Solving Cubic Equations by Iteration

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Abstract

We give algorithms to find the roots of a cubic polynomial. Instead of thinking of this note as a contribution to this age-old subject, it is better to think of it as a didactic piece on the behaviour of different iterative procedures in a simple example.

Contents

Note: This working paper, inspired by Strohbach (2011), will be expanded/updated frequently. All suggestions for improvement are welcome.

1 Introduction

An arbitrary monic cubic polynomial with real coefficients has always at least one real root. This implies that any cubic can be written as a product of the form

$$x^3 + px^2 + qx + r = (x + \gamma)(x^2 + \alpha x + \beta) \tag{1}$$

with (α, β, γ) real. One way to think about this is that we need three numbers to describe monic cubics. One system is to use the coefficients (p, q, r), another is to use (α, β, γ) defined implicitly by (1). A third system is to use the roots of the polynomial, as in

$$x^3 + px^2 + qx + r = (x - \xi_1)(x - \xi_2)(x - \xi_3) \tag{2}$$

In this third system two of the roots may be complex numbers and the roots can be ordered arbitrarily

Of the mappings of one system of three coordinates into another one is classical. The map $(p, q, r) \Rightarrow (\xi_1, \xi_2, \xi_3)$ is the Cardano map that describes the algebraic steps to solve a cubic equation, or the computational step to find the eigenvalues of the companion matrix. Also the map is not one-to-one because the three roots (ξ_1, ξ_2, ξ_3) can be permuted, and thus define $3! = 6$ solutions.

The map $(\alpha, \beta, \gamma) \Rightarrow (\xi_1, \xi_2, \xi_3)$ is much simpler. In fact

$$\xi_1 = -\gamma, \tag{3}$$

$$\xi_2 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}, \tag{4}$$

$$\xi_3 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}. \tag{5}$$

In this paper we are interested, following Strohbach (2011), in computer implementations of $(p, q, r) \Rightarrow (\alpha, \beta, \gamma)$. Comparing coefficients on both sides of (1) gives the equations

$$\gamma + \alpha = p, \tag{6}$$

$$\beta + \alpha\gamma = q, \tag{7}$$

$$\beta\gamma = r, \tag{8}$$

which is actually the map $(\alpha, \beta, \gamma) \Rightarrow (p, q, r)$.

(6), (7), and (8) are three equations in three unknowns, which always have at least one solution with (α, β, γ) real. If the cubic has only one real root the solution is unique, $\ldots$ complex roots .. If there are three distinct real roots, for example, then there are three solutions for γ and three corresponding solutions for α and β.

Vieta's formulas. $(\xi_1, \xi_2, \xi_3) \Rightarrow (p, q, r)$

$$p = -(\xi_1 + \xi_2 + \xi_3), \tag{9}$$
$$q = \xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3, \tag{10}$$
$$r = -\xi_1\xi_2\xi_3. \tag{11}$$

The Jacobian is

$$\begin{bmatrix} -1 & -1 & -1 \\ \xi_2 + \xi_3 & \xi_1 + \xi_3 & \xi_1 + \xi_2 \\ -\xi_2\xi_3 & -\xi_1\xi_3 & -\xi_1\xi_2 \end{bmatrix}$$

2 Algorithms

Initial

2.1 Newton

The most obvious way to solve the three equations (6), (7), and (8) numerically is Newton's method.

Consider the loss function
$$\sigma(\theta) = \frac{1}{2}\sum_{k=1}^{K} f_k^2(\theta)$$
which we want to minimize over $\theta \in \mathbb{R}^K$. It is known a priori that there is at least one $\theta_0 \in \Theta$ such that $f_k(\theta_0) = 0$. Thus
$$\min_{\theta\in\Theta} \sigma(\theta) = 0,$$
and
$$\theta_0 \in \operatorname*{argmin}_{\theta\in\Theta} \sigma(\theta)$$
Newton-Raphson = Gauss-Newton

$$f_k(\theta + \epsilon) \approx f_k(\theta) + \sum_{\ell=1}^{K} \mathcal{D}_\ell f_k(\theta)\epsilon_\ell$$

$$\sigma(\theta+\epsilon) \approx \frac{1}{2}\sum_{k=1}^{K}\{f_k(\theta)+\sum_{\ell=1}^{K}\mathcal{D}_\ell f_k(\theta)\epsilon_\ell\}^2 = \sigma(\theta)+\sum_{\ell=1}^{K}\left\{\sum_{k=1}^{K} f_k(\theta)\mathcal{D}_\ell f_k(\theta)\right\}\epsilon_\ell+\frac{1}{2}\sum_{\ell=1}^{K}\sum_{\nu=1}^{K}\left\{\sum_{k=1}^{K}\mathcal{D}_\ell f_k(\theta)\right.$$

$$\sigma(\theta+\epsilon) \approx \sigma(\theta)+\sum_{\ell=1}^{K}\left\{\sum_{k=1}^{K} f_k(\theta)\mathcal{D}_\ell f_k(\theta)\right\}\epsilon_\ell+\frac{1}{2}\sum_{\ell=1}^{K}\sum_{\nu=1}^{K}\left\{\sum_{k=1}^{K}\mathcal{D}_\ell f_k(\theta)\mathcal{D}_\nu f_k(\theta)+f_k(\theta)\mathcal{D}_{\ell\nu} f_k(\theta)\right\}\epsilon_\ell\epsilon_\nu$$

Gathering

$$F(\theta+\epsilon) \approx F(\theta)+\mathcal{J}(\theta)\epsilon$$

thus

$$\sigma(\theta+\epsilon)=\|F(\theta+\epsilon)\|^2 \approx \|F(\theta)+\mathcal{J}(\theta)\epsilon\|^2$$

which is minimized over ϵ at

$$\epsilon=-\{\mathcal{J}'(\theta)\mathcal{J}(\theta)\}^{-1}\mathcal{J}'(\theta)F(\theta)$$

and if $J(\theta)$ is square and non-singular

$$\epsilon=-\mathcal{J}^{-1}(\theta)F(\theta)$$

so that the update of θ is

$$\theta^+=\theta-\mathcal{J}^{-1}(\theta)F(\theta)$$

One application: $f_k(\theta)=\mathcal{D}_k g(\theta)$. $\mathcal{D}_\ell f_k(\theta)=\mathcal{D}_{k\ell} g(\theta)$ $\theta^+=\theta-\{\mathcal{D}^2 g(\theta)\}^{-1}\mathcal{D}g(\theta)$

This is different from NewtonLS applied to σ

$$\theta^+=\theta-\{\mathcal{D}^2\sigma(\theta)\}^{-1}\mathcal{D}\sigma(\theta)$$

2.1.1 abc

$$f_1(\alpha,\beta,\gamma):=\alpha+\gamma-p=0, \tag{12}$$
$$f_2(\alpha,\beta,\gamma):=\beta+\alpha\gamma-q=0, \tag{13}$$
$$f_3(\alpha,\beta,\gamma):=\beta\gamma-r=0. \tag{14}$$

The Jacobian is

$$\begin{bmatrix}1 & 0 & 1\\ \gamma & 1 & \alpha\\ 0 & \gamma & \beta\end{bmatrix}$$

and its inverse is

$$\frac{1}{\beta+\gamma^2-\alpha\gamma}\begin{bmatrix}\beta-\alpha\gamma & \gamma & -1\\ -\beta\gamma & \beta & \gamma-\alpha\\ \gamma^2 & -\gamma & 1\end{bmatrix}.$$

Thus the Newton correction is

$$\frac{1}{\beta+\gamma^2-\alpha\gamma}\begin{bmatrix}\beta-\alpha\gamma & \gamma & -1\\ -\beta\gamma & \beta & \gamma-\alpha\\ \gamma^2 & -\gamma & 1\end{bmatrix}\begin{bmatrix}\alpha+\gamma-p\\ \beta+\alpha\gamma-q\\ \beta\gamma-r\end{bmatrix}.$$

For an initial estimate of our iterations we guess a value for $\gamma^{(0)}$. Next $\alpha^{(0)}$ and $\beta^{(0)}$ are computed as the least squares solution of the over-determined system

$$\begin{bmatrix} 1 & 0 \\ \gamma^{(0)} & 1 \\ 0 & \gamma^{(0)} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} p - \gamma^{(0)} \\ q \\ r \end{bmatrix}$$

This is actually also a step in the ALS algorithm, which we discuss next.

