

# Entropy-based viscous regularization for the multi-dimensional Euler equations in low-Mach and transonic flows

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## Abstract

We present a new version of the entropy viscosity method, a viscous regularization technique for hyperbolic conservation laws, that is well-suited to low-Mach flows. By means of a low-Mach asymptotic study, new expressions for the entropy viscosity coefficients are derived. These definitions are valid for a wide range of Mach numbers, from subsonic flows (with very low Mach numbers) to supersonic flows, and no longer depend on an analytical expression for the entropy function. In addition, the entropy viscosity method is extended to Euler equations with variable area for nozzle flow problems. The effectiveness of the method is demonstrated using various 1-D and 2-D benchmark tests: flow in a converging-diverging nozzle; Leblanc shock tube; slow moving shock; strong shock for liquid phase; low-Mach flows around a cylinder and over a circular hump; and supersonic flow in a compression corner. Convergence studies are performed for both smooth solutions and solutions with shocks present.

*Key words:* entropy viscosity method, artificial viscosity, low-Mach regime, shock capturing, Euler equations with variable area.

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## 1. Introduction

Solving accurately compressible fluid equations in the low-Mach limit is an ongoing topic of research. In many engineering applications, compressibility effects require the solution of the compressible fluid equations in nearly incompressible regimes and/or for low-Mach flow problems. For example, such flows are encountered in aerodynamics in the study of airships. In the nuclear industry, flows are nearly in the incompressible regime but compressible effects cannot be neglected because of the intense heat source, and because of some postulated accident scenarios, and thus need to be accurately resolved. Hence, there is a strong interest to develop computational methods that can solve both compressible and incompressible flow problems.

When solving Euler equations for a wide range of Mach numbers, multiple questions must be addressed: stability, accuracy and solution convergence in the low-Mach regime. Because of the hyperbolic nature of the equations, shocks can form during transonic and supersonic flows and require the use of adequate numerical techniques to stabilize solution and correctly resolve the discontinuities. A wide range of stabilization methods are available in the literature: approximate Riemann solvers [1], flux-limiter techniques [2, 3], pressure-based viscosity methods [4], Lapidus method [5, 6, 7], and the entropy-viscosity method [8, 9], among others. These numerical methods are usually developed using simple equations of state and tested for transonic and supersonic flows where the disparity between the acoustic wave speed and the fluid speed is not excessively large and thus the Mach number is of order one. This approach, however, leads to a well-known accuracy problem in the low-Mach regime where the fluid velocity is smaller than the speed of sound by multiple orders of magnitude. The numerical dissipative terms become ill-scaled in the low-Mach regime and lead to the wrong numerical solution by changing the nature of the equations

28 solved. This behavior is well documented in the literature [10, 11, 12]. In [10], a  
 29 low-Mach asymptotic study has demonstrated convergence of the compressible  
 30 Euler equations to the incompressible ones. Many well-known stabilization tech-  
 31 niques, for instance, the Roe scheme and the SUPG technique, do not yield the  
 32 correct solution in the low-Mach regime and suitable modifications have been  
 33 proposed (see [13] for the Roe scheme and [12] for the SUPG method) to en-  
 34 sure the convergence to the correct solution while preserving the original shock  
 35 stabilization properties of these schemes. Additionally, the time step size may  
 36 be severely restricted when solving compressible fluid equations with an explicit  
 37 time discretization because of the large disparity between the fluid velocity and  
 38 the speed of sound. To avoid an excessive number of explicit time steps, time  
 39 preconditioners have been proposed and proved efficient [11]; however, because  
 40 they modify the time derivatives in the governing equations, such acceleration  
 41 techniques can only be used to obtain steady-state solutions for low-Mach flows  
 42 using explicit schemes. To avoid modifying the time derivatives, the temporal  
 43 implicit capabilities of the MOOSE multiphysics framework [14] are used. With  
 44 such a choice, low-Mach steady-state solutions can be obtained effectively while  
 45 preserving the accuracy of the transient solution; however, it requires the use of  
 46 nonlinear solvers.

47 In this paper, we employ the entropy viscosity method as a numerical sta-  
 48 bilization for the inviscid Euler equation and assess its performance in the low-  
 49 Mach regime. The entropy viscosity method is a viscous regularization technique  
 50 introduced by Guermond et al. to solve hyperbolic systems of equations and  
 51 has successfully been applied to multi-dimensional supersonic flows with various  
 52 spatial discretization schemes [15]. It is fairly straightforward to implement, can  
 53 be used with unstructured grids, and has dissipative terms that are consistent  
 54 with the entropy minimum principle. However, it has not been evaluated in the

55 low-Mach regime.

56 This paper is organized as follows: in Section 2 the current definition of the  
57 entropy viscosity method is recalled and its ill-scaled nature in the low-Mach  
58 regime is discussed. In Section 3, a new formulation of the viscosity residual  
59 is derived. This formulation no longer requires an analytical expression for the  
60 entropy function. A low-Mach asymptotic study is carried out to adapt the  
61 definition of the entropy viscosity coefficients in the incompressible limit while  
62 ensuring that the viscosity coefficients scale appropriately for all flow speeds  
63 (from low-Mach to supersonic). In Section 4, we extend the entropy viscosity  
64 method to Euler equations with variable area in order to model nozzle flows:  
65 the viscous dissipative terms are adapted so that the entropy minimum principle  
66 remains satisfied. Spatial and temporal discretizations and solution techniques  
67 are presented in Section 5. 1-D and 2-D numerical results are provided in  
68 Section 6 for a wide range of Mach numbers: liquid and gas nozzle flow problems,  
69 low-Mach flows over a cylinder and a circular bump (with Mach numbers as  
70 low as  $10^{-7}$ ), and supersonic flows in a compression corner [16]. Convergence  
71 studies are performed in 1-D in order to demonstrate the accuracy of the solution  
72 technique.

## 73 2. The Entropy Viscosity Method

### 74 2.1. Background

75 Euler equations in conservative form are given by

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \quad (1a)$$

76

$$\partial_t (\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u} + P \mathbb{I}) = 0 \quad (1b)$$

77

$$\partial_t (\rho E) + \vec{\nabla} \cdot [\vec{u} (\rho E + P)] = 0 \quad (1c)$$

78 where  $\rho$ ,  $\rho\vec{u}$  and  $E$  are the density, the momentum and the total specific energy,  
 79 respectively, and will be referred to as the conservative variables.  $\vec{u}$  is the fluid  
 80 velocity and its specific internal energy is denoted by  $e = E - \frac{u^2}{2}$ . An equation  
 81 of state, dependent upon  $\rho$  and  $e$ , is used to compute the pressure  $P$ . The tensor  
 82 product  $\vec{a} \otimes \vec{b}$  is such that  $(\vec{a} \otimes \vec{b})_{i,j} = a_i b_j$ . The identity tensor is denoted by  $\mathbb{I}$ .

83 Next, the entropy viscosity method [8, 9, 17, 18] applied to Eq. (1) is recalled.  
 84 The method consists of adding dissipative terms with a viscosity coefficient mod-  
 85 ulated by the entropy production; this allows for a high-order accuracy when  
 86 the solution is smooth (provided that the spatial and temporal discretizations  
 87 also are high order). The derivation of the viscous regularization (or dissipa-  
 88 tive terms) is carried out to be consistent with the entropy minimum principle;  
 89 details and proofs of the derivation can be found in [15]. The viscous regular-  
 90 ization thus obtained is valid for any equation of state as long as the physical  
 91 entropy function  $s$  is concave (or  $-s$  is a convex function) with respect to the  
 92 internal energy  $e$  and the specific volume  $1/\rho$ . The Euler equations with viscous  
 93 regularization become

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{u}) = \vec{\nabla} \cdot (\kappa \vec{\nabla} \rho) \quad (2a)$$

$$\partial_t (\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u} + P \mathbb{I}) = \vec{\nabla} \cdot (\mu \rho \vec{\nabla}^s \vec{u} + \kappa \vec{u} \otimes \vec{\nabla} \rho) \quad (2b)$$

$$\partial_t (\rho E) + \vec{\nabla} \cdot [\vec{u} (\rho E + P)] = \vec{\nabla} \cdot \left( \kappa \vec{\nabla} (\rho e) + \frac{1}{2} \|\vec{u}\|^2 \kappa \vec{\nabla} \rho + \rho \mu \vec{u} \vec{\nabla}^s \vec{u} \right) \quad (2c)$$

96 where  $\kappa$  and  $\mu$  are positive viscosity coefficients (in units of  $\text{length}^2/\text{time}$ ).  $\vec{\nabla}^s \vec{u}$   
 97 denotes the symmetric gradient operator and guarantees the method to be ro-  
 98 tationally invariant [15]. The viscosity coefficients are key ingredients in the  
 99 viscous regularization of Eq. (2). Other stabilization approaches have been pro-  
 100 posed in the literature, for instance, the Lapidus method [7, 5] or pressure-based  
 101 viscosity methods [4]. Here, we follow the work of Guermond et al. and define

102 the viscosity coefficients,  $\kappa$  and  $\mu$ , based on the local entropy production. These  
 103 coefficients are numerically evaluated using the local entropy residual  $R_{\text{ent}}(\vec{r}, t)$   
 104 defined in Eq. (3);  $R_{\text{ent}}(\vec{r}, t)$  is known to be peaked in shocks and vanishingly  
 105 small elsewhere [1].

$$R_{\text{ent}}(\vec{r}, t) := \partial_t s + \vec{u} \cdot \vec{\nabla} s \quad (3)$$

106 In the current version of the method, the ratio of  $\kappa$  to  $\mu$  is defined through  
 107 a numerical Prandtl number,  $\text{Pr} = \kappa/\mu$ .  $\text{Pr}$  is a user-defined parameter and  
 108 is usually taken in the range  $[0.001; 1]$ . Since the entropy residual  $R_{\text{ent}}(\vec{r}, t)$   
 109 may be extremely large in shocks, the definition of the viscosity coefficients  
 110 also includes a first-order viscosity coefficient that serves as an upper bound for  
 111 the entropy-based viscosity coefficients. The first-order viscosity coefficients,  
 112 denoted by  $\mu_{\text{max}}$  and  $\kappa_{\text{max}}$ , are chosen so that the numerical scheme becomes  
 113 equivalent to an upwind scheme when the first-order coefficients are employed.  
 114 The upwind scheme is known to be over-dissipative but guarantees monotonicity  
 115 [1]. In practice, the viscosity coefficients only saturate to the first-order viscosity  
 116 coefficients in shocks and are much smaller elsewhere, hence avoiding the over-  
 117 dissipation of the upwind method. The first-order viscosity coefficients  $\mu_{\text{max}}$   
 118 and  $\kappa_{\text{max}}$  are equal and set proportional to the largest local eigenvalue  $\|\vec{u}\| + c$ :

$$\mu_{\text{max}}(\vec{r}, t) = \kappa_{\text{max}}(\vec{r}, t) = \frac{h}{2} (\|\vec{u}(t, \vec{r})\| + c(t, \vec{r})), \quad (4)$$

119 where  $h$  denotes the local grid size (for higher than linear finite element rep-  
 120 resentations,  $h$  is defined as the ratio of the grid size to the polynomial order  
 121 of the test functions used, see Eq. 2.4 in [18]). For simplicity, the first-order  
 122 viscosity coefficients will only be referred to as  $\kappa_{\text{max}}(\vec{r}, t)$ . In practice, these

quantities are evaluated within a given cell  $K$  at quadrature points:

$$\kappa_{\max}^K(\vec{r}_q, t) = \frac{h_K}{2} \left( \|\vec{u}(t, \vec{r}_q)\| + c(t, \vec{r}_q) \right), \quad (5)$$

where  $\vec{r}_q$  denotes the position of a quadrature point. As stated earlier, the entropy viscosity coefficients, which we denote by  $\kappa_e$  and  $\mu_e$ , are set proportional to the entropy production evaluated by computing the local entropy residual  $R_{\text{ent}}$ . The definitions also include the inter-element jump  $J[s]$  of the entropy flux, allowing for the detection of discontinuities other than shocks (e.g., contact).  $\kappa_e$  and  $\mu_e$  are computed as follows

$$\mu_e^K(\vec{r}_q, t) = h_K^2 \frac{\max(|R_{\text{ent}}^K(\vec{r}_q, t)|, J^K[s](t))}{\|s - \bar{s}\|_\infty} \quad (6a)$$

$$\kappa_e^K(\vec{r}_q, t) = \text{Pr} \mu_e^K(\vec{r}_q, t), \quad (6b)$$

where  $\|\cdot\|_\infty$  and  $\bar{\cdot}$  denote the  $L_\infty$ -norm and the average operator over the entire computational domain, respectively. The definition of the entropy jump  $J[s]$  is spatial discretization-dependent and examples of definitions can be found in [18] for discontinuous Galerkin discretization. For continuous finite element methods (FEM), the jump of a given quantity is defined as the change of its normal derivative ( $\partial_n(\cdot) = \vec{n} \cdot \vec{\nabla}(\cdot)$ ) across the common face separating the two elements, and will be further referred to as the inter-element jump. We take the largest value over all faces  $f$  present on the boundary  $\partial K$  of element  $K$ :

$$J^K[s](t) = \max_{f \in \partial K} \max_{\vec{r}_q \in f} \left( \|\vec{u}(\vec{r}_q, t)\| \|\vec{\nabla} s(\vec{r}_q, t) \cdot \vec{n}(\vec{r}_q)\|_f \right), \quad (7)$$

where  $\llbracket a(\vec{r}_q) \rrbracket_f$  denotes the inter-element jump in  $a(\vec{r})$  at quadrature point  $\vec{r}_q$  on face  $f$  (the quadrature points  $\vec{r}_q$  are taken on the faces  $f$  of the element  $K$ ). With the definition given in Eq. (7), the jump is constant over each el-

ement  $K$  of the computational domain. The denominator  $\|s - \bar{s}\|_\infty$  is used for dimensionality purposes. Currently, there are no theoretical justifications for choosing the denominator beyond a dimensionality argument. Finally, the viscosity coefficients  $\mu$  and  $\kappa$  are as follows:

$$\mu(\vec{r}, t) = \min\left(\mu_e(\vec{r}, t), \mu_{\max}(\vec{r}, t)\right)$$

$$\text{and } \kappa(\vec{r}, t) = \min\left(\kappa_e(\vec{r}, t), \kappa_{\max}(\vec{r}, t)\right). \quad (8)$$

139 Given these definitions, we have the following properties. In shock regions, the  
 140 entropy viscosity coefficients will experience a peak because of entropy produc-  
 141 tion and thus will saturate to the first-order viscosity. The first-order coefficients  
 142 are known to be over-dissipative and will smooth out any oscillatory behavior.  
 143 Elsewhere in the domain, entropy production will be small and the viscosity  
 144 coefficients  $\mu$  and  $\kappa$  will remain small. High-order accuracy for entropy-based  
 145 viscous stabilization has been demonstrated using several 1-D shock tube ex-  
 146 amples and various 2-D tests [8, 9, 18].

