

Entropy-based viscous regularization for the multi-dimensional Euler equations in low-Mach and transonic flows

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Abstract

We present a new version of the entropy viscosity method, a viscous regularization technique for hyperbolic conservation laws, that is well-suited to low-Mach flows. By means of a low-Mach asymptotic study, new expressions for the entropy viscosity coefficients are derived. These definitions are valid for a wide range of Mach numbers, from subsonic flows (with very low Mach numbers) to supersonic flows, and no longer depend on an analytical expression for the entropy function. In addition, the entropy viscosity method is extended to Euler equations with variable area for nozzle flow problems. The effectiveness of the method is demonstrated using various 1-D and 2-D benchmark tests: flow in a converging-diverging nozzle; Leblanc shock tube; slow moving shock; strong shock for liquid phase; low-Mach flows around a cylinder and over a circular hump; and supersonic flow in a compression corner. Convergence studies are performed for both smooth solutions and solutions with shocks present.

Key words: entropy viscosity method, artificial viscosity, low-Mach regime, shock capturing, Euler equations with variable area.

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1. Introduction

Solving accurately compressible fluid equations in the low-Mach limit is an ongoing topic of research. In many engineering applications, compressibility effects require the solution of the compressible fluid equations in nearly incompressible regimes and/or for low-Mach flow problems. For example, such flows are encountered in aerodynamics in the study of airships. In the nuclear industry, flows are nearly in the incompressible regime but compressible effects cannot be neglected because of the intense heat source, and because of some postulated accident scenarios, and thus need to be accurately resolved. Hence, there is a strong interest to develop computational methods that can solve both compressible and incompressible flow problems.

When solving Euler equations for a wide range of Mach numbers, multiple questions must be addressed: stability, accuracy and solution convergence in the low-Mach regime. Because of the hyperbolic nature of the equations, shocks can form during transonic and supersonic flows and require the use of adequate numerical techniques to stabilize solution and correctly resolve the discontinuities. A wide range of stabilization methods are available in the literature: approximate Riemann solvers [1], flux-limiter techniques [2, 3], pressure-based viscosity methods [4], Lapidus method [5, 6, 7], and the entropy-viscosity method [8, 9], among others. These numerical methods are usually developed using simple equations of state and tested for transonic and supersonic flows where the disparity between the acoustic wave speed and the fluid speed is not excessively large and thus the Mach number is of order one. This approach, however, leads to a well-known accuracy problem in the low-Mach regime where the fluid velocity is smaller than the speed of sound by multiple orders of magnitude. The numerical dissipative terms become ill-scaled in the low-Mach regime and lead to the wrong numerical solution by changing the nature of the equations

28 solved. This behavior is well documented in the literature [10, 11, 12]. In [10], a
 29 low-Mach asymptotic study has demonstrated convergence of the compressible
 30 Euler equations to the incompressible ones. Many well-known stabilization tech-
 31 niques, for instance, the Roe scheme and the SUPG technique, do not yield the
 32 correct solution in the low-Mach regime and suitable modifications have been
 33 proposed (see [13] for the Roe scheme and [12] for the SUPG method) to en-
 34 sure the convergence to the correct solution while preserving the original shock
 35 stabilization properties of these schemes. Additionally, the time step size may
 36 be severely restricted when solving compressible fluid equations with an explicit
 37 time discretization because of the large disparity between the fluid velocity and
 38 the speed of sound. To avoid an excessive number of explicit time steps, time
 39 preconditioners have been proposed and proved efficient [11]; however, because
 40 they modify the time derivatives in the governing equations, such acceleration
 41 techniques can only be used to obtain steady-state solutions for low-Mach flows
 42 using explicit schemes. To avoid modifying the time derivatives, the temporal
 43 implicit capabilities of the MOOSE multiphysics framework [14] are used. With
 44 such a choice, low-Mach steady-state solutions can be obtained effectively while
 45 preserving the accuracy of the transient solution; however, it requires the use of
 46 nonlinear solvers.

47 In this paper, we employ the entropy viscosity method as a numerical sta-
 48 bilization for the inviscid Euler equation and assess its performance in the low-
 49 Mach regime. The entropy viscosity method is a viscous regularization technique
 50 introduced by Guermond et al. to solve hyperbolic systems of equations and
 51 has successfully been applied to multi-dimensional supersonic flows with various
 52 spatial discretization schemes [15]. It is fairly straightforward to implement, can
 53 be used with unstructured grids, and has dissipative terms that are consistent
 54 with the entropy minimum principle. However, it has not been evaluated in the

55 low-Mach regime.

56 This paper is organized as follows: in Section 2 the current definition of the
57 entropy viscosity method is recalled and its ill-scaled nature in the low-Mach
58 regime is discussed. In Section 3, a new formulation of the viscosity residual
59 is derived. This formulation no longer requires an analytical expression for the
60 entropy function. A low-Mach asymptotic study is carried out to adapt the
61 definition of the entropy viscosity coefficients in the incompressible limit while
62 ensuring that the viscosity coefficients scale appropriately for all flow speeds
63 (from low-Mach to supersonic). In Section 4, we extend the entropy viscosity
64 method to Euler equations with variable area in order to model nozzle flows:
65 the viscous dissipative terms are adapted so that the entropy minimum principle
66 remains satisfied. Spatial and temporal discretizations and solution techniques
67 are presented in Section 5. 1-D and 2-D numerical results are provided in
68 Section 6 for a wide range of Mach numbers: liquid and gas nozzle flow problems,
69 low-Mach flows over a cylinder and a circular bump (with Mach numbers as
70 low as 10^{-7}), and supersonic flows in a compression corner [16]. Convergence
71 studies are performed in 1-D in order to demonstrate the accuracy of the solution
72 technique.

73 2. The Entropy Viscosity Method

74 2.1. Background

75 Euler equations in conservative form are given by

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \quad (1a)$$

76

$$\partial_t (\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u} + P \mathbb{I}) = 0 \quad (1b)$$

77

$$\partial_t (\rho E) + \vec{\nabla} \cdot [\vec{u} (\rho E + P)] = 0 \quad (1c)$$

78 where ρ , $\rho\vec{u}$ and E are the density, the momentum and the total specific energy,
 79 respectively, and will be referred to as the conservative variables. \vec{u} is the fluid
 80 velocity and its specific internal energy is denoted by $e = E - \frac{u^2}{2}$. An equation
 81 of state, dependent upon ρ and e , is used to compute the pressure P . The tensor
 82 product $\vec{a} \otimes \vec{b}$ is such that $(\vec{a} \otimes \vec{b})_{i,j} = a_i b_j$. The identity tensor is denoted by \mathbb{I} .

83 Next, the entropy viscosity method [8, 9, 17, 18] applied to Eq. (1) is recalled.
 84 The method consists of adding dissipative terms with a viscosity coefficient mod-
 85 ulated by the entropy production; this allows for a high-order accuracy when
 86 the solution is smooth (provided that the spatial and temporal discretizations
 87 also are high order). The derivation of the viscous regularization (or dissipa-
 88 tive terms) is carried out to be consistent with the entropy minimum principle;
 89 details and proofs of the derivation can be found in [15]. The viscous regular-
 90 ization thus obtained is valid for any equation of state as long as the physical
 91 entropy function s is concave (or $-s$ is a convex function) with respect to the
 92 internal energy e and the specific volume $1/\rho$. The Euler equations with viscous
 93 regularization become

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{u}) = \vec{\nabla} \cdot (\kappa \vec{\nabla} \rho) \quad (2a)$$

$$\partial_t (\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u} + P \mathbb{I}) = \vec{\nabla} \cdot (\mu \rho \vec{\nabla}^s \vec{u} + \kappa \vec{u} \otimes \vec{\nabla} \rho) \quad (2b)$$

$$\partial_t (\rho E) + \vec{\nabla} \cdot [\vec{u} (\rho E + P)] = \vec{\nabla} \cdot \left(\kappa \vec{\nabla} (\rho e) + \frac{1}{2} \|\vec{u}\|^2 \kappa \vec{\nabla} \rho + \rho \mu \vec{u} \vec{\nabla}^s \vec{u} \right) \quad (2c)$$

96 where κ and μ are positive viscosity coefficients (in units of $\text{length}^2/\text{time}$). $\vec{\nabla}^s \vec{u}$
 97 denotes the symmetric gradient operator and guarantees the method to be ro-
 98 tationally invariant [15]. The viscosity coefficients are key ingredients in the
 99 viscous regularization of Eq. (2). Other stabilization approaches have been pro-
 100 posed in the literature, for instance, the Lapidus method [7, 5] or pressure-based
 101 viscosity methods [4]. Here, we follow the work of Guermond et al. and define

102 the viscosity coefficients, κ and μ , based on the local entropy production. These
 103 coefficients are numerically evaluated using the local entropy residual $R_{\text{ent}}(\vec{r}, t)$
 104 defined in Eq. (3); $R_{\text{ent}}(\vec{r}, t)$ is known to be peaked in shocks and vanishingly
 105 small elsewhere [1].

$$R_{\text{ent}}(\vec{r}, t) := \partial_t s + \vec{u} \cdot \vec{\nabla} s \quad (3)$$

106 In the current version of the method, the ratio of κ to μ is defined through
 107 a numerical Prandtl number, $\text{Pr} = \kappa/\mu$. Pr is a user-defined parameter and
 108 is usually taken in the range $[0.001; 1]$. Since the entropy residual $R_{\text{ent}}(\vec{r}, t)$
 109 may be extremely large in shocks, the definition of the viscosity coefficients
 110 also includes a first-order viscosity coefficient that serves as an upper bound for
 111 the entropy-based viscosity coefficients. The first-order viscosity coefficients,
 112 denoted by μ_{max} and κ_{max} , are chosen so that the numerical scheme becomes
 113 equivalent to an upwind scheme when the first-order coefficients are employed.
 114 The upwind scheme is known to be over-dissipative but guarantees monotonicity
 115 [1]. In practice, the viscosity coefficients only saturate to the first-order viscosity
 116 coefficients in shocks and are much smaller elsewhere, hence avoiding the over-
 117 dissipation of the upwind method. The first-order viscosity coefficients μ_{max}
 118 and κ_{max} are equal and set proportional to the largest local eigenvalue $\|\vec{u}\| + c$:

$$\mu_{\text{max}}(\vec{r}, t) = \kappa_{\text{max}}(\vec{r}, t) = \frac{h}{2} (\|\vec{u}(t, \vec{r})\| + c(t, \vec{r})), \quad (4)$$

119 where h denotes the local grid size (for higher than linear finite element rep-
 120 resentations, h is defined as the ratio of the grid size to the polynomial order
 121 of the test functions used, see Eq. 2.4 in [18]). For simplicity, the first-order
 122 viscosity coefficients will only be referred to as $\kappa_{\text{max}}(\vec{r}, t)$. In practice, these

quantities are evaluated within a given cell K at quadrature points:

$$\kappa_{\max}^K(\vec{r}_q, t) = \frac{h_K}{2} \left(\|\vec{u}(t, \vec{r}_q)\| + c(t, \vec{r}_q) \right), \quad (5)$$

where \vec{r}_q denotes the position of a quadrature point. As stated earlier, the entropy viscosity coefficients, which we denote by κ_e and μ_e , are set proportional to the entropy production evaluated by computing the local entropy residual R_{ent} . The definitions also include the inter-element jump $J[s]$ of the entropy flux, allowing for the detection of discontinuities other than shocks (e.g., contact). κ_e and μ_e are computed as follows

$$\mu_e^K(\vec{r}_q, t) = h_K^2 \frac{\max(|R_{\text{ent}}^K(\vec{r}_q, t)|, J^K[s](t))}{\|s - \bar{s}\|_\infty} \quad (6a)$$

$$\kappa_e^K(\vec{r}_q, t) = \text{Pr} \mu_e^K(\vec{r}_q, t), \quad (6b)$$

where $\|\cdot\|_\infty$ and $\bar{\cdot}$ denote the L_∞ -norm and the average operator over the entire computational domain, respectively. The definition of the entropy jump $J[s]$ is spatial discretization-dependent and examples of definitions can be found in [18] for discontinuous Galerkin discretization. For continuous finite element methods (FEM), the jump of a given quantity is defined as the change of its normal derivative ($\partial_n(\cdot) = \vec{n} \cdot \vec{\nabla}(\cdot)$) across the common face separating the two elements, and will be further referred to as the inter-element jump. We take the largest value over all faces f present on the boundary ∂K of element K :

$$J^K[s](t) = \max_{f \in \partial K} \max_{\vec{r}_q \in f} \left(\|\vec{u}(\vec{r}_q, t)\| \|\vec{\nabla} s(\vec{r}_q, t) \cdot \vec{n}(\vec{r}_q)\|_f \right), \quad (7)$$

where $\llbracket a(\vec{r}_q) \rrbracket_f$ denotes the inter-element jump in $a(\vec{r})$ at quadrature point \vec{r}_q on face f (the quadrature points \vec{r}_q are taken on the faces f of the element K). With the definition given in Eq. (7), the jump is constant over each el-

ement K of the computational domain. The denominator $\|s - \bar{s}\|_\infty$ is used for dimensionality purposes. Currently, there are no theoretical justifications for choosing the denominator beyond a dimensionality argument. Finally, the viscosity coefficients μ and κ are as follows:

$$\mu(\vec{r}, t) = \min\left(\mu_e(\vec{r}, t), \mu_{\max}(\vec{r}, t)\right)$$

$$\text{and } \kappa(\vec{r}, t) = \min\left(\kappa_e(\vec{r}, t), \kappa_{\max}(\vec{r}, t)\right). \quad (8)$$

139 Given these definitions, we have the following properties. In shock regions, the
 140 entropy viscosity coefficients will experience a peak because of entropy produc-
 141 tion and thus will saturate to the first-order viscosity. The first-order coefficients
 142 are known to be over-dissipative and will smooth out any oscillatory behavior.
 143 Elsewhere in the domain, entropy production will be small and the viscosity
 144 coefficients μ and κ will remain small. High-order accuracy for entropy-based
 145 viscous stabilization has been demonstrated using several 1-D shock tube ex-
 146 amples and various 2-D tests [8, 9, 18].

