Entropy-based artificial viscosity stabilization for non-equilibrium Grey Radiation-Hydrodynamics

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Outline

Grey Radiation-Hydrodynamics (GRHD)

GRHD system of equations

$$\partial_t(\rho) + \partial_x(\rho u) = 0$$

$$\partial_t(\rho u) + \partial_x\left(\rho u^2 + P\right) = -\partial_x\left(\frac{\epsilon}{3}\right)$$

$$\partial_t(\rho E) + \partial_x\left[u(\rho E + P)\right] = -\frac{u}{3}\partial_x\epsilon - \sigma_a c\left(aT^4 - \epsilon\right)$$

$$\partial_t\epsilon + \frac{4}{3}\partial_x(u\epsilon) = \frac{u}{3}\partial_x\epsilon + \partial_x\left(\frac{c}{3\sigma_t}\partial_x\epsilon\right) + \sigma_a c\left(aT^4 - \epsilon\right)$$

- ρ material density
- u material velocity
- E material specific total energy
- ϵ radiation energy density
- P material pressure
- T material temperature

A few remarks:

- Relaxation term in the energy and radiation equations: $\sigma_a c\left(aT^4 - \epsilon\right)$.
- Diffusion term: $\partial_x\left(\frac{c}{3\sigma_t}\partial_x\epsilon\right)$.
- The above system of equations is NOT hyperbolic.

Proposed goal

To stabilize the above system with a high-order artificial viscosity method based on the local entropy production

Quick overview of the entropy-based artificial viscosity formalism

General scalar conservation law: $\boxed{\partial_t u + \vec{\nabla}\cdot\vec{f}(u) = 0}$.

1. Add viscous fluxes $\boxed{\partial_t u + \vec{\nabla}\cdot\vec{f}(u) = \vec{\nabla}\cdot\mu\vec{\nabla}u}$
2. Let the amount of **artificial viscosity** μ be $\propto$ the **local entropy production**
   - Determine an entropy pair $(s(u),\ \vec{\Psi}(u))$ for the PDE under consideration
   - Compute the entropy residual $\boxed{R_e := \partial_t s(u_h) + \vec{\nabla}\cdot\Psi(u_h)}$, in each cell K, at each quadrature point x_q
   - Compute the speed and kinematic **entropy viscosity** associated with this residual

$$v_e^K(x_q) := h_K \frac{|R_e(x_q)|_K}{|s-\bar{s}|_\infty} \quad \text{and} \quad \mu_e^K(x_q) := h_K v_e^K(x_q) \tag{1}$$

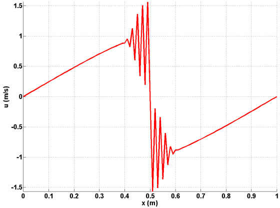
3. Limit the viscosity: upper bound = **Local Lax-Friedrichs** (LLF) viscosity

$$\mu^K(x_q) := \min\left(\frac{h_K}{2}\max_{x\in K}|\vec{f}'(u(x))|,\ \mu_e^K(x_q)\right) \tag{2}$$

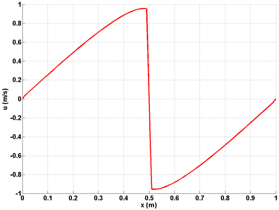
4. Plug in the standard Galerkin weak form as a viscous regularization

$$\int_V \left(\partial_t u_h + \vec{\nabla}\cdot\vec{f}(u_h)\right) b\,dx + \sum_K \int_K \mu^K \vec{\nabla}u_h\cdot\vec{\nabla}b\,dx = 0 \quad \forall b \tag{3}$$

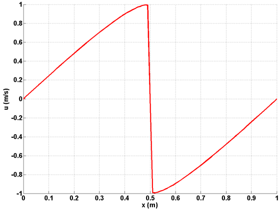
Example: Burgers equation $\partial_t u + \frac{1}{2}\partial_x u^2 = 0$



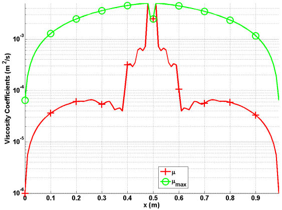
(a) Without stabilization



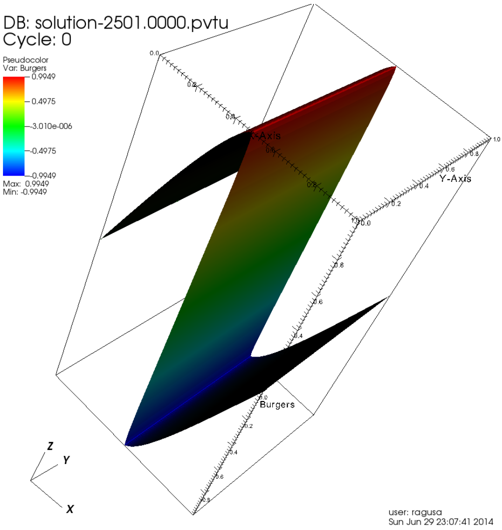
(b) With first-order viscosity



(c) With entropy viscosity



(d) Viscosity coefficient profiles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Pseudocolor
Var: Burgers
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-0.4975
-0.9949
Max: 0.9949
Min: -0.9949
X-Axis
Y-Axis
Burgers
user: ragusa
Sun Jun 29 23:07:41 2014

Viscous regularization of Euler equations

Regularized Euler equations

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{u}) = \vec{\nabla} \cdot \vec{f}$$
$$\partial_t (\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u} + P \mathbb{I}) = \vec{\nabla} \cdot \mathbb{g}$$
$$\partial_t (\rho E) + \vec{\nabla} \cdot [\vec{u} (\rho E + P)] = \vec{\nabla} \cdot \vec{h}$$

How to select the artificial viscous fluxes?

