Extension of the entropy viscosity method to the low Mach multi-D Euler equations and the seven-equation model.

by
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Outline:

introduction $\rightarrow$ hyperbolic system of equations and real life

Mathematical propoerties

Let us consider a hyperbolic conservation law of the form:

$$\partial_t U + \nabla \cdot F(U) = \mathbf{0} \text{ (strong form)}$$

- wave-dominated problem $\rightarrow$ eigenvalues.
- known to form shocks even with smooth initial conditions: characteristic equation and variable (REFS)
- require a numerical stabilization method to resolve shocks: approximate Riemann solver, artificial dissipative method, flux-limiting method, ...
- uniqueness of the weak solution is ensured by an entropy condition: the associated entropy equation satisfies an inequality and is peaked in the shock region.

$$\partial_t S(U) + \nabla \Phi(U) \geq 0$$

The general idea behind the entropy viscosity method

It is an artificial dissipation method with smart viscosity coefficient capable of tracking the shock so that dissipation is only added into the shock region.

- it requires a viscous regularization: the dissipative terms are consistent with the entropy inequality.
- each viscosity coefficient is function of a high-order viscosity coefficient and an upper bound called first-order viscosity coefficient.
- the high-order viscosity coefficient is defined proportional to the entropy residual.
- the first-order viscosity coefficient is function of the local maximum eigenvalue.
- also accounts for the inter element jumps $\rightarrow$ make the definition of the viscosity coefficients also sensitive to all discontinuities.

The Multiphysics Object-Oriented Simulations Environment (MOOSE)

- Explicit and implicit temporal integrators (first and second-order accuracy)
- Continuous and Discontinuous Galerkin Finite Element Method
- Mesh adaptivity, time step adaptivity
- Support parallel runs
- Buildt on PETSc and Libmesh
- For implicit solve, requires a preconditioner $\rightarrow$ either FDP or hard-coded preconditioner
- C++ language, opened source

All numerical solutions were run with BDF2 and linear test functions $\rightarrow$ second-order accuracy in time and space.

BurRer's Equation
(BadGEr)

A simple example of application of the EVM: the 1-D Burger's equation

The 1-D Burger's equation with its viscous regularization

$$\partial_t u(x,t) + \partial_x \left(\frac{u(x,t)^2}{2} \right) = \partial_x \left(\mu(x,t)\, \partial_x u(x,t) \right)$$

Definition of the local viscosity coefficient $\mu(x,t)$

$$\mu(x,t) = \min\left(\mu_e(x,t), \mu_{max}(x,t)\right)$$

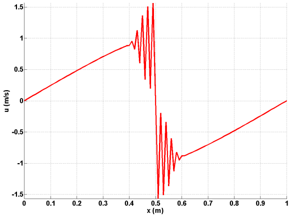
$$\mu_{max}(x,t) = \frac{h}{2}|u(x,t)|$$

$$\mu_e(x,t) = h^2 \frac{\max\left(R(x,t), J\right)}{||\eta(u) - \bar{\eta}(t)||_\infty}$$

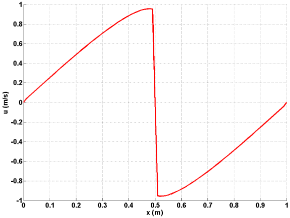
Entropy residual: $R(x,t) = \partial_t \eta(u) + \partial_x \Phi(u) \leq 0$ with $\eta(u) = u^2/2$

Jump: $J = [[\partial_x \Phi(u)]]$

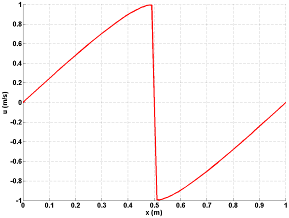
1-D numerical results (100 cells and CFL = 1)



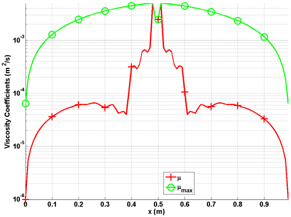
(a) Without stabilization.



(b) With first-order viscosity.



(c) With the EVM.



(d) Viscosity coefficient profiles.

The MULti-D Euler Equations with variable area (MULe DEEr)

Numerical methods for continuous and discontinuous schemes

- multi-wave problem, can develop shock waves and other types of discontinuities.
- Numerical methods for both continuous and discontinuous schemes: approximate Riemann solvers (HLL, HLLC, Roe scheme, $\cdots$), flux limiters , Lapidus viscosity, Pressure-based viscosity, SUPG, C-method, Entropy Viscosity Method (EVM).
- Numerical method can be ill-scaled in the low-Mach limit, yielding the wrong incompressible system $\rightarrow$ use of a Mach-based preconditioner for the dissipative terms to obtain the correct behavior in the low Mach limit.
- Low-Mach steady-state solution: time-dependent term preconditioner to accelerate the convergence of the solution to the steady-state (Turkel) when using an explicit scheme $\rightarrow$ the transient is no longer accurate. *Implicit solvers do not have this issue*.

- The isentropic compressible multi-D Euler equations degenerate into an incompressible system in the low-Mach asymptotic limit.

$$
\begin{aligned}
&\partial_t \rho + \vec{u} \cdot \boldsymbol{\nabla} \rho = 0 \\
&\partial_t \vec{u} + \vec{u}\ \boldsymbol{\nabla} \cdot \vec{u} + \frac{1}{\rho} \boldsymbol{\nabla} P = 0 \\
&\boldsymbol{\nabla} \cdot \vec{u} = 0 \\
&P(\vec{r}, t) = P_0(t) + M^2 P_2(\vec{r}, t)
\end{aligned}
$$

$\rightarrow$ no energy equation since the flow is assumed isentropic
$\rightarrow$ pressure fluctuations of the order of the Mach number square

The multi-D Euler equations with variable area

Mass conservation

$$\partial_t(\rho A) + \boldsymbol{\nabla}\cdot(\rho A\vec{u}) = 0$$

Momentum conservation

$$\partial_t(\rho\vec{u}A) + \boldsymbol{\nabla}\cdot(\rho A\vec{u}\otimes\vec{u} + P\mathbb{I}) = P\boldsymbol{\nabla}A$$

Energy conservation

$$\partial_t(\rho EA) + \boldsymbol{\nabla}\cdot[\vec{u}(\rho E + P)\,A] = 0$$

Equation of state

$$P = eos\,(\rho, e)$$

Multi-wave problem: $\lambda_1 = \vec{u}\cdot\vec{n} - c$, $\lambda_2 = \vec{u}\cdot\vec{n} + c$ and $\lambda_{2,\ldots,2+D} = \vec{u}\cdot\vec{n}$.
The area A is only a function of space.

