

# Entropy-based artificial viscosity stabilization for non-equilibrium Grey Radiation-Hydrodynamics

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# Outline

- 1 Background and Motivation
  - Grey Radiation-Hydrodynamics
  - A Brief Review of the Entropy Viscosity Method for Conservation Laws
- 2 Development of entropy-based artificial viscosity for the GRHD
  - Questions to answer
  - Previous results (JCP 2015)
  - New developments
- 3 Numerical results
  - Constant opacities
  - Temperature-dependent opacities
- 4 Conclusions

# Grey Radiation-Hydrodynamics (GRHD)

## GRHD system of equations

$$\partial_t (\rho) + \partial_x (\rho u) = 0$$

$$\partial_t (\rho u) + \partial_x (\rho u^2 + P) = -\partial_x \left( \frac{\epsilon}{3} \right)$$

$$\partial_t (\rho E) + \partial_x [u (\rho E + P)] = -\frac{u}{3} \partial_x \epsilon - \sigma_a c (a T^4 - \epsilon)$$

$$\partial_t \epsilon + \frac{4}{3} \partial_x (u \epsilon) = \frac{u}{3} \partial_x \epsilon + \partial_x \left( \frac{c}{3 \sigma_t} \partial_x \epsilon \right) + \sigma_a c (a T^4 - \epsilon)$$

- $\rho$  material density
- $u$  material velocity
- $E$  material specific total energy
- $\epsilon$  radiation energy density
- $P$  material pressure
- $T$  material temperature

## A few remarks:

- Relaxation term in the energy and radiation equations:  $\sigma_a c (a T^4 - \epsilon)$ .
- Diffusion term:  $\partial_x \left( \frac{c}{3 \sigma_t} \partial_x \epsilon \right)$ .
- The above system of equations is NOT hyperbolic.

## Proposed goal

To stabilize the above system with a high-order artificial viscosity method based on the local entropy production

# Quick overview of the entropy-based artificial viscosity formalism

General scalar conservation law:  $\partial_t u + \vec{\nabla} \cdot \vec{f}(u) = 0$ .

① Add viscous fluxes  $\partial_t u + \vec{\nabla} \cdot \vec{f}(u) = \vec{\nabla} \cdot \mu \vec{\nabla} u$

② Let the amount of **artificial viscosity**  $\mu$  be  $\propto$  the **local entropy production**

- Determine an entropy pair  $(s(u), \vec{\Psi}(u))$  for the PDE under consideration
- Compute the entropy residual  $R_e := \partial_t s(u_h) + \vec{\nabla} \cdot \vec{\Psi}(u_h)$ , in each cell  $K$ , at each quadrature point  $x_q$
- Compute the speed and kinematic **entropy viscosity** associated with this residual

$$v_e^K(x_q) := h_K \frac{|R_e(x_q)|_K}{|s - \bar{s}|_\infty} \quad \text{and} \quad \mu_e^K(x_q) := h_K v_e^K(x_q) \quad (1)$$

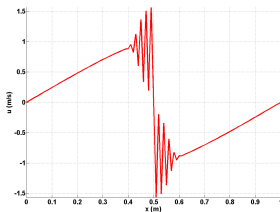
③ Limit the viscosity: upper bound = **Local Lax-Friedrichs** (LLF) viscosity

$$\mu^K(x_q) := \min \left( \frac{h_K}{2} \max_{x \in K} |\vec{f}'(u(x))|, \mu_e^K(x_q) \right) \quad (2)$$

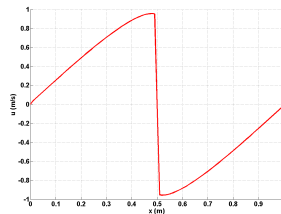
④ Plug in the standard Galerkin weak form as a **viscous regularization**

$$\int_V (\partial_t u_h + \vec{\nabla} \cdot \vec{f}(u_h)) b \, dx + \sum_K \int_K \mu^K \vec{\nabla} u_h \cdot \vec{\nabla} b \, dx = 0 \quad \forall b \quad (3)$$

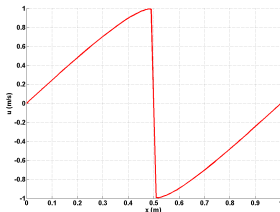
# Example: Burgers equation $\partial_t u + \frac{1}{2} \partial_x u^2 = 0$



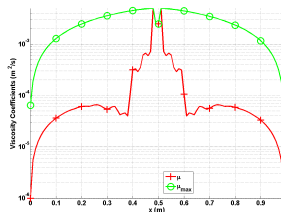
(a) Without stabilization



(b) With first-order viscosity



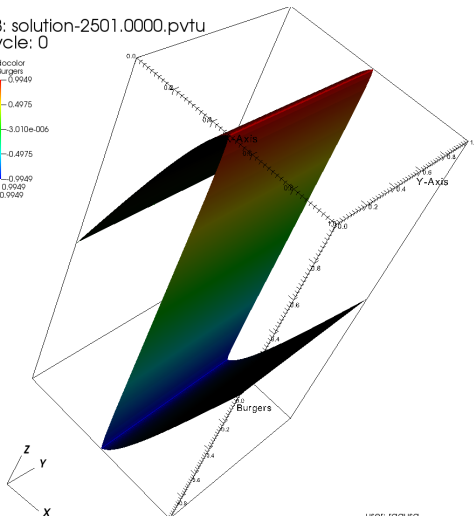
(c) With entropy viscosity



(d) Viscosity coefficient profiles

DB: solution-2501.0000.pvtu  
Cycle: 0

Pseudocolor  
Var: Burgers  
0.9949  
-0.4975  
-3.010e-006  
-0.4975  
-0.9949  
Max: 0.9949  
Min: -0.9949



user: ragusa  
Sun Jun 29 23:07:41 2014

# Viscous regularization of Euler equations

## Regularized Euler equations

$$\begin{aligned}\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{u}) &= \vec{\nabla} \cdot \vec{f} \\ \partial_t (\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u} + P \mathbb{I}) &= \vec{\nabla} \cdot \vec{g} \\ \partial_t (\rho E) + \vec{\nabla} \cdot [\vec{u} (\rho E + P)] &= \vec{\nabla} \cdot \vec{h}\end{aligned}$$

How to select the **artificial viscous fluxes**?

