Preliminary exam:

Extension of the entropy viscosity method to low Mach regime and multi-phase flows,

and

A Multi-Physics Example: application of the entropy viscosity method to the 1-D grey Radiation-Hydrodynamic equations (RHD).

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Outline:

1. Extension of the entropy viscosity method to low Mach regime and multi-phase flows.
   - Computational method: background
   - The multi-D Euler equations (with variable area).
     - Proposal.
     - Preliminary numerical results.
     - Conclusions: the multi-D Euler equations.
   - The multi-D seven-equation model (with variable area).
     - Proposal.
     - Preliminary numerical results.
     - Conclusions: the multi-D seven-equation model.
2. A Multi-Physics Example: application of the entropy viscosity method to the 1-D grey Radiation-Hydrodynamic equations (RHD).
   - The 1-D grey Radiation-Hydrodynamic equations (RHD).
   - Background.
   - Proposal.
   - Numerical results.
   - Conclusions: RHD

Extension of the entropy viscosity method to low Mach regime and multi-phase flows.

Background (1/2):

- Hyperbolic system of equations → nuclear reactors, oil/gas models, aerospace, aeronautic, . . .
- Single- and multi-phases flow.
- Numerical methods required for stabilization purpose: shocks/discontinuities.
- Systems of equations describing the physics must be well-posed → real eigenvalues.
- Entropy condition: ensures convergence of the numerical solution to the physical solution.

Background (2/2):

Numerical methods for continuous and discontinuous schemes:

- Approximate Riemann solvers [13]: HLL, HLLC, Roe scheme, $\cdots$
- Flux limiters [7, 8, 9, 10].
- Lapidus viscosity [25, 27, 26], Pressure-based viscosity [28].
- C-method [14].
- Streamline Upwind Petrov-Galerkin (SUPG) [11].
- Entropy viscosity method [1, 2, 3].

These numerical methods are mainly used with high-order explicit temporal integrators.

Low Mach regime:

The numerical methods developed to resolve shocks for supersonic flows are usually ill-scaled in the low Mach regime [30, 31, 32].

- The dissipative terms become dominant and change the nature of the equations.
- A fix is often required in the low Mach regime in order to yield the correct behavior $\rightarrow$ add enough viscosity to stabilize the scheme but not too much so that the physical solution is not altered.
- Example with Roe scheme [29]: a fix was designed in order to obtain the correct asymptotic behavior in the low Mach regime.

$$\mathbf{F}(\mathbf{U}) = (1 - M) \cdot \mathbf{F}_{\text{Low Mach}}(\mathbf{U}) + M \cdot \mathbf{F}_{Roe}(\mathbf{U})$$

- Example of scaling analysis with artificial dissipative terms: upwind scheme.

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{u}) = \frac{1}{2} \left(1 + \frac{1}{M^*} \right) \vec{\nabla} \cdot \left(\kappa \vec{\nabla} \rho \right)$$

Fluid solver requirements:

1. Compressible single and two-phase flow equations.
2. Being able to resolve shocks: numerical method.
3. Valid for any equation of state of type $P = eos(\rho, e)$.
4. Valid for a wide range of Mach numbers: compressible and incompressible flows.
5. Account for source terms: wall friction, wall heat source, gravity force, ...
6. Use *implicit* temporal integrators: Backward Euler and BDF2.

The entropy viscosity method, a good candidate?

- 1 and 2 are met by the entropy viscosity method ONLY for single-phase flow.
- 3: there exists a theoretical proof that states the method can be applied to any EOS with a convex entropy function.
- 4, 5 and 6 are investigated here.

The entropy viscosity method [19]:

It is based on the entropy production that occurs in shock and the entropy minimum principle.