147 2.2. Issues in the Low-Mach Regime

148 In the low-Mach Regime, a smooth flow is known to approach the isentropic
 149 limit, resulting in very little entropy production. Since the entropy viscosity
 150 method is directly based on the evaluation of the local entropy production, it
 151 is of interest to study how the entropy viscosity coefficients μ_e and κ_e scale
 152 in the low-Mach regime. In practice, the entropy residual R_{ent} will be very
 153 small in that regime and so will be the denominator $\|s - \bar{s}\|_\infty$, thus making
 154 the definition of the viscosity coefficients in Eq. (6) undetermined and likely ill-
 155 scaled. One possible approach would consist of expanding the numerator and
 156 denominator in terms of the Mach number and deriving its limit when the Mach
 157 number goes to zero. Such derivation may not be straightforward, especially

for general equations of state. However, this can be avoided by noting that the entropy residual R_{ent} can be recast as a function of pressure, density, velocity, and speed of sound as will be shown in Eq. (9) of Section 3.1. This alternate entropy residual definition is the basis for the low-Mach analysis carried out in this paper and possesses several advantages that are detailed next.

3. An All-speed Reformulation of the Entropy Viscosity Method

In this section, the entropy residual R_{ent} is recast as a function of pressure, density, velocity and speed of sound. Then, a low-Mach asymptotic study is carried out for the Euler equations with viscous regularization in order to derive an appropriate normalization parameter that is valid in the isentropic low-Mach regime as well as for transonic and supersonic flows.

3.1. New Definition of the Entropy Production Residual

The first step in defining viscosity coefficients that behave well in the low-Mach limit is to recast the entropy residual in terms of thermodynamic variables. This provides physical insight on possible normalization choices that can be valid in both low-Mach and transonic flows. The alternate definition of the entropy residual, the derivation of which is given in Appendix A, is the following:

$$R_{\text{ent}}(\vec{r}, t) := \partial_t s + \vec{u} \cdot \vec{\nabla} s = \frac{Ds}{Dt} = \frac{s_e}{P_e} \left(\underbrace{\frac{DP}{Dt} - c^2 \frac{D\rho}{Dt}}_{\tilde{R}_{\text{ent}}(\vec{r}, t)} \right), \quad (9)$$

where $\frac{D}{Dt}$ denotes the material derivative ($\frac{D}{Dt} := \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}$), and x_y is the standard shorthand notation for the partial derivative of x with respect to y , e.g., $P_e := \frac{\partial P}{\partial e}$. The entropy residuals R_{ent} and \tilde{R}_{ent} are proportional to one another and will experience similar variations in space and time. When using the SGEOS, the coefficient of proportionality $\frac{s_e}{P_e}$ can be recast under the form $\frac{s_e}{P_e} =$

180 $\frac{C_v(\gamma-1)}{P+\gamma P_\infty}$ which remains positive. Thus, one may elect to employ \tilde{R}_{ent} instead
 181 of R_{ent} for the evaluation of the local entropy residual. The new expression
 182 presents several advantages which include:

- 183 • An analytical expression of the entropy function s is no longer needed: the
 184 residual \tilde{R}_{ent} is evaluated using the local values of pressure, density, ve-
 185 locity and speed of sound. Deriving an entropy function for some complex
 186 equations of state may be difficult;
- 187 • Suitable normalizations for the residual \tilde{R}_{ent} can be devised. Examples
 188 include the pressure itself or combinations of the density, the speed of
 189 sound and the norm of the velocity, i.e., ρc^2 , $\rho c||\vec{u}||$ or $\rho||\vec{u}||^2$.

190 Denoting the normalization of \tilde{R}_{ent} by norm_P , the entropy-based viscosity co-
 191 efficients μ_e and κ_e can be re-defined as follows:

$$\mu_e^K(\vec{r}_q, t) = h_K^2 \frac{\max\left(|\tilde{R}_{\text{ent}}^K(\vec{r}_q, t)|, J^K(t)\right)}{\text{norm}_P^\mu}, \quad (10a)$$

192 and

$$\kappa_e^K(\vec{r}_q, t) = h_K^2 \frac{\max\left(|\tilde{R}_{\text{ent}}^K(\vec{r}_q, t)|, J^K(t)\right)}{\text{norm}_P^\kappa}, \quad (10b)$$

193 where

$$J^K(t) = \max_{f \in \partial K} \max_{\vec{r}_q \in f} \left(||\vec{u}(\vec{r}_q, t)|| \max\left(J^K[P](t), c^2(\vec{r}_q, t) ||J^K[\rho](t)||\right) \right) \quad (10c)$$

194 Note that now the jump operator J^K acts on the variables appearing in \tilde{R}_{ent} ,
 195 namely, pressure and density. The μ and κ coefficients are kinematic viscosities
 196 (units of m^2/s); the normalization parameters norm_P are thus in units of pres-
 197 sure, hence the use of the subscript P . Note also that we are not requiring the
 198 same normalization for both μ_e and κ_e so the entropy viscosity coefficients can

be different. The low-Mach asymptotic study presented next will determine the proper normalization.

3.2. Asymptotic Study in the Low-Mach Regime

The Euler equations with viscous stabilization, Eq. (6), bear some similarities with the Navier-Stokes equations in the sense that dissipative terms (containing second-order spatial derivatives) are present in both sets of equations. An abundant literature exists regarding the low-Mach asymptotic of the Navier-Stokes equations [10, 11, 12, 19]. The asymptotic study presented here is inspired by the work of Muller et al. [19] where an asymptotic derivation for the Navier-Stokes was presented. We remind the reader that the objective is to determine appropriate scaling for the entropy viscosity coefficients so that the dissipative terms remain well-scaled for two limit cases: (i) the isentropic low-Mach limit where Euler equations degenerate to an incompressible system of equations in the low-Mach limit and (ii) the non-isentropic limit with formation of shocks. The isentropic limit of the Euler equations with viscous regularization should yield incompressible fluid flow solutions in the low-Mach limit, namely, that the pressure fluctuations are of the order M^2 and that the velocity satisfies the divergence constraint $\vec{\nabla} \cdot \vec{u}_0 = 0$ [10, 11, 12]. For non-isentropic situations, shocks may form for any value of Mach number and the minimum entropy principle should still be satisfied so that numerical oscillations, if any, be controlled by the entropy viscosity method independently of the value of the Mach number. Our objective is to determine the appropriate scaling for norm_P^κ and norm_P^μ in these two limit cases.

The first step in the study of the limit cases (i) and (ii) is to re-write Eq. (2)

in a non-dimensional manner. To do so, the following variables are introduced:

$$\begin{aligned} \rho^* &= \frac{\rho}{\rho_\infty}, \quad u^* = \frac{u}{u_\infty}, \quad P^* = \frac{P}{\rho_\infty c_\infty^2}, \quad E^* = \frac{E}{c_\infty^2}, \\ x^* &= \frac{x}{L_\infty}, \quad t^* = \frac{t}{L_\infty/u_\infty}, \quad \mu^* = \frac{\mu}{\mu_\infty}, \quad \kappa^* = \frac{\kappa}{\kappa_\infty}, \end{aligned} \quad (11)$$

where the subscript ∞ denote the far-field or stagnation quantities and the superscript $*$ stands for the non-dimensional variables. The far-field reference quantities are chosen such that the dimensionless flow quantities are of order 1. The reference Mach number is given by

$$M_\infty = \frac{u_\infty}{c_\infty}, \quad (12)$$

where c_∞ is a reference value for the speed of sound. Then, the scaled Euler equations with viscous regularization are:

$$\partial_{t^*} \rho^* + \vec{\nabla}^* \cdot (\rho^* \vec{u}^*) = \frac{1}{\text{Pe}_\infty} \vec{\nabla}^* \cdot (\kappa^* \vec{\nabla}^* \rho^*) \quad (13a)$$

$$\begin{aligned} \partial_{t^*} (\rho^* \vec{u}^*) + \vec{\nabla}^* \cdot (\rho^* \vec{u}^* \otimes \vec{u}^*) + \frac{1}{M_\infty^2} \vec{\nabla}^* P^* &= \frac{1}{\text{Re}_\infty} \vec{\nabla}^* \cdot (\rho^* \mu^* \vec{\nabla}^{s,*} \vec{u}^*) \\ &+ \frac{1}{\text{Pe}_\infty} \vec{\nabla}^* \cdot (\vec{u}^* \otimes \kappa^* \vec{\nabla}^* \rho^*) \end{aligned} \quad (13b)$$

$$\begin{aligned} \partial_{t^*} (\rho^* E^*) + \vec{\nabla}^* \cdot [\vec{u}^* (\rho^* E^* + P^*)] &= \frac{1}{\text{Pe}_\infty} \vec{\nabla}^* \cdot (\kappa^* \vec{\nabla}^* (\rho^* e^*)) \\ &+ \frac{M_\infty^2}{\text{Re}_\infty} \vec{\nabla}^* \cdot (\vec{u}^* \rho^* \mu^* \vec{\nabla}^{s,*} \vec{u}^*) + \frac{M_\infty^2}{2\text{Pe}_\infty} \vec{\nabla}^* \cdot (\kappa^* (u^*)^2 \vec{\nabla}^* \rho^*) \end{aligned} \quad (13c)$$

228 where the numerical Reynolds (Re_∞) and Péclet (Pé_∞) numbers are defined as:

$$\text{Re}_\infty = \frac{u_\infty L_\infty}{\mu_\infty} \text{ and } \text{Pé}_\infty = \frac{u_\infty L_\infty}{\kappa_\infty} . \quad (14)$$

229 Note that the Prandtl number used in the original version of the entropy vis-
230 cosity method is simply given by

$$\text{Pr}_\infty = \text{Pé}_\infty / \text{Re}_\infty . \quad (15)$$

231 The numerical Reynolds and Péclet numbers defined in Eq. (14) are related to
232 the entropy viscosity coefficients μ_∞ and κ_∞ . Thus, once a scaling (in powers of
233 M_∞) is obtained for Re_∞ and Pé_∞ , the corresponding normalization parameters
234 norm_P^μ and norm_P^κ will automatically be set. For brevity, the superscripts * are
235 omitted in the remainder of this section.

236 For simplicity, we use here the ideal gas equation of state; its non-dimensionalized
237 expression is given by

$$P^* = (\gamma - 1) \rho^* \left(E^* - \frac{1}{2} M_\infty^2 (u^*)^2 \right) = (\gamma - 1) \rho^* e^* . \quad (16)$$

238 In the low-Mach isentropic limit, shocks cannot form and the compressible
239 Euler equations are known to converge to the incompressible equations when the
240 Mach number tends to zero. When adding dissipative terms, as is the case with
241 the entropy viscosity method, the main properties of the low-Mach asymptotic
242 limit must be preserved. We begin by expanding each variable in powers of the
243 Mach number. As an example, the expansion for the pressure is given by:

$$P(\vec{r}, t) = P_0(\vec{r}, t) + P_1(\vec{r}, t) M_\infty + P_2(\vec{r}, t) M_\infty^2 + \dots \quad (17)$$

244 By studying the resulting momentum equations for various powers of M_∞ , it is

245 observed that the leading order and first-order pressure terms, P_0 and P_1 , are
 246 spatially constant if and only if $\text{Re}_\infty = \text{Pe}_\infty = 1$. In this case, we have at order
 247 M_∞^{-2} :

$$\vec{\nabla} P_0 = 0 \quad (18a)$$

248 and at order M_∞^{-1}

$$\vec{\nabla} P_1 = 0. \quad (18b)$$

249 Using the scaling $\text{Re}_\infty = \text{Pe}_\infty = 1$, the leading-order expressions for the conti-
 250 nuity, momentum, and energy equations are:

$$\partial_t \rho_0 + \vec{\nabla} \cdot (\rho \vec{u})_0 = \vec{\nabla} \cdot (\kappa \vec{\nabla} \rho)_0 \quad (19a)$$

251

$$\partial_t (\rho \vec{u})_0 + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u})_0 + \vec{\nabla} P_2 = \vec{\nabla} \cdot (\rho \mu \vec{\nabla}^s \vec{u} + \kappa \vec{u} \otimes \vec{\nabla} \rho)_0 \quad (19b)$$

252

$$\partial_t (\rho E)_0 + \vec{\nabla} \cdot [\vec{u} (\rho E + P)]_0 = \vec{\nabla} \cdot (\kappa \vec{\nabla} (\rho e))_0 \quad (19c)$$

253 where the notation $(fg)_0$ means that we only keep the 0th-order terms in the
 254 product fg . The leading-order of the equation of state is given by

$$P_0 = (\gamma - 1)(\rho E)_0. \quad (20)$$

255 Using Eq. (20), the energy equation can be recast as a function of the leading-
 256 order pressure, P_0 , as follows:

$$\partial_t P_0 + \gamma \vec{\nabla} \cdot (\vec{u} P)_0 = \vec{\nabla} \cdot (\kappa \vec{\nabla} (P))_0. \quad (21)$$

257 From Eq. (18a), we infer that P_0 is spatially constant. Thus, Eq. (21) becomes

$$\frac{1}{\gamma P_0} \frac{dP_0}{dt} = -\vec{\nabla} \cdot \vec{u}_0 \quad (22)$$

258 and, at steady state, we have

$$\vec{\nabla} \cdot \vec{u}_0 = 0. \quad (23)$$

259 That is, the leading-order of velocity is divergence-free. The same reasoning can
 260 be applied to the leading-order of the continuity equation (Eq. (19a)) to show
 261 that the material derivative of the density variable is zero:

$$\frac{D\rho_0}{Dt} := \partial_t \rho_0 + \vec{u}_0 \cdot \vec{\nabla} \rho_0 = 0. \quad (24)$$

262 Therefore, we conclude that by setting the Reynolds and Péclet numbers to
 263 one, the incompressible fluid results are retrieved in the low-Mach limit when
 264 employing the compressible Euler equations with viscous regularization terms
 265 present. In addition, the scaling of the Prandtl number can also be obtained
 266 using Eq. (15), hence clarifying the use of the numerical Prandtl in the original
 267 entropy viscosity method [8].