By proving that the **regularized** equations satisfy a minimum principle on the specific entropy,  $s(\rho, e)$  [Guermond/Popov/Pasquetti (JCP 2011)]

## Minimum entropy principle

$$\inf_{x \in \mathbb{R}^d} s(x, t) \geq \inf_{x \in \mathbb{R}^d} s_0(x) \quad \forall t \geq 0 \quad (5)$$

# General idea of the derivation

Goal: To obtain an entropy relationship:  $\rho(\partial_t s + \vec{u} \cdot \vec{\nabla} s) = \dots \geq 0$

Entropy is a function of density  $\rho$  and internal energy  $e$ . Using chain rule, we have

$$\partial_\alpha s = s_\rho \partial_\alpha \rho + s_e \partial_\alpha e \quad \text{with } \alpha = t, x$$

Now, re-write Euler equations in non-conservative form as a function of  $\rho$ ,  $u$ , and  $e$ .

## Entropy equation

The following choice of viscous fluxes,  $\vec{f} = \kappa \vec{\nabla} \rho$ ,  $\mathfrak{g} = \mu \rho \vec{\nabla}^s \vec{u} + \vec{u} \otimes \vec{f}$ , and  $\vec{h} = \kappa \vec{\nabla}(\rho e) - \frac{1}{2} u^2 \vec{f} + \mathfrak{g} \cdot \vec{u}$ , yields:

$$\rho(\partial_t s + \vec{u} \cdot \vec{\nabla} s) = \vec{\nabla} \cdot (\rho \kappa \vec{\nabla} s) - \kappa \rho \mathbf{Q} + s_e \mu \vec{\nabla}^s \vec{u} : \vec{\nabla} \vec{u}$$

## Quadratic form

$$\mathbf{Q} = X^t \mathbb{X} X \quad \text{with } X = \begin{bmatrix} \vec{\nabla} \rho \\ \vec{\nabla} e \end{bmatrix} \quad \text{and } \mathbb{X} = \begin{bmatrix} \partial_\rho(\rho^2 \partial_\rho s) & \partial_{\rho,e} s \\ \partial_{\rho,e} s & \partial_{e,e} s \end{bmatrix}$$

The form  $\mathbf{Q}$  is negative definite if and only if  $-s$  is convex with respect to  $e$  and  $\rho^{-1}$ .

**QED** (recall:  $s_e = 1/T > 0$ )



## Euler equations with viscous regularization (final form)

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{u}) = \vec{\nabla} \cdot (\kappa \vec{\nabla} \rho)$$

$$\partial_t (\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u} + P \mathbb{I}) = \vec{\nabla} \cdot (\mu \rho \vec{\nabla}^s \vec{u} + \kappa \vec{u} \otimes \vec{\nabla} \rho)$$

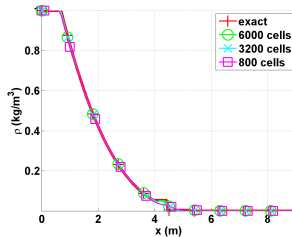
$$\partial_t (\rho E) + \vec{\nabla} \cdot [\vec{u} (\rho E + P)] = \vec{\nabla} \cdot \left( \kappa \vec{\nabla} (\rho e) + \frac{1}{2} \|\vec{u}\|^2 \kappa \vec{\nabla} \rho + \rho \mu \vec{u} \vec{\nabla} \vec{u} \right)$$

where  $\kappa$  and  $\mu$  are positive viscosity coefficients.

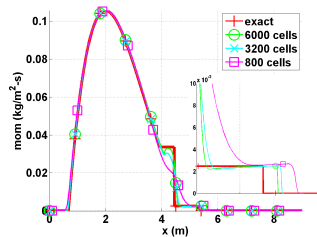
## Definition of the viscosity coefficients

- As before,  $\mu = \min(\mu^{LLF}, \mu^{entr})$  and  $\kappa = \min(\kappa^{LLF}, \kappa^{entr})$
- Entropy viscosities  $\propto$  **entropy production**
- All-speed (from low-Mach to supersonic) extension by [Delchini/Ragusa/Berry in Computers & Fluids, 2015](#)