Objectives: extend the EVM to low-Mach flows while maintaining its capabilities of solving for transonic and supersonic flows, and use an implicit solver.

How to do it?

1. recast the entropy equation as a function of the pressure, the density, the velocity and the speed of sound.
2. derive a viscous regularization for the multi-D Euler equations (already done).
3. work with the non-dimensionalized version of the multi-D Euler equations in order to understand how the different terms scale $\rightarrow$ will define non-dimensionalized numbers (Mach number, numerical Reynolds number, $\ldots$)
4. derive a definition for the viscosity coefficients that ensures well-scaled dissipative terms for a wide range of Mach numbers $\rightarrow$ will consider two cases: isentropic and non-isentropic (with shocks) flows.

Recast the entropy residual

New entropy residual

$$D_e(\vec{r},t) = \partial_t s + \vec{u}\cdot\nabla s = \frac{s_e}{P_e}\underbrace{\left(\frac{dP}{dt} - c^2\frac{d\rho}{dt}\right)}_{\tilde{D}_e(\vec{r},t)}$$

The viscosity coefficients

- The viscosity coefficient will be set proportional to $\tilde{D}_e(\vec{r},t)$ (instead of $D_e(\vec{r},t)$):

$$\mu_e(\vec{r},t) = h^2\frac{\max\left(\tilde{D}_e(\vec{r},t), J\right)}{\mathrm{norm}_P^{\mu}} \text{ and } \kappa_e(\vec{r},t) = h^2\frac{\max\left(\tilde{D}_e(\vec{r},t), J\right)}{\mathrm{norm}_P^{\kappa}}$$

- $\tilde{D}_e(\vec{r},t)$ is an alternative way of computing the local entropy production
- This new expression offers more diversity in the choice of the **normalization parameter** $norm_P$: P, ρc^2, $\rho c||\vec{u}||$ or $\rho||\vec{u}||^2$

A viscous regularization for the multi-D Euler equations with variable area

Mass conservation

$$\partial_t(\rho A) + \boldsymbol{\nabla}\cdot(\rho A\vec{u}) = \boldsymbol{\nabla}\cdot\vec{f}$$

Momentum conservation

$$\partial_t(\rho\vec{u}A) + \boldsymbol{\nabla}\cdot(\rho A\vec{u}\otimes\vec{u} + P\mathbb{I}) = P\boldsymbol{\nabla}A + \boldsymbol{\nabla}\cdot\left(\mathbb{F}(\vec{u}) + \vec{u}\otimes\vec{f}\right)$$

Energy conservation

$$\partial_t(\rho EA) + \boldsymbol{\nabla}\cdot[\vec{u}\,(\rho E + P)\,A] = \boldsymbol{\nabla}\cdot\left(\vec{h} + \vec{u}\cdot\mathbb{F}(\vec{u}) + \frac{||\vec{u}||^2}{2}f\right)$$

Dissipative terms

$\vec{f} = A\kappa\boldsymbol{\nabla}\rho$, $\vec{h} = A\kappa\boldsymbol{\nabla}(\rho e)$ and $\mathbb{F}(\vec{u}) = A\mu\boldsymbol{\nabla}^s\vec{u}$ or $\mathbb{F}(\vec{u}) = A\mu\boldsymbol{\nabla}\vec{u}$

$\rightarrow$ two positive viscosity coefficients μ and κ. Requires a concave physical entropy $s(\rho, e)$.

Non-dimensionalized multi-D Euler equation

We define some reference variables denoted by subscript ∞:

$$\rho^* = \frac{\rho}{\rho_\infty},\ u^* = \frac{u}{u_\infty},\ P^* = \frac{P}{\rho_\infty c_\infty^2},\ E^* = \frac{E}{c_\infty^2},$$
$$x^* = \frac{x}{L_\infty},\ t^* = \frac{t}{L_\infty/u_\infty},\ \mu^* = \frac{\mu}{\mu_\infty},\ \kappa^* = \frac{\kappa}{\kappa_\infty}$$

$\rightarrow$ μ_∞ and κ_∞ are function of the normalization parameters $norm_P^\mu$ and $norm_P^\kappa$, respectively. We also define the following reference numbers:

$$\text{Mach number: } M_\infty = \frac{u_\infty}{c_\infty},$$
$$\text{Numerical Reynolds number: } \mathrm{Re}_\infty = \frac{u_\infty L_\infty}{\mu_\infty},$$
$$\text{Numerical Péchlet number: } \mathrm{Pé}_\infty = \frac{u_\infty L_\infty}{\kappa_\infty},$$
$$\text{Numerical Prandlt number: } \mathrm{Pr}_\infty = \mathrm{Pé}_\infty/\mathrm{Re}_\infty$$

Non-dimensionalized multi-D Euler equation

$$\partial_{t^*}\rho^* + \boldsymbol{\nabla}^*\cdot(\rho^*\vec{u}^*) = \frac{1}{\mathrm{Pé}_\infty}\boldsymbol{\nabla}^*\cdot(\kappa^*\vec{\nabla}^*\rho^*)$$

$$\begin{aligned}\partial_{t^*}(\rho^*\vec{u}^*) + \boldsymbol{\nabla}^*\cdot(\rho^*\vec{u}^*\otimes\vec{u}^*) + \frac{1}{M_\infty^2}\vec{\nabla}^* P^* &= \frac{1}{\mathrm{Re}_\infty}\boldsymbol{\nabla}^*\cdot\left(\rho^*\mu^*\vec{\nabla}^{s,*}\vec{u}^*\right)\\ &+ \frac{1}{\mathrm{Pé}_\infty}\boldsymbol{\nabla}^*\cdot\left(\vec{u}^*\otimes\kappa^*\vec{\nabla}^*\rho^*\right)\end{aligned}$$