2.2 Alternating Least Squares

An alternative way to solve the system (6)-(8) is to minimize the function

$$\sigma(\alpha, \beta, \gamma) := \frac{1}{2}(\gamma + \alpha - p)^2 + \frac{1}{2}(\beta + \alpha\gamma - q)^2 + \frac{1}{2}(\beta\gamma - r)^2 \tag{15}$$

over (α, β, γ). In the ALS (Alternating Least Squares) algorithm we alternate minimizing over γ for fixed (α, β) and over (α, β) for fixed γ. Thus

$$(\alpha^{(k+1)}, \beta^{(k+1)}) = \underset{(\alpha,\beta)}{\text{argmin}}\, \sigma(\alpha, \beta, \gamma^{(k)}), \tag{16}$$

$$\gamma^{(k+1)} = \underset{\gamma}{\text{argmin}}\, \sigma(\alpha^{(k+1)}, \beta^{(k+1)}, \gamma). \tag{17}$$

The partials are

$$\mathcal{D}_1\sigma(\alpha, \beta, \gamma) = (\gamma + \alpha - p) + \gamma(\beta + \alpha\gamma - q), \tag{18}$$

$$\mathcal{D}_2\sigma(\alpha, \beta, \gamma) = (\beta + \alpha\gamma - q) + \gamma(\beta\gamma - r), \tag{19}$$

$$\mathcal{D}_3\sigma(\alpha, \beta, \gamma) = (\gamma + \alpha - p) + \alpha(\beta + \alpha\gamma - q) + \beta(\beta\gamma - r). \tag{20}$$

Write $\mathcal{D}_1\sigma(\alpha, \beta, \gamma) = 0$ and $\mathcal{D}_2\sigma(\alpha, \beta, \gamma) = 0$ as

$$\begin{bmatrix} (1 + \gamma^2) & \gamma \\ \gamma & (1 + \gamma^2) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} p - \gamma(1 - q) \\ q + \gamma r \end{bmatrix} \tag{21}$$

and $\mathcal{D}_3\sigma(\alpha, \beta, \gamma) = 0$ as

$$(1 + \alpha^2 + \beta^2)\gamma = (p - \alpha) + \alpha(q - \beta) + \beta r$$

$$\gamma^{(k+1)} = \frac{}{1 + (\alpha^{(k+1)})^2 + (\beta^{(k+1)})^2}$$

$$\alpha^{(k+1)} = \beta^{(k+1)} =$$

The speed of convergence ρ of the ALS iterations to a solution (α, β, γ) can be computed from the second derivatives of σ (De Leeuw (1994)).

$$\rho(\alpha, \beta, \gamma) := \frac{\begin{Bmatrix} \frac{\partial^2\sigma}{\partial\gamma\partial\alpha} & \frac{\partial^2\sigma}{\partial\gamma\partial\alpha} \end{Bmatrix} \begin{Bmatrix} \frac{\partial^2\sigma}{\partial\alpha\partial\alpha} & \frac{\partial^2\sigma}{\partial\alpha\partial\beta} \\ \frac{\partial^2\sigma}{\partial\beta\partial\alpha} & \frac{\partial^2\sigma}{\partial\beta\partial\beta} \end{Bmatrix}^{-1} \begin{Bmatrix} \frac{\partial^2\sigma}{\partial\gamma\partial\alpha} \\ \frac{\partial^2\sigma}{\partial\gamma\partial\alpha} \end{Bmatrix}}{\begin{Bmatrix} \frac{\partial^2\sigma}{\partial\gamma\partial\gamma} \end{Bmatrix}} \tag{22}$$

Differentiating .., .., .. once again gives the Hessian

$$\begin{bmatrix} 1+\gamma^2 & \gamma & (1+\beta-q)+2\alpha\gamma \\ \gamma & 1+\gamma^2 & (\alpha-r)+2\beta\gamma \\ (1+\beta-q)+2\alpha\gamma & (\alpha-r)+2\beta\gamma & 1+\alpha^2+\beta^2 \end{bmatrix}$$

2.3 Newton Least Squares

$$\begin{bmatrix} 1+\gamma^2 & \gamma & (1+\beta-q)+2\alpha\gamma \\ \gamma & 1+\gamma^2 & (\alpha-r)+2\beta\gamma \\ (1+\beta-q)+2\alpha\gamma & (\alpha-r)+2\beta\gamma & 1+\alpha^2+\beta^2 \end{bmatrix}$$

3 Examples

In this section we analyze the three algorithms (Newton, ALS, and Newton on the LS loss function) on four cubics which all have their roots between zero and three.

All runs use the same initial estimate. We select a value for $\gamma^{(0)}$. Next $\alpha^{(0)}$ and $\beta^{(0)}$ are computed as the least squares solution of the over-determined system

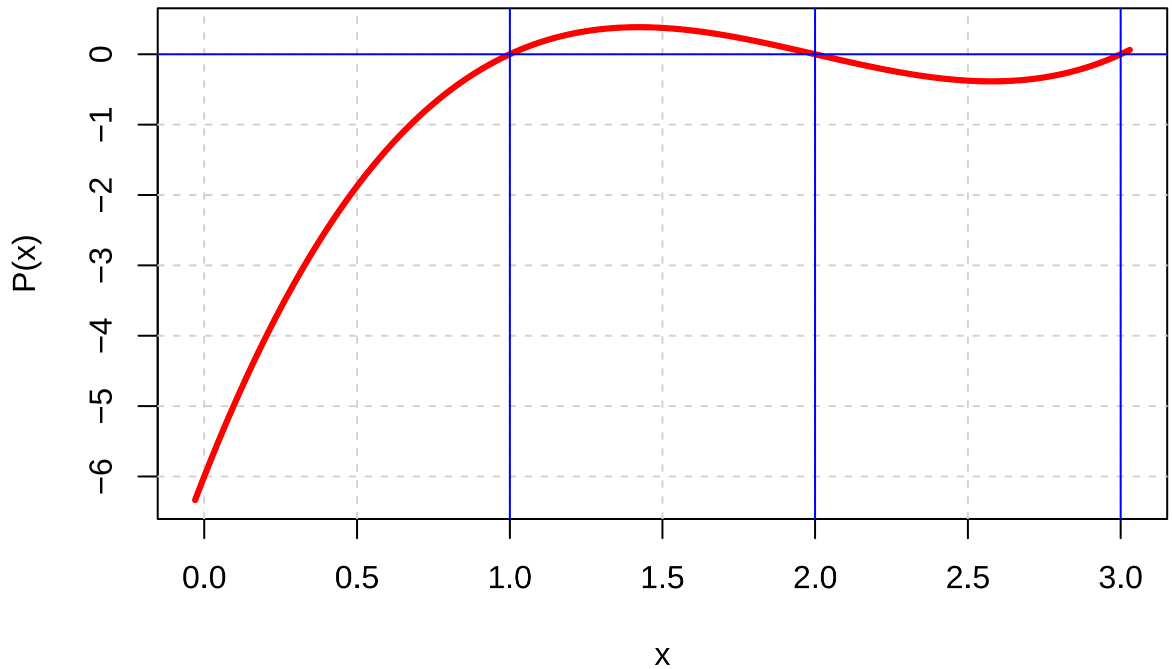
$$\begin{bmatrix} 1 & 0 \\ \gamma^{(0)} & 1 \\ 0 & \gamma^{(0)} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} p-\gamma^{(0)} \\ q \\ r \end{bmatrix} \tag{23}$$

Each of the four polynomial $\times$ three algorithms combination is run multiple times by using different values for $\gamma^{(0)}$. We varied $\gamma^{(0)}$ from 0.0 to 3.0 in steps of 0.1. This gives 31 runs, the output of which is summarized in a table. There are twelve such tables.

Venables, Hornik, and Mächler (2022)

3.1 Example 1: $f(x) = (x-1)(x-2)(x-3)$

The cubic $f(x) = (x-1)(x-2)(x-3)$, with (p, q, r) equal to -6, 11, -6 and $(\alpha, \beta, \gamma) = (-5, 6, -1)$, has three distinct real roots.



