

Extension of the entropy viscosity method to the  
multi-D 7-equation two-phase flow model.  
I do not know if we should have 'multi-D' in the title  
since we will only present 1-D results

Marc O. Delchini<sup>a</sup>, Jean C. Ragusa<sup>\*,a</sup>, Ray A. Berry<sup>b</sup>

<sup>a</sup>*Department of Nuclear Engineering, Texas A&M University, College Station, TX 77843,  
USA*

<sup>b</sup>*Idaho National Laboratory, Idaho Falls, ID 83415, USA*

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## Abstract

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*Key words:* two-phase flow model, with variable area, entropy viscosity method, stabilization method, low Mach regime, shocks.

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## 1. Introduction

- a few lines about the need for accurately resolving two-phase flows
- background on the different two-phase flow models: 5, 6 and 7-equation two-phase flow models
- then, focus on the different types of 7-equation two-phase flow models: they mostly differ because of the closure relaxations used
- discuss the different numerical solvers developed for the 7-equation two-phase flow model: HLL, HLLC, and approximated Riemann solvers accounting for the source terms
- emphasize the fact that the above numerical solvers only works on discontinuous schemes
- then, introduce the entropy viscosity method and details the organization of the paper

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\*Corresponding author

Email addresses: `delchmo@tamu.edu` (Marc O. Delchini), `jean.ragusa@tamu.edu` (Jean C. Ragusa), `ray.berry@inl.gov` (Ray A. Berry)

Compressible two-phase flows are found in numerous industrial applications and are an ongoing area of research in modeling and simulation over many years. A variety of models with different levels of complexity has been developed such as: five-equation model [1], six-equation model [2], and more recently the seven-equation model [3]. These models are all obtained by integrating the single-phase flow balance equations weighed by a characteristic or indicator function for each phase. The resulting system of equations contains non-conservative terms that describe the interaction between phases but also an equation for the volume fraction. Once a system of equations describing the physics is derived, the next challenging step is to develop a robust and accurate discretization to obtain a numerical solution. Assuming that the system of equations is hyperbolic under some conditions, a Riemann solver could be used but is often ruled out because of the complexity due to the number of equations involved. Furthermore, careless approximation for the treatment of the non-conservative terms can lead to failure in computing the numerical solution [4]. An alternative is to use an approximate Riemann solver, a well-established approach for single-phase flows, while deriving a consistent discretization scheme for the non-conservative terms.

This methodology was applied to the seven-equation model (SEM) introduced by Berry et al. in [3]. This model is known to be unconditionally hyperbolic which is highly desirable when working with approximate Riemann solvers and can treat a wide range of applications. Its particularity comes from the pressure and velocity relaxation terms in the volume fraction, momentum and energy equations that can bring the two phases in equilibrium when using large values of the relaxation parameters. In other words, the seven-equation model can degenerate into the six- and five-equation models. Alike for the other two-phase flow models, solving for the seven-equation model requires a numerical solver and significant effort was dedicated to this task for spatially discontinuous schemes. Because each phase is assumed to obey the Euler equations, most of the numerical solvers are adapted from the single-phase approximate Riemann solvers. For example, Saurel et al. [5, 6] employed a HLL-type scheme to solve for the SEM but noted that excessive dissipation was added to the contact discontinuity. A more advanced HLLC-type scheme was developed in [7] but only for the subsonic case and then extended to supersonic flows in [8]. More recently, Ambroso et al. [9] proposed an approximate Riemann solver accounting for source terms such as gravity and drag forces, but with no interphase mass transfer.

## 2. The multi-D 7-equation two-phase flow model

- give the equations and detail the different terms
- include the relaxation terms, the mass and heat exchange terms
- eigenvalues

- entropy equation WITHOUT the dissipative terms and five the details of the derivation in the appendix

The multi-D seven-equation two-phase model is obtained by assuming that each phase obeys the single-phase Euler equations (with phase-exchange terms) and by integrating over a control volume after multiplying by a characteristic function. The detailed derivation can be found in [3]. In this section, the governing equations are recalled for each phase (liquid and vapor) and the source terms are described.

The liquid phase obeys the following mass, momentum and energy balance equations, supplemented by a non-conservative volume-fraction equation:

$$\frac{\partial (\alpha \rho)_{liq} A}{\partial t} + \vec{\nabla} \cdot (\alpha \rho \mathbf{u} A)_{liq} = -\Gamma A_{int} A \quad (1a)$$

$$\begin{aligned} \frac{\partial (\alpha \rho \mathbf{u})_{liq} A}{\partial t} + \vec{\nabla} \cdot [\alpha_{liq} A (\rho \mathbf{u} \otimes \mathbf{u} + P \mathbb{I})_{liq}] &= P_{int} A \vec{\nabla} \alpha_{liq} + P_{liq} \alpha_{liq} \vec{\nabla} A \\ &+ A \lambda_u (\mathbf{u}_{vap} - \mathbf{u}_{liq}) - \Gamma A_{int} \mathbf{u}_{int} A \end{aligned} \quad (1b)$$