## 147 2.2. Issues in the Low-Mach Regime

148 In the low-Mach Regime, a smooth flow is known to approach the isentropic  
 149 limit, resulting in very little entropy production. Since the entropy viscosity  
 150 method is directly based on the evaluation of the local entropy production, it  
 151 is of interest to study how the entropy viscosity coefficients  $\mu_e$  and  $\kappa_e$  scale  
 152 in the low-Mach regime. In practice, the entropy residual  $R_{\text{ent}}$  will be very  
 153 small in that regime and so will be the denominator  $\|s - \bar{s}\|_\infty$ , thus making  
 154 the definition of the viscosity coefficients in Eq. (6) undetermined and likely ill-  
 155 scaled. One possible approach would consist of expanding the numerator and  
 156 denominator in terms of the Mach number and deriving its limit when the Mach  
 157 number goes to zero. Such derivation may not be straightforward, especially



for general equations of state. However, this can be avoided by noting that the entropy residual  $R_{\text{ent}}$  can be recast as a function of pressure, density, velocity, and speed of sound as will be shown in Eq. (9) of Section 3.1. This alternate entropy residual definition is the basis for the low-Mach analysis carried out in this paper and possesses several advantages that are detailed next.

### 3. An All-speed Reformulation of the Entropy Viscosity Method

In this section, the entropy residual  $R_{\text{ent}}$  is recast as a function of pressure, density, velocity and speed of sound. Then, a low-Mach asymptotic study is carried out for the Euler equations with viscous regularization in order to derive an appropriate normalization parameter that is valid in the isentropic low-Mach regime as well as for transonic and supersonic flows.

#### 3.1. New Definition of the Entropy Production Residual

The first step in defining viscosity coefficients that behave well in the low-Mach limit is to recast the entropy residual in terms of thermodynamic variables. This provides physical insight on possible normalization choices that can be valid in both low-Mach and transonic flows. The alternate definition of the entropy residual, the derivation of which is given in Appendix A, is the following:

$$R_{\text{ent}}(\vec{r}, t) := \partial_t s + \vec{u} \cdot \vec{\nabla} s = \frac{Ds}{Dt} = \frac{s_e}{P_e} \left( \underbrace{\frac{DP}{Dt} - c^2 \frac{D\rho}{Dt}}_{\tilde{R}_{\text{ent}}(\vec{r}, t)} \right), \quad (9)$$

where  $\frac{D}{Dt}$  denotes the material derivative ( $\frac{D}{Dt} := \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}$ ), and  $x_y$  is the standard shorthand notation for the partial derivative of  $x$  with respect to  $y$ , e.g.,  $P_e := \frac{\partial P}{\partial e}$ . When using the SGEOS, the coefficient of proportionality  $\frac{s_e}{P_e}$  can be recast under the form  $\frac{s_e}{P_e} = \frac{C_v(\gamma-1)}{P+\gamma P_\infty}$  which remains positive. Note that the definition of the speed of sound,  $c^2 = \left. \frac{\partial P}{\partial \rho} \right|_s$ , can be used to show that

180  $\tilde{R}_{\text{ent}}$  is zero in isentropic flow regions. The entropy residuals  $R_{\text{ent}}$  and  $\tilde{R}_{\text{ent}}$  are  
 181 proportional to one another and will experience similar variations in space and  
 182 time. Thus, one may elect to employ  $\tilde{R}_{\text{ent}}$  instead of  $R_{\text{ent}}$  for the evaluation  
 183 of the local entropy residual. The new expression presents several advantages  
 184 which include:

- 185 • An analytical expression of the entropy function  $s$  is no longer needed: the  
 186 residual  $\tilde{R}_{\text{ent}}$  is evaluated using the local values of pressure, density, ve-  
 187 locity and speed of sound. Deriving an entropy function for some complex  
 188 equations of state may be difficult;
- 189 • Suitable normalizations for the residual  $\tilde{R}_{\text{ent}}$  can be devised. Examples  
 190 include the pressure itself or combinations of the density, the speed of  
 191 sound and the norm of the velocity, i.e.,  $\rho c^2$ ,  $\rho c||\vec{u}||$  or  $\rho||\vec{u}||^2$ .

192 Denoting the normalization of  $\tilde{R}_{\text{ent}}$  by  $\text{norm}_P$ , the entropy-based viscosity co-  
 193 efficients  $\mu_e$  and  $\kappa_e$  can be re-defined as follows:

$$\mu_e^K(\vec{r}_q, t) = h_K^2 \frac{\max(|\tilde{R}_{\text{ent}}^K(\vec{r}_q, t)|, J^K(t))}{\text{norm}_P^\mu}, \quad (10a)$$

194 and

$$\kappa_e^K(\vec{r}_q, t) = h_K^2 \frac{\max(|\tilde{R}_{\text{ent}}^K(\vec{r}_q, t)|, J^K(t))}{\text{norm}_P^\kappa}, \quad (10b)$$

195 where

$$J^K(t) = \max_{f \in \partial K} \max_{\vec{r}_q \in f} ||\left(\vec{u}(\vec{r}_q, t) || \max(J^K[P](t), c^2(\vec{r}_q, t) || J^K[\rho](t))\right) \quad (10c)$$

196 Note that now the jump operator  $J^K$  acts on the variables appearing in  $\tilde{R}_{\text{ent}}$ ,  
 197 namely, pressure and density. The residual  $\tilde{R}_{\text{ent}}$  and the pressure jump will  
 198 only contribute to the viscosity coefficients in the shock region. On the other  
 199 hand, the density jump will contribute both in contact and shock regions. The

200  $\mu$  and  $\kappa$  coefficients are kinematic viscosities (units of  $m^2/s$ ); the normalization  
 201 parameters  $\text{norm}_P$  are thus in units of pressure, hence the use of the subscript  $P$ .  
 202 Note also that we are not requiring the same normalization for both  $\mu_e$  and  $\kappa_e$   
 203 so the entropy viscosity coefficients can be different. The low-Mach asymptotic  
 204 study presented next will determine the proper normalization.

### 205 *3.2. Asymptotic Study in the Low-Mach Regime*

206 The Euler equations with viscous stabilization, Eq. (6), bear some simi-  
 207 larities with the Navier-Stokes equations in the sense that dissipative terms  
 208 (containing second-order spatial derivatives) are present in both sets of equa-  
 209 tions. An abundant literature exists regarding the low-Mach asymptotic of the  
 210 Navier-Stokes equations [10, 11, 12, 19]. The asymptotic study presented here  
 211 is inspired by the work of Muller et al. [19] where an asymptotic derivation for  
 212 the Navier-Stokes was presented. We remind the reader that the objective is to  
 213 determine appropriate scaling for the entropy viscosity coefficients so that the  
 214 dissipative terms remain well-scaled for two limit cases: (i) the isentropic low-  
 215 Mach limit where Euler equations degenerate to an incompressible system of  
 216 equations in the low-Mach limit and (ii) the non-isentropic limit with formation  
 217 of shocks. The isentropic limit of the Euler equations with viscous regular-  
 218 ization should yield incompressible fluid flow solutions in the low-Mach limit,  
 219 namely, that the spatial steady-state pressure variations are of the order  $M^2$   
 220 and that the velocity satisfies the divergence constraint  $\vec{\nabla} \cdot \vec{u}_0 = 0$  [10, 11, 12].  
 221 For non-isentropic situations, shocks may form for any value of Mach number  
 222 and the minimum entropy principle should still be satisfied so that numerical  
 223 oscillations, if any, be controlled by the entropy viscosity method independently  
 224 of the value of the Mach number. Our objective is to determine the appropriate  
 225 scaling for  $\text{norm}_P^\kappa$  and  $\text{norm}_P^\mu$  in these two limit cases.

The first step in the study of the limit cases (i) and (ii) is to re-write Eq. (2)

in a non-dimensional manner. To do so, the following variables are introduced:

$$\begin{aligned} \rho^* &= \frac{\rho}{\rho_\infty}, \quad u^* = \frac{u}{u_\infty}, \quad P^* = \frac{P}{\rho_\infty c_\infty^2}, \quad E^* = \frac{E}{c_\infty^2}, \\ x^* &= \frac{x}{L_\infty}, \quad t^* = \frac{t}{L_\infty/u_\infty}, \quad \mu^* = \frac{\mu}{\mu_\infty}, \quad \kappa^* = \frac{\kappa}{\kappa_\infty}, \end{aligned} \quad (11)$$

where the subscript  $\infty$  denote the far-field or stagnation quantities and the superscript  $*$  stands for the non-dimensional variables. The far-field reference quantities are chosen such that the dimensionless flow quantities are of order 1. The reference Mach number is given by

$$M_\infty = \frac{u_\infty}{c_\infty}, \quad (12)$$

where  $c_\infty$  is a reference value for the speed of sound. Then, the scaled Euler equations with viscous regularization are:

$$\partial_{t^*} \rho^* + \vec{\nabla}^* \cdot (\rho^* \vec{u}^*) = \frac{1}{\text{Pé}_\infty} \vec{\nabla}^* \cdot (\kappa^* \vec{\nabla}^* \rho^*) \quad (13a)$$

$$\begin{aligned} \partial_{t^*} (\rho^* \vec{u}^*) + \vec{\nabla}^* \cdot (\rho^* \vec{u}^* \otimes \vec{u}^*) + \frac{1}{M_\infty^2} \vec{\nabla}^* P^* &= \frac{1}{\text{Re}_\infty} \vec{\nabla}^* \cdot (\rho^* \mu^* \vec{\nabla}^{s,*} \vec{u}^*) \\ &+ \frac{1}{\text{Pé}_\infty} \vec{\nabla}^* \cdot (\vec{u}^* \otimes \kappa^* \vec{\nabla}^* \rho^*) \end{aligned} \quad (13b)$$

$$\begin{aligned} \partial_{t^*} (\rho^* E^*) + \vec{\nabla}^* \cdot [\vec{u}^* (\rho^* E^* + P^*)] &= \frac{1}{\text{Pé}_\infty} \vec{\nabla}^* \cdot (\kappa^* \vec{\nabla}^* (\rho^* e^*)) \\ &+ \frac{M_\infty^2}{\text{Re}_\infty} \vec{\nabla}^* \cdot (\vec{u}^* \rho^* \mu^* \vec{\nabla}^{s,*} \vec{u}^*) + \frac{M_\infty^2}{2\text{Pé}_\infty} \vec{\nabla}^* \cdot (\kappa^* (u^*)^2 \vec{\nabla}^* \rho^*) \end{aligned} \quad (13c)$$

232 where the numerical Reynolds ( $\text{Re}_\infty$ ) and Péclet ( $\text{Pé}_\infty$ ) numbers are defined as:

$$\text{Re}_\infty = \frac{u_\infty L_\infty}{\mu_\infty} \text{ and } \text{Pé}_\infty = \frac{u_\infty L_\infty}{\kappa_\infty} . \quad (14)$$

233 Note that the Prandtl number used in the original version of the entropy vis-  
234 cosity method is simply given by

$$\text{Pr}_\infty = \text{Pé}_\infty / \text{Re}_\infty . \quad (15)$$

235 The numerical Reynolds and Péclet numbers defined in Eq. (14) are related to  
236 the entropy viscosity coefficients  $\mu_\infty$  and  $\kappa_\infty$ . Thus, once a scaling (in powers of  
237  $M_\infty$ ) is obtained for  $\text{Re}_\infty$  and  $\text{Pé}_\infty$ , the corresponding normalization parameters  
238  $\text{norm}_P^\mu$  and  $\text{norm}_P^\kappa$  will automatically be set. For brevity, the superscripts \* are  
239 omitted in the remainder of this section.