3.1.1 Newton.

```
##      itel alpha beta gamma root1 root2 root3     sigma      rate
## -3      1    -3    2    -3  3+0i  2+0i  1+0i 0.000e+00 0.000e+00
## -2.9    5    -3    2    -3  3+0i  2+0i  1+0i 0.000e+00 2.366e-08
## -2.8    6    -3    2    -3  3+0i  2+0i  1+0i 0.000e+00 0.000e+00
## -2.7    6    -3    2    -3  3+0i  2+0i  1+0i 0.000e+00 6.649e-07
## -2.6    7    -3    2    -3  3+0i  2+0i  1+0i 3.944e-31 7.202e-07
## -2.5   11    -3    2    -3  3+0i  2+0i  1+0i 0.000e+00 1.406e-07
## -2.4    7    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 8.929e-07
## -2.3    6    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 1.350e-06
## -2.2    6    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 0.000e+00
## -2.1    5    -4    3    -2  2+0i  3+0i  1+0i 3.944e-31 1.568e-07
## -2      1    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 0.000e+00
## -1.9    5    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 0.000e+00
## -1.8    5    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 7.151e-07
## -1.7    5    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 3.216e-06
## -1.6    5    -4    3    -2  2+0i  3+0i  1+0i 3.944e-31 4.855e-08
## -1.5    6    -4    3    -2  2+0i  3+0i  1+0i 3.944e-31 2.766e-08
## -1.4    8    -4    3    -2  2+0i  3+0i  1+0i 3.944e-31 6.522e-08
## -1.3    9    -5    6    -1  1+0i  3+0i  2+0i 1.578e-30 1.608e-07
## -1.2    7    -5    6    -1  1+0i  3+0i  2+0i 3.944e-31 1.688e-05
## -1.1    5    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 3.214e-06
## -1      1    -5    6    -1  1+0i  3+0i  2+0i 1.578e-30 0.000e+00
## -0.9    5    -5    6    -1  1+0i  3+0i  2+0i 1.578e-30 3.194e-07
## -0.8    6    -5    6    -1  1+0i  3+0i  2+0i 3.944e-31 9.274e-07
## -0.7    6    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 9.439e-08
## -0.6    6    -5    6    -1  1+0i  3+0i  2+0i 3.944e-31 1.448e-06
## -0.5    7    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 0.000e+00
## -0.4    7    -5    6    -1  1+0i  3+0i  2+0i 3.944e-31 7.855e-07
## -0.3    7    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 9.486e-08
## -0.2    7    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 2.885e-08
```

```
## -0.1    7    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 1.774e-08
## 0       7    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 6.352e-08
## 0.1     7    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 1.268e-07
## 0.2     7    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 1.685e-07
## 0.3     7    -5    6    -1  1+0i  3+0i  2+0i 3.944e-31 2.291e-07
## 0.4     7    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 2.290e-07
## 0.5     7    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 8.484e-08
## 0.6     7    -5    6    -1  1+0i  3+0i  2+0i 3.944e-31 2.596e-06
## 0.7     9    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 0.000e+00
## 0.8     8    -3    2    -3  3+0i  2+0i  1+0i 0.000e+00 7.182e-06
## 0.9     9    -3    2    -3  3+0i  2+0i  1+0i 3.944e-31 7.801e-07
## 1      11    -3    2    -3  3+0i  2+0i  1+0i 0.000e+00 0.000e+00
## 1.1    13    -3    2    -3  3+0i  2+0i  1+0i 3.944e-31 1.062e-06
## 1.2    11    -3    2    -3  3+0i  2+0i  1+0i 0.000e+00 5.242e-07
## 1.3    11    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 0.000e+00
## 1.4     8    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 8.669e-07
## 1.5    10    -4    3    -2  2+0i  3+0i  1+0i 3.944e-31 2.346e-07
## 1.6     9    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 0.000e+00
## 1.7    10    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 1.473e-06
## 1.8    13    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 0.000e+00
## 1.9    11    -5    6    -1  1+0i  3+0i  2+0i 1.972e-30 4.994e-07
## 2      10    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 2.360e-07
## 2.1    12    -5    6    -1  1+0i  3+0i  2+0i 3.944e-31 6.719e-07
## 2.2    14    -5    6    -1  1+0i  3+0i  2+0i 3.944e-31 1.749e-07
## 2.3    15    -3    2    -3  3+0i  2+0i  1+0i 0.000e+00 0.000e+00
## 2.4    10    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 5.486e-07
## 2.5    16    -4    3    -2  2+0i  3+0i  1+0i 3.944e-31 3.300e-07
## 2.6    12    -5    6    -1  1+0i  3+0i  2+0i 3.944e-31 5.861e-07
## 2.7    18    -5    6    -1  1+0i  3+0i  2+0i 3.944e-31 2.792e-07
## 2.8    12    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 0.000e+00
## 2.9    13    -5    6    -1  1+0i  3+0i  2+0i 0.000e+00 7.289e-08
## 3      13    -4    3    -2  2+0i  3+0i  1+0i 0.000e+00 0.000e+00
```

3.1.2 ALS

```
##     itel  alpha   beta  gamma     root1     root2     root3     sigma   rate
## 0    859 -5.000  6.000 -1.000  1.000+0i 3.0000+0i  2.000+0i 3.992e-19 0.9786
## 0.1  859 -5.000  6.000 -1.000  1.000+0i 3.0000+0i  2.000+0i 3.987e-19 0.9785
## 0.2  859 -5.000  6.000 -1.000  1.000+0i 3.0000+0i  2.000+0i 3.958e-19 0.9784
## 0.3  859 -5.000  6.000 -1.000  1.000+0i 3.0000+0i  2.000+0i 3.881e-19 0.9785
## 0.4  858 -5.000  6.000 -1.000  1.000+0i 3.0000+0i  2.000+0i 3.844e-19 0.9785
## 0.5  853 -5.000  6.000 -1.000  1.000+0i 3.0000+0i  2.000+0i 3.864e-19 0.9786
## 0.6  768 -5.000  6.000 -1.000  1.000+0i 3.0000+0i  2.000+0i 3.988e-19 0.9786
## 0.7 8921 -4.000  3.000 -2.000  2.000+0i 3.0000+0i  1.000+0i 3.983e-17 0.9982
## 0.8 8051 -4.000  3.000 -2.000  2.000+0i 3.0000+0i  1.000+0i 3.991e-17 0.9982
## 0.9 9031 -4.000  3.000 -2.000  2.000+0i 3.0000+0i  1.000+0i 3.985e-17 0.9982
## 1   8477 -4.000  3.000 -2.000  2.000+0i 3.0000+0i  1.000+0i 3.989e-17 0.9981
## 1.1  859 -5.000  6.000 -1.000  1.000+0i 3.0000+0i  2.000+0i 3.851e-19 0.9785
## 1.2   25  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3653
## 1.3   24  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 1.4   24  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 1.5   23  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 1.6   24  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
```

```
## 1.7   24  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3653
## 1.8   24  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 1.9   24  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 2     24  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3653
## 2.1   23  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 2.2   23  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 2.3   23  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 2.4   23  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 2.5   23  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 2.6   23  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 2.7   23  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 2.8   23  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 2.9   22  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
## 3     22  2.738 -1.508  3.208 -3.208+0i 0.4702+0i -3.208+0i 7.896e+01 0.3654
##      trate
## 0   0.9785
## 0.1 0.9785
## 0.2 0.9785
## 0.3 0.9785
## 0.4 0.9785
## 0.5 0.9785
## 0.6 0.9785
## 0.7 0.9982
## 0.8 0.9982
## 0.9 0.9982
## 1   0.9982
## 1.1 0.9785
## 1.2 0.3654
## 1.3 0.3654
## 1.4 0.3654
## 1.5 0.3654
## 1.6 0.3654
## 1.7 0.3654
## 1.8 0.3654
## 1.9 0.3654
## 2   0.3654
## 2.1 0.3654
## 2.2 0.3654
## 2.3 0.3654
## 2.4 0.3654
## 2.5 0.3654
## 2.6 0.3654
## 2.7 0.3654
## 2.8 0.3654
## 2.9 0.3654
## 3   0.3654
```

3.1.3 newtonLS

```
##      itel  alpha    beta  gamma     root1      root2      root3     sigma
## -3      1 -3.000  2.0000 -3.000  3.000+0i  2.0000+0i  1.0000+0i 0.000e+00
## -2.9    7 -3.000  2.0000 -3.000  3.000+0i  2.0000+0i  1.0000+0i 1.578e-30
## -2.8   15 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
```

