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Extension of the Entropy Viscosity Method to Flows with Friction and Source Terms

Marco Delchini, Jean C. Ragusa
Department of Nuclear Engineering
Texas A&M University, College Station, USA
marco.delchini@gmail.com, jean.ragusa@tamu.edu
Ray Berry
Idaho National Laboratory
Idaho Falls, ID, USA
ray.berry@inl.gov

ABSTRACT

This paper aims at extending the entropy viscosity method [2–4] when solving the 1-D Euler equations [8] with source terms. The entropy viscosity method has been successfully applied to hyperbolic systems of equations such as Burgers equation and Euler equations. The method consists in adding dissipative terms to the governing equations, where a viscosity coefficient modulates the amount of dissipation. This viscosity coefficient is based on the entropy production that occurs in discontinuities and shocks present in hyperbolic systems.
By adding source terms to Euler equations, the entropy viscosity method needs to be modified in order to account for the entropy production due to the source terms. Tests are run for Pressurized Nuclear Reactor type under the Moose framework [1] using Continuous Galerkin Finite Element Method and a second-order temporal implicit scheme, and allow to validate our approach.

Key Words: **Entropy-based viscosity scheme, Euler equations**

1. Introduction

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2. Entropy viscosity method: a review for single-phase Euler equations

In this section, the entropy-based viscosity method [2–4] is recalled for the multi-D Euler equations [8]. No source or friction terms are added at this stage. The entropy viscosity method consists of adding appropriate dissipative terms, with a viscosity coefficient modulated by the entropy production. The scheme detects shocks (entropy production is large in shocks) and mitigates the unphysical oscillatory behavior usually associated with high-order schemes for conservation equations. The derivation of the artificial dissipative terms to be added to Euler equations can be found in [5] (they

are recalled below, in Eq. (3)). Formally, solutions to Euler equations should satisfy the entropy minimum principle, i.e.,

$$\partial_t s + u \cdot \nabla s \geq 0 \tag{1}$$

where s is an entropy function. The viscous regularization of [5] reproduces this property such that

$$\rho\big(\partial_t s + u \cdot \nabla s\big) + \text{conservative terms} \geq 0. \tag{2}$$

The viscous regularization yields conservative terms of the form $-\nabla \cdot K \nabla s$ and a right-hand-side contribution that is positive. The Euler equation with viscous regularization are:

$$\begin{cases} \partial_t(\rho) + \nabla \cdot (\rho \vec{u}) = \nabla \cdot (\kappa \nabla \rho) \\ \partial_t(\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u} + PI) = \nabla \left(\mu \rho \nabla \vec{u} + \kappa \vec{u} \otimes \nabla \rho\right) \\ \partial_t(\rho E) + \nabla \cdot [\vec{u}(\rho E + P)] = \nabla \cdot \left(\kappa \nabla(\rho e) + \frac{1}{2}||\vec{u}||^2 \kappa \nabla \rho + \rho \vec{u} \mu \nabla \vec{u}\right) \\ P = P(\rho, e) \end{cases} \tag{3}$$

where κ and μ are local positive viscosity coefficients.
The existence of a specific entropy s, function of the density ρ and the internal energy e is assumed. Convexity of $-s$ with respect to e and $1/\rho$ is required, along with the following equality verified by the partial derivatives of s : $P\partial_e s + \rho^2 \partial_\rho s = 0$.

The definition for the local viscosity coefficients μ and κ is given next. In the current version of the method, κ and μ are set equal, so that the above viscous regularization (Eq. (3)) is equivalent to the parabolic regularization [6]. The current definition includes a first-order viscosity coefficient referred to with the subscript max, and a high-order viscosity coefficient referred to with the subscript e. The first-order viscosity coefficients μ_{max} and κ_{max} are proportional to the largest local eigenvalue $||\vec{u}|| + c$ and equivalent to a first-order upwind-scheme. Such scheme are known to to be over-dissipative and monotone [8]:

$$\mu_{max}(\vec{r}, t) = \kappa_{max}(\vec{r}, t) = \frac{h}{2}\left(||\vec{u}|| + c\right), \tag{4}$$

where h is the grid size.

