The Covariance of Covariances

Jan de Leeuw

February 21, 2025

We derive the classical formula for the first and second order moments of covariance using discrete variable calculations. In addition the equally classical first order asymptotics for covariances and correlations is discussed. And finally we derive an exact expression for the covariances of covariances.

Table of contents

Note

This is a working manuscript which will be expanded/updated frequently. All suggestions for improvement are welcome. All Rmd, tex, html, pdf, R, and C files are in the public domain. Attribution will be appreciated, but is not required. The files can be found at https://github.com/deleeuw/covcor

Notation

- We use the "Dutch Convention" of underlining random variables (Hemelrijk (1966)).
- The symbol $:=$ is used for definitions.

1 Introduction

Discrete, continuous. Holland (1979), Gifi (1990), Ouimet (2021)

Suppose we have p real vectors $x_1, \cdots, x_p$ of length m, which we call *profiles*. Profiles are the possible outcomes of a measurement of m variables, and in our framework the number of profiles is finite.

Define a random vector $\underline{e}$, which takes as its values the p unit vectors $e_1, \cdots, e_p$. Unit vector e_r has all its elements equal to zero, except for element r, which is equal to one. Define $\pi_r := \text{prob}(\underline{e} = e_r)$.

$$\underline{x} = \sum_{r=1}^{p} \underline{e}_r x_r$$

random variable $\underline{x}$ which take value x_r with probability π_r.

Next define the raw product moments, or product moments about zero, of order t as

$$\mu_{i_1 \cdots i_t} := \sum_{r=1}^{p} \pi_r \prod_{s=1}^{t} x_{ri_s},$$

and the centered product moments, or product moments around the mean, by

$$\sigma_{i_1 \cdots i_t} := \sum_{r=1}^{p} \pi_r \prod_{s=1}^{t} (x_{ri_s} - \mu_{i_s}).$$

Note that some of the subscripts $i_1, \cdots, i_t$ can be equal

Next, suppose we have n independent copies $\underline{e}_1, \cdots \underline{e}_n$ of $\underline{e}$. Suppose

$$\underline{n} = \sum_{\nu=1}^{n} \underline{e}_\nu \underline{p} = \frac{1}{n}\underline{n}$$

are the vectors of, respectively, *frequencies* and *proportions* of the profiles. the covariance of variables i and j as

$$\underline{c}_{ij} := \sum_{\nu} \underline{p}_\nu x_{\nu i} x_{\nu j} - \sum_{\nu} \underline{p}_\nu x_{\nu i} \sum_{\eta} \underline{p}_\eta x_{\eta j}$$

We want to compute the expected value of the covariances $\underline{c}_{ij}$ and the covariance of pairs of covariances $\underline{c}_{ij}$ and $\underline{c}_{kl}$.

1.1 Computation

Let $\underline{\epsilon}_\nu := \underline{p}_\nu - \pi_\nu$ so that

$$\mathcal{E}(\underline{\epsilon}_\nu) = 0,$$

and

$$\mathcal{E}(\underline{\epsilon}_\nu\underline{\epsilon}_\eta) = n^{-1}(\delta^{\nu\eta}\pi_\nu - \pi_\nu\pi_\eta).$$

Now we can write

$$\underline{s}_{ij} = \sigma_{ij} + \sum_\nu \underline{\epsilon}_\nu(x_{\nu i} - \mu_i)(x_{\nu j} - \mu_j) - \sum_\nu\sum_\eta \underline{\epsilon}_\nu\underline{\epsilon}_\eta(x_{\nu i} - \mu_i)(x_{\eta j} - \mu_j),$$

with

$$\sigma_{ij} = \sum_\nu \pi_\nu(x_{\nu i} - \mu_i)(x_{\nu j} - \mu_j).$$

and

$$\mu_i = \sum_\nu \pi_\nu x_{\nu i}$$

It follows that

$$\mathcal{E}(\underline{c}_{ij}) = \sigma_{ij} - \frac{1}{n}\sum_\nu\sum_\eta x_{\nu i}x_{\eta j}(\delta^{\nu\eta}\pi_\nu - \pi_\nu\pi_\eta) = \frac{n-1}{n}\sigma_{ij}.$$