- Add artificial dissipative terms consistent with the entropy minimum principle.
- Define a local and smooth viscosity coefficient function of the grid size.
- The viscosity coefficient is defined proportional to the entropy production.
- The entropy production is locally computed by evaluating the entropy residual:

$$D_e(\vec{r}, t) = \partial_t s + \vec{u} \cdot \vec{\nabla} s \geq 0$$

(Euler equations)

Application to the multi-D Euler equations [15]:

The multi-D Euler Equations: [13]

$$\begin{cases} \partial_t \rho + \vec{\nabla} \cdot (\rho \vec{u}) = \vec{\nabla} \cdot \left(\kappa \vec{\nabla} \rho \right) \\ \partial_t \left(\rho \vec{u} \right) + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u}) + \vec{\nabla} P = \vec{\nabla} \cdot \left[\left(\mu \rho \vec{\nabla}^s \vec{u} + \vec{u} \otimes \left(\kappa \vec{\nabla} \rho \right) \right) \right] \\ \partial_t \left(\rho E \right) + \vec{\nabla} \cdot [\vec{u}(\rho E + P)] = \vec{\nabla} \cdot \left[\left(\kappa \vec{\nabla} (\rho e) + \frac{||\vec{u}||^2}{2} \kappa \vec{\nabla} \rho + \mu \rho \vec{u} \cdot \vec{\nabla}^s \vec{u} \right) \right] \\ P = eos(\rho, e) \end{cases}$$

Definition of the local viscosity coefficients μ and κ: $(\mu = \kappa)$

$$\begin{cases} \mu(\vec{r}, t) = \min(\mu_e(\vec{r}, t), \mu_{max}(\vec{r}, t)) \\ \mu_e(\vec{r}, t) = h^2 \frac{\max(D_e(\vec{r}, t), J)}{||s(\vec{r}, t) - \bar{s}(t)||_\infty} \\ \mu_{max}(\vec{r}, t) = \frac{h}{2} \left(||\vec{u}|| + c \right) \\ D_e(\vec{r}, t) = \partial_t s + \vec{u} \cdot \vec{\nabla} \cdot (s) \geq 0 \text{ with } s\left(\rho, e\right) \geq 0 \end{cases}$$

The multi-D Euler equations (with variable area) [13]:

The multi-D Euler Equations (with variable area):

$$
\begin{cases}
\partial_t(\rho A) + \vec{\nabla} \cdot (\rho \vec{u} A) = 0 \\
\partial_t(\rho \vec{u} A) + \vec{\nabla} \cdot (A \rho \vec{u} \otimes \vec{u}) + \vec{\nabla}(AP) = \underbrace{P \vec{\nabla} A}_{\text{extra term}} \\
\partial_t(\rho E A) + \vec{\nabla} \cdot [\vec{u} A (\rho E + P)] = 0 \\
P = eos(\rho, e)
\end{cases}
$$

ρ is the density, P the pressure, E the specific total energy, $\vec{u}$ the velocity, e the internal energy, and A the area.

A few remarks:

- The area A is assumed to be spatially dependent only: $A(\vec{r})$.
- For the purpose of this proposal, $A(\vec{r})$ is assumed to be smooth: 1-D nozzle.
- The eigenvalues are unchanged.

Proposal (1/3): theoretical approach.

1. Derive the dissipative terms using the entropy minimum principle: this step is straightforward with help of [19].
2. Investigate the low Mach regime $\rightarrow$ the entropy residual will be recast under the following form:

$$D_e(\vec{r}, t) = \partial_t s + \vec{u} \cdot \vec{\nabla} s = \frac{s_e}{P_e} \underbrace{\left(\frac{dP}{dt} - c^2 \frac{d\rho}{dt} \right)}_{\tilde{D}_e(\vec{r}, t)}$$

   - $\tilde{D}_e(\vec{r}, t)$ is an alternative way of computing the local entropy production.
   - The viscosity coefficient will be set proportional to $\tilde{D}_e(\vec{r}, t)$ (instead of $D_e(\vec{r}, t)$):

$$\mu_e(\vec{r}, t) = h^2 \frac{\tilde{D}_e(\vec{r}, t)}{f(P)}$$

   - This new expression offers more diversity in the choice of the normalization parameter $f(P)$: P, ρc^2, $\rho c||\vec{u}||$ or $\rho||\vec{u}||^2$.
   - $f(P)$ will be determined by doing a low Mach asymptotic study of the multi-D Euler equations.

