Extension of the entropy viscosity method to the low Mach multi-D Euler equations and the seven-equation model.

by
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# Outline:

1. Introduction

2. The multi-D Euler equations with variable area

introduction $\rightarrow$ hyperbolic system of equations and real life

# Mathematical propoerties

Let us consider a hyperbolic conservation law of the form:

$$\partial_t U + \boldsymbol{\nabla} \cdot F(U) = \mathbf{0} \text{ (strong form)}$$

- wave-dominated problem $\rightarrow$ eigenvalues.
- known to form shocks even with smooth initial conditions: characteristic equation and variable (REFS)
- require a numerical stabilization method to resolve shocks: approximate Riemann solver, artificial dissipative method, flux-limiting method, ...
- uniqueness of the weak solution is ensured by an entropy condition: the associated entropy equation satisfies an inequality and is peaked in the shock region.

$$\partial_t S(U) + \boldsymbol{\nabla} \Phi(U) \geq 0$$

# The general idea behind the entropy viscosity method

*It is an artificial dissipation method with smart viscosity coefficient capable of tracking the shock so that dissipation is only added into the shock region.*

- it requires a viscous regularization: the dissipative terms are consistent with the entropy inequality.

- each viscosity coefficient is function of a high-order viscosity coefficient and an upper bound called first-order viscosity coefficient.

- the high-order viscosity coefficient is defined proportional to the entropy residual.

- the first-order viscosity coefficient is function of the local maximum eigenvalue.

- also accounts for the inter element jumps $\rightarrow$ make the definition of the viscosity coefficients also sensitive to all discontinuities.

# The Multiphysics Object-Oriented Simulations Environment (MOOSE)

- Explicit and implicit temporal integrators (first and second-order accuracy)

- Continuous and Discontinuous Galerkin Finite Element Method

- Mesh adaptivity, time step adaptivity

- Support parallel runs

- Buildt on PETSc and Libmesh

- For implicit solve, requires a preconditioner $\rightarrow$ either FDP or hard-coded preconditioner

- C++ language, opened source

All numerical solutions were run with BDF2 and linear test functions $\rightarrow$ second-order accuracy in time and space.

# A simple example of application of the EVM: the 1-D Burger's equation

## The 1-D Burger's equation with its viscous regularization

$$\partial_t u(x, t) + \partial_x \left( \frac{u(x, t)^2}{2} \right) = \partial_x \left( \mu\left( x, t \right) \partial_x u(x, t) \right)$$

## Definition of the local viscosity coefficient $\mu(x, t)$

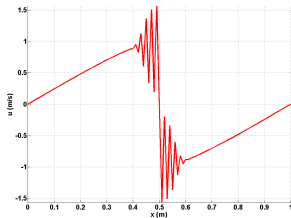$\mu(x, t) = \min\left( \mu_e(x, t), \mu_{max}(x, t) \right)$

$\mu_{max}(x, t) = \frac{h}{2} |u(x, t)|$

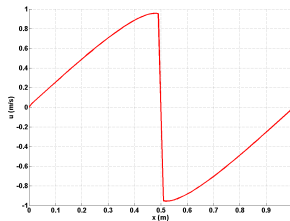$\mu_e(x, t) = h^2 \frac{\max\left( R(x, t), J \right)}{||\eta(u) - \bar{\eta}(t)||_\infty}$

Entropy residual: $R(x, t) = \partial_t \eta(u) + \partial_x \Phi(u) \leq 0$ with $\eta(u) = u^2/2$
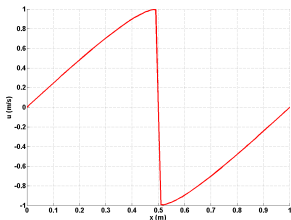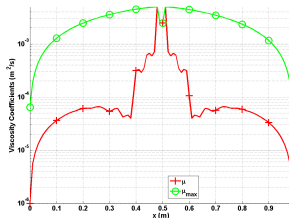
Jump: $J = [[\partial_x \Phi(u)]]$

(a) Without stabilization.



