

# Application of the entropy viscosity method to the 1-D grey Radiation-Hydrodynamic equations (RHD).

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Figure: Rhea

# Outline:

- 1 Objective and background.
- 2 The 1-D grey Radiation-Hydrodynamic equations (RHD).
- 3 The theoretical approach.
  - A viscous regularization for the RHD
  - Definition of the viscosity coefficient.
- 4 1-D numerical results
- 5 Conclusions and future work.

## Objective:

Apply the entropy viscosity method to the 1-D grey radiation-hydrodynamic equations → multi-physic problem with source terms.

## Background:

- RHD are a wave-dominated problem with source terms.
- They are known to develop shocks due to the nature of Euler equations.
- Great amount of work available in the literature on how to solve RHD: approximate Riemann solver [13, 4], flux limiter [10], ...
- Have a common approach: focus on the hyperbolic part of the RHD.
- Attempts to derive a Riemann solver accounting for the source terms.
- Use of semi-implicit schemes because of the difference of characteristic time scale between the two physics → implicit scheme has some advantage.

# The 1-D grey Radiation-Hydrodynamic equations (RHD):

RHD system of equations:

$$\partial_t (\rho) + \partial_x (\rho u) = 0$$

$$\partial_t (\rho u) + \partial_x (\rho u^2 + P) = -\partial_x \left( \frac{\epsilon}{3} \right)$$

$$\partial_t (\rho E) + \partial_x [u (\rho E + P)] = -\frac{u}{3} \partial_x \epsilon - \sigma_a c (a T^4 - \epsilon)$$

$$\partial_t \epsilon + \frac{4}{3} \partial_x (u \epsilon) = \frac{u}{3} \partial_x \epsilon + \partial_x \left( \frac{c}{3\sigma_t} \partial_x \epsilon \right) + \sigma_a c (a T^4 - \epsilon)$$

A few remarks:

- Relaxation term in the energy and radiation equations:  $\sigma_a c (a T^4 - \epsilon)$ .
- Diffusion term:  $\partial_x \left( \frac{c}{3\sigma_t} \partial_x \epsilon \right)$ .
- The above system of equations is NOT hyperbolic ????

# The entropy viscosity method (EVM) [16]:

*It is an artificial dissipation method with smart viscosity coefficient(s) capable of tracking the shock so that dissipation is only added into the shock region.*

- The method requires a *hyperbolic system of equations*.
- Add artificial dissipative terms consistent with the entropy minimum principle.
- Define a local and smooth viscosity coefficient function of the grid size.
- The viscosity coefficient is defined proportional to the entropy production.
- The entropy production is locally computed by evaluating the entropy residual:

$$R_e = \partial_t s + \vec{u} \cdot \vec{\nabla} s \geq 0$$

## Our approach:

- Consider the hyperbolic part of the system of equations (no source terms).
- Keep in mind that the source terms may affect the entropy viscosity method and their effect will have to be studied later.

## Our hyperbolic system of equations:

$$\begin{cases} \partial_t (\rho) + \partial_x (\rho u) = 0 \\ \partial_t (\rho u) + \partial_x (\rho u^2 + P + \frac{\epsilon}{3}) = 0 \\ \partial_t (\rho E) + \partial_x [u (\rho E + P)] = 0 \\ \partial_t \epsilon + \frac{4}{3} \partial_x (u \epsilon) - \frac{u}{3} \partial_x \epsilon = 0 \end{cases}$$

Eigenvalues:  $\lambda_1 = u - c$ ,  $\lambda_{2,3} = u$  and  $\lambda_4 = u + c$  where  $c$  is the speed of sound defined as:

$$c^2 = \underbrace{P_\rho + \frac{P}{\rho^2} P_e}_{c_{\text{Euler}}^2} + \underbrace{\frac{4\epsilon}{9\rho}}_{c_{\text{rad}}^2}$$

## Viscous regularization and viscosity coefficient(s):

- 1 Derive the dissipative terms using the modified system of equations (no source term).

An entropy  $s$  function of the material density  $\rho$  and internal energy  $e$ , and the radiation density energy  $\epsilon$ .