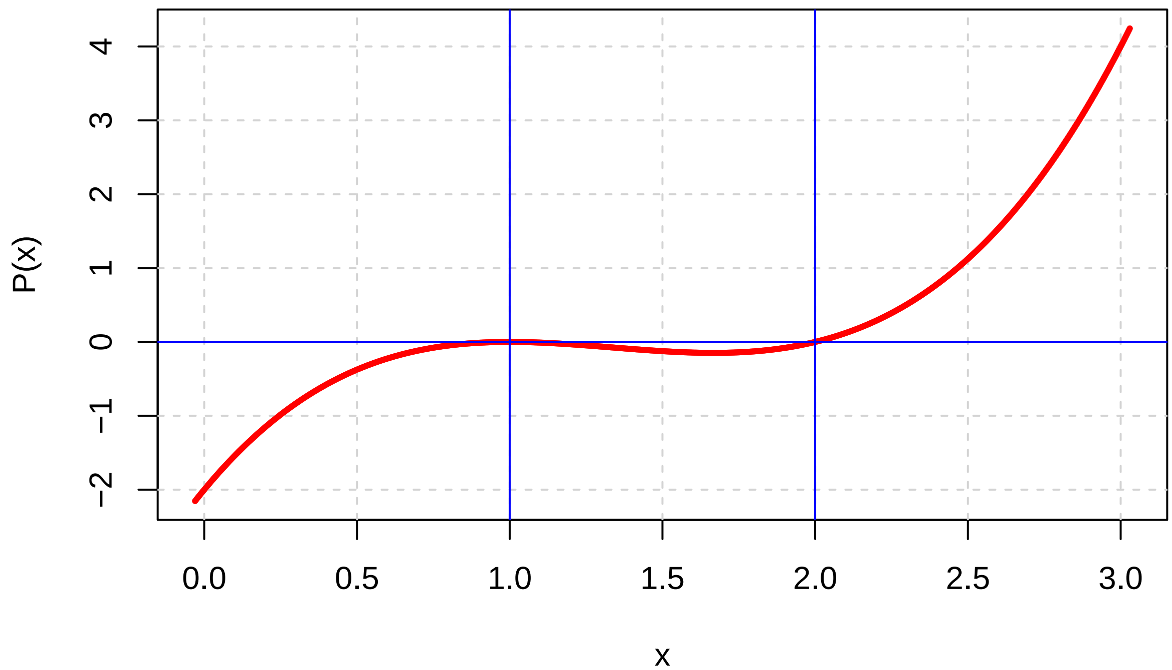
```
## -2.7    9 -3.457  2.4040 -2.492  2.492+0i  2.4924+0i  0.9645+0i 1.519e-03
## -2.6    7 -3.457  2.4040 -2.492  2.492+0i  2.4924+0i  0.9645+0i 1.519e-03
## -2.5    4 -3.457  2.4040 -2.492  2.492+0i  2.4924+0i  0.9645+0i 1.519e-03
## -2.4    7 -3.457  2.4040 -2.492  2.492+0i  2.4924+0i  0.9645+0i 1.519e-03
## -2.3    8 -3.457  2.4040 -2.492  2.492+0i  2.4924+0i  0.9645+0i 1.519e-03
## -2.2   16  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## -2.1    7 -4.000  3.0000 -2.000  2.000+0i  3.0000+0i  1.0000+0i 0.000e+00
## -2      1 -4.000  3.0000 -2.000  2.000+0i  3.0000+0i  1.0000+0i 0.000e+00
## -1.9    6 -4.000  3.0000 -2.000  2.000+0i  3.0000+0i  1.0000+0i 0.000e+00
## -1.8    7 -4.000  3.0000 -2.000  2.000+0i  3.0000+0i  1.0000+0i 0.000e+00
## -1.7    7 -4.000  3.0000 -2.000  2.000+0i  3.0000+0i  1.0000+0i 0.000e+00
## -1.6    9 -3.457  2.4040 -2.492  2.492+0i  2.4924+0i  0.9645+0i 1.519e-03
## -1.5    7 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
## -1.4   14 -4.000  3.0000 -2.000  2.000+0i  3.0000+0i  1.0000+0i 0.000e+00
## -1.3    5 -4.788  4.5833 -1.323  1.323+0i  3.4647+0i  1.3229+0i 1.156e-02
## -1.2    7 -4.788  4.5833 -1.323  1.323+0i  3.4647+0i  1.3229+0i 1.156e-02
## -1.1   11 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
## -1      1 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
## -0.9    7 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 3.944e-31
## -0.8    8 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
## -0.7    9 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
## -0.6    9 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 3.944e-31
## -0.5   10 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
## -0.4   10 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
## -0.3   11 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
## -0.2   11 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
## -0.1   11 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
## 0      11 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 3.944e-31
## 0.1    10 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
## 0.2    11  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 0.3    15 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 0.000e+00
## 0.4    15  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 0.5     8  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 0.6     6  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 0.7     5  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 0.8     5  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 0.9     5  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 1       5  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 1.1     4  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 1.2     5  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 1.3     5  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 1.4     6  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 1.5    14 -5.000  6.0000 -1.000  1.000+0i  3.0000+0i  2.0000+0i 7.889e-31
## 1.6    22  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 1.7    10  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 1.8    13  1.908  0.8762  1.137 -1.137+0i -0.7706+0i -1.1370+0i 9.702e+01
## 1.9     8  2.738 -1.5084  3.208 -3.208+0i  0.4702+0i -3.2083+0i 7.896e+01
## 2       7  2.738 -1.5084  3.208 -3.208+0i  0.4702+0i -3.2083+0i 7.896e+01
## 2.1     6  2.738 -1.5084  3.208 -3.208+0i  0.4702+0i -3.2083+0i 7.896e+01
## 2.2     6  2.738 -1.5084  3.208 -3.208+0i  0.4702+0i -3.2083+0i 7.896e+01
## 2.3     6  2.738 -1.5084  3.208 -3.208+0i  0.4702+0i -3.2083+0i 7.896e+01
## 2.4     6  2.738 -1.5084  3.208 -3.208+0i  0.4702+0i -3.2083+0i 7.896e+01
## 2.5     6  2.738 -1.5084  3.208 -3.208+0i  0.4702+0i -3.2083+0i 7.896e+01
## 2.6     5  2.738 -1.5084  3.208 -3.208+0i  0.4702+0i -3.2083+0i 7.896e+01
```

```
## 2.7     5  2.738 -1.5084  3.208 -3.208+0i  0.4702+0i -3.2083+0i 7.896e+01
## 2.8     5  2.738 -1.5084  3.208 -3.208+0i  0.4702+0i -3.2083+0i 7.896e+01
## 2.9     5  2.738 -1.5084  3.208 -3.208+0i  0.4702+0i -3.2083+0i 7.896e+01
## 3       5  2.738 -1.5084  3.208 -3.208+0i  0.4702+0i -3.2083+0i 7.896e+01
##           rate
## -3   0.000e+00
## -2.9 4.454e-07
## -2.8 1.360e-06
## -2.7 8.862e-07
## -2.6 5.317e-07
## -2.5 4.778e-08
## -2.4 2.939e-07
## -2.3 6.374e-08
## -2.2 9.645e-08
## -2.1 2.877e-05
## -2   0.000e+00
## -1.9 6.432e-06
## -1.8 7.312e-07
## -1.7 0.000e+00
## -1.6 1.416e-07
## -1.5 2.414e-07
## -1.4 2.276e-06
## -1.3 1.305e-07
## -1.2 8.908e-07
## -1.1 4.193e-08
## -1   0.000e+00
## -0.9 1.926e-06
## -0.8 0.000e+00
## -0.7 0.000e+00
## -0.6 1.663e-06
## -0.5 0.000e+00
## -0.4 1.177e-06
## -0.3 0.000e+00
## -0.2 0.000e+00
## -0.1 3.224e-07
## 0    5.365e-07
## 0.1  1.304e-06
## 0.2  5.200e-07
## 0.3  0.000e+00
## 0.4  1.127e-08
## 0.5  2.856e-08
## 0.6  5.847e-07
## 0.7  5.050e-07
## 0.8  8.388e-07
## 0.9  1.735e-07
## 1    8.765e-09
## 1.1  8.847e-07
## 1.2  4.798e-07
## 1.3  5.500e-07
## 1.4  1.467e-08
## 1.5  3.719e-07
## 1.6  5.217e-08
## 1.7  2.017e-08
## 1.8  1.073e-06
```

```
## 1.9  2.556e-08
## 2    6.649e-08
## 2.1  7.936e-07
## 2.2  4.851e-08
## 2.3  3.461e-08
## 2.4  1.927e-07
## 2.5  7.392e-07
## 2.6  2.648e-06
## 2.7  5.202e-07
## 2.8  1.184e-07
## 2.9  1.985e-08
## 3    1.509e-07
```

3.2 Example 2: $f(x) = (x-1)^2(x-2)$

The cubic $f(x) = (x-1)^2(x-2)$ with (p,q,r) equal to -4, 5, -2, has two distinct real roots, one of which is a double root. $(\alpha,\beta,\gamma) = (-3,2,-1)$