$$\begin{aligned}\partial_{t^*}(\rho^* E^*) + \boldsymbol{\nabla}^*\cdot[\vec{u}^*(\rho^* E^* + P^*)] &= \frac{1}{\mathrm{Pé}_\infty}\boldsymbol{\nabla}^*\cdot\left(\kappa^*\vec{\nabla}^*(\rho^* e^*)\right)\\ +\frac{M_\infty^2}{\mathrm{Re}_\infty}\boldsymbol{\nabla}^*\cdot\left(\vec{u}^*\rho^*\mu^*\vec{\nabla}^{s,*}\vec{u}^*\right) &+ \frac{M_\infty^2}{2\mathrm{Pé}_\infty}\boldsymbol{\nabla}^*\cdot\left(\kappa^*(u^*)^2\vec{\nabla}^*\rho^*\right)\end{aligned}$$

The above equations are valid for both isentropic and non-isentropic flows and for all Mach numbers.

For an isentropic flow

$\rightarrow$ choose Re_∞ and $\mathrm{Pé}_\infty$ so that we recover the incompressible equations.

- assume Ideal gas equation of state for simplicity: $P = (\gamma - 1)\rho e$
- choose $\mathrm{Re}_\infty = \mathrm{Pé}_\infty = 1$
- expand each variable in power of the Mach number:
  $P(\vec{r}, t) = P_0(\vec{r}, t) + P_1(\vec{r}, t) M_\infty + P_2(\vec{r}, t) M_\infty^2 + \ldots$
- derive the leading, first and second-order momentum equations
- derive the leading-order energy and mass equations

For an isentropic flow: momentum equation

$$\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \frac{1}{M_\infty^2} \nabla P = \frac{1}{\mathrm{Re}_\infty} \nabla \cdot \left(\rho \mu \vec{\nabla}^s \vec{u} \right) \quad + \frac{1}{\mathrm{Pé}_\infty} \nabla \cdot (\vec{u} \otimes \kappa \nabla \rho)$$

Leading and first-order momentum equation:

$$\nabla P_0 = \nabla P_1 = 0 \longrightarrow P(\vec{r}, t) = \tilde{P}_0(t) + P_2(\vec{r}, t) M_\infty^2$$

Second-order momentum equation:

$$\partial_t (\rho \vec{u})_0 + \nabla \cdot (\rho \vec{u} \otimes \vec{u})_0 + P_2 = \nabla \cdot \left(\rho \mu \vec{\nabla}^s \vec{u} \right)_0 + \nabla \cdot (\vec{u} \otimes \kappa \nabla \rho)_0$$

For an isentropic flow: energy and mass equations

$$\partial_{t^*}\left(\rho^* E^*\right) + \boldsymbol{\nabla}^* \cdot \left[\vec{u}^* \left(\rho^* E^* + P^*\right)\right] = \frac{1}{\mathrm{Pé}_\infty} \boldsymbol{\nabla}^* \cdot \left(\kappa^* \vec{\nabla}^* (\rho^* e^*)\right)$$
$$+ \frac{M_\infty^2}{\mathrm{Re}_\infty} \boldsymbol{\nabla}^* \cdot \left(\vec{u}^* \rho^* \mu^* \vec{\nabla}^{s,*} \vec{u}^*\right) + \frac{M_\infty^2}{2\mathrm{Pé}_\infty} \boldsymbol{\nabla}^* \cdot \left(\kappa^* (u^*)^2 \vec{\nabla}^* \rho^*\right)$$

Leading-order energy equation:

$$\partial_t(\rho E)_0 + \boldsymbol{\nabla} \cdot [\vec{u}(\rho E + P)]_0 = \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla}(\rho e))_0 \longrightarrow \boldsymbol{\nabla} \cdot \vec{u}_0 = 0$$

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \vec{u}) = \frac{1}{\mathrm{Pé}_\infty} \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} \rho)$$

Leading-order mass equation:

$$\partial_t \rho_0 + \boldsymbol{\nabla} \cdot (\rho \vec{u})_0 = \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} \rho)_0 \rightarrow \partial_t \rho_0 + \vec{u}_0 \boldsymbol{\nabla} \rho_0 = \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} \rho)_0$$

For an isentropic flow: derivation of $\mathrm{norm}_P^{\mu,\kappa}$

We were able to recover the incompressible equations by choosing $\mathrm{Re}_\infty = \mathrm{Pé}_\infty = 1$. What does that imply for norm_P^{μ} and norm_P^{κ}?

Derive an expression for κ_∞:

$$\kappa_e(\vec{r}, t) = h^2 \frac{\max\left(\tilde{D}_e(\vec{r}, t), J\right)}{\mathrm{norm}_P^{\kappa}} \longrightarrow \kappa_\infty = \frac{\rho_\infty c_\infty^2 u_\infty L}{\mathrm{norm}_{P,\infty}^{\kappa}}$$

Then, use the substitute the above expression for norm_P^{κ} into $\mathrm{Pé}_\infty$ to obtain:

$$\mathrm{norm}_{P,\infty}^{\kappa} = \mathrm{Pé}_\infty \rho_\infty c_\infty^2 = \rho_\infty c_\infty^2$$

Same derivation using μ_e and Re_∞ leads to:

$$\mathrm{norm}_{P,\infty}^{\mu} = \mathrm{Re}_\infty \rho_\infty c_\infty^2 = \rho_\infty c_\infty^2$$

For a non-isentropic flow i.e. with shocks

- the flow can experience shocks and other waves $\rightarrow$ discontinuities
- the low-Mach asymptotic study is no longer valid
- directly work with the non-dimensionalized Euler equations
- determine the scaling of Re_∞ and $\mathrm{Pé}_\infty$ to stabilize the equations
- look at the low-Mach limit

For a non-isentropic flow: momentum equation

$$\partial_t (\rho \vec{u}) + \boldsymbol{\nabla} \cdot (\rho^* \vec{u} \otimes \vec{u}) + \frac{1}{M_\infty^2} \boldsymbol{\nabla} P = \frac{1}{\mathrm{Re}_\infty} \boldsymbol{\nabla} \cdot \left(\rho \mu \vec{\nabla}^s \vec{u} \right) + \frac{1}{\mathrm{Pé}_\infty} \boldsymbol{\nabla} \cdot (\vec{u} \otimes \kappa \boldsymbol{\nabla} \rho)$$

In the shock region, the term $\frac{1}{M_\infty^2} \boldsymbol{\nabla} P$ will become dominant and will need to be stabilize by a dissipative term of the same scaling:

- (a) $\mathrm{Re}_\infty = M_\infty^2$ and $\mathrm{Pé}_\infty = 1$
- (b) $\mathrm{Re}_\infty = 1$ and $\mathrm{Pé}_\infty = M_\infty^2$
- (c) $\mathrm{Re}_\infty = \mathrm{Pé}_\infty = M_\infty^2$

$\longrightarrow$ each of the above option will affect the other equations (mass and energy equations).

