

Extension of the entropy viscosity method to the
multi-D 7-equation two-phase flow model.
I do not know if we should have 'multi-D' in the title
since we will only present 1-D results

Marc O. Delchini^a, Jean C. Ragusa^{*,a}, Ray A. Berry^b

^a*Department of Nuclear Engineering, Texas A&M University, College Station, TX 77843,
USA*

^b*Idaho National Laboratory, Idaho Falls, ID 83415, USA*

Abstract

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Key words: two-phase flow model, with variable area, entropy viscosity method, stabilization method, low Mach regime, shocks.

1. Introduction

- a few lines about the need for accurately resolving two-phase flows
- background on the different two-phase flow models: 5, 6 and 7-equation two-phase flow models
- then, focus on the different types of 7-equation two-phase flow models: they mostly differ because of the closure relaxations used
- discuss the different numerical solvers developed for the 7-equation two-phase flow model: HLL, HLLC, and approximated Riemann solvers accounting for the source terms
- emphasize the fact that the above numerical solvers only works on discontinuous schemes
- then, introduce the entropy viscosity method and details the organization of the paper

*Corresponding author

Email addresses: `delchmo@tamu.edu` (Marc O. Delchini), `jean.ragusa@tamu.edu` (Jean C. Ragusa), `ray.berry@inl.gov` (Ray A. Berry)

14 2. The multi-D 7-equation two-phase flow model

The multi-D seven-equation two-phase model presented in this paper is obtained by assuming that each phase obeys the single-phase Euler equations (with phase-exchange terms) and by integrating over a control volume after multiplying by a characteristic function. The detailed derivation can be found in [1]. In this section, the governing multi-dimensional equations are recalled for a phase k in interaction with a phase j . Each phase obeys the following mass, momentum and energy balance equations, supplemented by a non-conservative volume-fraction equation:

$$\frac{\partial \alpha_k A}{\partial t} + A \mathbf{u}_{int} \cdot \nabla \alpha_k = A \mu_P (P_k - P_j) - \frac{\Gamma A_{int} A}{\rho_{int}} \quad (1a)$$

$$\frac{\partial (\alpha \rho)_k A}{\partial t} + \nabla \cdot (\alpha \rho \mathbf{u} A)_k = -\Gamma A_{int} A \quad (1b)$$

$$\begin{aligned} \frac{\partial (\alpha \rho \mathbf{u})_k A}{\partial t} + \nabla \cdot [\alpha_k A (\rho \mathbf{u} \otimes \mathbf{u} + P \mathbb{I})_k] &= P_{int} A \nabla \alpha_k + P_k \alpha_k \nabla A \\ &+ A \lambda_u (\mathbf{u}_j - \mathbf{u}_k) - \Gamma A_{int} \mathbf{u}_{int} A \end{aligned} \quad (1c)$$

$$\begin{aligned} \frac{\partial (\alpha \rho E)_k A}{\partial t} + \nabla \cdot [\alpha_k \mathbf{u}_k A (\rho E + P)_k] &= P_{int} A \mathbf{u}_{int} \cdot \nabla \alpha_k - \bar{P}_{int} A \mu_P (P_k - P_j) \\ &+ A \lambda_u \bar{\mathbf{u}}_{int} \cdot (\mathbf{u}_j - \mathbf{u}_k) + \Gamma A_{int} \left(\frac{P_{int}}{\rho_{int}} - H_{k,int} \right) A \end{aligned} \quad (1d)$$

where α_k , ρ_k , \mathbf{u}_k and E_k denote the volume fraction, the density, the velocity vector and the total specific energy of phase k , respectively. The phasic pressure P_k is computed from an equation of state. The interfacial pressure and velocity and their corresponding average values are denoted by P_{int} , \mathbf{u}_{int} , \bar{P}_{int} and $\bar{\mathbf{u}}_{int}$, respectively, and are defined in Eq. (2).

$$P_{int} = \bar{P}_{int} + \frac{Z_k Z_j}{Z_k + Z_j} \frac{\nabla \alpha_k}{\|\nabla \alpha_k\|} \cdot (\mathbf{u}_j - \mathbf{u}_k) \quad (2a)$$

$$\bar{P}_{int} = \frac{Z_j P_k + Z_k P_j}{Z_k + Z_j} \quad (2b)$$

$$\mathbf{u}_{int} = \bar{\mathbf{u}}_{int} + \frac{\nabla \alpha_k}{\|\nabla \alpha_k\|} \frac{P_j - P_k}{Z_k + Z_j} \quad (2c)$$

$$\bar{\mathbf{u}}_{int} = \frac{Z_k \mathbf{u}_k + Z_j \mathbf{u}_j}{Z_k + Z_j}. \quad (2d)$$

The interfacial specific total enthalpy of phase k , $H_{k,int}$, is defined as follows: $H_{k,int} = h_{k,int} + 0.5 \|\mathbf{u}_{int}\|^2$, where $h_{k,int}$ is the phasic enthalpy evaluated at the interface conditions ($P_{int}, T_{int} = T_{sat}(\bar{P}_{int})$). Following [1], the pressure

and velocity relaxation coefficients, μ_P and λ_u respectively, are function of the acoustic impedance $Z_k = \rho_k c_k$ and the specific interfacial area A_{int} as shown in Eq. (3).

$$\lambda_u = \frac{1}{2} \mu_P Z_k Z_j \quad (3a)$$

$$\mu_P = \frac{A_{int}}{Z_k + Z_j} \quad (3b)$$

15 The specific interfacial area (i.e., the interfacial surface area per unit volume
16 of two-phase mixture), A_{int} , must be specified from some type of flow regime
17 map or function under the form of a correlation. In [1], A_{int} is chosen to be a
18 function of the liquid volume fraction:

$$A_{int} = A_{int}^{max} \left[6.75 (1 - \alpha_k)^2 \alpha_k \right], \quad (4)$$

where $A_{int}^{max} = 5100 \text{ m}^2/\text{m}^3$. With such definition, the interfacial area is zero in the limits $\alpha_k = 0$ and $\alpha_k = 1$. Lastly, Γ is the net mass transfer rate per unit interfacial area from phase j to phase k . Its expression, given in Eq. (5), is obtained by considering a vaporization/condensation process that is dominated by heat diffusion at the interface [1, 2]:

$$\begin{aligned} \Gamma = \Gamma_j &= \frac{h_{T,k} (T_k - T_{int}) + h_{T,j} (T_j - T_{int})}{h_{j,int} - h_{k,int}} \\ &= \frac{h_{T,k} (T_k - T_{int}) + h_{T,j} (T_j - T_{int})}{L_v (T_{int})}, \end{aligned} \quad (5)$$

19 where $L_v (T_{int}) = h_{j,int} - h_{k,int}$ represents the latent heat of vaporization. The
20 interface temperature is determined by the saturation constraint $T_{int} = T_{sat}(P)$
21 with the appropriate pressure $P = \bar{P}_{int}$ determined above. The interfacial heat
22 transfer coefficients for phases k and j are denoted by $h_{T,k}$ and $h_{T,j}$, respectively,
23 and computed from correlations [1].

The set of equations obeyed by phase j are simply obtained by substituting k by j and j by k in Eq. (1), keeping the same definition of the interfacial variables and remembering that $\Gamma_j = -\Gamma_k$. In the case of two-phase flows, the equation for the volume fraction of phase j is simply replaced by the algebraic relation

$$\alpha_j = 1 - \alpha_k,$$

24 which reduces the number of equations from eight to seven and yields the seven-
25 equation two-phase flow model.

The seven-equation model has interesting properties that are discussed next. A set of seven waves is present in such a model: two acoustic waves and a contact wave for each phase supplanted by a volume fraction wave propagating at the interfacial velocity \mathbf{u}_{int} . Considering a domain of dimension \mathbb{D} , the

corresponding eigenvalues are the following for each phase k :

$$\begin{aligned}
\lambda_1 &= \mathbf{u}_{int} \cdot \bar{\mathbf{n}} \\
\lambda_{2,k} &= \mathbf{u}_k \cdot \bar{\mathbf{n}} - c_k \\
\lambda_{3,k} &= \mathbf{u}_k \cdot \bar{\mathbf{n}} + c_k \\
\lambda_{d+3,k} &= \mathbf{u}_k \cdot \bar{\mathbf{n}} \text{ for } d = 1 \dots \mathbb{D},
\end{aligned} \tag{6}$$

where $\bar{\mathbf{n}}$ is a unit vector pointing to a given direction. The eigenvalues given in Eq. (6) are unconditionally real (as long as the chosen equation of state yields a real sound speed) which presents an interesting property for the development of numerical methods since the system is hyperbolic and well-posed. To relax the seven-equation model to the ill-posed classical six-equation model, only the pressures should be relaxed toward a single pressure for both phases. This is accomplished by specifying the pressure relaxation coefficient to be very large, i.e., letting it approach infinity. But if the pressure relaxation coefficient goes to infinity, so does the velocity relaxation rate also approach infinity. This then relaxes the seven-equation model not to the classical six-equation model but to the mechanical equilibrium five-equation model of Kapila [3]. This reduced five-equation model is also hyperbolic and well-posed. The five-equation model provides a very useful starting point for constructing multi-dimensional interface resolving methods which dynamically captures evolving and spontaneously generated interfaces [4]. Thus the seven-equation model can be relaxed locally to couple seamlessly with such a multi-dimensional, interface resolving code. Numerically, the mechanical relaxation coefficients μ_P (pressure) and λ_u (velocity) can be relaxed independently to yield solutions to useful, reduced models. It is noted, however, that relaxation of pressure only by making μ_P large without relaxing velocity will indeed give ill-posed and unstable numerical solutions, just as the classical six-equation two-phase model does, with sufficiently fine spatial resolution, as confirmed in [1, 5]. For each phase k , an entropy equation can be derived and its sign proved positive when accounting only for the pressure and velocity relaxation terms (all of the terms proportional to the net mass transfer term Γ are removed). The entropy function for a phase k is denoted by s_k and function of the density ρ_k and the internal energy e_k . The derivation is detailed in Appendix A and only the final result is recalled here:

$$\begin{aligned}
(s_e)_k^{-1} \alpha_k \rho_k A \frac{Ds_k}{Dt} &= \mu_P \frac{Z_k}{Z_k + Z_j} (P_j - P_k)^2 + \lambda_u \frac{Z_j}{Z_k + Z_j} (\mathbf{u}_j - \mathbf{u}_k)^2 \\
&\quad \frac{Z_k}{(Z_k + Z_j)^2} \left[Z_j (\mathbf{u}_j - \mathbf{u}_k) + \frac{\nabla \alpha_k}{\|\nabla \alpha_k\|} (P_k - P_j) \right]^2. \tag{7}
\end{aligned}$$

26 The partial derivative of the entropy function s_k with respect to the internal
27 energy e_k , $(s_e)_k$, is shown to be proportional to the inverse of the temperature
28 of phase k , alike for the single phase Euler equations [6, 7]. The right hand-side
29 of Eq. (7) is unconditionally positive since all terms are squared and thus, is
30 used to demonstrate the entropy minimum principle. Furthermore, Eq. (7) is

valid for both phases $\{k, j\}$ and ensures positivity of the total entropy equation that is obtained by summing over the phases:

$$\sum_k (s_e)_k^{-1} \alpha_k \rho_k A \frac{Ds_k}{Dt} = \sum_k (s_e)_k^{-1} \alpha_k \rho_k A (\partial_t s_k + \mathbf{u}_k \cdot \nabla s_k) \geq 0. \quad (8)$$

Note that when one phase disappears, Eq. (8) degenerates into the single phase entropy equation obtained from the multi-D Euler equations [1, 7].