268 3.3. Scaling of Re_∞ and $Pé_\infty$ for non-isentropic flows

269 Next, we consider the non-isentropic case. Recall that even subsonic flows
 270 can present shocks (for instance, a step initial condition in the pressure will trig-
 271 ger shock formation, independently of the Mach number). The non-dimensional
 272 form of the Euler equations given in Eq. (13) provides some insight on the dom-
 273 inant terms as a function of the Mach number. This is particular obvious in
 274 the momentum equation, Eq. (13b), where the gradient of pressure is scaled by
 275 $1/M_\infty^2$. In the non-isentropic case, we no longer have $\frac{\vec{\nabla} P}{M_\infty^2} = \vec{\nabla} P_2$ and there-
 276 fore the pressure gradient term may need to be stabilized by some dissipative
 277 terms of the same scaling so as to prevent spurious oscillations from forming.
 278 By inspecting the dissipative terms presents in the the momentum equation,
 279 having a dissipative term that scales as $1/M_\infty^2$ leads to the following three op-
 280 tions: (a) $Re_\infty = M_\infty^2$ and $Pé_\infty = 1$, (b) $Re_\infty = 1$ and $Pé_\infty = M_\infty^2$, or (c)

281 $\text{Re}_\infty = \text{Pé}_\infty = M_\infty^2$. Any of these choices will also affect the stabilization of
 282 the continuity and energy equations. For instance, using a Péclet number equal
 283 to M_∞^2 may effectively stabilize the continuity equation in the shock region but
 284 this may also add an excessive amount of dissipation for subsonic flows at the
 285 location of the contact wave. Such a behavior may not be suitable for accuracy
 286 purpose, making options (b) and (c) inappropriate. The same reasoning, left to
 287 the reader, can be carried out for the energy equation (Eq. (13c)) and results in
 288 the same conclusion. The remaining choice, option (a), has the proper scaling:
 289 in this case, only the dissipation terms involving $\vec{\nabla}^{s,*} \vec{u}^*$ scale as $1/M_\infty^2$ since
 290 $\text{Re}_\infty = M_\infty^2$, leaving the regularization of the continuity equation unaffected
 291 because $\text{Pé}_\infty = 1$.

292 *3.4. An All-speed normalization of the entropy residual*

293 The study of the above limit cases yields two different possible scalings for
 294 the Reynolds number: $\text{Re}_\infty = 1$ in the low-Mach limit and $\text{Re}_\infty = M_\infty^2$ for
 295 non-isentropic flows, whereas the numerical Péclet number always scales as one.
 296 In order to have a stabilization method valid for a wide range of Mach numbers,
 297 from very low-Mach to supersonic flows, these two scalings should be combined
 298 in a unique definition.

299 We begin with the normalization parameter norm_P^κ . Using the definition of
 300 the viscosity coefficients given in Eq. (10) and the scaling of Eq. (11), it can be
 301 shown that:

$$\kappa_\infty = \frac{\rho_\infty c_\infty^2 u_\infty L}{\text{norm}_{P,\infty}^\kappa}, \quad (25)$$

302 where $\text{norm}_{P,\infty}$ is the reference far-field quantity for the normalization parame-
 303 ter norm_P . Substituting Eq. (25) into Eq. (14) and recalling that the numerical
 304 Péclet number scales as unity, we obtain:

$$\text{norm}_{P,\infty}^\kappa = \text{Pé}_\infty \rho_\infty c_\infty^2 = \rho_\infty c_\infty^2. \quad (26)$$

Eq. (26) provides a proper normalization factor to define the κ viscosity coefficient. The derivation for norm_P^μ is similar and yields

$$\text{norm}_P^\mu = \text{Re}_\infty \rho_\infty c_\infty^2 = \begin{cases} \rho \|\vec{u}\|^2 & \text{for non-isentropic flows} \\ \rho c^2 = \text{norm}_P^\kappa & \text{for low-Mach flows} \end{cases}. \quad (27)$$

A smooth function to transition between these two states is as follows:

$$\sigma(M) = \frac{\tanh(a(M - M^{\text{thresh}})) + |\tanh(a(M - M^{\text{thresh}}))|}{2}, \quad (28)$$

where M^{thresh} is a threshold Mach number value beyond which the flow is no longer considered to be low-Mach (we use $M^{\text{thresh}} = 0.05$), M is the local Mach number, and the scalar a determines how rapidly the transition from $\text{norm}_P^\mu = \rho c^2$ to $\text{norm}_P^\mu = \rho \|\vec{u}\|^2$ occurs in the vicinity of M^{thresh} (we use $a = 3$). It is easy to verify that

$$\text{norm}_P^\mu = (1 - \sigma(M))\rho c^2 + \sigma(M)\rho \|\vec{u}\|^2 \quad (29)$$

satisfies Eq. (27). Finally, we summarize the definition of the viscosity coefficients μ and κ for completeness:

$$\kappa(\vec{r}, t) = \min(\mu_{\text{max}}(\vec{r}, t), \kappa_e(\vec{r}, t)), \quad (30a)$$

$$\mu(\vec{r}, t) = \min(\mu_{\text{max}}(\vec{r}, t), \mu_e(\vec{r}, t)), \quad (30b)$$

where the first-order viscosity is given by

$$\kappa_{\text{max}}(\vec{r}, t) = \mu_{\text{max}}(\vec{r}, t) = \frac{h}{2}(\|\vec{u}\| + c) \quad (30c)$$

317 and the entropy viscosity coefficients by

$$\kappa_e(\vec{r}, t) = \frac{h^2 \max(\tilde{R}_{\text{ent}}, J)}{\rho c^2} \text{ and } \mu_e(\vec{r}, t) = \frac{h^2 \max(\tilde{R}_{\text{ent}}, J)}{\text{norm}_P^\mu} \quad (30d)$$

318 with the jumps given by

$$J = \max \left(\|\vec{u}\| [|\vec{\nabla} P \cdot \vec{n}|], \|\vec{u}\| c^2 [|\vec{\nabla} \rho \cdot \vec{n}|] \right) \quad (30e)$$

319 where norm_P^κ is computed from Eq. (29). The jump J is a function of the jump
 320 of pressure and density gradients across the face with respect to its normal vector
 321 \vec{n} . Then, the largest value over all faces is determined and used in the definition
 322 of the viscosity coefficients. With the definition of the viscosity coefficients μ and
 323 κ proposed in Eq. (30), the dissipative terms are expected to scale appropriately
 324 for very low-Mach regimes as well for transonic and supersonic flows.

325 **Remark 1.** *When using the definition of the first-order viscosity coefficient*
 326 *μ_{\max} , it can be shown that the Reynolds and Péclet numbers scale as the in-*
 327 *verse of the Mach number leading to ill-scaled dissipative terms in the low-Mach*
 328 *limit. This behavior is undesirable but in the shock region where the second-order*
 329 *viscosity coefficient saturates to the first-order viscosity coefficient.*

330 **4. Extension of the entropy viscosity technique to Euler equations** 331 **with variable area**

332 Fluid flows in nozzles and in pipes of varying cross-sectional area can be
 333 modeled using the variable-area variant of the Euler equations, where the con-
 334 servative variables are now multiplied by the area A . In addition, these equa-
 335 tions differ from the standard Euler equations in that the momentum equation
 336 Eq. (31b) contains a non-conservative term proportional to the area gradient.

337 Here, the variable area is assumed to be a smooth function of space only.

$$\partial_t (\rho A) + \vec{\nabla} \cdot (\rho \vec{u} A) = 0, \quad (31a)$$

338

$$\partial_t (\rho \vec{u} A) + \vec{\nabla} \cdot [A (\rho \vec{u} \otimes \vec{u} + P \mathbb{I})] = P \vec{\nabla} A, \quad (31b)$$

339

$$\partial_t (\rho E A) + \vec{\nabla} \cdot [\vec{u} A (\rho E + P)] = 0. \quad (31c)$$

340 The application of the entropy viscosity method to the Euler equations with
 341 variable area is not fundamentally different to its application to the standard
 342 Euler equations. However, we need to derive the associated dissipative terms
 343 and verify that the entropy minimum principle is still satisfied. The variable-
 344 area Euler equations with viscous regularization are given below; details of the
 345 derivation are provided in Appendix B.

$$\partial_t (\rho A) + \vec{\nabla} \cdot (\rho \vec{u} A) = \vec{\nabla} \cdot (A \kappa \vec{\nabla} \rho), \quad (32a)$$

346

$$\partial_t (\rho \vec{u} A) + \vec{\nabla} \cdot [A (\rho \vec{u} \otimes \vec{u} + P \mathbb{I})] = P \vec{\nabla} A + \vec{\nabla} \cdot \left[A \left(\mu \rho \vec{\nabla}^s \vec{u} + \kappa \vec{u} \otimes \vec{\nabla} \rho \right) \right], \quad (32b)$$

$$\begin{aligned} \partial_t (\rho A E) + \vec{\nabla} \cdot [\vec{u} A (\rho E + P)] = \\ \vec{\nabla} \cdot \left[A \left(\kappa \vec{\nabla} (\rho e) + \frac{1}{2} ||\vec{u}||^2 \kappa \vec{\nabla} \rho + \rho \mu \vec{u} \vec{\nabla}^s \vec{u} \right) \right]. \end{aligned} \quad (32c)$$

347 The dissipative terms are quite similar to the ones obtained for the standard
 348 Euler equations: each dissipative flux is simply multiplied by the variable area
 349 A in order to ensure conservation of the dissipative flux. When assuming a
 350 constant area, Eqs. 2 are recovered.

351 A low-Mach asymptotic limit of the Euler equations with variable area on the
 352 same model as in Section 3.2 will lead to the divergence constraint $\vec{\nabla} \cdot (\vec{u} A) = 0$
 353 that can be recast as $\vec{\nabla} \cdot \vec{u} = -\vec{u} \cdot \vec{\nabla} A / A$. The gradient of the area acts as a

source term and will force the fluid to accelerate or decelerate, depending on its sign.

5. Discretizations and Solution Techniques

In this section, we briefly describe the spatial and temporal discretizations and the solution techniques used to solve the system of equations Eq. (32). For conciseness, we re-write the system of equations in the following form:

$$\partial_t \mathbf{U} + \vec{\nabla} \cdot \vec{\mathbf{F}}(\mathbf{U}) = \mathbf{S} + \vec{\nabla} \cdot \mathbf{D}(\mathbf{U}) \vec{\nabla} \mathbf{U} \quad (33)$$

where $\mathbf{U} = [\rho A, \rho \vec{u} A, \rho E A]^T$ is the solution vector, \mathbf{F} denotes the inviscid flux

$$\vec{\mathbf{F}} \equiv \begin{bmatrix} \rho u A \\ (\rho u^2 + p) A \\ u(\rho E + P) A \end{bmatrix} \quad (34)$$

and \mathbf{S} is a source term that contains the non-conservative term $P \vec{\nabla} A$. The term $\vec{\nabla} \cdot \mathbf{D}(\mathbf{U}) \vec{\nabla} \mathbf{U}$ stands for the artificial dissipative terms.

5.1. Spatial and Temporal Discretizations

The system of equations given in Eq. (33) is discretized using a continuous Galerkin finite element method and temporal integrators available through the MOOSE multiphysics framework [14].

5.1.1. Continuous Finite Elements

In order to apply the continuous finite element method, Eq. (33) is multiplied by a test function $\mathbf{W}(\vec{r})$, integrated by parts and each integral is decomposed into a sum of integrals over each element K of the discrete mesh Ω . The following

weak form is obtained:

$$\begin{aligned} \sum_K \int_K \partial_t \mathbf{U} \mathbf{W} - \sum_K \int_K \vec{\mathbf{F}}(\mathbf{U}) \cdot \vec{\nabla} \mathbf{W} + \int_{\partial\Omega} \vec{\mathbf{F}}(\mathbf{U}) \cdot \vec{n} \mathbf{W} - \sum_K \int_K \mathbf{S} \mathbf{W} \\ + \sum_K \int_K D(\mathbf{U}) \vec{\nabla} \mathbf{U} \cdot \vec{\nabla} \mathbf{W} - \int_{\partial\Omega} D(\mathbf{U}) \vec{\nabla} \mathbf{U} \cdot \vec{n} \mathbf{W} = 0. \end{aligned} \quad (35)$$

368 The integrals over the elements K are evaluated using a numerical quadrature.
 369 The MOOSE framework provides a wide range of test functions and quadrature
 370 rules. Linear Lagrange polynomials are employed as test functions in the re-
 371 sults section. Second-order spatial convergence will be demonstrated for smooth
 372 solutions.