For a non-isentropic flow: mass equation

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \vec{u}) = \frac{1}{\mathrm{Pé}_\infty} \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} \rho)$$

choice (a) $\mathrm{Pé}_\infty = 1$

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \vec{u}) = \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} \rho)$$

$\rightarrow$ the dissipative term is *well-scaled*

choice (b) and (c) $\mathrm{Pé}_\infty = M_\infty^2$

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \vec{u}) = \frac{1}{M_\infty^2} \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} \rho)$$

$\rightarrow$ the dissipative term is *ill-scaled*

Options (b) and (c) are not inappropriate. Thus, we are left with option (a): $\mathrm{Re}_\infty = M_\infty^2$ and $\mathrm{Pé}_\infty = 1$

For a non-isentropic flow: energy equation

$$\partial_{t^*}\left(\rho^* E^*\right) + \boldsymbol{\nabla}^* \cdot \left[\vec{u}^*\left(\rho^* E^* + P^*\right)\right] = \frac{1}{\mathrm{Pé}_\infty} \boldsymbol{\nabla}^* \cdot \left(\kappa^* \vec{\nabla}^* (\rho^* e^*)\right)$$
$$+ \frac{M_\infty^2}{\mathrm{Re}_\infty} \boldsymbol{\nabla}^* \cdot \left(\vec{u}^* \rho^* \mu^* \vec{\nabla}^{s,*} \vec{u}^*\right) + \frac{M_\infty^2}{2\mathrm{Pé}_\infty} \boldsymbol{\nabla}^* \cdot \left(\kappa^* (u^*)^2 \vec{\nabla}^* \rho^*\right)$$

With option (a) it yields:

$$\partial_{t^*}\left(\rho^* E^*\right) + \boldsymbol{\nabla}^* \cdot \left[\vec{u}^*\left(\rho^* E^* + P^*\right)\right] = \boldsymbol{\nabla}^* \cdot \left(\kappa^* \vec{\nabla}^* (\rho^* e^*)\right)$$
$$+ \boldsymbol{\nabla}^* \cdot \left(\vec{u}^* \rho^* \mu^* \vec{\nabla}^{s,*} \vec{u}^*\right) + \frac{M_\infty^2}{2} \boldsymbol{\nabla}^* \cdot \left(\kappa^* (u^*)^2 \vec{\nabla}^* \rho^*\right)$$

$\rightarrow$ all dissipative terms are *well-scaled*.

For a non-isentropic flow: derivation of $\text{norm}_P^{\mu,\kappa}$

We derive the scaling of norm_P^{μ} and norm_P^{κ} when $\text{Re}_\infty = M_\infty^2$ and $\text{Pé}_\infty = 1$:

Derive an expression for κ_∞:

$$\kappa_e(\vec{r}, t) = h^2 \frac{\max\left(\tilde{D}_e(\vec{r}, t), J\right)}{\text{norm}_P^{\kappa}} \longrightarrow \kappa_\infty = \frac{\rho_\infty c_\infty^2 u_\infty L}{\text{norm}_{P,\infty}^{\kappa}}$$

Then, use the substitute the above expression for norm_P^{κ} into Pé_∞ to obtain:

$$\text{norm}_{P,\infty}^{\kappa} = \text{Pé}_\infty \rho_\infty c_\infty^2 = \rho_\infty c_\infty^2 \rightarrow \text{ same as for isentropic flow}$$

Same derivation using μ_e and Re_∞ leads to:

$$\text{norm}_{P,\infty}^{\mu} = \text{Re}_\infty \rho_\infty c_\infty^2 = \rho_\infty u_\infty^2 \rightarrow \text{ different from isentropic flow}$$

How to merge the two cases?

$$\mu(\vec{r},t) = \min\Big(\mu_{\max}(\vec{r},t), \mu_e(\vec{r},t)\Big) \text{ and } \kappa(\vec{r},t) = \min\Big(\mu_{\max}(\vec{r},t), \kappa_e(\vec{r},t)\Big)$$

where the first-order viscosity is given by

$$\kappa_{\max}(\vec{r},t) = \mu_{\max}(\vec{r},t) = \frac{h}{2}\Big(||u|| + c\Big)$$

and the entropy viscosity coefficients by

$$\kappa_e(\vec{r},t) = \frac{h^2 \max(\widetilde{R}, J)}{\rho c^2} \text{ and } \mu_e(\vec{r},t) = \frac{h^2 \max(\widetilde{R}, J)}{\mathrm{norm}_P^{\mu}}$$

where

$$\mathrm{norm}_P^{\mu} = \mathrm{Re}_{\infty}\rho_{\infty}c_{\infty}^2 = \begin{cases} \rho||u||^2 & \text{if } \left|\widetilde{R}^*\right| \geq M \text{ (i.e., non-isentropic flow)} \\ \rho c^2 = \mathrm{norm}_P^{\kappa} & \text{otherwise} \end{cases}$$

with the jumps given by

$$J = ||\vec{u}|| \max\Big([[\nabla P \cdot \vec{n}]], c^2[[\nabla \rho \cdot \vec{n}]]\Big)$$

Table: Subsonic flow inlet boundary conditions.

boundary type	U_1^{bc}	U_2^{bc}	U_3^{bc}	U_4^{ℓ}
static pressure	P	T	θ	u
mass flow rate	$\rho\|\|u\|\|$	h	θ	u
stagnation pressure	P_0	T_0	θ	u