$$\begin{aligned} \frac{\partial (\alpha \rho E)_{liq} A}{\partial t} + \vec{\nabla} \cdot [\alpha_{liq} \mathbf{u}_{liq} A (\rho E + P)_{liq}] &= P_{int} \mathbf{u}_{int} A \vec{\nabla} \alpha_{liq} - \bar{P}_{int} A \mu_P (P_{liq} - P_{vap}) \\ &+ \bar{\mathbf{u}}_{int} A \lambda_u (\mathbf{u}_{vap} - \mathbf{u}_{liq}) + \Gamma A_{int} \left( \frac{P_{int}}{\rho_{int}} - H_{liq,int} \right) A \\ &+ Q_{wall,liq} + Q_{int,liq} \end{aligned} \quad (1c)$$

$$\frac{\partial \alpha_{liq} A}{\partial t} + A \mathbf{u}_{int} \cdot \vec{\nabla} \alpha_{liq} = A \mu_P (P_{liq} - P_{vap}) - \frac{\Gamma A_{int} A}{\rho_{int}} \quad (1d)$$

On the same model, the equations for the vapor phase are:

$$\frac{\partial (\alpha \rho A)_{vap}}{\partial t} + \vec{\nabla} \cdot (\alpha \rho \mathbf{u})_{vap} A = \Gamma A_{int} A \quad (2a)$$

$$\begin{aligned} \frac{\partial (\alpha \rho \mathbf{u})_{vap} A}{\partial t} + \vec{\nabla} \cdot [\alpha_{vap} A (\rho \mathbf{u} \otimes \mathbf{u} + P \mathbb{I})_{vap}] &= P_{int} A \vec{\nabla} \alpha_{vap} + P_{vap} \alpha_{vap} \vec{\nabla} A \\ &+ A \lambda_u (\mathbf{u}_{liq} - \mathbf{u}_{vap}) + \Gamma A_{int} \mathbf{u}_{int} A \end{aligned} \quad (2b)$$

$$\begin{aligned} \frac{\partial (\alpha \rho E)_{vap} A}{\partial t} + \vec{\nabla} \cdot [\alpha_{vap} \mathbf{u}_{vap} A (\rho E + P)_{vap}] &= P_{int} \mathbf{u}_{int} A \vec{\nabla} \alpha_{vap} - \bar{P}_{int} A \mu_P (P_{vap} - P_{liq}) \\ &+ \bar{\mathbf{u}}_{int} A \lambda_u (\mathbf{u}_{liq} - \mathbf{u}_{vap}) - \Gamma A_{int} \left( \frac{P_{int}}{\rho_{int}} - H_{vap,int} \right) A \\ &+ Q_{wall,vap} + Q_{int,vap} \end{aligned} \quad (2c)$$

$$\frac{\partial \alpha_{vap} A}{\partial t} + A \mathbf{u}_{int} \cdot \vec{\nabla} \alpha_{vap} = A \mu_P (P_{vap} - P_{liq}) + \frac{\Gamma A_{int} A}{\rho_{int}} \quad (2d)$$

where  $\alpha_k$ ,  $\rho_k$ ,  $\mathbf{u}_k$  and  $E_k$  denote the volume fraction, the density, the velocity vector and the total specific energy of phase  $k = \{liq, vap\}$ , respectively. The phase pressure  $P_k$  is computed from an equation of state. The interfacial variables are denoted by the subscript *int* and their definition will be given in Eq. (4). The interfacial pressure and velocity and their corresponding average values are denoted by  $P_{int}$ ,  $\mathbf{u}_{int}$ ,  $\bar{P}_{int}$  and  $\bar{\mathbf{u}}_{int}$ , respectively.  $\Gamma$  is the net mass transfer rate per unit interfacial area from the liquid to the vapor phase and  $A_{int}$  is the interfacial area per unit volume of mixture. Also,  $H_{liq,int}$  and  $H_{vap,int}$  are the liquid and gas total specific enthalpies at the interface, respectively, with the following definition:  $H_k = h_k + 0.5 \|\mathbf{u}\|^2$ .  $\mu_P$  is the pressure relaxation coefficient and  $\lambda_u$  denotes the velocity relaxation coefficient. The wall and interfacial heat sources are denoted by  $Q_{wall,k}$  and  $Q_{int,k}$ , respectively, and are detailed in Section ???. Lastly, the cross section  $A$  is assumed spatially dependent. In the case of two-phase flows, the equation for the vapor volume fraction, Eq. (2d), is simply replaced by the algebraic relation

$$\alpha_{vap} = 1 - \alpha_{liq} \quad (3)$$

$$P_{int} = \bar{P}_{int} + \frac{Z_{liq} Z_{vap}}{Z_{liq} + Z_{vap}} \frac{\vec{\nabla} \alpha_{liq}}{\|\vec{\nabla} \alpha_{liq}\|} \cdot (\mathbf{u}_{vap} - \mathbf{u}_{liq}) \quad (4a)$$

$$\bar{P}_{int} = \frac{Z_{vap} P_{liq} + Z_{liq} P_{vap}}{Z_{liq} + Z_{vap}} \quad (4b)$$

$$\mathbf{u}_{int} = \bar{\mathbf{u}}_{int} + \frac{\vec{\nabla} \alpha_{liq}}{\|\vec{\nabla} \alpha_{liq}\|} \frac{P_{vap} - P_{liq}}{Z_{liq} + Z_{vap}} \quad (4c)$$

$$\bar{\mathbf{u}}_{int} = \frac{Z_{liq} \mathbf{u}_{liq} + Z_{vap} \mathbf{u}_{vap}}{Z_{liq} + Z_{vap}}. \quad (4d)$$