3. A viscous regularization for the multi-D seven-equation two-phase flow model

We now propose to derive a viscous regularization for the seven-equation model given in Eq. (1) by using the same methodology as for the multi-D Euler equations with/without variable area [6, 8]. The method consists in adding perturbation terms to the system of equation under consideration, and re-derive the entropy equation whose sign is known to be positive to ensure uniqueness of the numerical solution [9]. Because of the addition of perturbation terms, the entropy equation is modified and contains extra terms of unknown sign. By carefully choosing a definition for each of the perturbation term, the sign of the entropy equation can be determined and proved positive. For the seven-equation model, derivation of a viscous regularization can be achieved by considering either the phasic entropy equation (Eq. (7)) or the total entropy equation (Eq. (8)). In the later case, the entropy minimum principle is verified for the whole system which may not ensure positivity of the entropy equation for each phase. However, positivity of the total entropy equation can be also achieved by assuming that the entropy minimum principle holds for each phase. This stronger requirement will also ensure consistency with the single phase Euler equations when one of the phase disappears in the limits $\alpha_k \rightarrow 0$. Thus, it is chosen to work with the phasic entropy equations given in Eq. (7).

For the purpose of this section, the system of equations given in Eq. (9) is considered, which is obtained by simply omitting the mass source terms (terms proportional to Γ) in Eq. (1).

$$\partial_t (\alpha_k A) + A \mathbf{u}_{int} \cdot \nabla \alpha_k = A \mu_P (P_k - P_j) \quad (9a)$$

$$\partial_t (\alpha_k \rho_k A) + \nabla \cdot (\alpha_k \rho_k \mathbf{u}_k A) = 0 \quad (9b)$$

$$\begin{aligned} \partial_t (\alpha_k \rho_k u_k A) + \nabla \cdot [\alpha_k A (\rho_k \mathbf{u}_k \otimes \mathbf{u}_k + P_k \mathbb{I})] = \\ \alpha_k P_k \nabla A + P_{int} A \nabla \alpha_k + A \lambda_u (\mathbf{u}_j - \mathbf{u}_k) \end{aligned} \quad (9c)$$

$$\begin{aligned} \partial_t (\alpha_k \rho_k E_k A) + \nabla \cdot [\alpha_k A \mathbf{u}_k (\rho_k E_k + P_k)] = \\ A P_{int} \mathbf{u}_{int} \cdot \nabla \alpha_k - \mu_P \bar{P}_{int} (P_k - P_j) + A \lambda_u \bar{\mathbf{u}}_{int} \cdot (\mathbf{u}_j - \mathbf{u}_k) \end{aligned} \quad (9d)$$

In order to apply the entropy viscosity method, perturbation terms are added to each equation of Eq. (9), which yields:

$$\partial_t (\alpha_k A) + A \mathbf{u}_{int} \cdot \nabla \alpha_k = A \mu_P (P_k - P_j) + \nabla \cdot \mathbf{l}_k \quad (10a)$$

$$\partial_t (\alpha_k \rho_k A) + \nabla \cdot (\alpha_k \rho_k \mathbf{u}_k A) = \nabla \cdot \mathbf{f}_k \quad (10b)$$

$$\begin{aligned} \partial_t (\alpha_k \rho_k \mathbf{u}_k A) + \nabla \cdot [\alpha_k A (\rho_k \mathbf{u}_k \otimes \mathbf{u}_k + P_k \mathbb{I})] = \\ \alpha_k P_k \nabla A + P_{int} A \nabla \alpha_k + A \lambda_u (\mathbf{u}_j - \mathbf{u}_k) + \nabla \cdot \mathbf{g}_k \end{aligned} \quad (10c)$$

$$\begin{aligned} \partial_t (\alpha_k \rho_k E_k A) + \nabla \cdot [\alpha_k A \mathbf{u}_k (\rho_k E_k + P_k)] = \\ P_{int} A \mathbf{u}_{int} \cdot \nabla \alpha_k - \mu_P \bar{P}_{int} (P_k - P_j) + A \lambda_u \bar{\mathbf{u}}_{int} \cdot (\mathbf{u}_j - \mathbf{u}_k) \\ + \nabla \cdot (\mathbf{h}_k + \mathbf{u} \cdot \mathbf{g}_k) \end{aligned} \quad (10d)$$

where \mathbf{f}_k , \mathbf{g}_k , \mathbf{h}_k and \mathbf{l}_k are the phasic perturbation terms to determine. The next step consists in deriving the entropy equation for the phase k , on the same model as what is done in Appendix A for the system of equations (Eq. (9)) that does not contain the perturbation terms.

1. derive the phasic density and internal energy equations from Eq. (10).
2. assuming that the phasic entropy, s_k , is function of the density, ρ_k and the internal energy, e_k , derive the entropy equation by using the chain rule:

$$\frac{Ds_k}{Dt} = (s_\rho)_k \frac{D\rho_k}{Dt} + (s_e)_k \frac{De_k}{Dt} \quad (11)$$

where $\frac{D(\cdot)}{Dt} = \partial_t(\cdot) + \mathbf{u} \cdot \nabla(\cdot)$ is the material derivative. The terms $(s_e)_k$ and $(s_\rho)_k$ denote the partial derivative of the entropy s_k with respect to e_k and ρ_k , respectively.

3. isolate the terms of interest and choose an appropriate expression for each of the perturbation terms in order to ensure positivity of the new term in the right-hand side.

We first derive the phasic density equation for the primitive variable ρ_k by combining Eq. (10a) and Eq. (10b) to obtain:

$$\alpha_k A \left[\partial_t \rho_k + (\mathbf{u}_k - \underline{\mathbf{u}_{int}}) \cdot \nabla \rho_k \right] = \underline{\underline{A \rho_k \mu_P (P_k - P_j)}} + \nabla \cdot \mathbf{f}_k - \rho_k \nabla \cdot \mathbf{l}_k \quad (12)$$

In order to derive the phasic internal energy equation, the phasic velocity equation is obtained by subtracting the phasic density equation from the phasic momentum equation:

$$\begin{aligned} \alpha_k \rho_k A [\partial_t \mathbf{u}_k + \mathbf{u}_k \cdot \nabla \mathbf{u}_k] + \nabla \cdot (\alpha_k \rho_k A P_k \mathbb{I}) = \\ \alpha_k P_k \nabla A + P_{int} A \nabla \alpha_k + A \lambda (\mathbf{u}_j - \mathbf{u}_k) + \nabla \cdot \mathbf{g}_k - \mathbf{u}_k \otimes \mathbf{f}_k \end{aligned} \quad (13)$$

After multiplying Eq. (13) by the phasic velocity vector \mathbf{u}_k , the resulting phasic kinetic energy equation is subtracted from the phasic total energy equation to obtain the internal energy equation for phase k :

$$\begin{aligned} \alpha_k \rho_k A [\partial_t \mathbf{e}_k + \mathbf{u}_k \cdot \nabla \cdot \mathbf{e}_k] + \alpha_k \rho_k A P_k \nabla \mathbf{u}_k = \\ \underline{\underline{P_{int} A (\mathbf{u}_{int} - \mathbf{u}_k) \cdot \nabla \alpha_k - \alpha_k P_k \mathbf{u}_k \cdot \nabla A}} \\ \underline{\underline{-\bar{P}_{int} A \mu_P (P_k - P_j) + A \lambda_u (\mathbf{u}_j - \mathbf{u}_k) \cdot (\bar{\mathbf{u}}_{int} - \mathbf{u}_k)}} \\ + \nabla \cdot \mathbf{h}_k + \mathbf{g}_k : \nabla \mathbf{u}_k + \|\mathbf{u}\|_k^2 \mathbf{f}_k \end{aligned} \quad (14)$$

The underline terms in Eq. (12) and Eq. (14) yield the positive terms in the right-hand-side of Eq. (7) and thus are ignored in the remaining of the derivation for brevity. The phasic entropy equation is now obtained by combining the phasic density equation (Eq. (12)) and the phasic internal energy equation (Eq. (14)) through the chain rule given in Eq. (11) to yield:

$$\begin{aligned} \alpha_k \rho_k A \frac{Ds_k}{Dt} = (\rho s_\rho)_k [\nabla \cdot \mathbf{f}_k - \rho_k \nabla \cdot \mathbf{l}_k] + \\ (s_e)_k [\nabla \cdot \mathbf{h}_k + \mathbf{g}_k : \nabla \mathbf{u}_k + (\|\mathbf{u}\|_k^2 - e_k) \nabla \cdot \mathbf{f}_k], \end{aligned} \quad (15)$$

where it was assumed that the entropy of phase k satisfies the second thermodynamic law:

$$\begin{aligned} T_k ds_k = de_k - P_k \frac{d\rho_k}{\rho_k^2} \\ \text{which implies } P_k (s_e)_k + \rho_k (s_\rho)_k = 0, \\ (s_e)_k = T_k^{-1} \text{ and } (s_\rho)_k = -(s_e)_k P_k \frac{d\rho_k}{\rho_k^2}. \end{aligned} \quad (16)$$

Eq. (16) is also used to compute the partial derivative of the entropy with respect to the density, $(s_\rho)_k$, and the internal energy, $(s_e)_k$, if needed.

Following the methodology applied in [6, 8], the right-hand side of Eq. (15) can be further simplified by using the following expression for the dissipative terms \mathbf{f}_k , \mathbf{g}_k and \mathbf{h}_k :

$$\mathbf{f}_k = \tilde{\mathbf{f}}_k + \rho_k \mathbf{l}_k \quad (17a)$$

$$\mathbf{g}_k = \alpha_k \rho_k A \mu_k \mathbb{F}(\mathbf{u}_k) + \mathbf{f}_k \otimes \mathbf{u}_k \quad (17b)$$

$$\mathbf{h}_k = \tilde{\mathbf{h}}_k - \frac{\|\mathbf{u}_k\|^2}{2} \mathbf{f}_k + (\rho e)_k \mathbf{l}_k, \quad (17c)$$

where μ_k is a positive viscosity coefficient for phase k . Note the area function A in the definition of \mathbf{g}_k . Substituting the expression of the dissipative term

given in Eq. (17) into Eq. (15), it yields:

$$\begin{aligned}
\alpha_k \rho_k A \frac{Ds_k}{Dt} &= \underbrace{\nabla \cdot \left[(s_e)_k \tilde{\mathbf{h}}_k + \left(e_k (s_e)_k - \rho_k (s_\rho)_k \right) \tilde{\mathbf{f}}_k \right]}_{\mathcal{R}_0} \\
&\quad \underbrace{(s_e)_k \alpha_k \rho_k A \mu_k \mathbb{F}(\mathbf{u}_k) : \nabla \mathbf{u}_k}_{\mathcal{R}_1} - \underbrace{\tilde{\mathbf{h}}_k \cdot \nabla (s_e)_k - \tilde{\mathbf{f}}_k \cdot \nabla [(e s_e)_k - (\rho s_\rho)_k]}_{\mathcal{R}_2} + \\
&\quad \underbrace{(s_e)_k \nabla \cdot (\rho_k e_k \mathbf{l}_k) - (s_e)_k e_k \nabla \cdot (\rho_k \mathbf{l}_k) + \rho_k (s_\rho)_k \nabla \cdot (\rho_k \mathbf{l}_k) - \rho_k^2 (s_\rho)_k \nabla \cdot \mathbf{l}_k}_{\mathcal{R}_3}. \quad (18)
\end{aligned}$$

We now split the right-hand-side of Eq. (18) into three residuals denoted by \mathcal{R}_1 , \mathcal{R}_2 and \mathcal{R}_3 and will study the sign of each of them. Since $(s_e)_k$ is defined as the inverse of the temperature and thus positive, the sign of the first term, \mathcal{R}_1 , is conditioned by the choice of the function $\mathbb{F}(\mathbf{u}_k)$ so that the product with the tensor $\nabla \mathbf{u}_k$ is positive. As in [6, 8], $\mathbb{F}(\mathbf{u}_k)$ is chosen proportional to the symmetric gradient of the velocity vector $\nabla^s \mathbf{u}_k$, whose entries are given by $((\nabla^s \mathbf{u})_{i,j})_k = \frac{1}{2} (\partial_{x_i} u_j + \partial_{x_j} u_i)_k$. With such a choice, the viscous regularization is rotation invariant. After a few lines of algebra, the third term \mathcal{R}_3 can be recast as a function of the gradient of the entropy as follows:

$$\mathcal{R}_3 = \rho_k A \mathbf{l}_k \cdot \nabla s_k. \quad (19)$$

One of the assumptions made in the entropy minimum principle is that the entropy is at a minimum which implies that its gradient is null. Because of this, it follows that the term \mathcal{R}_3 is zero at the minimum and thus, the entropy minimum principle is verified independently of the definition of the perturbation term \mathbf{l}_k used in the volume fraction equation Eq. (10a). It will be explained later in this section how to derive a definition for \mathbf{l}_k .