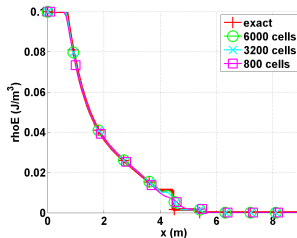
# Leblanc shock tube



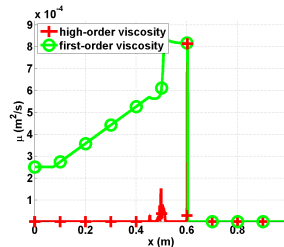
(a) Density



(b) Momentum

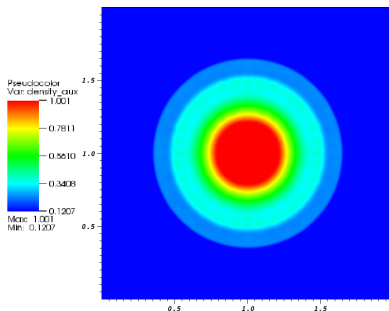


(c) Total energy

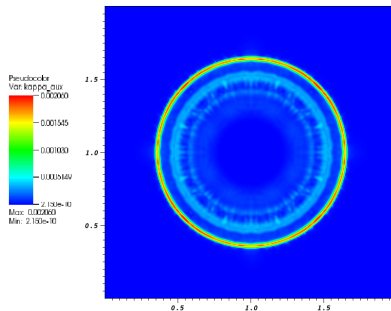


(d) Viscosity

## 2-D explosion test



(a) Density



(b) Viscosity

# Entropy-based artificial viscosity technique for the GRHD

## Questions to answer:

- 1 The GRHD equations are **not hyperbolic**. Can we apply the entropy viscosity method (EVM)?
  - Our initial attempt: apply the EVM to the hyperbolic part of the GRHD [an idea similar to [Balsara JQSRT 1999](#), [Lowrie&Morel JQSRT 2001](#)]
- 2 What is an appropriate functional form for the entropy of the GRHD,  
 $s(\rho, e, \epsilon) = \dots???$
- 3 What is an appropriate expression for the viscous fluxes so that the **regularized** GRHD eqs satisfy the minimum entropy principle?
- 4 Is the viscous regularization well-behaved in the **equilibrium-diffusion limit**?

## Hyperbolic part of the GRHD

$$\partial_t (\rho) + \partial_x (\rho u) = 0 \quad (7a)$$

$$\partial_t (\rho u) + \partial_x \left( \rho u^2 + P + \frac{\epsilon}{3} \right) = 0 \quad (7b)$$

$$\partial_t (\rho E) + \partial_x [u (\rho E + P)] + \frac{u}{3} \partial_x \epsilon = 0 \quad (7c)$$

$$\partial_t \epsilon + \frac{4}{3} \partial_x (u \epsilon) - \frac{u}{3} \partial_x \epsilon = 0 \quad (7d)$$

## Eigenvalues

$$\lambda_{1,4} = u \pm c_m$$

$$\lambda_{2,3} = u$$

with

$$c_m^2 = \underbrace{P_\rho + \frac{P}{\rho^2} P_e}_{c_{Euler}^2} + \underbrace{\frac{4\epsilon}{9\rho}}_{c_{rad}^2}$$



# Entropy-based artificial viscosity for the GRHD: derivation

## Study of the hyperbolic parts of the GRHD: process

- 1 Add viscous regularization (fluxes) to the equations

$$\partial_t (\rho) + \partial_x (\rho u) = \partial_x f \quad (8a)$$

$$\partial_t (\rho u) + \partial_x \left( \rho u^2 + P + \frac{\epsilon}{3} \right) = \partial_x g \quad (8b)$$

$$\partial_t (\rho E) + \partial_x [u (\rho E + P)] + \frac{u}{3} \partial_x \epsilon = \partial_x h \quad (8c)$$

$$\partial_t \epsilon + \frac{4}{3} \partial_x (u \epsilon) - \frac{u}{3} \partial_x \epsilon = \partial_x \ell \quad (8d)$$

- 2 With  $s(\rho, e, \epsilon)$ , use chain rule to obtain the entropy relationship

$$\partial_\alpha s = \partial_\rho s \partial_\alpha \rho + \partial_e s \partial_\alpha e + \partial_\epsilon s \partial_\alpha \epsilon \quad (9)$$

- 3 An observation: we can greatly simplify the expression by assuming  
 $s(\rho, e, \epsilon) = s_{Euler}(\rho, e) + s_{rad}(\rho, \epsilon)$

## Study of the hyperbolic parts of the GRHD: results

①

$$s(\rho, e, \epsilon) = s_{Euler}(\rho, e) + \frac{4a^{1/4}}{3\rho} \epsilon^{3/4} \quad (10)$$

- ② Using the Eulerian viscous fluxes, supplemented by an radiation energy viscous flux

$$\begin{cases} f &= \kappa \partial_x \rho \\ g &= \rho \mu \partial_x u + u f \\ h &= \kappa \partial_x (\rho e) - \frac{1}{2} u^2 f + g u \\ \ell &= \kappa \partial_x \epsilon \end{cases} \quad (11)$$

we get the following result:

- ③ Entropy conservation statement:

$$\rho \frac{Ds}{Dt} = \partial_x (\rho \kappa \partial_x s) + (\kappa \partial_x \rho)(\partial_x s) - \rho \kappa X^T A X + s_e \rho \mu (\partial_x u)^2 \geq 0$$

$$X = \begin{bmatrix} \partial_x \rho \\ \partial_x e \\ \partial_x \epsilon \end{bmatrix} \text{ and } A = \begin{bmatrix} \partial_\rho (\rho^2 \partial_\rho s_{Euler}) & \partial_{\rho,e} s_{Euler} & 0 \\ \partial_{\rho,e} s_{Euler} & \partial_{e,e} s_{Euler} & 0 \\ 0 & 0 & -\frac{a^{1/4}}{4\rho} \epsilon^{-5/4} \end{bmatrix}$$

The form  $X^T A X$  is negative -definite ([Delchini/Ragusa/Morel, JCP 2015](#))

# New developments

## Entropy conservation statement for the full GRHD equations

Recently, we have been able to show:

$$\begin{aligned} \rho \frac{Ds}{Dt} = & \partial_x (\rho \kappa \partial_x s) + (\kappa \partial_x \rho) (\partial_x s) - \rho \kappa X^T A X + s_e \rho \mu (\partial_x u)^2 \\ & + \left( \rho s_\epsilon - s_e \right) \sigma_a c \left( a T^4 - \epsilon \right) + \rho s_\epsilon \partial_x \left( \frac{c}{3 \sigma_t} \partial_x \epsilon \right) \geq 0 \end{aligned} \quad (12)$$

where the terms in **red** are unconditionally entropy-producing (**unpublished, in preparation**)

