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Extension of the Entropy Viscosity Method to Flows with Friction Forces and Source Terms

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ABSTRACT

In this paper, we extend the entropy viscosity method [3–5] to 1-D Euler equations with source terms present. The entropy viscosity method has been successfully applied to hyperbolic systems of equations such as Burgers equation and Euler equations. This method consists in adding dissipative terms to the governing equations so as to ensure the entropy minimum principle. The dissipative terms contain a viscosity coefficient that modulates locally the amount of dissipation. This viscosity coefficient is based on the entropy production that occurs in discontinuities and shocks present in hyperbolic systems. By adding source terms to Euler equations (friction and gravity forces to the momentum equation and heat source/sink int he energy equation), the entropy viscosity method needs to be modified in order to account for the entropy production due to these terms. Tests are run for a 1D Pressurized Water Reactor channel using the MOOSE framework [1]. The equation are discretized with a standard continuous Finite Element Method (linear polynomials) and a second-order temporal implicit scheme (BDF2).

Key Words: **Entropy-based viscosity scheme, Euler equations with source terms, viscous regularization, Low Mach flow**

1. Introduction

Hyperbolic systems of equations are challenging to solve numerically, due to the presence of shocks. Godunov upwind schemes are overly dissipative and only first-order accurate, while higher order methods need flux limiters to prevent unphysical oscillations. Recently, a new technique to stabilize hyperbolic equations has been introduced [3–5] . It requires adding artificial dissipation terms to the equations while ensuring that the entropy minimum principle remains satisfied. This technique, coined entropy viscosity method, is independent of the spatial discretization and thus can be used with a standard Galerkin Finite Element Method (FEM). Many FEM software possess advanced functionalities for continuous FEM (e.g., deal.ii, libmesh). We have developed our fluid solver with Idaho National Laboratory's MOOSE framework [1], with the goal of applying the entropy viscosity method to fluid flow problems for nuclear reactor applications. Therefore, it is crucial to stress the capabilities of the method beyond Euler equations. Here, we analyze the implication of

adding wall friction, gravity, and heat source/sink to the conservation equations. A 1D channel, with conditions representative of the ones of a PWR, is used in our numerical tests. This test case also presents the challenge of being in the low-Mach limit, an asymptotic region not yet tested with the entropy viscosity method. In Section 2, we recall the fundamentals of the entropy viscosity method. Its extension to Euler equations with friction forces, gravity forces, and heat sources/sinks is presented in Section 3. Numerical results are provided in Section 4 with the entropy viscosity method and also with the SUPG stabilization method [2] for comparison.

2. Entropy viscosity method: a review for single-phase Euler equations

In this section, the entropy-based viscosity method [3–5] is recalled for the multi-D Euler equations [10]. No source or friction terms are added at this stage. The entropy viscosity method consists of adding appropriate dissipative terms, with a viscosity coefficient modulated by the entropy production. The scheme detects shocks (entropy production is large in shocks) and mitigates the unphysical oscillatory behavior usually associated with high-order schemes for conservation equations. The derivation of the artificial dissipative terms to be added to Euler equations can be found in [6] (they are recalled below, in Eq. (3)). Formally, solutions to Euler equations should satisfy the entropy minimum principle, i.e.,

$$\partial_t s + u \cdot \nabla s \geq 0 \tag{1}$$

where s is an entropy function. The viscous regularization of [6] reproduces this property such that

$$\rho\big(\partial_t s + u \cdot \nabla s\big) + \text{conservative terms} \geq 0. \tag{2}$$