The interfacial velocities  $\mathbf{u}_{int}$  and its average value  $\bar{\mathbf{u}}_{int}$  are computed from: The pressure,  $\mu_P$ , and velocity,  $\lambda_u$ , relaxation coefficients are proportional to each other and function of the interfacial area  $A_{int}$ :

$$\lambda_u = \frac{1}{2} \mu_P Z_{liq} Z_{vap} \quad (5)$$

$$\mu_P = \frac{A_{int}}{Z_{liq} + Z_{vap}} \quad (6)$$

63 The specific interfacial area (i.e., the interfacial surface area per unit volume  
64 of two-phase mixture),  $A_{int}$ , must be specified from some type of flow regime  
65 map or function under the form of a correlation. In [3],  $A_{int}$  is chosen to be a  
66 function of the liquid volume fraction:

$$A_{int} = A_{int}^{max} \left[ 6.75 (1 - \alpha_{liq})^2 \alpha_{liq} \right], \quad (7)$$

where  $A_{int}^{max} = 5100 \text{ m}^2/\text{m}^3$ . With such definition, the interfacial area is zero in the limits  $\alpha_{liq} = 0$  and  $\alpha_{liq} = 1$ . To relax the seven-equation model to the ill-posed classical six-equation model, only the pressures should be relaxed toward a single pressure for both phases. This is accomplished by specifying the pressure relaxation coefficient to be very large, i.e., letting it approach infinity. But if the pressure relaxation coefficient goes to infinity, so does the velocity relaxation rate also approach infinity. This then relaxes the seven-equation model not to the classical six-equation model but to the mechanical equilibrium five-equation model of Kapila [1]. This reduced five-equation model is also hyperbolic and well-posed. The five-equation model provides a very useful starting point for constructing multi-dimensional interface resolving methods which dynamically captures evolving and spontaneously generated interfaces [10]. Thus the seven-equation model can be relaxed locally to couple seamlessly with such a multi-dimensional, interface resolving code.

Numerically, the mechanical relaxation coefficients  $\mu_P$  (pressure) and  $\lambda_u$  (velocity) can be relaxed independently to yield solutions to useful, reduced models (as explained previously). It is noted, however, that relaxation of pressure only by making  $\mu_P$  large without relaxing velocity will indeed give ill-posed and unstable numerical solutions, just as the classical six-equation two-phase model does, with sufficiently fine spatial resolution, as confirmed in [3, 11].

Even though the implementation of the seven-equation two-phase model does not use the generalized approach of DEM [3], the interfacial pressure and velocity closures as well as the pressure and velocity relaxation coefficients of Equations (4a) to (6) are utilized.

A simple expression for the interphase mass flow rate is obtained from [3]:

$$\begin{aligned}\Gamma &= \Gamma_{vap} = \frac{h_{T,liq}(T_{liq} - T_{int}) + h_{T,vap}(T_{vap} - T_{int})}{h_{vap,int} - h_{liq,int}} \\ &= \frac{h_{T,liq}(T_{liq} - T_{int}) + h_{T,vap}(T_{vap} - T_{int})}{L_v(T_{int})}\end{aligned}\quad (8)$$

where  $L_v(T_{int}) = h_{vap,int} - h_{liq,int}$  represents the latent heat of vaporization. The interface temperature is determined by the saturation constraint  $T_{int} = T_{sat}(P)$  with the appropriate pressure  $P = \bar{P}_{int}$  determined above, the interphase mass flow rate is thus determined.

The set of eight equations given in Eq. (1) and in Eq. (2) is now reduced to seven which yields the multi-D seven-equation model. A set of seven waves is present in such a model: two acoustic waves and a contact wave for each phase supplanted by a volume fraction wave propagating at the interfacial velocity  $\mathbf{u}_{int}$ . Considering a domain of dimension  $\mathbb{D}$ , the corresponding eigenvalues are

the following for each phase  $k$ :

$$\begin{aligned}\lambda_1 &= \mathbf{u}_{int} \cdot \bar{\mathbf{n}} \\ \lambda_{2,k} &= \mathbf{u}_k \cdot \bar{\mathbf{n}} - c_k \\ \lambda_{3,k} &= \mathbf{u}_k \cdot \bar{\mathbf{n}} + c_k \\ \lambda_{d+3,k} &= \mathbf{u}_k \cdot \bar{\mathbf{n}} \text{ for } d = 1 \dots \mathbb{D},\end{aligned}$$

where  $\bar{\mathbf{n}}$  is a unit vector pointing to a given direction. For each phase  $k$ , an entropy equation can be derived when accounting only for the pressure and velocity relaxation terms (all of the terms proportional to the net mass transfer term  $\Gamma$  and the interfacial heat transfer  $Q_{int,k}$  are removed). The entropy function for a phase  $k$  is denoted by  $s_k$  and function of the density  $\rho_k$  and the internal energy  $e_k$ . The derivation is detailed in APPENDIX and only the final result is recalled here when assuming that the phase  $k$  is in interaction with a phase  $j$ :