Proposal (2/3): perform 1-D tests.

3. Sod shock tubes [13] with Ideal Gas equation of state [23]:

$$P = (\gamma - 1)\rho e$$

4. 1-D nozzle with Stiffened Gas equation of state [24]: vapor phase (supersonic flow) and liquid phase (subsonic flow) NEW.

$$P = (\gamma - 1)\rho(e - q) - \gamma P_\infty$$

   There is an analytical solution for steady-state.

5. Tait equation of state [39] (?) NEW

$$P = P_0 \left(1 - \frac{\rho}{\rho_0}\right)^{\gamma - 1}$$

6. Investigate the effect of source terms on the method. Tests will be performed for a 1-D core channel of a PWR using RELAP-7 code NEW.
   $\rightarrow$ derive the entropy residual with the source terms.

Proposal (3/3): perform 2-D tests.

5. Perform 2-D tests for supersonic flows:
   - Forward facing step [35].
   - Riemann problem 12 [34].
   - Sedov test [13].
   - Compression and decompression corner [36].
6. Perform 2-D tests for subsonic flow: hump (no analytical solution) and cylinder (analytical solution). NEW
   - Hump [37].
   - Cylinder [32].

1-D numerical results: Leblanc shock tube [38].

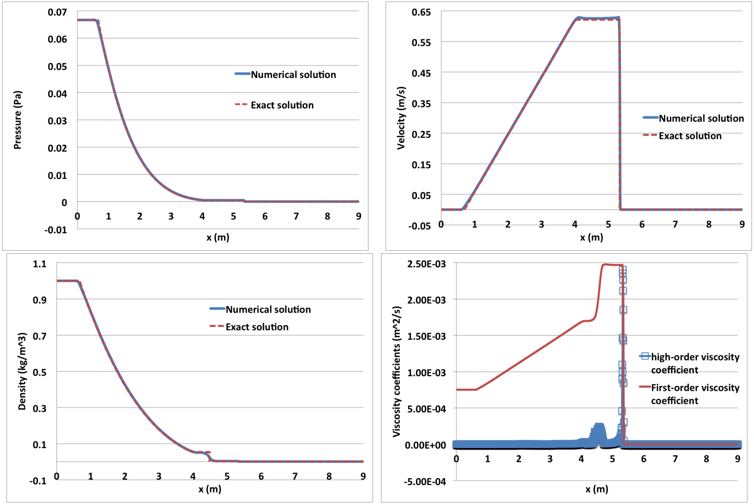

Figure: Numerical solution at $t = 4.s$ with 2000 cells, linear polynomials and BDF2.

2-D numerical results: compression corner ($M = 2.5$).

Conclusions: the multi-D Euler equations.

What we have done so far:

- Derived a definition for the viscosity coefficient that is consistent for supersonic and subsonic flows (hump).
- Tested the new definition of the viscosity coefficients with 1- and 2-D tests.
- Investigated the effect of the source terms on the method and performed a test on a 1-D PWR core channel (RELAP-7).

What is left to do:

- Remains to test the method for a subsonic flow around a cylinder: good test since an analytical steady-state solution is available (appendix).

The multi-D seven-equation model (with variable area) [33]:

The model:

- Each phase obeys the single-phase Euler equations: two continuity equations, two momentum equations and two energy equations.
- Seventh equation: void fraction equation $\rightarrow$ an internal boundary condition between the two phases at the interface.
- Exchange terms between phases: relaxation terms. These terms were derived using *rational thermodynamic* [22] $\rightarrow$ consistent with the entropy minimum principle.
- The system of equations is well-posed, it has 7 real eigenvalues.
- The seven-equation model degenerates to single-phase Euler equations when one phase disappears.