The viscous regularization yields conservative terms of the form $-\nabla \cdot K \nabla s$ and a right-hand-side contribution that is positive. The Euler equation with viscous regularization are:

$$\begin{cases} \partial_t(\rho) + \nabla \cdot (\rho \vec{u}) = \nabla \cdot (\kappa \nabla \rho) \\ \partial_t(\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u} + PI) = \nabla \left(\mu \rho \nabla \vec{u} + \kappa \vec{u} \otimes \nabla \rho\right) \\ \partial_t(\rho E) + \nabla \cdot [\vec{u}(\rho E + P)] = \nabla \cdot \left(\kappa \nabla(\rho e) + \frac{1}{2}||\vec{u}||^2 \kappa \nabla \rho + \rho \vec{u} \mu \nabla \vec{u}\right) \\ P = P(\rho, e) \end{cases} \tag{3}$$

where κ and μ are local positive viscosity coefficients.
The existence of a specific entropy s, function of the density ρ and the internal energy e is assumed. Convexity of $-s$ with respect to e and $1/\rho$ is required, along with the following equality verified by the partial derivatives of s : $P\partial_e s + \rho^2 \partial_\rho s = 0$.

The definition for the local viscosity coefficients μ and κ is given next. In the current version of the method, κ and μ are set equal, so that the above viscous regularization (Eq. (3)) is equivalent to the parabolic regularization [7]. The current definition includes a first-order viscosity coefficient referred to with the subscript max, and a high-order viscosity coefficient referred to with the subscript e. The first-order viscosity coefficients μ_{max} and κ_{max} are proportional to the largest local eigenvalue $||\vec{u}|| + c$ and equivalent to a first-order upwind-scheme. Such scheme are known to to be over-dissipative and monotone [10]:

$$\mu_{max}(\vec{r}, t) = \kappa_{max}(\vec{r}, t) = \frac{h}{2}\left(||\vec{u}|| + c\right), \tag{4}$$

where h is the grid size.

The higher-order viscosity coefficients κ_e and μ_e are set proportional to the entropy production that is evaluated by computing the local entropy residual D_e. It also includes the jump J of the entropy flux (this enables the detections of discontinuities others than shocks):

$$\mu_e(\vec{r},t) = \kappa_e(\vec{r},t) = h^2 \frac{\max\left(|D_e(\vec{r},t)|, J\right)}{||s-\bar{s}||_{max}} \text{ with } D_e(\vec{r},t) = \partial_t s + \vec{u}\cdot\nabla s \tag{5}$$

where $||\cdot||_{max}$ and $\bar{\cdot}$ denote the infinite norm operator and the average operator over the entire computational domain, respectively. The definition of the jump J is discretization-dependent and examples of definition can be found in [3] for DGFEM. The denominator $||s-\bar{s}||_{max}$ is used for dimensionality purposes. The definition of the viscosity coefficients μ and κ is function of the first- and higher-order viscosity coefficients as follows:

$$\mu(\vec{r},t) = \min\left(\mu_e(\vec{r},t), \mu_{max}(\vec{r},t)\right) \text{ and } \kappa(\vec{r},t) = \min\left(\kappa_e(\vec{r},t), \kappa_{max}(\vec{r},t)\right); \tag{6}$$

This definition possesses the following properties. In shock regions, the higher-order viscosity coefficient exhibits large values because of the entropy production, and thus, saturates to the over-dissipative first-order viscosity and will smooth out oscillations. Everywhere else, the entropy production being negligible, the viscosity coefficients μ and κ are of order h^2 and small.
Using the above definition of the entropy-based viscosity method, high-order accuracy was demonstrated and good results were obtained with 1-D Sod shock tubes and various 2-D tests [3–5].

In this paper, we modified the expression for the entropy residual (using chain rule) so as to make it function of pressure and density Eq. (7).

$$D_e(\vec{r},t) = \partial_t s + \vec{u}\cdot\nabla s = \frac{s_e}{P_e}\underbrace{\left(\frac{dP}{dt} - c^2\frac{d\rho}{dt}\right)}_{\tilde{D}_e(\vec{r},t)}, \tag{7}$$

where $\frac{d\cdot}{dt}$ denotes the material or total derivative, and P_e is the partial derivative of the pressure P with respect to the internal energy e. This is motivated by the following observation: the current definition of the viscosity coefficients requires an analytical expression of the entropy function s which can be difficult to obtain when dealing with complex equations of states, and, does not seem to be adapted to low Mach flows that are known to be isentropic: the entropy residual $D_e(\vec{r},t)$ and the denominator $||s-\bar{s}||_{max}$ will both tend to zero leading to an undetermined form. Since $D_e(\vec{r},t)$ and $\tilde{D}_e(\vec{r},t)$ are proportional to each other, the definition of the viscosity coefficients μ and κ can rely on $\tilde{D}_e(\vec{r},t)$ without affecting the heart of the entropy viscosity method as follows:

$$\mu_e(\vec{r},t) = \kappa_e(\vec{r},t) = h^2 \frac{\max\left(|\tilde{D}_e(\vec{r},t)|, J\right)}{(1-M)\rho c^2 + M\rho|\vec{u}|^2} \tag{8}$$

The denominator is now changed as shown in Eq. (8) and is of the same dimension as the pressure. It is function of the Mach number M, the speed of sound c, the density ρ and the norm of the velocity vector $|\vec{u}|^2$, and ensure consistency when dealing with low Mach flows. The jump J, is chosen to be proportional to the jump of the pressure and density gradients at the interfaces:

$$J_{i+1/2} = |\vec{u}|_{i+1/2} \max\left([[\nabla P\cdot\vec{n}]]_{i+1/2}, c^2[[\nabla\rho\cdot\vec{n}]]_{i+1/2}\right) \text{ with } [[\cdot]] = |(\nabla\cdot)_i - (\nabla\cdot)_{i+1}|\cdot\vec{n}, \tag{9}$$

where $i+1/2$ denotes the interface between cells i and $i+1$, and $\vec{n}$ its outward normal. The definition of the viscosity coefficients μ and κ remain unchanged, as well as the dissipative terms.

3. Extension of the entropy viscosity method to include friction, gravity, and sink/source terms

Next, we show that the entropy viscosity method can be modified to retain it validity when adding source terms to the 1-D Euler equations. Gravity and the wall friction forces are added in the momentum equation and a wall-heat source is supplied in the energy equation, as shown in Eq. (10). The artificial dissipative terms were omitted.

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u^2 + P) = -\frac{A}{2D_h} \rho f |u| u - \rho g \\ \partial_t (\rho E) + \nabla_x \cdot [u (\rho E + P)] = a_w h_w (T - T_w) - \rho g u \end{cases} \tag{10}$$

f is a positive friction factor, g is the gravity constant, and D_h is the hydraulic diameter given by $D_h = 4A/p$ (A and p being the geometry cross-section and wetted perimeter, respectively). The variables a_w, T_w and h_w denote the heated surface, the wall temperature, and the wall heat transfer, respectively. Wall friction does not affect the total energy equation because the velocity at the wall is zero and thus no power is induced by the wall friction forces.

Because the entropy residual method relies on the sign of the entropy residual, we need to understand how source terms may affect the entropy residual. In order to achieve this, one can repeat the derivation of the entropy residual, with source terms included. Following the same steps detailed in [6], the following entropy residual is obtained (the gravity dependent terms cancel out):

$$\rho \frac{ds}{dt} = \rho \frac{s_e}{P_e} \left(\frac{dP}{dt} - c^2 \frac{d\rho}{dt} \right) = s_e \left(a_w h_w (T - T_w) + \frac{\rho}{2D_h} f |u| u^2 \right). \tag{11}$$

In order to prove the entropy minimum principle, the sign of the right-hand side of Eq. (11) needs to be studied. Since s_e is positive by definition [6], the sign will depend upon the terms inside the brackets. We observe that the friction term is always positive and, therefore, will not affect the sign of the entropy residual. However, the sign of the wall heat source term can be either positive or negative and thus needs to be included in the definition of the entropy viscosity coefficients μ_e and κ_e in order to account for the entropy production due to heating/cooling:

$$\mu_e(\vec{r},t) = \kappa_e(\vec{r},t) = h^2 \frac{\max\left(|\tilde{D}_e(\vec{r},t)|, |D_w(\vec{r},t)|, J \right)}{(1-M)\rho c^2 + M\rho|\vec{u}|^2} \text{ with } D_w(\vec{r},t) = a_w h_w (T - T_w) \tag{12}$$

The definitions of the first order viscosity coefficients given in Eq. (4) remain unchanged, as well as the ones for the viscosity coefficients μ and κ (Eq. (6)).