Finally, the Regularized **full** GRHD equations are: (shown with  $\kappa = \mu$ )

$$\partial_t (\rho) + \partial_x (\rho u) = \partial_x (\kappa \partial_x \rho) \quad (13a)$$

$$\partial_t (\rho u) + \partial_x \left( \rho u^2 + P + \frac{\epsilon}{3} \right) = \partial_x (\kappa \partial_x (\rho u)) \quad (13b)$$

$$\partial_t (\rho E) + \partial_x [u (\rho E + P)] = -\frac{u}{3} \partial_x \epsilon - \sigma_a c \left( a T^4 - \epsilon \right) + \partial_x (\kappa \partial_x (\rho E)) \quad (13c)$$

$$\partial_t \epsilon + \frac{4}{3} \partial_x (u \epsilon) = \frac{u}{3} \partial_x \epsilon + \partial_x \left( \frac{c}{3 \sigma_t} \partial_x \epsilon \right) + \sigma_a c \left( a T^4 - \epsilon \right) + \partial_x (\kappa \partial_x \epsilon) \quad (13d)$$

## Equilibrium Diffusion Limit

### non-dimensionalization:

$$\partial_{t'} (\rho') + \partial_{x'} (\rho' u') = \mathbb{V}_\infty \partial_x (\kappa' \partial_{x'} \rho') \quad (14a)$$

$$\partial_{t'} (\rho' u') + \partial_{x'} \left( \rho u^{2'} + P' + \mathbb{P}_\infty \frac{\epsilon'}{3} \right) = \mathbb{V}_\infty \partial_{x'} (\kappa' \partial_{x'} (\rho' u')) \quad (14b)$$

$$\begin{aligned} \partial_{t'} (\rho' E') + \partial_{x'} [u' (\rho' E' + P')] &= -\mathbb{P}_\infty \frac{u'}{3} \partial_{x'} \epsilon' \\ &\quad - \mathbb{P}_\infty \mathbb{C}_\infty^{-1} \mathbb{L}_\infty (\sigma'_t - \mathbb{L}_{s,\infty} \sigma'_s) (T'^4 - \epsilon') + \mathbb{V}_\infty \partial_{x'} (\kappa' \partial_{x'} (\rho' E')) \end{aligned} \quad (14c)$$

$$\begin{aligned} \partial_{t'} \epsilon' + \frac{4}{3} \partial_{x'} (u' \epsilon') &= \frac{u'}{3} \partial_{x'} \epsilon' + \mathbb{L}_\infty^{-1} \mathbb{C}_\infty^{-1} \partial_{x'} \left( \frac{1}{3 \sigma'_t} \partial_{x'} \epsilon' \right) \\ &\quad + \mathbb{C}_\infty^{-1} \mathbb{L}_\infty (\sigma'_t - \mathbb{L}_{s,\infty} \sigma'_s) (T'^4 - \epsilon') + \mathbb{V}_\infty \partial_{x'} (\kappa' \partial_{x'} \epsilon') \end{aligned} \quad (14d)$$

### non-dimensional parameters

$$\begin{aligned} \mathbb{L}_\infty = L_\infty \sigma_{t,\infty} &= \mathcal{O}(\varepsilon^{-1}), \quad \mathbb{L}_{s,\infty} = \frac{\sigma_{s,\infty}}{\sigma_{t,\infty}} = \mathcal{O}(\varepsilon), \quad \mathbb{C}_\infty = \frac{c_{m,\infty}}{c} = \mathcal{O}(\varepsilon) \\ \mathbb{P}_\infty &= \frac{a T_\infty^4}{\rho_\infty c_{m,\infty}^2} = \mathcal{O}(1), \quad \mathbb{V}_\infty = \frac{\kappa_\infty}{c_{m,\infty} L_\infty} = \mathcal{O}(1) \end{aligned}$$



The variables are expanded in a power series in  $\varepsilon$

Equilibrium Diffusion Limit results:

$$\partial_t \rho_0 + \partial_x (\rho u)_0 = \partial_x (\kappa \partial_x \rho)_0 \quad (15a)$$

$$\partial_t (\rho u)_0 + \partial_x (\rho u^2 + P^*)_0 = \partial_x (\kappa \partial_x (\rho u))_0 \quad (15b)$$

$$\partial_x (\rho E^*)_0 + \partial_x [u (\rho E^* + P^*)]_0 = \partial_x \left( \frac{1}{3\sigma_t} \partial_x T^4 \right)_0 + \partial_x (\kappa \partial_x \rho E^*)_0 \quad (15c)$$

$$P^* = P + \mathbb{P}_\infty \frac{T^4}{3} \text{ and } E^* = E + \mathbb{P}_\infty \frac{T^4}{\rho}$$

Leading order of entropy

$$s_0(\rho, e) = s_{Euler,0}(\rho, e) + \frac{4}{3} \frac{T_0^3}{\rho_0} \quad (16)$$

We recover the EDL results ([Lowrie/Morel, JQSRT, 2001](#)).

Viscous regularization scales adequately with  $\mathbb{V}_\infty = \mathcal{O}(1)$ .

## Numerical solution

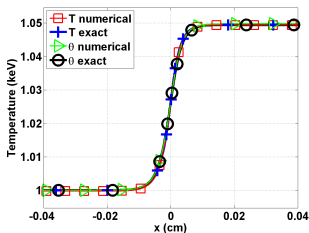
- spatial discretization: CFEM
- temporal discretization: fully implicit (BDF2)
- solution technique: JFNK with finite-difference approximation of the Jacobian as preconditioner
- semi-analytical solutions provided by Jim Ferguson (LANL)
- Two sets of results presented today:
  - ① with **constant opacities** (Mach 1.05, 2, 5, 50)
  - ② with temperature-dependent opacities (Mach 3)

## Additional results in the back-up slides

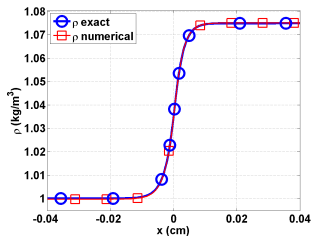
Second-order convergence for manufactured solutions in