The multi-D seven-equation model (with variable area) [33]:

We consider two phases j, k. Phase k obeys the following system of equations:

$$\begin{cases} \partial_t(\alpha_k A) & + \quad \vec{u}_I A\vec{\nabla}\alpha_k = A\mu_{rel}(P_k - P_j) \\ \partial_t(\alpha_k\rho_k A) & + \quad \vec{\nabla}\cdot(\alpha_k\rho_k\vec{u}_k A) = 0 \\ \partial_t(\alpha_k\rho_k\vec{u}_k A) & + \quad \vec{\nabla}\cdot[\alpha_k A(\rho_k\vec{u}_k\otimes\vec{u}_k)] + \vec{\nabla}(\alpha_k A P_k) = \\ & \qquad \alpha_k P_k\vec{\nabla}A + P_I A\vec{\nabla}\alpha_k + A\lambda_{rel}(\vec{u}_j - \vec{u}_k) \\ \partial_t(\alpha_k\rho_k E_k A) & + \quad \vec{\nabla}\cdot[\alpha_k A\vec{u}_j(\rho_k E_k + P_k)] = \\ & \qquad P_I\vec{u}_I A\vec{\nabla}\alpha_k - \mu_{rel}\bar{P}_I(P_k - P_j) + \bar{\vec{u}}_k A\lambda_{rel}(\vec{u}_j - \vec{u}_k) \end{cases}$$

$$\begin{cases} P_I = \bar{P}_I - \frac{\vec{\nabla}\alpha_k}{|\vec{\nabla}\alpha_k|}\frac{Z_k Z_j}{Z_k+Z_j}\cdot(\vec{u}_k - \vec{u}_j) \\ \bar{P}_I = \frac{Z_k P_j + Z_j P_k}{Z_k+Z_j} \\ \vec{u}_I = \bar{\vec{u}}_I - \frac{\vec{\nabla}\alpha_k}{|\vec{\nabla}\alpha_k|}\frac{P_k-P_j}{Z_k+Z_j} \\ \bar{\vec{u}}_I = \frac{Z_k\vec{u}_k + Z_j\vec{u}_j}{Z_k+Z_j} \end{cases} \quad \text{and} \quad \begin{cases} \mu_{rel} = \frac{A_{int}}{Z_k+Z_j} \\ \lambda_{rel} = \frac{\mu_{rel}}{2}Z_k Z_j \\ A_{int} = 6.25\cdot A_{int,max}\cdot\alpha_k(1-\alpha_k)^2 \end{cases}$$

Proposal: theoretical approach (1/3) NEW.

1. Derive the dissipative terms using the entropy minimum principle. For continuity, momentum and energy equations, the dissipative terms should be similar to what is obtained for Euler equations. The dissipative term of the void-fraction equation should be of the form $\vec{\nabla} \cdot \left(\beta \vec{\nabla} \alpha_k \right)$.
2. Define the viscosity coefficients: μ, κ and β.
3. Investigate the low Mach regime $\rightarrow$ results from Euler equations hold. What about β?

Proposal: 1-D tests (2/3) NEW.

5. 1-D nozzle with Stiffened Gas equation of state [33].

Proposal: 2-D tests (3/3) NEW.

6. Vapor + liquid shock tubes.

1-D numerical results: nozzle.