The higher-order viscosity coefficients κ_e and μ_e are set proportional to the entropy production that is evaluated by computing the local entropy residual D_e. It also includes the jump J of the entropy flux (this enables the detections of discontinuities others than shocks):

$$\mu_e(\vec{r}, t) = \kappa_e(\vec{r}, t) = h^2 \frac{\max\left(|D_e(\vec{r}, t)|, J\right)}{||s - \bar{s}||_{max}} \text{ with } D_e(\vec{r}, t) = \partial_t s + \vec{u} \cdot \nabla s \tag{5}$$

where $||\cdot||_{max}$ and $\bar{\cdot}$ denote the infinite norm operator and the average operator over the entire computational domain, respectively. The definition of the jump J is discretization-dependent and examples of definition can be found in [2] for DGFEM. The denominator $||s - \bar{s}||_{max}$ is used for dimensionality purposes. The definition of the viscosity coefficients μ and κ is function of the first- and higher-order viscosity coefficients as follows:

$$\mu(\vec{r}, t) = \min\left(\mu_e(\vec{r}, t), \mu_{max}(\vec{r}, t)\right) \text{ and } \kappa(\vec{r}, t) = \min\left(\kappa_e(\vec{r}, t), \kappa_{max}(\vec{r}, t)\right); \tag{6}$$

This definition possesses the following properties. In shock regions, the higher-order viscosity coefficient exhibits large values because of the entropy production, and thus, saturates to the over-dissipative first-order viscosity and will smooth out oscillations. Everywhere else, the entropy

production being negligible, the viscosity coefficients μ and κ are of order h^2 and small.
Using the above definition of the entropy-based viscosity method, high-order accuracy was demonstrated and good results were obtained with 1-D Sod shock tubes and various 2-D tests [2–4].

In this paper, we modified the expression for the entropy residual (usign chain rule) so as to make it function of pressure and density Eq. (7).

$$D_e(\vec{r},t) = \partial_t s + \vec{u}\cdot\nabla s = \frac{s_e}{P_e}\underbrace{\left(\frac{dP}{dt} - c^2\frac{d\rho}{dt}\right)}_{\tilde{D}_e(\vec{r},t)}, \tag{7}$$

where $\frac{d\cdot}{dt}$ denotes the material or total derivative, and P_e is the partial derivative of the pressure P with respect to the internal energy e. This is motivated by the following observation: the current definition of the viscosity coefficients requires an analytical expression of the entropy function s which can be difficult to obtain when dealing with complex equations of states, and, does not seem to be adapted to low Mach flows that are known to be isentropic: the entropy residual $D_e(\vec{r},t)$ and the denominator $||s-\bar{s}||_{max}$ will both tend to zero leading to an undetermined form. Since $D_e(\vec{r},t)$ and $\tilde{D}_e(\vec{r},t)$ are proportional to each other, the definition of the viscosity coefficients μ and κ can rely on $\tilde{D}_e(\vec{r},t)$ without affecting the heart of the entropy viscosity method as follows:

$$\mu_e(\vec{r},t) = \kappa_e(\vec{r},t) = h^2\frac{\max\left(|\tilde{D}_e(\vec{r},t)|, J\right)}{(1-M)\rho c^2 + M\rho|\vec{u}|^2} \tag{8}$$

The denominator is now changed as shown in Eq. (8) and is of the same dimension as the pressure. It is function of the Mach number M, the speed of sound c, the density ρ and the norm of the velocity vector $|\vec{u}|^2$, and ensure consistency when dealing with low Mach flows. The jump J, is chosen to be proportional to the jump of the pressure and density gradients at the interfaces:

$$J_{i+1/2} = |\vec{u}|_{i+1/2}\max\left([[\nabla P\cdot\vec{n}]]_{i+1/2}, c^2[[\nabla\rho\cdot\vec{n}]]_{i+1/2}\right) \text{ with } [[\cdot]] = |(\nabla\cdot)_i - (\nabla\cdot)_{i+1}|\cdot\vec{n}, \tag{9}$$

where $i+1/2$ denotes the interface between cells i and $i+1$, and $\vec{n}$ its outward normal. The definition of the viscosity coefficients μ and κ remain unchanged, as well as the dissipative terms.