373 5.1.2. Temporal integration

374 The MOOSE framework offers both first- and second-order explicit and im-
 375 plicit temporal integrators. In all of the numerical examples presented in Sec-
 376 tion 6, the temporal derivative will be evaluated using the second-order, back-
 377 ward difference temporal integrator BDF2. By considering three consecutive
 378 solutions, \mathbf{U}^{n-1} , \mathbf{U}^n and \mathbf{U}^{n+1} , at times t^{n-1} , t^n and t^{n+1} , respectively, BDF2
 379 can be expressed as:

$$\int_K \partial_t \mathbf{U} \mathbf{W} = \int_K (\omega_0 \mathbf{U}^{n+1} + \omega_1 \mathbf{U}^n + \omega_2 \mathbf{U}^{n-1}) \mathbf{W}, \quad (36)$$

with

$$\begin{aligned} \omega_0 = \frac{2\Delta t^{n+1} + \Delta t^n}{\Delta t^{n+1}(\Delta t^{n+1} + \Delta t^n)}, \quad \omega_1 = -\frac{\Delta t^{n+1} + \Delta t^n}{\Delta t^{n+1}\Delta t^n}, \\ \text{and } \omega_2 = \frac{\Delta t^{n+1}}{\Delta t^n(\Delta t^{n+1} + \Delta t^n)} \end{aligned}$$

380 where $\Delta t^n = t^n - t^{n-1}$ and $\Delta t^{n+1} = t^{n+1} - t^n$.

381 5.2. Boundary conditions

382 Boundary conditions are implemented by performing a characteristic decom-
 383 position to compute the appropriate flux at the boundaries. Our implementation
 384 of the subsonic boundary conditions is inspired by the method described in [20]
 385 and was adapted for a time implicit solver. Neumann boundary conditions are
 386 used for all of the boundary types, except for the inlet supersonic boundary that
 387 are strongly imposed with Dirichlet boundary conditions.

388 For each numerical solution presented in Section 6, the type of boundary con-
 389 ditions used will be specified and taken among the following: supersonic inlet,
 390 subsonic inlet (stagnation pressure boundary), subsonic outlet, and supersonic
 391 outlet. The artificial diffusion coefficient $D(\mathbf{U})$ is set to zero at the boundary of
 392 the computational domain so that the boundary term $\int_{\partial\Omega} D(\mathbf{U}) \vec{\nabla} \mathbf{U} \cdot \vec{n} \mathbf{W}$ stem-
 393 ming from the integration by parts of the artificial dissipative terms in Eq. (35)
 394 is ignored.

395 5.3. Solver

396 A Jacobian-free-Newton-Krylov (JFNK) method is used to solve for the so-
 397 lution at the end of each time step. An approximate Jacobian matrix of the
 398 discretized equations was derived and implemented. Obtaining the matrix en-
 399 tries requires that the partial derivatives of pressure with respect to the conser-
 400 vative variables be known (this is relatively simple for the stiffened and ideal
 401 gas equations of state but may be more complex for general equations of state).
 402 The contributions of the artificial dissipative terms to the Jacobian matrix are
 403 approximated by lagging the viscosity coefficients (computing them with the
 404 previous solution). For instance, this is shown in Eq. (37) for the dissipative
 405 terms present in the continuity equation:

$$\frac{\partial}{\partial \mathbf{U}} \left(\kappa \vec{\nabla} \cdot \rho \vec{\nabla} W \right) \simeq \kappa \frac{\partial}{\partial \mathbf{U}} \left(\vec{\nabla} \cdot \rho \vec{\nabla} W \right), \quad (37)$$

where \mathbf{U} denotes any of the conservative variables and W denotes the component of \mathbf{W} associated with the continuity equation. In the above, we have neglected $\frac{\partial \kappa}{\partial \mathbf{U}}$.

6. Numerical Results

1-D and 2-D numerical solutions for the Euler equations with viscous regularization solved using the entropy viscosity method are presented here. Our results show that the new definitions for the viscosity coefficients are robust in the low-Mach limit as well as for transonic and supersonic flows and that shocks are appropriately resolved.

The first set of 1-D simulations consist of liquid water and steam flowing in a converging-diverging nozzle. This test is of interest for multiple reasons: (a) a steady state can be reached (some stabilization methods are known to have difficulties reaching a steady state, [2, 3]), (b) an analytical solution is available and a space-time convergence study can be performed, (c) it can be performed for liquid and gas phases, wherein the gas phase simulation presents a shock while the liquid-phase simulation has a significantly lower Mach number. Next, a 1-D shock tube test (in a straight pipe), taken from the Leblanc test-case suite [21], is performed. This test is known to be more challenging than Sod shock tubes and the fluid's Mach number varies spatially between 0 and 5. A convergence study is also performed to demonstrate convergence of the numerical solution to the exact solution. A slow moving shock is also investigated [22]. This test helps in assessing the ability of the method to damp the post-shock low frequency noise (oscillations). Finally, a strong shock for a liquid phase (Mach number around 0.1) is also performed [23].

The initial conditions for the aforementioned 1-D test cases are given in Table 1.

ρ_{left}	u_{left}	P_{left}	ρ_{right}	u_{right}	P_{right}
Leblanc shock tube (Section 6.3)					
1	0	$4 \cdot 10^{-2}$	10^{-3}	0	$4 \cdot 10^{-11}$
Strong shock for liquid phase (Section 6.4)					
1000	0	10^9	1000	0	10^5
Slow moving shock (Section 6.5)					
1	-0.81	1	3.86	-3.44	10.33

Table 1: Initial conditions for the 1-D test cases (density in kg/m^3 , velocity in m/s , pressure in Pa).

432 The 2-D simulations are outlined next. First, 2-D subsonic flows around a
433 cylinder [12] and over a circular hump [24] are presented for various far-field
434 Mach numbers (as low of 10^{-7}). Numerical results of a supersonic flow over
435 a compression corner are provided to illustrate the ability of the new viscosity
436 definitions to handle supersonic flows. Convergence studies are performed when
437 analytical solutions are available.

438 For each simulation, data relative to the boundary conditions, the Courant-
439 Friedrichs-Lewy number (CFL), mesh and equation of state are provided. All of
440 the numerical solutions presented are obtained using BDF2 as temporal integra-
441 tor and linear (1-D mesh), \mathbb{P}_1 (2-D triangular mesh), and \mathbb{Q}_1 (2-D quadrangular
442 mesh) finite elements. The spatial integrals are numerically computed using a
443 second-order Gauss quadrature rule. Steady-state is detected in a transient sim-
444 ulation by monitoring the nonlinear residual before proceeding with the Newton
445 solves for a given time step. The ideal gas [25] or stiffened gas equations of state
446 [26] are used; a generic expression is given in Eq. (38).

$$P = (\gamma - 1)\rho(e - q) - \gamma P_\infty \quad (38)$$

where the parameters γ , q , and P_∞ are fluid-dependent and are given in Table 2.
The ideal gas equation of state is recovered by setting $q = P_\infty = 0$ in Eq. (38).

The entropy function for the stiffened gas equation of state is concave and given

Table 2: Stiffened Gas Equation of State parameters for steam and liquid water.

fluid	γ	C_v ($J.kg^{-1}.K^{-1}$)	P_∞ (Pa)	q ($J.kg^{-1}$)
liquid water (Section 6.1)	2.35	1816	10^9	$-1167 \cdot 10^3$
steam (Section 6.2)	1.43	1040	0	$2030 \cdot 10^3$
liquid water (Section 6.4)	4.4	1000	$6 \cdot 10^8$	0

by

$$s = C_v \ln \left(\frac{P + P_\infty}{\rho^{\gamma-1}} \right),$$

where C_v is the heat capacity at constant volume.

Finally, the convergence rates are computed using the following relation

$$\text{rate}_h = \ln \left(\frac{\|U_{2h} - U_{\text{exact}}\|}{\|U_h - U_{\text{exact}}\|} \right) / \ln 2 \quad (39)$$

where $\|\cdot\|$ denotes either the L_1 or L_2 norms and h is the characteristic grid

size.

6.1. Liquid water in a 1-D converging-diverging nozzle

A simulation for liquid flow through a 1-D converging-diverging nozzle is performed. The variable area expression is given by $A(x) = 1 + 0.5 \cos(2\pi x/L)$ with length $L = 1m$. At the inlet, the stagnation pressure and temperature are set to $P_0 = 1MPa$ and $T_0 = 453K$, respectively. At the outlet, only the static pressure is specified: $P_s = 0.5MPa$. Initially, the liquid is at rest, the temperature is uniform and equal to the stagnation temperature and the pressure linearly decreases from the stagnation pressure inlet value to the static pressure outlet value. The stiffened gas equation of state is used to model the liquid water with the parameters provided in Table 2. Because of the low pressure

462 difference between the inlet and the outlet, the smooth initial conditions, and
 463 the large value of P_∞ in Eq. (38), the flow remains subsonic and thus displays
 464 no shock. A detailed derivation of the exact steady-state solution can be found
 465 in [27]. A uniform mesh of 50 cells was used to obtain the numerical solution
 466 and the time step size was computed using a CFL number of 750. Plots of
 467 the Mach number, density, and pressure are given at steady state in Fig. 1 for
 468 the numerical and exact solutions. The viscosity coefficients are also graphed
 469 in Fig. 1d.

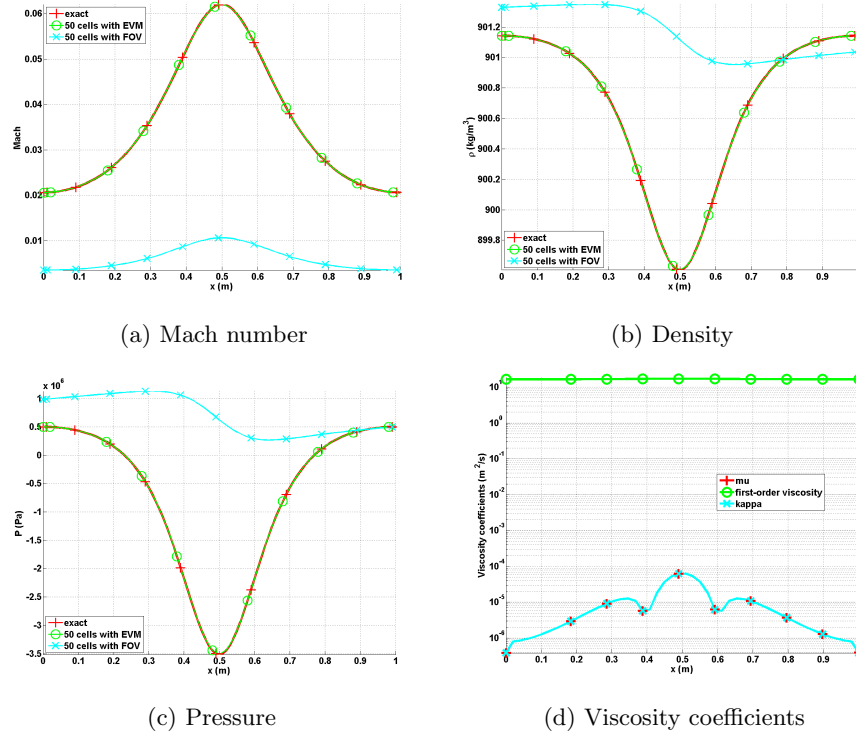


Figure 1: Steady-state solution for a liquid flowing through a 1-D converging-diverging nozzle.

470 In Fig. 1, the numerical solutions obtained using the first-order viscosity
 471 (FOV) and the entropy viscosity method (EVM) are plotted against the exact

472 solution. The numerical solution obtained with the EVM and the exact solution
 473 overlap, even for a fairly coarse mesh (50 cells). On the other hand, the numer-
 474 ical solution obtained with the FOV does not give the correct steady state: this
 475 is an illustration of the effect of ill-scaled dissipative terms in the low-Mach limit
 476 that was pointed out in Remark 1. Note that the entropy viscosity coefficient
 477 is very small compared to the first-order one (Fig. 1d): (i) the numerical solu-
 478 tion is smooth as shown in Fig. 1 and (ii) the flow is in a isentropic low-Mach
 479 regime A convergence study was performed using the exact solution as a refer-
 480 ence: the L_1 and L_2 norms of the error and the corresponding convergence rates
 481 are computed at steady state on various uniform meshes from 4 to 256 cells.
 482 Spatial convergence results using linear finite elements are reported in Table 3
 483 and Table 4 for the primitive variables: density, velocity and pressure.