$$\mu_{rel} \text{ and } \lambda_{rel} \to \infty$$

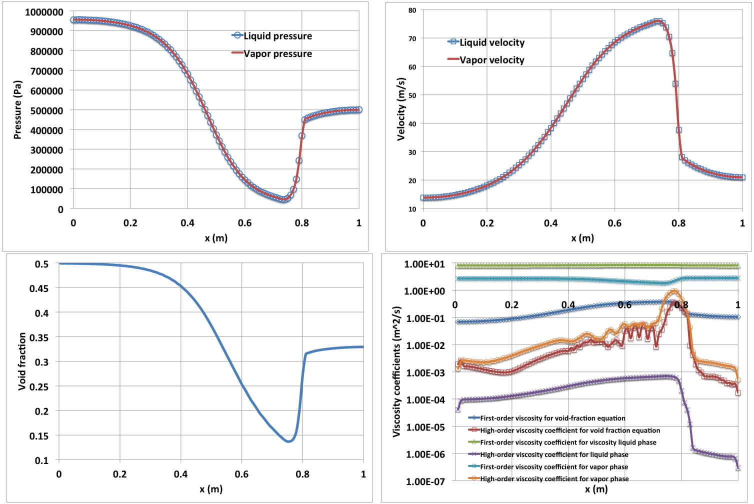

Figure: Steady-state numerical solution with 100 cells, linear polynomials and BDF2.

Conclusions: the multi-D seven-equation model.

What we have done so far:

- Derived the dissipative terms using the entropy minimum principle $\rightarrow$ valid for any equation of state under the same assumptions as for Euler equations.
- Derived a definition for the viscosity coefficients: μ, κ and β.
- Performed tests with a 1-D nozzle.

What is left to do:

- Perform some more tests in 1-D: include mass and heat transfer between phases [33].
- Tests in 2-D.

A Multi-Physics Example: application of the entropy viscosity method to the 1-D grey Radiation-Hydrodynamic equations (RHD).

The 1-D grey Radiation-Hydrodynamic equations (RHD):

RHD system of equations:

$$
\begin{cases}
\partial_t (\rho) + \partial_x (\rho u) = 0 \\
\partial_t (\rho u) + \partial_x \left(\rho u^2 + P\right) = -\partial_x \left(\frac{\epsilon}{3}\right) \\
\partial_t (\rho E) + \partial_x [u (\rho E + P)] = -\frac{u}{3}\partial_x \epsilon - \sigma_a c \left(a T^4 - \epsilon\right) \\
\partial_t \epsilon + \frac{4}{3}\partial_x (u\epsilon) = \frac{u}{3}\partial_x \epsilon + \partial_x \left(\frac{c}{3\sigma_t}\partial_x \epsilon\right) + \sigma_a c \left(a T^4 - \epsilon\right)
\end{cases}
$$

A few remarks:

- The total energy (material+radiation energy density) is conserved.
- The relaxation term $\sigma_a c \left(a T^4 - \epsilon\right)$ behaves like a diffusion term when $\sigma_a \to \infty$ [18].
- The above system of equations is NOT hyperbolic ????

Background:

- RHD are a wave-dominated problem.
- They are known to develop shocks due to the nature of Euler equations.
- Great amount of work available in the literature on how to solve RHD: approximate Riemann solver [16, 5], flux limiter [12], $\cdots$
- Have a common approach: focus on the hyperbolic part of the RHD.
- Attempts to derive a Riemann solver accounting for the source terms.
- Use of semi-implicit schemes because of the difference of characteristic time scale between the two physics $\rightarrow$ implicit scheme has some advantage.

Our approach:

- Consider the hyperbolic part of the system of equations (no source terms).
- Keep in mind that the source terms may affect the entropy viscosity method and their effect will have to be studied later.

Our hyperbolic system of equations:

$$\begin{cases} \partial_t(\rho) + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x\left(\rho u^2 + P + \frac{\epsilon}{3}\right) = 0 \\ \partial_t(\rho E) + \partial_x\left[u(\rho E + P)\right] = 0 \\ \partial_t \epsilon + \frac{4}{3}\partial_x(u\epsilon) - \frac{u}{3}\partial_x \epsilon = 0 \end{cases}$$

Eigenvalues: $\lambda_1 = u - c$, $\lambda_{2,3} = u$ and $\lambda_4 = u + c$ where c is the speed of sound defined as:

$$c^2 = \underbrace{P_\rho + \frac{P}{\rho^2}P_e}_{c^2_{\text{Euler}}} + \underbrace{\frac{4\epsilon}{9\rho}}_{c^2_{\text{rad}}}$$

Proposal: theoretical approach (1/2) NEW.

