Application of the entropy viscosity method to the 1-D grey Radiation-Hydrodynamic equations (RHD).

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Figure: Rhea

Outline:

1. Objective and background.
2. The 1-D grey Radiation-Hydrodynamic equations (RHD).
3. The theoretical approach.
   - A viscous regularization for the RHD
   - Definition of the viscosity coefficient.
4. 1-D numerical results
5. Conclusions and future work.

Objective:

Apply the entropy viscosity method to the 1-D grey radiation-hydrodynamic equations $\rightarrow$ multiphysics problem with source terms.

Background:

- RHD are a wave-dominated problem with source terms.
- They are known to develop shocks due to the nature of Euler equations.
- Great amount of work available in the literature on how to solve RHD: approximate Riemann solver [13, 4], flux limiter [10], $\cdots$
- Have a common approach: focus on the hyperbolic part of the RHD.
- Attempts to derive a Riemann solver accounting for the source terms.
- Use of semi-implicit schemes because of the difference of characteristic time scale between the two physics $\rightarrow$ implicit scheme has some advantage.

The 1-D grey Radiation-Hydrodynamic equations (RHD):

RHD system of equations:

$$
\begin{aligned}
&\partial_t(\rho) + \partial_x(\rho u) = 0 \\
&\partial_t(\rho u) + \partial_x\left(\rho u^2 + P\right) = -\partial_x\left(\frac{\epsilon}{3}\right) \\
&\partial_t(\rho E) + \partial_x\left[u(\rho E + P)\right] = -\frac{u}{3}\partial_x \epsilon - \sigma_a c\left(aT^4 - \epsilon\right) \\
&\partial_t \epsilon + \frac{4}{3}\partial_x(u\epsilon) = \frac{u}{3}\partial_x \epsilon + \partial_x\left(\frac{c}{3\sigma_t}\partial_x \epsilon\right) + \sigma_a c\left(aT^4 - \epsilon\right)
\end{aligned}
$$

A few remarks:

- Relaxation term in the energy and radiation equations: $\sigma_a c\left(aT^4 - \epsilon\right)$.
- Diffusion term: $\partial_x\left(\frac{c}{3\sigma_t}\partial_x \epsilon\right)$.
- The above system of equations is NOT hyperbolic ????

The entropy viscosity method (EVM) [16]:

It is an artificial dissipation method with smart viscosity coefficient(s) capable of tracking the shock so that dissipation is only added into the shock region.

- The method requires a *hyperbolic system of equations*.
- Add artificial dissipative terms consistent with the entropy minimum principle.
- Define a local and smooth viscosity coefficient function of the grid size.
- The viscosity coefficient is defined proportional to the entropy production.
- The entropy production is locally computed by evaluating the entropy residual:

$$R_e = \partial_t s + \vec{u} \cdot \vec{\nabla} s \geq 0$$

Our approach:

- Consider the hyperbolic parts of the system of equations (no source terms).
- Keep in mind that the source terms may affect the entropy viscosity method and their effect will have to be studied later.

Our hyperbolic system of equations:

$$\begin{cases} \partial_t(\rho) + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x\left(\rho u^2 + P + \frac{\epsilon}{3}\right) = 0 \\ \partial_t(\rho E) + \partial_x\left[u(\rho E + P)\right] + u\partial_x\frac{\epsilon}{3} = 0 \\ \partial_t\epsilon + \frac{4}{3}\partial_x(u\epsilon) - \frac{u}{3}\partial_x\epsilon = 0 \end{cases}$$

Eigenvalues: $\lambda_1 = u - c_{sp}$, $\lambda_{2,3} = u$ and $\lambda_4 = u + c_{sp}$ where c_{sp} is the speed of sound defined as:

$$c_{sp}^2 = \underbrace{P_\rho + \frac{P}{\rho^2}P_e}_{c^2_{\text{Euler}}} + \underbrace{\frac{4\epsilon}{9\rho}}_{c^2_{\text{rad}}}$$

Viscous regularization and viscosity coefficient(s):

1. Derive the dissipative terms using the modified system of equations (no source term).
   An entropy s function of the material density ρ and internal energy e, and the radiation density energy ϵ.