Table 3: L_1 norm of the error for the liquid phase in a 1-D converging-diverging nozzle at steady state.

cells	density	rate	pressure	rate	velocity	rate
4	$2.8037 \cdot 10^{-1}$	—	$8.4705 \cdot 10^5$	—	7.2737	—
8	$1.3343 \cdot 10^{-1}$	1.07	$4.7893 \cdot 10^5$	0.82	6.1493	0.24
16	$2.9373 \cdot 10^{-2}$	2.18	$1.0613 \cdot 10^5$	2.17	1.2275	2.32
32	$5.1120 \cdot 10^{-3}$	2.52	$1.8446 \cdot 10^4$	2.52	$1.8943 \cdot 10^{-1}$	2.69
64	$1.0558 \cdot 10^{-3}$	2.28	$3.7938 \cdot 10^3$	2.28	$3.7919 \cdot 10^{-2}$	2.32
128	$2.3712 \cdot 10^{-4}$	2.15	$8.4471 \cdot 10^2$	2.17	$8.5517 \cdot 10^{-3}$	2.15
256	$5.6058 \cdot 10^{-5}$	2.08	$1.9839 \cdot 10^2$	2.09	$2.0475 \cdot 10^{-3}$	2.06
512	$1.3278 \cdot 10^{-5}$	2.08	$4.6622 \cdot 10^1$	2.09	$4.9516 \cdot 10^{-4}$	2.04
1024	$3.1193 \cdot 10^{-6}$	2.08	$1.1755 \cdot 10^1$	1.99	$1.2379 \cdot 10^{-4}$	2.00

Table 4: L_2 norm of the error for the liquid phase in a 1-D converging-diverging nozzle at steady state.

cells	density	rate	pressure	rate	velocity	rate
4	$3.106397 \cdot 10^{-1}$	—	$5.254445 \cdot 10^5$	—	3.288543	—
8	$7.491623 \cdot 10^{-2}$	2.05	$1.636966 \cdot 10^5$	1.68	1.823880	0.85
16	$2.079858 \cdot 10^{-2}$	1.85	$4.627338 \cdot 10^4$	1.49	$4.990605 \cdot 10^{-1}$	0.87
32	$5.329627 \cdot 10^{-3}$	1.96	$1.180287 \cdot 10^4$	1.97	$1.261018 \cdot 10^{-1}$	1.98
64	$1.341583 \cdot 10^{-3}$	1.99	$2.967104 \cdot 10^3$	1.99	$3.160914 \cdot 10^{-2}$	1.99
128	$3.359766 \cdot 10^{-4}$	1.99	$7.428087 \cdot 10^2$	1.99	$7.907499 \cdot 10^{-3}$	1.99
256	$8.403859 \cdot 10^{-5}$	1.99	$1.857861 \cdot 10^2$	1.99	$1.977292 \cdot 10^{-3}$	1.99
512	$2.10075 \cdot 10^{-5}$	2.00	$4.7024 \cdot 10^1$	1.98	$4.9516 \cdot 10^{-4}$	1.99

We note that the convergence rates measured in both the L_1 and L_2 norm of the error are equal to 2; the entropy viscosity method preserves the high-order accuracy of the discretization used when the numerical solution is smooth. The new definition of the entropy viscosity coefficients behaves appropriately in the low-Mach limit.

6.2. Steam in a 1-D converging-diverging nozzle

We use the same nozzle geometry, initial conditions and boundary conditions as in the previously example but replace liquid water with steam and use the steam parameters of the stiffened gas equation of state, Table 2. In this example, compressible effects will become dominant. The pressure difference between the inlet and outlet is large enough to accelerate the steam through the nozzle, leading to the formation of a shock in the diverging portion of the nozzle. The behavior is different from the one observed for the liquid water phase in Section 6.1 because of the liquid to gas density ratio is about 1,000. An exact solution at steady state is available for the gas phase [27]. The aim of this section is to show that when using the new definitions of the viscosity coefficients (Eq. (30)), the shock can be correctly resolved without spurious oscillations. The steady-state numerical solution, obtained using a uniform mesh

with 500 cells, is shown in Fig. 2. The CFL was set to 80 (a high CFL value can be used because the shock is stationary).

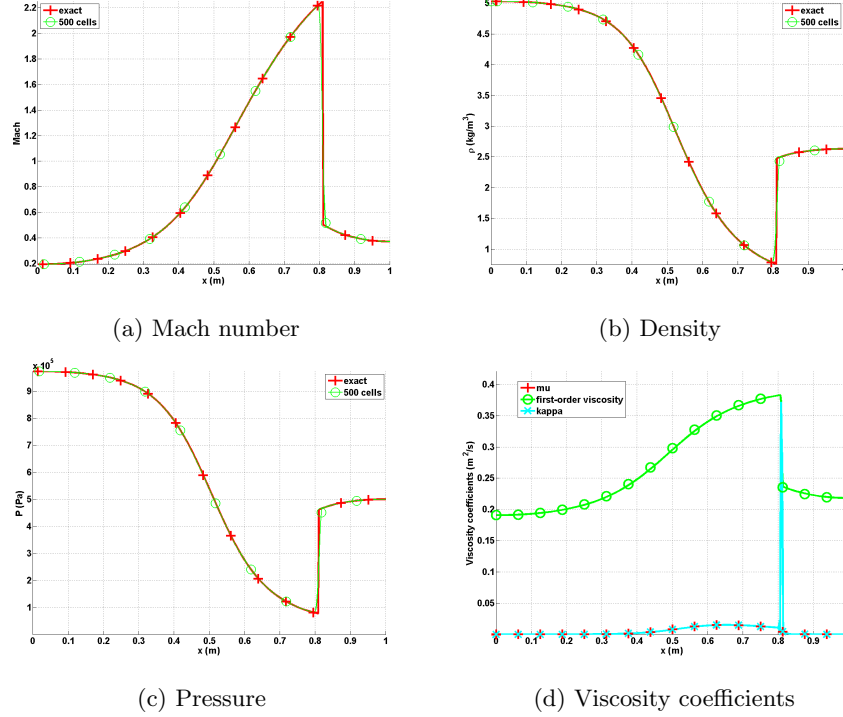


Figure 2: Steady-state solution for vapor phase flowing in a 1-D converging-diverging nozzle.

The steady-state solution of the density, Mach number and pressure are given in Fig. 2. The steady-state solution exhibits a shock around $x = 0.8m$ and matches the exact solution. In Fig. 2d, the first-order and entropy viscosity coefficients are plotted at steady state (on a log scale): the entropy viscosity coefficient is peaked in the shock region around $x = 0.8m$ where it saturates to the first-order viscosity coefficient. Elsewhere, the entropy viscosity coefficient is small. In order to prove convergence of the numerical solution to the exact solution, a convergence study is performed. Because of the presence of a shock, second-order accuracy is not expected and the convergence rate of a numerical

513 solution should be 1 and $1/2$ when measured in the L_1 and L_2 norms, respec-
514 tively (see Theorem 9.3 in [28]). Results are reported in Table 5 and Table 6
515 for the primitive variables: density, velocity and pressure. The convergence
516 rates for the L_1 and L_2 norms of the error computed using Eq. (39) are in good
agreement with the theoretical values.

Table 5: L_1 norm of the error for the vapor phase in a 1-D converging-diverging nozzle at steady state.

cells	density	rate	pressure	rate	velocity	rate
5	$0.72562 \cdot 10^{-1}$	—	$1.5657 \cdot 10^5$	—	173.69	—
10	$0.4165 \cdot 10^{-1}$	0.80	$9.6741 \cdot 10^4$	0.63	120.69	0.53
20	$0.20675 \cdot 10^{-1}$	1.01	$4.9193 \cdot 10^4$	0.97	72.149	0.74
40	$0.093703 \cdot 10^{-1}$	1.14	$2.0103 \cdot 10^4$	0.73	34.716	1.06
80	$0.047328 \cdot 10^{-1}$	0.99	$1.0208 \cdot 10^4$	0.98	16.082	1.11
160	$0.023965 \cdot 10^{-2}$	0.98	$5.1969 \cdot 10^3$	0.97	7.9573	1.02
320	$0.020768 \cdot 10^{-2}$	1.03	$2.5116 \cdot 10^3$	1.05	3.7812	1.07
640	$0.0059715 \cdot 10^{-2}$	0.98	$1.2754 \cdot 10^3$	0.98	1.8353	1.04

517

Table 6: L_2 norm of the error for the vapor phase in a 1-D converging-diverging nozzle at steady state.

cells	density	rate	pressure	rate	velocity	rate
5	$9.7144 \cdot 10^{-1}$	—	$2.0215 \cdot 10^5$	—	236.94	—
10	$5.9718 \cdot 10^{-1}$	0.70	$1.3024 \cdot 10^5$	0.63	166.56	0.51
20	$2.9503 \cdot 10^{-1}$	1.02	$6.6503 \cdot 10^4$	0.97	103.36	0.69
40	$1.8193 \cdot 10^{-1}$	0.69	$4.0171 \cdot 10^4$	0.73	66.374	0.64
80	$1.3366 \cdot 10^{-1}$	0.44	$2.3163 \cdot 10^4$	0.44	42.981	0.63
160	$9.6638 \cdot 10^{-2}$	0.47	$1.7263 \cdot 10^4$	0.42	31.717	0.44
320	$7.0896 \cdot 10^{-2}$	0.45	$1.2763 \cdot 10^4$	0.44	23.138	0.45
640	$5.2191 \cdot 10^{-2}$	0.44	$9.4217 \cdot 10^3$	0.44	16.910	0.45

518 6.3. Leblanc shock tube

519 The 1-D Leblanc shock tube is a Riemann problem designed to test the
520 robustness and the accuracy of stabilization methods. The initial conditions
521 are given in Table 1. The ideal gas equation of state (with $\gamma = 5/3$) is used to
522 compute the pressure. This test is computationally challenging because of the

523 large pressure ratio at the initial interface. The computational domain consists
 524 of a 1-D straight pipe of length $L = 9m$ with the initial interface located at
 525 $x = 2m$. At $t = 0s$, the interface is removed. The numerical solution is run
 526 until $t = 4s$ and the density, momentum and total energy profiles are given in
 527 Fig. 3, along with the exact solution. The viscosity coefficients are also plotted
 528 in Fig. 3d. These plots were run with three different uniform meshes of 800,
 529 3200, and 6000 cells and a constant $CFL = 1$.

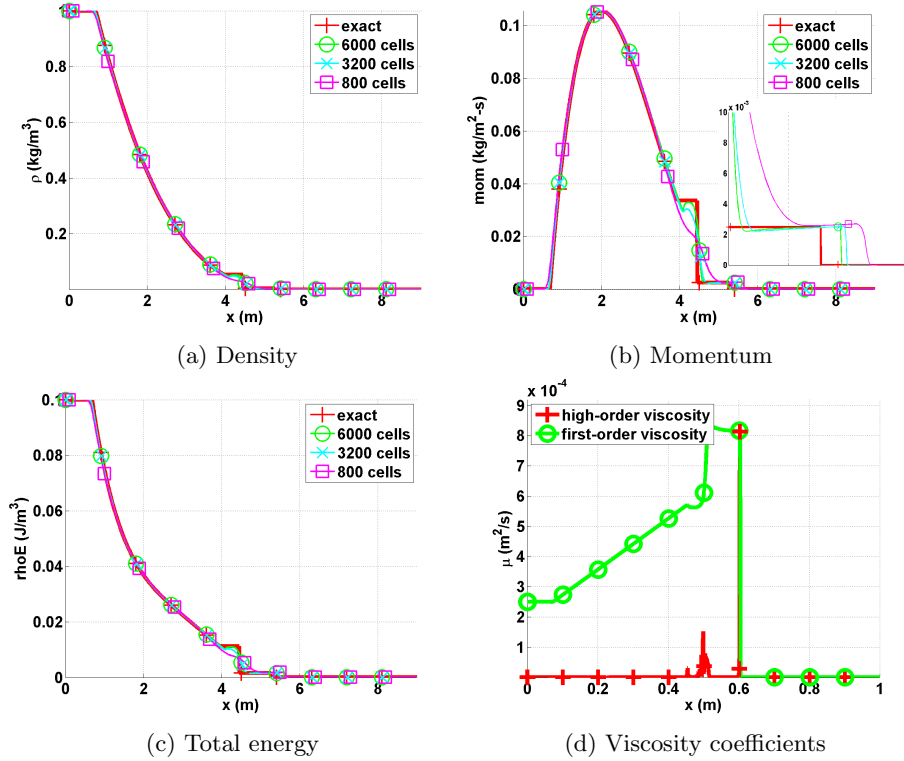


Figure 3: Exact and Numerical solutions for the 1-D Leblanc shock tube at $t = 4s$.

530 The density, momentum and total energy profiles are provided in Fig. 3.
 531 In Fig. 3b, the shock region is zoomed in for better resolution: the shock is
 532 well resolved. We also observe that the shock position computed numerically

converges to the exact position under mesh refinement. The contact wave at $x = 4.5m$ can be seen in Fig. 3b. The entropy viscosity coefficient profile is shown in Fig. 3d and behaves as expected: it saturates to the first-order viscosity in the shock region, thus preventing oscillations from forming. At the location of the contact wave, a smaller peak is observed and is due to the presence of the jump terms in the definition of the entropy viscosity coefficient (Eq. (30)). The Mach number, not plotted, is of the order of 1.3 just before the shock and reaches a maximum value close to 5 in the contact region.

Once again, a convergence study is performed in order to prove convergence of the numerical solution to the exact solution. As in the previous example (vapor phase in the 1-D nozzle, Section 6.2), the expected convergence rates in the L_1 and L_2 norms are 1 and $1/2$, respectively. The exact solution was obtained by running a 1-D Riemann solver and used as the reference solution to compute the L_1 and L_2 -norms that are reported in Table 7 and Table 8 for the conservative variables: density, momentum and total energy. The convergence rates are again approaching their theoretical values.