- ① streaming limit
- ② diffusion limit

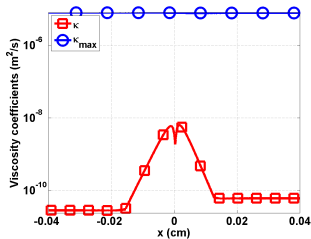
# Steady-state solution for Mach 1.05



(a) Temperatures

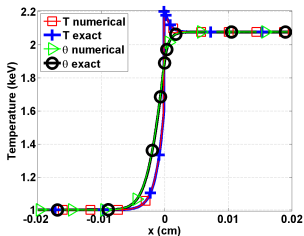


(b) Material density

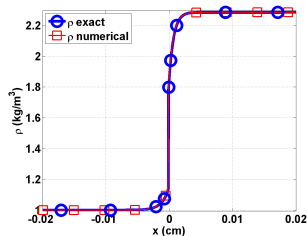


(c) Viscosity coefficients

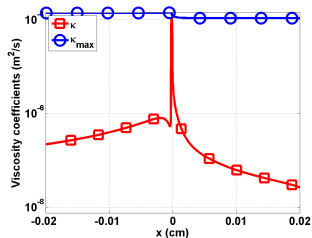
# Steady-state solution for Mach 2



(a) Temperatures

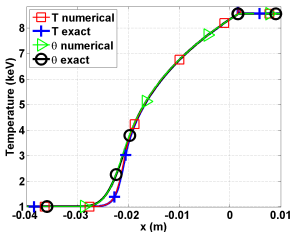


(b) Material density

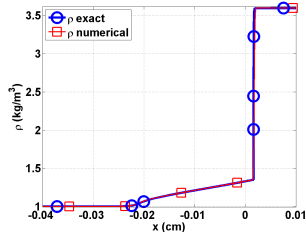


(c) Viscosity coefficients

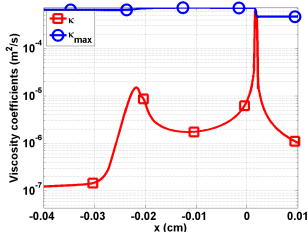
# Steady-state solution for Mach 5



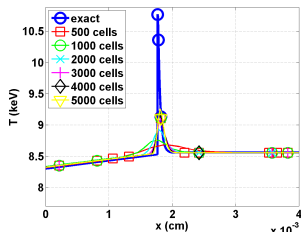
(a) Temperatures



(b) Material density

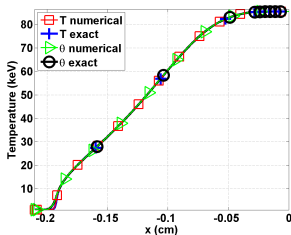


(c) Viscosity coefficients

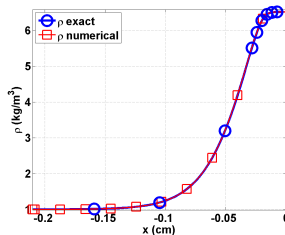


(d) Zoom at the Z spike

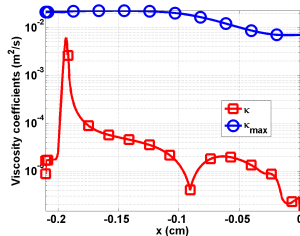
# Steady-state solution for Mach 50



(a) Temperatures



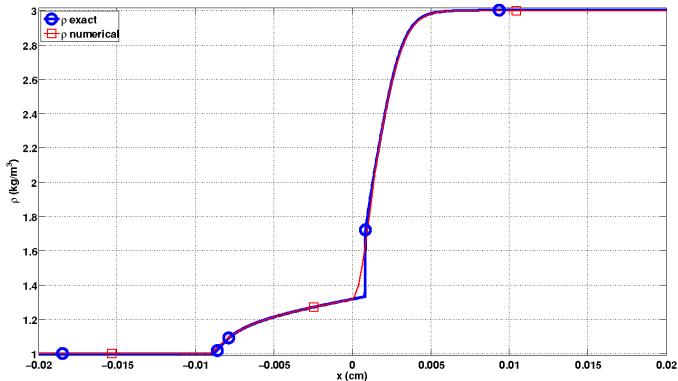
(b) Material density



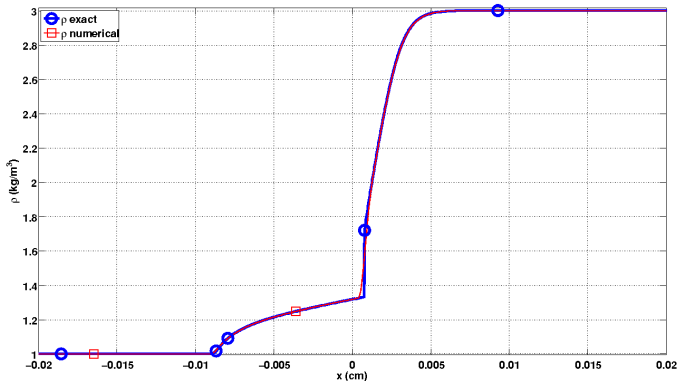
(c) Viscosity coefficients

## Steady-state solution for Mach 3: density, 500 cells

Now, results with temperature-dependent opacities:

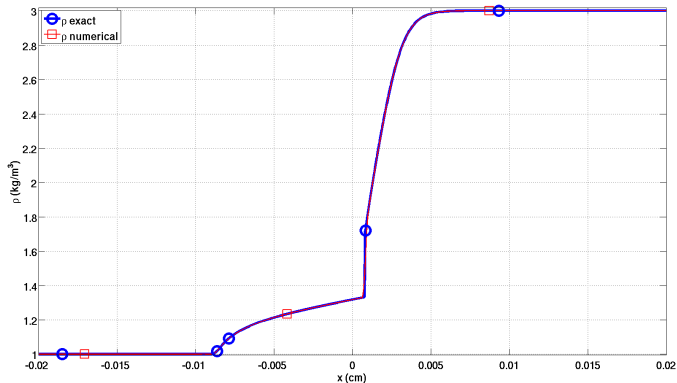


# Steady-state solution for Mach 3: density, 1000 cells

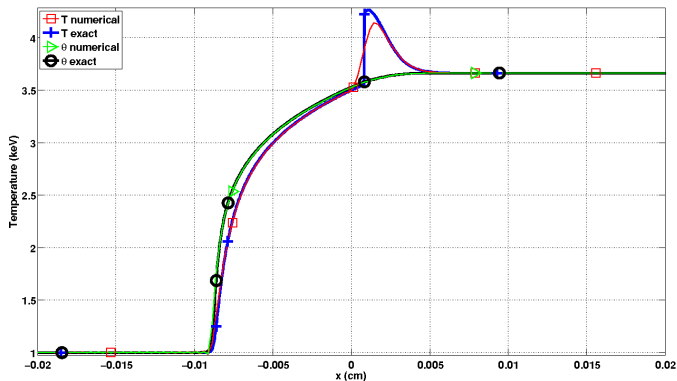




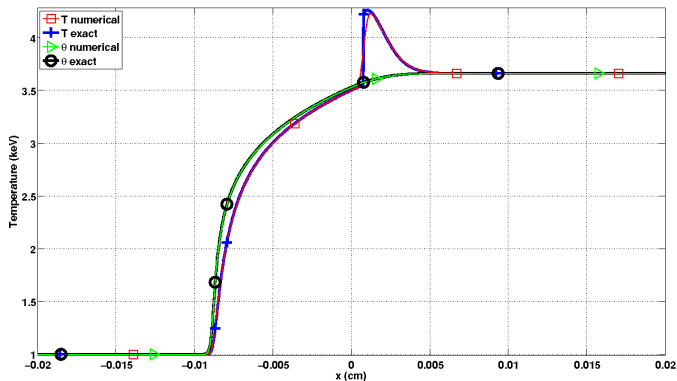
# Steady-state solution for Mach 3: density, 2000 cells



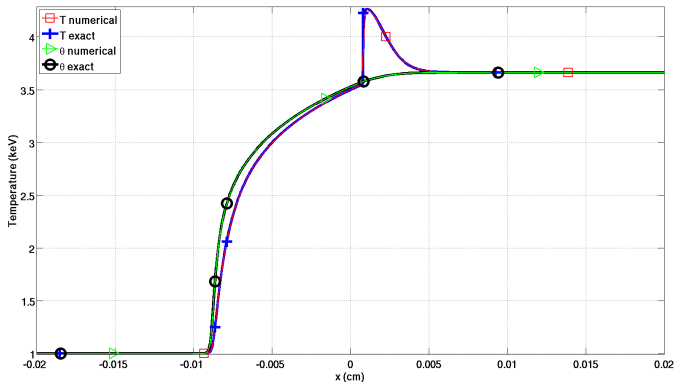
# Steady-state solution for Mach 3: temperature, 500 cells



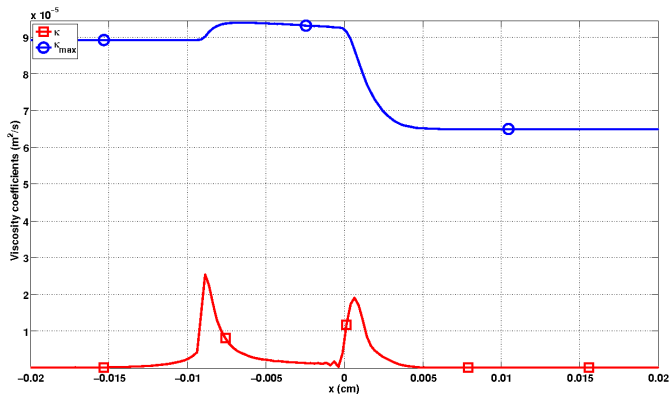
# Steady-state solution for Mach 3: temperature, 1000 cells



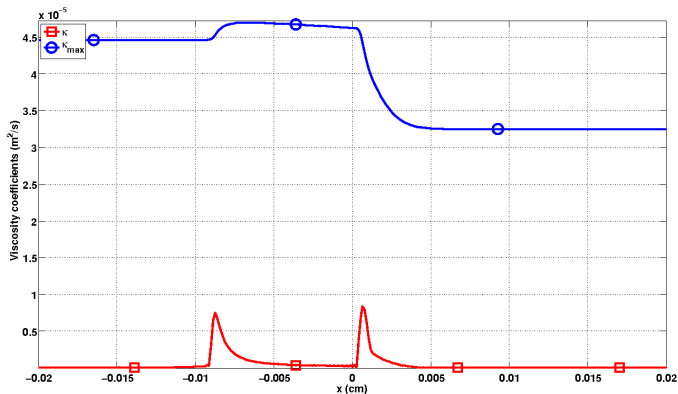
## Steady-state solution for Mach 3: temperature, 2000 cells



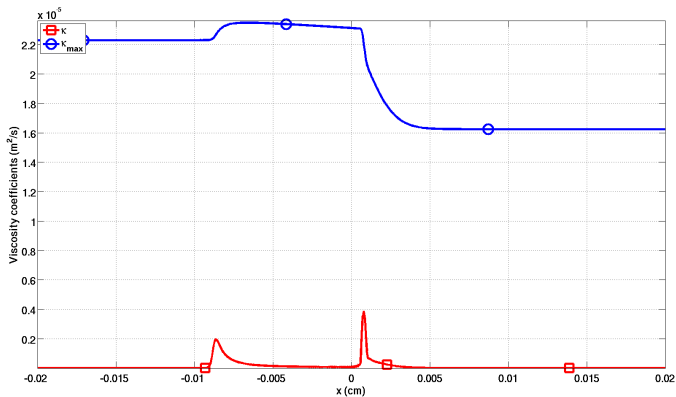
# Steady-state solution for Mach 3: viscosity, 500 cells



# Steady-state solution for Mach 3: viscosity, 1000 cells



## Steady-state solution for Mach 3: viscosity, 2000 cells



## Conclusions

- Extended the entropy-viscosity method to the full Grey Radiation-Hydrodynamic equations.
- Verified the entropy minimum principle for the regularized equations GRHD.
- Viscous regularization scales appropriately in the equilibrium-diffusion limit.
- Numerical results are in excellent agreement with semi-analytical solutions.