2. Define the viscosity coefficient(s).

Method of Manufactured Solutions and 1-D results:

1. Study the effect of the source terms, $\partial_x \left(\frac{c}{3\sigma_t} \partial_x \epsilon \right)$ and $\sigma_a c \left(a T^4 - \epsilon \right)$, on the entropy viscosity method using the Method of Manufactured Solutions (MMS), and show high-order convergence.

2. Perform 1-D tests for different Mach numbers: Mach$= 1.05$, 1.2, 2, 5, and 50. Semi-analytical solutions are available.

A viscous regularization for the RHD:

$$\partial_t(\rho) + \partial_x(\rho u) = \partial_x(\kappa \partial_x \rho)$$
$$\partial_t(\rho u) + \partial_x\left(\rho u^2 + P + \frac{\epsilon}{3}\right) = \partial_x(\mu \partial_x u) + \partial_x(\kappa \partial_x \rho)$$
$$\partial_t(\rho E) + \partial_x[u(\rho E + P)] = \partial_x(\kappa \partial_x \rho e) + 0.5\partial_x\left(\kappa u^2 \partial_x \rho\right) + \partial_x(\mu \rho u \partial_x u)$$
$$\partial_t \epsilon + \frac{4}{3}\partial_x(u\epsilon) - \frac{u}{3}\partial_x \epsilon = \partial_x(\kappa \partial_x \epsilon)$$

The entropy inequality:

$$R_e = \partial_t s + u\partial_x s = \frac{s_e}{P_e}\underbrace{\left(\frac{DP}{Dt} - c_{Euler}^2 \frac{D\rho}{Dt}\right)}_{\tilde{R}_e} \geq 0$$

where $s(\rho, e, \alpha) = s_{Euler}(\rho, e) + \frac{\rho_0}{\rho}\hat{s}(\alpha)$, s_{Euler} and $\hat{s}$ being concave.

two viscosity coefficients: κ and μ. In the remaining of this presentation, we will assume that $\mu = \kappa \rightarrow$ parabolic regularization.

Definition of the viscosity coefficient μ:

$$\mu(x,t) = \min\left(\mu_{max}(x,t), \mu_e(x,t)\right)$$
$$\mu_{max}(x,t) = 0.5h\left(|u| + c\right)$$
$$\mu_e(x,t) = h^2 \frac{\max\left(\tilde{R}_e(x,t), J\right)}{norm_P}$$

Entropy residual:

$$\tilde{R}_e = \frac{DP}{Dt} - c_{Euler}^2 \frac{D\rho}{Dt}$$

The jump J and the normalization parameter $norm_P$:

$$J = \max\left(|u|[[\partial_x P]], |u| c_{Euler}^2 [[\partial_x \rho]]\right)$$
$$norm_P = \rho c_{Euler}^2$$

EVM and the diffusion term $\partial_x \left(\frac{c}{3\sigma_t} \partial_x \epsilon \right)$:

In the radiation equation, there are two second-order terms: diffusion and dissipative terms.

$$\partial_x \left(\frac{c}{3\sigma_t} \partial_x \epsilon \right) + \partial_x \left(\kappa \partial_x \epsilon \right) \to \partial_x \left(\max \left(\frac{c}{3\sigma_t}, \kappa \right) \partial_x \epsilon \right)$$

EVM and the relaxation term

Consider the system of conservation laws:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x p(u) = \frac{1}{\psi}\left(v - f(u)\right) \end{cases}$$

as $\psi \to 0$, it can be shown that:

$$\begin{cases} v \sim f(u) \\ \partial_t u + \partial_x f(u) = \partial_x\left(\beta \partial_x u\right) \end{cases}$$

where β is function of ψ, $\partial_u p(u)$ and $f(u)$.