Table 7: L_1 norm of the error for the 1-D Leblanc test at $t = 4s$.

cells	density	rate	momentum	rate	total energy	rate
100	$1.0354722 \cdot 10^{-2}$	—	$3.5471714 \cdot 10^{-3}$	—	$1.4033046 \cdot 10^{-3}$	—
200	$7.2680512 \cdot 10^{-3}$	0.51	$2.5933119 \cdot 10^{-3}$	0.45	$9.8611746 \cdot 10^{-4}$	0.51
400	$5.0825628 \cdot 10^{-3}$	0.52	$2.0668092 \cdot 10^{-3}$	0.33	$7.7844421 \cdot 10^{-4}$	0.34
800	$3.4025056 \cdot 10^{-3}$	0.58	$1.4793838 \cdot 10^{-3}$	0.48	$5.5702549 \cdot 10^{-4}$	0.48
1600	$2.1649953 \cdot 10^{-3}$	0.65	$9.7152832 \cdot 10^{-4}$	0.61	$3.5720171 \cdot 10^{-4}$	0.64
3200	$1.2465433 \cdot 10^{-3}$	0.79	$5.5937409 \cdot 10^{-4}$	0.79	$2.0491799 \cdot 10^{-4}$	0.80
6400	$6.4476928 \cdot 10^{-4}$	0.95	$3.0244198 \cdot 10^{-4}$	0.89	$1.0914891 \cdot 10^{-4}$	0.91
12800	$3.3950948 \cdot 10^{-4}$	0.93	$1.5958118 \cdot 10^{-4}$	0.92	$5.7909794 \cdot 10^{-5}$	0.91

6.4. 1-D shock tube with a liquid phase

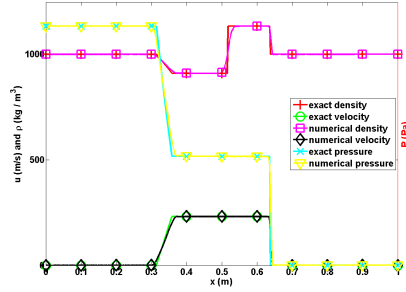
The purpose of this test is to investigate the ability of the entropy viscosity method to stabilize a strong shock with a small Mach number [23] (this reference

Table 8: L_2 norm of the error for the 1-D Leblanc test at $t = 4s$.

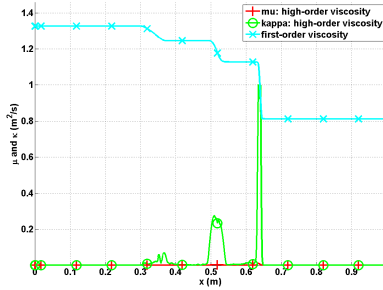
cells	density	rate	momentum	rate	total energy	rate
100	$5.7187851 \cdot 10^{-3}$	—	$1.7767236 \cdot 10^{-3}$	—	$7.6112265 \cdot 10^{-4}$	—
200	$3.8995238 \cdot 10^{-3}$	0.55	$1.4913161 \cdot 10^{-3}$	0.25	$5.5497308 \cdot 10^{-4}$	0.46
400	$2.8103526 \cdot 10^{-3}$	0.47	$1.3305301 \cdot 10^{-3}$	0.16	$4.6063172 \cdot 10^{-4}$	0.27
800	$2.1081933 \cdot 10^{-3}$	0.41	$1.1398931 \cdot 10^{-3}$	0.22	$3.7798953 \cdot 10^{-4}$	0.29
1600	$1.5731052 \cdot 10^{-3}$	0.42	$9.0394227 \cdot 10^{-4}$	0.33	$2.9584646 \cdot 10^{-4}$	0.35
3200	$1.0610667 \cdot 10^{-3}$	0.57	$6.2735595 \cdot 10^{-4}$	0.53	$2.054455 \cdot 10^{-4}$	0.53
6400	$7.3309974 \cdot 10^{-4}$	0.53	$4.4545754 \cdot 10^{-4}$	0.49	$1.4670834 \cdot 10^{-4}$	0.49
12800	$5.1020991 \cdot 10^{-4}$	0.52	$3.1266758 \cdot 10^{-4}$	0.51	$1.0299897 \cdot 10^{-5}$	0.51

is for a two-phase flow model but we are only interested in the initial conditions for the liquid phase): the Mach number in the shock region is of the order of 0.1. In this case, as explained in Section 3.2, the viscosity coefficients are required to have different order of magnitude in order to ensure the correct scaling of the dissipative terms. The purpose of this test is to validate the approach presented in Section 3.2.

The stiffened gas equation of state is used to model a liquid flow with the parameters given in Table 2. The computational domain of length $L = 1m$ is uniformly discretized using 500 cells. The step initial conditions are given in Table 1. The simulation is run with a $CFL = 1$ until the final time $t_{\text{final}} = 7 \cdot 10^{-5}s$. Results for pressure, density, velocity and the viscosity coefficients are given in Fig. 4 along with the exact solution for comparison purposes. The numerical solution is in good agreement with exact solution in Fig. 4a. The viscosity coefficients μ and κ are not equal in the shock because the Mach number is of order 0.1. The viscosity coefficient κ saturates to the first-order viscosity in the shock region around $x = 0.65m$ and is sufficient to stabilize the numerical scheme.



(a) Density, velocity and pressure profiles.



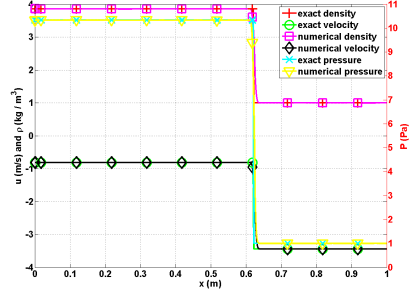
(b) Viscosity coefficients profile.

Figure 4: Numerical solution for the 1-D liquid shock tube at $t_{\text{final}} = 7 \cdot 10^{-5} \text{ s}$.

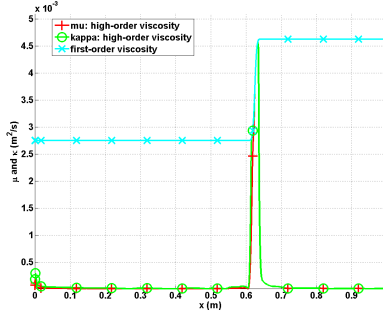
6.5. 1-D slow moving shock

Slow moving shocks are known to produce post-shock noise of low frequency that is not damped by some numerical dissipation methods [22]. The aim of this simulation is to test the ability of the entropy viscosity method to dampen the low frequency waves. The 1-D slow moving shock consists of a shock wave moving from left to right with the initial conditions given in Table 1. The ideal gas equation of state is used with a heat capacity ratio $\gamma = 1.4$. In order to make the shock travel a significant distance, the final time is taken equal to $t = 1.1 \text{ s}$. A pressure boundary condition is used at the left boundary to let the rarefaction and contact waves exit the domain. The numerical solution, obtained with 200 equally-spaced cells, is given in Fig. 5 and is compared to the exact solution obtained from a Riemann solver. We use a CFL of 1. With this CFL value,

581 it takes about 50 time steps for the shock to traverse one cell. The numerical
 582 results are in good agreement with the exact solution and do not display any
 583 post-shock noise. The rarefaction and contact waves are not visible on Fig. 5a
 584 since they exited the computational domain through the left pressure boundary
 585 condition earlier. As explained in [29], Godunov's type methods usually fail to
 586 resolve a slow moving shock because of the nature of the stabilization method:
 587 the method scales as the eigenvalue of the appropriate field. In the case of a slow
 588 moving shock, the dissipation added to the system is under-estimated and leads
 589 to post-shock noise. In the case of the entropy viscosity method, the entropy
 590 residual detects the shock position and the viscosity coefficients saturate to the
 591 first-order viscosity values in the shock region. The main difference between a
 592 Godunov's type method and the entropy viscosity method lies in the definition of
 593 the first-order viscosity coefficients that are proportional to the *local maximum*
 594 *eigenvalue* $||\vec{u}|| + c$ and not to the eigenvalue of the characteristic field.



(a) Velocity, density and pressure



(b) Viscosity coefficients

Figure 5: Slow moving shock profiles at $t = 1.1s$.

6.6. Subsonic flow over a 2-D cylinder

Fluid flow over a 2-D cylinder is often used as a benchmark case to test numerical schemes in the low-Mach regime [10, 11, 12]. For this test, an analytical solution is available in the incompressible limit and is often referred to as the potential steady-state flow solution. The main features of the potential flow are the following:

- The solution is symmetric: the iso-Mach contour lines are used to assess the symmetry of the numerical solution;
- The velocity at the top of the cylinder is twice the incoming velocity set at the inlet;

- The steady-state pressure fluctuations are proportional to the square of inlet Mach number, i.e.,

$$\delta P = \frac{\max(P(\vec{r})) - \min(P(\vec{r}))}{\max(P(\vec{r}))} \propto M_\infty^2 \quad (40)$$

where δP and M_∞ denote the steady-state pressure fluctuations and the inlet Mach number, respectively.

The computational domain consists of a 1×1 square with a circular hole of radius 0.05 in its center. A \mathbb{P}_1 triangular mesh with 4008 triangular elements is employed to discretize the geometry. The ideal gas equation of state, with $\gamma = 1.4$ is used. At the inlet, a subsonic stagnation boundary condition is used: the stagnation pressure and temperature are computed using the following relations:

$$\begin{cases} P_0 = P \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} \\ T_0 = T \left(1 + \frac{\gamma-1}{2} M^2\right) \end{cases} \quad (41)$$

A static pressure boundary condition, with static pressure $P_s = 101,325 \text{ Pa}$, is set at the outlet boundary. The implementation of the pressure boundary conditions is based on [20]. A solid wall boundary condition is set for the top and bottom walls of the computational domain. The simulations are run until a steady state is reached (with a CFL of 40). When the residual norm (for all equations) is less than 10^{-12} the steady state is considered to have been reached.

Several simulations are performed, with inlet Mach numbers M_{inlet} ranging from 10^{-3} to 10^{-7} , and are shown in Fig. 6. The iso-Mach contour lines are drawn using 30 equally-spaced intervals, from 2×10^{-10} to M_{inlet} .

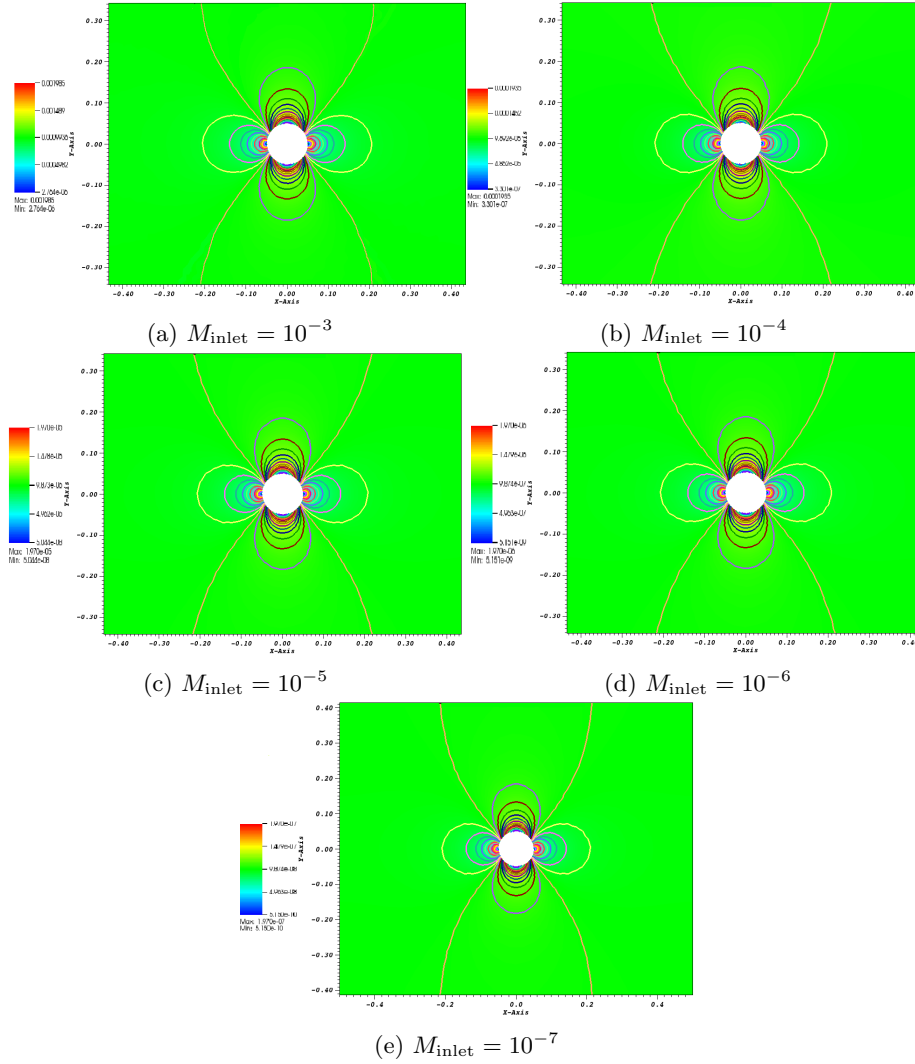


Figure 6: Iso-Mach lines for a subsonic flow over a 2-D cylinder with inlet Mach number values from ranging from 10^{-3} to 10^{-7} (steady-state solution).

624 The velocity at the top of the cylinder and at the inlet are given for different
625 Mach-number values (ranging from 10^{-3} to 10^{-7}) in Table 9. The ratio of the
626 inlet velocity to the velocity at the top of cylinder is also computed and is very
627 close to the theoretical value of 2 that is expected in the incompressible limit.

Table 9: Velocity ratio for different Mach numbers.

Mach number	inlet velocity	velocity at the top of the cylinder	ratio
10^{-3}	$2.348 \cdot 10^{-3}$	$1.176 \cdot 10^{-3}$	1.99
10^{-4}	$2.285 \cdot 10^{-4}$	$1.145 \cdot 10^{-4}$	1.99
10^{-5}	$2.283 \cdot 10^{-5}$	$1.144 \cdot 10^{-5}$	1.99
10^{-6}	$2.283 \cdot 10^{-6}$	$1.144 \cdot 10^{-6}$	1.99
10^{-7}	$2.283 \cdot 10^{-7}$	$1.144 \cdot 10^{-7}$	1.99

628 In Fig. 7, the fluctuations in pressure and velocity are plotted as a function
 629 of the Mach number (on a log-log scale). The pressure fluctuations are expected
 630 to be of the order of M^2 in the incompressible limit, which we observe. From
 631 Bernoulli's principle, this implies that the velocity fluctuations should be of
 632 order M in the incompressible limit, which we also observe in Fig. 7. It is
 633 known that some stabilization methods, e.g., [10, 11, 12], can produce pressure
 634 fluctuations with the wrong Mach-number order. Here, the entropy viscosity
 635 method yields the correct orders in the low-Mach limit. For ease of comparison,
 636 reference lines with slope values of 1 and 2 are also plotted.