By proving that the regularized equations satisfy a minimum principle on the specific entropy, $s(\rho, e)$ [Guermond/Popov/Pasquetti (JCP 2011)]

Minimum entropy principle

$$\inf_{x \in \mathbb{R}^d} s(x, t) \geq \inf_{x \in \mathbb{R}^d} s_0(x) \qquad \forall t \geq 0 \tag{5}$$

General idea of the derivation

Goal: To obtain an entropy relationship: $\rho(\partial_t s + \vec{u} \cdot \vec{\nabla} s) = \ldots \geq 0$

Entropy is a function of density ρ and internal energy e. Using chain rule, we have

$$\partial_\alpha s = s_\rho \partial_\alpha \rho + s_e \partial_\alpha e \quad \text{with } \alpha = t, x$$

Now, re-write Euler equations in non-conservative form as a function of ρ, u, and e.

Entropy equation

The following choice of viscous fluxes, $\vec{f} = \kappa \vec{\nabla} \rho$, $\mathbb{g} = \mu \rho \vec{\nabla}^s \vec{u} + \vec{u} \otimes \vec{f}$,and $\vec{h} = \kappa \vec{\nabla}(\rho e) - \frac{1}{2} u^2 \vec{f} + \mathbb{g} \cdot \vec{u}$, yields:

$$\rho \left(\partial_t s + \vec{u} \cdot \vec{\nabla} s \right) = \vec{\nabla} \cdot \left(\rho \kappa \vec{\nabla} s \right) - \kappa \rho \mathbf{Q} + s_e \mu \vec{\nabla}^s \vec{u} : \vec{\nabla} \vec{u}$$

Quadratic form

$$\mathbf{Q} = X^t \Sigma X \quad \text{with } X = \begin{bmatrix} \vec{\nabla} \rho \\ \vec{\nabla} e \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \partial_\rho(\rho^2 \partial_\rho s) & \partial_{\rho,e} s \\ \partial_{\rho,e} s & \partial_{e,e} s \end{bmatrix}$$

The form $\mathbf{Q}$ is negative definite if and only if $-s$ is convex with respect to e and ρ^{-1}.

QED (recall: $s_e = 1/T > 0$)

Euler equations with viscous regularization (final form)

$$\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{u}) = \vec{\nabla}\cdot\left(\kappa\vec{\nabla}\rho\right)$$

$$\partial_t\left(\rho\vec{u}\right) + \vec{\nabla}\cdot(\rho\vec{u}\otimes\vec{u} + P\mathbb{I}) = \vec{\nabla}\cdot\left(\mu\rho\vec{\nabla}^s\vec{u} + \kappa\vec{u}\otimes\vec{\nabla}\rho\right)$$

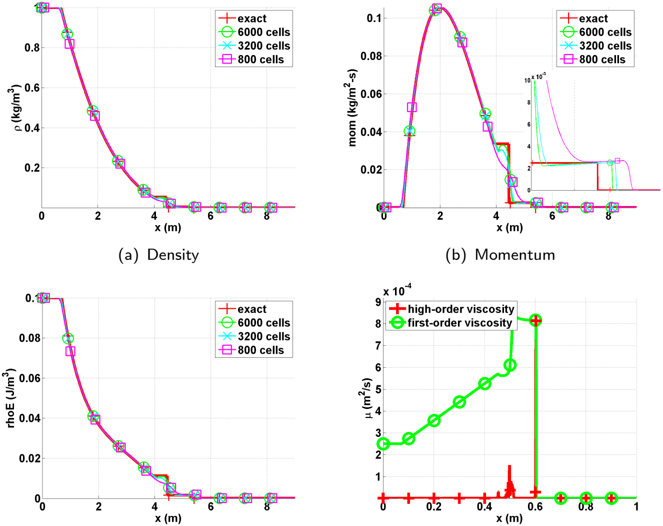
$$\partial_t\left(\rho E\right) + \vec{\nabla}\cdot[\vec{u}\left(\rho E + P\right)] = \vec{\nabla}\cdot\left(\kappa\vec{\nabla}\left(\rho e\right) + \frac{1}{2}||\vec{u}||^2\kappa\vec{\nabla}\rho + \rho\mu\vec{u}\vec{\nabla}\vec{u}\right)$$

where κ and μ are positive viscosity coefficients.