3.2.1 Newton

```
##      itel alpha beta gamma root1 root2 root3     sigma      rate
## -3      7    -2    1    -2  2+0i  1+0i  1+0i 9.861e-32 4.474e-07
## -2.9    7    -2    1    -2  2+0i  1+0i  1-0i 0.000e+00 1.458e-05
## -2.8    6    -2    1    -2  2+0i  1+0i  1+0i 0.000e+00 4.599e-06
## -2.7    6    -2    1    -2  2+0i  1+0i  1-0i 0.000e+00 1.601e-06
## -2.6    6    -2    1    -2  2+0i  1+0i  1-0i 0.000e+00 4.275e-07
## -2.5    6    -2    1    -2  2+0i  1+0i  1+0i 0.000e+00 9.685e-08
## -2.4    6    -2    1    -2  2+0i  1+0i  1+0i 0.000e+00 0.000e+00
## -2.3    6    -2    1    -2  2+0i  1+0i  1+0i 0.000e+00 4.619e-06
## -2.2    5    -2    1    -2  2+0i  1+0i  1+0i 2.465e-32 1.693e-06
## -2.1    5    -2    1    -2  2+0i  1+0i  1+0i 3.944e-31 1.104e-07
```

```
## -2       1    -2    1    -2  2+0i  1+0i  1-0i 0.000e+00 0.000e+00
## -1.9     5    -2    1    -2  2+0i  1+0i  1-0i 4.930e-31 1.207e-07
## -1.8     6    -2    1    -2  2+0i  1+0i  1+0i 0.000e+00 0.000e+00
## -1.7     7    -2    1    -2  2+0i  1+0i  1+0i 0.000e+00 1.804e-07
## -1.6    10    -2    1    -2  2+0i  1+0i  1+0i 2.465e-32 4.890e-07
## -1.5    22    -3    2    -1  1+0i  2+0i  1+0i 1.817e-29 4.843e-08
## -1.4    22    -3    2    -1  1+0i  2+0i  1+0i 6.040e-30 3.443e-08
## -1.3    22    -3    2    -1  1+0i  2+0i  1+0i 7.124e-30 3.667e-08
## -1.2    22    -3    2    -1  1+0i  2+0i  1+0i 4.733e-30 3.472e-08
## -1.1    21    -3    2    -1  1+0i  2+0i  1+0i 8.677e-30 4.151e-08
## -1       1    -3    2    -1  1+0i  2+0i  1+0i 0.000e+00 0.000e+00
## -0.9    21    -3    2    -1  1+0i  2+0i  1+0i 2.061e-29 5.038e-08
## -0.8    22    -3    2    -1  1+0i  2+0i  1+0i 2.507e-29 5.443e-08
## -0.7    23    -3    2    -1  1+0i  2+0i  1+0i 1.008e-29 4.417e-08
## -0.6    23    -3    2    -1  1+0i  2+0i  1+0i 4.144e-29 6.049e-08
## -0.5    24    -3    2    -1  1+0i  2+0i  1+0i 6.040e-30 3.947e-08
## -0.4    24    -3    2    -1  1+0i  2+0i  1+0i 1.817e-29 4.764e-08
## -0.3    24    -3    2    -1  1+0i  2+0i  1+0i 3.306e-29 5.660e-08
## -0.2    25    -3    2    -1  1+0i  2+0i  1+0i 4.363e-30 3.356e-08
## -0.1    25    -3    2    -1  1+0i  2+0i  1+0i 5.152e-30 3.617e-08
## 0       25    -3    2    -1  1+0i  2+0i  1+0i 7.593e-30 3.917e-08
## 0.1     25    -3    2    -1  1+0i  2+0i  1+0i 1.008e-29 4.284e-08
## 0.2     25    -3    2    -1  1+0i  2+0i  1+0i 1.469e-29 4.525e-08
## 0.3     25    -3    2    -1  1+0i  2+0i  1+0i 1.597e-29 4.763e-08
## 0.4     25    -3    2    -1  1+0i  2+0i  1+0i 1.745e-29 4.935e-08
## 0.5     25    -3    2    -1  1+0i  2+0i  1+0i 1.893e-29 4.984e-08
## 0.6     25    -3    2    -1  1+0i  2+0i  1+0i 1.403e-29 4.754e-08
## 0.7     25    -3    2    -1  1+0i  2+0i  1+0i 6.508e-30 3.602e-08
## 0.8     23    -3    2    -1  1+0i  2+0i  1+0i 3.673e-30 3.301e-08
## 0.9     24    -3    2    -1  1+0i  2+0i  1+0i 3.599e-29 5.972e-08
## 1        3    -2    1    -2  2+0i  1+0i  1+0i 0.000e+00 1.404e-15
## 1.1      9    -2    1    -2  2+0i  1+0i  1+0i 9.861e-32 3.168e-05
## 1.2      9    -2    1    -2  2+0i  1+0i  1+0i 2.465e-32 2.174e-06
## 1.3     10    -2    1    -2  2+0i  1+0i  1+0i 2.465e-32 6.430e-08
## 1.4     10    -2    1    -2  2+0i  1+0i  1-0i 0.000e+00 2.685e-06
## 1.5     11    -2    1    -2  2+0i  1+0i  1+0i 9.861e-32 1.152e-07
## 1.6     11    -2    1    -2  2+0i  1+0i  1+0i 2.465e-32 6.381e-08
## 1.7     11    -2    1    -2  2+0i  1+0i  1+0i 0.000e+00 4.854e-07
## 1.8     11    -2    1    -2  2+0i  1+0i  1+0i 2.465e-32 1.519e-06
## 1.9     11    -2    1    -2  2+0i  1+0i  1+0i 0.000e+00 2.571e-06
## 2       11    -2    1    -2  2+0i  1+0i  1+0i 2.465e-32 2.822e-06
## 2.1     11    -2    1    -2  2+0i  1+0i  1-0i 0.000e+00 2.280e-06
## 2.2     11    -2    1    -2  2+0i  1+0i  1-0i 4.930e-31 1.483e-06
## 2.3     11    -2    1    -2  2+0i  1+0i  1+0i 0.000e+00 8.236e-07
## 2.4     11    -2    1    -2  2+0i  1+0i  1+0i 2.465e-32 4.130e-07
## 2.5     11    -2    1    -2  2+0i  1+0i  1+0i 9.861e-32 1.861e-07
## 2.6     11    -2    1    -2  2+0i  1+0i  1-0i 0.000e+00 9.310e-08
## 2.7     11    -2    1    -2  2+0i  1+0i  1-0i 0.000e+00 3.127e-08
## 2.8     11    -2    1    -2  2+0i  1+0i  1+0i 2.465e-32 1.164e-07
## 2.9     11    -2    1    -2  2+0i  1+0i  1+0i 3.944e-31 3.766e-08
## 3       11    -2    1    -2  2+0i  1+0i  1+0i 9.861e-32 1.407e-06
```

3.2.2 ALS

```
##      itel  alpha  beta   gamma     root1         root2          root3     sigma
## 0   10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 0.1 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 0.2 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 0.3 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 0.4 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 0.5 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 0.6 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.373e-07
## 0.7 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.371e-07
## 0.8 10000 -3.029 2.059 -0.9710 0.9710+0i 1.998+0.0000i 1.0308+0.0000i 1.322e-07
## 0.9 10000 -2.963 1.929 -1.0364 1.0364+0i 1.997+0.0000i 0.9661+0.0000i 2.526e-07
## 1    2532 -2.000 1.000 -2.0000 2.0000+0i 1.000+0.0000i 0.9999+0.0000i 8.391e-18
## 1.1  1966 -2.000 1.000 -2.0000 2.0000+0i 1.000+0.0001i 1.0000-0.0001i 8.442e-18
## 1.2  2006 -2.000 1.000 -2.0000 2.0000+0i 1.000+0.0001i 1.0000-0.0001i 8.376e-18
## 1.3  2010 -2.000 1.000 -2.0000 2.0000+0i 1.000+0.0001i 1.0000-0.0001i 8.425e-18
## 1.4  2008 -2.000 1.000 -2.0000 2.0000+0i 1.000+0.0001i 1.0000-0.0001i 8.490e-18
## 1.5  1991 -2.000 1.000 -2.0000 2.0000+0i 1.000+0.0001i 1.0000-0.0001i 8.444e-18
## 1.6  2290 -2.000 1.000 -2.0000 2.0000+0i 1.000+0.0000i 0.9999+0.0000i 8.450e-18
## 1.7 10000 -2.965 1.932 -1.0351 1.0351+0i 1.997+0.0000i 0.9673+0.0000i 2.189e-07
## 1.8 10000 -3.029 2.060 -0.9708 0.9708+0i 1.998+0.0000i 1.0310+0.0000i 1.369e-07
## 1.9 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 2   10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 2.1 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 2.2 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 2.3 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 2.4 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 2.5 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 2.6 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 2.7 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 2.8 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 2.9 10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
## 3   10000 -3.029 2.060 -0.9707 0.9707+0i 1.998+0.0000i 1.0311+0.0000i 1.374e-07
##       rate  trate
## 0   0.9999 0.9999
## 0.1 0.9999 0.9999
## 0.2 0.9999 0.9999
## 0.3 0.9999 0.9999
## 0.4 0.9999 0.9999
## 0.5 0.9999 0.9999
## 0.6 0.9999 0.9999
## 0.7 0.9999 0.9999
## 0.8 0.9999 0.9999
## 0.9 0.9998 0.9998
## 1   0.9920 0.9921
## 1.1 0.9921 0.9921
## 1.2 0.9921 0.9921
## 1.3 0.9920 0.9921
## 1.4 0.9921 0.9921
## 1.5 0.9921 0.9921
## 1.6 0.9921 0.9921
## 1.7 0.9999 0.9999
## 1.8 0.9999 0.9999
```

```
## 1.9 0.9999 0.9999
## 2   0.9999 0.9999
## 2.1 0.9999 0.9999
## 2.2 0.9999 0.9999
## 2.3 0.9999 0.9999
## 2.4 0.9999 0.9999
## 2.5 0.9999 0.9999
## 2.6 0.9999 0.9999
## 2.7 0.9999 0.9999
## 2.8 0.9999 0.9999
## 2.9 0.9999 0.9999
## 3   0.9999 0.9999
```