3. Extension of the entropy viscosity method to include friction, gravity, and sink/source terms

Next, we show that the entropy viscosity method can be modified to retain it validity when adding source terms to the 1-D Euler equations. Gravity and the wall friction forces are added in the momentum equation and a wall-heat source is supplied in the energy equation, as shown in Eq. (10). The artificial dissipative terms were omitted.

$$\begin{cases} \partial_t\rho + \nabla_x\cdot(\rho u) = 0 \\ \partial_t(\rho u) + \nabla_x\cdot(\rho u^2 + P) = -\frac{A}{2D_h}\rho f|u|u + \rho g \\ \partial_t(\rho E) + \nabla_x\cdot[u(\rho E + P)] = a_w h_w(T - T_w) \end{cases} \tag{10}$$

f is a positive friction factor, g is the gravity constant, and D_h is the hydraulic diameter given by $D_h = 4A/p$ (A and p being the geometry cross-section and wetted perimeter, respectively). The

variables a_w, T_w and h_w denote the heated surface, the wall temperature, and the wall heat transfer, respectively. Wall friction does not affect the total energy equation because the velocity at the wall is zero and thus no power is induced by the wall friction forces.

Because the entropy residual method relies on the sign of the entropy residual, we need to understand how source terms may affect the entropy residual. In order to achieve this, one can repeat the derivation of the entropy residual, with source terms included. Following the same steps detailed in [5], the following entropy residual is obtained:

$$\rho\frac{ds}{dt} = \rho\frac{s_e}{P_e}\left(\frac{dP}{dt} - c^2\frac{d\rho}{dt}\right) = s_e\left(a_w h_w(T - T_w) + \frac{\rho}{2D_h}f|u|u^2\right). \tag{11}$$

In order to prove the entropy minimum principle, the sign of the right-hand side of Eq. (11) needs to be studied. Since s_e is positive by definition [5], the sign will depend upon the terms inside the brackets. We observe that the friction term is always positive and, therefore, will not affect the sign of the entropy residual. However, the sign of the wall heat source term can be either positive or negative and thus needs to be included in the definition of the entropy viscosity coefficients μ_e and κ_e in order to account for the entropy production due to heating/cooling:

$$\mu_e(\vec{r},t) = \kappa_e(\vec{r},t) = h^2\frac{\max\left(|\tilde{D}_e(\vec{r},t)|, |D_w(\vec{r},t)|, J\right)}{(1-M)\rho c^2 + M\rho|\vec{u}|^2} \text{ with } D_w(\vec{r},t) = a_w h_w(T - T_w) \tag{12}$$

The definitions of the first order viscosity coefficients given in Eq. (4) remain unchanged, as well as the ones for the viscosity coefficients μ and κ (Eq. (6)).

4. Numerical results for a Pressurized Nuclear Reactor PWR

Numerical tests are performed for a 1-D pipe of cross-section $A = 7.854E-05\ m^2$ and length $L = 3.865\ m$ with the following parameters: the wall temperature T_w is set to a constant value of 600 K, the heat transfer coefficient h_w is time and space dependent, given in Eq. (13), the heated surface a_w is computed from A and L and set to 0.0298 m, and the friction factor is constant and equal to 0.01.

$$h_w(x,t) = h_{w,0}(1 - e^{-t})\sin\left(\frac{\pi}{2L}x\right) \text{ with } h_{w,0} = 5.33 \cdot 10^4 W/m. \tag{13}$$

For boundary conditions, we impose, at the inlet, a mass inflow ($\rho u = 3600\ kg/(m^2 s)$) and a fluid temperature ($T = 559.15$ K) and employ a static pressure condition at the exit ($P_s = 155$ bar) . The Stiffened Gas Equation of State (SGEOS) is used [7] with following parameters: $P_\infty = 8.5 \cdot 10^8$ Pa, $q = -1151e3\ J/m^3$, $\gamma = 2.04$ and $C_v = 2069\ J/(K \cdot kg)$. The steady-state is reached at around $t = 30$ s with a time step of $\Delta t = 1$ sec. Figure 1 through Figure 4 present the results obtained using 20 mesh cells and using either the over-dissipative first-order viscosity or the higher-order entropy viscosity to stabilize the numerical scheme. From these figures, it is clear that the first-order viscosity yields erroneous answers, while the entropy-viscosity results are correct. Results with a finer spatial resolution are also shown.