- 2 Define the viscosity coefficient(s).

## Method of Manufactured Solutions and 1-D results:

- 1 Study the effect of the source terms,  $\partial_x \left( \frac{c}{3\sigma_t} \partial_x \epsilon \right)$  and  $\sigma_a c (aT^4 - \epsilon)$ , on the entropy viscosity method using the method of manufactured solutions, and show high-order convergence.
- 2 Perform 1-D tests for different Mach numbers: Mach= 1.05, 1.2, 2, 5, and 50. Semi-analytical solutions are available.

## A viscous regularization for the RHD:

$$\partial_t(\rho) + \partial_x(\rho u) = \partial_x(\kappa \partial_x \rho)$$

$$\partial_t(\rho u) + \partial_x\left(\rho u^2 + P + \frac{\epsilon}{3}\right) = \partial_x(\mu \partial_x u) + \partial_x(\kappa \partial_x \rho)$$

$$\partial_t(\rho E) + \partial_x[u(\rho E + P)] = \partial_x(\kappa \partial_x \rho e) + 0.5 \partial_x(\kappa u^2 \partial_x \rho) + \partial_x(\mu \rho u \partial_x u)$$

$$\partial_t \epsilon + \frac{4}{3} \partial_x(u \epsilon) - \frac{u}{3} \partial_x \epsilon = \partial_x(\kappa \partial_x \epsilon)$$

## The entropy inequality:

$$R_e = \partial_t s + u \partial_x s = \frac{s_e}{P_e} \underbrace{\left( \frac{DP}{Dt} - c_{Euler}^2 \frac{D\rho}{Dt} \right)}_{\tilde{R}_e} \geq 0$$

where  $s(\rho, e, \alpha) = s_{Euler}(\rho, e) + \frac{\rho_0}{\rho} \hat{s}(\alpha)$ ,  $s_{Euler}$  and  $\hat{s}$  being concave.

two viscosity coefficients:  $\kappa$  and  $\mu$ . In the rest of this presentation, we will assume that  $\kappa = \mu \rightarrow$  parabolic regularization.



## Definition of the viscosity coefficient $\kappa$ :

$$\mu(x, t) = \min(\mu_{\max}(x, t), \mu_e(x, t))$$

$$\mu_{\max}(x, t) = 0.5h(|u| + c)$$

$$\mu_e(x, t) = h^2 \frac{\max(\tilde{R}_e(x, t), J)}{norm_P}$$

## Entropy residual:

$$\tilde{R}_e = \frac{DP}{Dt} - c_{Euler}^2 \frac{D\rho}{Dt}$$

## The jump $J$ and the normalization parameter $norm_P$ :

$$J = \max(|u|[[\partial_x P]], |u|c_{Euler}^2[[\partial_x \rho]])$$

$$norm_P = \rho c_{Euler}^2$$

## EVM and the diffusion term $\partial_x \left( \frac{c}{3\sigma_t} \partial_x \epsilon \right)$ :

In the radiation equation, there are two second-order terms: diffusion and dissipative terms.

$$\partial_x \left( \frac{c}{3\sigma_t} \partial_x \epsilon \right) + \partial_x (\kappa \partial_x \epsilon) \rightarrow \partial_x \left( \max \left( \frac{c}{3\sigma_t}, \kappa \right) \partial_x \epsilon \right)$$

## Relaxation term:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x p(u) = \frac{1}{\psi} (v - f(u)) \end{cases}$$

as  $\psi \rightarrow 0$ , it can be shown that:

$$\begin{cases} v \sim f(u) \\ \partial_t u + \partial_x f(u) = \partial_x (\beta \partial_x u) \end{cases}$$

where  $\beta$  is function of  $\psi$ ,  $p(u)$  and  $f(u)$ .

## Problem:

$\Rightarrow$  the relaxation term behaves as a diffusion term in the asymptotic limit and thus, will compete with the dissipative term.

## Solution:

MMS to investigate the behavior of the dissipative terms in the asymptotic limit.