4. Numerical results for a Pressurized Nuclear Reactor PWR

Numerical tests are performed for a 1-D pipe of cross-section $A = 7.854E-05\ m^2$ and length $L = 3.865\ m$ with the following parameters: the heat transfer coefficient h_w is set to a constant

$5.33e4\ W/m$, the heated surface a_w is computed from A and L and set to $0.0298\ m$, and the friction factor is constant and equal to 0.01. The wall temperature T_w is spatial dependent and computed using the model available in RELAP7 for PWR [8]. For boundary conditions, we impose, at the inlet, a mass inflow ($\rho u = 3982\ kg/(m^2 s)$) and a fluid temperature ($T = 559.15$ K) and employ a static pressure condition at the exit ($P_s = 155$ bar) . The Stiffened Gas Equation of State (SGEOS) is used [9] with following parameters: $P_\infty = 8.5 \cdot 10^8$ Pa, $q = -1151E + 03\ J/m^3$, $\gamma = 2.04$ and $C_v = 2069\ J/(K \cdot kg)$. The steady-state is reached at around $t = 150$ sec with a time step of $\Delta t = 0.5$ sec. Figure 1 through Figure 4 present the results obtained using 20 mesh cells and using either the over-dissipative first-order viscosity or the higher-order entropy viscosity or SUPG method to stabilize the numerical scheme. From these figures, it is clear that employing the first-order viscosity leads to erroneous answers, while the entropy-viscosity results are correct and look similar to what is obtained with SUPG method. Plots with a finer spatial resolution (100 cells) are also shown.

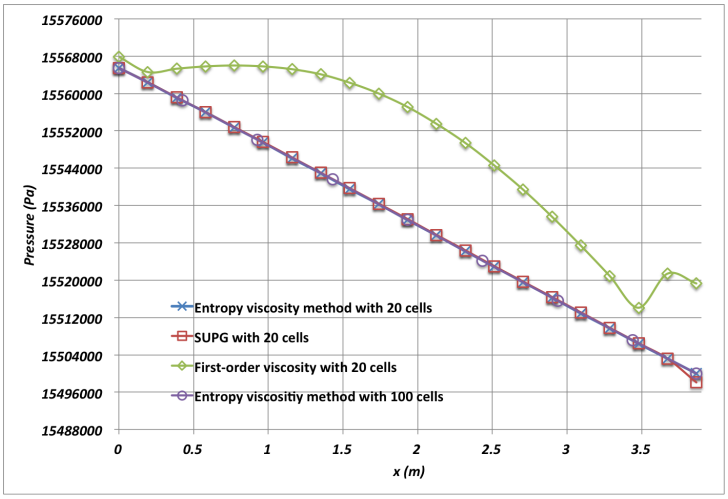

Figure 1: PWR test case: pressure axial profile

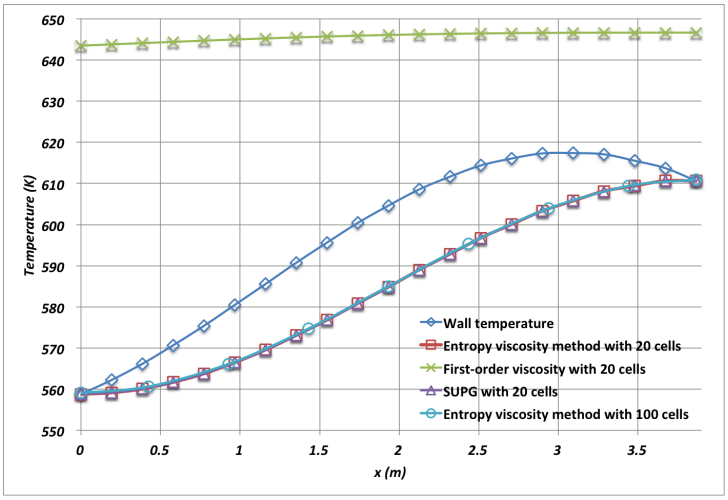

Figure 2: PWR test case: temperature axial profile

In Fig. 1, the steady-state pressure profile obtained with SUPG method shows a small non-physical change of slope at the outlet that disappear when a finer mesh is used. This artifact is not seen when