Problem:

$\Rightarrow$ the relaxation term behaves as a diffusion term in the asymptotic limit and thus, will compete with the dissipative term $\partial_x\left(\mu \partial_x u\right)$.

Solution:

MMS to investigate the behavior of the dissipative terms in the asymptotic limit.

The code

- Idaho National Laboratory MOOSE framework.
- Fully implicit $\rightarrow$ non-linear solver with a preconditioner.
- Continuous Galerkin Finite Element Method (CGFEM).
- Second-order accuracy in time (BDF2) and space (linear polynomials).
- Gauss quadrature rule.
- RHEA: Radiation-Hydrodynamic EquAtions (1-D code).

MMS: the equilibrium-diffusion limit

$$\rho = \sin(x - t) + 2$$
$$u = \cos(x - t) + 2$$
$$T = \frac{0.5\gamma(\cos(x - t) + 2)}{\sin(x - t) + 2}$$
$$\epsilon = aT^4$$

$\sigma_a = \sigma_t = 1000\ cm^{-1}$.

Table: L_2 norms of the error for the equilibrium diffusion limit case using a manufactured solution.

# of cells	time step size (sh)	ρ	ratio	$\rho\mathbf{E}$	ratio
20	10^{-1}	0.590766	NA	1.333774	NA
40	$5\ 10^{-1}$	0.290626	2.03	0.478819	2.79
80	$2.5\ 10^{-2}$	0.0959801	3.021	0.154119	3.11
160	$1.25\ 10^{-2}$	0.02593738	3.70	0.0405175	3.80
320	$6.25\ 10^{-3}$	$6.471444\ 10^{-3}$	4.00	$9.90446\ 10^{-3}$	4.09
640	$3.125\ 10^{-3}$	$1.584158\ 10^{-3}$	4.01	$2.44727\ 10^{-3}$	4.04
# of cells	time step size (sh)	ϵ	**ratio**	$\rho\mathbf{u}$	**ratio**
20	10^{-1}	0.00650085	NA	0.910998	NA
40	$5\ 10^{-1}$	0.00124983	5.20	0.4090946	2.23
80	$2.5\ 10^{-2}$	0.000262797	4.76	0.125943	3.25
160	$1.25\ 10^{-2}$	$6.17726\ 10^{-5}$	4.25	$3.381042\ 10^{-3}$	3.72
320	$6.25\ 10^{-3}$	$1.509184\ 10^{-5}$	4.09	$8.373657\ 10^{-3}$	4.04
640	$3.125\ 10^{-3}$	$3.72548\ 10^{-6}$	4.05	$2.070538\ 10^{-3}$	4.04

The second manufactured solution is used to test the method in the streaming limit: the radiation streaming dominates the absorption/re-emission term and evolves at a fast time scale.

MMS: the streaming limit

$$\rho = \sin(x - t) + 2$$
$$u = (\sin(x - t) + 2)^{-1}$$
$$T = 0.5\gamma$$
$$\epsilon = \sin(x - 1000t) + 2$$

$\sigma_a = \sigma_t = 1\ cm^{-1}$.

Table: L_2 norms of the error for the streaming limit case using a manufactured solution.