# The code

- Idaho National Laboratory MOOSE framework.
- Fully implicit  $\rightarrow$  non-linear solver with a preconditioner.
- Continuous Galerkin Finite Element Method (CGFEM).
- Second-order in time (BDF2) and space (linear polynomials).
- Gauss quadrature rule.
- RHEA: Radiation-Hydrodynamic EquAtions (1-D code).

## MMS: the equilibrium diffusion limit

$$\rho = \sin(x - t) + 2$$

$$u = \cos(x - t) + 2$$

$$T = \frac{0.5\gamma(\cos(x - t) + 2)}{\sin(x - t) + 2}$$

$$\epsilon = aT^4$$

$$\sigma_a = \sigma_t = 1000 \text{ cm}^{-1}.$$

**Table:**  $L_2$  norms of the error for for the equilibrium diffusion limit case using a manufactured solution.

# of cells	time step size ( $sh$ )	$\rho$	ratio	$\rho E$	ratio
20	$10^{-1}$	0.590766	NA	1.333774	NA
40	$5 \cdot 10^{-1}$	0.290626	2.03	0.478819	2.79
80	$2.5 \cdot 10^{-2}$	0.0959801	3.021	0.154119	3.11
160	$1.25 \cdot 10^{-2}$	0.02593738	3.70	0.0405175	3.80
320	$6.25 \cdot 10^{-3}$	$6.471444 \cdot 10^{-3}$	4.00	$9.90446 \cdot 10^{-3}$	4.09
640	$3.125 \cdot 10^{-3}$	$1.584158 \cdot 10^{-3}$	4.01	$2.44727 \cdot 10^{-3}$	4.04

# of cells	time step size ( $sh$ )	$\epsilon$	ratio	$\rho u$	ratio
20	$10^{-1}$	0.00650085	NA	0.910998	NA
40	$5 \cdot 10^{-1}$	0.00124983	5.20	0.4090946	2.23
80	$2.5 \cdot 10^{-2}$	0.000262797	4.76	0.125943	3.25
160	$1.25 \cdot 10^{-2}$	$6.17726 \cdot 10^{-5}$	4.25	$3.381042 \cdot 10^{-3}$	3.72
320	$6.25 \cdot 10^{-3}$	$1.509184 \cdot 10^{-5}$	4.09	$8.373657 \cdot 10^{-3}$	4.04
640	$3.125 \cdot 10^{-3}$	$3.72548 \cdot 10^{-6}$	4.05	$2.070538 \cdot 10^{-3}$	4.04

The second manufactured solution is used to test the method in the streaming limit: the radiation streaming dominates the absorption/re-emission term and evolves at a fast time scale.

