $\boldsymbol{u}_{int}$, and the following relations:

$$\begin{aligned}\boldsymbol{u}_{int} - \boldsymbol{u}_k &= \frac{Z_j}{Z_k + Z_j}(\boldsymbol{u}_j - \boldsymbol{u}_k) - \frac{\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||}\frac{Pk - P_j}{Z_k + Z_j} \\ P_{int} - P_k &= \frac{Z_k}{Z_k + Z_j}(P_j - P_k) - \frac{\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||}\frac{Z_k Z_j}{Z_k + Z_j}(\boldsymbol{u}_k - \boldsymbol{u}_j),\end{aligned}$$

term (b) becomes:

$$\begin{aligned}&[P_{int}(\boldsymbol{u}_{int} - \boldsymbol{u}_k) + P_k(\boldsymbol{u}_k - \boldsymbol{u}_{int})]\cdot\boldsymbol{\nabla}\alpha_k = (P_{int} - P_k)(\boldsymbol{u}_{int} - \boldsymbol{u}_k)\cdot\boldsymbol{\nabla}\alpha_k = \\ &\quad \frac{Z_k}{(Z_k + Z_j)^2}\boldsymbol{\nabla}\alpha_k \cdot \left[Z_j(\boldsymbol{u}_j - \boldsymbol{u}_k)(P_j - P_k) + \frac{\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||}Z_j^2(\boldsymbol{u}_j - \boldsymbol{u}_k)^2 + \right. \\ &\quad\quad \left. \frac{\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||}(P_k - P_j)^2 + \frac{\boldsymbol{\nabla}\alpha_k\cdot\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||^2}(P_k - P_j)Z_j(\boldsymbol{u}_k - \boldsymbol{u}_j)\right]\end{aligned} \tag{53}$$

The above equation is factorized by $||\boldsymbol{\nabla}\alpha_k||$ and then recast under a quadratic form using $\frac{\boldsymbol{\nabla}\alpha_k\cdot\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||^2} = 1$. This yields:

$$\begin{aligned}&[(\boldsymbol{u}_{int} - \boldsymbol{u}_k)P_{int} + (\boldsymbol{u}_k - \boldsymbol{u}_{int})P_k]\,\boldsymbol{\nabla}\alpha_k = \\ &\quad ||\boldsymbol{\nabla}\alpha_k||\frac{Z_k}{(Z_k + Z_j)^2}\left[Z_j(\boldsymbol{u}_j - \boldsymbol{u}_k) + \frac{\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||}(P_k - P_j)\right]^2\end{aligned} \tag{54}$$

Thus, using Eq. (51), Eq. (52), Eq. (53) and Eq. (54), the entropy equation obtained in [3] holds and is recalled here for convenience:

$$\begin{aligned}(s_e)_k^{-1}\alpha_k\rho_k A\frac{Ds_k}{Dt} &= \mu_P\frac{Z_k}{Z_k + Z_j}(P_j - P_k)^2 + \lambda_u\frac{Z_j}{Z_k + Z_j}(\boldsymbol{u}_j - \boldsymbol{u}_k)^2 \\ &\quad \frac{Z_k}{(Z_k + Z_j)^2}\left[Z_j(\boldsymbol{u}_j - \boldsymbol{u}_k) + \frac{\boldsymbol{\nabla}\alpha_k}{||\boldsymbol{\nabla}\alpha_k||}(P_k - P_j)\right]^2.\end{aligned}$$

Appendix B Compatibility of the viscous regularization for the seven-equation two-phase model with the generalized Harten entropies

We investigate in this appendix whether the viscous regularization of the seven-equation two-phase model derived in Section 3 is compatible with some or all generalized entropy identified in Harten et al. [20]. Considering the single-phase Euler equations, Harten et al. [20] demonstrated that a function $\rho\mathscr{H}(s)$ is called a generalized entropy and strictly concave if $\mathscr{H}$ is twice differential and

$$\mathscr{H}'(s) \geq 0, \quad \mathscr{H}'(s)c_p^{-1} - \mathscr{H}'' \geq 0, \ \forall\,(\rho, e) \in \mathbb{R}_+^2 \ , \tag{55}$$

where $c_p\,(\rho, e) = T\partial_T s\,(\rho, e)$ is the specific heat at constant pressure (T is a function of e and ρ through the equation of state). Because the seven-equation two-phase model was initially derived by assuming that each phase obeys the single-phase Euler equation, we want to investigate whether the above property still holds when considering the Seven-Equation two-phase flow Model with viscous regularization. To do so, we consider a phasic generalized entropy, $\mathscr{H}_k(s_k)$ and a phasic specific heat at constant pressure, $c_{p,k}\,(\rho_k, e_k) = T_k\partial_{T_k} s_k\,(\rho_k, T_k)$ characterized by Eq. (55). The objective is to find an entropy inequality verified by $\rho_k\mathscr{H}_k(s_k)$.