1. Derive the dissipative terms using the modified system of equations (no source term). The entropy s is assumed to be a function of three variables: the material density ρ and internal energy e, and the radiation density energy ϵ.
2. Define the viscosity coefficient(s).

Proposal: manufactured solutions and 1-D results (2/2) NEW.

1. Study the effect of the source terms, $\partial_x \left(\frac{c}{3\sigma_t} \partial_x \epsilon \right)$ and $\sigma_a c \left(aT^4 - \epsilon \right)$, on the entropy viscosity method using the method of manufactured solutions, and show high-order convergence.
2. Perform 1-D tests for different Mach numbers: Mach= 1.05, 1.2, 2, 5, and 50. Semi-analytical solutions are available.

Numerical results: Mach 1.05.

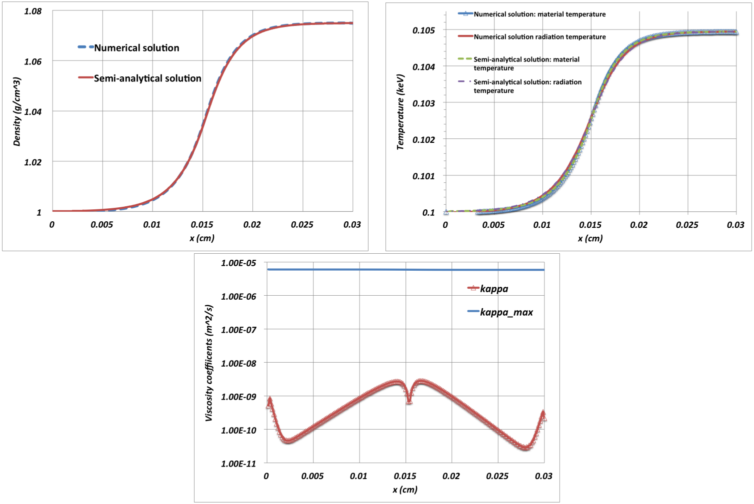

Figure: Steady-state numerical solution with 500 cells, linear polynomials and BDF2.

Numerical results: Mach 5.

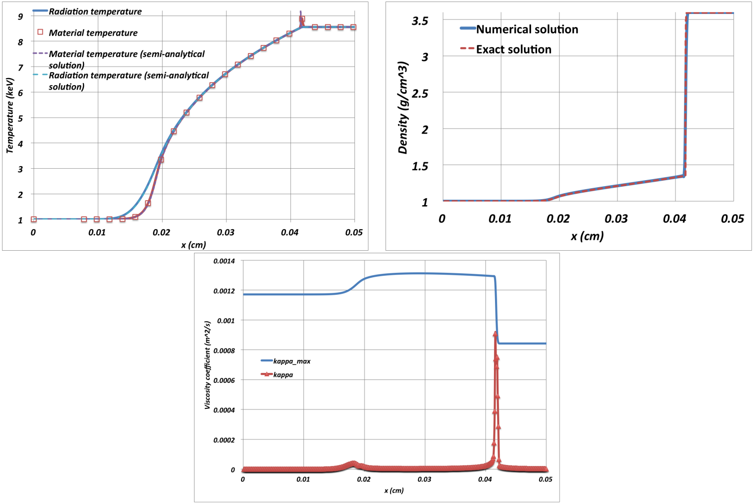

Figure: Steady-state numerical solution with 500 cells, linear polynomials and BDF2.

Conclusions: RHD.

- The entropy viscosity method was successfully applied to the 1-D RHD.
- 1-D numerical results show good agreement with semi-analytical solutions.
- Demonstrated high-order accuracy and correct behavior in the equilibrium diffusion limit.