(b) With first-order viscosity.



(c) With the EVM.



(d) Viscosity coefficient profiles.

The multi-D Euler equations with variable area

## Numerical methods for continuous and discontinuous schemes

- multi-wave problem, can develop shock waves and other types of discontinuities.

- Numerical methods for both continuous and discontinuous schemes: approximate Riemann solvers (HLL, HLLC, Roe scheme, $\cdots$), flux limiters , Lapidus viscosity, Pressure-based viscosity, SUPG, C-method, Entropy Viscosity Method (EVM).

- Numerical method can be ill-scaled in the low-Mach limit, yielding the wrong incompressible system $\rightarrow$ use of a Mach-based preconditioner for the dissipative terms to obtain the correct behavior in the low Mach limit.

- Low-Mach steady-state solution: time-dependent term preconditioner to accelerate the convergence of the solution to the steady-state (Turkel) when using an explicit scheme $\rightarrow$ the transient is no longer accurate. *Implicit solvers do not have this issue*.

- The isentropic compressible multi-D Euler equations degenerate into an incompressible system in the low-Mach asymptotic limit.

$$\partial_t \rho + \vec{u} \cdot \boldsymbol{\nabla} \rho = 0$$

$$\partial_t \vec{u} + \vec{u} \, \boldsymbol{\nabla} \cdot \vec{u} + \frac{1}{\rho} \boldsymbol{\nabla} P = 0$$

$$P(\vec{r}, t) = P_0(t) + M^2 P_2(\vec{r}, t)$$

$\rightarrow$ no energy equation since the flow is assumed isentropic
$\rightarrow$ pressure fluctuations of the order of the Mach number square

## The multi-D Euler equations with variable area

Mass conservation
$$\partial_t(\rho A) + \boldsymbol{\nabla}\cdot(\rho A \vec{u}) = 0$$
Momentum conservation
$$\partial_t(\rho \vec{u} A) + \boldsymbol{\nabla}\cdot(\rho A \vec{u} \otimes \vec{u} + P\mathbb{I}) = P\boldsymbol{\nabla} A$$
Energy conservation
$$\partial_t(\rho E A) + \boldsymbol{\nabla}\cdot[\vec{u}(\rho E + P) A] = 0$$
Equation of state
$$P = eos(\rho, e)$$

Multi-wave problem: $\lambda_1 = \vec{u}\cdot\vec{n} - c$, $\lambda_2 = \vec{u}\cdot\vec{n} + c$ and $\lambda_{2,\dots,2+D} = \vec{u}\cdot\vec{n}$.
The area $A$ is only a function of space.

*Objectives: extend the EVM to low-Mach flows while maintaining its capabilities of solving for transonic and supersonic flows, and use an implicit solver.*

## How to do it?

1. derive a viscous regularization for the multi-D Euler equations (already done).

2. work with the non-dimensionalized version of the multi-D Euler equations in order to understand how the different terms scale $\rightarrow$ will define non-dimensionalized numbers (Mach number, numerical Reynolds number, . . . )

3. derive a definition for the viscosity coefficients that ensures well-scaled dissipative terms for a wide range of Mach numbers $\rightarrow$ will consider two cases: isentropic and non-isentropic.

# A viscous regularization for the multi-D Euler equations with variable area

Mass conservation

$$\partial_t(\rho A) + \boldsymbol{\nabla} \cdot (\rho A \vec{u}) = \boldsymbol{\nabla} \cdot \vec{f}$$

Momentum conservation

$$\partial_t(\rho \vec{u} A) + \boldsymbol{\nabla} \cdot (\rho A \vec{u} \otimes \vec{u} + P \mathbb{I}) = P \boldsymbol{\nabla} A + \boldsymbol{\nabla} \cdot \left( \mathbb{F}(\vec{u}) + \vec{u} \otimes \vec{f} \right)$$