# of cells	time step size (sh)	ρ	ratio	$\rho\mathbf{E}$	ratio
20	10^{-1}	$1.4373\ 10^{-2}$	NA	$5.88521\ 10^{-1}$	NA
40	$5.\ 10^{-2}$	$3.760208\ 10^{-3}$	3.82	$1.4244\ 10^{-1}$	4.13
80	$2.5\ 10^{-2}$	$9.91724\ 10^{-4}$	3.79	$3.2047\ 10^{-2}$	4.44
160	$1.25\ 10^{-2}$	$2.4455\ 10^{-4}$	4.06	$7.4886\ 10^{-3}$	4.28
320	$6.25\ 10^{-3}$	$6.280715\ 10^{-5}$	3.89	$1.82327\ 10^{-3}$	4.11
640	$3.125\ 10^{-3}$	$1.57920\ 10^{-5}$	3.98	$4.50463\ 10^{-4}$	4.05
1280	$1.5625\ 10^{-4}$	$3.96096\ 10^{-6}$	3.99	$1.12061\ 10^{-4}$	4.02
# of cells	time step size (sh)	ϵ	ratio	$\rho\mathbf{u}$	ratio
20	10^{-1}	$3.82001\ 10^{-1}$	NA	$2.354671\ 10^{-3}$	NA
40	$5.\ 10^{-2}$	$1.21500\ 10^{-1}$	3.14	$6.138814\ 10^{-4}$	3.84
80	$2.5\ 10^{-2}$	$3.27966\ 10^{-2}$	3.70	$1.74974\ 10^{-4}$	3.51
160	$1.25\ 10^{-2}$	$8.38153\ 10^{-3}$	3.91	$3.61297\ 10^{-5}$	4.84
320	$6.25\ 10^{-3}$	$2.10925\ 10^{-3}$	3.97	$9.03866\ 10^{-6}$	3.99
640	$3.125\ 10^{-3}$	$5.28472\ 10^{-4}$	3.99	$2.25649\ 10^{-6}$	4.01
1280	$1.5625\ 10^{-4}$	$1.322268\ 10^{-4}$	3.99	$5.69984\ 10^{-7}$	3.95

1-D numerical results:

- Inlet Mach number: 1.05, 1.2, 2, 5 and 50.
- Step initial conditions.
- All tests reach a steady-state solution.
- BDF2 and linear polynomials.
- Uniform mesh.
- Semi-analytical solutions are plotted for comparison.
- Ideal Gas Equation of State: $P = (\gamma - 1)\rho e$.

Numerical results: Mach 1.05.

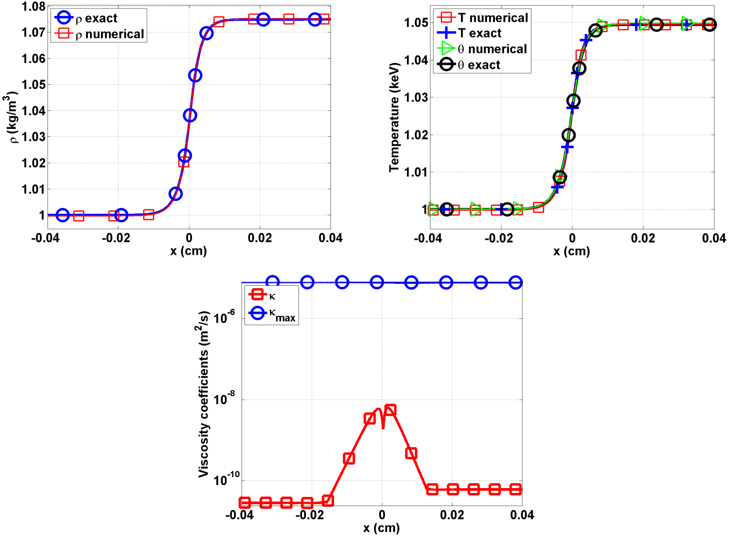


Figure: Steady-state numerical solution with 500 cells, linear polynomials and BDF2.