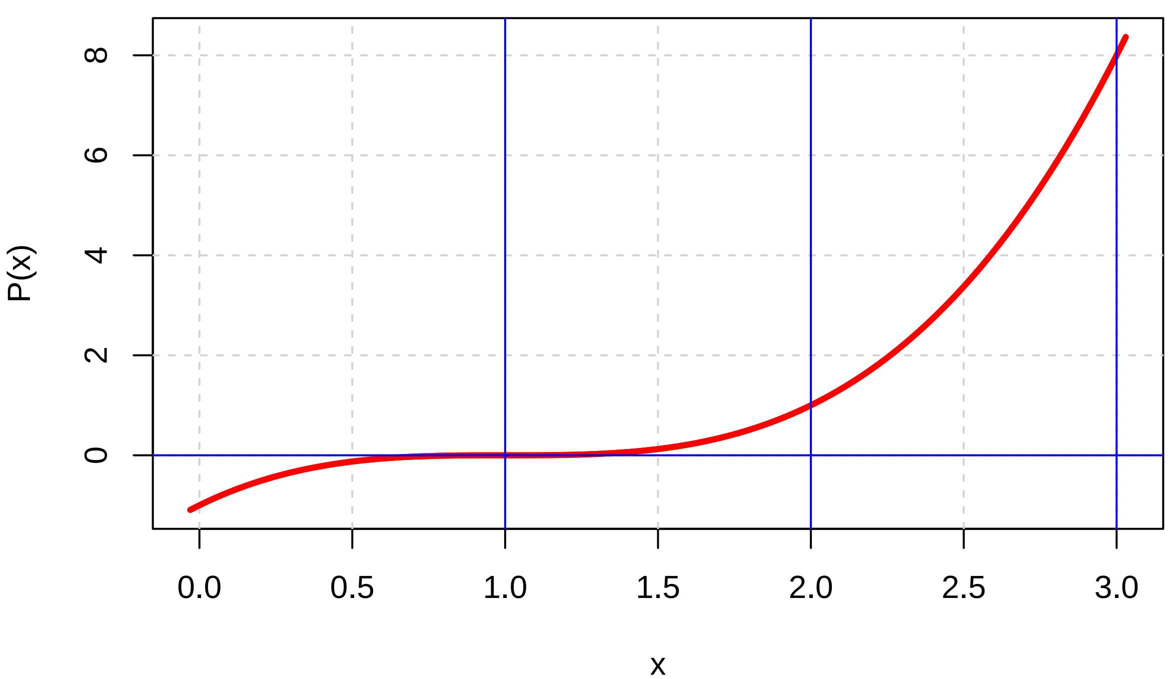
3.2.3 newtonLS

```
##      itel  alpha  beta   gamma     root1    root2     root3     sigma      rate
## -3     10 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000+0i 4.191e-31 4.400e-07
## -2.9   10 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000+0i 2.465e-32 4.332e-07
## -2.8   10 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000+0i 3.944e-31 1.336e-06
## -2.7    9 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000+0i 9.861e-32 4.512e-06
## -2.6    9 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000+0i 0.000e+00 2.528e-07
## -2.5    9 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000-0i 0.000e+00 5.688e-07
## -2.4    8 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000+0i 9.861e-32 2.083e-06
## -2.3    8 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000-0i 3.944e-31 3.529e-06
## -2.2    7 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000+0i 9.861e-32 5.488e-07
## -2.1    6 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000+0i 0.000e+00 1.120e-06
## -2      1 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000-0i 0.000e+00 0.000e+00
## -1.9    7 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000-0i 0.000e+00 1.902e-05
## -1.8   24 -2.999 1.999 -1.0007 1.0007+0i 2.000+0i 0.9993+0i 3.316e-14 2.438e-09
## -1.7    7 -2.375 1.250 -1.5883 1.5883+0i 1.588+0i 0.7871+0i 1.027e-03 1.092e-05
## -1.6    4 -2.375 1.250 -1.5883 1.5883+0i 1.588+0i 0.7871+0i 1.027e-03 1.293e-06
## -1.5    9 -2.375 1.250 -1.5883 1.5883+0i 1.588+0i 0.7871+0i 1.027e-03 1.291e-06
## -1.4    8 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000+0i 0.000e+00 0.000e+00
## -1.3   21 -3.001 2.001 -0.9993 0.9993+0i 2.000+0i 1.0007+0i 4.566e-14 5.290e-09
## -1.2   22 -2.999 1.999 -1.0006 1.0006+0i 2.000+0i 0.9994+0i 2.413e-14 3.573e-09
## -1.1   20 -2.999 1.999 -1.0007 1.0007+0i 2.000+0i 0.9993+0i 4.092e-14 7.120e-09
## -1      1 -3.000 2.000 -1.0000 1.0000+0i 2.000+0i 1.0000+0i 0.000e+00 0.000e+00
## -0.9   21 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 2.378e-14 6.677e-09
## -0.8   24 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 1.700e-14 6.684e-09
## -0.7   25 -3.001 2.001 -0.9993 0.9993+0i 2.000+0i 1.0007+0i 3.146e-14 1.472e-09
## -0.6   26 -3.001 2.001 -0.9993 0.9993+0i 2.000+0i 1.0007+0i 3.581e-14 8.480e-09
## -0.5   27 -3.001 2.001 -0.9993 0.9993+0i 2.000+0i 1.0007+0i 3.094e-14 3.182e-09
## -0.4   28 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 2.229e-14 9.683e-09
## -0.3   28 -3.001 2.001 -0.9993 0.9993+0i 2.000+0i 1.0007+0i 4.071e-14 3.011e-09
## -0.2   29 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 2.323e-14 4.019e-09
## -0.1   29 -3.001 2.001 -0.9993 0.9993+0i 2.000+0i 1.0007+0i 3.520e-14 1.598e-09
## 0      30 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 1.698e-14 1.731e-09
## 0.1    30 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 2.120e-14 3.959e-09
## 0.2    30 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 2.031e-14 5.871e-09
## 0.3    26 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 2.967e-14 1.795e-09
## 0.4    32 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 1.642e-14 1.478e-09
## 0.5   115 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000+0i 9.861e-32 3.717e-07
## 0.6   229 -2.999 1.999 -1.0006 1.0006+0i 2.000+0i 0.9994+0i 2.392e-14 5.901e-09
```

```
## 0.7    23 -2.375 1.250 -1.5883 1.5883+0i 1.588+0i 0.7871+0i 1.027e-03 1.178e-06
## 0.8    35 -2.375 1.250 -1.5883 1.5883+0i 1.588+0i 0.7871+0i 1.027e-03 1.715e-06
## 0.9    79 -2.999 1.999 -1.0007 1.0007+0i 2.000+0i 0.9993+0i 4.307e-14 5.263e-09
## 1      81 -2.375 1.250 -1.5883 1.5883+0i 1.588+0i 0.7871+0i 1.027e-03 1.650e-05
## 1.1    36 -2.999 1.999 -1.0006 1.0006+0i 2.000+0i 0.9994+0i 2.838e-14 3.732e-09
## 1.2    28 -2.375 1.250 -1.5883 1.5883+0i 1.588+0i 0.7871+0i 1.027e-03 2.894e-08
## 1.3    40 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 2.594e-14 2.468e-09
## 1.4    23 -2.375 1.250 -1.5883 1.5883+0i 1.588+0i 0.7871+0i 1.027e-03 2.350e-07
## 1.5    26 -2.375 1.250 -1.5883 1.5883+0i 1.588+0i 0.7871+0i 1.027e-03 4.958e-07
## 1.6    26 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 2.436e-14 1.018e-08
## 1.7    30 -2.375 1.250 -1.5883 1.5883+0i 1.588+0i 0.7871+0i 1.027e-03 5.439e-08
## 1.8    59 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 2.040e-14 3.735e-09
## 1.9    75 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 1.790e-14 1.188e-08
## 2      19 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000-0i 0.000e+00 0.000e+00
## 2.1    13 -2.375 1.250 -1.5883 1.5883+0i 1.588+0i 0.7871+0i 1.027e-03 6.309e-06
## 2.2    52 -2.999 1.999 -1.0005 1.0005+0i 2.000+0i 0.9995+0i 1.484e-14 1.200e-09
## 2.3    28 -2.000 1.000 -2.0000 2.0000+0i 1.000+0i 1.0000-0i 0.000e+00 2.642e-07
## 2.4    47 -2.999 1.999 -1.0007 1.0007+0i 2.000+0i 0.9993+0i 3.066e-14 2.722e-09
## 2.5    38 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 2.928e-14 4.692e-10
## 2.6    40 -3.001 2.001 -0.9993 0.9993+0i 2.000+0i 1.0007+0i 4.300e-14 8.868e-09
## 2.7    32 -3.001 2.001 -0.9993 0.9993+0i 2.000+0i 1.0007+0i 3.665e-14 3.271e-09
## 2.8    66 -2.375 1.250 -1.5883 1.5883+0i 1.588+0i 0.7871+0i 1.027e-03 3.670e-08
## 2.9    43 -3.001 2.001 -0.9993 0.9993+0i 2.000+0i 1.0007+0i 3.054e-14 8.981e-09
## 3      33 -3.001 2.001 -0.9994 0.9994+0i 2.000+0i 1.0006+0i 2.634e-14 2.374e-09
```

3.3 Example 3: $f(x) = (x-1)^3$

The cubic $f(x) = (x-1)^3$, with (p, q, r) equal to -3, 3, -1, has a single triple real root. $(\alpha, \beta, \gamma) = (-2, 1, -1)$