1-D converging-diverging nozzle

Stiffened gas equation of state

$$P = (\gamma - 1)\rho(e - q) - \gamma P_\infty$$

Equation of state parameters

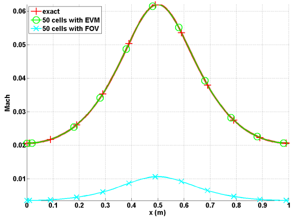
fluid	γ	$C_v\ (J.kg^{-1}.K^{-1})$	$P_\infty\ (Pa)$	$q\ (J.kg^{-1})$
liquid water	2.35	1816	10^9	$-1167\ 10^3$
steam	1.43	1040	0	$2030\ 10^3$

Cross-section A

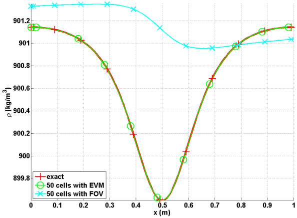
$$A(x) = 1 + 0.5\cos(2\pi x)$$

$\rightarrow$ a steady state is reached
$\rightarrow$ low-Mach flow for liquid water
$\rightarrow$ supersonic flow for steam

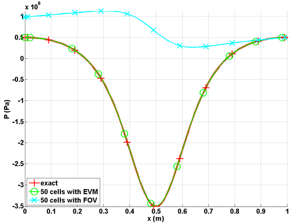
1-D converging-diverging nozzle: liquid water



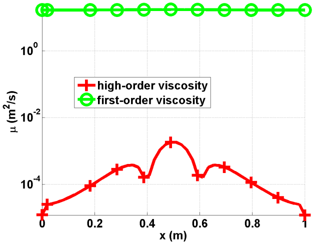
(e) Mach number



(f) Density



(g) Pressure



(h) Viscosity coefficients

1-D converging-diverging nozzle: liquid water

Convergence rates for the L_1 norm of the error:

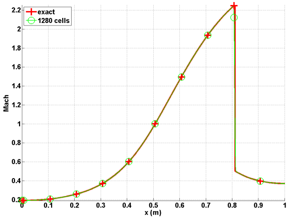
cells	density	rate	pressure	rate	velocity	rate
4	$2.8037\ 10^{-1}$	–	$8.4705\ 10^{5}$	–	7.2737	–
8	$1.3343\ 10^{-1}$	0.495	$4.7893\ 10^{5}$	0.24	6.1493	0.0747
16	$2.9373\ 10^{-2}$	2.10	$1.0613\ 10^{5}$	2.09	1.2275	2.25
32	$5.1120\ 10^{-3}$	2.58	$1.8446\ 10^{4}$	2.58	$1.8943\ 10^{-1}$	2.78
64	$1.0558\ 10^{-3}$	2.31	$3.7938\ 10^{3}$	2.31	$3.7919\ 10^{-2}$	2.37
128	$2.3712\ 10^{-4}$	2.18	$8.4471\ 10^{2}$	2.19	$8.5517\ 10^{-3}$	2.17
256	$5.6058\ 10^{-5}$	2.08	$1.9839\ 10^{2}$	2.09	$2.0475\ 10^{-3}$	2.07
512	$1.3278\ 10^{-5}$	2.07	$4.6622\ 10^{1}$	2.08	$4.9516\ 10^{-4}$	2.06
1024	$3.1193\ 10^{-6}$	–	$1.1755\ 10^{1}$	–	$1.2379\ 10^{-4}$	–

1-D converging-diverging nozzle: liquid water

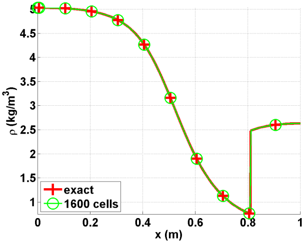
Convergence rates for the L_2 norm of the error:

cells	density	rate	pressure	rate	velocity	rate
4	$3.106397\ 10^{-1}$	–	$5.254445\ 10^{5}$	–	3.288543	–
8	$7.491623\ 10^{-2}$	2.06	$1.636966\ 10^{5}$	1.62	1.823880	0.14
16	$2.079858\ 10^{-2}$	1.81	$4.627338\ 10^{4}$	1.77	$4.990605\ 10^{-1}$	1.83
32	$5.329627\ 10^{-3}$	1.96	$1.180287\ 10^{4}$	1.96	$1.261018\ 10^{-1}$	1.98
64	$1.341583\ 10^{-3}$	1.99	$2.967104\ 10^{3}$	1.99	$3.160914\ 10^{-2}$	1.99
128	$3.359766\ 10^{-4}$	1.99	$7.428087\ 10^{2}$	1.99	$7.907499\ 10^{-3}$	1.99
256	$8.403859\ 10^{-5}$	1.99	$1.857861\ 10^{2}$	2.01	$1.977292\ 10^{-3}$	2.00
512	$2.10075\ 10^{-5}$	–	$4.7024\ 10^{1}$	–	$4.9516\ 10^{-4}$	–

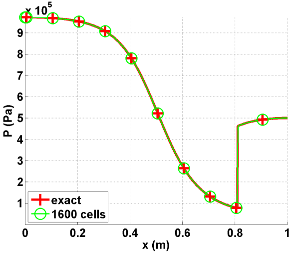
1-D converging-diverging nozzle: vapor



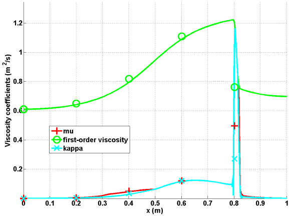
(i) Mach number



(j) Density



(k) Pressure



(l) Viscosity coefficients

1-D converging-diverging nozzle: vapor

Convergence rates for the L_1 norm of the error:

cells	density	rate	pressure	rate	velocity	rate
5	$0.72562\ 10^{-1}$	–	$1.5657\ 10^{5}$	–	173.69	–
10	$0.4165\ 10^{-1}$	0.80088	$9.6741\ 10^{4}$	0.63425	120.69	0.52519
20	$0.20675\ 10^{-1}$	1.0104	$4.9193\ 10^{4}$	0.96971	72.149	0.74228
40	$0.093703\ 10^{-1}$	1.1417	$2.0103\ 10^{4}$	0.72728	34.716	1.0554
80	$0.047328\ 10^{-1}$	0.9854	$1.0208\ 10^{4}$	0.9777	16.082	1.1101
160	$0.023965\ 10^{-2}$	0.9817	$5.1969\ 10^{3}$	0.9739	7.9573	1.0150
320	$0.020768\ 10^{-2}$	0.9886	$2.5116\ 10^{3}$	1.0490	3.7812	1.0734
640	$0.0059715\ 10^{-2}$	1.0160	$1.2754\ 10^{3}$	0.9776	1.8353	1.0428