We start from the entropy inequality verified by s_k,

$$\begin{aligned}\alpha_k\rho_k A\frac{Ds_k}{Dt} &= \boldsymbol{f}_k \cdot \boldsymbol{\nabla} s_k + \boldsymbol{\nabla}\cdot(\alpha_k A\rho_k\kappa_k\boldsymbol{\nabla} s_k)\\ &- \alpha_k\rho_k A\kappa_k\mathbf{Q}_k + (s_e)_k\alpha_k A\rho_k\mu_k\boldsymbol{\nabla}^s\boldsymbol{u}_k : \boldsymbol{\nabla}\boldsymbol{u}_k.\end{aligned} \tag{56}$$

Eq. (56) is multiplied by $\mathscr{H}_k'(s_k)$ to yield:

$$\begin{aligned}\alpha_k\rho_k A\frac{D\mathscr{H}_k(s_k)}{Dt} = \boldsymbol{\nabla}\cdot(\alpha_k A\rho_k\kappa_k\boldsymbol{\nabla}\mathscr{H}_k(s_k)) - \mathscr{H}_k''(s_k)\alpha_k A\kappa_k\rho_k||\boldsymbol{\nabla} s_k||^2 +\\ \mathscr{H}_k'(s_k)\boldsymbol{f}_k \cdot \boldsymbol{\nabla} s_k - \mathscr{H}_k'(s_k)\alpha_k\rho_k A\kappa_k\mathbf{Q}_k +\\ \mathscr{H}_k'(s_k)(s_e)_k\alpha_k A\rho_k\mu_k\boldsymbol{\nabla}^s\boldsymbol{u}_k : \boldsymbol{\nabla}\boldsymbol{u}_k.\end{aligned} \tag{57}$$

Let us now multiply the continuity equation of phase k by $\mathscr{H}_k(s_k)$ and add the result to the above equation to obtain:

$$\begin{aligned}\partial_t\,(\alpha_k\rho_k A\mathscr{H}_k(s_k)) + \boldsymbol{\nabla}\cdot(\alpha_k\rho_k\boldsymbol{u}_k A\mathscr{H}_k(s_k)) -\\ \boldsymbol{\nabla}\cdot[\alpha_k A\rho_k\kappa_k\boldsymbol{\nabla}\mathscr{H}_k(s_k) + \alpha_k A\kappa_k\mathscr{H}_k(s_k)\boldsymbol{\nabla}\rho_k \ + A\kappa_k\rho_k\mathscr{H}_k(s_k)\boldsymbol{\nabla}\alpha_k] =\\ \underbrace{-\mathscr{H}_k''(s_k)\alpha_k A\kappa_k\rho_k||\boldsymbol{\nabla} s_k||^2 - \mathscr{H}_k'(s_k)\alpha_k A\kappa_k\rho_k\mathbf{Q}_k}_{\mathbb{T}_0} +\\ \underbrace{\mathscr{H}_k'(s_k)(s_e)_k\alpha_k A\rho_k\mu_k\boldsymbol{\nabla}^s\boldsymbol{u}_k : \boldsymbol{\nabla}\boldsymbol{u}_k}_{\mathbb{T}_1}.\end{aligned} \tag{58}$$

As in Section 3, the left-hand side of Eq. (58) is split into two residuals denoted by $\mathbb{T}_0$ and $\mathbb{T}_1$ in order to study the sign of each of them. We start by studying the sign of $\mathbb{T}_1$ that is positive since it is assumed that $\mathscr{H}_k'(s_k) \geq 0$. We now investigate the sign of $\mathbb{T}_0$. Using Eq. (55), it is obtained:

$$-\mathbb{T}_0 \leq \mathscr{H}_k'(s_k)\alpha_k A\kappa_k\rho_k\left(c_{p,k}^{-1}||\boldsymbol{\nabla} s_k||^2 + \mathbf{Q}_k\right) \ . \tag{59}$$

The right-hand side of Eq. (59) is a quadratic form that was already defined in Appendix 5 of [11] and recast under the matricial form $X_k^t\mathbb{S}X_k$ where $\mathbb{S}$ is a

2×2 matrix and the vector X_k is defined in Section 3. In [11], the matrix $\mathbb{S}$ is proved to be negative semi-definite which allows us to conclude that $-\mathbb{T}_0$ is of the same sign using Eq. (59). Then, knowing the sign of the two residuals $\mathbb{T}_0$ and $\mathbb{T}_1$, we conclude that:

$$\partial_t \left(\alpha_k \rho_k A \mathscr{H}_k(s_k)\right) + \boldsymbol{\nabla} \cdot \left(\alpha_k \rho_k \boldsymbol{u}_k A \mathscr{H}_k(s_k)\right) - \\ \boldsymbol{\nabla} \cdot \left[\alpha_k A \rho_k \kappa_k \boldsymbol{\nabla} \mathscr{H}_k(s_k) + \alpha_k A \kappa_k \mathscr{H}_k(s_k) \boldsymbol{\nabla} \rho_k + A \kappa_k \rho_k \mathscr{H}_k(s_k) \boldsymbol{\nabla} \alpha_k\right] \geq 0 \ ,$$

which allows us to conclude that an entropy inequality is satisfied for all generalized entropies $\rho_k \mathscr{H}_k(s_k)$ when using the viscous regularization derived in Section 3 for the seven-equation two-phase model. Note that the above inequality holds for the total entropy of the system when summing over the phases.