Definition of the viscosity coefficients

- As before, $\mu = \min(\mu^{LLF}, \mu^{entr})$ and $\kappa = \min(\kappa^{LLF}, \kappa^{entr})$
- Entropy viscosities $\propto$ entropy production
- All-speed (from low-Mach to supersonic) extension by Delchini/Ragusa/Berry in Computers & Fluids, 2015

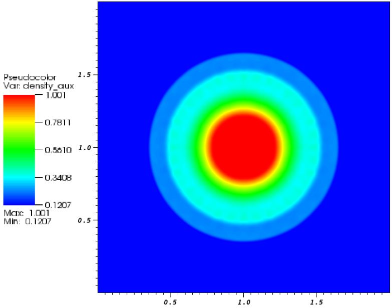
Leblanc shock tube



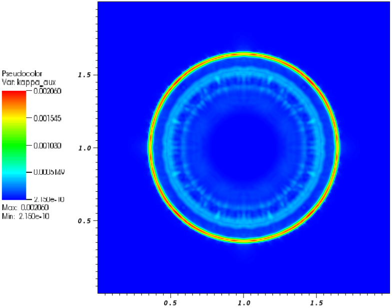
(a) Density (b) Momentum

(c) Total energy (d) Viscosity

2-D explosion test



(a) Density



(b) Viscosity

Entropy-based artificial viscosity technique for the GRHD

Questions to answer:

1. The GRHD equations are not hyperbolic. Can we apply the entropy viscosity method (EVM)?
   - Our initial attempt: apply the EVM to the hyperbolic part of the GRHD [an idea similar to Balsara JQSRT 1999, Lowrie&Morel JQSRT 2001]
2. What is an appropriate functional form for the entropy of the GRHD, $s(\rho, e, \epsilon) = \ldots ???$
3. What is an appropriate expression for the viscous fluxes so that the regularized GRHD eqs satisfy the minimum entropy principle?
4. Is the viscous regularization well-behaved in the equilibrium-diffusion limit?

Hyperbolic part of the GRHD

$$\partial_t(\rho) + \partial_x(\rho u) = 0 \tag{7a}$$

$$\partial_t(\rho u) + \partial_x\left(\rho u^2 + P + \frac{\epsilon}{3}\right) = 0 \tag{7b}$$

$$\partial_t(\rho E) + \partial_x\left[u(\rho E + P)\right] + \frac{u}{3}\partial_x \epsilon = 0 \tag{7c}$$

$$\partial_t \epsilon + \frac{4}{3}\partial_x(u\epsilon) - \frac{u}{3}\partial_x \epsilon = 0 \tag{7d}$$

Eigenvalues

$$\lambda_{1,4} = u \pm c_m$$
$$\lambda_{2,3} = u$$

with

$$c_m^2 = \underbrace{P_\rho + \frac{P}{\rho^2}P_e}_{c_{Euler}^2} + \underbrace{\frac{4\epsilon}{9\rho}}_{c_{rad}^2}$$

Entropy-based artificial viscosity for the GRHD: derivation

Study of the hyperbolic parts of the GRHD: process

1. Add viscous regularization (fluxes) to the equations

$$\partial_t(\rho) + \partial_x(\rho u) = \partial_x f \tag{8a}$$

$$\partial_t(\rho u) + \partial_x\left(\rho u^2 + P + \frac{\epsilon}{3}\right) = \partial_x g \tag{8b}$$

$$\partial_t(\rho E) + \partial_x\left[u(\rho E + P)\right] + \frac{u}{3}\partial_x \epsilon = \partial_x h \tag{8c}$$

$$\partial_t \epsilon + \frac{4}{3}\partial_x(u\epsilon) - \frac{u}{3}\partial_x \epsilon = \partial_x \ell \tag{8d}$$

2. With $s(\rho, e, \epsilon)$, use chain rule to obtain the entropy relationship

$$\partial_\alpha s = \partial_\rho s \partial_\alpha \rho + \partial_e s \partial_\alpha e + \partial_\epsilon s \partial_\alpha \epsilon \tag{9}$$

3. An observation: we can greatly simplify the expression by assuming $s(\rho, e, \epsilon) = s_{Euler}(\rho, e) + s_{rad}(\rho, \epsilon)$

Study of the hyperbolic parts of the GRHD: results

1.
$$s(\rho, e, \epsilon) = s_{Euler}(\rho, e) + \frac{4a^{1/4}}{3\rho}\epsilon^{\frac{3}{4}} \tag{10}$$

2. Using the Eulerian viscous fluxes, supplemented by an radiation energy viscous flux
$$\begin{cases} f & = \kappa \partial_x \rho \\ g & = \rho\mu\partial_x u + uf \\ h & = \kappa\partial_x(\rho e) - \frac{1}{2}u^2 f + gu \\ \ell & = \kappa \partial_x \epsilon \end{cases} \tag{11}$$
we get the following result:

3. Entropy conservation statement:

$$\boxed{\rho \frac{Ds}{Dt} = \partial_x\left(\rho\kappa\partial_x s\right) + (\kappa\partial_x\rho)(\partial_x s) - \rho\kappa X^T A X + s_e \rho\mu(\partial_x u)^2 \geq 0}$$

$$X = \begin{bmatrix} \partial_x \rho \\ \partial_x e \\ \partial_x \epsilon \end{bmatrix} \text{ and } A = \begin{bmatrix} \partial_\rho\left(\rho^2 \partial_\rho s_{Euler}\right) & \partial_{\rho,e} s_{Euler} & 0 \\ \partial_{\rho,e} s_{Euler} & \partial_{e,e} s_{Euler} & 0 \\ 0 & 0 & -\frac{a^{1/4}}{4\rho}\epsilon^{-5/4} \end{bmatrix}$$

The form $X^T A X$ is negative -definite (Delchini/Ragusa/Morel, JCP 2015)

New developments

Entropy conservation statement for the full GRHD equations

Recently, we have been able to show:

$$\rho \frac{Ds}{Dt} = \partial_x \left(\rho\kappa\partial_x s\right) + (\kappa\partial_x\rho)(\partial_x s) - \rho\kappa X^T A X + s_e \rho\mu(\partial_x u)^2 + \left(\rho s_\epsilon - s_e\right)\sigma_a c\left(aT^4 - \epsilon\right) + \rho s_\epsilon \partial_x\left(\frac{c}{3\sigma_t}\partial_x\epsilon\right) \geq 0 \quad (12)$$

where the terms in red are unconditionally entropy-producing (unpublished, in preparation)

Finally, the Regularized full GRHD equations are:

$$\partial_t\left(\rho\right) + \partial_x\left(\rho u\right) = \partial_x\left(\kappa\partial_x\rho\right) \quad (13a)$$

$$\partial_t\left(\rho u\right) + \partial_x\left(\rho u^2 + P + \frac{\epsilon}{3}\right) = \partial_x\left(\kappa\partial_x(\rho u)\right) \quad (13b)$$

$$\partial_t\left(\rho E\right) + \partial_x\left[u\left(\rho E + P\right)\right] = -\frac{u}{3}\partial_x\epsilon - \sigma_a c\left(aT^4 - \epsilon\right) + \partial_x\left(\kappa\partial_x(\rho E)\right) \quad (13c)$$

$$\partial_t\epsilon + \frac{4}{3}\partial_x\left(u\epsilon\right) = \frac{u}{3}\partial_x\epsilon + \partial_x\left(\frac{c}{3\sigma_t}\partial_x\epsilon\right) + \sigma_a c\left(aT^4 - \epsilon\right) + \partial_x\left(\kappa\partial_x\epsilon\right) \quad (13d)$$

Equilibrium Diffusion Limit

non-dimensionalization:

$$\partial_{t'}\left(\rho'\right)+\partial_{x'}\left(\rho' u'\right)=\mathbb{V}_\infty \partial_x\left(\kappa' \partial_{x'} \rho'\right) \tag{14a}$$

$$\partial_{t'}\left(\rho' u'\right)+\partial_{x'}\left(\rho u^{2'}+P'+\mathbb{P}_\infty \frac{\epsilon'}{3}\right)=\mathbb{V}_\infty \partial_{x'}\left(\kappa' \partial_{x'}(\rho' u')\right) \tag{14b}$$

$$\begin{aligned}\partial_{t'}\left(\rho' E'\right)+\partial_{x'}\left[u'\left(\rho' E'+P'\right)\right]=&-\mathbb{P}_\infty \frac{u'}{3}\partial_{x'}\epsilon' \\ &-\mathbb{P}_\infty \mathbb{C}_\infty^{-1}\mathbb{L}_\infty\left(\sigma_t'-\mathbb{L}_{s,\infty}\sigma_s'\right)\left(T'^{,4}-\epsilon'\right)+\mathbb{V}_\infty \partial_{x'}\left(\kappa' \partial_{x'}(\rho' E')\right)\end{aligned} \tag{14c}$$

$$\begin{aligned}\partial_{t'}\epsilon'+\frac{4}{3}\partial_{x'}\left(u'\epsilon'\right)=&\frac{u'}{3}\partial_{x'}\epsilon'+\mathbb{L}_\infty^{-1}\mathbb{C}_\infty^{-1}\partial_{x'}\left(\frac{1}{3\sigma_t'}\partial_{x'}\epsilon'\right) \\ &+\mathbb{C}_\infty^{-1}\mathbb{L}_\infty\left(\sigma_t'-\mathbb{L}_{s,\infty}\sigma_s'\right)\left(T'^{,4}-\epsilon'\right)+\mathbb{V}_\infty \partial_{x'}\left(\kappa' \partial_{x'}\epsilon'\right)\end{aligned} \tag{14d}$$

non-dimensional parameters

$$\mathbb{L}_\infty=L_\infty \sigma_{t,\infty}=\mathcal{O}(\varepsilon^{-1}),\ \mathbb{L}_{s,\infty}=\frac{\sigma_{s,\infty}}{\sigma_{t,\infty}}=\mathcal{O}(\varepsilon),\ \mathbb{C}_\infty=\frac{c_{m,\infty}}{c}=\mathcal{O}(\varepsilon)$$

$$\mathbb{P}_\infty=\frac{aT_\infty^4}{\rho_\infty c_{m,\infty}^2}=\mathcal{O}(1),\ \mathbb{V}_\infty=\frac{\kappa_\infty}{c_{m,\infty}L_\infty}=\mathcal{O}(1)$$