We now focus on the term denoted by \mathcal{R}_2 , that is found identical to the right-hand-side of the single phase entropy equation obtained from the multi-D Euler equations (see [6, 8]). Thus, the term \mathcal{R}_2 is known to be positive when (i) assuming concavity of the entropy function s_k with respect to the internal energy e_k and the specific volume $1/\rho_k$ (or convexity of $-s_k$) and (ii) choosing the following definitions for the dissipative terms $\tilde{\mathbf{h}}_k$ and $\tilde{\mathbf{f}}_k$:

$$\tilde{\mathbf{f}}_k = \alpha_k A \kappa_k \nabla \rho_k \quad (20a)$$

$$\tilde{\mathbf{h}}_k = \alpha_k A \kappa_k \nabla (\rho e)_k, \quad (20b)$$

where κ_k is another positive viscosity coefficient. In addition, using Eq. (20a), the term \mathcal{R}_0 can be recast as a function of the phasic entropy as follows:

$$\mathcal{R}_0 = \nabla \cdot (\alpha_k A \kappa_k \rho_k \nabla s_k) \quad (21)$$

The entropy equation can now be written in its final form:

$$\begin{aligned}
\alpha_k \rho_k A \frac{Ds_k}{Dt} &= \mathbf{f}_k \cdot \nabla s_k + \nabla \cdot (\alpha_k A \rho_k \kappa_k \nabla s_k) \\
&\quad - \alpha_k \rho_k A \kappa_k \mathbf{Q}_k + (s_e)_k \alpha_k A \rho_k \mu_k \nabla^s \mathbf{u}_k : \nabla \mathbf{u}_k, \quad (22)
\end{aligned}$$

80 where \mathbf{Q}_k is a negative semi-definite quadratic form under the assumption of s_k
 81 being concave with respect to e_k and $1/\rho_k$, and defined as:

$$\begin{aligned} \mathbf{Q}_k &= X_k^t \Sigma_k X_k \\ \text{with } X_k &= \begin{bmatrix} \nabla \rho_k \\ \nabla e_k \end{bmatrix} \text{ and } \Sigma_k = \begin{bmatrix} \rho_k^{-2} \partial_{\rho_k} (\rho_k^2 \partial_{\rho_k} s_k) & \partial_{\rho_k, e_k} s_k \\ \partial_{\rho_k, e_k} s_k & \partial_{e_k, e_k} s_k \end{bmatrix}. \end{aligned}$$

82 Eq. (22) is used to prove the entropy minimum principle: assuming that s_k
 83 reaches its minimum value in $\mathbf{r}_{min}(t)$ at each time t , the gradient, ∇s_k , and
 84 Laplacian, Δs_k , of the entropy are null and positive at this particular point,
 85 respectively. Furthermore, it is recalled that the viscosity coefficients μ_k and
 86 κ_k are positive by definition. Then, because the terms in the right-hand-side of
 87 Eq. (22) are proven either positive or null when the entropy reaches a minimum
 88 value, the entropy minimum principle holds for each phase k , **independently**
 89 **of the definition of the dissipative term \mathbf{l}_k** , such as:

$$\alpha_k \rho_k A \partial_t s_k(\mathbf{r}_{min}, t) \geq 0 \Rightarrow \partial_t s_k(\mathbf{r}_{min}, t) \geq 0$$

90 [Do we need to make the above statement a theorem or property?](#)

91 It remains to obtain a definition for the dissipative term \mathbf{l}_k used in the
 92 volume fraction equation Eq. (10a). A way to achieve this is to consider the
 93 volume fraction equation, by itself and notice that it is an hyperbolic equation
 94 with eigenvalue \mathbf{u}_{int} . An entropy equation can be derived and used to prove the
 95 entropy minimum principle by properly choosing the dissipative term [9]. The
 96 objective is to ensure positivity of the volume fraction and also uniqueness of
 97 the weak solution. Following the work of Guermond et al. in [10, 11], it can be
 98 shown that a dissipative term ensuring positivity and uniqueness of the weak
 99 solution for the volume fraction equation, is of the form $\mathbf{l}_k = \beta_k A \nabla \alpha_k$, where
 100 β_k is a positive viscosity coefficient. The dissipative term is proportional to the
 101 area A for consistency with the other terms of the volume fraction equation
 102 Eq. (10a).

All of the dissipative terms are now defined and recalled here:

$$\mathbf{l}_k = \beta_k A \nabla \alpha_k \quad (23a)$$

$$\mathbf{f}_k = \alpha_k A \kappa_k \nabla \rho_k + \rho_k A \mathbf{l}_k \quad (23b)$$

$$\mathbf{g}_k = \alpha_k A \mu_k \rho \nabla^s \mathbf{u}_k \quad (23c)$$

$$\mathbf{h}_k = \alpha_k A \kappa_k \nabla (\rho e)_k + \mathbf{u}_k : \mathbf{g}_k - \frac{\|\mathbf{u}_k\|^2}{2} \mathbf{f}_k + (\rho e)_k \mathbf{l}_k \quad (23d)$$

103 At this point, some remarks are in order:

- 104 1. The viscous regularization given in Eq. (23) for the multi-D seven-equation
105 model, is equivalent to the parabolic regularization [12] when assuming
106 $\beta_k = \kappa_k = \mu_k$ and $\mathbb{F}(\mathbf{u}_k) = \alpha_k \rho_k \kappa_k \nabla \mathbf{u}_k$, but is no longer rotation in-
107 variant. However, decoupling between the regularization on the velocity
108 and on the density in the momentum equation is important to make the
109 regularization rotation invariant but also to ensure well-scaled dissipative
110 terms for a wide range of Mach number as was shown in [8] for the multi-D
111 Euler equations.
- 112 2. The dissipative term \mathbf{l}_k requires the definition of a new viscosity coefficient
113 β_k . It was shown that this viscosity coefficient is independent of the
114 other viscosity coefficients μ_k and κ_k . Its definition should account for
115 the eigenvalue \mathbf{u}_{int} and the entropy equation associated with the volume
116 fraction equation.
- 117 3. The dissipative term \mathbf{f}_k is a function of \mathbf{l}_k . Thus, all of the other dissipa-
118 tive terms are also functions of \mathbf{l}_k .
- 119 4. The partial derivatives $(s_e)_k$ and $(s_{\rho_k})_k$ can be computed using the defini-
120 tion provided in Eq. (16) and are functions of the phasic thermodynamic
121 variables: pressure, temperature and density.
- 122 5. All of the dissipative terms are chosen to be proportional to the void frac-
123 tion α_k and the cross-sectional area A , but the one in the volume fraction
124 equation that is only proportional to A . For instance, $\alpha_k A \nabla \rho_k$ is the flux
125 of the dissipative term in the continuity equation through the pseudo-area,
126 $\alpha_k A$, seen by the phase k . When one of the phases disappears, the dissipa-
127 tive terms must go to zero for consistency. On the other hand, when α_k
128 goes to one, the single-phase Euler equations with variable area and with
129 proper viscous regularization must be recovered.
- 130 6. Compatibility of the viscous regularization proposed in Eq. (23) with the
131 generalized entropies identified in Harten et al. [13] is demonstrated in
132 Appendix B.

133 At this point in the paper, we have derived a viscous regularization for the
134 multi-D seven-equation two-phase flow model that ensures positivity of the en-
135 tropy residual, uniqueness of the numerical solution when assuming concavity
136 of the phasic entropy s_k , and is consistent with the viscous regularization de-
137 rived for the multi-D Euler equations [6, 8] in the limit $\alpha_k \rightarrow 1$. The viscous
138 regularization involves a set of three viscosity coefficients for each phase, μ_k , κ_k
139 and β_k , that are assumed positive. Definition of the viscosity coefficients is now
140 required to complete the numerical stabilization method. Since the focus of this
141 paper is the entropy viscosity method, the viscosity coefficients will be defined
142 function of entropy residuals in Section 4. However, one can also devise a def-
143 inition for the viscosity coefficients μ_k and κ_k by analogy to Lapidus [14, 15]
144 or some pressure-based methods [16] used for the single-phase Euler equations.
145 On the other hand, the viscosity coefficient, β_k , for the volume fraction equa-
146 tion should rely on artificial dissipation stabilization methods used for scalar
147 hyperbolic equations.

Remark. *Through the derivations of the viscous regularization, it was noted*

that another set of dissipative terms \mathbf{f}_k and \mathbf{l}_k would also ensures positivity of the entropy residual:

$$\mathbf{l}_k = \beta_k T_k \left[\frac{\rho_k}{P_k + \rho_k e_k} \nabla \left(\frac{P_k}{\rho_k e_k} \right) - \frac{1}{P_k} \nabla \rho_k \right] \quad (24a)$$

$$\mathbf{f}_k = \kappa_k \nabla \rho_k + \frac{\rho_k^2 (s_\rho)_k}{(\rho s_\rho - e s_e)_k} \mathbf{l}_k \quad (24b)$$

148 However, the definition of \mathbf{l}_k proposed in Eq. (24a) was not considered as valid
 149 for the following reasons: positivity of the volume fraction cannot be achieved
 150 and the parabolic regularization is not retrieved when assuming equal viscosity
 151 coefficients.

152 4. A all-speed formulation of the Entropy Viscosity Method

153 When working with artificial dissipative numerical stabilization methods,
 154 great care needs to be carried to the definition of the viscosity coefficients that
 155 will determine the accuracy of the method. Generally speaking, sufficient arti-
 156 ficial viscosity should be added into the shock and discontinuity regions to pre-
 157 vent spurious oscillations from forming, while little dissipation is added when
 158 the numerical solution is smooth. Such requirements can be achieved by track-
 159 ing shocks and discontinuities in the numerical solutions. When dealing with
 160 fluid equations, the low-mach asymptotic limit also has to be accounted for in
 161 the definition of the viscosity coefficients in order to ensure well-scaled dissipa-
 162 tive terms [17, 18, 19]. Also, because each phase can experience different flow
 163 regime (the gas phase is supersonic whereas the liquid phase remains subsonic),
 164 it is chosen to work with three distinct viscosity coefficients for each phase.
 165 The purpose of this section is to derive a definition for the phasic viscosity
 166 coefficients, μ_k , κ_k and β_k , that ensures the correct numerical solution in the
 167 low-mach limit, can accurately resolves shocks in transonic and supersonic flows
 168 and is also consistent with the definition of the viscosity coefficients devised for
 169 the single-phase Euler equations in the limit $\alpha_k \rightarrow 1$. As a result, the approach
 170 used in [8] will be applied here in this section.