General conclusions:

- The method can be applied with any equation of state with a convex entropy.
- The method is simple to implement and the viscosity coefficient is computed on the fly.

QUESTIONS/COMMENTS ?

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Viscous regularization for the multi-D Euler equations (with variable area):

$$
\begin{cases}
\partial_t\left(\rho A\right) + \vec{\nabla}\cdot\left(\rho\vec{u}A\right) = \vec{\nabla}\cdot\left(A\kappa\vec{\nabla}\rho\right) \\
\partial_t\left(\rho\vec{u}A\right) + \vec{\nabla}\cdot\left(\rho\vec{u}\otimes\vec{u}A\right) + \vec{\nabla}(PA) = P\vec{\nabla}A + \vec{\nabla}\cdot\left[A\left(\mu\rho\vec{\nabla}^s\vec{u} + \vec{u}\otimes\left(\kappa\vec{\nabla}\rho\right)\right)\right] \\
\partial_t\left(\rho E A\right) + \vec{\nabla}\cdot\left[\vec{u}(\rho E + P)A\right] = \vec{\nabla}\cdot\left[A\left(\kappa\vec{\nabla}(\rho e) + \frac{||\vec{u}||^2}{2}\kappa\vec{\nabla}\rho + \mu\rho\vec{u}\cdot\vec{\nabla}^s\vec{u}\right)\right] \\
P = eos(\rho, e)
\end{cases}
$$

Viscous regularization for the multi-D seven-equation model:

$$
\begin{cases}
\partial_t(\alpha_k A) & +\vec{u}_I A\vec{\nabla}\alpha_k = A\mu_{rel}(P_k - P_j) + \vec{\nabla}\cdot\vec{l} \\
\partial_t(\alpha_k\rho_k A) & +\vec{\nabla}\cdot(\alpha_k\rho_k\vec{u}_k A) = \vec{\nabla}\cdot\vec{f} \\
\partial_t(\alpha_k\rho_k\vec{u}_k A) & +\vec{\nabla}\cdot[\alpha_k A(\rho_k\vec{u}_k\otimes\vec{u}_k)] + \vec{\nabla}(\alpha_k A P_k) = \\
 & \alpha_k P_k\vec{\nabla}A + P_I A\vec{\nabla}\alpha_k + A\lambda_{rel}(\vec{u}_j - \vec{u}_k) + \vec{\nabla}\cdot\bar{g} \\
\partial_t(\alpha_k\rho_k E_k A) & +\vec{\nabla}\cdot[\alpha_k A\vec{u}_j(\rho_k E_k + P_k)] = \\
 & P_I\vec{u}_I A\vec{\nabla}\alpha_k - \mu_{rel}\bar{P}_I(P_k - P_j) + \bar{\vec{u}}_k A\lambda_{rel}(\vec{u}_j - \vec{u}_k) + \vec{\nabla}\cdot\vec{h}
\end{cases}
$$

$$
\begin{cases}
\vec{l} = A\beta_k\vec{\nabla}\alpha_k \\
\vec{f} = \alpha_k A\kappa_k\vec{\nabla}\rho_k + \rho_k\vec{l} \\
\bar{g} = \alpha_k A\rho_k\mu_k\vec{\nabla}\vec{u} + \vec{u}\otimes\vec{f} \\
\vec{h} = \alpha_k A\kappa_k\vec{\nabla}(\rho_k e_k) - \frac{||\vec{u}||^2}{2}\vec{f} + \vec{u}\cdot\bar{g} + \rho_k e_k\vec{l}
\end{cases}
$$