The variables are expanded in a power series in ε

Equilibrium Diffusion Limit results:

$$\partial_t \rho_0 + \partial_x (\rho u)_0 = \partial_x (\kappa \partial_x \rho)_0 \tag{15a}$$

$$\partial_t (\rho u)_0 + \partial_x \left(\rho u^2 + P^*\right)_0 = \partial_x (\kappa \partial_x (\rho u))_0 \tag{15b}$$

$$\partial_x (\rho E^*)_0 + \partial_x [u (\rho E^* + P^*)]_0 = \partial_x \left(\frac{1}{3\sigma_t} \partial_x T^4\right)_0 + \partial_x (\kappa \partial_x \rho E^*)_0 \tag{15c}$$

$P^* = P + \mathbb{P}_\infty \frac{T^4}{3}$ and $E^* = E + \mathbb{P}_\infty \frac{T^4}{\rho}$

Leading order of entropy

$$s_0 (\rho, e) = s_{Euler,0} (\rho, e) + \frac{4}{3} \frac{T_0^3}{\rho_0} \tag{16}$$

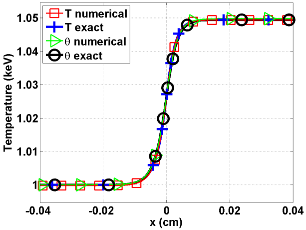
We recover the EDL results (Lowrie/Morel, JQSRT, 2001).

Viscous regularization scales adequately with $\mathbb{V}_\infty = \mathcal{O}(1)$.

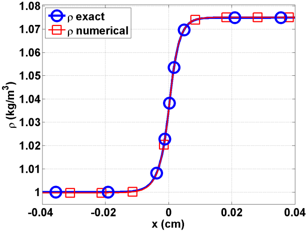
Numerical solution

- spatial discretization: CFEM
- temporal discretization: fully implicit (BDF2)
- solution technique: JFNK with finite-difference approximation of the Jacobian as preconditioner
- semi-analytical solutions provided by Jim Ferguson (LANL)
- Two sets of results presented today:
  1. with constant opacities (Mach 1.05, 2, 5, 50)
  2. with temperature-dependent opacities (Mach 3)