3.3.1 Newton

```
##      itel alpha   beta   gamma     root1          root2          root3
## -3     22    -2 0.9999 -1.0001 1.0001+0i 0.9999+0.0001i 0.9999-0.0001i
## -2.9   22    -2 0.9999 -1.0001 1.0001+0i 0.9999+0.0001i 0.9999-0.0001i
## -2.8   20    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## -2.7   20    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## -2.6   20    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## -2.5   20    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## -2.4   20    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## -2.3   20    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0001i 0.9999-0.0001i
## -2.2   20    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0001i 0.9999-0.0001i
## -2.1   20    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0001i 0.9999-0.0001i
## -2     20    -2 0.9999 -1.0001 1.0001+0i 0.9999+0.0001i 0.9999-0.0001i
## -1.9   21    -2 0.9999 -1.0001 1.0001+0i 0.9999+0.0001i 0.9999-0.0001i
## -1.8   19    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## -1.7   19    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## -1.6   19    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0001i 0.9999-0.0001i
## -1.5   20    -2 0.9999 -1.0001 1.0001+0i 0.9999+0.0001i 0.9999-0.0001i
## -1.4   18    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## -1.3   18    -2 0.9999 -1.0001 1.0001+0i 0.9999+0.0001i 0.9999-0.0001i
## -1.2   17    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0001i 0.9999-0.0001i
## -1.1   17    -2 0.9999 -1.0001 1.0001+0i 0.9999+0.0001i 0.9999-0.0001i
## -1      1    -2 1.0000 -1.0000 1.0000+0i 1.0000+0.0000i 1.0000+0.0000i
## -0.9   17    -2 1.0001 -0.9999 0.9999+0i 1.0001+0.0000i 1.0001-0.0000i
## -0.8   17    -2 1.0002 -0.9998 0.9998+0i 1.0001+0.0000i 1.0001-0.0000i
## -0.7   18    -2 1.0002 -0.9998 0.9998+0i 1.0001+0.0000i 1.0001-0.0000i
## -0.6   20    -2 1.0001 -0.9999 0.9999+0i 1.0001+0.0000i 1.0001-0.0000i
## -0.5   19    -2 1.0002 -0.9998 0.9998+0i 1.0001+0.0000i 1.0001-0.0000i
## -0.4   19    -2 1.0002 -0.9998 0.9998+0i 1.0001+0.0000i 1.0001-0.0000i
## -0.3   20    -2 1.0001 -0.9999 0.9999+0i 1.0001+0.0000i 1.0001-0.0000i
## -0.2   20    -2 1.0002 -0.9998 0.9998+0i 1.0001+0.0000i 1.0001-0.0000i
## -0.1   20    -2 1.0002 -0.9998 0.9998+0i 1.0001+0.0000i 1.0001-0.0000i
## 0      20    -2 1.0002 -0.9998 0.9998+0i 1.0001+0.0000i 1.0001-0.0000i
## 0.1    20    -2 1.0002 -0.9998 0.9998+0i 1.0001+0.0000i 1.0001-0.0000i
## 0.2    22    -2 1.0001 -0.9999 0.9999+0i 1.0001+0.0000i 1.0001-0.0000i
## 0.3    21    -2 1.0001 -0.9999 0.9999+0i 1.0001+0.0000i 1.0001-0.0000i
## 0.4    21    -2 1.0002 -0.9998 0.9998+0i 1.0001+0.0000i 1.0001-0.0000i
## 0.5    21    -2 1.0002 -0.9998 0.9998+0i 1.0001+0.0000i 1.0001-0.0000i
## 0.6    21    -2 1.0002 -0.9998 0.9998+0i 1.0001+0.0000i 1.0001-0.0000i
## 0.7    21    -2 1.0002 -0.9998 0.9998+0i 1.0001+0.0000i 1.0001-0.0000i
## 0.8    22    -2 1.0001 -0.9999 0.9999+0i 1.0001+0.0000i 1.0001-0.0000i
## 0.9    20    -2 1.0001 -0.9999 0.9999+0i 1.0001+0.0000i 1.0001-0.0000i
## 1       3    -2 1.0000 -1.0000 1.0000+0i 1.0000+0.0000i 1.0000-0.0000i
## 1.1    20    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 1.2    21    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 1.3    22    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0001i 0.9999-0.0001i
## 1.4    22    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 1.5    23    -2 0.9999 -1.0001 1.0001+0i 0.9999+0.0001i 0.9999-0.0001i
## 1.6    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0001i 0.9999-0.0001i
## 1.7    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0001i 0.9999-0.0001i
## 1.8    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 1.9    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 2      23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
```

```
## 2.1    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 2.2    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 2.3    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 2.4    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 2.5    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 2.6    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 2.7    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 2.8    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 2.9    23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
## 3      23    -2 0.9998 -1.0002 1.0002+0i 0.9999+0.0002i 0.9999-0.0002i
##          sigma      rate
## -3   1.600e-24 2.455e-08
## -2.9 1.367e-24 3.314e-08
## -2.8 1.313e-23 1.781e-08
## -2.7 1.090e-23 1.093e-08
## -2.6 8.903e-24 1.701e-08
## -2.5 7.126e-24 2.925e-08
## -2.4 5.575e-24 1.515e-08
## -2.3 4.245e-24 2.735e-08
## -2.2 3.132e-24 7.646e-09
## -2.1 2.224e-24 1.519e-08
## -2   1.508e-24 1.576e-08
## -1.9 1.521e-24 2.904e-08
## -1.8 1.033e-23 2.287e-08
## -1.7 5.618e-24 4.817e-08
## -1.6 2.704e-24 7.804e-09
## -1.5 1.732e-24 2.537e-08
## -1.4 6.272e-24 1.031e-08
## -1.3 1.349e-24 3.035e-08
## -1.2 2.556e-24 8.346e-09
## -1.1 1.345e-24 7.798e-08
## -1   0.000e+00 0.000e+00
## -0.9 1.828e-24 3.663e-08
## -0.8 4.670e-24 1.877e-08
## -0.7 3.249e-24 1.373e-08
## -0.6 1.698e-24 4.460e-08
## -0.5 4.195e-24 2.172e-08
## -0.4 1.229e-23 7.699e-09
## -0.3 1.631e-24 2.069e-08
## -0.2 3.314e-24 3.404e-08
## -0.1 5.951e-24 3.749e-08
## 0    9.691e-24 9.869e-09
## 0.1  1.461e-23 1.365e-08
## 0.2  1.815e-24 6.255e-08
## 0.3  1.544e-24 1.863e-08
## 0.4  1.956e-24 3.150e-08
## 0.5  2.298e-24 1.116e-08
## 0.6  2.388e-24 8.209e-09
## 0.7  1.948e-24 1.319e-08
## 0.8  1.421e-24 2.859e-08
## 0.9  1.330e-24 1.777e-08
## 1    0.000e+00 1.766e-15
## 1.1  5.074e-24 3.244e-08
## 1.2  1.077e-23 2.855e-08
```

```
## 1.3  3.561e-24 2.406e-08
## 1.4  1.104e-23 1.432e-08
## 1.5  1.372e-24 1.544e-08
## 1.6  2.467e-24 8.805e-09
## 1.7  3.779e-24 1.112e-08
## 1.8  5.118e-24 1.550e-08
## 1.9  6.295e-24 1.553e-08
## 2    7.186e-24 2.175e-08
## 2.1  7.746e-24 1.827e-08
## 2.2  7.999e-24 2.296e-08
## 2.3  7.996e-24 1.557e-08
## 2.4  7.807e-24 1.495e-08
## 2.5  7.495e-24 1.769e-08
## 2.6  7.108e-24 1.944e-08
## 2.7  6.684e-24 1.607e-08
## 2.8  6.251e-24 1.548e-08
## 2.9  5.827e-24 1.955e-08
## 3    5.423e-24 9.154e-09
```