$$\begin{aligned}(s_e)_k^{-1} \alpha_k \rho_k A \frac{Ds_k}{Dt} &= \mu_P \frac{Z_k}{Z_k + Z_j} (P_j - P_k)^2 + \lambda_u \frac{Z_j}{Z_k + Z_j} (\mathbf{u}_j - \mathbf{u}_k)^2 \\ &\quad \frac{Z_k}{(Z_k + Z_j)^2} \left[ Z_j (\mathbf{u}_j - \mathbf{u}_k) + \frac{\vec{\nabla} \alpha_k}{\|\vec{\nabla} \alpha_k\|} (P_k - P_j) \right]^2, \quad (9)\end{aligned}$$

where  $Z_k$  denotes the phasic acoustic impedance and is defined as the product of the density and the speed of sound:  $Z_k = \rho_k c_k$ . The partial derivative of the entropy function  $s_k$  with respect to the internal energy  $e_k$ ,  $(s_e)_k$ , is defined proportional to the inverse of the temperature of phase  $k$  as for the single phase Euler equations. The right hand-side of Eq. (9) is unconditionally positive since all terms are squared. Furthermore, Eq. (9) is valid for each phase  $k = \{liq, vap\}$  and ensures positivity of the total entropy equation that is obtained by summing over the phases:

$$\sum_k (s_e)_k^{-1} \alpha_k \rho_k A \frac{Ds_k}{Dt} = \sum_k (s_e)_k^{-1} \alpha_k \rho_k A \left( \partial_t s_k + \mathbf{u}_k \cdot \vec{\nabla} s_k \right) \geq 0. \quad (10)$$

Note that when one phase disappears, Eq. (10) degenerates into the single phase entropy equation.

### 3. A viscous regularization for the multi-D 7-equation two-phase flow model

- explain why we work with the phase entropy equation instead of considering the total entropy residual by summing over the two phases
- viscous regularization must be consistent with single-phase flow equation
- recall the notion of entropy condition and entropy inequality  $\rightarrow$  require dissipative terms in order to get a sign

- 114 • give the system of equations with the dissipative terms
- 115 • guide the reader through the derivation of the dissipative terms
- 116 • give the entropy residual with all terms in the right hand-side
- 117 • make the link with the single-phase flow equations
- 118 • explain how to derive the dissipative term for the volume fraction equation
- 119 • emphasizes the fact that the regularization is valid for any EOS with
- 120 convex entropy
- 121 • a few words about the parabolic regularization

In this section, the dissipative terms for the multi-D seven-equation model *with pressure and velocity relaxation source terms* are derived (the mass and energy transfer terms are omitted). The methodology proposed in SECTION is followed. For clarity purpose, the seven-equation model with pressure and velocity relaxation terms is recalled when considering a phase  $k$  in interaction with a second phase  $j$ :

$$\partial_t (\alpha_k A) + A \mathbf{u}_{int} \cdot \vec{\nabla} \alpha_k = A \mu_P (P_k - P_j) \quad (11a)$$

$$\partial_t (\alpha_k \rho_k A) + \vec{\nabla} \cdot (\alpha_k \rho_k \mathbf{u}_k A) = 0 \quad (11b)$$

$$\begin{aligned} \partial_t (\alpha_k \rho_k u_k A) + \vec{\nabla} \cdot [\alpha_k A (\rho_k \mathbf{u}_k \otimes \mathbf{u}_k + P_k \mathbb{I})] = \\ \alpha_k P_k \vec{\nabla} A + P_{int} A \vec{\nabla} \alpha_k + A \lambda_u (\mathbf{u}_j - \mathbf{u}_k) \end{aligned} \quad (11c)$$

$$\begin{aligned} \partial_t (\alpha_k \rho_k E_k A) + \vec{\nabla} \cdot [\alpha_k A \mathbf{u}_k (\rho_k E_k + P_k)] = \\ A P_{int} \mathbf{u}_{int} \cdot \vec{\nabla} \alpha_k - \mu_P \bar{P}_{int} (P_k - P_j) + A \lambda_u \bar{\mathbf{u}}_{int} \cdot (\mathbf{u}_j - \mathbf{u}_k) \end{aligned} \quad (11d)$$