171 4.1. Definition of the viscosity coefficients

In the entropy viscosity method, each viscosity coefficient is function of an upper and a lower bound that are referred to as first-order viscosity coefficient and entropy viscosity coefficient (high-order coefficient), respectively, as shown in Eq. (25). The first-order viscosity coefficient is denoted by the subscript *max* and is defined proportional to the largest local eigenvalue so that the stabilization scheme becomes over-dissipative and smooth out all discontinuities. The entropy viscosity coefficient is set proportional to an entropy residual and

jumps of quantities to determine, and denoted by the subscript e .

$$\begin{aligned}\beta_k(\mathbf{r}, t) &= \min(\beta_{e,k}(\mathbf{r}, t), \beta_{max,k}(\mathbf{r}, t)), \\ \mu_k(\mathbf{r}, t) &= \min(\mu_{e,k}(\mathbf{r}, t), \mu_{max,k}(\mathbf{r}, t)), \\ \kappa_k(\mathbf{r}, t) &= \min(\kappa_{e,k}(\mathbf{r}, t), \kappa_{max,k}(\mathbf{r}, t)),\end{aligned}\tag{25}$$

where all of the variables are locally defined. We now define the first-order viscosity coefficients and will focus first on the phasic viscosity coefficients κ_k and μ_k that are untimely linked to the mass, momentum and energy equations. These two viscosity coefficients are involved in dissipative terms that identical to the ones obtained for the single-phase Euler equations [6, 8] when seeing the term $\alpha_k A$ as a pseudo cross-section and assuming an uniform volume fraction profile. Thus, it is chosen to define the corresponding first-order viscosity coefficients proportional to the local largest eigenvalue $\|\mathbf{u}_k\| + c_k$ as follows:

$$\kappa_{max,k}(\mathbf{r}, t) = \mu_{max,k}(\mathbf{r}, t) = \frac{h}{2} (\|\mathbf{u}_k\|(\mathbf{r}, t) + c_k(\mathbf{r}, t)),\tag{26}$$

where h is the grid size (each phase is solved on the same mesh). It remains to define the first-order viscosity coefficient, $\beta_{max,k}$, used in the volume fraction equation. Because the volume fraction equation can be treated as a hyperbolic scalar equation with an unique eigenvalue \mathbf{u}_{int} , the first-order viscosity coefficient is defined by analogy with Burger's equation [10, 11] as follows:

$$\beta_{max,k}(\mathbf{r}, t) = \frac{h}{2} \|\mathbf{u}_{int}(\mathbf{r}, t)\|.\tag{27}$$

After defining the first-order viscosity coefficients for each phase, we focus our attention to the entropy viscosity coefficients denoted by the subscript e in Eq. (25). We first choose to investigate the definitions of $\mu_{e,k}$ and $\kappa_{e,k}$. The entropy viscosity coefficients are set proportional to the entropy residual given in Eq. (28), that is known to be positive and peaked in the shock region.

$$R_k(\mathbf{r}, t) := \frac{Ds_k}{Dt} = \partial_t s_k + \mathbf{u}_k \cdot \nabla s_k\tag{28}$$

It is also accounted for the jumps of quantities that will be determined further. The objective is to be able to track spatially and temporally any shock and discontinuity forming in the computational domain. In [8], it was demonstrated the usefulness of recasting the entropy residual as a function of pressure, velocity, density and speed of sound as shown in Eq. (29). The alternative expression of the entropy residual denoted by $\tilde{R}_k(\mathbf{r}, t)$, no longer requires an analytical expression of the entropy s_k and experiences the same variations (in absolute value) as the original definition of the entropy residual (Eq. (28)).

$$R_k(\mathbf{r}, t) = \frac{Ds_k}{Dt} = \frac{(s_e)_k}{(P_e)_k} \underbrace{\left(\frac{DP_k}{Dt} - c_k^2 \frac{D\rho_k}{Dt} \right)}_{\tilde{R}_k(\mathbf{r}, t)},\tag{29}$$

Using the new expression of the entropy residual \tilde{R}_k , we now propose a definition, given in Eq. (30), for the phasic entropy viscosity coefficients $\mu_{e,k}$ and $\kappa_{e,k}$ that also accounts for jumps, J_k , of some function of the pressure and density for generality purpose. The jump helps at tracking contact waves or discontinuities other than shock that are not seen by the entropy residual. Its definition will be detailed in Section 4. A distinct normalization parameter is also introduced for each viscosity coefficient that is used for dimensionality purpose: a quick dimensional study of the dissipative terms shows that the viscosity coefficients are kinematic viscosity ($m^2 \cdot s^{-1}$). Thus, the normalization parameters has units in pressure and its final definition will be determined by a low-Mach asymptotic limit of Eq. (10) in order to ensure well-scaled dissipative terms for all-Mach flows. We see here the advantage of using the new expression for the entropy residual \tilde{R}_k that offers more diversity in the choice of the normalization parameters: the pressure itself and combination of the density, the sound speed and the norm of the velocity.

$$\mu_{e,k}(\mathbf{r}, t) = h^2 \frac{\max \left(|\tilde{R}_k(\mathbf{r}_q, t)|, ||J_k^\mu|| \right)}{\text{norm}_{P,k}^\mu}, \quad (30a)$$

and

$$\kappa_{e,k}(\mathbf{r}, t) = h^2 \frac{\max \left(|\tilde{R}_k(\mathbf{r}_q, t)|, ||J_k^\kappa|| \right)}{\text{norm}_{P,k}^\kappa}. \quad (30b)$$

It remains to define the entropy viscosity coefficient $\beta_{e,k}$. For the purpose of this paragraph, let us consider the scalar volume fraction equation and assume that the interface velocity \mathbf{u}_{int} is given. Because it is a scalar hyperbolic equation, it is proposed to define the entropy viscosity coefficients on the same model as what is done for Burger's equation [10, 11]. Thus, the entropy viscosity coefficient β_e is defined as a function of an entropy residual, R_k^α , derived from the volume fraction equation for phase k , and the jump of a function of the volume fraction, J_k^α , as shown in Eq. (31).

$$\beta_{e,k}(\mathbf{r}, t) = h^2 \frac{\max \left(|R_k^\alpha(\mathbf{r}_q, t)|, ||J_k^\alpha|| \right)}{\text{norm}_{\alpha,k}^\beta} \quad (31)$$

We also introduce a normalization parameter, $\text{norm}_{\alpha,k}^\beta$, whose expression will be further investigated in Section 4.2. To derive the entropy residual, $R_{\alpha,k}$, we consider the volume fraction equation for phase k with its viscous regularization and assume the existence of an entropy denoted by $\eta_k(\alpha_k)$ [9]:

$$\partial_t (A\alpha_k) + A\mathbf{u}_{int} \cdot \nabla \alpha_k = \nabla \cdot (\beta_k A \nabla \alpha_k) \quad (32)$$

After multiplying by $\frac{d\eta(\alpha_k)}{d\alpha_k}$ and using the chain rule, an expression for the entropy equation is obtained:

$$\underbrace{\partial_t (A\eta(\alpha_k)) + A\mathbf{u}_{int} \cdot \nabla \eta(\alpha_k)}_{R_k^\alpha} = \frac{d\eta(\alpha_k)}{d\alpha_k} \nabla \cdot (\beta_k A \nabla \alpha_k) \quad (33)$$

220 The entropy residual, R_k^α , is defined as the left hand side of Eq. (33) and is
 221 known to be peaked in the shock region and positive when assuming convexity
 222 of the entropy η_k with respect to α_k [9]. Such a behavior is identical to the
 223 entropy residual \tilde{R}_k defined in Eq. (29), and will allow detection of the shock
 224 wave in the volume fraction profile when used in the definition of the entropy
 225 viscosity coefficient $\beta_{e,k}$.

226 At this point of the paper, the definition of the viscosity coefficients are
 227 not finalized: the jumps and normalization parameters still have to be defined.
 228 Details regarding the definition of the jump will be given in Section 5. The
 229 normalization parameters are derived from a low-Mach asymptotic limit analysis
 230 which is the purpose of the next section.

231 4.2. Asymptotic study in the low-Mach regime

232 Developing a numerical method for fluid equations require to investigate
 233 the low-Mach asymptotic limit. In this particular limit, numerical methods
 234 developed for transonic and supersonic flows usually fail due to ill-scaled dissipa-
 235 tive terms. A fix can be found by performing a low-Mach asymptotic limit
 236 to ensure well-scaled dissipative terms [17, 18, 19]. Then, it is proposed to
 237 perform a low-Mach asymptotic limit to derive a definition for the phasic nor-
 238 malization parameters introduced in Section 4.1. We consider the case where
 239 the relaxation coefficients are set to zero: the two phases do not interact and
 240 the seven-equation model degenerates into two sets of Euler equations with a
 241 pseudo cross-section $\alpha_k A$. Two limit cases (a) and (b) will be considered to
 242 determine appropriate scaling for the entropy viscosity coefficients so that the
 243 dissipative terms remain well-scaled for: (a) the isentropic low-Mach limit where
 244 the seven-equation model degenerate to an incompressible system of equations
 245 in the low-Mach limit and (b) the non-isentropic limit with formation of shocks.
 246 In the low-Mach limit, the isentropic limit of the seven-equation model with vis-
 247 cous regularization should yield incompressible fluid flow solutions (the seven-
 248 equation model was derived by assuming that each phase obeys the multi-D
 249 Euler equations), namely, that the phasic pressure fluctuations are of the order
 250 M_k^2 and that the velocity satisfies the divergence constraint $\nabla \cdot (\vec{u}A)_k = 0$
 251 [17, 18, 19]. For non-isentropic situations, shocks may form for any value of
 252 Mach number (a step initial pressure will always yield a shock wave) and the
 253 minimum entropy principle should still be satisfied so that numerical oscillations,
 254 if any, be controlled by the entropy viscosity method independently of
 255 the value of the Mach number. For each case the scaling of the numerical ad-
 256 dimensional numbers will be given along with the definition of the normalization
 257 parameters defined in Section 4.1 for each viscosity coefficients. The asymptotic
 258 study is performed on the multi-D version of the seven-equation model with the
 259 Stiffened Gas Equation of State (SGEOS) given in Eq. (34).

$$P_k = (\gamma_k - 1) \rho_k e_k - \gamma_k P_{k,\infty} \quad (34)$$

The first step in the study of the two limit cases (a) and (b) is to re-write
 each system of equations in a non-dimensional manner. To do so, the following

variables are introduced for each phase k :

$$\begin{aligned}\rho_k^* &= \frac{\rho_k}{\rho_{k,\infty}}, \quad u_k^* = \frac{\mathbf{u}_k}{u_{k,\infty}}, \quad P_k^* = \frac{P_k}{\rho_{k,\infty} c_{k,\infty}^2}, \quad E_k^* = \frac{E_k}{c_{k,\infty}^2}, \quad x^* = \frac{x}{L_\infty}, \\ t_k^* &= \frac{t_k}{L_\infty / u_{k,\infty}}, \quad \mu_k^* = \frac{\mu_k}{\mu_{k,\infty}}, \quad \kappa_k^* = \frac{\kappa_k}{\kappa_{k,\infty}}, \quad P_{int}^* = \frac{P_{int}}{P_{int,\infty}}, \\ u_{int}^* &= \frac{u_{int}}{u_{int,\infty}}, \quad \bar{P}_{int}^* = \frac{\bar{P}_{int}}{\bar{P}_{int,\infty}}, \quad \bar{u}_{int}^* = \frac{\bar{u}_{int}}{\bar{u}_{int,\infty}},\end{aligned}\quad (35)$$

where the subscript ∞ denote the far-field or stagnation quantities and the superscript $*$ stands for the non-dimensional variables. The far-field reference quantities are chosen such that the dimensionless flow quantities are of order 1. The stagnation quantities for the pressure and velocity interfacial variables will be specified for each case. The reference Mach number is given by

$$M_{k,\infty} = \frac{u_{k,\infty}}{c_{k,\infty}}. \quad (36)$$

Because we consider that phases do not interact with each other, it is assumed that the interfacial pressure and velocity scale as the phasic pressure and velocity, respectively: $P_{int,\infty} = \rho_{k,\infty} c_{k,\infty}^2$ and $u_{int,\infty} = u_{k,\infty}$. Under these assumptions, the interfacial pressure and velocity are simply replaced by P_k and \mathbf{u}_k in the equations. Then, the system of equations with viscous regularization becomes:

$$\partial_t (\alpha_k A) + A \mathbf{u}_k \cdot \nabla \alpha_k = \nabla \cdot (A \beta_k \nabla \alpha_k) \quad (37a)$$

$$\partial_t (\alpha_k \rho_k A) + \nabla \cdot (\alpha_k \rho_k \mathbf{u}_k A) = \nabla \cdot (A \alpha_k \kappa_k \nabla \rho_k) + \nabla \cdot (A \beta_k \rho_k \nabla \alpha_k) \quad (37b)$$

$$\begin{aligned}\partial_t (\alpha_k \rho_k u_k A) + \nabla \cdot [\alpha_k A (\rho_k \mathbf{u}_k \otimes \mathbf{u}_k + P_k)] = \\ \alpha_k P_k \nabla A + P_k A \nabla \alpha_k + \nabla \cdot (A \mu_k \alpha_k \rho_k \nabla^s \mathbf{u}_k) + \\ \nabla \cdot (A \kappa_k \alpha_k \mathbf{u}_k \otimes \nabla \rho_k) + \nabla \cdot (A \beta_k \rho_k \mathbf{u}_k \otimes \nabla \alpha_k)\end{aligned}\quad (37c)$$

$$\begin{aligned}\partial_t (\alpha_k \rho_k E_k A) + \nabla \cdot [\alpha_k A \mathbf{u}_k (\rho_k E_k + P_k)] = \\ P_k A \mathbf{u}_k \cdot \nabla \alpha_k + \nabla \cdot (A \kappa_k \alpha_k \nabla (\rho_k e_k)) + \\ \nabla \cdot \left(A \kappa_k \alpha_k \frac{\|\mathbf{u}_k\|^2}{2} \nabla \rho_k \right) + \nabla \cdot (A \mu_k \alpha_k \rho_k \mathbf{u}_k : \nabla^s \mathbf{u}_k) + \\ \nabla \cdot (A \beta_k \rho_k e_k \nabla \alpha_k)\end{aligned}\quad (37d)$$