### MMS: the streaming limit

$$\rho = \sin(x - t) + 2$$

$$u = (\sin(x - t) + 2)^{-1}$$

$$T = 0.5\gamma$$

$$\epsilon = \sin(x - 1000t) + 2$$

$$\sigma_a = \sigma_t = 1 \text{ cm}^{-1}.$$

**Table:**  $L_2$  norms of the error for for the streaming limit case using a manufactured solution.

# of cells	time step size ( $sh$ )	$\rho$	ratio	$\rho E$	ratio
20	$10^{-1}$	$1.4373 \cdot 10^{-2}$	NA	$5.88521 \cdot 10^{-1}$	NA
40	$5 \cdot 10^{-2}$	$3.760208 \cdot 10^{-3}$	3.82	$1.4244 \cdot 10^{-1}$	4.13
80	$2.5 \cdot 10^{-2}$	$9.91724 \cdot 10^{-4}$	3.79	$3.2047 \cdot 10^{-2}$	4.44
160	$1.25 \cdot 10^{-2}$	$2.4455 \cdot 10^{-4}$	4.06	$7.4886 \cdot 10^{-3}$	4.28
320	$6.25 \cdot 10^{-3}$	$6.280715 \cdot 10^{-5}$	3.89	$1.82327 \cdot 10^{-3}$	4.11
640	$3.125 \cdot 10^{-3}$	$1.57920 \cdot 10^{-5}$	3.98	$4.50463 \cdot 10^{-4}$	4.05
1280	$1.5625 \cdot 10^{-4}$	$3.96096 \cdot 10^{-6}$	3.99	$1.12061 \cdot 10^{-4}$	4.02
# of cells	time step size ( $sh$ )	$\epsilon$	ratio	$\rho u$	ratio
20	$10^{-1}$	$3.82001 \cdot 10^{-1}$	NA	$2.354671 \cdot 10^{-3}$	NA
40	$5 \cdot 10^{-2}$	$1.21500 \cdot 10^{-1}$	3.14	$6.138814 \cdot 10^{-4}$	3.84
80	$2.5 \cdot 10^{-2}$	$3.27966 \cdot 10^{-2}$	3.70	$1.74974 \cdot 10^{-4}$	3.51
160	$1.25 \cdot 10^{-2}$	$8.38153 \cdot 10^{-3}$	3.91	$3.61297 \cdot 10^{-5}$	4.84
320	$6.25 \cdot 10^{-3}$	$2.10925 \cdot 10^{-3}$	3.97	$9.03866 \cdot 10^{-6}$	3.99
640	$3.125 \cdot 10^{-3}$	$5.28472 \cdot 10^{-4}$	3.99	$2.25649 \cdot 10^{-6}$	4.01
1280	$1.5625 \cdot 10^{-4}$	$1.322268 \cdot 10^{-4}$	3.99	$5.69984 \cdot 10^{-7}$	3.95



## 1-D numerical results:

- Inlet Mach number: 1.05, 1.2, 2, 5 and 50.
- Step initial conditions.
- All tests reach a steady-state solution.
- BDF2 and linear polynomials.
- Uniform mesh.
- Semi-analytical solutions are plotted for comparison.

# Numerical results: Mach 1.05.

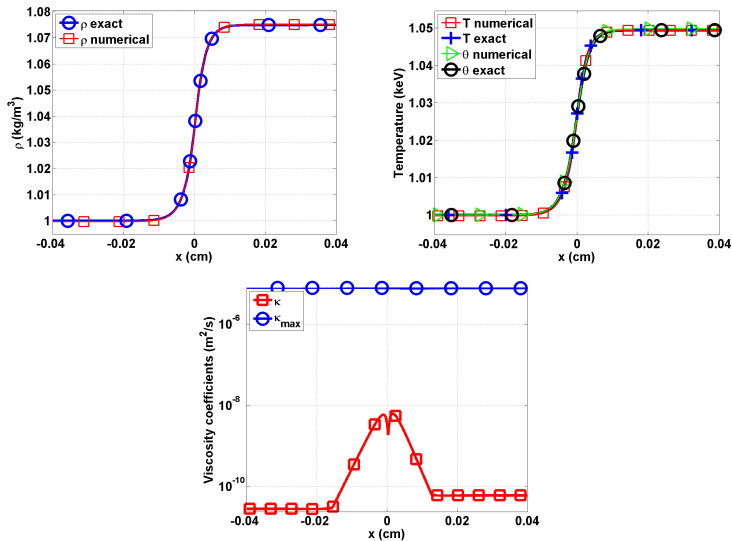


Figure: Steady-state numerical solution with 500 cells, linear polynomials and BDF2.

# Numerical results: Mach 1.2.

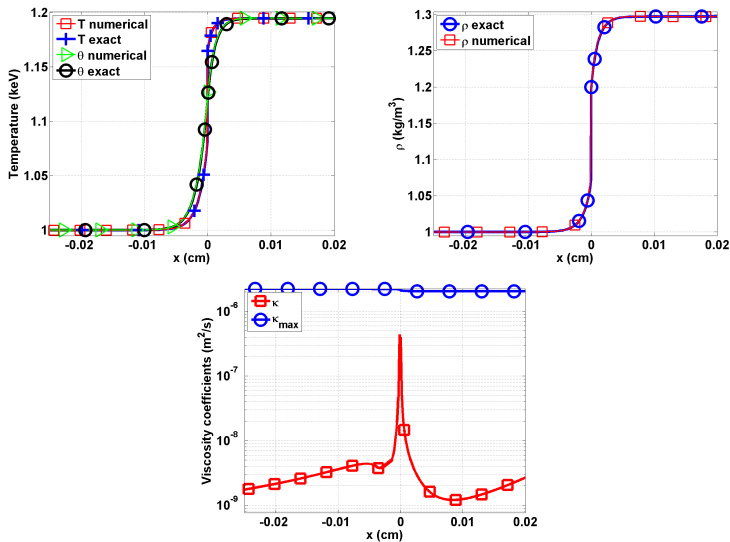


Figure: Steady-state numerical solution with 1000 cells, linear polynomials and BDF2.