In order to apply the EVM, dissipative terms are added to each equation of the system given in Eq. (11), which yields:

$$\partial_t (\alpha_k A) + \mathbf{u}_{int} A \vec{\nabla} \alpha_k = A \mu_P (P_k - P_j) + \vec{\nabla} \cdot \mathbf{l}_k \quad (12a)$$

$$\partial_t (\alpha_k \rho_k A) + \vec{\nabla} \cdot (\alpha_k \rho_k \mathbf{u}_k A) = \vec{\nabla} \cdot \mathbf{f}_k \quad (12b)$$

$$\begin{aligned} \partial_t (\alpha_k \rho_k \mathbf{u}_k A) + \vec{\nabla} \cdot [\alpha_k A (\rho_k \mathbf{u}_k \otimes \mathbf{u}_k + P_k \mathbb{I})] = \\ \alpha_k P_k \vec{\nabla} A + P_{int} A \vec{\nabla} \alpha_k + A \lambda_u (\mathbf{u}_j - \mathbf{u}_k) + \vec{\nabla} \cdot \mathbf{g}_k \end{aligned} \quad (12c)$$

$$\begin{aligned} \partial_t (\alpha_k \rho_k E_k A) + \vec{\nabla} \cdot [\alpha_k A \mathbf{u}_k (\rho_k E_k + P_k)] = \\ P_{int} A \mathbf{u}_{int} \cdot \vec{\nabla} \alpha_k - \mu_P \bar{P}_{int} (P_k - P_j) + A \lambda_u \bar{\mathbf{u}}_{int} \cdot (\mathbf{u}_j - \mathbf{u}_k) + \vec{\nabla} \cdot (\mathbf{h}_k + \mathbf{u} \cdot \mathbf{g}_k) \end{aligned} \quad (12d)$$

where  $\mathbf{f}_k$ ,  $\mathbf{g}_k$ ,  $\mathbf{h}_k$  and  $\mathbf{l}_k$  are the dissipative terms. The next step consists of deriving the entropy equation for the phase  $k$ , on the same model as what is done in APPENDIX. Extra terms will appear in the right-hand-side of the entropy equation due to the dissipative terms. By choosing properly the definition of the dissipative terms, the sign of these extra terms can be controlled in order to ensure positivity of the entropy residual:

1. recast the system of equation given in Eq. (12) in terms of the primitive variables  $(\alpha_k, \rho_k, \mathbf{u}_k, e_k)$ .
2. derive the entropy equation by using the chain rule:

$$\frac{Ds_k}{Dt} = (s_\rho)_k \frac{D\rho_k}{Dt} + (s_e)_k \frac{De_k}{Dt} \quad (13)$$

where  $\frac{D}{Dt}$  is the material derivative. The terms  $(s_e)_k$  and  $(s_\rho)_k$  denote the partial derivative of the entropy  $s_k$  with respect to  $e_k$  and  $\rho_k$ , respectively.

3. isolate the terms of interest and choose an appropriate expression for each of the dissipative terms in order to ensure positivity of the right-hand side.

We first recast Eq. (12) in terms of the primitive variables: the volume fraction equation remains unchanged. The equation for the primitive variable  $\rho_k$  is derived by combining Eq. (12a) and Eq. (12b):

$$\alpha_k A \left[ \partial_t \rho_k + (\mathbf{u}_k - \mathbf{u}_{int}) \cdot \vec{\nabla} \rho_k \right] = A \rho_k \mu_P (P_k - P_j) + \vec{\nabla} \cdot \mathbf{f}_k - \rho_k \vec{\nabla} \cdot \mathbf{l}_k \quad (14)$$

The velocity equation is obtained by subtracting the density equation from the momentum equation:

$$\begin{aligned} \alpha_k \rho_k A \left[ \partial_t \mathbf{u}_k + \mathbf{u}_k \cdot \vec{\nabla} \mathbf{u}_k \right] + \vec{\nabla} \cdot (\alpha_k \rho_k A P_k \mathbb{I}) = \\ \alpha_k P_k \vec{\nabla} A + P_{int} A \vec{\nabla} \alpha_k + A \lambda (\mathbf{u}_j - \mathbf{u}_k) + \vec{\nabla} \cdot \mathbf{g}_k - \mathbf{u}_k \otimes \mathbf{f}_k \end{aligned} \quad (15)$$

After multiplying Eq. (15) by the velocity vector  $\mathbf{u}_k$ , the resulting kinetic energy equation is subtracted from the total energy equation to obtain the internal energy equation for phase  $k$ :

$$\begin{aligned} \alpha_k \rho_k A \left[ \partial_t e_k + \mathbf{u}_k \cdot \vec{\nabla} e_k \right] + \alpha_k \rho_k A P_k \vec{\nabla} \mathbf{u}_k = \\ P_{int} A (\mathbf{u}_{int} - \mathbf{u}_k) \cdot \vec{\nabla} \alpha_k - \alpha_k P_k \mathbf{u}_k \vec{\nabla} A \\ - \bar{P}_{int} A \mu_P (P_k - P_j) + A \lambda_u (\mathbf{u}_j - \mathbf{u}_k) \cdot (\bar{\mathbf{u}}_{int} - \mathbf{u}_k) \\ + \vec{\nabla} \cdot \mathbf{h}_k + \mathbf{g}_k : \vec{\nabla} \mathbf{u}_k + \|\mathbf{u}\|_k^2 \mathbf{f}_k \end{aligned} \quad (16)$$