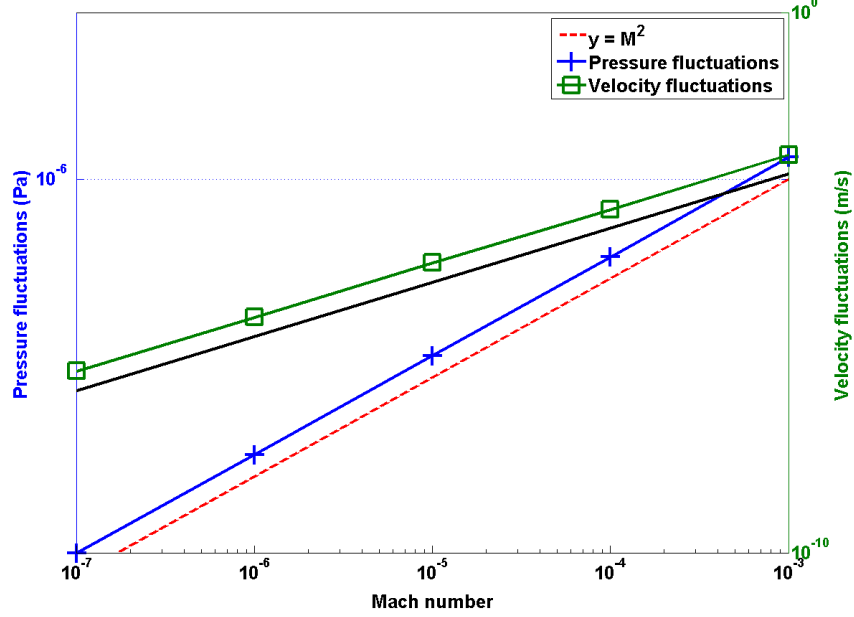
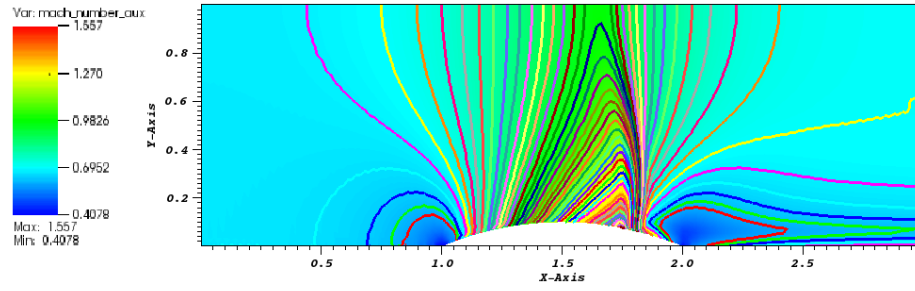


Figure 7: Log-log plot of the steady-state pressure and velocity fluctuations as a function of the far-field Mach number.

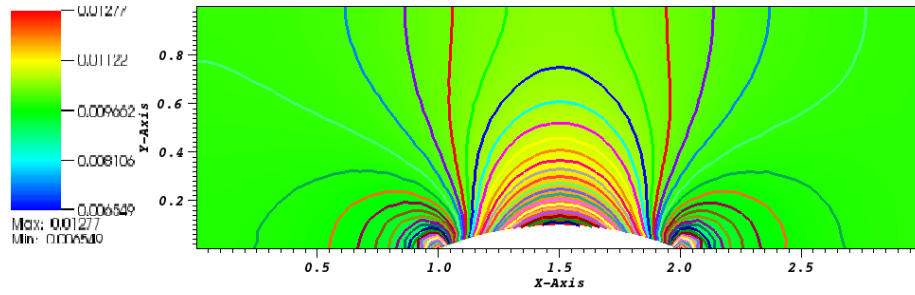
6.7. Subsonic flow over a 2-D hump

This is a another example of an internal flow configuration. It consists of a channel of height $L = 1 \text{ m}$ and length $3L$, with a circular bump of length L and thickness $0.1L$. The bump is located on the bottom wall at a distance L from the inlet. The system is initialized with an uniform pressure $P = 101,325 \text{ Pa}$ and temperature $T = 300 \text{ K}$. The initial velocity is computed from the inlet Mach number, the pressure, the temperature and the ideal gas equation (with $\gamma = 1.4$). Here, $C_v = 717 \text{ J/kg} - \text{K}$. At the inlet, a subsonic stagnation boundary condition is used and the stagnation pressure and temperature are computed using Eq. (41). The static pressure $P_s = 101,325 \text{ Pa}$ is set at the subsonic outlet. The results are shown in Fig. 8a, Fig. 8b, Fig. 8c and Fig. 8d for the inlet Mach numbers $M_\infty = 0.7$, $M_\infty = 0.01$, $M_\infty = 10^{-4}$ and $M_\infty = 10^{-7}$,

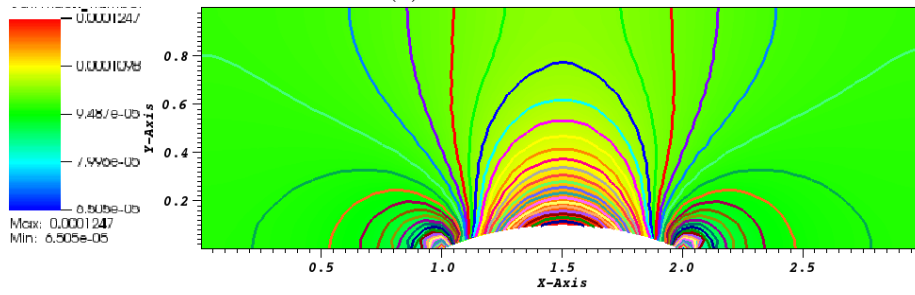
649 respectively. It is expected that, for low Mach numbers, the solution does not
650 depend on the Mach number and is identical to the incompressible flow solution.
651 On the other hand, for a flow with $M = 0.7$, the compressible effects become
652 non negligible and a shock can form. An uniform grid of 3352 Q_1 elements was
653 used to obtain the numerical solution for Mach numbers less than and equal
654 to $M_\infty = 0.01$. A spatial mesh, once refined, was employed for the $M_\infty = 0.7$
655 simulation in order to better resolve the shock. A CFL of 20 was employed and
656 the simulations were run until steady state.



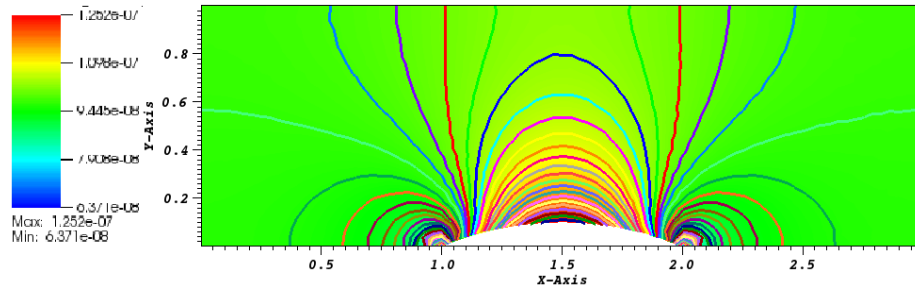
(a) Mach 0.7



(b) Mach 10^{-2}



(c) Mach 10^{-5}



(d) Mach 10^{-7}

Figure 8: Iso-Mach lines for a 2-D flow over a circular bump (steady-state solution).

657 The results shown in Fig. 8b, Fig. 8c and Fig. 8d correspond to the low-
 658 Mach regime. The iso-Mach lines are drawn ranging from the minimum and the
 659 maximum values (provided in each legend) using 50 equally-spaced intervals.
 660 The steady-state solution is symmetric and does not depend on the value of the
 661 inlet Mach number, as expected in the incompressible limit.

662 In Fig. 8a, the steady-state numerical solution develops a shock: the com-
 663 pressibility effects are no longer small. The iso-Mach lines are also plotted with
 664 50 intervals and range from 0.4 to 1.6. The shock is well resolved and does not
 665 display any instabilities or spurious oscillations.

666 *6.8. Supersonic flow in a compression corner*

667 In this last example, we consider a supersonic flow at Mach 2.5 impinging
 668 on a corner with an angle of 15° . From the oblique shock theory [16], an
 669 analytical solution for this supersonic flow is available and gives the downstream-
 670 to-upstream pressure, entropy and Mach number ratios. The initial conditions
 671 are chosen to be spatially uniform: the pressure and temperature are set to $P =$
 672 $101,325 \text{ Pa}$ and $T = 300 \text{ K}$, respectively. The ideal gas equation of state is used
 673 with the same parameters as in Section 6.7. The initial velocity is computed from
 674 the upstream Mach number. The inlet is supersonic and therefore, the pressure,
 675 temperature and velocity are specified using Dirichlet boundary conditions. The
 676 outlet is also supersonic and none of the characteristics enter the domain through
 677 this boundary; the values are computed by the solver.

678 The simulation is run with $CFL = 2$ until steady state is reached. A 2-D
 679 mesh made of 16,109 Q_1 elements is used. The ratios for pressure, entropy and
 680 Mach number computed using the analytical (published with only two significant
 681 digits) and the numerical solutions are given in Table 10; they are in excellent
 682 agreement. The shock wave angle at steady state is also known and given by

the so-called $\theta - \beta - M$ relation:

$$\tan \theta = 2 \cot \beta \frac{M^2 \sin^2 \beta - 1}{M^2 (\gamma + \cos^2(2\beta)) + 2}, \quad (42)$$

where θ , β and M denote the corner angle, the shock wave angle, and the upstream Mach number, respectively. For Mach 2.5 and a 15° corner angle, the analytical value for the shock wave angle is 36.94° at steady state. From Fig. 9a, the numerical value of the shock wave angle can be measured and is found to be equal to 36.9° and thus is in excellent agreement with the theory.

	analytical	numerical
Pressure	2.47	2.467
Mach number	0.74	0.741
Entropy	1.03	1.026

Table 10: Ratio of analytical and numerical downstream to upstream quantities for the compression corner problems (corner angle of 15° and inlet $M = 2.5$ (analytical values from [16])).

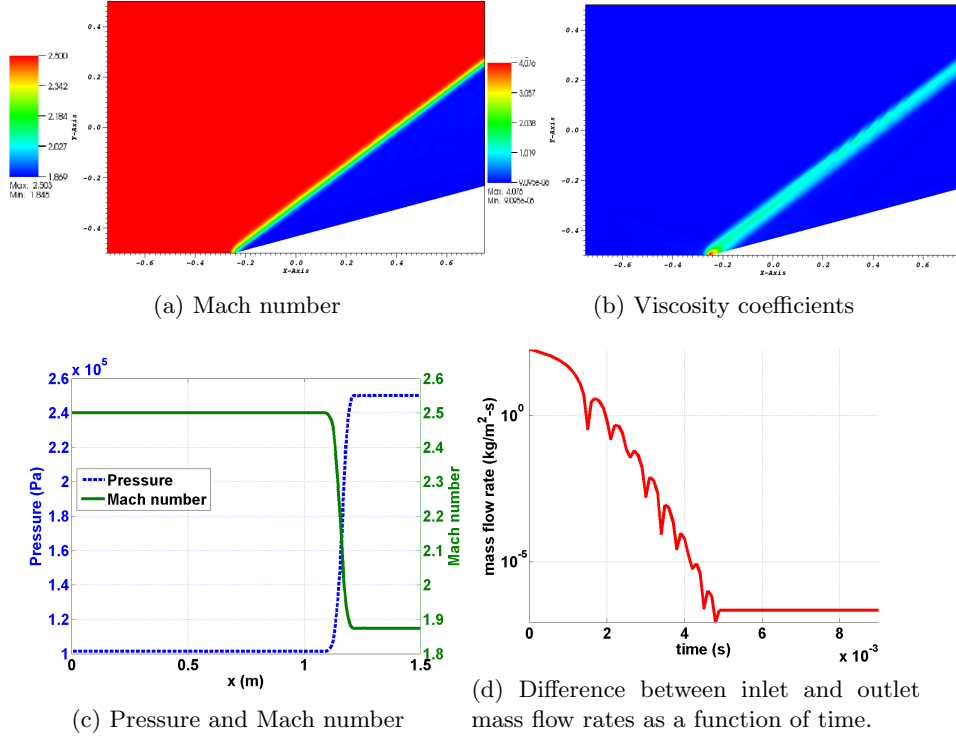


Figure 9: Steady-state solution for a flow in a 2-D compression corner.

689 The steady-state numerical solution is given in Fig. 9; the Mach number and
 690 the viscosity coefficients are plotted in Fig. 9a and Fig. 9b, respectively. The
 691 steady-state solution is composed of two regions of constant state separated by
 692 an oblique shock. Fig. 9b shows that the viscosity coefficient is large in the shock
 693 and small elsewhere, as expected. At the location of the corner ($x = -0.25m$,
 694 $y = -0.5m$), the viscosity coefficient is peaked because of the treatment of the
 695 wall boundary condition: at this particular node, the normal is not well defined
 696 and may cause some numerical errors. The 1-D graphs at $y = 0$ for the pressure
 697 and the Mach number are given in Fig. 9c: no spurious oscillations are observed
 698 and the shock is well resolved. Finally, the difference between the inlet and
 699 outlet mass flow rates is plotted in Fig. 9d and shows that a steady state has
 700 indeed been reached.