# Numerical results: Mach 2.

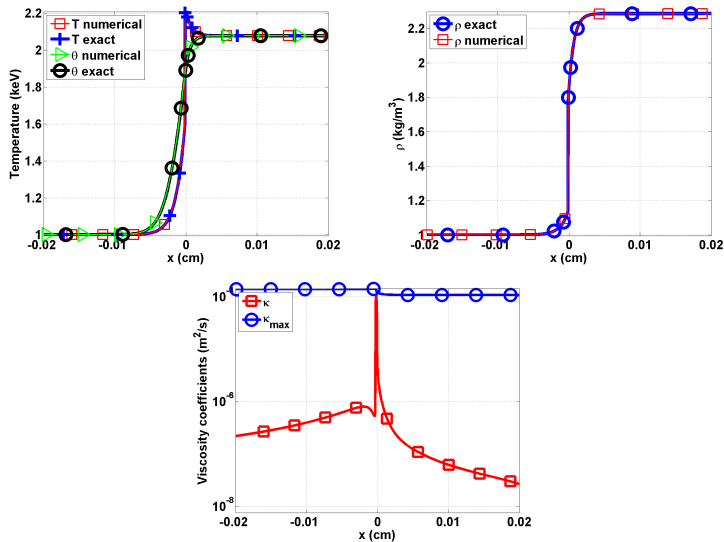


Figure: Steady-state numerical solution with 1000 cells, linear polynomials and BDF2.

# Numerical results: Mach 5.

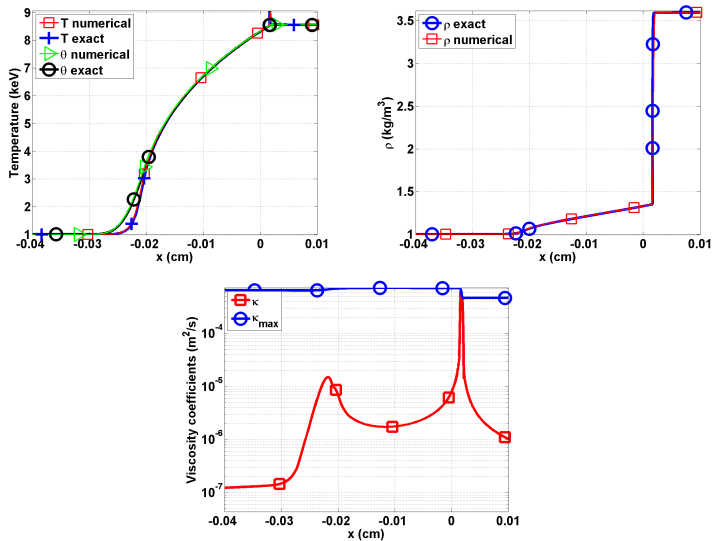
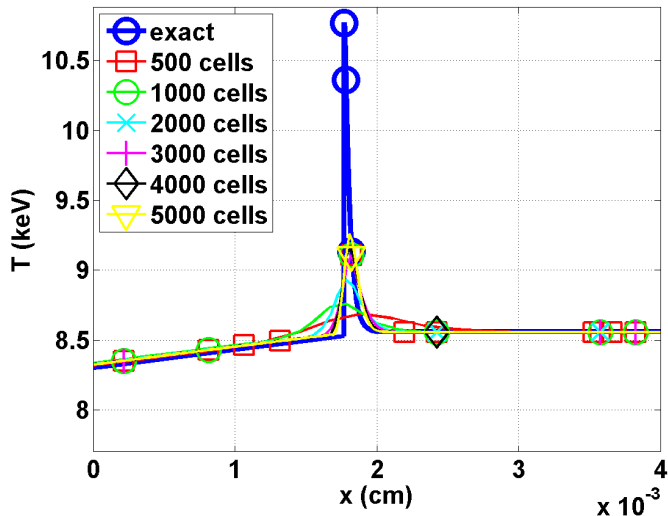


Figure: Steady-state numerical solution with 1000 cells, linear polynomials and BDF2.