138 The blue terms in Eq. (14) and Eq. (16) yield the positive terms in the right-  
 139 hand-side of Eq. (9) and thus are ignored in the remaining of the derivation. The  
 140 entropy equation is now obtained by combining the density equation (Eq. (14))  
 141 and the internal energy equation (Eq. (16)) through the chain rule given in  
 142 Eq. (13) to yield:

$$\alpha_k \rho_k A \frac{Ds_k}{Dt} = (s_e)_k \left[ \vec{\nabla} \cdot \mathbf{h}_k + \mathbf{g}_k : \vec{\nabla} \mathbf{u}_k + (||\mathbf{u}||_k^2 - e_k) \vec{\nabla} \cdot \mathbf{f}_k \right] + (\rho s_\rho)_k \left[ \vec{\nabla} \cdot \mathbf{f}_k - \rho_k \vec{\nabla} \cdot \mathbf{l}_k \right]. \quad (17)$$

where it was assumed that the entropy of phase  $k$  satisfies the second thermodynamic law:

$$\begin{aligned} T_k ds_k &= de_k - P_k \frac{d\rho_k}{\rho_k^2} \\ \text{which implies } P_k(s_e)_k + \rho_k(s_\rho)_k &= 0, \\ (s_e)_k &= T_k^{-1} \text{ and } (s_\rho)_k = -(s_e)_k P_k \frac{d\rho_k}{\rho_k^2}. \end{aligned} \quad (18)$$

143 From this point, two options are available in order to derive the dissipative  
 144 terms: either we consider the total entropy residual of the system by summing  
 145 Eq. (17) over each phase, or we can consider each phase independently. This  
 146 dilemma can be answered by remembering that the seven-equation model de-  
 147 generates into the single phase flow equations in the limits  $\alpha_k = 0, 1$ . Thus, the  
 148 dissipative terms also have to be consistent with the single-phase flow limits. As  
 149 a result, it is chosen to derive the dissipative terms by considering each phase  
 150 independently which will automatically ensure positivity of the total entropy  
 151 residual as well.

The right-hand side of Eq. (17) can be further simplified by using the following expression for the dissipative terms  $\mathbf{f}_k$ ,  $\mathbf{g}_k$  and  $\mathbf{h}_k$ :

$$\mathbf{f}_k = \tilde{\mathbf{f}}_k + \rho_k \mathbf{l}_k \quad (19)$$

$$\mathbf{g}_k = \alpha_k \rho_k A \mu_k \mathbb{F}(\mathbf{u}_k) + \mathbf{f}_k \otimes \mathbf{u}_k \quad (20)$$

$$\mathbf{h}_k = \tilde{\mathbf{h}}_k - \frac{||\mathbf{u}||^2}{2} \mathbf{f}_k + (\rho e)_k \mathbf{l}_k. \quad (21)$$

Note the area function  $A$  in the definition of  $\mathbf{g}$ . It yields:

$$\begin{aligned} \alpha_k \rho_k A \frac{Ds_k}{Dt} &= \\ &\underbrace{(s_e)_k \alpha_k \rho_k A \mu_k \mathbb{F}(\mathbf{u}_k) : \vec{\nabla} \mathbf{u}_k}_{\mathcal{R}_1} + \underbrace{\left[ \vec{\nabla} \cdot \tilde{\mathbf{h}}_k - e_k \vec{\nabla} \cdot \tilde{\mathbf{f}}_k \right] + (\rho s_\rho)_k \vec{\nabla} \cdot \tilde{\mathbf{f}}_k}_{\mathcal{R}_2} + \\ &\underbrace{(s_e)_k \vec{\nabla} \cdot (\rho_k e_k \mathbf{l}_k) - (s_e)_k e_k \vec{\nabla} \cdot (\rho_k \mathbf{l}_k) + \rho_k (s_\rho)_k \vec{\nabla} \cdot (\rho_k \mathbf{l}_k) - \rho_k^2 (s_\rho)_k \vec{\nabla} \cdot \mathbf{l}_k}_{\mathcal{R}_3}, \end{aligned} \quad (22)$$

152 where  $\mu_k$  is a positive viscosity coefficient for phase  $k$ . For simplicity, the right-  
 153 hand-side of Eq. (22) is split into three terms denoted by  $\mathcal{R}_1$ ,  $\mathcal{R}_2$  and  $\mathcal{R}_3$ . Since

( $s_e$ ) $_k$  is defined as the inverse of the temperature and thus positive, the sign of the first term,  $\mathcal{R}_1$ , is conditioned by the choice of the function  $\mathbb{F}(\mathbf{u}_k)$  so that the product with the tensor  $\vec{\nabla}\mathbf{u}_k$  is positive. As in [12],  $\mathbb{F}(\mathbf{u}_k)$  is chosen proportional to the symmetric gradient of the velocity vector  $\vec{\nabla}^s\mathbf{u}_k$ , whose entries are given by  $(\vec{\nabla}^s\mathbf{u})_{i,j} = \frac{1}{2}(\partial_{x_i}u_j + \partial_{x_j}u_i)$ . Such a choice ensures the associated dissipative terms to be rotationally invariant and also positivity of  $\mathcal{R}_1$ . An other option would be to simply set  $\mathbb{F}(\mathbf{u}_k)$  proportional to  $\vec{\nabla}\mathbf{u}_k$  which allows to recover the parabolic regularization.