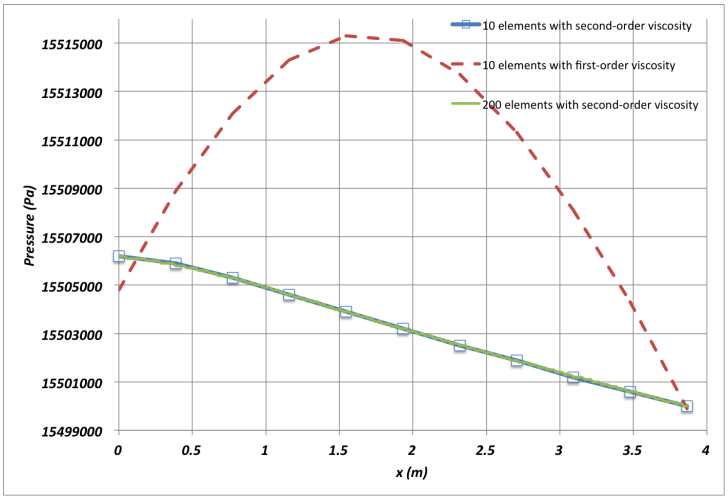

Figure 1: PWR test case: pressure axial profile

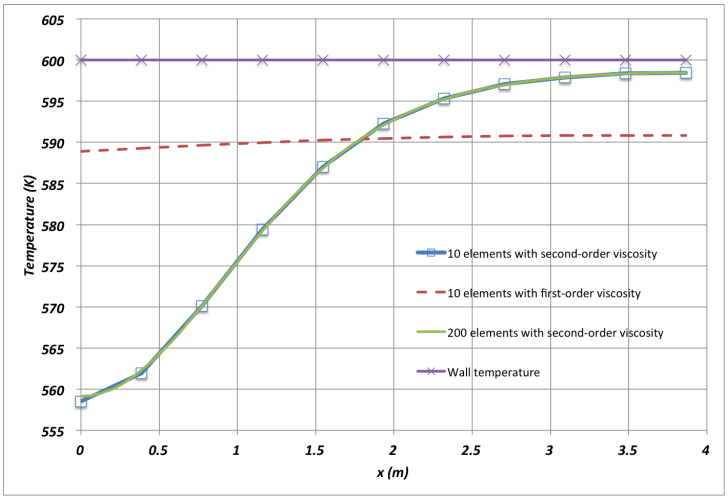

Figure 2: PWR test case: Temperature axial profile

This paragraph aims at showing why the first-order viscosity becomes ill-scaled in the low Mach limit by considering the continuity equation with its artificial dissipative term. The same reasoning can be applied to the momentum and energy equations. The first step consists of defining dimensionless variables (with a $\tilde{\cdot}$) as follows:

$$\rho = \tilde{\rho}\rho^*,\ u = \tilde{u}u^*,\ x = \tilde{x}L,\ t = \tilde{t}\frac{u^*}{L} \text{ and } \kappa = \tilde{\kappa}\kappa^* \tag{14}$$

Using the above definition, the continuity equation yields:

$$\partial_{\tilde{t}}\tilde{\rho} + \tilde{\nabla}\cdot\left(\tilde{\rho}\vec{\tilde{u}}\right) = \frac{k^*}{L\vec{u^*}}\tilde{\nabla}\cdot\left(\tilde{\kappa}\tilde{\nabla}\tilde{\rho}\right) \tag{15}$$