1-D converging-diverging nozzle: vapor

Convergence rates for the L_2 norm of the error:

cells	density	rate	pressure	rate	velocity	rate
5	$9.7144\ 10^{-1}$	—	$2.0215\ 10^{5}$	—	236.94	—
10	$5.9718\ 10^{-1}$	0.70195	$1.3024\ 10^{5}$	0.63425	166.56	0.50854
20	$2.9503\ 10^{-1}$	1.0173	$6.6503\ 10^{4}$	0.96971	103.36	0.68831
40	$1.8193\ 10^{-1}$	0.69747	$4.0171\ 10^{4}$	0.72728	66.374	0.6390
80	$1.3366\ 10^{-1}$	0.44485	$2.3163\ 10^{4}$	0.43576	42.981	0.62692
160	$9.6638\ 10^{-2}$	0.46790	$1.7263\ 10^{4}$	0.42413	31.717	0.43844
320	$7.0896\ 10^{-2}$	0.44688	$1.2763\ 10^{4}$	0.43571	23.138	0.45499
640	$5.2191\ 10^{-2}$	0.44190	$9.4217\ 10^{3}$	0.43790	16.910	0.45238

2-D low-Mach flow over a cylinder

Typical benchmark problem for low-Mach flow:

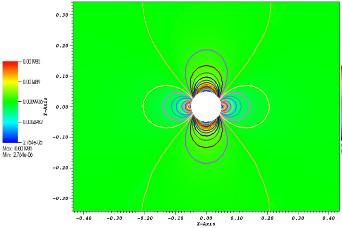
- The steady-state solution is symmetric: the iso-Mach contour lines are used to asses the symmetry of the numerical solution
- The velocity at the top of the cylinder is twice the incoming velocity set at the inlet
- The pressure fluctuations are proportional to the square of inlet Mach number, i.e.,

$$\delta P = \frac{\max(P(\vec{r},t)) - \min(P(\vec{r},t))}{\max(P(\vec{r},t))} \propto M_\infty^2$$

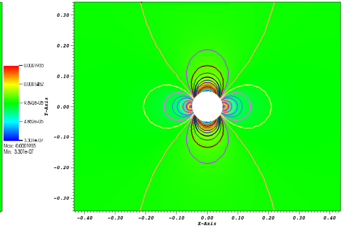
  where δP and M_∞ denote the pressure fluctuations and the inlet Mach number, respectively.

- triangular mesh with 4008 triangular elements
- Ideal Gas equation of state with $\gamma = 1.4$
- CFL $= 20$