701 The results presented in this paper demonstrate the ability of the entropy
702 viscosity method with the new definitions of the viscosity coefficients to correctly
703 simulate several types of flows (from very low Mach subsonic to transonic flows)
704 without tuning parameters.

705 7. Conclusions

706 A new version of the entropy viscosity method that is valid for a wide range
707 of Mach numbers has been derived and presented for the inviscid Euler equa-
708 tions. The definition of the viscosity coefficients is now consistent with the
709 low-Mach asymptotic limit, does not require an analytical expression for the
710 entropy function, and is therefore applicable to a larger variety of flow regimes,
711 from very low-Mach flows to supersonic flows. The method has also been ex-
712 tended to Euler equation with variable area to solve nozzle flow problems. In
713 1-D, convergence of the numerical solution to the exact solution was demon-
714 strated by computing the convergence rates of the L1 and L2 norms for flows
715 in a converging-diverging nozzle and in straight pipes. For smooth solutions,
716 second-order convergence was verified; solutions with shocks converged with the
717 expected theoretical rates of 1 (L_1 -norm) and 0.5 (L_2 -norm).

718 The effectiveness of the method was also demonstrated in 2-D using a se-
719 ries of benchmark problems for both subsonic and supersonic flows in various
720 geometries, with Mach numbers ranging from 10^{-7} to 2.5. For very low-Mach
721 flows, we numerically verified that the pressure fluctuations were proportional
722 to the square of the Mach number, as expected in the incompressible limit.

723 In the future, we plan to further extend the entropy viscosity method to the
724 seven-equation two-phase flow fluid model [20]. This two-phase flow system of
725 equations is a good candidate for two reasons: it is unconditionally hyperbolic
726 and degenerates to the standard Euler equations when one phase disappears.

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804 ing, Massachusetts, 1990.

805 **A. Derivation of the entropy residual as a function of density, pres-**
806 **sure and speed of sound**

The entropy residual is defined as follows:

$$R_{\text{ent}}(\vec{r}, t) = \partial_t s(\vec{r}, t) + \vec{u} \cdot \vec{\nabla} s(\vec{r}, t),$$

where all variables were defined previously. This form of the entropy residual is not suitable for the low-Mach limit as explained in Section 2.1. In this appendix, we recast the entropy residual $R_{\text{ent}}(\vec{r}, t)$ as a function of the primitive variables (pressure, velocity and density) and the speed of sound. The first step of this derivation is to use the chain rule, recalling that the entropy is a function of the internal energy e and the density ρ , yielding

$$R_{\text{ent}}(\vec{r}, t) = s_e \frac{De}{Dt} + s_\rho \frac{D\rho}{Dt},$$

807 where s_e denotes the partial derivative of s with respect to the variable e . We
808 recall that $\frac{D}{Dt}$ denotes the material derivative. Since the internal energy e is a
809 function of pressure P and density ρ (through the equation of state), we use
810 again the chain rule to re-express the previous equation as a function of the
811 material derivatives in P and ρ :

$$\begin{aligned} R_{\text{ent}}(\vec{r}, t) &= s_e e_P \frac{DP}{Dt} + (s_e e_\rho + s_\rho) \frac{D\rho}{Dt} \\ &= s_e e_P \left(\frac{DP}{Dt} + \frac{1}{s_e e_P} (s_e e_\rho + s_\rho) \frac{D\rho}{Dt} \right) \\ &= s_e e_P \left(\frac{DP}{Dt} + \left(\frac{e_\rho}{e_P} + \frac{s_\rho}{s_e e_P} \right) \frac{D\rho}{Dt} \right). \end{aligned}$$

To prove that the term multiplying the material derivative of the density is indeed equal to the square of the speed of sound, we recall that the speed of sound is defined as the partial derivative of pressure with respect to density at

constant entropy, which can be recast as a function of the entropy as follows (see Appendix A.2 of [15]):

$$c^2 := \left. \frac{\partial P}{\partial \rho} \right|_{s=cst} = P_\rho - \frac{s_\rho}{s_e} P_e .$$

Using the following relations (see Appendix A.1 of [15])

$$P_e = \frac{1}{e_P} \text{ and } P_\rho = -\frac{e_\rho}{e_P} .$$

Substitution of these expressions into the entropy residual equation above gives Eq. (9), which is recalled below for completeness:

$$R_{\text{ent}}(\vec{r}, t) := \partial_t s + \vec{u} \cdot \vec{\nabla} s = \frac{Ds}{Dt} = \frac{s_e}{P_e} \left(\underbrace{\frac{DP}{Dt} - c^2 \frac{D\rho}{Dt}}_{\tilde{R}_{\text{ent}}(\vec{r}, t)} \right) .$$

812 **B. Derivation of the dissipative terms for the Euler equations with**
813 **variable area using the entropy minimum principle**

814 The Euler equations (without viscous regularization) with variable area are
815 recalled here

$$\partial_t (\rho A) + \vec{\nabla} \cdot (\rho \vec{u} A) = 0 \quad (43a)$$

$$\partial_t (\rho \vec{u} A) + \vec{\nabla} \cdot [A (\rho \vec{u} \otimes \vec{u} + P \mathbb{I})] = P \vec{\nabla} A \quad (43b)$$

$$\partial_t (\rho E A) + \vec{\nabla} \cdot [\vec{u} A (\rho E + P)] = 0. \quad (43c)$$

818 The specific entropy is a function of the density ρ and the internal energy e , i.e.,
819 $s(e, \rho)$. The above system of equations satisfies the minimum entropy principle
820 [30],

$$A \rho \left(\partial_t s + \vec{u} \cdot \vec{\nabla} s \right) \geq 0. \quad (44)$$

821 The entropy function s satisfies the second law of thermodynamics, $T ds =$
822 $de - \frac{P}{\rho^2} d\rho$, which implies $s_e := T^{-1}$ and $s_\rho := -PT^{-1}\rho^{-2}$. One can show that
823 [15]

$$s_e = T^{-1} \geq 0 \text{ and } P s_e + \rho^2 s_\rho = 0. \quad (45)$$

824 In order to apply the entropy viscosity method to the variable-area Euler equa-
825 tions, dissipative terms need to be added to each equation in Eq. (43). The
826 functional forms of these terms need to be such that the entropy residual de-
827 rived with these terms present also satisfies the minimum entropy principle. To
828 prove the minimum entropy principle, the extra terms appearing in the entropy
829 residual are either recast as conservative terms or shown to be positive. The
830 rest of this appendix presents this demonstration. Following [15], we first write
831 the variable-area equations with dissipative terms:

$$\partial_t (\rho A) + \vec{\nabla} \cdot (\rho \vec{u} A) = \vec{\nabla} \cdot f \quad (46a)$$

$$\partial_t (\rho \vec{u} A) + \vec{\nabla} \cdot [A (\rho \vec{u} \otimes \vec{u} + P \mathbb{I})] = P \vec{\nabla} A + \vec{\nabla} \cdot g \quad (46b)$$

832

$$\partial_t (\rho E A) + \vec{\nabla} \cdot [\vec{u} A (\rho E + P)] = \vec{\nabla} \cdot (h + \vec{u} \cdot g). \quad (46c)$$

833 where f , g and h are dissipative fluxes to be determined. Starting from the
 834 modified system of equations given in Eq. (46), the entropy residual is derived
 835 again. The derivation requires the following steps : express the governing laws
 836 in terms of primitive variables (ρ, \vec{u}, e) , multiply the continuity equation by ρs_ρ
 837 and the internal energy equation by s_e , and invoke multivariate chain rule, e.g.,
 838 $\partial s / \partial x = s_e \partial e / \partial x + s_\rho \partial \rho / \partial x$. These steps are similar to those used for the
 839 standard Euler equations [15]. Some of the lengthy algebra is omitted here.
 840 The above steps yield:

$$A \rho \left(\partial_t s + \vec{u} \cdot \vec{\nabla} s \right) = s_e \left[\vec{\nabla} \cdot h + g : \vec{\nabla} u + \left(\frac{u^2}{2} - e \right) \vec{\nabla} \cdot f \right] + \rho s_\rho \vec{\nabla} \cdot f. \quad (47)$$

841 The next step consists of choosing a definition for each of the dissipative terms
 842 so that the left hand-side is positive. The right hand-side of Eq. (47) can be
 843 simplified using the relations $g = A \mu \vec{\nabla}^s \vec{u} + f \otimes \vec{u}$ and $h = \tilde{h} - 0.5 \|\vec{u}\|^2 f$ to give

$$A \rho \left(\partial_t s + \vec{u} \cdot \vec{\nabla} s \right) = s_e \left[\vec{\nabla} \cdot \tilde{h} - e \vec{\nabla} \cdot f \right] + \rho s_\rho \vec{\nabla} \cdot f + A s_e \mu \vec{\nabla}^s \vec{u} : \vec{\nabla} \vec{u}. \quad (48)$$

The right hand-side is now integrated by parts:

$$\begin{aligned} A \rho \left(\partial_t s + \vec{u} \cdot \vec{\nabla} s \right) &= \vec{\nabla} \cdot \left[s_e \tilde{h} - s_e e f + \rho s_\rho f \right] \\ &\quad - \vec{\nabla} \cdot \tilde{h} \vec{\nabla} s_e + f \cdot \vec{\nabla} (e s_e) - f \cdot \vec{\nabla} (\rho s_\rho) + A s_e \mu \vec{\nabla}^s \vec{u} : \vec{\nabla} \vec{u} \end{aligned} \quad (49)$$

844 where $\vec{\nabla}^s$ is the symmetric gradient. The term $A s_e \mu \vec{\nabla}^s \vec{u} : \vec{\nabla} \vec{u}$ is positive and
 845 thus, does not need any further modification. It remains to treat the other

846 terms of the right hand-side that we now call rhs :

$$rhs = \vec{\nabla} \cdot [s_e \tilde{h} - s_e e f + \rho s_\rho f] - \tilde{h} \cdot \vec{\nabla} s_e + f \cdot \vec{\nabla} (e s_e) - f \cdot \vec{\nabla} (\rho s_\rho).$$

847 The first term in rhs is a conservative term. By carefully choosing a definition
 848 for \tilde{h} and f , the conservative term can be expressed as a function of the entropy
 849 s . The inclusion of the variable area in the choice of the dissipative terms is also
 850 required so that, when assuming constant area, the standard Euler equations
 851 are recovered. The following definitions for \tilde{h} and f are chosen:

$$\tilde{h} = A\kappa \vec{\nabla}(\rho e) \text{ and } f = A\kappa \vec{\nabla} \rho,$$

852 which yields, using the chain rule,

$$rhs = \vec{\nabla} \cdot (\rho A \kappa \vec{\nabla} s) - A \kappa \underbrace{\left[\vec{\nabla}(\rho e) \vec{\nabla} s_e - \vec{\nabla} \rho \vec{\nabla} (e s_e) + \vec{\nabla} \rho \vec{\nabla} (\rho s_\rho) \right]}_{\mathbf{Q}}$$

853 It remains to treat the term \mathbf{Q} that can be recast under a quadratic form.

854 Following [15], one obtain:

$$\begin{aligned} \mathbf{Q} &= \rho X^t \Sigma X \\ \text{with } X &= \begin{bmatrix} \vec{\nabla} \rho \\ \vec{\nabla} e \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \rho^{-2} \partial_\rho (\rho^2 \partial_\rho s) & \partial_{\rho, es} \\ \partial_{\rho, es} & \partial_{e, es} \end{bmatrix} \end{aligned}$$

855 The matrix Σ is symmetric and identical to the matrix obtained in [15]. The sign
 856 of the quadratic form can be simply determined by studying the positiveness of
 857 the matrix Σ . In this particular case, it is required to prove that the matrix is
 858 negative definite: the quadratic form is on the right hand-side and is preceded by
 859 a negative sign. According to [15], the convexity of the opposite of the entropy

860 function, i.e., $-s$, with respect to the internal energy e and the specific volume
 861 $1/\rho$ is sufficient to ensure that the matrix Σ is negative definite.
 862 Thus, the right hand-side of the entropy residual Eq. (47) is now either recast
 863 as conservative terms, or known to be positive. Thus, the entropy minimum
 864 principle holds.

865 **C. Entropy residual for isentropic flows**

866 This appendix shows that the entropy residual is zero for isentropic flows.
 867 For convenience, we recall here the entropy residual as a function of the pressure,
 868 density, velocity, and speed of sound:

$$\tilde{R}_{\text{ent}} = \frac{DP}{Dt} - c^2 \frac{D\rho}{Dt}. \quad (50)$$

869 Assuming an isentropic flow, pressure is only a function of density, i.e., $P =$
 870 $f(\rho)$ or equivalently $\rho = f^{-1}(P)$. Using the definition of the speed of sound
 871 $c^2 = \left. \frac{\partial P}{\partial \rho} \right)_s$ and the above form of the equation of state, we have

$$c^2 = \left. \frac{\partial P}{\partial \rho} \right)_s = \frac{dP}{d\rho} = \frac{df(\rho)}{d\rho}. \quad (51)$$

872 Using the chain rule, the entropy residual in Eq. (50) can be recast as follows
 873 and proven equal to zero:

$$\tilde{R}_{\text{ent}} = \frac{df(\rho)}{d\rho} \frac{D\rho}{Dt} - c^2 \frac{D\rho}{Dt} = c^2 \frac{D\rho}{Dt} - c^2 \frac{D\rho}{Dt} = 0. \quad (52)$$