After a few lines of algebra, the third term  $\mathcal{R}_3$  can be recast as a function of the gradient of the entropy as follows:

$$\mathcal{R}_2 = \rho_k A \mathbf{l}_k \cdot \vec{\nabla} s_k. \quad (23)$$

One of the assumptions made in the entropy minimum principle is to that the entropy is at a minimum which implies that its gradient is null. Because of this, it follows that the term  $\mathcal{R}_3$  is zero at the minimum and thus, the entropy minimum principle is verified independently of the definition of the dissipative term  $\mathbf{l}_k$  used in the volume fraction equation. It will be explained later in this section how to derive a definition for  $\mathbf{l}_k$ .

We now focus on the term denoted by  $\mathcal{R}_2$ , that is found identical to the right-hand-side of the single phase entropy equation obtained from the multi-D Euler equations (see Eq. (??) in APPENDIX). Thus, following [12] and also APPENDIX, the term  $\mathcal{R}_2$  is known to be positive when (i) assuming concavity of the entropy function  $s_k$  with respect to the internal energy  $e_k$  and the specific volume  $1/\rho_k$  (or convexity of  $-s_k$ ) and (ii) choosing the following definitions for the dissipative terms  $\tilde{\mathbf{h}}_k$  and  $\tilde{\mathbf{f}}_k$ :

$$\tilde{\mathbf{f}}_k = \alpha_k A \kappa_k \vec{\nabla} \rho_k \quad (24)$$

$$\tilde{\mathbf{h}}_k = \alpha_k A \kappa_k \vec{\nabla} (\rho e)_k, \quad (25)$$

where  $\kappa_k$  is another positive viscosity coefficient. The entropy equation can now be written in its final form:

$$\begin{aligned} \alpha_k \rho_k A \frac{Ds_k}{Dt} - \mathbf{f}_k \cdot \vec{\nabla} s_k - \vec{\nabla} \cdot (\alpha_k \rho_k A \vec{\nabla} s_k) = \\ - \alpha_k A \kappa_k \mathbf{Q}_k + (s_e)_k \alpha_k A \rho_k \mu_k \vec{\nabla}^s \mathbf{u}_k : \vec{\nabla} \mathbf{u}_k, \end{aligned} \quad (26)$$

where  $\mathbf{Q}_k$  is a negative semi-definite quadratic form defined as:

$$\begin{aligned} \mathbf{Q}_k &= X_k^t \Sigma_k X_k \\ \text{with } X_k &= \begin{bmatrix} \vec{\nabla} \rho_k \\ \vec{\nabla} e_k \end{bmatrix} \text{ and } \Sigma_k = \begin{bmatrix} \partial_{\rho_k} (\rho_k^2 \partial_{\rho_k} s_k) & \partial_{\rho_k, e_k} s_k \\ \partial_{\rho_k, e_k} s_k & \partial_{e_k, e_k} s_k \end{bmatrix}. \end{aligned}$$

Eq. (26) is used to prove the entropy minimum principle: assuming that  $s_k$  reaches its minimum value in  $\mathbf{r}_{min}(t)$  at each time  $t$ , the gradient,  $\vec{\nabla} s_k$ , and Laplacian,  $\Delta s_k$ , of the entropy are null and positive at this particular point,

172 respectively. Furthermore, it is recalled that the viscosity coefficients  $\mu_k$  and  
 173  $\kappa_k$  are positive by definition. Then, because the right-hand-side of Eq. (26) is  
 174 proven positive, the entropy minimum principle holds for each phase  $k$ , **inde-**  
 175 **pendently of the definition of the dissipative term  $\mathbf{l}_k$** , such as:

$$\alpha_k \rho_k A \partial_t s_k(\mathbf{r}_{min}, t) \geq 0 \Rightarrow \partial_t s_k(\mathbf{r}_{min}, t) \geq 0$$

176 It remains to obtain a definition for the dissipative term  $\mathbf{l}_k$  used in the  
 177 volume fraction equation. A way to achieve this is to consider the volume  
 178 fraction equation, Eq. (12a), by itself and notice that it is an hyperbolic equation  
 179 with eigenvalue  $\mathbf{u}_{int}$ . An entropy equation can be derived and used to prove  
 180 the entropy minimum principle by properly choosing the dissipative term. The  
 181 objective is to ensure positivity of the volume fraction and also uniqueness of  
 182 the weak solution. Following the work of Guermond et al. in [13, 14] and by  
 183 analogy with Burger's equation described in SECTION, it can be shown that a  
 184 dissipative term ensuring positivity and uniqueness of the weak solution for the  
 185 volume fraction equation, is of the form  $\mathbf{l}_k = \beta_k A \vec{\nabla} \alpha_k$  where  $\beta_k$  is a positive  
 186 viscosity coefficient.