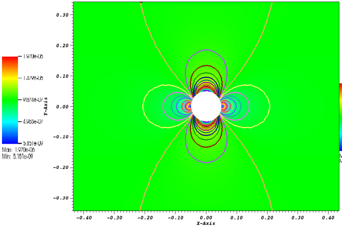
2-D low-Mach flow over a cylinder



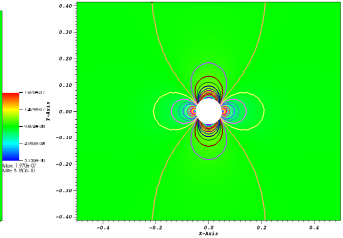
(m) $M_{\text{inlet}} = 10^{-3}$



(n) $M_{\text{inlet}} = 10^{-4}$

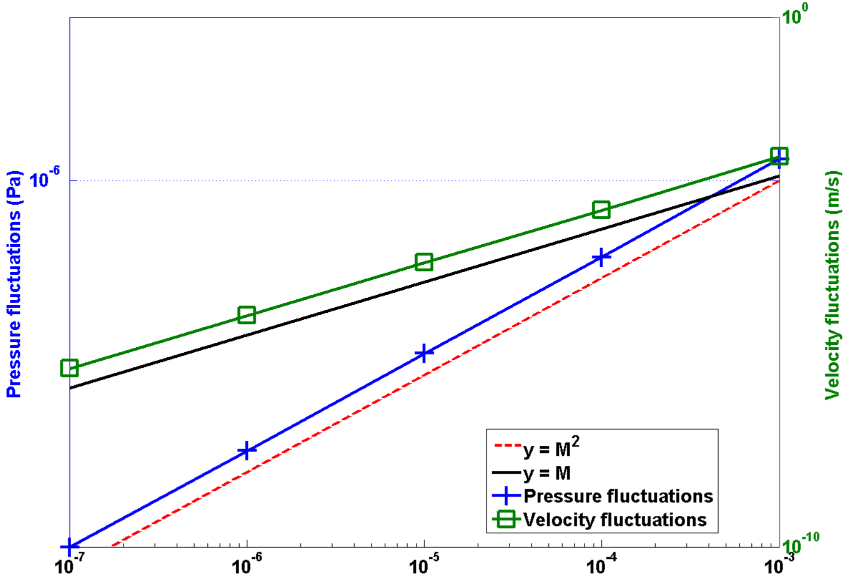


(o) $M_{\text{inlet}} = 10^{-6}$



(p) $M_{\text{inlet}} = 10^{-7}$

2-D low-Mach flow over a cylinder



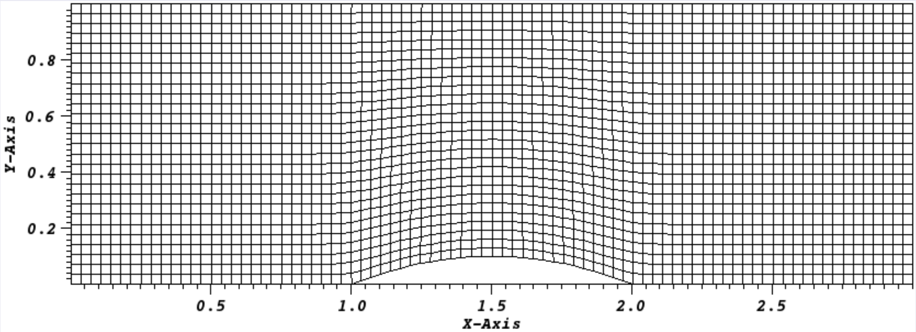
2-D low-Mach flow over a cylinder

Velocity ratio for different Mach numbers

Mach number	inlet velocity	velocity at the top of the cylinder	ratio
10^{-3}	$2.348\ 10^{-3}$	$1.176\ 10^{-3}$	1.99
10^{-4}	$2.285\ 10^{-4}$	$1.145\ 10^{-4}$	1.99
10^{-5}	$2.283\ 10^{-5}$	$1.144\ 10^{-5}$	1.99
10^{-6}	$2.283\ 10^{-6}$	$1.144\ 10^{-6}$	1.99
10^{-7}	$2.283\ 10^{-7}$	$1.144\ 10^{-7}$	1.99

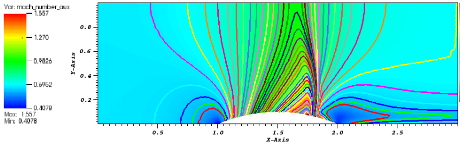
2-D low-Mach flow over a bump

Geometry: an uniform grid of 3352 Q_1 elements

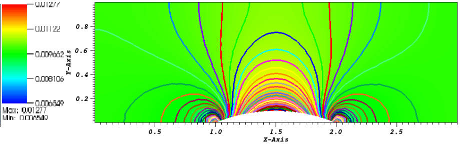


- Ideal Gas equation of state with $\gamma = 1.4$
- CFL $= 20$
- Inlet flow for different Mach numbers, static pressure and wall-boundary conditions.

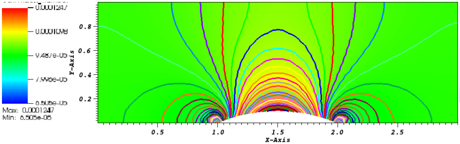
2-D low-Mach flow over a bump



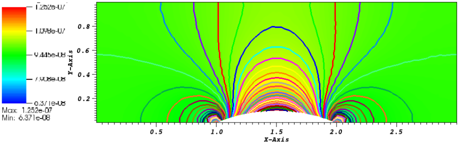
(q) $M_{\text{inlet}} = 0.7$



(r) $M_{\text{inlet}} = 10^{-2}$



(s) $M_{\text{inlet}} = 10^{-4}$



(t) $M_{\text{inlet}} = 10^{-7}$

2-D Mach 3 flow over a forward facing step

The SEVEN-EquAtion Model with variable area
(SEVEN-bandEd ArMadillo)

The seven-equation model (SEM)

- Each phase obeys the single-phase Euler equations: two continuity equations, two momentum equations and two energy equations.
- Seventh equation: void fraction equation $\rightarrow$ an internal boundary condition between the two phases at the interface.
- Exchange terms between phases: relaxation terms. These terms were derived using *rational thermodynamic* $\rightarrow$ consistent with the entropy minimum principle.
- The system of equations is well-posed, and has seven waves.
- The seven-equation model degenerates to single-phase Euler equations when one phase disappears.

The multi-D seven-equation model (with variable area)

We consider two phases j, k. Phase k obeys the following system of equations:

$$\begin{cases} \partial_t(\alpha_k A) & + \quad \vec{u}_{int} A \boldsymbol{\nabla} \alpha_k = A\mu_{rel}(P_k - P_j) \\ \partial_t(\alpha_k \rho_k A) & + \quad \boldsymbol{\nabla} \cdot (\alpha_k \rho_k \vec{u}_k A) = 0 \\ \partial_t(\alpha_k \rho_k \vec{u}_k A) & + \quad \boldsymbol{\nabla} \cdot [\alpha_k A (\rho_k \vec{u}_k \otimes \vec{u}_k)] + \boldsymbol{\nabla}(\alpha_k A P_k) = \\ & \quad \alpha_k P_k \boldsymbol{\nabla} A + P_{int} A \boldsymbol{\nabla} \alpha_k + A\lambda_{rel}(\vec{u}_j - \vec{u}_k) \\ \partial_t(\alpha_k \rho_k E_k A) & + \quad \boldsymbol{\nabla} \cdot [\alpha_k A \vec{u}_j (\rho_k E_k + P_k)] = \\ & \quad P_{int} \vec{u}_{int} A \boldsymbol{\nabla} \alpha_k - \mu_{rel} \bar{P}_{int} (P_k - P_j) + \bar{\vec{u}}_k A \lambda_{rel} (\vec{u}_j - \vec{u}_k) \end{cases}$$

$$\begin{cases} P_{int} = \bar{P}_{int} - \frac{\boldsymbol{\nabla}\alpha_k}{|\boldsymbol{\nabla}\alpha_k|} \frac{Z_k Z_j}{Z_k + Z_j} \cdot (\vec{u}_k - \vec{u}_j) \\ \bar{P}_{int} = \frac{Z_k P_j + Z_j P_k}{Z_k + Z_j} \\ \vec{u}_{int} = \bar{\vec{u}}_{int} - \frac{\boldsymbol{\nabla}\alpha_k}{|\boldsymbol{\nabla}\alpha_k|} \frac{P_k - P_j}{Z_k + Z_j} \\ \bar{\vec{u}}_{int} = \frac{Z_k \vec{u}_k + Z_j \vec{u}_j}{Z_k + Z_j} \end{cases} \quad \text{and} \quad \begin{cases} \mu_{rel} = \frac{A_{int}}{Z_k + Z_j} \\ \lambda_{rel} = \frac{\mu_{rel}}{2} Z_k Z_j \\ A_{int} = 6.25 \cdot A_{int,max} \cdot \alpha_k (1 - \alpha_k)^2 \end{cases}$$