240 For simplicity, we use here the ideal gas equation of state; its non-dimensionalized  
241 expression is given by

$$P^* = (\gamma - 1) \rho^* \left( E^* - \frac{1}{2} M_\infty^2 (u^*)^2 \right) = (\gamma - 1) \rho^* e^* . \quad (16)$$

242 In the low-Mach isentropic limit, shocks cannot form and the compressible  
243 Euler equations are known to converge to the incompressible equations when the  
244 Mach number tends to zero. When adding dissipative terms, as is the case with  
245 the entropy viscosity method, the main properties of the low-Mach asymptotic  
246 limit must be preserved. We begin by expanding each variable in powers of the  
247 Mach number. As an example, the expansion for the pressure is given by:

$$P(\vec{r}, t) = P_0(\vec{r}, t) + P_1(\vec{r}, t) M_\infty + P_2(\vec{r}, t) M_\infty^2 + \dots \quad (17)$$

248 By studying the resulting momentum equations for various powers of  $M_\infty$ , it is

249 observed that the leading order and first-order pressure terms,  $P_0$  and  $P_1$ , are  
 250 spatially constant if and only if  $\text{Re}_\infty = \text{Pe}_\infty = 1$ . In this case, we have at order  
 251  $M_\infty^{-2}$ :

$$\vec{\nabla} P_0 = 0 \quad (18a)$$

252 and at order  $M_\infty^{-1}$

$$\vec{\nabla} P_1 = 0. \quad (18b)$$

253 Using the scaling  $\text{Re}_\infty = \text{Pe}_\infty = 1$ , the leading-order expressions for the conti-  
 254 nuity, momentum, and energy equations are:

$$\partial_t \rho_0 + \vec{\nabla} \cdot (\rho \vec{u})_0 = \vec{\nabla} \cdot (\kappa \vec{\nabla} \rho)_0 \quad (19a)$$

255

$$\partial_t (\rho \vec{u})_0 + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u})_0 + \vec{\nabla} P_2 = \vec{\nabla} \cdot (\rho \mu \vec{\nabla}^s \vec{u} + \kappa \vec{u} \otimes \vec{\nabla} \rho)_0 \quad (19b)$$

256

$$\partial_t (\rho E)_0 + \vec{\nabla} \cdot [\vec{u} (\rho E + P)]_0 = \vec{\nabla} \cdot (\kappa \vec{\nabla} (\rho e))_0 \quad (19c)$$

257 where the notation  $(fg)_0$  means that we only keep the 0<sup>th</sup>-order terms in the  
 258 product  $fg$ . The leading-order of the equation of state is given by

$$P_0 = (\gamma - 1)(\rho E)_0. \quad (20)$$

259 Using Eq. (20), the energy equation can be recast as a function of the leading-  
 260 order pressure,  $P_0$ , as follows:

$$\partial_t P_0 + \gamma \vec{\nabla} \cdot (\vec{u} P)_0 = \vec{\nabla} \cdot (\kappa \vec{\nabla} (P))_0. \quad (21)$$

261 From Eq. (18a), we infer that  $P_0$  is spatially constant. Thus, Eq. (21) becomes

$$\frac{1}{\gamma P_0} \frac{dP_0}{dt} = -\vec{\nabla} \cdot \vec{u}_0 \quad (22)$$

262 and, at steady state, we have

$$\vec{\nabla} \cdot \vec{u}_0 = 0. \quad (23)$$

263 That is, the leading-order of velocity is divergence-free. The same reasoning can  
 264 be applied to the leading-order of the continuity equation (Eq. (19a)) to show  
 265 that the material derivative of the density variable is zero:

$$\frac{D\rho_0}{Dt} := \partial_t \rho_0 + \vec{u}_0 \cdot \vec{\nabla} \rho_0 = 0. \quad (24)$$

266 Therefore, we conclude that by setting the Reynolds and Péclet numbers to  
 267 one, the incompressible fluid results are retrieved in the low-Mach limit when  
 268 employing the compressible Euler equations with viscous regularization terms  
 269 present. In addition, the scaling of the Prandtl number can also be obtained  
 270 using Eq. (15), hence clarifying the use of the numerical Prandtl in the original  
 271 entropy viscosity method [8].

### 272 3.3. Scaling of $Re_\infty$ and $Pé_\infty$ for non-isentropic flows

273 Next, we consider the non-isentropic case. Recall that even subsonic flows  
 274 can present shocks (for instance, a step initial condition in the pressure will trig-  
 275 ger shock formation, independently of the Mach number). The non-dimensional  
 276 form of the Euler equations given in Eq. (13) provides some insight on the dom-  
 277 inant terms as a function of the Mach number. This is particular obvious in  
 278 the momentum equation, Eq. (13b), where the gradient of pressure is scaled by  
 279  $1/M_\infty^2$ . In the non-isentropic case, we no longer have  $\frac{\vec{\nabla} P}{M_\infty^2} = \vec{\nabla} P_2$  and there-  
 280 fore the pressure gradient term may need to be stabilized by some dissipative  
 281 terms of the same scaling so as to prevent spurious oscillations from forming.  
 282 By inspecting the dissipative terms presents in the the momentum equation,  
 283 having a dissipative term that scales as  $1/M_\infty^2$  leads to the following three op-  
 284 tions: (a)  $Re_\infty = M_\infty^2$  and  $Pé_\infty = 1$ , (b)  $Re_\infty = 1$  and  $Pé_\infty = M_\infty^2$ , or (c)

285  $\text{Re}_\infty = \text{Pé}_\infty = M_\infty^2$ . Any of these choices will also affect the stabilization of  
 286 the continuity and energy equations. For instance, using a Péclet number equal  
 287 to  $M_\infty^2$  may effectively stabilize the continuity equation in the shock region but  
 288 this may also add an excessive amount of dissipation for subsonic flows at the  
 289 location of the contact wave. Such a behavior may not be suitable for accuracy  
 290 purpose, making options (b) and (c) inappropriate. The same reasoning, left to  
 291 the reader, can be carried out for the energy equation (Eq. (13c)) and results in  
 292 the same conclusion. The remaining choice, option (a), has the proper scaling:  
 293 in this case, only the dissipation terms involving  $\vec{\nabla}^{s,*} \vec{u}^*$  scale as  $1/M_\infty^2$  since  
 294  $\text{Re}_\infty = M_\infty^2$ , leaving the regularization of the continuity equation unaffected  
 295 because  $\text{Pé}_\infty = 1$ .

#### 296 *3.4. An All-speed normalization of the entropy residual*

297 The study of the above limit cases yields two different possible scalings for  
 298 the Reynolds number:  $\text{Re}_\infty = 1$  in the low-Mach limit and  $\text{Re}_\infty = M_\infty^2$  for  
 299 non-isentropic flows, whereas the numerical Péclet number always scales as one.  
 300 In order to have a stabilization method valid for a wide range of Mach numbers,  
 301 from very low-Mach to supersonic flows, these two scalings should be combined  
 302 in a unique definition.

303 We begin with the normalization parameter  $\text{norm}_P^\kappa$ . Using the definition of  
 304 the viscosity coefficients given in Eq. (10) and the scaling of Eq. (11), it can be  
 305 shown that:

$$\kappa_\infty = \frac{\rho_\infty c_\infty^2 u_\infty L}{\text{norm}_{P,\infty}^\kappa}, \quad (25)$$

306 where  $\text{norm}_{P,\infty}$  is the reference far-field quantity for the normalization param-  
 307 eter  $\text{norm}_P$ . Substituting Eq. (25) into Eq. (14) and recalling that the numerical  
 308 Péclet number scales as unity, we obtain:

$$\text{norm}_{P,\infty}^\kappa = \text{Pé}_\infty \rho_\infty c_\infty^2 = \rho_\infty c_\infty^2. \quad (26)$$



Eq. (26) provides a proper normalization factor to define the  $\kappa$  viscosity coefficient. The derivation for  $\text{norm}_P^\mu$  is similar and yields

$$\text{norm}_P^\mu = \text{Re}_\infty \rho_\infty c_\infty^2 = \begin{cases} \rho \|\vec{u}\|^2 & \text{for non-isentropic flows} \\ \rho c^2 = \text{norm}_P^\kappa & \text{for low-Mach flows} \end{cases}. \quad (27)$$

A smooth function to transition between these two states is as follows:

$$\sigma(M) = \frac{\tanh(a(M - M^{\text{thresh}})) + |\tanh(a(M - M^{\text{thresh}}))|}{2}, \quad (28)$$

where  $M^{\text{thresh}}$  is a threshold Mach number value beyond which the flow is no longer considered to be low-Mach (we use  $M^{\text{thresh}} = 0.05$ ),  $M$  is the local Mach number, and the scalar  $a$  determines how rapidly the transition from  $\text{norm}_P^\mu = \rho c^2$  to  $\text{norm}_P^\mu = \rho \|\vec{u}\|^2$  occurs in the vicinity of  $M^{\text{thresh}}$  (we use  $a = 3$ ). It is easy to verify that

$$\text{norm}_P^\mu = (1 - \sigma(M))\rho c^2 + \sigma(M)\rho \|\vec{u}\|^2 \quad (29)$$

satisfies Eq. (27). Finally, we summarize the definition of the viscosity coefficients  $\mu$  and  $\kappa$  for completeness:

$$\kappa(\vec{r}, t) = \min(\mu_{\text{max}}(\vec{r}, t), \kappa_e(\vec{r}, t)), \quad (30a)$$

$$\mu(\vec{r}, t) = \min(\mu_{\text{max}}(\vec{r}, t), \mu_e(\vec{r}, t)), \quad (30b)$$

where the first-order viscosity is given by

$$\kappa_{\text{max}}(\vec{r}, t) = \mu_{\text{max}}(\vec{r}, t) = \frac{h}{2}(\|\vec{u}\| + c) \quad (30c)$$

321 and the entropy viscosity coefficients by

$$\kappa_e(\vec{r}, t) = \frac{h^2 \max(\tilde{R}_{\text{ent}}, J)}{\rho c^2} \text{ and } \mu_e(\vec{r}, t) = \frac{h^2 \max(\tilde{R}_{\text{ent}}, J)}{\text{norm}_P^\mu} \quad (30d)$$

322 with the jumps given by

$$J = \max \left( ||\vec{u}|| [|\vec{\nabla} P \cdot \vec{n}|], ||\vec{u}|| c^2 [|\vec{\nabla} \rho \cdot \vec{n}|] \right) \quad (30e)$$

323 where  $\text{norm}_P^\kappa$  is computed from Eq. (29). The jump  $J$  is a function of the jump  
 324 of pressure and density gradients across the face with respect to its normal vector  
 325  $\vec{n}$ . Then, the largest value over all faces is determined and used in the definition  
 326 of the viscosity coefficients. With the definition of the viscosity coefficients  $\mu$  and  
 327  $\kappa$  proposed in Eq. (30), the dissipative terms are expected to scale appropriately  
 328 for very low-Mach regimes as well for transonic and supersonic flows.

#### 329 **4. Extension of the entropy viscosity technique to Euler equations** 330 **with variable area**

331 Fluid flows in nozzles and in pipes of varying cross-sectional area can be  
 332 modeled using the variable-area variant of the Euler equations, where the con-  
 333 servative variables are now multiplied by the area  $A$ . In addition, these equa-  
 334 tions differ from the standard Euler equations in that the momentum equation  
 335 Eq. (31b) contains a non-conservative term proportional to the area gradient.  
 336 Here, the variable area is assumed to be a smooth function of space only.

$$\partial_t (\rho A) + \vec{\nabla} \cdot (\rho \vec{u} A) = 0, \quad (31a)$$

$$\partial_t (\rho \vec{u} A) + \vec{\nabla} \cdot [A (\rho \vec{u} \otimes \vec{u} + P \mathbb{I})] = P \vec{\nabla} A, \quad (31b)$$

$$\partial_t (\rho E A) + \vec{\nabla} \cdot [\vec{u} A (\rho E + P)] = 0. \quad (31c)$$

339 The application of the entropy viscosity method to the Euler equations with  
 340 variable area is not fundamentally different to its application to the standard  
 341 Euler equations. However, we need to derive the associated dissipative terms  
 342 and verify that the entropy minimum principle is still satisfied. The variable-  
 343 area Euler equations with viscous regularization are given below; details of the  
 344 derivation are provided in Appendix B.

$$\partial_t (\rho A) + \vec{\nabla} \cdot (\rho \vec{u} A) = \vec{\nabla} \cdot (A \kappa \vec{\nabla} \rho) , \quad (32a)$$

$$345 \quad \partial_t (\rho \vec{u} A) + \vec{\nabla} \cdot [A (\rho \vec{u} \otimes \vec{u} + P \mathbb{I})] = P \vec{\nabla} A + \vec{\nabla} \cdot \left[ A \left( \mu \rho \vec{\nabla}^s \vec{u} + \kappa \vec{u} \otimes \vec{\nabla} \rho \right) \right] , \quad (32b)$$

$$\begin{aligned} \partial_t (\rho A E) + \vec{\nabla} \cdot [\vec{u} A (\rho E + P)] = \\ \vec{\nabla} \cdot \left[ A \left( \kappa \vec{\nabla} (\rho e) + \frac{1}{2} ||\vec{u}||^2 \kappa \vec{\nabla} \rho + \rho \mu \vec{u} \vec{\nabla}^s \vec{u} \right) \right] . \end{aligned} \quad (32c)$$

346 The dissipative terms are quite similar to the ones obtained for the standard  
 347 Euler equations: each dissipative flux is simply multiplied by the variable area  
 348  $A$  in order to ensure conservation of the dissipative flux. When assuming a  
 349 constant area, Eqs. 2 are recovered.

350 A low-Mach asymptotic limit of the Euler equations with variable area on the  
 351 same model as in Section 3.2 will lead to the divergence constraint  $\vec{\nabla} \cdot (\vec{u} A) = 0$   
 352 that can be recast as  $\vec{\nabla} \cdot \vec{u} = -\vec{u} \cdot \vec{\nabla} A / A$ . The gradient of the area acts as a  
 353 source term and will force the fluid to accelerate or decelerate, depending on its  
 354 sign.

## 355 5. Discretizations and Solution Techniques

356 In this section, we briefly describe the spatial and temporal discretizations  
 357 and the solution techniques used to solve the system of equations Eq. (32). For

conciseness, we re-write the system of equations in the following form:

$$\partial_t \mathbf{U} + \vec{\nabla} \cdot \vec{\mathbf{F}}(\mathbf{U}) = \mathbf{S} + \vec{\nabla} \cdot \mathbf{D}(\mathbf{U}) \vec{\nabla} \mathbf{U} \quad (33)$$

where  $\mathbf{U} = [\rho A, \rho \vec{u} A, \rho E A]^T$  is the solution vector,  $\mathbf{F}$  denotes the inviscid flux

$$\vec{\mathbf{F}} \equiv \begin{bmatrix} \rho u A \\ (\rho u^2 + p) A \\ u(\rho E + P) A \end{bmatrix} \quad (34)$$

and  $\mathbf{S}$  is a source term that contains the non-conservative term  $P \vec{\nabla} A$ . The term  $\vec{\nabla} \cdot \mathbf{D}(\mathbf{U}) \vec{\nabla} \mathbf{U}$  stands for the artificial dissipative terms.

### 5.1. Spatial and Temporal Discretizations

The system of equations given in Eq. (33) is discretized using a continuous Galerkin finite element method and temporal integrators available through the MOOSE multiphysics framework [14].

#### 5.1.1. Continuous Finite Elements

In order to apply the continuous finite element method, Eq. (33) is multiplied by a test function  $\mathbf{W}(\vec{r})$ , integrated by parts and each integral is decomposed into a sum of integrals over each element  $K$  of the discrete mesh  $\Omega$ . The following weak form is obtained:

$$\begin{aligned} \sum_K \int_K \partial_t \mathbf{U} \mathbf{W} - \sum_K \int_K \vec{\mathbf{F}}(\mathbf{U}) \cdot \vec{\nabla} \mathbf{W} + \int_{\partial\Omega} \vec{\mathbf{F}}(\mathbf{U}) \cdot \vec{n} \mathbf{W} - \sum_K \int_K \mathbf{S} \mathbf{W} \\ + \sum_K \int_K \mathbf{D}(\mathbf{U}) \vec{\nabla} \mathbf{U} \cdot \vec{\nabla} \mathbf{W} - \int_{\partial\Omega} \mathbf{D}(\mathbf{U}) \vec{\nabla} \mathbf{U} \cdot \vec{n} \mathbf{W} = 0. \end{aligned} \quad (35)$$

The integrals over the elements  $K$  are evaluated using a numerical quadrature.