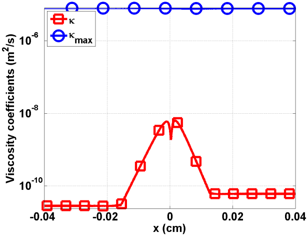
Steady-state solution for Mach 1.05



(a) Temperatures

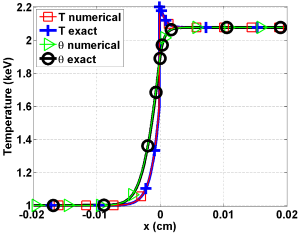


(b) Material density

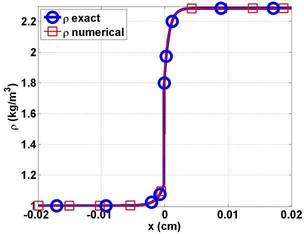


(c) Viscosity coefficients

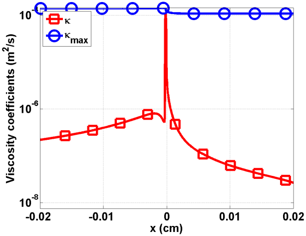
Steady-state solution for Mach 2



(a) Temperatures

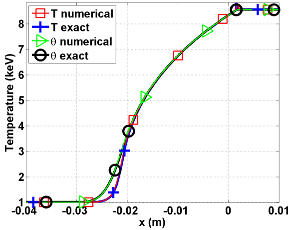


(b) Material density

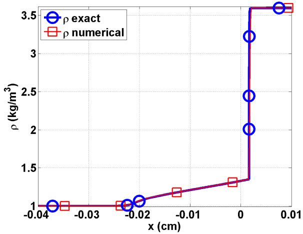


(c) Viscosity coefficients

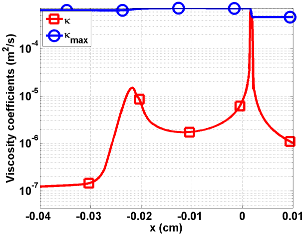
Steady-state solution for Mach 5



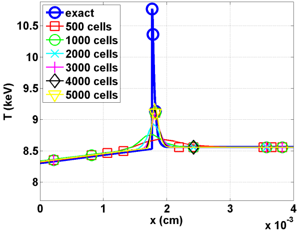
(a) Temperatures



(b) Material density

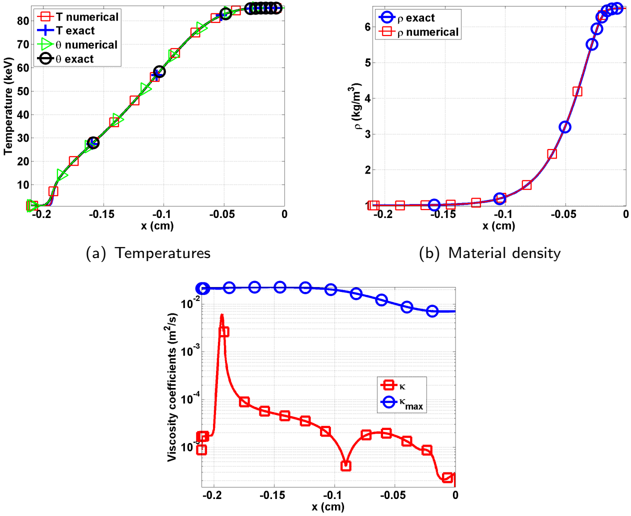


(c) Viscosity coefficients



(d) Zoom at the Z spike

Steady-state solution for Mach 50

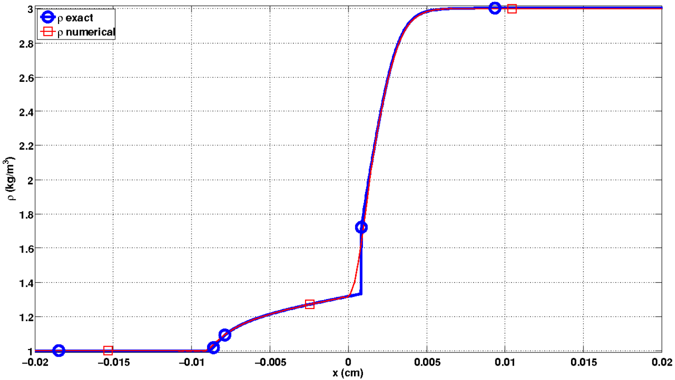


(a) Temperatures (b) Material density

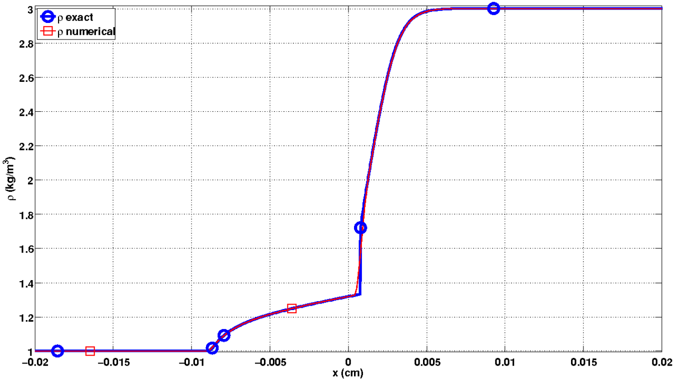
(c) Viscosity coefficients

Steady-state solution for Mach 3: density, 500 cells

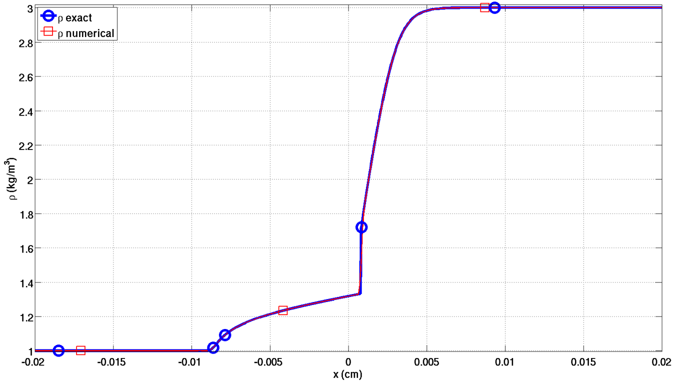
Now, results with temperature-dependent opacities:



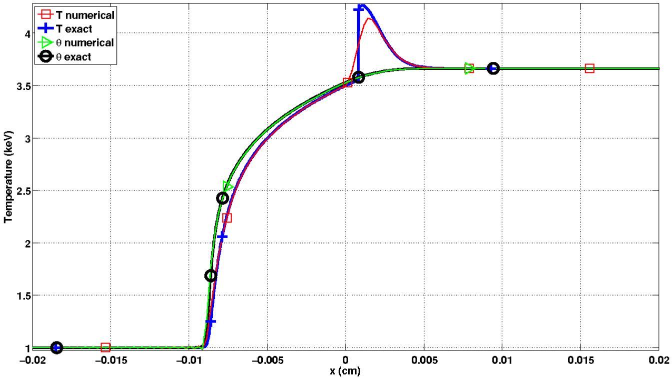
Steady-state solution for Mach 3: density, 1000 cells