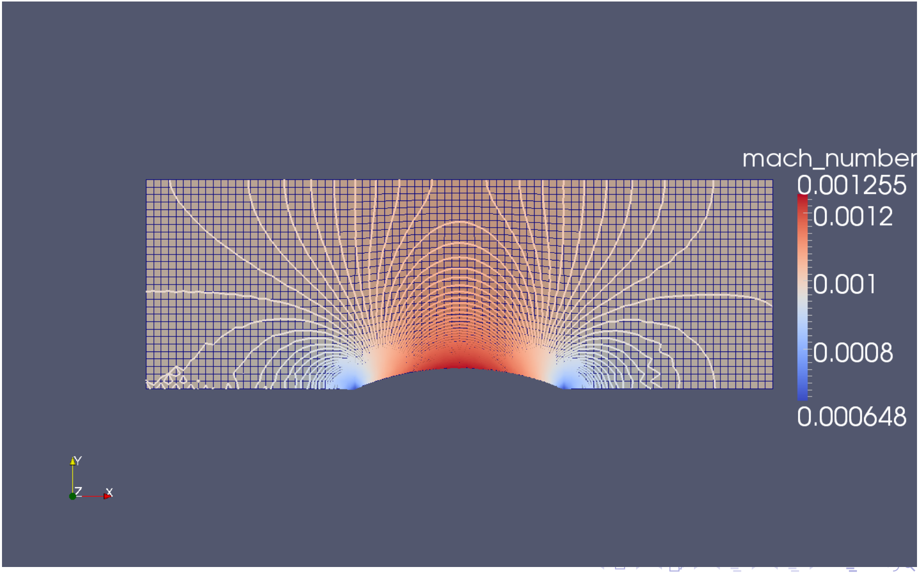
Entropy equation:

$$
\alpha_k A\rho_k\frac{ds_k}{dt} + (\alpha_k A\rho_k\kappa_k + \rho_k l_k)\vec{\nabla}s_k - \vec{\nabla}\cdot\left(\alpha_k A\rho_k\vec{\nabla}s_k\right) =
$$

$$
\partial_e s_k\alpha_k A\rho_k\mu_k\vec{\nabla}^s\vec{u} : \vec{\nabla}\vec{u} - X_k\Sigma_k X_k^t + Q
$$

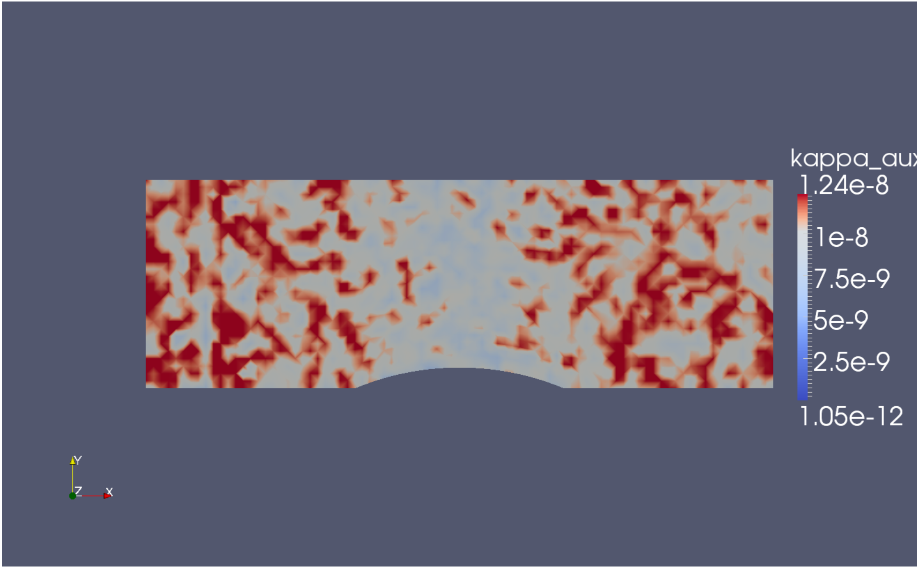
2-D numerical results: hump.

Isomachs for $M = 0.001$ with 2352 elements.



2-D numerical results: hump.

Viscosity for $M = 0.001$ with 2352 elements.



2-D numerical results: cylinder.

Mach number for $M_\infty = 0.05$.