The MOOSE framework provides a wide range of test functions and quadrature

rules. Linear Lagrange polynomials are employed as test functions in the results section. Second-order spatial convergence will be demonstrated for smooth solutions.

### 5.1.2. Temporal integration

The MOOSE framework offers both first- and second-order explicit and implicit temporal integrators. In all of the numerical examples presented in Section 6, the temporal derivative will be evaluated using the second-order, backward difference temporal integrator BDF2. By considering three consecutive solutions,  $\mathbf{U}^{n-1}$ ,  $\mathbf{U}^n$  and  $\mathbf{U}^{n+1}$ , at times  $t^{n-1}$ ,  $t^n$  and  $t^{n+1}$ , respectively, BDF2 can be expressed as:

$$\int_K \partial_t \mathbf{U} \mathbf{W} = \int_K (\omega_0 \mathbf{U}^{n+1} + \omega_1 \mathbf{U}^n + \omega_2 \mathbf{U}^{n-1}) \mathbf{W}, \quad (36)$$

with

$$\omega_0 = \frac{2\Delta t^{n+1} + \Delta t^n}{\Delta t^{n+1} (\Delta t^{n+1} + \Delta t^n)}, \quad \omega_1 = -\frac{\Delta t^{n+1} + \Delta t^n}{\Delta t^{n+1} \Delta t^n},$$

$$\text{and } \omega_2 = \frac{\Delta t^{n+1}}{\Delta t^n (\Delta t^{n+1} + \Delta t^n)}$$

where  $\Delta t^n = t^n - t^{n-1}$  and  $\Delta t^{n+1} = t^{n+1} - t^n$ .

### 5.2. Boundary conditions

Boundary conditions are implemented by performing a characteristic decomposition to compute the appropriate flux at the boundaries. Our implementation of the subsonic boundary conditions is inspired by the method described in [20] and was adapted for a time implicit solver. Neumann boundary conditions are used for all of the boundary types, except for the inlet supersonic boundary that are strongly imposed with Dirichlet boundary conditions.

For each numerical solution presented in Section 6, the type of boundary conditions used will be specified and taken among the following: supersonic inlet, subsonic inlet (stagnation pressure boundary), subsonic outlet, and supersonic outlet. The artificial diffusion coefficient  $D(\mathbf{U})$  is set to zero at the boundary of the computational domain so that the boundary term  $\int_{\partial\Omega} D(\mathbf{U}) \vec{\nabla} \mathbf{U} \cdot \vec{n} \mathbf{W}$  stemming from the integration by parts of the artificial dissipative terms in Eq. (35) is ignored.

### 5.3. Solver

A Jacobian-free-Newton-Krylov (JFNK) method is used to solve for the solution at the end of each time step. An approximate Jacobian matrix of the discretized equations was derived and implemented. Obtaining the matrix entries requires that the partial derivatives of pressure with respect to the conservative variables be known (this is relatively simple for the stiffened and ideal gas equations of state but may be more complex for general equations of state). The contributions of the artificial dissipative terms to the Jacobian matrix are approximated by lagging the viscosity coefficients (computing them with the previous solution). For instance, this is shown in Eq. (37) for the dissipative terms present in the continuity equation:

$$\frac{\partial}{\partial \mathbf{U}} \left( \kappa \vec{\nabla} \cdot \rho \vec{\nabla} W \right) \simeq \kappa \frac{\partial}{\partial \mathbf{U}} \left( \vec{\nabla} \cdot \rho \vec{\nabla} W \right), \quad (37)$$

where  $\mathbf{U}$  denotes any of the conservative variables and  $W$  denotes the component of  $\mathbf{W}$  associated with the continuity equation. In the above, we have neglected  $\frac{\partial \kappa}{\partial \mathbf{U}}$ .

## 408 6. Numerical Results

409 1-D and 2-D numerical solutions for the Euler equations with viscous reg-  
410 ularization solved using the entropy viscosity method are presented here. Our  
411 results show that the new definitions for the viscosity coefficients are robust in  
412 the low-Mach limit as well as for for transonic and supersonic flows and that  
413 shocks are appropriately resolved.

414 The first set of 1-D simulations consist of liquid water and steam flowing in  
415 a converging-diverging nozzle. This test is of interest for multiple reasons: (a)  
416 a steady state can be reached (some stabilization methods are known to have  
417 difficulties reaching a steady state, [2, 3]), (b) an analytical solution is available  
418 and a space-time convergence study can be performed, (c) it can be performed  
419 for liquid and gas phases, wherein the gas phase simulation presents a shock  
420 while the liquid-phase simulation has a significantly lower Mach number. Next,  
421 a 1-D shock tube test (in a straight pipe), taken from the Leblanc test-case suite  
422 [21], is performed. This test is known to be more challenging than Sod shock  
423 tubes and the fluid’s Mach number varies spatially between 0 and 5. A con-  
424 vergence study is also performed to demonstrate convergence of the numerical  
425 solution to the exact solution. A slow moving shock is also investigated [22].  
426 This test helps in assessing the ability of the method to damp the post-shock  
427 low frequency noise (oscillations). Finally, a strong shock for a liquid phase  
428 (Mach number around 0.1) is also performed [23].

429 The initial conditions for the aforementioned 1-D test cases are given in  
430 Table 1.

431 The 2-D simulations are outlined next. First, 2-D subsonic flows around a  
432 cylinder [12] and over a circular hump [24] are presented for various far-field  
433 Mach numbers (as low of  $10^{-7}$ ). Numerical results of a supersonic flow over  
434 a compression corner are provided to illustrate the ability of the new viscosity

$\rho_{\text{left}}$	$u_{\text{left}}$	$P_{\text{left}}$	$\rho_{\text{right}}$	$u_{\text{right}}$	$P_{\text{right}}$
Leblanc shock tube (Section 6.3)					
1	0	$4 \cdot 10^{-2}$	$10^{-3}$	0	$4 \cdot 10^{-11}$
Strong shock for liquid phase (Section 6.4)					
1000	0	$10^9$	1000	0	$10^5$
Slow moving shock (Section 6.5)					
1	-0.81	1	3.86	-3.44	10.33

Table 1: Initial conditions for the 1-D test cases (density in  $kg/m^3$ , velocity in  $m/s$ , pressure in  $Pa$ ).

435 definitions to handle supersonic flows. Convergence studies are performed when  
436 analytical solutions are available.

437 For each simulation, data relative to the boundary conditions, the Courant-  
438 Friedrichs-Lewy number ( $CFL$ ), mesh and equation of state are provided. All of  
439 the numerical solutions presented are obtained using BDF2 as temporal integra-  
440 tor and linear (1-D mesh),  $\mathbb{P}_1$  (2-D triangular mesh), and  $\mathbb{Q}_1$  (2-D quadrangular  
441 mesh) finite elements. The spatial integrals are numerically computed using a  
442 second-order Gauss quadrature rule. Steady-state is detected in a transient sim-  
443 ulation by monitoring the nonlinear residual before proceeding with the Newton  
444 solves for a given time step. The ideal gas [25] or stiffened gas equations of state  
445 [26] are used; a generic expression is given in Eq. (38).

$$P = (\gamma - 1)\rho(e - q) - \gamma P_\infty \quad (38)$$

where the parameters  $\gamma$ ,  $q$ , and  $P_\infty$  are fluid-dependent and are given in Table 2.  
The ideal gas equation of state is recovered by setting  $q = P_\infty = 0$  in Eq. (38).  
The entropy function for the stiffened gas equation of state is concave and given  
by

$$s = C_v \ln \left( \frac{P + P_\infty}{\rho^{\gamma-1}} \right),$$

446 where  $C_v$  is the heat capacity at constant volume.



Table 2: Stiffened Gas Equation of State parameters for steam and liquid water.

fluid	$\gamma$	$C_v$ ( $J.kg^{-1}.K^{-1}$ )	$P_\infty$ (Pa)	$q$ ( $J.kg^{-1}$ )
liquid water (Section 6.1)	2.35	1816	$10^9$	$-1167 \cdot 10^3$
steam (Section 6.2)	1.43	1040	0	$2030 \cdot 10^3$
liquid water (Section 6.4)	4.4	1000	$6 \cdot 10^8$	0

447

448 Finally, the convergence rates are computed using the following relation

$$\text{rate}_h = \ln \left( \frac{\|U_{2h} - U_{\text{exact}}\|}{\|U_h - U_{\text{exact}}\|} \right) / \ln 2 \quad (39)$$

449 where  $\|\cdot\|$  denotes either the  $L_1$  or  $L_2$  norms and  $h$  is the characteristic grid  
450 size.

#### 451 6.1. Liquid water in a 1-D converging-diverging nozzle

452 A simulation for liquid flow through a 1-D converging-diverging nozzle is  
453 performed. The variable area expression is given by  $A(x) = 1 + 0.5 \cos(2\pi x/L)$   
454 with length  $L = 1m$ . At the inlet, the stagnation pressure and temperature are  
455 set to  $P_0 = 1MPa$  and  $T_0 = 453K$ , respectively. At the outlet, only the static  
456 pressure is specified:  $P_s = 0.5MPa$ . Initially, the liquid is at rest, the tem-  
457 perature is uniform and equal to the stagnation temperature and the pressure  
458 linearly decreases from the stagnation pressure inlet value to the static pressure  
459 outlet value. The stiffened gas equation of state is used to model the liquid  
460 water with the parameters provided in Table 2. Because of the low pressure  
461 difference between the inlet and the outlet, the smooth initial conditions, and  
462 the large value of  $P_\infty$  in Eq. (38), the flow remains subsonic and thus displays  
463 no shock. A detailed derivation of the exact steady-state solution can be found  
464 in [27]. A uniform mesh of 50 cells was used to obtain the numerical solution  
465 and the time step size was computed using a  $CFL$  number of 750. Plots of

the Mach number, density, and pressure are given at steady state in Fig. 1 for the numerical and exact solutions. The viscosity coefficients are also graphed in Fig. 1d.

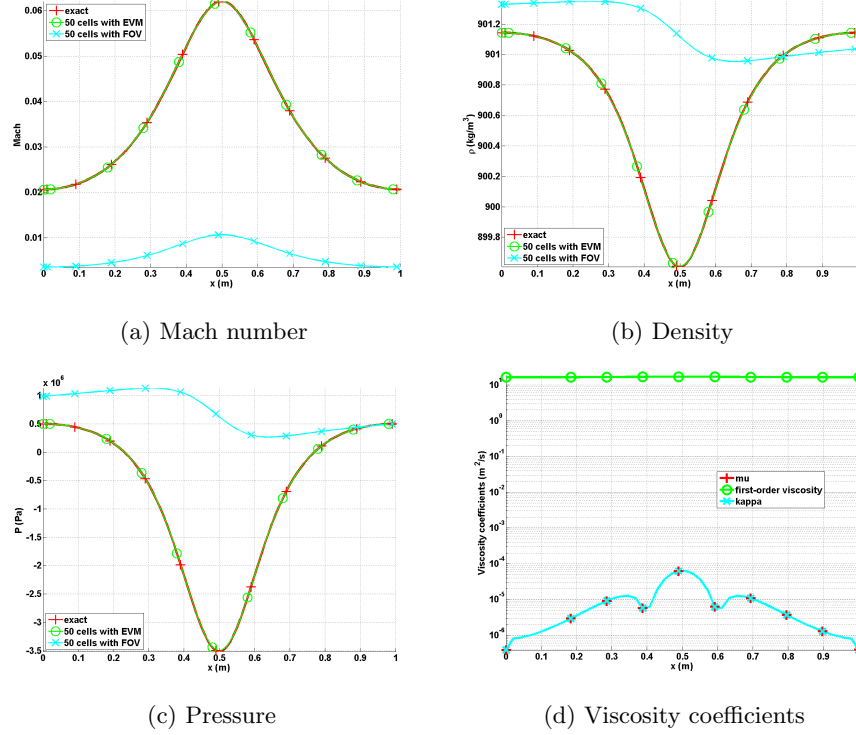


Figure 1: Steady-state solution for a liquid flowing through a 1-D converging-diverging nozzle.

In Fig. 1, the numerical solutions obtained using the first-order viscosity (FOV) and the entropy viscosity method (EVM) are plotted against the exact solution. The numerical solution obtained with the EVM and the exact solution overlap, even for a fairly coarse mesh (50 cells). On the other hand, the numerical solution obtained with the FOV does not give the correct steady state: this is an illustration of the effect of ill-scaled dissipative terms in the low-Mach limit when using the FOV. Note that the entropy viscosity coefficient is very small

476 compared to the first-order one (Fig. 1d): (i) the numerical solution is smooth  
477 as shown in Fig. 1 and (ii) the flow is in a isentropic low-Mach regime. A conver-  
478 gence study was performed using the exact solution as a reference: the  $L_1$  and  
479  $L_2$  norms of the error and the corresponding convergence rates are computed at  
480 steady state on various uniform meshes from 4 to 256 cells. Spatial convergence  
481 results using linear finite elements are reported in Table 3 and Table 4 for the  
482 primitive variables: density, velocity and pressure.