## Outlook

- Multi-D.
- Replace radiation diffusion with  $S_n$  radiation transport.
- Switch solution technique to IMEX (implicit for radiation, explicit for hydro).
- Other spatial discretization (DGFEM).
- FCT  
→ poster tomorrow on FCT for radiation transport



# Seven-equation two-phase flow model

with viscous regularization

$$\frac{\partial \alpha_k A}{\partial t} + A \vec{u}_{int} \cdot \vec{\nabla} \alpha_k - A \mu_P (P_k - P_j) = \vec{\nabla} \cdot \vec{l}_k \quad (17a)$$

$$\frac{\partial (\alpha \rho)_k A}{\partial t} + \vec{\nabla} \cdot [(\alpha \rho \vec{u})_k A] = \vec{\nabla} \cdot \vec{f}_k \quad (17b)$$

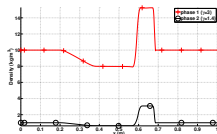
$$\begin{aligned} \frac{\partial (\alpha \rho \vec{u})_k A}{\partial t} + \vec{\nabla} \cdot [\alpha_k A (\rho \vec{u} \otimes \vec{u} + P \mathbb{I})_k] - P_{int} A \vec{\nabla} \alpha_k + P_k \alpha_k \vec{\nabla} A \\ - A \lambda_u (\vec{u}_j - \vec{u}_k) = \vec{\nabla} \cdot \vec{g}_k \end{aligned} \quad (17c)$$

$$\begin{aligned} \frac{\partial (\alpha \rho E)_k A}{\partial t} + \vec{\nabla} \cdot [\alpha_k \vec{u}_k A (\rho E + P)_k] - P_{int} A \vec{u}_{int} \cdot \vec{\nabla} \alpha_k + \bar{P}_{int} A \mu_P (P_k - P_j) \\ - A \lambda_u \vec{u}_{int} \cdot (\vec{u}_j - \vec{u}_k) = \vec{\nabla} \cdot (\vec{h}_k + \vec{u}_k \cdot \vec{g}_k) \end{aligned} \quad (17d)$$

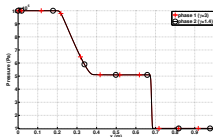
Viscous fluxes:

$$\begin{aligned} \vec{l}_k &= \beta_k A \vec{\nabla} \alpha_k, \quad \vec{f}_k = \alpha_k A \kappa_k \vec{\nabla} \rho_k + \rho_k \vec{l}_k \\ \vec{g}_k &= \alpha_k A \mu_k \rho_k \vec{\nabla}^s \vec{u}_k + \vec{f}_k \otimes \vec{u}_k, \quad \vec{h}_k = \alpha_k A \kappa_k \vec{\nabla} (\rho e)_k - \frac{\|\vec{u}_k\|^2}{2} \vec{f}_k + (\rho e)_k \vec{l}_k \end{aligned}$$

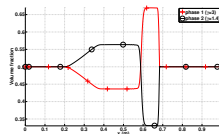
# 7-equation two-phase flow: shock tube with large relaxation



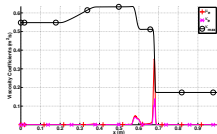
(a) Densities



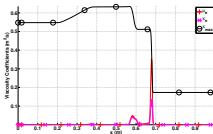
(b) Pressures



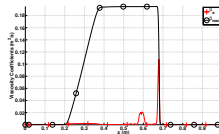
(c) Volume fractions



(d) Viscosity  $\kappa$ ,  $\mu$  phase 1



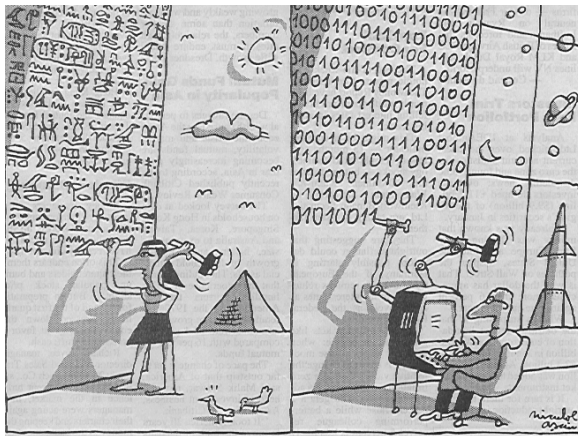
(e) Viscosity  $\kappa$ ,  $\mu$  phase 1



(f) Viscosity  $\beta$  (volume fraction)

## Thank you

Thanks to Jim Ferguson (LANL) for the semi-analytical solutions.  
Thanks to Jean-Luc Guermond and Bojan Popov (Texas A&M) for fruitful discussions.