Next, define

$$\underline{\delta}_{ij} := \underline{s}_{ij} - \frac{n-1}{n}\sigma_{ij}.$$

The $\underline{\delta}_{ij}$ have expectation zero, and

$$\text{COV}(\underline{c}_{ij}, \underline{c}_{kl}) = \mathcal{E}(\underline{\delta}_{ij}\underline{\delta}_{kl}).$$

We have

$$\underline{\delta}_{ij} = \frac{1}{n}\gamma_{ij} + \sum_\nu \underline{\epsilon}_\nu(x_{\nu i} - \mu_i)(x_{\eta j} - \mu_j) - \sum_\nu\sum_\eta \underline{\epsilon}_\nu\underline{\epsilon}_\eta x_{\nu i}x_{\eta j}, \tag{1a}$$

$$\underline{\delta}_{kl} = \frac{1}{n}\gamma_{kl} + \sum_\nu \underline{\epsilon}_\nu(x_{\nu k} - \mu_k)(x_{\eta l} - \mu_l) - \sum_\nu\sum_\eta \underline{\epsilon}_\nu\underline{\epsilon}_\eta x_{\nu k}x_{\eta l}. \tag{1b}$$

1.2 Asymptotics

From

$$\mathcal{E}(\underline{\delta}_{ij}\underline{\delta}_{kl}) = \frac{1}{n}\sum_{\nu}\sum_{\eta}(\delta^{\nu\eta}\pi_{\nu}-\pi_{\nu}\pi_{\eta})\{x_{\nu i}x_{\nu j}-\mu_{j}x_{\nu i}-\mu_{i}x_{\nu j}\}\{x_{\eta k}x_{\eta l}-\mu_{l}x_{\eta k}-\mu_{k}x_{\eta l}\} = \frac{1}{n}\mu_{ijkl}-$$

$$x_{\nu i}x_{\nu j} = (x_{\nu i}-\mu_{i})(x_{\nu j}-\mu_{j}) + x_{\nu i}\mu_{j} + x_{\nu j}\mu_{i} - \mu_{i}\mu_{j}$$

$$x_{\nu i}x_{\nu j} - \mu_{j}x_{\nu i} - \mu_{i}x_{\nu j} = (x_{\nu i}-\mu_{i})(x_{\nu j}-\mu_{j}) - \mu_{i}\mu_{j}$$

$$\mathcal{E}(\underline{\delta}_{ij}\underline{\delta}_{kl}) = \frac{1}{n}\sum_{\nu}\sum_{\eta}(\delta^{\nu\eta}\pi_{\nu}-\pi_{\nu}\pi_{\eta})\{(x_{\nu i}-\mu_{i})(x_{\nu j}-\mu_{j})-\mu_{i}\mu_{j}\}\{(x_{\eta k}-\mu_{k})(x_{\eta l}-\mu_{l})-\mu_{k}\mu_{l}\} =$$
$$\sigma_{ijkl} - \mu_{k}\mu_{l}\sigma_{ij} - \mu_{i}\mu_{j}\sigma_{kl} + \mu_{i}\mu_{j}\mu_{k}\mu_{l} - (\sigma_{ij}-\mu_{i}\mu_{j})(\sigma_{kl}-\mu_{k}\mu_{l}) = \sigma_{ijkl} - \sigma_{ij}\sigma_{kl} \tag{2}$$