Table 3:  $L_1$  norm of the error for the liquid phase in a 1-D converging-diverging nozzle at steady state.

cells	density	rate	pressure	rate	velocity	rate
4	$2.8037 \cdot 10^{-1}$	—	$8.4705 \cdot 10^5$	—	7.2737	—
8	$1.3343 \cdot 10^{-1}$	1.07	$4.7893 \cdot 10^5$	0.82	6.1493	0.24
16	$2.9373 \cdot 10^{-2}$	2.18	$1.0613 \cdot 10^5$	2.17	1.2275	2.32
32	$5.1120 \cdot 10^{-3}$	2.52	$1.8446 \cdot 10^4$	2.52	$1.8943 \cdot 10^{-1}$	2.69
64	$1.0558 \cdot 10^{-3}$	2.28	$3.7938 \cdot 10^3$	2.28	$3.7919 \cdot 10^{-2}$	2.32
128	$2.3712 \cdot 10^{-4}$	2.15	$8.4471 \cdot 10^2$	2.17	$8.5517 \cdot 10^{-3}$	2.15
256	$5.6058 \cdot 10^{-5}$	2.08	$1.9839 \cdot 10^2$	2.09	$2.0475 \cdot 10^{-3}$	2.06
512	$1.3278 \cdot 10^{-5}$	2.08	$4.6622 \cdot 10^1$	2.09	$4.9516 \cdot 10^{-4}$	2.04
1024	$3.1193 \cdot 10^{-6}$	2.08	$1.1755 \cdot 10^1$	1.99	$1.2379 \cdot 10^{-4}$	2.00

Table 4:  $L_2$  norm of the error for the liquid phase in a 1-D converging-diverging nozzle at steady state.

cells	density	rate	pressure	rate	velocity	rate
4	$3.106397 \cdot 10^{-1}$	—	$5.254445 \cdot 10^5$	—	3.288543	—
8	$7.491623 \cdot 10^{-2}$	2.05	$1.636966 \cdot 10^5$	1.68	1.823880	0.85
16	$2.079858 \cdot 10^{-2}$	1.85	$4.627338 \cdot 10^4$	1.49	$4.990605 \cdot 10^{-1}$	0.87
32	$5.329627 \cdot 10^{-3}$	1.96	$1.180287 \cdot 10^4$	1.97	$1.261018 \cdot 10^{-1}$	1.98
64	$1.341583 \cdot 10^{-3}$	1.99	$2.967104 \cdot 10^3$	1.99	$3.160914 \cdot 10^{-2}$	1.99
128	$3.359766 \cdot 10^{-4}$	1.99	$7.428087 \cdot 10^2$	1.99	$7.907499 \cdot 10^{-3}$	1.99
256	$8.403859 \cdot 10^{-5}$	1.99	$1.857861 \cdot 10^2$	1.99	$1.977292 \cdot 10^{-3}$	1.99
512	$2.10075 \cdot 10^{-5}$	2.00	$4.7024 \cdot 10^1$	1.98	$4.9516 \cdot 10^{-4}$	1.99

483 We note that the convergence rates measured in both the  $L_1$  and  $L_2$  norm of  
484 the error are equal to 2; the entropy viscosity method preserves the high-order

485 accuracy of the discretization used when the numerical solution is smooth. The  
 486 new definition of the entropy viscosity coefficients behaves appropriately in the  
 487 low-Mach limit.

## 488 *6.2. Steam in a 1-D converging-diverging nozzle*

489 We use the same nozzle geometry, initial conditions and boundary condi-  
 490 tions as in the previously example but replace liquid water with steam and use  
 491 the steam parameters of the stiffened gas equation of state, Table 2. In this  
 492 example, compressible effects will become dominant. The pressure difference  
 493 between the inlet and outlet is large enough to accelerate the steam through  
 494 the nozzle, leading to the formation of a shock in the diverging portion of the  
 495 nozzle. The behavior is different from the one observed for the liquid water  
 496 phase in Section 6.1 because of the liquid to gas density ratio is about 1,000.  
 497 An exact solution at steady state is available for the gas phase [27]. The aim  
 498 of this section is to show that when using the new definitions of the viscosity  
 499 coefficients (Eq. (30)), the shock can be correctly resolved without spurious os-  
 500 cillations. The steady-state numerical solution, obtained using a uniform mesh  
 501 with 500 cells, is shown in Fig. 2. The  $CFL$  was set to 80 (a high  $CFL$  value  
 502 can be used because the shock is stationary).

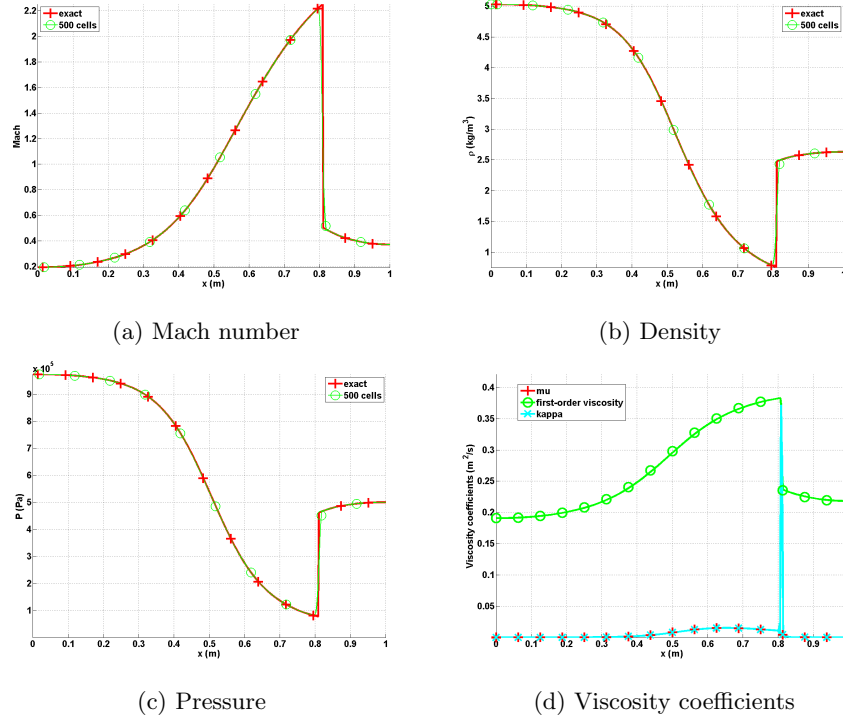


Figure 2: Steady-state solution for vapor phase flowing in a 1-D converging-diverging nozzle.

503 The steady-state solution of the density, Mach number and pressure are  
 504 given in Fig. 2. The steady-state solution exhibits a shock around  $x = 0.8m$   
 505 and matches the exact solution. In Fig. 2d, the first-order and entropy viscosity  
 506 coefficients are plotted at steady state (on a log scale): the entropy viscosity  
 507 coefficient is peaked in the shock region around  $x = 0.8m$  where it saturates to  
 508 the first-order viscosity coefficient. Elsewhere, the entropy viscosity coefficient  
 509 is small. In order to prove convergence of the numerical solution to the exact  
 510 solution, a convergence study is performed. Because of the presence of a shock,  
 511 second-order accuracy is not expected and the convergence rate of a numerical  
 512 solution should be 1 and  $1/2$  when measured in the  $L_1$  and  $L_2$  norms, respec-  
 513 tively (see Theorem 9.3 in [28]). Results are reported in Table 5 and Table 6

514 for the primitive variables: density, velocity and pressure. The convergence  
515 rates for the  $L_1$  and  $L_2$  norms of the error computed using Eq. (39) are in good  
agreement with the theoretical values.

Table 5:  $L_1$  norm of the error for the vapor phase in a 1-D converging-diverging  
nozzle at steady state.

cells	density	rate	pressure	rate	velocity	rate
5	$0.72562 \cdot 10^{-1}$	—	$1.5657 \cdot 10^5$	—	173.69	—
10	$0.4165 \cdot 10^{-1}$	0.80	$9.6741 \cdot 10^4$	0.63	120.69	0.53
20	$0.20675 \cdot 10^{-1}$	1.01	$4.9193 \cdot 10^4$	0.97	72.149	0.74
40	$0.093703 \cdot 10^{-1}$	1.14	$2.0103 \cdot 10^4$	0.73	34.716	1.06
80	$0.047328 \cdot 10^{-1}$	0.99	$1.0208 \cdot 10^4$	0.98	16.082	1.11
160	$0.023965 \cdot 10^{-2}$	0.98	$5.1969 \cdot 10^3$	0.97	7.9573	1.02
320	$0.020768 \cdot 10^{-2}$	1.03	$2.5116 \cdot 10^3$	1.05	3.7812	1.07
640	$0.0059715 \cdot 10^{-2}$	0.98	$1.2754 \cdot 10^3$	0.98	1.8353	1.04

516

Table 6:  $L_2$  norm of the error for the vapor phase in a 1-D converging-diverging  
nozzle at steady state.

cells	density	rate	pressure	rate	velocity	rate
5	$9.7144 \cdot 10^{-1}$	—	$2.0215 \cdot 10^5$	—	236.94	—
10	$5.9718 \cdot 10^{-1}$	0.70	$1.3024 \cdot 10^5$	0.63	166.56	0.51
20	$2.9503 \cdot 10^{-1}$	1.02	$6.6503 \cdot 10^4$	0.97	103.36	0.69
40	$1.8193 \cdot 10^{-1}$	0.69	$4.0171 \cdot 10^4$	0.73	66.374	0.64
80	$1.3366 \cdot 10^{-1}$	0.44	$2.3163 \cdot 10^4$	0.44	42.981	0.63
160	$9.6638 \cdot 10^{-2}$	0.47	$1.7263 \cdot 10^4$	0.42	31.717	0.44
320	$7.0896 \cdot 10^{-2}$	0.45	$1.2763 \cdot 10^4$	0.44	23.138	0.45
640	$5.2191 \cdot 10^{-2}$	0.44	$9.4217 \cdot 10^3$	0.44	16.910	0.45

### 517 6.3. Leblanc shock tube

518 The 1-D Leblanc shock tube is a Riemann problem designed to test the  
519 robustness and the accuracy of stabilization methods. The initial conditions  
520 are given in Table 1. The ideal gas equation of state (with  $\gamma = 5/3$ ) is used to  
521 compute the pressure. This test is computationally challenging because of the  
522 large pressure ratio at the initial interface. The computational domain consists  
523 of a 1-D straight pipe of length  $L = 9m$  with the initial interface located at

524  $x = 2m$ . At  $t = 0s$ , the interface is removed. The numerical solution is run  
 525 until  $t = 4s$  and the density, momentum and total energy profiles are given in  
 526 Fig. 3, along with the exact solution. The viscosity coefficients are also plotted  
 527 in Fig. 3d. These plots were run with three different uniform meshes of 800,  
 528 3200, and 6000 cells and a constant  $CFL = 1$ .

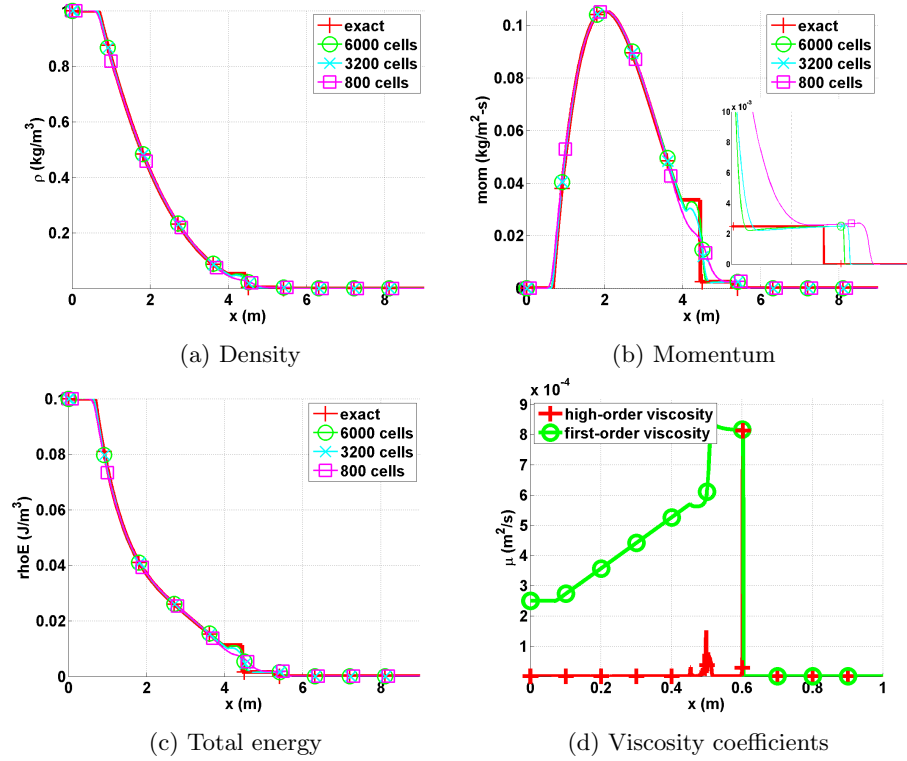


Figure 3: Exact and Numerical solutions for the 1-D Leblanc shock tube at  $t = 4s$ .

529 The density, momentum and total energy profiles are provided in Fig. 3.  
 530 In Fig. 3b, the shock region is zoomed in for better resolution: the shock is  
 531 well resolved. We also observe that the shock position computed numerically  
 532 converges to the exact position under mesh refinement. The contact wave at  
 533  $x = 4.5m$  can be seen in Fig. 3b. The entropy viscosity coefficient profile is

shown in Fig. 3d and behaves as expected: it saturates to the first-order viscosity in the shock region, thus preventing oscillations from forming. At the location of the contact wave, a smaller peak is observed and is due to the presence of the jump terms in the definition of the entropy viscosity coefficient (Eq. (30)). The Mach number, not plotted, is of the order of 1.3 just before the shock and reaches a maximum value close to 5 in the contact region.

Once again, a convergence study is performed in order to prove convergence of the numerical solution to the exact solution. As in the previous example (vapor phase in the 1-D nozzle, Section 6.2), the expected convergence rates in the  $L_1$  and  $L_2$  norms are 1 and  $1/2$ , respectively. The exact solution was obtained by running a 1-D Riemann solver and used as the reference solution to compute the  $L_1$  and  $L_2$ -norms that are reported in Table 7 and Table 8 for the conservative variables: density, momentum and total energy. The convergence rates are again approaching their theoretical values.