All of the dissipative terms are now defined and recalled here:

$$\mathbf{l}_k = \beta_k A \vec{\nabla} \alpha_k \quad (27a)$$

$$\mathbf{f}_k = \alpha_k A \kappa_k \vec{\nabla} \rho_k + \rho_k A \mathbf{l}_k \quad (27b)$$

$$\mathbf{g}_k = \alpha_k A \mu_k \rho \vec{\nabla}^s \mathbf{u}_k \quad (27c)$$

$$\mathbf{h}_k = \alpha_k A \kappa_k \vec{\nabla} (\rho e)_k + \mathbf{u}_k : \mathbf{g}_k - \frac{\|\mathbf{u}_k\|^2}{2} \mathbf{f}_k + (\rho e)_k \mathbf{l}_k \quad (27d)$$

187 At this point, some remarks are in order:

- 188 1. The viscous regularization given in Eq. (27) for the multi-D seven-equation  
 189 model, is equivalent to the parabolic regularization [15] when assuming  
 190  $\beta_k = \kappa_k$  and  $\mathbb{F}(\mathbf{u}_k) = \alpha_k \rho_k \kappa_k \vec{\nabla} \mathbf{u}_k$ . However, decoupling between the reg-  
 191 ularization on the velocity and on the density in the momentum equation  
 192 is important to make the regularization rotation invariant but also to en-  
 193 sure well-scaled dissipative terms for a wide range of Mach number as was  
 194 shown in SECTION for the multi-D Euler equations.
- 195 2. The dissipative term  $\mathbf{l}_k$  requires the definition of a new viscosity coefficient  
 196  $\beta_k$ . It was shown that this viscosity coefficient is independent of the  
 197 other viscosity coefficients  $\mu_k$  and  $\kappa_k$ . Its definition should account for the  
 198 eigenvalue associated with the void fraction equation  $\mathbf{u}_{int}$ . In addition,  
 199 an entropy residual can be determined by analogy to Burger's equation.
- 200 3. The dissipative term  $\mathbf{f}_k$  is a function of  $\mathbf{l}_k$ . Thus, all of the other dissipa-  
 201 tive terms are also functions of  $\mathbf{l}_k$ .

- 202 4. The partial derivatives  $(s_e)_k$  and  $(s_{\rho_k})_k$  can be computed using the defi-  
 203 nition provided in Eq. (18) and are functions of the thermodynamic vari-  
 204 ables: pressure, temperature and density.
- 205 5. All of the dissipative terms are chosen to be proportional to the the void  
 206 fraction  $\alpha_k$  and the cross-sectional area  $A$ , but the one in the volume  
 207 fraction equation that is only proportional to  $A$ . For instance,  $\alpha_k A \vec{\nabla} \rho_k$   
 208 is the flux of the dissipative term in the continuity equation through the  
 209 area seen by the phase  $\alpha_k A$ . When one of the phases disappears, the  
 210 dissipative terms must to go to zero for consistency. On the other hand,  
 211 when  $\alpha_k$  goes to one, the single-phase equation must be recovered.
- 212 6. Compatibility of the viscous regularization proposed in Eq. (27) with the  
 213 generalized entropies identified in Harten et al. [16] has not been investi-  
 214 gated yet. However, it is believed that the entropy inequalities still holds  
 215 because of the similarities of the entropy residual for the multi-D seven-  
 216 equation model with the entropy residual derived in the single phase flow  
 217 case [12].

Through the derivations of the viscous regularization, it was noted that another set of dissipative terms  $\mathbf{f}_k$  and  $\mathbf{l}_k$  would also ensures positivity of the entropy residual:

$$\mathbf{l}_k = \beta_k T_k \left[ \frac{\rho_k}{P_k + \rho_k e_k} \vec{\nabla} \left( \frac{P_k}{\rho_k e_k} \right) - \frac{1}{P_k} \vec{\nabla} \rho_k \right] \quad (28a)$$

$$\mathbf{f}_k = \kappa_k \vec{\nabla} \rho_k + \frac{\rho_k^2 (s_\rho)_k}{(\rho s_\rho - e s_e)_k} \mathbf{l}_k \quad (28b)$$

218 However, the definition of  $\mathbf{l}_k$  proposed in Eq. (28a) was not considered as valid  
 219 for the following reasons: positivity of the volume fraction cannot be achieved  
 220 and the parabolic regularization is not retrieved.

221  
 222 A rotation invariant viscous regularization for the multi-D seven-equation  
 223 model is now available involving three viscosity coefficients  $\beta_k$ ,  $\mu_k$  and  $\kappa_k$ , for  
 224 each phase  $k$ . Definition of these viscosity coefficients is the purpose of the next  
 225 section (SECTION).

#### 226 4. A definition of the viscosity coefficients for all Mach flows

- 227 • non-dimensionalize the equations but use  $P_\infty$  for the pressure instead of  
 228  $(\rho c^2)_\infty$
- 229 • introduce a new Pechlet number for  $\beta$ : its behavior should be the same  
 230 as the Pechlet number for  $\kappa$
- 231 • two cases: zero and infinite relaxation coefficients
- 232 • derive the normalization parameters for the isentropic and non-isentropic  
 233 flows
- 234 • discussion about the

## 235 5. 1-D numerical results

- 236 • simple advection problem
- 237 • shock tube with two independent fluids: exact solution and could do con-
- 238     vergence test for this particular test
- 239 • shock tube with infinite relaxation coefficients
- 240 • 1-D nozzle with two independent fluids
- 241 • 1-D nozzle with infinite relaxation coefficients
- 242 • 1-D nozzle with infinite relaxation coefficients, mass and heat transfer

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