Then using the scaling introduced in Eq. (35), the scaled equations for the phase k with viscous regularization are: [The following set of equations is very painful to read. I guess we can improve the format but I cannot think of a better way](#)

of presenting the scaled equations, unless we include all of this in an appendix (I am not for it)

$$\partial_{t^*} (\alpha_k A)^* + A^* \mathbf{u}_k^* \cdot \nabla^* \alpha_k^* = \frac{1}{\text{Pé}_{k,\infty}^\beta} \nabla^* \cdot (A \beta_k \nabla^* \alpha_k)^* \quad (38a)$$

$$\begin{aligned} \partial_{t^*} (\alpha_k \rho_k A)^* + \nabla^* \cdot (\alpha_k \rho_k \mathbf{u}_k A)^* &= \frac{1}{\text{Pé}_{k,\infty}^\kappa} \nabla^* \cdot (A \kappa_k \nabla^* \rho_k)^* + \\ &\frac{1}{\text{Pé}_{k,\infty}^\beta} \nabla^* \cdot (A \beta_k \rho_k \nabla^* \alpha_k)^* \end{aligned} \quad (38b)$$

$$\begin{aligned} \partial_{t^*} (\alpha_k \rho_k \mathbf{u}_k A)^* + \nabla^* \cdot [\alpha_k A (\rho_k \mathbf{u}_k \otimes \mathbf{u}_k)]^* &+ \frac{A \alpha_k^*}{M_{k,\infty}^2} \nabla^* P_k^* = \\ \frac{1}{M_{k,\infty}^2} \alpha_k^* P_k^* \nabla^* A^* + \frac{1}{M_{k,\infty}^2} P_k^* A^* \nabla^* \alpha_k^* &+ \frac{1}{\text{Re}_{k,\infty}} \nabla^* \cdot (A \alpha_k \mu_k \rho_k \nabla^s \mathbf{u}_k)^* + \\ \frac{1}{\text{Pé}_{k,\infty}^\kappa} \nabla^* \cdot (A \alpha_k \kappa_k \mathbf{u}_k \otimes \nabla^* \rho_k)^* &+ \frac{1}{\text{Pé}_{k,\infty}^\beta} \nabla^* \cdot (A \beta_k \rho_k \mathbf{u}_k \otimes \nabla \alpha_k)^* \end{aligned} \quad (38c)$$

$$\begin{aligned} \alpha_k^* A^* [\partial_t (\rho_k E_k) + \mathbf{u}_k \cdot \nabla (\rho_k E_k)]^* &+ \alpha_k \nabla^* \cdot (A \mathbf{u}_k P_k) + \rho_k^* E_k^* \alpha_k^* \nabla^* \cdot (\mathbf{u} A)_k^* = \\ \frac{1}{\text{Pé}_{k,\infty}^\kappa} \nabla^* \cdot (A \alpha_k \kappa_k \nabla (\rho_k e_k))^* &+ \frac{M_{k,\infty}^2}{\text{Pé}_{k,\infty}^\kappa} \nabla^* \cdot \left(A \alpha_k \kappa_k \frac{\|\mathbf{u}_k\|^2}{2} \nabla \rho \right)^* + \\ \frac{M_{k,\infty}^2}{\text{Re}_{k,\infty}} \nabla^* \cdot (A \alpha_k \mu_k \rho_k \mathbf{u}_k : \nabla^s \mathbf{u}_k)^* &+ \\ \frac{1}{\text{Pé}_{k,\infty}^\beta} \nabla (\rho_k e_k)^* \cdot (A \beta_k \nabla \alpha_k)^* &- \frac{M_{k,\infty}^2}{\text{Pé}_{k,\infty}^\beta} \rho_k \frac{\|\mathbf{u}_k^2\|}{2} \nabla \cdot (\beta_k A \nabla \alpha_k) \end{aligned} \quad (38d)$$

where the phasic numerical Reynolds ($\text{Re}_{k,\infty}$) and Péclet ($\text{Pé}_{k,\infty}^\kappa$ and $\text{Pé}_{k,\infty}^\beta$) numbers are defined as:

$$\text{Re}_{k,\infty} = \frac{u_{k,\infty} L_\infty}{\mu_{k,\infty}}, \text{Pé}_{k,\infty}^\kappa = \frac{u_{k,\infty} L_\infty}{\kappa_{k,\infty}} \text{ and } \text{Pé}_{k,\infty}^\beta = \frac{u_{k,\infty} L_\infty}{\beta_{k,\infty}}. \quad (39)$$

Note that the phasic energy equation was recast under a non-conservative form by using the volume fraction (Eq. (38a)) to facilitate the derivations when trying to recover the divergence constraint onto the velocity. The numerical Reynolds and Péclet numbers defined in Eq. (39) are related to the phasic entropy viscosity coefficients $\mu_{k,\infty}$, $\kappa_{k,\infty}$ and $\beta_{k,\infty}$. Thus, once a scaling (in powers of $M_{k,\infty}$) is obtained for $\text{Re}_{k,\infty}$, $\text{Pé}_{k,\infty}^\kappa$ and $\text{Pé}_{k,\infty}^\mu$, the corresponding normalization parameters $\text{norm}_{P,k}^\mu$, $\text{norm}_{P,k}^\kappa$ and $\text{norm}_{\alpha,k}^\beta$ will automatically be set. For brevity, the superscripts * are omitted in the remainder of this section.

In the low-Mach isentropic limit, the seven-equation model converges to an incompressible system of equations when the Mach number tends to zero, that

277 is characterized with pressure fluctuations of order $M_{k,\infty}^2$ and the divergent
 278 constraint on the velocity: $\nabla \cdot (A\mathbf{u}_k) = 0$. When adding dissipative terms, as is
 279 the case with the entropy viscosity method, the main properties of the low-Mach
 280 asymptotic limit must be preserved. We begin by expanding each variable in
 281 powers of the Mach number. As an example, the expansion for the pressure is
 282 given by:

$$P_k(\mathbf{r}, t) = P_{k,0}(\mathbf{r}, t) + P_{k,1}(\mathbf{r}, t)M_{k,\infty} + P_{k,2}(\mathbf{r}, t)M_{k,\infty}^2 + \dots \quad (40)$$

283 By studying the resulting momentum equations for various powers of M_∞ , it
 284 is observed that the leading- and first-order pressure terms, $P_{k,0}$ and $P_{k,1}$, are
 285 spatially constant if and only if $\text{Re}_{k,\infty} = \text{Pe}_{k,\infty}^\kappa = \text{Pe}_{k,\infty}^\beta = 1$. In this case, we
 286 have at order $M_{k,\infty}^{-2}$:

$$\nabla P_{k,0} = 0 \quad (41a)$$

287 and at order $M_{k,\infty}^{-1}$

$$\nabla P_{k,1} = 0. \quad (41b)$$

288 From Eq. (41) we infer that the leading- and first-order pressure terms are
 289 spatially independent which ensures pressure fluctuations of order Mach num-
 290 ber square, as expected in the low-Mach asymptotic limit. Using the scaling
 291 $\text{Re}_{k,\infty} = \text{Pe}_{k,\infty}^\kappa = \text{Pe}_{k,\infty}^\beta = 1$, the second-order momentum equations and the
 292 leading-order expressions for the volume fraction, continuity and energy equa-
 293 tions are:

$$\partial_t (A\alpha_k)_0 + \mathbf{u}_{k,0} \cdot \nabla \alpha_{k,0} = \nabla \cdot (A\beta_k \nabla \alpha_k)_0 \quad (42a)$$

$$\partial_t (A\alpha_k \rho_k)_0 + \nabla \cdot (A\alpha_k \rho_k \mathbf{u}_k)_0 = \nabla \cdot (A\alpha_k \kappa_k \nabla \rho_k)_0 + \nabla \cdot (A\beta_k \nabla \alpha_k)_0 \quad (42b)$$

$$\begin{aligned} \partial_t (\alpha_k A \rho_k \mathbf{u}_k)_0 + \nabla \cdot (A\alpha_k \rho_k \mathbf{u}_k \otimes \mathbf{u}_k)_0 + A\alpha_k \nabla P_{k,2} = \\ \nabla \cdot [A\alpha_k (\mu_k \rho_k \nabla^s \mathbf{u}_k + \kappa_k \mathbf{u}_k \otimes \nabla \rho_k)]_0 + \nabla \cdot (A\beta_k \rho \mathbf{u} \nabla \alpha_k)_0 \end{aligned} \quad (42c)$$

$$\begin{aligned} \alpha_{k,0} A [\partial_t (\rho_k E_k) + \mathbf{u}_k \cdot \nabla (\rho_k E_k)]_0 + \alpha_{k,0} \nabla \cdot [A\mathbf{u}_k P_k]_0 + \alpha_{k,0} \rho_{k,0} E_{k,0} \nabla \cdot (\mathbf{u}_k A)_0 = \\ \nabla \cdot [A\alpha_k \kappa_k \nabla (\rho_k e_k)] + A\beta_{k,0} \nabla (\rho_k e_k)_0 \cdot \nabla \alpha_{k,0} \end{aligned} \quad (42d)$$

295 where the notation $(fg)_0$ means that we only keep the 0th-order terms in the
 296 product fg . The set of equations given in Eq. (42) are similar to the multi-D
 297 single-phase Euler equations with variable area when seeing $A\alpha_k$ as a pseudo-
 298 area [8]. The leading-order of the Stiffened Gas Equation of State (Eq. (34)) is
 299 given by

$$P_{k,0} = (\gamma_k - 1)\rho_{k,0}E_{k,0} - \gamma_k P_{k,\infty} = (\gamma_k - 1)\rho_0 e_{k,0} - \gamma_k P_{k,\infty}. \quad (43)$$

Using Eq. (43), the energy equation can be recast as a function of the leading-
 order pressure, P_0 , as follows:

$$\begin{aligned} A\alpha_{k,0} [\partial_t (P_k) + (\gamma_k - 1)\mathbf{u}_k \cdot \nabla P_k]_0 + (\gamma_k - 1)\alpha_{k,0} \nabla \cdot [A\mathbf{u}_k P_k]_0 + \\ \alpha_{k,0} (P_{k,0} + \gamma_k P_{k,\infty}) \nabla \cdot (\mathbf{u}_k A)_0 = \\ [\nabla \cdot (A\alpha_k \kappa_k \nabla (P_k))_0 + A\beta_{k,0} \nabla P_{k,0} \cdot \nabla \alpha_{k,0}]. \end{aligned} \quad (44)$$

300 From Eq. (41a), we infer that P_0 is spatially constant. Thus, Eq. (44) becomes

$$\frac{A}{\gamma(P_{k,0} + P_{k,\infty})} \frac{dP_0}{dt} = -\nabla \cdot (\mathbf{u}_k A)_0 \quad (45)$$

301 and, at steady state, we have

$$\nabla \cdot (\mathbf{u}_k A)_0 = 0. \quad (46)$$

302 That is, the leading-order of the product of velocity and cross section is divergence-
 303 free which corresponds to what is obtained when dealing with the multi-D Euler
 304 equations with variable area. Note that when assuming a constant cross sec-
 305 tion A , the usual divergence constraint, $\nabla \cdot \mathbf{u}_{k,0}$ is recovered. Also, Eq. (45) is
 306 slightly modified due to the use of the Stiffened Gas Equation of State in the
 307 asymptotic limit. However, the Ideal Gas Equation of State degenerates from
 308 the Stiffened Gas Equation of State by simply assuming $P_{k,\infty}$ which yields the
 309 usual leading-order single-phase energy equation with constant cross section:

$$\frac{1}{\gamma P_{k,0}} \frac{dP_0}{dt} = -\nabla \cdot \mathbf{u}_{k,0} \quad (47)$$

The same reasoning can be applied to the leading-order of the continuity equa-
 tion (Eq. (42b)) to show that the material derivative of the density variable is
 stabilized by well-scaled dissipative terms:

$$\left. \frac{D\alpha_k \rho_k}{Dt} \right|_0 := \partial_t (\alpha_k \rho)_0 + \mathbf{u}_{k,0} \cdot \nabla \cdot (\alpha_k \rho_k)_0 = \frac{1}{A} \nabla \cdot [\alpha_k A \kappa_k \nabla \rho + A \beta_k \rho_k \nabla \alpha_k]_0. \quad (48)$$

310 Therefore, we conclude that by setting the Reynolds and Péclet numbers to
 311 one, the incompressible fluid results are retrieved in the low-Mach limit when
 312 employing the compressible seven-equation model with viscous regularization
 313 and without relaxation terms.