Table 7:  $L_1$  norm of the error for the 1-D Leblanc test at  $t = 4s$ .

cells	density	rate	momentum	rate	total energy	rate
100	$1.0354722 \cdot 10^{-2}$	—	$3.5471714 \cdot 10^{-3}$	—	$1.4033046 \cdot 10^{-3}$	—
200	$7.2680512 \cdot 10^{-3}$	0.51	$2.5933119 \cdot 10^{-3}$	0.45	$9.8611746 \cdot 10^{-4}$	0.51
400	$5.0825628 \cdot 10^{-3}$	0.52	$2.0668092 \cdot 10^{-3}$	0.33	$7.7844421 \cdot 10^{-4}$	0.34
800	$3.4025056 \cdot 10^{-3}$	0.58	$1.4793838 \cdot 10^{-3}$	0.48	$5.5702549 \cdot 10^{-4}$	0.48
1600	$2.1649953 \cdot 10^{-3}$	0.65	$9.7152832 \cdot 10^{-4}$	0.61	$3.5720171 \cdot 10^{-4}$	0.64
3200	$1.2465433 \cdot 10^{-3}$	0.79	$5.5937409 \cdot 10^{-4}$	0.79	$2.0491799 \cdot 10^{-4}$	0.80
6400	$6.4476928 \cdot 10^{-4}$	0.95	$3.0244198 \cdot 10^{-4}$	0.89	$1.0914891 \cdot 10^{-4}$	0.91
12800	$3.3950948 \cdot 10^{-4}$	0.93	$1.5958118 \cdot 10^{-4}$	0.92	$5.7909794 \cdot 10^{-5}$	0.91

#### 6.4. 1-D shock tube with a liquid phase

The purpose of this test is to investigate the ability of the entropy viscosity method to stabilize a strong shock with a small Mach number [23] (this reference is for a two-phase flow model but we are only interested in the initial conditions for the liquid phase): the Mach number in the shock region is of the order of 0.1.



Table 8:  $L_2$  norm of the error for the 1-D Leblanc test at  $t = 4s$ .

cells	density	rate	momentum	rate	total energy	rate
100	$5.7187851 \cdot 10^{-3}$	—	$1.7767236 \cdot 10^{-3}$	—	$7.6112265 \cdot 10^{-4}$	—
200	$3.8995238 \cdot 10^{-3}$	0.55	$1.4913161 \cdot 10^{-3}$	0.25	$5.5497308 \cdot 10^{-4}$	0.46
400	$2.8103526 \cdot 10^{-3}$	0.47	$1.3305301 \cdot 10^{-3}$	0.16	$4.6063172 \cdot 10^{-4}$	0.27
800	$2.1081933 \cdot 10^{-3}$	0.41	$1.1398931 \cdot 10^{-3}$	0.22	$3.7798953 \cdot 10^{-4}$	0.29
1600	$1.5731052 \cdot 10^{-3}$	0.42	$9.0394227 \cdot 10^{-4}$	0.33	$2.9584646 \cdot 10^{-4}$	0.35
3200	$1.0610667 \cdot 10^{-3}$	0.57	$6.2735595 \cdot 10^{-4}$	0.53	$2.054455 \cdot 10^{-4}$	0.53
6400	$7.3309974 \cdot 10^{-4}$	0.53	$4.4545754 \cdot 10^{-4}$	0.49	$1.4670834 \cdot 10^{-4}$	0.49
12800	$5.1020991 \cdot 10^{-4}$	0.52	$3.1266758 \cdot 10^{-4}$	0.51	$1.0299897 \cdot 10^{-5}$	0.51

553 In this case, as explained in Section 3.2, the viscosity coefficients are required  
554 to have different order of magnitude in order to ensure the correct scaling of the  
555 dissipative terms. The purpose of this test is to validate the approach presented  
556 in Section 3.2.

557 The stiffened gas equation of state is used to model a liquid flow with the  
558 parameters given in Table 2. The computational domain of length  $L = 1m$  is  
559 uniformly discretized using 500 cells. The step initial conditions are given in  
560 Table 1. The simulation is run with a  $CFL = 1$  until the final time  $t_{\text{final}} =$   
561  $7 \cdot 10^{-5}s$ . Results for pressure, density, velocity and the viscosity coefficients  
562 are given in Fig. 4 along with the exact solution for comparison purposes. The  
563 numerical solution is in good agreement with exact solution in Fig. 4a. The  
564 viscosity coefficients  $\mu$  and  $\kappa$  are not equal in the shock because the Mach  
565 number is of order 0.1. The viscosity coefficient  $\kappa$  saturates to the first-order  
566 viscosity in the shock region around  $x = 0.65m$  and is sufficient to stabilize the  
567 numerical scheme.

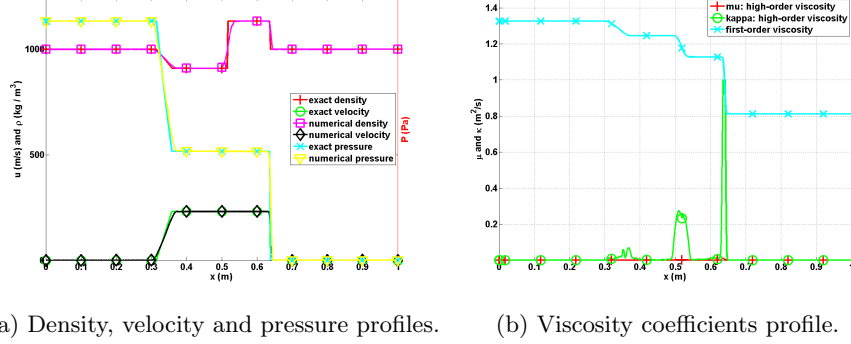


Figure 4: Numerical solution for the 1-D liquid shock tube at  $t_{\text{final}} = 7 \cdot 10^{-5} \text{ s}$ .

### 6.5. 1-D slow moving shock

Slow moving shocks are known to produce post-shock noise of low frequency that is not damped by some numerical dissipation methods [22]. The aim of this simulation is to test the ability of the entropy viscosity method to dampen the low frequency waves. The 1-D slow moving shock consists of a shock wave moving from left to right with the initial conditions given in Table 1. The ideal gas equation of state is used with a heat capacity ratio  $\gamma = 1.4$ . In order to make the shock travel a significant distance, the final time is taken equal to  $t = 1.1 \text{ s}$ . A pressure boundary condition is used at the left boundary to let the rarefaction and contact waves exit the domain. The numerical solution, obtained with 200 equally-spaced cells, is given in Fig. 5 and is compared to the exact solution obtained from a Riemann solver. We use a  $CFL$  of 1. With this  $CFL$  value, it takes about 50 time steps for the shock to traverse one cell. The numerical results are in good agreement with the exact solution and do not display any post-shock noise. The rarefaction and contact waves are not visible on Fig. 5a since they exited the computational domain through the left pressure boundary condition earlier. As explained in [29], Godunov's type methods usually fail to resolve a slow moving shock because of the nature of the stabilization method:

586 the method scales as the eigenvalue of the appropriate field. In the case of a slow  
 587 moving shock, the dissipation added to the system is under-estimated and leads  
 588 to post-shock noise. In the case of the entropy viscosity method, the entropy  
 589 residual detects the shock position and the viscosity coefficients saturate to the  
 590 first-order viscosity values in the shock region. The main difference between a  
 591 Godunov's type method and the entropy viscosity method lies in the definition of  
 592 the first-order viscosity coefficients that are proportional to the *local maximum*  
 593 *eigenvalue*  $||\vec{u}|| + c$  and not to the eigenvalue of the characteristic field.

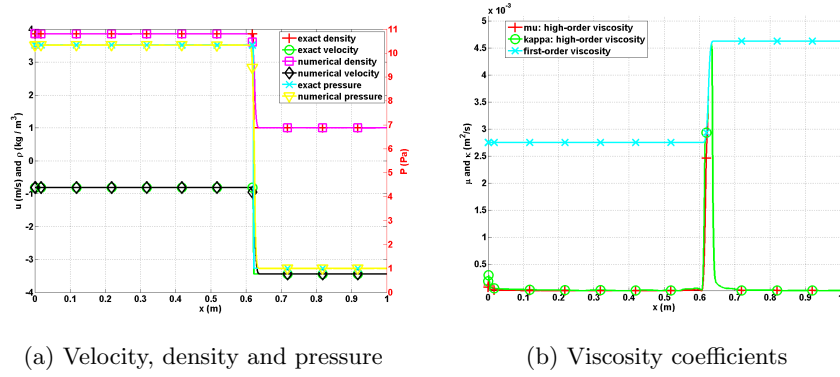


Figure 5: Slow moving shock profiles at  $t = 1.1s$ .

#### 6.6. Subsonic flow over a 2-D cylinder

595 Fluid flow over a 2-D cylinder is often used as a benchmark case to test nu-  
 596 merical schemes in the low-Mach regime [10, 11, 12]. For this test, an analytical  
 597 solution is available in the incompressible limit and is often referred to as the  
 598 potential steady-state flow solution. The main features of the potential flow are  
 599 the following:

- The solution is symmetric: the iso-Mach contour lines are used to assess the symmetry of the numerical solution;
- The steady-state velocity at the top of the cylinder is twice the incoming velocity set at the inlet;

- The spatial steady-state pressure variations are proportional to the square of inlet Mach number, i.e.,

$$\delta P = \frac{\max(P(\vec{r})) - \min(P(\vec{r}))}{\max(P(\vec{r}))} \propto M_\infty^2 \quad (40)$$

where  $\delta P$  and  $M_\infty$  denote the spatial steady-state pressure variations and the inlet Mach number, respectively.

The computational domain consists of a  $1 \times 1$  square with a circular hole of radius 0.05 in its center. A  $\mathbb{P}_1$  triangular mesh with 4008 triangular elements is employed to discretize the geometry. The ideal gas equation of state, with  $\gamma = 1.4$  is used. At the inlet, a subsonic stagnation boundary condition is used: the stagnation pressure and temperature are computed using the following relations:

$$\begin{cases} P_0 = P \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} \\ T_0 = T \left(1 + \frac{\gamma-1}{2} M^2\right) \end{cases} \quad (41)$$

A static pressure boundary condition, with static pressure  $P_s = 101,325 \text{ Pa}$ , is set at the outlet boundary. The implementation of the pressure boundary conditions is based on [20]. A solid wall boundary condition is set for the top and bottom walls of the computational domain. The simulations are run until a steady state is reached (with a  $CFL$  of 40). When the residual norm (for all equations) is less than  $10^{-12}$  the steady state is considered to have been reached.

Several simulations are performed, with inlet Mach numbers  $M_{\text{inlet}}$  ranging from  $10^{-3}$  to  $10^{-7}$ , and are shown in Fig. 6. The iso-Mach contour lines are drawn using 30 equally-spaced intervals, from  $2 \times 10^{-10}$  to  $M_{\text{inlet}}$ .

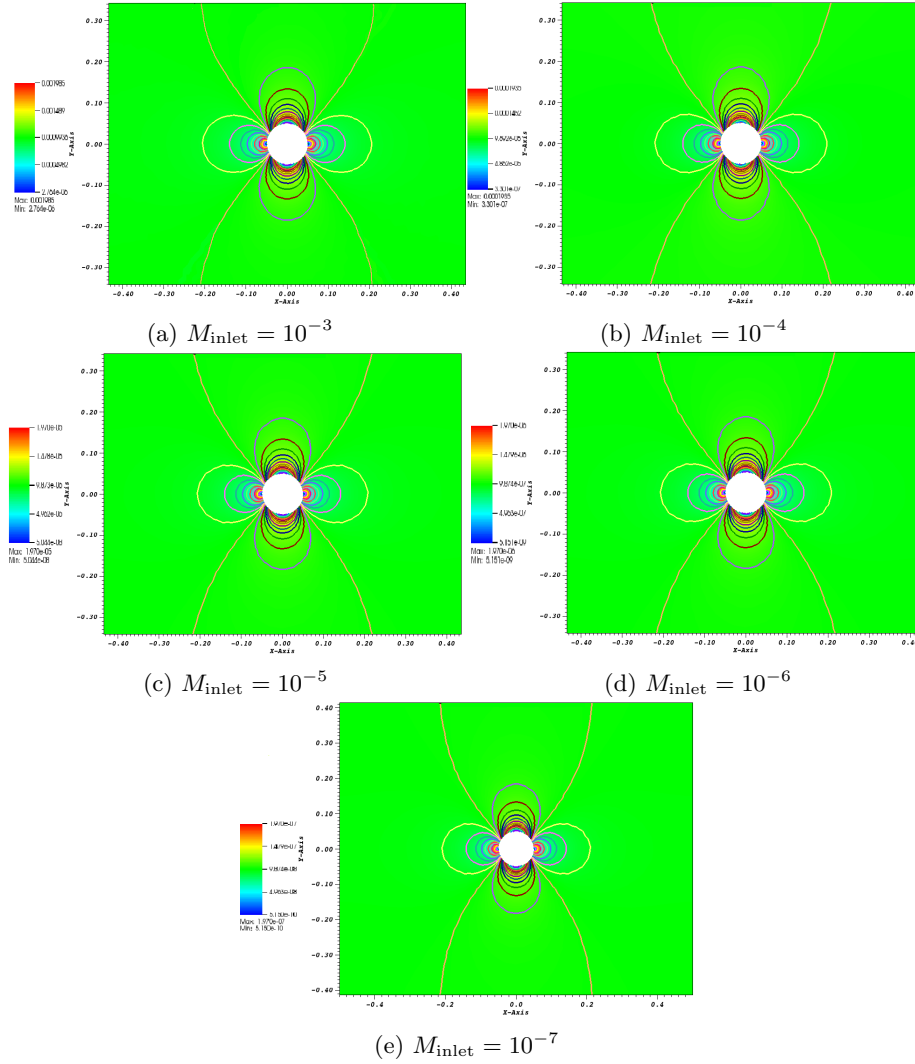


Figure 6: Iso-Mach lines for a subsonic flow over a 2-D cylinder with inlet Mach number values from ranging from  $10^{-3}$  to  $10^{-7}$  (steady-state solution).

623 The steady-state velocity at the top of the cylinder and at the inlet are  
624 given for different Mach-number values (ranging from  $10^{-3}$  to  $10^{-7}$ ) in Table 9.  
625 The ratio of the inlet velocity to the velocity at the top of cylinder is also  
626 computed and is very close to the theoretical value of 2 that is expected in the  
627 incompressible limit.

Table 9: Steady-state velocity ratio for different Mach numbers.