# Why an upper bound for viscosity?

Large entropy residual in shocks  $\longrightarrow$  large entropy viscosity  $\mu_e$

There is such a thing as too much of a good thing ...  
*Il ne faut point être plus royaliste que le Roy*

## Upper bound for $\mu$

First-order upwind scheme is monotone but over dissipative. We should not exceed the amount of stabilization that such a scheme provides.

upwinding = **centered approximation (Galerkin)** – **numerical diffusion**

Example: linear advection  $\partial_t u + \beta \partial_x u = 0$

$$\beta \frac{u_i - u_{i-1}}{h} = \beta \frac{u_{i+1} - u_{i-1}}{2h} - \frac{\beta h}{2} \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \quad (18)$$

So, the dissipative term is  $\frac{\beta h}{2} \partial_{xx} u$  and the first-order viscosity is  $\frac{\beta h}{2}$

## First-order viscosity

- scalar conservation law:  $\frac{h}{2} |f'(u)|$
- system:  $\frac{h}{2} \max(\text{eig}(\partial_u f))$

## Manufactured solution: equilibrium-diffusion limit

**Table:**  $L_2$  norms of the error for for the equilibrium diffusion limit case using a manufactured solution.

| # of cells | time step size ( $sh$ ) | $\rho$                   | ratio | $\rho E$                 | ratio |
|------------|-------------------------|--------------------------|-------|--------------------------|-------|
| 20         | $10^{-1}$               | 0.590766                 | NA    | 1.333774                 | NA    |
| 40         | $5 \cdot 10^{-2}$       | 0.290626                 | 2.03  | 0.478819                 | 2.79  |
| 80         | $2.5 \cdot 10^{-2}$     | 0.0959801                | 3.021 | 0.154119                 | 3.11  |
| 160        | $1.25 \cdot 10^{-2}$    | 0.02593738               | 3.70  | 0.0405175                | 3.80  |
| 320        | $6.25 \cdot 10^{-3}$    | $6.471444 \cdot 10^{-3}$ | 4.00  | $9.90446 \cdot 10^{-3}$  | 4.09  |
| 640        | $3.125 \cdot 10^{-3}$   | $1.584158 \cdot 10^{-3}$ | 4.01  | $2.44727 \cdot 10^{-3}$  | 4.04  |
| # of cells | time step size ( $sh$ ) | $\epsilon$               | ratio | $\rho u$                 | ratio |
| 20         | $10^{-1}$               | 0.00650085               | NA    | 0.910998                 | NA    |
| 40         | $5 \cdot 10^{-2}$       | 0.00124983               | 5.20  | 0.4090946                | 2.23  |
| 80         | $2.5 \cdot 10^{-2}$     | 0.000262797              | 4.76  | 0.125943                 | 3.25  |
| 160        | $1.25 \cdot 10^{-2}$    | $6.17726 \cdot 10^{-5}$  | 4.25  | $3.381042 \cdot 10^{-3}$ | 3.72  |
| 320        | $6.25 \cdot 10^{-3}$    | $1.509184 \cdot 10^{-5}$ | 4.09  | $8.373657 \cdot 10^{-3}$ | 4.04  |
| 640        | $3.125 \cdot 10^{-3}$   | $3.72548 \cdot 10^{-6}$  | 4.05  | $2.070538 \cdot 10^{-3}$ | 4.04  |

## Manufactured solution: streaming limit

**Table:**  $L_2$  norms of the error for the streaming limit case using a manufactured solution.

| # of cells | time step size ( $sh$ ) | $\rho$                   | ratio | $\rho E$                 | ratio |
|------------|-------------------------|--------------------------|-------|--------------------------|-------|
| 20         | $10^{-1}$               | $1.4373 \cdot 10^{-2}$   | NA    | $5.88521 \cdot 10^{-1}$  | NA    |
| 40         | $5 \cdot 10^{-2}$       | $3.760208 \cdot 10^{-3}$ | 3.82  | $1.4244 \cdot 10^{-1}$   | 4.13  |
| 80         | $2.5 \cdot 10^{-2}$     | $9.91724 \cdot 10^{-4}$  | 3.79  | $3.2047 \cdot 10^{-2}$   | 4.44  |
| 160        | $1.25 \cdot 10^{-2}$    | $2.4455 \cdot 10^{-4}$   | 4.06  | $7.4886 \cdot 10^{-3}$   | 4.28  |
| 320        | $6.25 \cdot 10^{-3}$    | $6.280715 \cdot 10^{-5}$ | 3.89  | $1.82327 \cdot 10^{-3}$  | 4.11  |
| 640        | $3.125 \cdot 10^{-3}$   | $1.57920 \cdot 10^{-5}$  | 3.98  | $4.50463 \cdot 10^{-4}$  | 4.05  |
| 1280       | $1.5625 \cdot 10^{-4}$  | $3.96096 \cdot 10^{-6}$  | 3.99  | $1.12061 \cdot 10^{-4}$  | 4.02  |
| # of cells | time step size ( $sh$ ) | $\epsilon$               | ratio | $\rho u$                 | ratio |
| 20         | $10^{-1}$               | $3.82001 \cdot 10^{-1}$  | NA    | $2.354671 \cdot 10^{-3}$ | NA    |
| 40         | $5 \cdot 10^{-2}$       | $1.21500 \cdot 10^{-1}$  | 3.14  | $6.138814 \cdot 10^{-4}$ | 3.84  |
| 80         | $2.5 \cdot 10^{-2}$     | $3.27966 \cdot 10^{-2}$  | 3.70  | $1.74974 \cdot 10^{-4}$  | 3.51  |
| 160        | $1.25 \cdot 10^{-2}$    | $8.38153 \cdot 10^{-3}$  | 3.91  | $3.61297 \cdot 10^{-5}$  | 4.84  |
| 320        | $6.25 \cdot 10^{-3}$    | $2.10925 \cdot 10^{-3}$  | 3.97  | $9.03866 \cdot 10^{-6}$  | 3.99  |
| 640        | $3.125 \cdot 10^{-3}$   | $5.28472 \cdot 10^{-4}$  | 3.99  | $2.25649 \cdot 10^{-6}$  | 4.01  |
| 1280       | $1.5625 \cdot 10^{-4}$  | $1.322268 \cdot 10^{-4}$ | 3.99  | $5.69984 \cdot 10^{-7}$  | 3.95  |