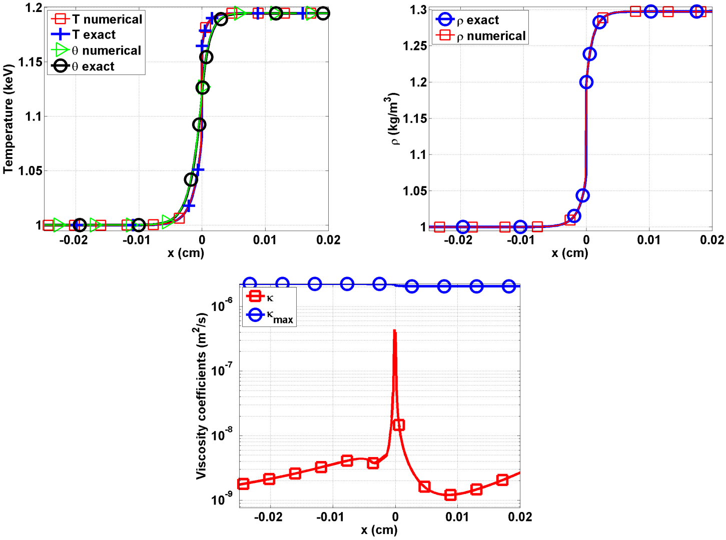


Figure: Steady-state numerical solution with 1000 cells, linear polynomials and BDF2.

Numerical results: Mach 2.

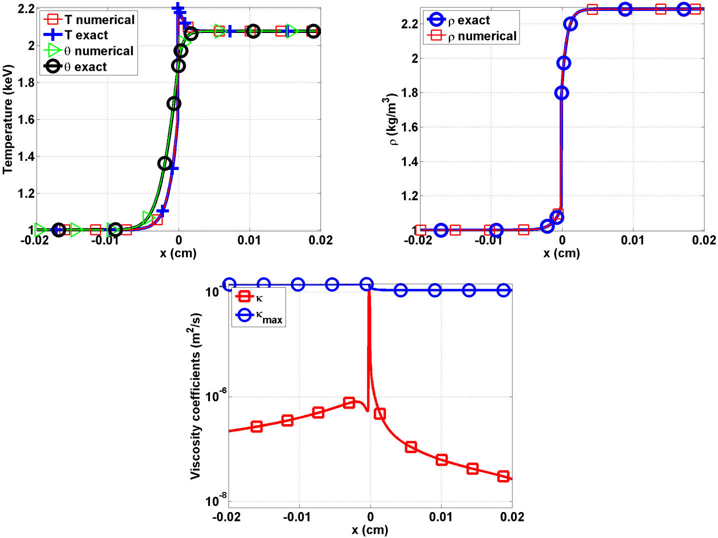


Figure: Steady-state numerical solution with 1000 cells, linear polynomials and BDF2.

Numerical results: Mach 5.

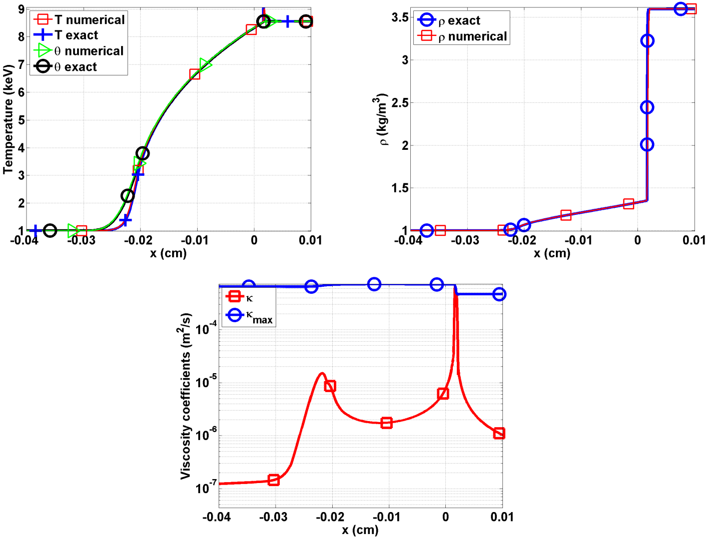
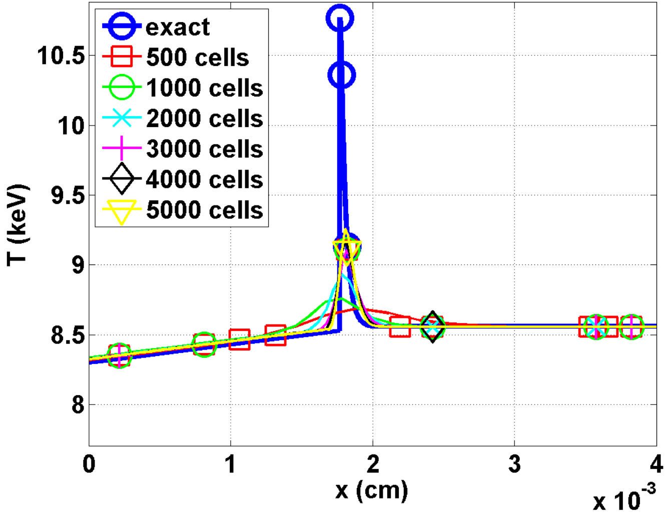