Energy conservation

$$\partial_t(\rho E A) + \boldsymbol{\nabla} \cdot [\vec{u} (\rho E + P) A] = \boldsymbol{\nabla} \cdot \left( \vec{h} + \vec{u} \cdot \mathbb{F}(\vec{u}) + \frac{||\vec{u}||^2}{2} f \right)$$

Dissipative terms

$$\vec{f} = A \kappa \boldsymbol{\nabla} \rho, \ \vec{h} = A \kappa \boldsymbol{\nabla}(\rho e) \text{ and } \mathbb{F}(\vec{u}) = A \mu \boldsymbol{\nabla}^s \vec{u} \text{ or } \mathbb{F}(\vec{u}) = A \mu \boldsymbol{\nabla} \vec{u}$$

$\rightarrow$ two positive viscosity coefficients $\mu$ and $\kappa$. Requires a concave physical entropy $s(\rho, e)$.

# Non-dimensionalized multi-D Euler equation

We define some reference variables denoted by subscript $\infty$:

$$\rho^* = \frac{\rho}{\rho_\infty}, \ u^* = \frac{u}{u_\infty}, \ P^* = \frac{P}{\rho_\infty c_\infty^2}, \ E^* = \frac{E}{c_\infty^2},$$

$$x^* = \frac{x}{L_\infty}, \ t^* = \frac{t}{L_\infty/u_\infty}, \ \mu^* = \frac{\mu}{\mu_\infty}, \ \kappa^* = \frac{\kappa}{\kappa_\infty}$$

We also define the following reference numbers:

Mach number: $M_\infty = \dfrac{u_\infty}{c_\infty}$,

Numerical Reynolds number: $\mathrm{Re}_\infty = \dfrac{u_\infty L_\infty}{\mu_\infty}$,

Numerical Péchlet number: $\mathrm{Pé}_\infty = \dfrac{u_\infty L_\infty}{\kappa_\infty}$,

Numerical Prandlt number: $\mathrm{Pr}_\infty = \mathrm{Pé}_\infty/\mathrm{Re}_\infty$

# Non-dimensionalized multi-D Euler equation

$$\partial_{t^*} \rho^* + \boldsymbol{\nabla}^* \cdot (\rho^* \vec{u}^*) = \frac{1}{\text{Pé}_\infty} \boldsymbol{\nabla}^* \cdot (\kappa^* \vec{\nabla}^* \rho^*)$$

$$\partial_{t^*} (\rho^* \vec{u}^*) + \boldsymbol{\nabla}^* \cdot (\rho^* \vec{u}^* \otimes \vec{u}^*) + \frac{1}{M_\infty^2} \vec{\nabla}^* P^* = \frac{1}{\text{Re}_\infty} \boldsymbol{\nabla}^* \cdot \left( \rho^* \mu^* \vec{\nabla}^{s,*} \vec{u}^* \right)$$
$$+ \frac{1}{\text{Pé}_\infty} \boldsymbol{\nabla}^* \cdot \left( \vec{u}^* \otimes \kappa^* \vec{\nabla}^* \rho^* \right)$$

$$\partial_{t^*} (\rho^* E^*) + \boldsymbol{\nabla}^* \cdot [\vec{u}^* (\rho^* E^* + P^*)] = \frac{1}{\text{Pé}_\infty} \boldsymbol{\nabla}^* \cdot \left( \kappa^* \vec{\nabla}^* (\rho^* e^*) \right)$$
$$+ \frac{M_\infty^2}{\text{Re}_\infty} \boldsymbol{\nabla}^* \cdot \left( \vec{u}^* \rho^* \mu^* \vec{\nabla}^{s,*} \vec{u}^* \right) + \frac{M_\infty^2}{2 \text{Pé}_\infty} \boldsymbol{\nabla}^* \cdot \left( \kappa^* (u^*)^2 \vec{\nabla}^* \rho^* \right)$$

The above equations are valid for both isentropic and non-isentropic flows and for all Mach numbers.