# Numerical results: Mach 5, Zeldovich spike.



# Numerical results: Mach 50.

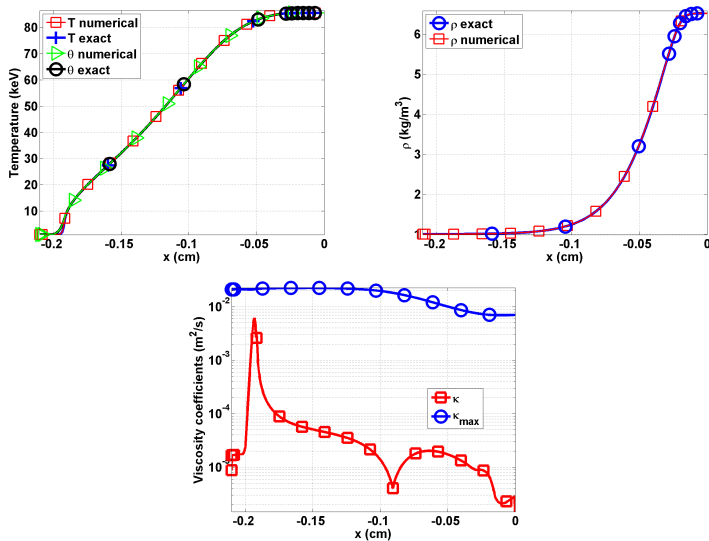


Figure: Steady-state numerical solution with 1000 cells, linear polynomials and BDF2.

## Conclusions:

- The entropy viscosity method was successfully applied to the 1-D RHD.
- 1-D numerical results show good agreement with semi-analytical solutions.
- Demonstrated high-order accuracy and correct behavior in the equilibrium diffusion limit.
- The method can be applied with any equation of state with a convex entropy.
- The method is simple to implement and the viscosity coefficient is computed on the fly.

## Future work:

- Extension to multi-D simulations: the theoretical approach holds.
- $S_n$  transport approximation (instead of the radiation-diffusion equation) coupled to Euler equations.



# QUESTIONS/COMMENTS ?

# Entropy residual with dissipative terms:

$$\frac{Ds}{Dt} + \underbrace{\left( P\partial_e s + \rho^2 \partial_\rho s + \frac{4}{3} \rho \epsilon \partial_\epsilon s \right)}_{(a)} \partial_x u =$$
$$\partial_x (\rho \kappa \partial_x s) + \kappa \partial_e s \partial_x s - \rho \kappa \underbrace{XAX^t}_{(b)} + \underbrace{s_e \rho \mu (\partial_x u)^2}_{(c)}$$







where  $X$  is a row vector defined as  $X = (\rho, e, \epsilon)$  and  $A$  is the 3x3 symmetric matrix:

$$A = \begin{bmatrix} \partial_\rho (\rho^2 \partial_\rho s) & \partial_{\rho,e} s & \partial_\rho (\rho \partial_\epsilon s) \\ \partial_{\rho,e} s & \partial_{e,e} s & \partial_{e,\epsilon} s \\ \partial_\rho (\rho \partial_\epsilon s) & \partial_{e,\epsilon} s & \partial_{\epsilon,\epsilon} s \end{bmatrix}$$

# Positivity of the matrix $A$ :

With  $s(\rho, e, \epsilon) = \tilde{s}(\rho, e) + \frac{\rho_0}{\rho} \hat{s}(\epsilon)$ :

$$A = \begin{bmatrix} \partial_\rho (\rho^2 \partial_\rho \tilde{s}) & \partial_{\rho, e} \tilde{s} & 0 \\ \partial_{\rho, e} \tilde{s} & \partial_{e, e} \tilde{s} & 0 \\ 0 & 0 & \rho^{-1} \partial_{\epsilon, \epsilon} \hat{s} \end{bmatrix}$$

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