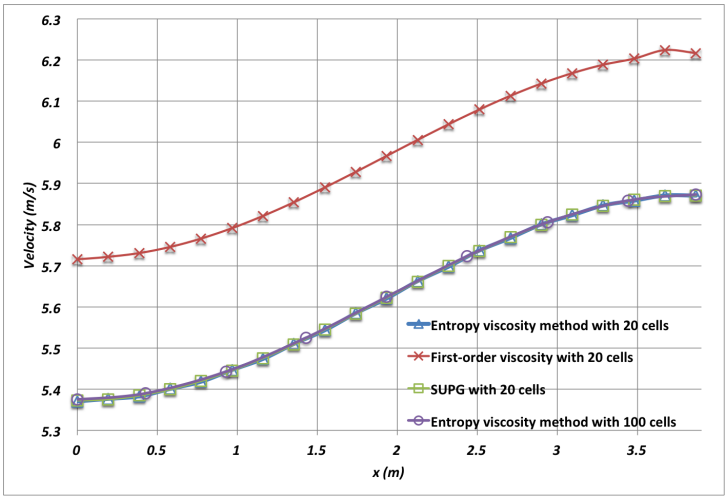

Figure 3: PWR test case: velocity axial profile

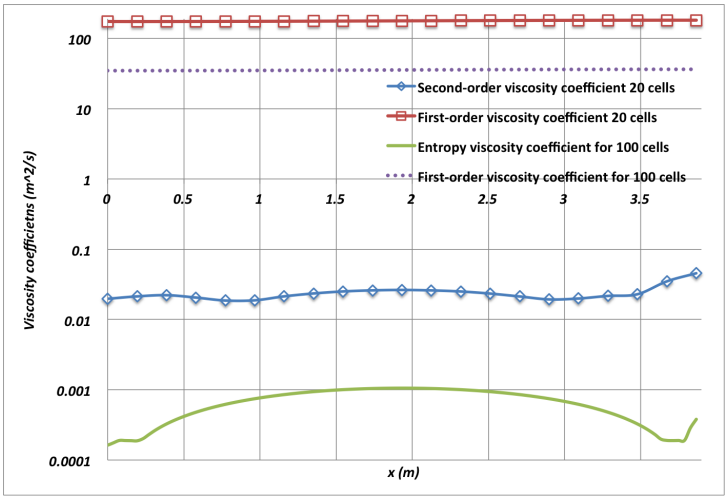

Figure 4: PWR test case: viscosity axial profile

using the entropy viscosity method.

This paragraph aims at explaining why the first-order viscosity becomes ill-scaled. This is due to the low-Mach nature of the flow under consideration (flow speed around 5 m/s while the speed of sound is around 1,500 m/s). We carry out a low Mach limit study for the continuity equation written with its artificial dissipative term. The same reasoning can be applied to the momentum and energy equations as well. The first step consists of defining dimensionless variables (with a $\tilde{\cdot}$) as follows:

$$\rho = \tilde{\rho}\rho^*, u = \tilde{u}u^*, x = \tilde{x}L, t = \tilde{t}\frac{u^*}{L} \text{ and } \kappa = \tilde{\kappa}\kappa^* \tag{13}$$

Using the above definition, the continuity equation yields:

$$\partial_{\tilde{t}}\tilde{\rho} + \tilde{\nabla}\cdot\left(\tilde{\rho}\vec{\tilde{u}}\right) = \frac{k^*}{Lu^*}\tilde{\nabla}\cdot\left(\tilde{\kappa}\tilde{\nabla}\tilde{\rho}\right) \tag{14}$$