Mach number	inlet velocity	velocity at the top of the cylinder	ratio
$10^{-3}$	$2.348 \cdot 10^{-3}$	$1.176 \cdot 10^{-3}$	1.99
$10^{-4}$	$2.285 \cdot 10^{-4}$	$1.145 \cdot 10^{-4}$	1.99
$10^{-5}$	$2.283 \cdot 10^{-5}$	$1.144 \cdot 10^{-5}$	1.99
$10^{-6}$	$2.283 \cdot 10^{-6}$	$1.144 \cdot 10^{-6}$	1.99
$10^{-7}$	$2.283 \cdot 10^{-7}$	$1.144 \cdot 10^{-7}$	1.99

628 In Fig. 7, the spatial steady-state variations in pressure and velocity are  
 629 plotted as a function of the Mach number (on a log-log scale). The spatial  
 630 steady-state pressure variations are expected to be of the order of  $M^2$  in the  
 631 incompressible limit, which we observe. From Bernoulli's principle, this implies  
 632 that the velocity spatial variations should be of order  $M$  in the incompressible  
 633 limit, which we also observe in Fig. 7. It is known that some stabilization  
 634 methods, e.g., [10, 11, 12], can produce spatial steady-state pressure variations  
 635 with the wrong Mach-number order. Here, the entropy viscosity method yields  
 636 the correct orders in the low-Mach limit. For ease of comparison, reference lines  
 637 with slope values of 1 and 2 are also plotted.

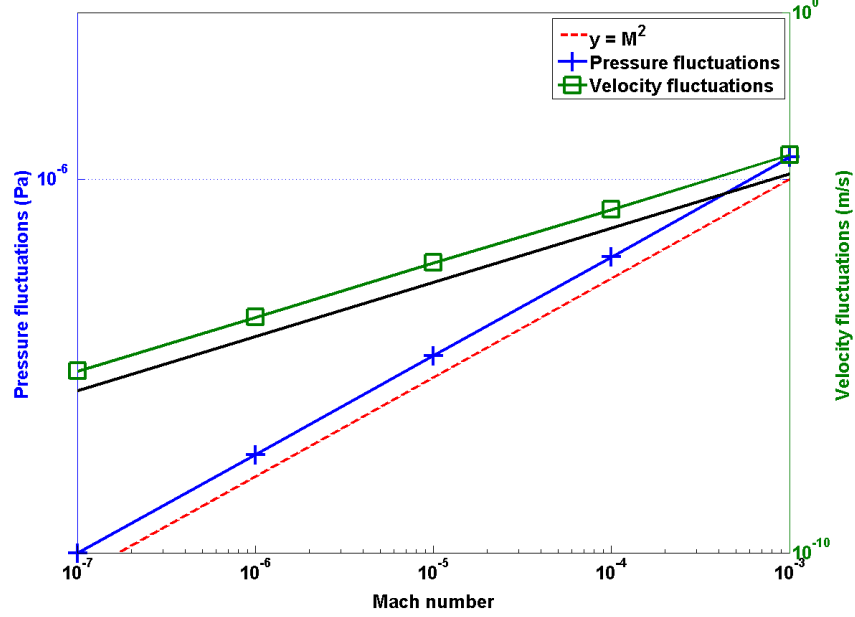


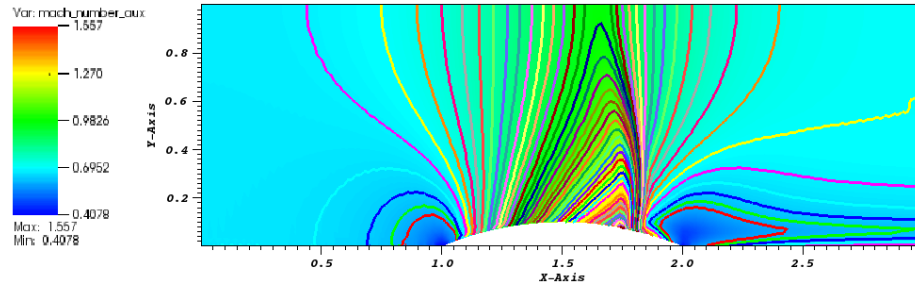
Figure 7: Log-log plot of the spatial steady-state pressure and velocity variations as a function of the far-field Mach number.

#### 6.7. Subsonic flow over a 2-D hump

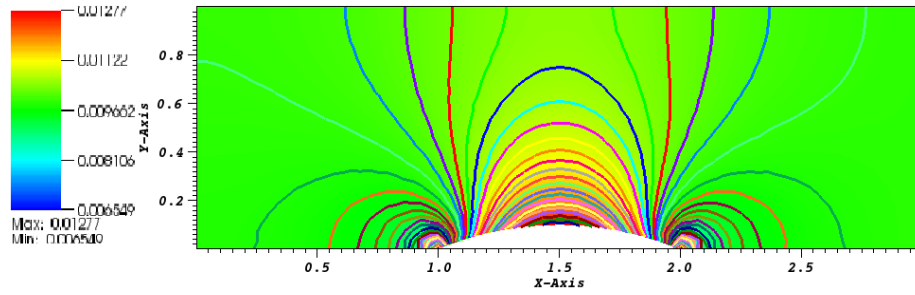
This is a another example of an internal flow configuration. It consists of a channel of height  $L = 1 \text{ m}$  and length  $3L$ , with a circular bump of length  $L$  and thickness  $0.1L$ . The bump is located on the bottom wall at a distance  $L$  from the inlet. The system is initialized with an uniform pressure  $P = 101,325 \text{ Pa}$  and temperature  $T = 300 \text{ K}$ . The initial velocity is computed from the inlet Mach number, the pressure, the temperature and the ideal gas equation (with  $\gamma = 1.4$ ). Here,  $C_v = 717 \text{ J/kg} - \text{K}$ . At the inlet, a subsonic stagnation boundary condition is used and the stagnation pressure and temperature are computed using Eq. (41). The static pressure  $P_s = 101,325 \text{ Pa}$  is set at the subsonic outlet. The results are shown in Fig. 8a, Fig. 8b, Fig. 8c and Fig. 8d for the inlet Mach numbers  $M_\infty = 0.7$ ,  $M_\infty = 0.01$ ,  $M_\infty = 10^{-4}$  and  $M_\infty = 10^{-7}$ ,

650 respectively. It is expected that, for low Mach numbers, the solution does not  
651 depend on the Mach number and is identical to the incompressible flow solution.  
652 On the other hand, for a flow with  $M = 0.7$ , the compressible effects become  
653 non negligible and a shock can form. An uniform grid of 3352  $Q_1$  elements was  
654 used to obtain the numerical solution for Mach numbers less than and equal  
655 to  $M_\infty = 0.01$ . A spatial mesh, once refined, was employed for the  $M_\infty = 0.7$   
656 simulation in order to better resolve the shock. A  $CFL$  of 20 was employed and  
657 the simulations were run until steady state.

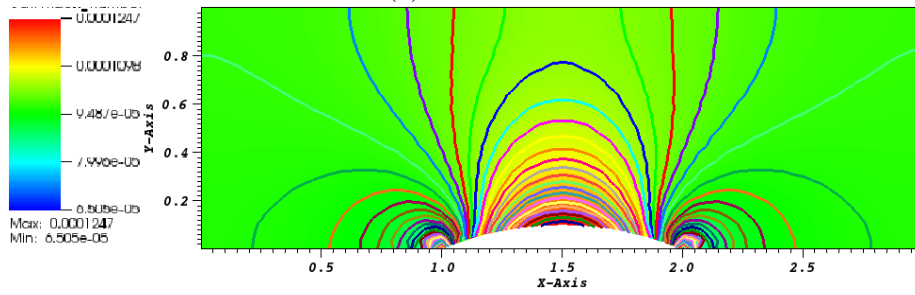




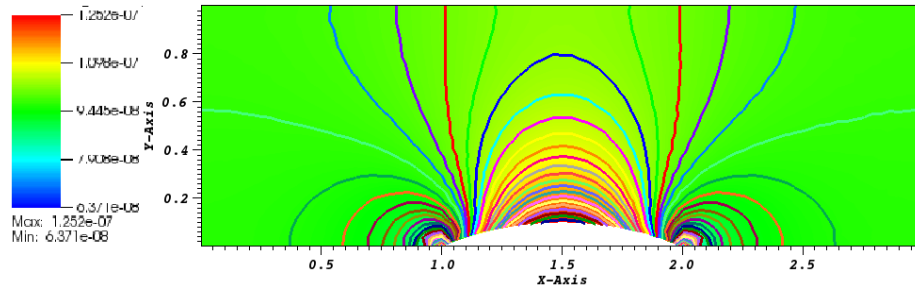
(a) Mach 0.7



(b) Mach  $10^{-2}$



(c) Mach  $10^{-5}$



(d) Mach  $10^{-7}$

Figure 8: Iso-Mach lines for a 2-D flow over a circular bump (steady-state solution).

658 The results shown in Fig. 8b, Fig. 8c and Fig. 8d correspond to the low-  
 659 Mach regime. The iso-Mach lines are drawn ranging from the minimum and the  
 660 maximum values (provided in each legend) using 50 equally-spaced intervals.  
 661 The steady-state solution is symmetric and does not depend on the value of the  
 662 inlet Mach number, as expected in the incompressible limit.

663 In Fig. 8a, the steady-state numerical solution develops a shock: the com-  
 664 pressibility effects are no longer small. The iso-Mach lines are also plotted with  
 665 50 intervals and range from 0.4 to 1.6. The shock is well resolved and does not  
 666 display any instabilities or spurious oscillations.

#### 667 *6.8. Supersonic flow in a compression corner*

668 In this last example, we consider a supersonic flow at Mach 2.5 impinging  
 669 on a corner with an angle of  $15^\circ$ . From the oblique shock theory [16], an  
 670 analytical solution for this supersonic flow is available and gives the downstream-  
 671 to-upstream pressure, entropy and Mach number ratios. The initial conditions  
 672 are chosen to be spatially uniform: the pressure and temperature are set to  $P =$   
 673  $101,325 \text{ Pa}$  and  $T = 300 \text{ K}$ , respectively. The ideal gas equation of state is used  
 674 with the same parameters as in Section 6.7. The initial velocity is computed from  
 675 the upstream Mach number. The inlet is supersonic and therefore, the pressure,  
 676 temperature and velocity are specified using Dirichlet boundary conditions. The  
 677 outlet is also supersonic and none of the characteristics enter the domain through  
 678 this boundary; the values are computed by the solver.

679 The simulation is run with  $CFL = 2$  until steady state is reached. A 2-D  
 680 mesh made of 16,109  $Q_1$  elements is used. The ratios for pressure, entropy and  
 681 Mach number computed using the analytical (published with only two significant  
 682 digits) and the numerical solutions are given in Table 10; they are in excellent  
 683 agreement. The shock wave angle at steady state is also known and given by

the so-called  $\theta - \beta - M$  relation:

$$\tan \theta = 2 \cot \beta \frac{M^2 \sin^2 \beta - 1}{M^2 (\gamma + \cos^2(2\beta)) + 2}, \quad (42)$$

where  $\theta$ ,  $\beta$  and  $M$  denote the corner angle, the shock wave angle, and the upstream Mach number, respectively. For Mach 2.5 and a  $15^\circ$  corner angle, the analytical value for the shock wave angle is  $36.94^\circ$  at steady state. From Fig. 9a, the numerical value of the shock wave angle can be measured and is found to be equal to  $36.9^\circ$  and thus is in excellent agreement with the theory.

	analytical	numerical
Pressure	2.47	2.467
Mach number	0.74	0.741
Entropy	1.03	1.026

Table 10: Ratio of analytical and numerical downstream to upstream quantities for the compression corner problems (corner angle of  $15^\circ$  and inlet  $M = 2.5$  (analytical values from [16])).

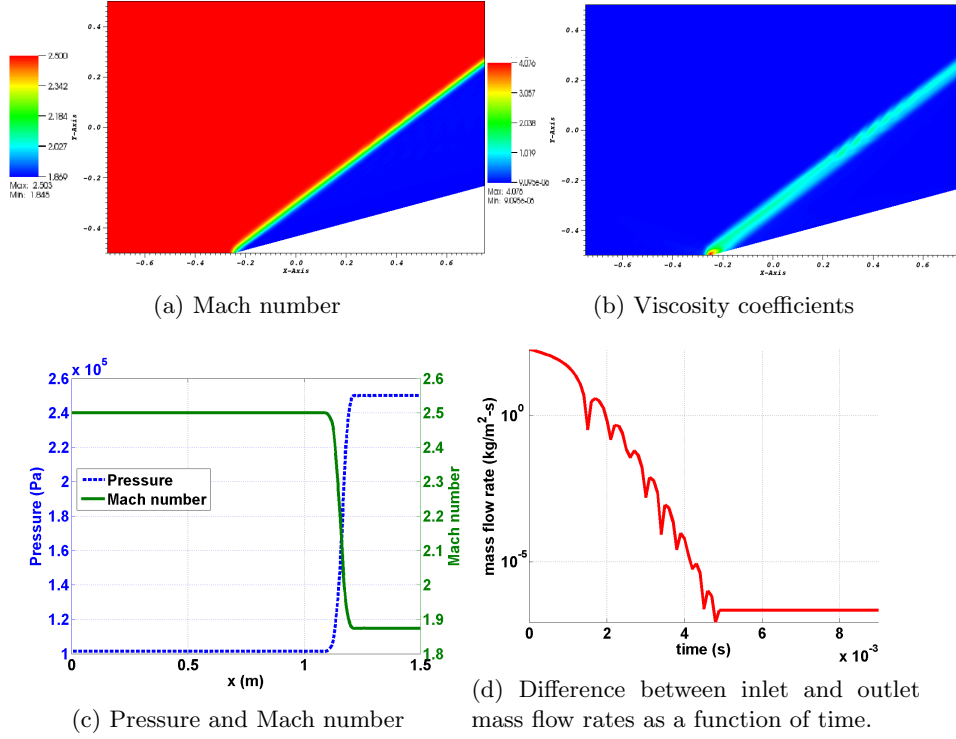


Figure 9: Steady-state solution for a flow in a 2-D compression corner.

690 The steady-state numerical solution is given in Fig. 9; the Mach number and  
 691 the viscosity coefficients are plotted in Fig. 9a and Fig. 9b, respectively. The  
 692 steady-state solution is composed of two regions of constant state separated by  
 693 an oblique shock. Fig. 9b shows that the viscosity coefficient is large in the shock  
 694 and small elsewhere, as expected. At the location of the corner ( $x = -0.25m$ ,  
 695  $y = -0.5m$ ), the viscosity coefficient is peaked because of the treatment of the  
 696 wall boundary condition: at this particular node, the normal is not well defined  
 697 and may cause some numerical errors. The 1-D graphs at  $y = 0$  for the pressure  
 698 and the Mach number are given in Fig. 9c: no spurious oscillations are observed  
 699 and the shock is well resolved. Finally, the difference between the inlet and  
 700 outlet mass flow rates is plotted in Fig. 9d and shows that a steady state has  
 701 indeed been reached.