1.2.1 Correlations

$$\underline{z}_{ij} = n^{\frac{1}{2}}(\underline{s}_{ij} - \sigma_{ij})$$

$$\underline{r}_{ij} = (\sigma_{ij} + n^{-\frac{1}{2}}z_{ij})(\sigma_{ii} + n^{-\frac{1}{2}}z_{ii})^{-\frac{1}{2}}(\sigma_{jj} + n^{-\frac{1}{2}}z_{jj})^{-\frac{1}{2}}$$

$$\underline{r}_{ij} = \rho_{ij}(1 + n^{-\frac{1}{2}}\frac{\underline{z}_{ij}}{\sigma_{ij}})(1 + n^{-\frac{1}{2}}\frac{\underline{z}_{ii}}{\sigma_{ii}})^{-\frac{1}{2}}(1 + n^{-\frac{1}{2}}\frac{\underline{z}_{jj}}{\sigma_{jj}})^{-\frac{1}{2}}$$

$$\underline{r}_{ij} = \rho_{ij} + n^{-\frac{1}{2}}\rho_{ij}\left\{\frac{\underline{z}_{ij}}{\sigma_{ij}} - \frac{1}{2}\frac{\underline{z}_{ii}}{\sigma_{ii}} - \frac{1}{2}\frac{\underline{z}_{jj}}{\sigma_{jj}}\right\}$$

$$
\begin{aligned}
n\text{ACOV}(\underline{r}_{ij},\underline{r}_{kl}) = \rho_{ij}\rho_{kl}\ \mathcal{E}\left\{\frac{\underline{z}_{ij}}{\sigma_{ij}} - \frac{1}{2}\frac{\underline{z}_{ii}}{\sigma_{ii}} - \frac{1}{2}\frac{\underline{z}_{jj}}{\sigma_{jj}}\right\}\left\{\frac{\underline{z}_{kl}}{\sigma_{kl}} - \frac{1}{2}\frac{\underline{z}_{kk}}{\sigma_{kk}} - \frac{1}{2}\frac{\underline{z}_{ll}}{\sigma_{ll}}\right\} = \\
\rho_{ij}\rho_{kl}\left\{\frac{\sigma_{ijkl}-\sigma_{ij}\sigma_{kl}}{\sigma_{ij}\sigma_{kl}} - \frac{1}{2}\frac{\sigma_{ijkk}-\sigma_{ij}\sigma_{kk}}{\sigma_{ij}\sigma_{kk}} - \frac{1}{2}\frac{\sigma_{ijll}-\sigma_{ij}\sigma_{ll}}{\sigma_{ij}\sigma_{ll}} + \right. \\
-\frac{1}{2}\frac{\sigma_{iikl}-\sigma_{ii}\sigma_{kl}}{\sigma_{ii}\sigma_{kl}} + \frac{1}{4}\frac{\sigma_{iikk}-\sigma_{ii}\sigma_{kk}}{\sigma_{ii}\sigma_{kk}} + \frac{1}{4}\frac{\sigma_{iill}-\sigma_{ii}\sigma_{ll}}{\sigma_{ii}\sigma_{ll}} + \\
\left. -\frac{1}{2}\frac{\sigma_{jjkl}-\sigma_{jj}\sigma_{kl}}{\sigma_{jj}\sigma_{kl}} + \frac{1}{4}\frac{\sigma_{jjkk}-\sigma_{jj}\sigma_{kk}}{\sigma_{jj}\sigma_{kk}} + \frac{1}{4}\frac{\sigma_{jjll}-\sigma_{jj}\sigma_{ll}}{\sigma_{jj}\sigma_{ll}}\right\} = \\
\rho_{ij}\rho_{kl}\left\{\frac{\sigma_{ijkl}}{\sigma_{ij}\sigma_{kl}} - \frac{1}{2}\left(\frac{\sigma_{ijkk}}{\sigma_{ij}\sigma_{kk}} + \frac{\sigma_{ijll}}{\sigma_{ij}\sigma_{ll}} + \frac{\sigma_{iikl}}{\sigma_{ii}\sigma_{kl}} + \frac{\sigma_{jjkl}}{\sigma_{jj}\sigma_{kl}}\right)\right. \\
\left. +\frac{1}{4}\left(\frac{\sigma_{iikk}}{\sigma_{ii}\sigma_{kk}} + \frac{\sigma_{iill}}{\sigma_{ii}\sigma_{ll}} + + \frac{\sigma_{jjkk}}{\sigma_{jj}\sigma_{kk}} + + \frac{\sigma_{jjll}}{\sigma_{jj}\sigma_{ll}}\right)\right\} \quad (3)
\end{aligned}
$$