Viscous regularization for the multi-D SEM:

$$\partial_t (\alpha_k A) + \vec{u}_{int} A \nabla \alpha_k = A\mu_{rel} (P_k - P_j) + \nabla \cdot \vec{l}$$

$$\partial_t (\alpha_k \rho_k A) + \nabla \cdot (\alpha_k \rho_k \vec{u}_k A) = \nabla \cdot \vec{f}$$

$$\partial_t (\alpha_k \rho_k \vec{u}_k A) + \nabla \cdot [\alpha_k A (\rho_k \vec{u}_k \otimes \vec{u}_k + P_k \mathbb{I})] =$$

$$\alpha_k P_k \nabla A + P_{int} A \nabla \alpha_k + A\lambda_{rel} (\vec{u}_j - \vec{u}_k) + \nabla \cdot \bar{g}$$

$$\partial_t (\alpha_k \rho_k E_k A) + \nabla \cdot [\alpha_k A \vec{u}_j (\rho_k E_k + P_k)] =$$

$$P_{int} \vec{u}_{int} A \nabla \alpha_k - \mu_{rel} \bar{P}_{int} (P_k - P_j) + \bar{\vec{u}}_k A \lambda_{rel} (\vec{u}_j - \vec{u}_k) + \nabla \cdot \vec{h}$$

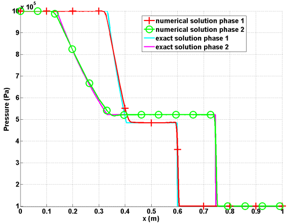
$$\begin{cases} \vec{l} = A\beta_k \nabla \alpha_k \\ \vec{f} = \alpha_k A \kappa_k \nabla \rho_k + \rho_k \vec{l} \\ \bar{g} = \alpha_k A \rho_k \mu_k \nabla \vec{u} + \vec{u} \otimes \vec{f} \\ \vec{h} = \alpha_k A \kappa_k \nabla (\rho_k e_k) - \frac{||\vec{u}||^2}{2} \vec{f} + \vec{u} \cdot \bar{g} + \rho_k e_k \vec{l} \end{cases}$$

if $\alpha_k \to 1$, the multi-D Euler equations are retrieved

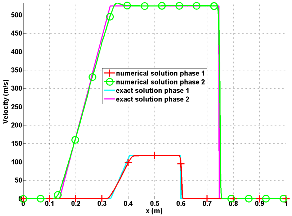
Numerical results

1-D shock tube with two independent fluids

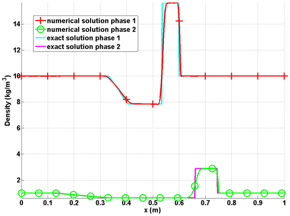
1-D shock tube with two independent fluids



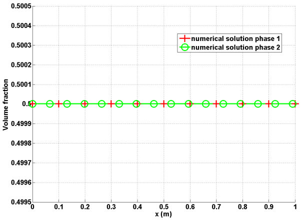
(u) Pressure at $t = 305\ \mu s$



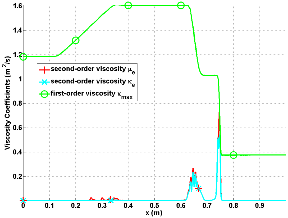
(v) Velocity at $t = 305\ \mu s$



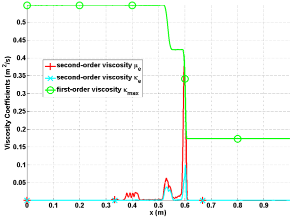
(w) Pressure at $t = 305\ \mu s$



(x) Velocity at $t = 305\ \mu s$



(y) Viscosity coefficients phase 1



(z) Viscosity coefficients phase 2

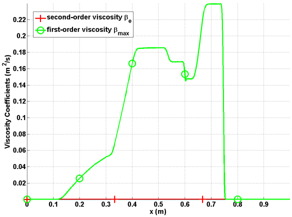
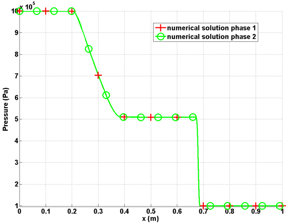


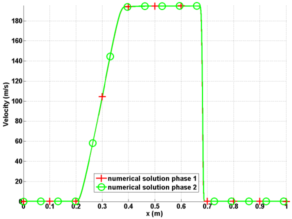
Figure: Viscosity coefficients volume fraction

1-D shock tube with infinite relaxation coefficients

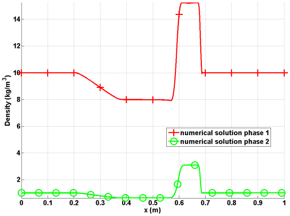
1-D shock tube with infinite relaxation coefficients



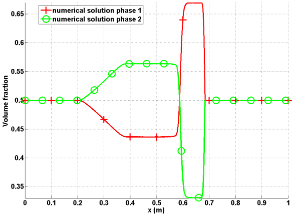
(a) Pressure at $t = 305\ \mu s$



(b) Velocity at $t = 305\ \mu s$

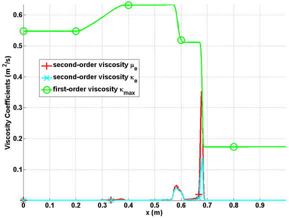


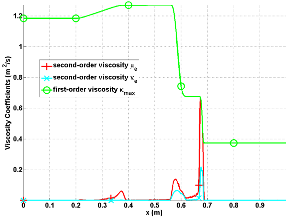
(c) Pressure at $t = 305\ \mu s$



(d) Velocity at $t = 305\ \mu s$

1-D shock tube with infinite relaxation coefficients





(e) Viscosity coefficients phase 1 (f) Viscosity coefficients phase 2

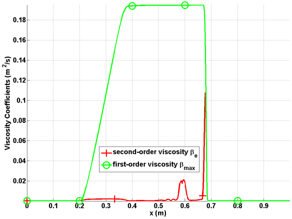


Figure: Viscosity coefficients volume fraction

Conclusion

The 1-D Radiation-Hydrodynamic EquAtion (RHEA)

QUESTIONS/COMMENTS ?