702 The results presented in this paper demonstrate the ability of the entropy  
703 viscosity method with the new definitions of the viscosity coefficients to correctly  
704 simulate several types of flows (from very low Mach subsonic to transonic flows)  
705 without tuning parameters.

## 706 7. Conclusions

707 A new version of the entropy viscosity method that is valid for a wide range  
708 of Mach numbers has been derived and presented for the inviscid Euler equa-  
709 tions. The definition of the viscosity coefficients is now consistent with the  
710 low-Mach asymptotic limit, does not require an analytical expression for the  
711 entropy function, and is therefore applicable to a larger variety of flow regimes,  
712 from very low-Mach flows to supersonic flows. The method has also been ex-  
713 tended to Euler equation with variable area to solve nozzle flow problems. In  
714 1-D, convergence of the numerical solution to the exact solution was demon-  
715 strated by computing the convergence rates of the L1 and L2 norms for flows  
716 in a converging-diverging nozzle and in straight pipes. For smooth solutions,  
717 second-order convergence was verified; solutions with shocks converged with the  
718 expected theoretical rates of 1 ( $L_1$ -norm) and 0.5 ( $L_2$ -norm).

719 The effectiveness of the method was also demonstrated in 2-D using a se-  
720 ries of benchmark problems for both subsonic and supersonic flows in various  
721 geometries, with Mach numbers ranging from  $10^{-7}$  to 2.5. For very low-Mach  
722 flows, we numerically verified that the spatial steady-state pressure variations  
723 were proportional to the square of the Mach number, as expected in the incom-  
724 pressible limit.

725 In the future, we plan to further extend the entropy viscosity method to the  
726 seven-equation two-phase flow fluid model [20]. This two-phase flow system of  
727 equations is a good candidate for two reasons: it is unconditionally hyperbolic

728 and degenerates to the standard Euler equations when one phase disappears.

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806 ing, Massachusetts, 1990.

807 **A. Derivation of the entropy residual as a function of density, pres-**  
808 **sure and speed of sound**

The entropy residual is defined as follows:

$$R_{\text{ent}}(\vec{r}, t) = \partial_t s(\vec{r}, t) + \vec{u} \cdot \vec{\nabla} s(\vec{r}, t),$$

where all variables were defined previously. This form of the entropy residual is not suitable for the low-Mach limit as explained in Section 2.1. In this appendix, we recast the entropy residual  $R_{\text{ent}}(\vec{r}, t)$  as a function of the primitive variables (pressure, velocity and density) and the speed of sound. The first step of this derivation is to use the chain rule, recalling that the entropy is a function of the internal energy  $e$  and the density  $\rho$ , yielding

$$R_{\text{ent}}(\vec{r}, t) = s_e \frac{De}{Dt} + s_\rho \frac{D\rho}{Dt},$$

809 where  $s_e$  denotes the partial derivative of  $s$  with respect to the variable  $e$ . We  
810 recall that  $\frac{D}{Dt}$  denotes the material derivative. Since the internal energy  $e$  is a  
811 function of pressure  $P$  and density  $\rho$  (through the equation of state), we use  
812 again the chain rule to re-express the previous equation as a function of the  
813 material derivatives in  $P$  and  $\rho$ :

$$\begin{aligned} R_{\text{ent}}(\vec{r}, t) &= s_e e_P \frac{DP}{Dt} + (s_e e_\rho + s_\rho) \frac{D\rho}{Dt} \\ &= s_e e_P \left( \frac{DP}{Dt} + \frac{1}{s_e e_P} (s_e e_\rho + s_\rho) \frac{D\rho}{Dt} \right) \\ &= s_e e_P \left( \frac{DP}{Dt} + \left( \frac{e_\rho}{e_P} + \frac{s_\rho}{s_e e_P} \right) \frac{D\rho}{Dt} \right). \end{aligned}$$

To prove that the term multiplying the material derivative of the density is indeed equal to the square of the speed of sound, we recall that the speed of sound is defined as the partial derivative of pressure with respect to density at

constant entropy, which can be recast as a function of the entropy as follows (see Appendix A.2 of [15]):

$$c^2 := \left. \frac{\partial P}{\partial \rho} \right|_{s=cst} = P_\rho - \frac{s_\rho}{s_e} P_e .$$

Using the following relations (see Appendix A.1 of [15])

$$P_e = \frac{1}{e_P} \text{ and } P_\rho = -\frac{e_\rho}{e_P} .$$

Substitution of these expressions into the entropy residual equation above gives Eq. (9), which is recalled below for completeness:

$$R_{\text{ent}}(\vec{r}, t) := \partial_t s + \vec{u} \cdot \vec{\nabla} s = \frac{Ds}{Dt} = \frac{s_e}{P_e} \left( \underbrace{\frac{DP}{Dt} - c^2 \frac{D\rho}{Dt}}_{\tilde{R}_{\text{ent}}(\vec{r}, t)} \right) .$$

814 **B. Derivation of the dissipative terms for the Euler equations with**  
815 **variable area using the entropy minimum principle**

816 The Euler equations (without viscous regularization) with variable area are  
817 recalled here

$$\partial_t (\rho A) + \vec{\nabla} \cdot (\rho \vec{u} A) = 0 \quad (43a)$$

$$\partial_t (\rho \vec{u} A) + \vec{\nabla} \cdot [A (\rho \vec{u} \otimes \vec{u} + P \mathbb{I})] = P \vec{\nabla} A \quad (43b)$$

$$\partial_t (\rho E A) + \vec{\nabla} \cdot [\vec{u} A (\rho E + P)] = 0. \quad (43c)$$

820 The specific entropy is a function of the density  $\rho$  and the internal energy  $e$ , i.e.,  
821  $s(e, \rho)$ . The above system of equations satisfies the minimum entropy principle  
822 [30],

$$A \rho \left( \partial_t s + \vec{u} \cdot \vec{\nabla} s \right) \geq 0. \quad (44)$$

823 The entropy function  $s$  satisfies the second law of thermodynamics,  $T ds =$   
824  $de - \frac{P}{\rho^2} d\rho$ , which implies  $s_e := T^{-1}$  and  $s_\rho := -PT^{-1}\rho^{-2}$ . One can show that  
825 [15]

$$s_e = T^{-1} \geq 0 \text{ and } P s_e + \rho^2 s_\rho = 0. \quad (45)$$

826 In order to apply the entropy viscosity method to the variable-area Euler equa-  
827 tions, dissipative terms need to be added to each equation in Eq. (43). The  
828 functional forms of these terms need to be such that the entropy residual de-  
829 rived with these terms present also satisfies the minimum entropy principle. To  
830 prove the minimum entropy principle, the extra terms appearing in the entropy  
831 residual are either recast as conservative terms or shown to be positive. The  
832 rest of this appendix presents this demonstration. Following [15], we first write  
833 the variable-area equations with dissipative terms:

$$\partial_t (\rho A) + \vec{\nabla} \cdot (\rho \vec{u} A) = \vec{\nabla} \cdot f \quad (46a)$$

$$\partial_t (\rho \vec{u} A) + \vec{\nabla} \cdot [A (\rho \vec{u} \otimes \vec{u} + P \mathbb{I})] = P \vec{\nabla} A + \vec{\nabla} \cdot g \quad (46b)$$

834

$$\partial_t (\rho E A) + \vec{\nabla} \cdot [\vec{u} A (\rho E + P)] = \vec{\nabla} \cdot (h + \vec{u} \cdot g). \quad (46c)$$

835 where  $f$ ,  $g$  and  $h$  are dissipative fluxes to be determined. Starting from the  
 836 modified system of equations given in Eq. (46), the entropy residual is derived  
 837 again. The derivation requires the following steps : express the governing laws  
 838 in terms of primitive variables  $(\rho, \vec{u}, e)$ , multiply the continuity equation by  $\rho s_\rho$   
 839 and the internal energy equation by  $s_e$ , and invoke multivariate chain rule, e.g.,  
 840  $\partial s / \partial x = s_e \partial e / \partial x + s_\rho \partial \rho / \partial x$ . These steps are similar to those used for the  
 841 standard Euler equations [15]. Some of the lengthy algebra is omitted here.  
 842 The above steps yield:

$$A \rho \left( \partial_t s + \vec{u} \cdot \vec{\nabla} s \right) = s_e \left[ \vec{\nabla} \cdot h + g : \vec{\nabla} u + \left( \frac{u^2}{2} - e \right) \vec{\nabla} \cdot f \right] + \rho s_\rho \vec{\nabla} \cdot f. \quad (47)$$

843 The next step consists of choosing a definition for each of the dissipative terms  
 844 so that the left hand-side is positive. The right hand-side of Eq. (47) can be  
 845 simplified using the relations  $g = A \mu \vec{\nabla}^s \vec{u} + f \otimes \vec{u}$  and  $h = \tilde{h} - 0.5 ||\vec{u}||^2 f$  to give

$$A \rho \left( \partial_t s + \vec{u} \cdot \vec{\nabla} s \right) = s_e \left[ \vec{\nabla} \cdot \tilde{h} - e \vec{\nabla} \cdot f \right] + \rho s_\rho \vec{\nabla} \cdot f + A s_e \mu \vec{\nabla}^s \vec{u} : \vec{\nabla} \vec{u}. \quad (48)$$

The right hand-side is now integrated by parts:

$$\begin{aligned} A \rho \left( \partial_t s + \vec{u} \cdot \vec{\nabla} s \right) &= \vec{\nabla} \cdot \left[ s_e \tilde{h} - s_e e f + \rho s_\rho f \right] \\ &\quad - \vec{\nabla} \cdot \tilde{h} \vec{\nabla} s_e + f \cdot \vec{\nabla} (e s_e) - f \cdot \vec{\nabla} (\rho s_\rho) + A s_e \mu \vec{\nabla}^s \vec{u} : \vec{\nabla} \vec{u} \end{aligned} \quad (49)$$

846 where  $\vec{\nabla}^s$  is the symmetric gradient. The term  $A s_e \mu \vec{\nabla}^s \vec{u} : \vec{\nabla} \vec{u}$  is positive and  
 847 thus, does not need any further modification. It remains to treat the other

848 terms of the right hand-side that we now call  $rhs$ :

$$rhs = \vec{\nabla} \cdot [s_e \tilde{h} - s_e e f + \rho s_\rho f] - \tilde{h} \cdot \vec{\nabla} s_e + f \cdot \vec{\nabla} (e s_e) - f \cdot \vec{\nabla} (\rho s_\rho).$$

849 The first term in  $rhs$  is a conservative term. By carefully choosing a definition  
 850 for  $\tilde{h}$  and  $f$ , the conservative term can be expressed as a function of the entropy  
 851  $s$ . The inclusion of the variable area in the choice of the dissipative terms is also  
 852 required so that, when assuming constant area, the standard Euler equations  
 853 are recovered. The following definitions for  $\tilde{h}$  and  $f$  are chosen:

$$\tilde{h} = A\kappa \vec{\nabla}(\rho e) \text{ and } f = A\kappa \vec{\nabla} \rho,$$

854 which yields, using the chain rule,

$$rhs = \vec{\nabla} \cdot (\rho A \kappa \vec{\nabla} s) - A \kappa \underbrace{\left[ \vec{\nabla}(\rho e) \vec{\nabla} s_e - \vec{\nabla} \rho \vec{\nabla} (e s_e) + \vec{\nabla} \rho \vec{\nabla} (\rho s_\rho) \right]}_{\mathbf{Q}}$$

855 It remains to treat the term  $\mathbf{Q}$  that can be recast under a quadratic form.

856 Following [15], one obtain:

$$\begin{aligned} \mathbf{Q} &= \rho X^t \Sigma X \\ \text{with } X &= \begin{bmatrix} \vec{\nabla} \rho \\ \vec{\nabla} e \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \rho^{-2} \partial_\rho (\rho^2 \partial_\rho s) & \partial_{\rho, e} s \\ \partial_{\rho, e} s & \partial_{e, e} s \end{bmatrix} \end{aligned}$$

857 The matrix  $\Sigma$  is symmetric and identical to the matrix obtained in [15]. The sign  
 858 of the quadratic form can be simply determined by studying the positiveness of  
 859 the matrix  $\Sigma$ . In this particular case, it is required to prove that the matrix is  
 860 negative definite: the quadratic form is on the right hand-side and is preceded by  
 861 a negative sign. According to [15], the convexity of the opposite of the entropy

862 function, i.e.,  $-s$ , with respect to the internal energy  $e$  and the specific volume  
863  $1/\rho$  is sufficient to ensure that the matrix  $\Sigma$  is negative definite.  
864 Thus, the right hand-side of the entropy residual Eq. (47) is now either recast  
865 as conservative terms, or known to be positive. Thus, the entropy minimum  
866 principle holds.

### 867 **C. Entropy residual for isentropic flows**

868 This appendix shows that the entropy residual is zero for isentropic flows.  
 869 For convenience, we recall here the entropy residual as a function of the pressure,  
 870 density, velocity, and speed of sound:

$$\tilde{R}_{\text{ent}} = \frac{DP}{Dt} - c^2 \frac{D\rho}{Dt}. \quad (50)$$

871 Assuming an isentropic flow, pressure is only a function of density, i.e.,  $P =$   
 872  $f(\rho)$  or equivalently  $\rho = f^{-1}(P)$ . Using the definition of the speed of sound  
 873  $c^2 = \left. \frac{\partial P}{\partial \rho} \right)_s$  and the above form of the equation of state, we have

$$c^2 = \left. \frac{\partial P}{\partial \rho} \right)_s = \frac{dP}{d\rho} = \frac{df(\rho)}{d\rho}. \quad (51)$$

874 Using the chain rule, the entropy residual in Eq. (50) can be recast as follows  
 875 and proven equal to zero:

$$\tilde{R}_{\text{ent}} = \frac{df(\rho)}{d\rho} \frac{D\rho}{Dt} - c^2 \frac{D\rho}{Dt} = c^2 \frac{D\rho}{Dt} - c^2 \frac{D\rho}{Dt} = 0. \quad (52)$$