It remains to define k^* that depends upon using the first- or second-order viscosity coefficient: when using the first-order viscosity (Eq. (4)), an expression for κ^* is: $\kappa^* = \frac{L}{2}\left(\vec{u^*} + c^*\right)$. By substituting the definition of κ^* into Eq. (15), the following expression is obtained:

$$\partial_{\tilde{t}}\tilde{\rho} + \tilde{\nabla}\cdot\left(\tilde{\rho}\vec{\tilde{u}}\right) = \left(1 + \frac{1}{M^*}\right)\tilde{\nabla}\cdot\left(\tilde{\kappa}\tilde{\nabla}\tilde{\rho}\right) \tag{16}$$

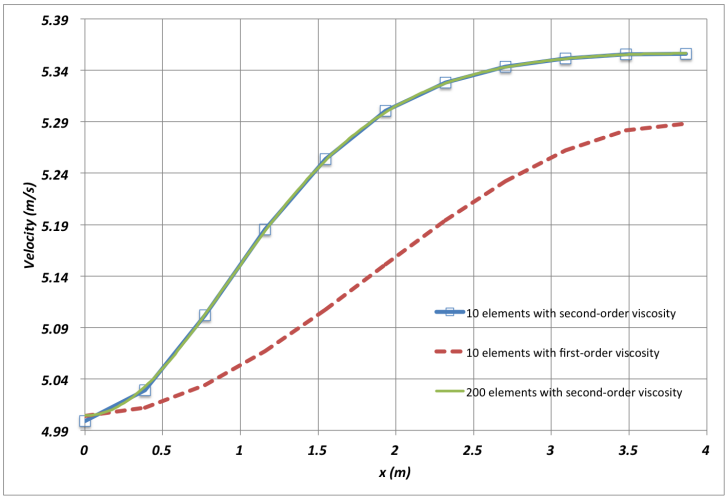

Figure 3: PWR test case: Velocity axial profile

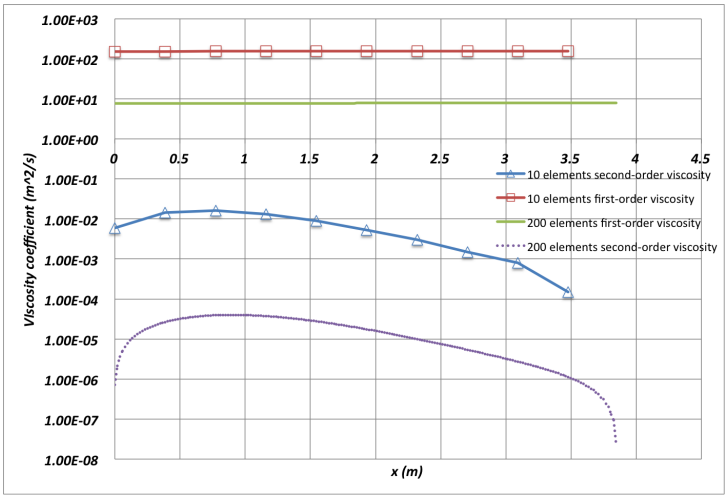

Figure 4: PWR test case: Viscosity axial profile

where M^* is a reference Mach number. Thus, in the low Mach limit, the dissipative term will become ill-scaled and alter the physical solution. On the other hand, when using the entropy-viscosity coefficient Eq. (5) in the low Mach limit with $\tilde{P} = \rho^*(c^2)^* P$, the dissipative term is well-scaled:

$$\partial_{\tilde{t}}\tilde{\rho} + \tilde{\nabla}\cdot\left(\tilde{\rho}\vec{\tilde{u}}\right) = \tilde{\nabla}\cdot\left(\tilde{\kappa}\tilde{\nabla}\tilde{\rho}\right) \tag{17}$$

5. Conclusions

show how the entropy residual is modified when accounting for source terms. Also modified the definition of the entropy viscosity coefficients accordingly. show that artificial dissipative terms become ill-scaled when using the first-order viscosity coefficient. Does not happen with the entropy viscosity coefficient. results are provided for PWR type and show good agreement with the prediction in the low Mach limit. The artificial dissipative terms do not alter the physical solution

when the second-order viscosity coefficient is used. However, as predicted, the solution with the first-order viscosity is no correct. the steady-state solution is correctly resolved even with few cells.

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