Figure: Steady-state numerical solution with 1000 cells, linear polynomials and BDF2.

Numerical results: Mach 5, Zeldovich spike.



Numerical results: Mach 50.

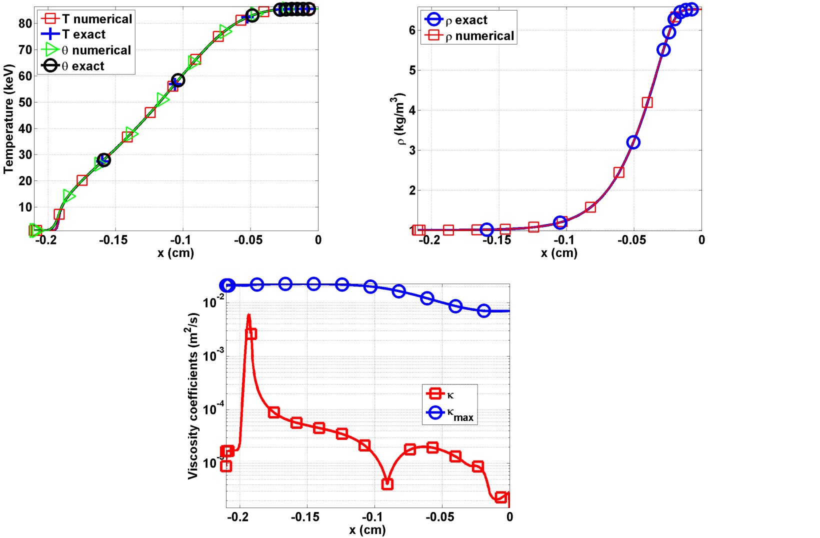


Figure: Steady-state numerical solution with 1000 cells, linear polynomials and BDF2.

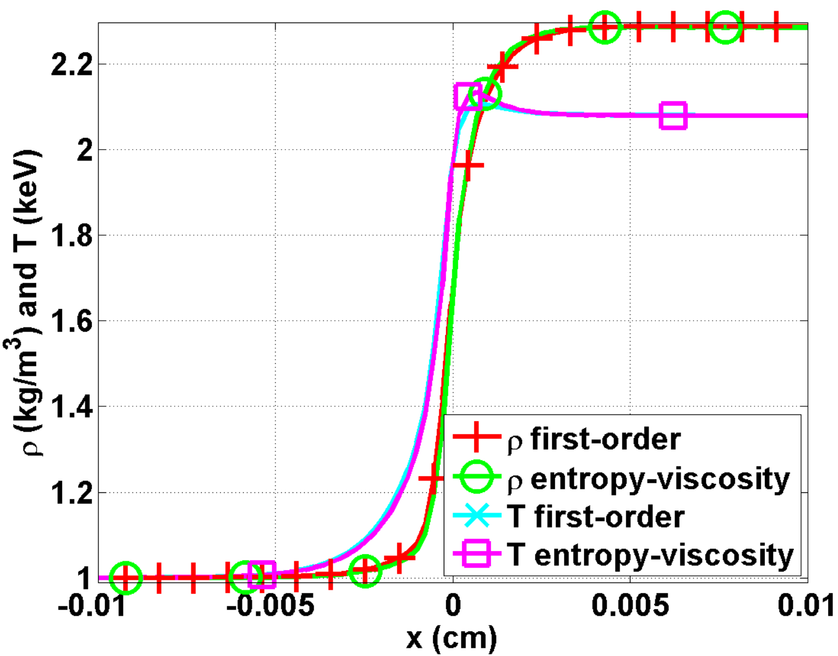
Conclusions:

- The entropy viscosity method was successfully applied to the 1-D RHD.
- 1-D numerical results show good agreement with semi-analytical solutions.
- Demonstrated high-order accuracy and correct behavior in the equilibrium-diffusion limit.
- The method can be applied to any equation of state with a convex entropy.
- The method is simple to implement and the viscosity coefficient is computed on the fly.

Future work:

- Extension to multi-D simulations: the theoretical approach holds.
- S_n transport approximation (instead of the radiation-diffusion equation) coupled to Euler equations.

QUESTIONS/COMMENTS ?


ρ (kg/m³) and T (keV)
2.2
2
1.8
1.6
1.4
1.2
1
-0.01
-0.005
0
0.005
0.01
x (cm)
ρ first-order
ρ entropy-viscosity
T first-order
T entropy-viscosity

Entropy residual with dissipative terms:

$$\frac{Ds}{Dt} + \underbrace{\left(P\partial_e s + \rho^2 \partial_\rho s + \frac{4}{3}\rho\epsilon\partial_\epsilon s \right)}_{(a)} \partial_x u =$$

$$\partial_x \left(\rho\kappa\partial_x s \right) + \kappa\partial_e s \partial_x s - \rho\kappa \underbrace{XAX^t}_{(b)} + \underbrace{s_e \rho\mu(\partial_x u)^2}_{(c)}$$

where X is a row vector defined as $X = (\rho, e, \epsilon)$ and A is the 3x3 symmetric matrix:

$$A = \begin{bmatrix} \partial_\rho \left(\rho^2 \partial_\rho s \right) & \partial_{\rho,e} s & \partial_\rho \left(\rho \partial_\epsilon s \right) \\ \partial_{\rho,e} s & \partial_{e,e} s & \partial_{e,\epsilon} s \\ \partial_\rho \left(\rho \partial_\epsilon s \right) & \partial_{e,\epsilon} s & \partial_{\epsilon,\epsilon} s \end{bmatrix}$$

Positivity of the matrix A:

With $s(\rho, e, \epsilon) = s_{Euler}(\rho, e) + \frac{\rho_0}{\rho}\hat{s}(\epsilon)$:

$$A = \begin{bmatrix} \partial_\rho\left(\rho^2 \partial_\rho \tilde{s}\right) & \partial_{\rho,e}\tilde{s} & 0 \\ \partial_{\rho,e}\tilde{s} & \partial_{e,e}\tilde{s} & 0 \\ 0 & 0 & \rho^{-1}\partial_{\epsilon,\epsilon}\hat{s} \end{bmatrix}$$

With $s(\rho, e, \epsilon) = s_{Euler}(\rho, e) + \frac{\rho_0}{\rho}\hat{s}(\epsilon)$, assuming that

$$P\partial_e s + \rho^2 \partial_\rho s + \frac{4}{3}\rho\epsilon\partial_\epsilon s = 0,$$

it yields

$$P\partial_e s_{Euler} + \rho^2 \partial_\rho s_{Euler} = \alpha \text{ and } \hat{s} - \frac{4\epsilon}{3}\partial_\epsilon \hat{s} = \alpha.$$

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