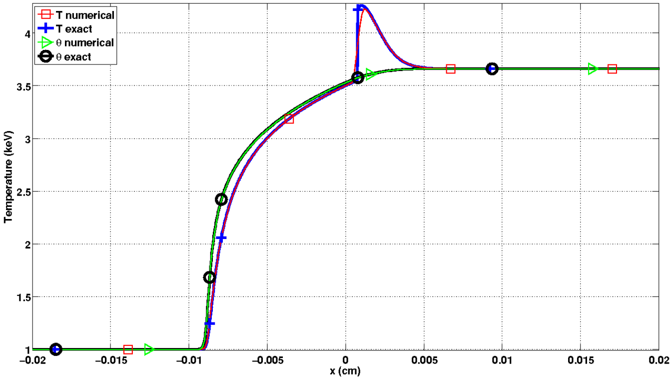
Steady-state solution for Mach 3: density, 2000 cells



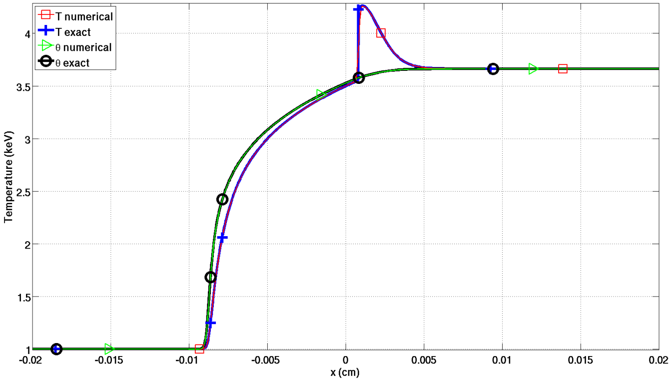
Steady-state solution for Mach 3: temperature, 500 cells



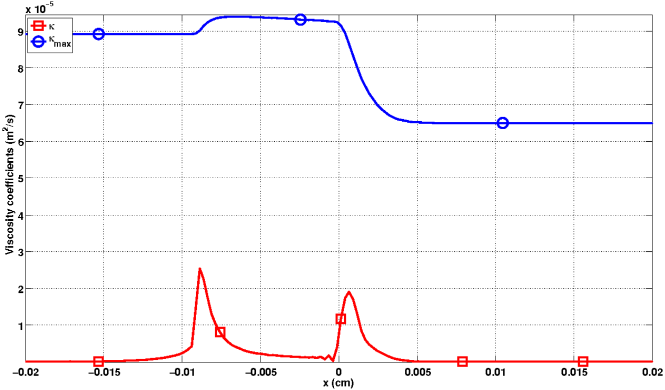
Steady-state solution for Mach 3: temperature, 1000 cells



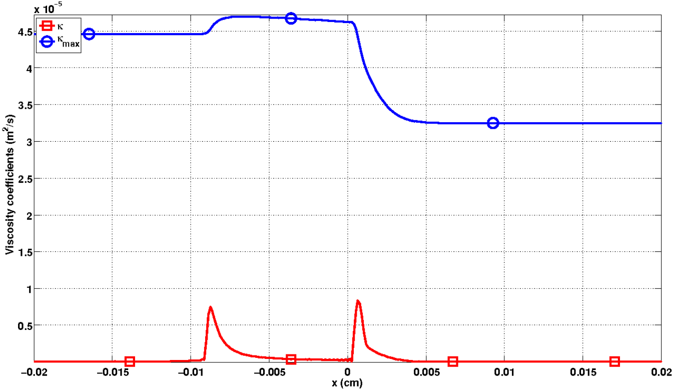
Steady-state solution for Mach 3: temperature, 2000 cells



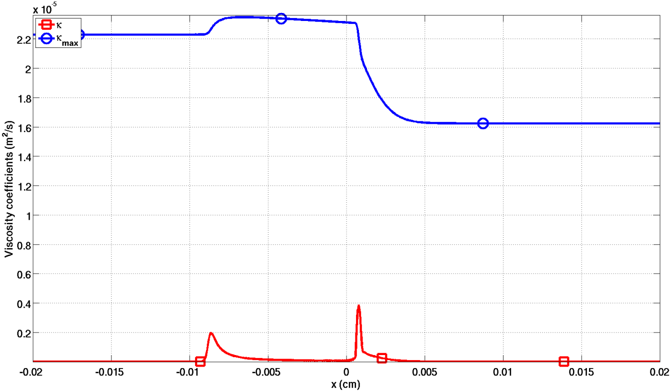
Steady-state solution for Mach 3: viscosity, 500 cells



Steady-state solution for Mach 3: viscosity, 1000 cells



Steady-state solution for Mach 3: viscosity, 2000 cells



Conclusions

- Extended the entropy-viscosity method to the **full** Grey Radiation-Hydrodynamic equations.
- Verified the entropy minimum principle for the regularized equations GRHD.
- Viscous regularization scales appropriately in the equilibrium-diffusion limit.
- Numerical results are in excellent agreement with semi-analytical solutions.