The coefficient k^* depends upon using the first- or entropy-order viscosity coefficient: when using the first-order viscosity (Eq. (4)), an expression for κ^* is: $\kappa^* = \frac{L}{2}\left(\vec{u^*} + c^*\right)$. By substituting this

definition into Eq. (14), the following expression for the scaled continuity equation is obtained:

$$\partial_{\tilde{t}}\tilde{\rho} + \tilde{\nabla}\cdot\left(\tilde{\rho}\vec{\tilde{u}}\right) = \frac{1}{2}\left(1+\frac{1}{M^*}\right)\tilde{\nabla}\cdot\left(\tilde{\kappa}\tilde{\nabla}\tilde{\rho}\right) \tag{15}$$

where M^* is a reference Mach number. Thus, in the low Mach limit, the dissipative term will become ill-scaled and will alter the solution greatly when using the first-order viscosity. However, when employing the entropy-viscosity coefficient Eq. (5) in the low Mach limit with $\tilde{P} = \rho^*(c^2)^* P$, the dissipative term is well-scaled:

$$\partial_{\tilde{t}}\tilde{\rho} + \tilde{\nabla}\cdot\left(\tilde{\rho}\vec{\tilde{u}}\right) = \tilde{\nabla}\cdot\left(\tilde{\kappa}\tilde{\nabla}\tilde{\rho}\right). \tag{16}$$

We can see that it is therefore critical to evaluate and, if needed adapt, the definition of the viscosity coefficients employed with the dissipative terms in a wide range of flow speeds.
A good way to assess the impact of the dissipative terms onto the steady-state solution is to plot the momentum variable that is expected to be constant in the low Mach limit without mass source and under the condition of having well-scaled dissipative terms (Eq. (15) and Eq. (16)).

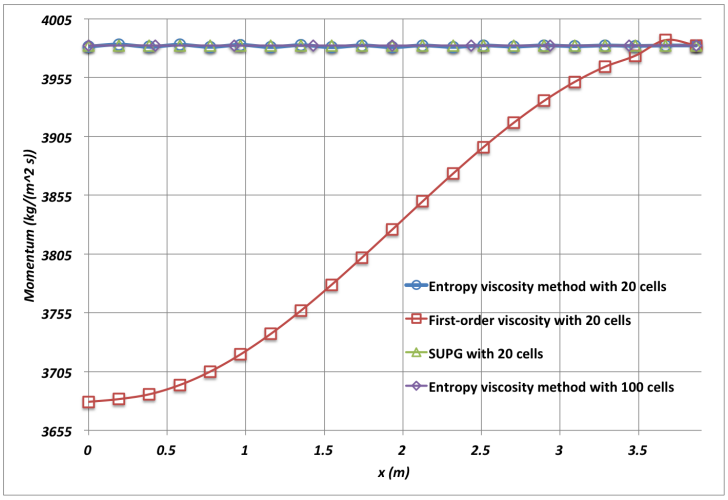

Figure 5: PWR test case: momentum axial profile

From Fig. 5, it is clear that the momentum variable remains constant ($3982\ m/s$) through the domain at steady-state when using either the entropy viscosity method or SUPG. When run with the first-order viscosity, the momentum variable displays a 7% variation over the domain because of the $\frac{1}{M^*}$ coefficient in the dissipative term as shown in Eq. (15).

5. Conclusions

We have extended the entropy viscosity method to fluid conservation equations (Euler equations) that contains terms pertaining to wall friction, gravity, and heat source/sink. This was achieved by verifying that the entropy minimum principle is still verified, provided that the definition of the entropy residual be amended to include the contribution of the heat source/sink terms.

We have applied the newly adapted technique to a 1D pipe problem representative of a PWR channel under normal operating conditions. The artificial dissipative terms do not alter the physical solution when the higher-order viscosity coefficient is used. However, the solution with the first-order viscosity is no correct. The first-order viscosity definition resulted in that term being ill-scaled in the low-Mach limit. However, the entropy viscosity coefficient did not show such ill-scaling as Mach $\rightarrow 0$.

The steady-state solution is correctly resolved even with few spatial cells using a continuous finite element approach implement in the MOOSE software.

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