314 4.3. Scaling of $Re_{k,\infty}$, $Pe_{k,\infty}^\kappa$ and $Pe_{k,\infty}^\beta$ for non-isentropic flows

Next, we consider the non-isentropic case. Recall that even subsonic flows
 can present shocks (for instance, a step initial condition in the pressure will trig-
 ger shock formation, independently of the Mach number). The non-dimensional
 form of the seven-equation model given in Eq. (38) provides some insight on
 the dominant terms as a function of the Mach number. This is particular ob-
 vious in the momentum equation, Eq. (38c), where the gradient of pressure is
 scaled by $1/M_{k,\infty}^2$. In the non-isentropic case, we no longer have $\frac{\nabla P_k}{M_{k,\infty}^2} = \nabla P_{k,2}$
 and therefore the pressure gradient term may need to be stabilized by some
 dissipative terms of the same scaling so as to prevent spurious oscillations from
 forming. By inspecting the dissipative terms presents in the momentum equa-
 tion, having a dissipative term that scales as $1/M_{k,\infty}^2$ leads to a total of eight

different options. Only three of them are investigated for brevity (note that the five other options can be ruled out by following the same reasoning as what is done next):

- (a) $\text{Re}_{k,\infty} = 1$, $\text{Pé}_{k,\infty}^\kappa = M_{k,\infty}^2$ and $\text{Pé}_{k,\infty}^\beta = 1$,
- (b) $\text{Re}_{k,\infty} = 1$, $\text{Pé}_{k,\infty}^\kappa = 1$ and $\text{Pé}_{k,\infty}^\beta = M_{k,\infty}^2$ or
- (c) $\text{Re}_{k,\infty} = M_{k,\infty}^2$, $\text{Pé}_{k,\infty}^\kappa = 1$ and $\text{Pé}_{k,\infty}^\beta = 1$.

Any of these choices will also affect the stabilization of the volume fraction, continuity and energy equations. For instance, using Péclet numbers equal to $M_{k,\infty}^2$ may effectively stabilize the volume fraction and continuity equation in the shock region but this may also add an excessive amount of dissipation for subsonic flows at the location of the contact wave. Such a behavior may not be suitable for accuracy purpose, making options (a) and (b) inappropriate. The same reasoning, left to the reader, can be carried out for the energy equation (Eq. (38d)) and results in the same conclusion. The remaining choice, option (c), has the proper scaling: in this case, only the dissipation terms involving $\nabla^{s,*} \mathbf{u}_k^*$ scale as $1/M_{k,\infty}^2$ since $\text{Re}_{k,\infty} = M_{k,\infty}^2$, leaving the regularization of the volume fraction and continuity equations unaffected because $\text{Pé}_{k,\infty}^\beta = \text{Pé}_{k,\infty}^\kappa = 1$.

4.4. An all-speed formulation of the viscosity coefficients

The study of the above limit cases yields two different possible scalings for the phasic Reynolds number: $\text{Re}_{k,\infty} = 1$ in the low-Mach limit and $\text{Re}_{k,\infty} = M_{k,\infty}^2$ for non-isentropic flows, whereas the phasic numerical Péclet numbers ($\text{Pé}_{k,\infty}^\kappa$ and $\text{Pé}_{k,\infty}^\beta$) always scales as one. In order to have a stabilization method valid for a wide range of Mach numbers, from very low-Mach to supersonic flows, these two scalings should be combined in a unique definition.

We begin with the normalization parameter $\text{norm}_{P,k}^\kappa$. Using the definition of the viscosity coefficients given in Eq. (30) and the scaling of Eq. (35), it can be shown that:

$$\kappa_{k,\infty} = \frac{\rho_{k,\infty} c_{k,\infty}^2 u_{k,\infty} L_{k,\infty}}{\text{norm}_{k,P,\infty}^\kappa} , \quad (49)$$

where $\text{norm}_{k,P,\infty}$ is the reference far-field quantity for the normalization parameter $\text{norm}_{k,P}$. Substituting Eq. (49) into Eq. (39) and recalling that the phasic numerical Péclet number scales as unity, we obtain:

$$\text{norm}_{k,P,\infty}^\kappa = \text{Pé}_{k,\infty} \rho_{k,\infty} c_{k,\infty}^2 = \rho_{k,\infty} c_{k,\infty}^2 . \quad (50)$$

Eq. (50) provides a proper normalization factor to define the κ_k viscosity coefficient. The derivation for $\text{norm}_{k,P}^\mu$ is similar and yields

$$\begin{aligned} \text{norm}_{k,P,\infty}^\mu &= \text{Re}_{k,\infty} \rho_{k,\infty} c_{k,\infty}^2 = \\ &\begin{cases} \rho_{k,\infty} \|u_{k,\infty}\|^2 & \text{for non-isentropic flows} \\ \rho_{k,\infty} c_{k,\infty}^2 = \text{norm}_{k,P,\infty}^\kappa & \text{for low-Mach flows} \end{cases} . \end{aligned} \quad (51)$$

339 A smooth function to transition between these two states is as follows:

$$\sigma(M_k) = \frac{\tanh(a_k(M_k - M_k^{\text{thresh}})) + |\tanh(a_k(M_k - M_k^{\text{thresh}}))|}{2}, \quad (52)$$

340 where M_k^{thresh} is a phasic threshold Mach number value beyond which the flow
 341 is no longer considered to be low-Mach (we use $M_k^{\text{thresh}} = 0.05$), M_k is the local
 342 Mach number, and the scalar a_k determines how rapidly the transition from
 343 $\text{norm}_{k,P,\infty}^\mu = \rho_k c_k^2$ to $\text{norm}_{k,P}^\mu = \rho_k \|\mathbf{u}_k\|^2$ occurs in the vicinity of M_k^{thresh} (we
 344 use $a_k = 3$). It is easy to verify that

$$\text{norm}_{k,P}^\mu = (1 - \sigma(M_k))\rho_k c_k^2 + \sigma(M_k)\rho_k \|\mathbf{u}_k\|^2 \quad (53)$$

345 satisfies Eq. (51).

346 It remains to determine the normalization parameter, $\text{norm}_{\alpha,k}^\beta$, for the vis-
 347 cosity coefficient β_k , by using the scaling of the Péclet number $\text{Pe}_{k,\infty}^\beta$ derived
 348 from the low-Mach asymptotic limit. Following the same reasoning as above, it
 349 yields:

$$\text{norm}_{k,\alpha,\infty}^\beta = 1, \quad (54)$$

350 where $\text{norm}_{\alpha,k,\infty}$ is the reference far-field quantity for the normalization pa-
 351 rameter $\text{norm}_{\alpha,k}$ used in the definition of the viscosity coefficient β_k (Eq. (31)).
 352 The normalization parameter scales as one. Then, it is chosen to use the same
 353 scaling as for Burger's equation [11] e.g.

$$\text{norm}_{k,\alpha}^\beta = \|\eta(\alpha_k) - \bar{\eta}(\alpha_k)\|_\infty, \quad (55)$$

354 where $\bar{\eta}$ is the average value of the entropy η over the entire computational
 355 domain.

356 At this point of the paper, we have derived a viscous regularization for the
 357 seven-equation model consistent with the entropy minimum principle, and de-
 358 fined viscosity coefficients for all-Mach flows. After describing the discretization
 359 scheme used (Section 5), 1-D numerical tests are performed in order to demon-
 360 strate the accuracy of our numerical method in Section 6.

361 5. Discretizations and Solution Techniques

362 In this section, we briefly describe the spatial and temporal discretizations
 363 and the solution techniques used to solve the system of equations Eq. (10). For
 364 conciseness, we re-write the system of equations in the following form:

$$\partial_t \mathbf{U}_k + \nabla \cdot \mathbf{F}_k(\mathbf{U}_k) = \mathbf{R}_k(\mathbf{U}_k) + \mathbf{N}_k(\mathbf{U}_k) + \nabla \cdot \mathbf{D}_k(\mathbf{U}_k) \nabla \mathbf{U}_k \quad (56)$$

365 where $\mathbf{U}_k = [(\alpha A)_k, (\alpha \rho A)_k, (\alpha \rho \mathbf{u} A)_k, (\alpha \rho E A)_k]^T$ is the solution vector, $\mathbf{F}_k(\mathbf{U}_k)$
 366 denotes the inviscid flux, $\nabla \cdot \mathbf{D}_k(\mathbf{U}_k) \nabla \mathbf{U}_k$ is the dissipative flux and $\mathbf{N}_k(\mathbf{U}_k)$
 367 and $\mathbf{R}_k(\mathbf{U}_k)$ contain the non-conservative and relaxation terms, respectively.

$$\mathbf{F} \equiv \begin{bmatrix} 0 \\ (\alpha \rho u A)_k \\ [\alpha (\rho u^2 + P) A]_k \\ [\alpha u (\rho E + P) A]_k \end{bmatrix}, \mathbf{N} \equiv \begin{bmatrix} -A \mathbf{u}_{int} \cdot \nabla \alpha_k \\ 0 \\ \alpha_k P_k \nabla A + P_{int} A \nabla \alpha_k \\ P_{int} A \mathbf{u}_{int} \cdot \nabla \alpha_k \end{bmatrix}$$

$$\text{and } \mathbf{R} \equiv \begin{bmatrix} A\mu_P (P_k - P_j) \\ 0 \\ A\lambda_u (\mathbf{u}_j - \mathbf{u}_k) \\ -\bar{P}_{int} A\mu_P (P_k - P_j) + \bar{u}_{int} A\lambda_u (\mathbf{u}_j - \mathbf{u}_k) \end{bmatrix}.$$

369 5.1. Spatial and Temporal Discretizations

370 The system of equations given in Eq. (56) is discretized using a continuous
371 Galerkin finite element method and temporal integrators available through the
372 MOOSE multiphysics framework [20].

373 5.1.1. Continuous Finite Elements

In order to apply the continuous finite element method, Eq. (56) is multiplied by a test function $\mathbf{W}(\mathbf{r})$, integrated by parts and each integral is decomposed into a sum of integrals over each element K of the discrete mesh Ω . The following weak form is obtained:

$$\begin{aligned} \sum_K \int_K \partial_t \mathbf{U} \mathbf{W} - \sum_K \int_K \mathbf{F}(\mathbf{U}) \cdot \nabla \mathbf{W} + \int_{\partial\Omega} \mathbf{F}(\mathbf{U}) \cdot \mathbf{n} \mathbf{W} - \sum_K \int_K (\mathbf{N}(\mathbf{U}) + \mathbf{R}(\mathbf{U})) \mathbf{W} \\ + \sum_K \int_K D(\mathbf{U}) \nabla \mathbf{U} \cdot \nabla \mathbf{W} - \int_{\partial\Omega} D(\mathbf{U}) \nabla \mathbf{U} \cdot \mathbf{n} \mathbf{W} = 0. \end{aligned} \quad (57)$$

374 The integrals over the elements K are evaluated using a numerical quadrature.
375 The MOOSE framework provides a wide range of test functions and quadrature
376 rules. Linear Lagrange polynomials are employed as test functions in the re-
377 sults section. Second-order spatial convergence will be demonstrated for smooth
378 solutions.