Outlook

- Multi-D.
- Replace radiation diffusion with S_n radiation transport.
- Switch solution technique to IMEX (implicit for radiation, explicit for hydro).
- Other spatial discretization (DGFEM).
- FCT
  $\rightarrow$ poster tomorrow on FCT for radiation transport

Seven-equation two-phase flow model

with viscous regularization

$$\frac{\partial \alpha_k A}{\partial t} + A\vec{u}_{int} \cdot \vec{\nabla}\alpha_k - A\mu_P(P_k - P_j) = \vec{\nabla}\cdot\vec{l}_k \tag{17a}$$

$$\frac{\partial (\alpha\rho)_k A}{\partial t} + \vec{\nabla}\cdot[(\alpha\rho\vec{u})_k A] = \vec{\nabla}\cdot\vec{f}_k \tag{17b}$$

$$\frac{\partial (\alpha\rho\vec{u})_k A}{\partial t} + \vec{\nabla}\cdot\left[\alpha_k A\left(\rho\vec{u}\otimes\vec{u} + P\mathbb{I}\right)_k\right] - P_{int}A\vec{\nabla}\alpha_k + P_k\alpha_k\vec{\nabla}A - A\lambda_u(\vec{u}_j - \vec{u}_k) = \vec{\nabla}\cdot\mathfrak{g}_k \tag{17c}$$

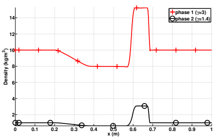
$$\frac{\partial (\alpha\rho E)_k A}{\partial t} + \vec{\nabla}\cdot\left[\alpha_k\vec{u}_k A\left(\rho E + P\right)_k\right] - P_{int}A\vec{u}_{int}\cdot\vec{\nabla}\alpha_k + \bar{P}_{int}A\mu_P(P_k - P_j) - A\lambda_u\bar{\vec{u}}_{int}\cdot(\vec{u}_j - \vec{u}_k) = \vec{\nabla}\cdot\left(\vec{h}_k + \vec{u}_k\cdot\mathfrak{g}_k\right) \tag{17d}$$

Viscous fluxes:

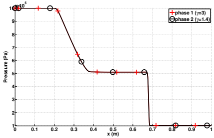
$$\vec{l}_k = \beta_k A\vec{\nabla}\alpha_k\,, \quad \vec{f}_k = \alpha_k A\kappa_k\vec{\nabla}\rho_k + \rho_k\vec{l}_k$$

$$\mathfrak{g}_k = \alpha_k A\mu_k\rho_k\vec{\nabla}^s\vec{u}_k + \vec{f}_k\otimes\vec{u}_k\,, \quad \vec{h}_k = \alpha_k A\kappa_k\vec{\nabla}(\rho e)_k - \frac{\|\vec{u}_k\|^2}{2}\vec{f}_k + (\rho e)_k\vec{l}_k$$

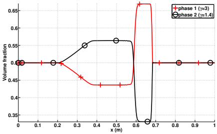
7-equation two-phase flow: shock tube with large relaxation



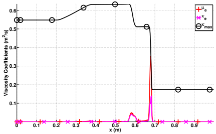
(a) Densities



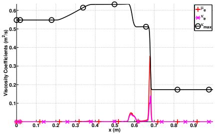
(b) Pressures



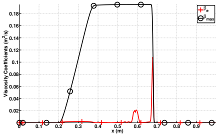
(c) Volume fractions



(d) Viscosity κ, μ phase 1



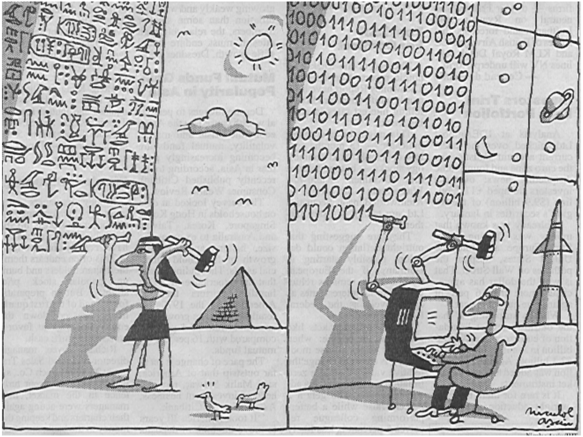
(e) Viscosity κ, μ phase 1



(f) Viscosity β (volume fraction)

Thank you

Thanks to Jim Ferguson (LANL) for the semi-analytical solutions.
Thanks Jean-Luc Guermond and Bojan Popov (Texas A&M) for fruitful discussions.



Why an upper bound for viscosity?

Large entropy residual in shocks $\longrightarrow$ large entropy viscosity μ_e

There is such a thing as too much of a good thing ...
Il ne faut point être plus royaliste que le Roy

Upper bound for μ

First-order upwind scheme is monotone but over dissipative. We should not exceed the amount of stabilization that such a scheme provides.

upwinding = centered approximation (Galerkin) − numerical diffusion
Example: linear advection $\partial_t u + \beta \partial_x u = 0$

$$\beta \frac{u_i - u_{i-1}}{h} = \beta \frac{u_{i+1} - u_{i-1}}{2h} - \frac{\beta h}{2} \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{18}$$

So, the dissipative term is $\frac{\beta h}{2} \partial_{xx} u$ and the first-order viscosity is $\frac{\beta h}{2}$

First-order viscosity

- scalar conservation law: $\frac{h}{2}|f'(u)|$
- system: $\frac{h}{2} \max\left(\text{eig}(\partial_u f)\right)$