1.3 Exact Calculations

If we multiply (1a) and (1b) we have nine terms. Taking expectations of each of these nine terms gives

$$\text{term I}: \frac{1}{n^2}\gamma_{ij}\gamma_{kl}$$

$$\text{term II}: \frac{1}{n}\gamma_{ij}\mathcal{E}(\sum_{\nu}\underline{\epsilon}_{\nu}\{x_{\nu k}x_{\nu l} - \mu_l x_{\nu k} - \mu_k x_{\nu l}\}) = 0$$

$$\text{term III}: -\frac{1}{n}\gamma_{ij}\mathcal{E}(\sum_{\nu}\sum_{\eta}\underline{\epsilon}_{\nu}\underline{\epsilon}_{\eta}x_{\nu k}x_{\eta l}) = -\frac{1}{n^2}\gamma_{ij}\gamma_{kl}$$

$$\text{term IV}: \frac{1}{n}\gamma_{kl}\mathcal{E}(\sum_{\nu}\underline{\epsilon}_{\nu}\{x_{\nu i}x_{\nu j} - \mu_j x_{\nu i} - \mu_i x_{\nu j}\}) = 0$$

$$\text{term V}: \frac{1}{n}\sum_{\nu}\sum_{\eta}(\delta^{\nu\eta}\pi_{\nu} - \pi_{\nu}\pi_{\eta})\{x_{\nu i}x_{\nu j} - \mu_j x_{\nu i} - \mu_i x_{\nu j}\}\{x_{\eta k}x_{\eta l} - \mu_l x_{\eta k} - \mu_k x_{\eta l}\}$$

$$\text{term VI}: -\sum_{\alpha}\sum_{\nu}\sum_{\eta}\mathcal{E}(\underline{\epsilon}_{\alpha}\underline{\epsilon}_{\nu}\underline{\epsilon}_{\eta})\{x_{\alpha i}x_{\alpha j} - \mu_j x_{\alpha i} - \mu_i x_{\alpha j}\}x_{\nu k}x_{\eta l}$$

$$\text{term VII}: -\frac{1}{n}\gamma_{kl}\sum_{\nu}\sum_{\eta}\mathcal{E}(\underline{\epsilon}_{\nu}\underline{\epsilon}_{\eta})x_{\nu i}x_{\eta j} = -\frac{1}{n^2}\gamma_{ij}\gamma_{kl}$$

$$\text{term VIII} : -\sum_{\nu}\sum_{\eta}\sum_{\alpha}\mathcal{E}(\underline{\epsilon}_{\nu}\underline{\epsilon}_{\eta}\underline{\epsilon}_{\alpha})x_{\nu i}x_{\eta j}\{x_{\alpha k}x_{\alpha l} - \mu_l x_{\alpha k} - \mu_k x_{\alpha l}\}$$

$$\text{term IX} : \sum_{\nu}\sum_{\eta}\sum_{\alpha}\sum_{\beta}\mathcal{E}(\underline{\epsilon}_{\nu}\underline{\epsilon}_{\eta}\underline{\epsilon}_{\alpha}\underline{\epsilon}_{\beta})x_{\nu i}x_{\eta j}x_{\alpha k}x_{\beta l}$$

References

Gifi, A. 1990. *Nonlinear Multivariate Analysis*. New York, N.Y.: Wiley.

Hemelrijk, J. 1966. "Underlining Random Variables." *Statistica Neerlandica* 20: 1–7.

Holland, P. W. 1979. "The Tyranny of Continuous Models in a World of Discrete Data." *IHS-Journal* 3: 29–42.

Ouimet, F. 2021. "General Formulas for the Central and Non-Central Moments of the Multinomial Distribution." *Stats* 4: 18–27.