379 5.1.2. Temporal integration

380 The MOOSE framework offers both first- and second-order explicit and im-
381 plicit temporal integrators. In all of the numerical examples presented in Sec-
382 tion 6, the temporal derivative will be evaluated using the second-order, back-
383 ward difference temporal integrator BDF2. By considering three consecutive
384 solutions, \mathbf{U}^{n-1} , \mathbf{U}^n and \mathbf{U}^{n+1} , at times t^{n-1} , t^n and t^{n+1} , respectively, BDF2
385 can be expressed as:

$$\int_K \partial_t \mathbf{U} \mathbf{W} = \int_K (\omega_0 \mathbf{U}^{n+1} + \omega_1 \mathbf{U}^n + \omega_2 \mathbf{U}^{n-1}) \mathbf{W}, \quad (58)$$

with

$$\omega_0 = \frac{2\Delta t^{n+1} + \Delta t^n}{\Delta t^{n+1} (\Delta t^{n+1} + \Delta t^n)}, \quad \omega_1 = -\frac{\Delta t^{n+1} + \Delta t^n}{\Delta t^{n+1} \Delta t^n},$$

$$\text{and } \omega_2 = \frac{\Delta t^{n+1}}{\Delta t^n (\Delta t^{n+1} + \Delta t^n)}$$

386 where $\Delta t^n = t^n - t^{n-1}$ and $\Delta t^{n+1} = t^{n+1} - t^n$.

5.2. Boundary conditions

Boundary conditions for the seven-equation model are challenging because of the wave-dominated nature of the equations but also because of the non-conservative form of the volume fraction equation. Unlike the continuity, momentum and energy equations of each phase, the flux of the volume fraction equation is not integrated by part because not under conservative form, and thus, does not stem a boundary flux. Then, the boundary condition for the volume fraction equation (Eq. (1a)) is treated independently of the other equations (continuity, momentum and energy for each phase) for two reasons: (i) it is a simple advection equation with the real eigenvalue \mathbf{u}_{int} , and (ii), the hyperbolic flux, $\mathbf{u}_{int} \cdot \nabla \alpha_k$, is not integrated by part since not under a conservative form. The sign of the dot product between the eigenvalue and the outward normal to the boundary, $\mathbf{u}_{int} \cdot \mathbf{n}$, determines the nature of the boundary: negative for an inlet and positive for an outlet. For the later case, the physical information exits the computational domain and does not require any particular treatment. In the former case, the physical information enters the computational domain which requires to specify a value for the volume fraction. Since there is no flux at the boundary coming from the integration by part of the hyperbolic flux, the boundary value is imposed by using a Dirichlet boundary condition in the volume fraction equation. Our implementation of the boundary conditions for the continuity, momentum and energy equations, is inspired by the method described in [1] and was adapted for a time implicit solver [7]. The boundary type is identified from the study of the sign of the eigenvalues that depends on the Mach number. The numerical results presented in Section 6 were all obtained by using subsonic stagnation and static pressure boundary conditions for the inlet and outlet, respectively. The boundary flux is computed from the supplied variables at the boundary and also by iterating on a given number of variables (depending on the sign of the eigenvalues) through the implicit solver to transmit information from inside the computational domain toward the boundary.

The artificial diffusion coefficient $D(\mathbf{U})$ is set to zero at the boundary of the computational domain so that the boundary term $\int_{\partial\Omega} D(\mathbf{U}) \nabla \mathbf{U} \cdot \mathbf{n} \mathbf{W}$ stemming from the integration by parts of the artificial dissipative terms in Eq. (57) is ignored.

5.3. Solver

A Jacobian-free-Newton-Krylov (JFNK) method is used to solve for the solution at the end of each time step. An approximate Jacobian matrix of the discretized equations was derived and implemented. Obtaining the matrix entries requires that the partial derivatives of pressure with respect to the conservative variables be known (this is relatively simple for the stiffened and ideal gas equations of state but may be more complex for general equations of state). The contributions of the artificial dissipative terms to the Jacobian matrix are approximated by lagging the viscosity coefficients (computing them with the previous solution). For instance, this is shown in Eq. (59) for the dissipative

terms present in the continuity equation:

$$\frac{\partial}{\partial \mathbf{U}} (\kappa \nabla \rho \cdot \nabla W) \simeq \kappa \nabla \cdot \frac{\partial \rho}{\partial \mathbf{U}} \nabla W, \quad (59)$$

where \mathbf{U} denotes any of the conservative variables and W denotes the component of \mathbf{W} associated with the continuity equation. In the above, we have neglected $\frac{\partial \kappa}{\partial \mathbf{U}}$.

6. 1-D numerical results

- simple advection problem
- shock tube with two independent fluids: exact solution and could do convergence test for this particular test
- shock tube with infinite relaxation coefficients
- 1-D nozzle with two independent fluids
- 1-D nozzle with infinite relaxation coefficients
- 1-D nozzle with infinite relaxation coefficients, mass and heat transfer

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492 **Appendix A Entropy equation for the multi-D seven equation model**
 493 **without viscous regularization**

This appendix provides the steps that lead to the derivation of the phasic entropy equation of the seven-equation model [1]. For the purpose of this dissertation, two phases are considered and denoted by the indexes j and k . In the seven-equation model, each phase obeys to the following set of equations (Eq. (60)):

$$\partial_t (\alpha_k A) + A \mathbf{u}_{int} \cdot \nabla \alpha_k = A \mu (P_k - P_j) \quad (60a)$$

$$\partial_t (\alpha_k \rho_k A) + \nabla \cdot (\alpha_k \rho_k \mathbf{u}_k A) = 0 \quad (60b)$$

$$\begin{aligned} \partial_t (\alpha_k \rho_k \mathbf{u}_k A) + \nabla \cdot [\alpha_k A (\rho_k \mathbf{u}_k \otimes \mathbf{u}_k + P_k \mathbb{I})] = \\ \alpha_k P_k \nabla A + P_{int} A \nabla \alpha_k + A \lambda (\mathbf{u}_j - \mathbf{u}_k) \end{aligned} \quad (60c)$$

$$\begin{aligned} \partial_t (\alpha_k \rho_k E_k A) + \nabla \cdot [\alpha_k A \mathbf{u}_k (\rho_k E_k + P_k)] = \\ P_{int} A \mathbf{u}_{int} \cdot \nabla \alpha_k - \mu \bar{P}_{int} (P_k - P_j) + \bar{\mathbf{u}}_{int} A \lambda (\mathbf{u}_j - \mathbf{u}_k) \end{aligned} \quad (60d)$$

494 where ρ_k , \mathbf{u}_k , E_k and P_k are the density, the velocity, the specific total energy
 495 and the pressure of k^{th} phase, respectively. The pressure and velocity relaxation
 496 parameters are denoted by μ_P and λ_u , respectively. The variables with index
 497 $_{int}$ correspond to the interfacial variables and a definition is given in Eq. (61).
 498 The cross section A is only function of space: $\partial_t A = 0$.

$$\left\{ \begin{array}{l} P_{int} = \bar{P}_{int} - \frac{\nabla \alpha_k}{\|\nabla \alpha_k\|} \frac{Z_k Z_j}{Z_k + Z_j} (\mathbf{u}_k - \mathbf{u}_j) \\ \bar{P}_{int} = \frac{Z_k P_j + Z_j P_k}{Z_k + Z_j} \\ \mathbf{u}_{int} = \bar{\mathbf{u}}_{int} - \frac{\nabla \alpha_k}{\|\nabla \alpha_k\|} \frac{P_k - P_j}{Z_k + Z_j} \\ \bar{\mathbf{u}}_{int} = \frac{Z_k \mathbf{u}_k + Z_j \mathbf{u}_j}{Z_k + Z_j} \end{array} \right. \quad (61)$$

499 where $Z_k = \rho_k c_k$ and $Z_j = \rho_j c_j$ are the impedance of the phase k and j , respec-
 500 tively. The speed of sound is denoted by the variable c . The function $sgn(x)$
 501 returns the sign of the variable x .

502 The first step consists of rearranging the equations given in Eq. (61) using the
 503 primitive variables $(\alpha_k, \rho_k, \mathbf{u}_k, e_k)$, where e_k is the specific internal energy of
 504 k^{th} phase. We introduce the material derivative $\frac{D(\cdot)}{Dt} = \partial_t(\cdot) + \mathbf{u}_k \cdot \nabla(\cdot)$ for
 505 simplicity.

506 The volume fraction is unchanged. The continuity equation is modified as fol-
 507 lows:

$$\alpha_k A \frac{D\rho_k}{Dt} + \rho_k A \mu (P_k - P_j) + \rho_k A (\mathbf{u}_k - \mathbf{u}_j) \cdot \nabla \alpha_k + \rho_k \alpha_k \nabla \cdot (A \mathbf{u}_k) = 0 \quad (62)$$

508 The momentum and continuity equations are combined to yield the velocity
509 equation:

$$\alpha_k \rho_k A \frac{D\mathbf{u}_k}{Dt} + \partial_x (\alpha_k A P_k) = \alpha_k P_k \nabla A + P_{int} A \nabla \alpha_k + A \lambda_u (\mathbf{u}_j - \mathbf{u}_k) \quad (63)$$

The internal energy is obtained from the total energy and the kinetic equation (\mathbf{u}_k *Eq. (63)):

$$\begin{aligned} \alpha_k \rho_k A \frac{De_k}{Dt} + \nabla \cdot (\alpha_k \mathbf{u}_k A P_k) - \mathbf{u}_k \cdot \nabla (\alpha_k A P_k) &= P_{int} A (\mathbf{u}_{int} - \mathbf{u}_k) \cdot \nabla \alpha_k \\ &\quad - \alpha_k P_k \mathbf{u}_k \cdot \nabla A - \bar{P}_{int} A \mu_P (P_k - P_j) + A \lambda_u (\mathbf{u}_j - \mathbf{u}_k) \cdot (\bar{\mathbf{u}}_{int} - \mathbf{u}_k) \end{aligned} \quad (64)$$

510 In the next step, we assume the existence of a phase wise entropy s_k function
511 of the density ρ_k and the internal energy e_k . Using the chain rule,

$$\frac{Ds_k}{Dt} = (s_\rho)_k \frac{D\rho_k}{Dt} + (s_e)_k \frac{De_k}{Dt}, \quad (65)$$

512 along with the internal energy and the continuity equations, the following en-
513 tropy equation is obtained:

$$\begin{aligned} \alpha_k \rho_k A \frac{Ds_k}{Dt} + \underbrace{A (P_k (s_e)_k + \rho_k^2 (s_\rho)_k) \mathbf{u}_k \cdot \nabla \alpha_k + \alpha_k (P_k (s_e)_k + \rho_k^2 (s_\rho)_k) \mathbf{u}_k \cdot \nabla A}_{(a)} &= \\ (s_e)_k P_{int} A [(\mathbf{u}_{int} - \mathbf{u}_k) \cdot \nabla \alpha_k - \bar{P}_{int} A \mu_P (P_k - P_j) + A \lambda_u (\bar{\mathbf{u}}_{int} - \mathbf{u}_k) \cdot (\mathbf{u}_j - \mathbf{u}_k)] &- \\ \rho_k^2 (s_\rho)_k [\mu_P A (P_k - P_j) + A (\mathbf{u}_k - \mathbf{u}_{int}) \cdot \nabla \alpha_k] & \quad (66) \end{aligned}$$

514 where $(s_e)_k$ and $(s_\rho)_k$ denote the partial derivatives of the entropy s_k with
515 respect to the internal energy e_k and the density ρ_k , respectively. The second
516 term, (a), in the left hand side of Eq. (66) can be set to zero by assuming the
517 following relation between the partial derivatives of the entropy s_k :

$$P_k (s_e)_k + \rho_k^2 (s_\rho)_k = 0. \quad (67)$$

518 The above equation is equivalent to the application of the second thermody-
519 namic law when assuming reversibility:

$$T_k ds_k = de_k - \frac{P_k}{\rho_k^2} d\rho_k \text{ with } (s_e)_k = \frac{1}{T_k} \text{ and } (s_\rho)_k = -\frac{P_k}{\rho_k^2} (s_e)_k \quad (68)$$

520 Thus, equation Eq. (66) can be rearranged using the relation $(s_\rho)_k = -\frac{P_k}{\rho_k^2} (s_e)_k$:

$$\begin{aligned} ((s_e)_k)^{-1} \alpha_k \rho_k \frac{Ds}{Dt} &= \underbrace{[P_{int} (\mathbf{u}_{int} - \mathbf{u}_k) + P_k (\mathbf{u}_k - \mathbf{u}_{int})] \cdot \nabla \alpha_k}_{(b)} + \\ &\quad \underbrace{\mu (P_k - P_j) (P_k - \bar{P}_{int})}_{(c)} + \underbrace{\lambda (\mathbf{u}_j - \mathbf{u}_k) \cdot (\bar{\mathbf{u}}_{int} - \mathbf{u}_k)}_{(d)} \end{aligned} \quad (69)$$

521 The right hand side of equation Eq. (69) is split into three terms (b), (c) and
 522 (d) that will be treated independently from each other. The terms (c) and (d)
 523 are simpler to start with and can be easily recast by using the definitions of $\bar{\mathbf{u}}_{int}$
 524 and \bar{P}_{int} given in equation Eq. (61):

$$\begin{aligned}\mu(P_k - P_j)(P_k - \bar{P}_{int}) &= \mu_P \frac{Z_k}{Z_k + Z_j} (P_j - P_k)^2 \\ \lambda(\mathbf{u}_j - \mathbf{u}_k) \cdot (\bar{\mathbf{u}}_{int} - \mathbf{u}_k) &= \lambda_u \frac{Z_j}{Z_k + Z_j} (\mathbf{u}_j - \mathbf{u}_k)^2\end{aligned}\quad (70)$$

525 By definition, μ_P , λ_u and Z_k are all positive. Thus, the above terms are uncon-
 526 ditionally positive.
 527 It remains to look at the last term (b). Once again, by using the definition of
 528 P_{int} and \mathbf{u}_{int} , and the following relations:

$$\begin{aligned}\mathbf{u}_{int} - \mathbf{u}_k &= \frac{Z_j}{Z_k + Z_j} (\mathbf{u}_j - \mathbf{u}_k) - \frac{\nabla \alpha_k}{\|\nabla \alpha_k\|} \frac{P_k - P_j}{Z_k + Z_j} \\ P_{int} - P_k &= \frac{Z_k}{Z_k + Z_j} (P_j - P_k) - \frac{\nabla \alpha_k}{\|\nabla \alpha_k\|} \frac{Z_k Z_j}{Z_k + Z_j} (\mathbf{u}_k - \mathbf{u}_j),\end{aligned}$$

529 (b) yields:

$$\begin{aligned}[P_{int}(\mathbf{u}_{int} - \mathbf{u}_k) + P_k(\mathbf{u}_k - \mathbf{u}_{int})] \cdot \nabla \alpha_k &= (P_{int} - P_k)(\mathbf{u}_{int} - \mathbf{u}_k) \cdot \nabla \alpha_k = \\ &= \frac{Z_k}{(Z_k + Z_j)^2} \nabla \alpha_k \cdot \left[Z_j(\mathbf{u}_j - \mathbf{u}_k)(P_j - P_k) + \frac{\nabla \alpha_k}{\|\nabla \alpha_k\|} Z_j^2 (\mathbf{u}_j - \mathbf{u}_k)^2 + \right. \\ &\quad \left. \frac{\nabla \alpha_k}{\|\nabla \alpha_k\|} (P_k - P_j)^2 + \frac{\nabla \alpha_k \cdot \nabla \alpha_k}{\|\nabla \alpha_k\|^2} (P_k - P_j) Z_j (\mathbf{u}_k - \mathbf{u}_j) \right]\end{aligned}\quad (71)$$

The above equation is factorized by $\|\nabla \alpha_k\|$ and then recast under a quadratic form when noticing that $\frac{\nabla \alpha_k \cdot \nabla \alpha_k}{\|\nabla \alpha_k\|^2} = 1$, which yields:

$$\begin{aligned}[(\mathbf{u}_{int} - \mathbf{u}_k)P_{int} + (\mathbf{u}_k - \mathbf{u}_{int})P_k] \cdot \nabla \alpha_k &= \\ \|\nabla \alpha_k\| \frac{Z_k}{(Z_k + Z_j)^2} \left[Z_j(\mathbf{u}_j - \mathbf{u}_k) + \frac{\nabla \alpha_k}{\|\nabla \alpha_k\|} (P_k - P_j) \right]^2\end{aligned}\quad (72)$$

Thus, using results from Eq. (69), Eq. (70), Eq. (71) and Eq. (72), the entropy equation obtained in [1] holds and is recalled here for convenience:

$$\begin{aligned}(s_e)_k^{-1} \alpha_k \rho_k A \frac{Ds_k}{Dt} &= \mu_P \frac{Z_k}{Z_k + Z_j} (P_j - P_k)^2 + \lambda_u \frac{Z_j}{Z_k + Z_j} (\mathbf{u}_j - \mathbf{u}_k)^2 \\ &\quad + \frac{Z_k}{(Z_k + Z_j)^2} \left[Z_j(\mathbf{u}_j - \mathbf{u}_k) + \frac{\nabla \alpha_k}{\|\nabla \alpha_k\|} (P_k - P_j) \right]^2.\end{aligned}$$

530 **Appendix B Compatibility of the viscous regularization for the seven-**
531 **equation two-phase model with the generalized Harten**
532 **entropies**

533 We investigate in this appendix whether the viscous regularization of the
534 seven-equation two-phase model derived in Section 3 is compatible with some
535 or all generalized entropy identified in Harten et al. [13]. Considering the single-
536 phase Euler equations, Harten et al. [13] demonstrated that a function $\rho \mathcal{H}(s)$
537 is called a generalized entropy and strictly concave if f is twice differential and

$$\mathcal{H}'(s) \geq 0, \quad \mathcal{H}'(s)c_p^{-1} - \mathcal{H}'' \geq 0, \quad \forall (\rho, e) \in \mathbb{R}_+^2, \quad (73)$$

538 where $c_p(\rho, e) = T \partial_T s(\rho, T)$ is the specific heat at constant pressure. Because
539 the seven-equation two-phase model was initially derived by assuming that each
540 phase obeys the single-phase Euler equation, we want to investigate whether
541 the above property still holds when considering the seven-equation model with
542 viscous regularization. To do so, we consider a phasic generalized entropy, $f_k(s_k)$
543 and a phasic specific heat at constant pressure, $c_{p,k}(\rho_k, e_k) = T_k \partial_{T_k} s_k(\rho_k, T_k)$
544 characterized by Eq. (73). The objective is to find an entropy inequality verified
545 by $\rho_k \mathcal{H}_k(s_k)$.

We start from the entropy inequality verified by s_k ,

$$\begin{aligned} \alpha_k \rho_k A \frac{Ds_k}{Dt} &= \mathbf{f}_k \cdot \nabla s_k + \nabla \cdot (\alpha_k A \rho_k \kappa_k \nabla s_k) \\ &\quad - \alpha_k \rho_k A \kappa_k \mathbf{Q}_k + (s_e)_k \alpha_k A \rho_k \mu_k \nabla^s \mathbf{u}_k : \nabla \mathbf{u}_k. \end{aligned} \quad (74)$$

Eq. (74) is multiplied by $\mathcal{H}'(s)$ to yield:

$$\begin{aligned} \alpha_k \rho_k A \frac{D\mathcal{H}_k(s_k)}{Dt} &= \nabla \cdot (\alpha_k A \rho_k \kappa_k \nabla \mathcal{H}_k(s_k)) - \mathcal{H}_k''(s_k) \alpha_k A \kappa_k \rho_k \|\nabla s_k\|^2 + \\ &\quad \mathcal{H}_k'(s_k) \mathbf{f}_k \cdot \nabla s_k - \mathcal{H}_k'(s_k) \alpha_k \rho_k A \kappa_k \mathbf{Q}_k + \\ &\quad \mathcal{H}_k'(s_k) (s_e)_k \alpha_k A \rho_k \mu_k \nabla^s \mathbf{u}_k : \nabla \mathbf{u}_k. \end{aligned} \quad (75)$$

Let us now multiply the continuity equation of phase k by $\mathcal{H}_k(s_k)$ and add the result to the above equation to obtain:

$$\begin{aligned} &\partial_t (\alpha_k \rho_k A \mathcal{H}_k(s_k)) + \nabla \cdot (\alpha_k \rho_k \mathbf{u}_k A \mathcal{H}_k(s_k)) - \\ &\nabla \cdot [\alpha_k A \rho_k \kappa_k \nabla \mathcal{H}_k(s_k) + \alpha_k A \kappa_k \mathcal{H}_k(s_k) \nabla \rho_k + A \kappa_k \rho_k \mathcal{H}_k(s_k) \nabla \alpha_k] = \\ &\underbrace{-\mathcal{H}_k''(s_k) \alpha_k A \kappa_k \rho_k \|\nabla s_k\|^2 - \mathcal{H}_k'(s_k) \alpha_k A \kappa_k \rho_k \mathbf{Q}_k +}_{\mathbb{T}_0} \\ &\quad \underbrace{\mathcal{H}_k'(s_k) (s_e)_k \alpha_k A \rho_k \mu_k \nabla^s \mathbf{u}_k : \nabla \mathbf{u}_k}_{\mathbb{T}_1}. \end{aligned} \quad (76)$$

546 Alike in Section 3, the left-hand side of Eq. (76) is split into two residuals
547 denoted by \mathbb{T}_0 and \mathbb{T}_1 in order to study the sign of each of them. We start by

548 studying the sign of \mathbb{T}_1 that is positive since it is assumed that $\mathcal{H}'_k(s_k) \geq 0$. We
 549 now investigate the sign of \mathbb{T}_0 . Using Eq. (73), it is obtained:

$$-\mathbb{T}_0 \leq \mathcal{H}'_k(s_k) \alpha_k A \kappa_k \rho_k \left(c_{p,k}^{-1} \|\nabla s_k\|^2 + \mathbf{Q}_k \right) . \quad (77)$$

The right-hand side of Eq. (77) is a quadratic form that was already defined in Appendix 5 of [6] and recast under the matricial form $X_k^t \mathbb{S} X_k$ where \mathbb{S} is a 2×2 matrix and the vector X_k is defined in Section 3. In [6], the matrix \mathbb{S} is proved to be negative semi-definite which allows us to conclude that $-\mathbb{T}_0$ is of the same sign using Eq. (77). Then, knowing the sign of the two residuals \mathbb{T}_0 and \mathbb{T}_1 , it is concluded that:

$$\begin{aligned} & \partial_t (\alpha_k \rho_k A \mathcal{H}_k(s_k)) + \nabla \cdot (\alpha_k \rho_k \mathbf{u}_k A \mathcal{H}_k(s_k)) - \\ & \nabla \cdot [\alpha_k A \rho_k \kappa_k \nabla \mathcal{H}_k(s_k) + \alpha_k A \kappa_k \mathcal{H}_k(s_k) \nabla \rho_k + A \kappa_k \rho_k \mathcal{H}_k(s_k) \nabla \alpha_k] \geq 0 , \end{aligned}$$

550 which allows us to conclude that an entropy inequality is satisfied for all gen-
 551 eralized entropies $\rho_k \mathcal{H}_k(s_k)$ when using the viscous regularization derived in
 552 Section 3 for the seven-equation two-phase model. Note that the above inequal-
 553 ity holds for the total entropy of the system when summing over the phases.