3.3.2 ALS

```
##      itel  alpha   beta   gamma     root1          root2          root3
## 0   10000 -2.106 1.1181 -0.8939 0.8939+0i 1.0528+0.0982i 1.0528-0.0982i
## 0.1 10000 -2.106 1.1181 -0.8939 0.8939+0i 1.0528+0.0982i 1.0528-0.0982i
## 0.2 10000 -2.106 1.1181 -0.8939 0.8939+0i 1.0528+0.0982i 1.0528-0.0982i
## 0.3 10000 -2.106 1.1181 -0.8939 0.8939+0i 1.0528+0.0982i 1.0528-0.0982i
## 0.4 10000 -2.106 1.1181 -0.8939 0.8939+0i 1.0528+0.0982i 1.0528-0.0982i
## 0.5 10000 -2.106 1.1181 -0.8939 0.8939+0i 1.0528+0.0982i 1.0528-0.0982i
## 0.6 10000 -2.106 1.1181 -0.8939 0.8939+0i 1.0528+0.0982i 1.0528-0.0982i
## 0.7 10000 -2.106 1.1181 -0.8939 0.8939+0i 1.0528+0.0982i 1.0528-0.0982i
## 0.8 10000 -2.106 1.1181 -0.8939 0.8939+0i 1.0528+0.0982i 1.0528-0.0982i
## 0.9 10000 -2.106 1.1179 -0.8941 0.8941+0i 1.0528+0.0980i 1.0528-0.0980i
## 1   10000 -2.075 1.0808 -0.9251 0.9251+0i 1.0374+0.0680i 1.0374-0.0680i
## 1.1 10000 -1.885 0.8970 -1.1153 1.1153+0i 0.9426+0.0921i 0.9426-0.0921i
## 1.2 10000 -1.884 0.8956 -1.1170 1.1170+0i 0.9418+0.0933i 0.9418-0.0933i
## 1.3 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 1.4 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 1.5 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 1.6 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 1.7 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 1.8 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 1.9 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 2   10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 2.1 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 2.2 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 2.3 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 2.4 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 2.5 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 2.6 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 2.7 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 2.8 10000 -1.883 0.8956 -1.1171 1.1171+0i 0.9417+0.0934i 0.9417-0.0934i
## 2.9 10000 -1.883 0.8956 -1.1170 1.1170+0i 0.9417+0.0934i 0.9417-0.0934i
## 3   10000 -1.883 0.8956 -1.1170 1.1170+0i 0.9417+0.0934i 0.9417-0.0934i
```

```
##         sigma   rate  trate
## 0   2.925e-07 0.9999 0.9999
## 0.1 2.925e-07 0.9999 0.9999
## 0.2 2.925e-07 0.9999 0.9999
## 0.3 2.925e-07 0.9999 0.9999
## 0.4 2.925e-07 0.9999 0.9999
## 0.5 2.925e-07 0.9999 0.9999
## 0.6 2.924e-07 0.9999 0.9999
## 0.7 2.923e-07 0.9999 0.9999
## 0.8 2.920e-07 0.9999 0.9999
## 0.9 2.893e-07 0.9999 0.9999
## 1   3.411e-08 1.0000 1.0000
## 1.1 3.094e-07 0.9999 0.9999
## 1.2 3.369e-07 0.9999 0.9999
## 1.3 3.381e-07 0.9999 0.9999
## 1.4 3.383e-07 0.9999 0.9999
## 1.5 3.384e-07 0.9999 0.9999
## 1.6 3.384e-07 0.9999 0.9999
## 1.7 3.384e-07 0.9999 0.9999
## 1.8 3.384e-07 0.9999 0.9999
## 1.9 3.384e-07 0.9999 0.9999
## 2   3.384e-07 0.9999 0.9999
## 2.1 3.383e-07 0.9999 0.9999
## 2.2 3.383e-07 0.9999 0.9999
## 2.3 3.383e-07 0.9999 0.9999
## 2.4 3.382e-07 0.9999 0.9999
## 2.5 3.382e-07 0.9999 0.9999
## 2.6 3.381e-07 0.9999 0.9999
## 2.7 3.381e-07 0.9999 0.9999
## 2.8 3.380e-07 0.9999 0.9999
## 2.9 3.380e-07 0.9999 0.9999
## 3   3.379e-07 0.9999 0.9999
```

3.3.3 newtonLS

```
##      itel  alpha   beta   gamma     root1           root2           root3
## -3     38 -1.985 0.9850 -1.0152 1.0152+0i 0.9924+0.0130i 0.9924-0.0130i
## -2.9   38 -1.985 0.9853 -1.0149 1.0149+0i 0.9925+0.0128i 0.9925-0.0128i
## -2.8   38 -1.985 0.9856 -1.0146 1.0146+0i 0.9927+0.0125i 0.9927-0.0125i
## -2.7   38 -1.986 0.9859 -1.0143 1.0143+0i 0.9929+0.0122i 0.9929-0.0122i
## -2.6   38 -1.986 0.9863 -1.0139 1.0139+0i 0.9931+0.0119i 0.9931-0.0119i
## -2.5   36 -1.983 0.9834 -1.0169 1.0169+0i 0.9916+0.0144i 0.9916-0.0144i
## -2.4   36 -1.984 0.9840 -1.0163 1.0163+0i 0.9918+0.0140i 0.9918-0.0140i
## -2.3   36 -1.984 0.9845 -1.0157 1.0157+0i 0.9922+0.0135i 0.9922-0.0135i
## -2.2   36 -1.985 0.9852 -1.0150 1.0150+0i 0.9925+0.0129i 0.9925-0.0129i
## -2.1   36 -1.986 0.9859 -1.0143 1.0143+0i 0.9928+0.0123i 0.9928-0.0123i
## -2     34 -1.983 0.9834 -1.0169 1.0169+0i 0.9915+0.0145i 0.9915-0.0145i
## -1.9   34 -1.984 0.9844 -1.0158 1.0158+0i 0.9921+0.0135i 0.9921-0.0135i
## -1.8   34 -1.985 0.9856 -1.0146 1.0146+0i 0.9927+0.0125i 0.9927-0.0125i
## -1.7   32 -1.983 0.9836 -1.0167 1.0167+0i 0.9917+0.0143i 0.9917-0.0143i
## -1.6   32 -1.985 0.9853 -1.0149 1.0149+0i 0.9926+0.0128i 0.9926-0.0128i
## -1.5   30 -1.984 0.9840 -1.0162 1.0162+0i 0.9919+0.0139i 0.9919-0.0139i
## -1.4   29 -1.986 0.9865 -1.0137 1.0137+0i 0.9932+0.0117i 0.9932-0.0117i
```

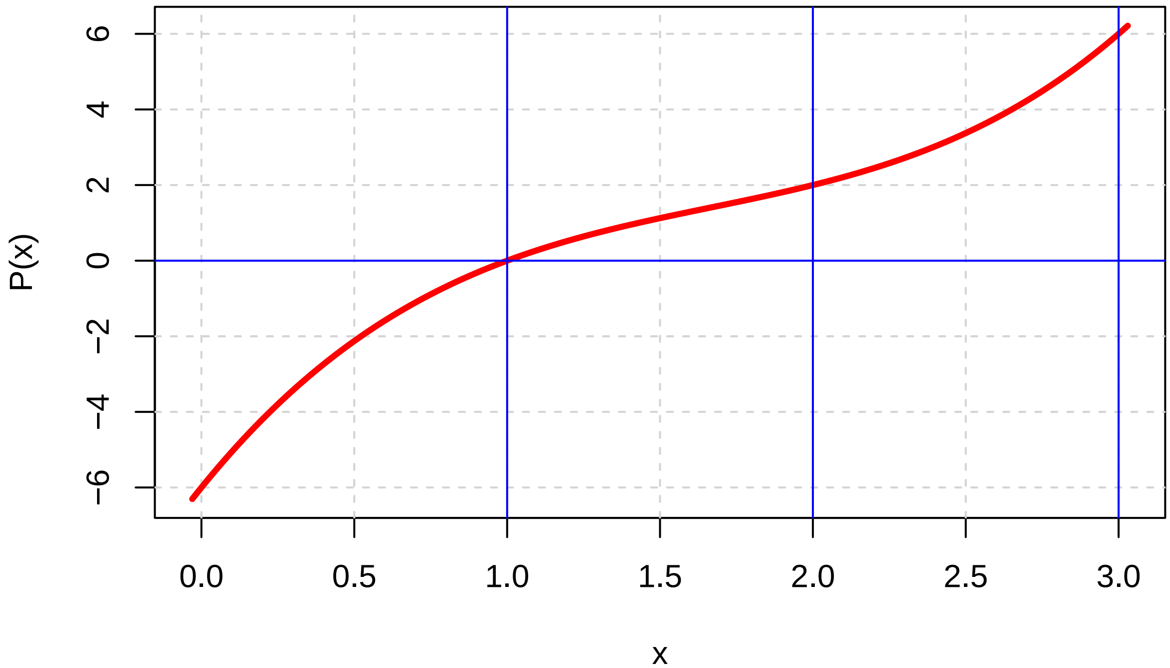
```
## -1.3   27 -1.986 0.9866 -1.0136 1.0136+0i 0.9932+0.0117i 0.9932-0.0117i
## -1.2   24 -1.985 0.9851 -1.0152 1.0152+0i 0.9924+0.0130i 0.9924-0.0130i
## -1.1   18 -1.984 0.9843 -1.0159 1.0159+0i 0.9920+0.0137i 0.9920-0.0137i
## -1      1 -2.000 1.0000 -1.0000 1.0000+0i 1.0000+0.0000i 1.0000+0.0000i
## -0.9   19 -2.014 1.0139 -0.9863 0.9863+0i 1.0068+0.0120i 1.0068-0.0120i
## -0.8   25 -2.014 1.0143 -0.9859 0.9859+0i 1.0070+0.0123i 1.0070-0.0123i
## -0.7   29 -2.014 1.0145 -0.9857 0.9857+0i 1.0072+0.0125i 1.0072-0.0125i
## -0.6   33 -2.014 1.0142 -0.9860 0.9860+0i 1.0070+0.0122i 1.0070-0.0122i
## -0.5   36 -2.017 1.0168 -0.9835 0.9835+0i 1.0083+0.0145i 1.0083-0.0145i
## -0.4   40 -2.015 1.0151 -0.9851 0.9851+0i 1.0074+0.0130i 1.0074-0.0130i
## -0.3   44 -2.015 1.0151 -0.9851 0.9851+0i 1.0074+0.0130i 1.0074-0.0130i
## -0.2   43 -2.015 1.0153 -0.9849 0.9849+0i 1.0075+0.0132i 1.0075-0.0132i
## -0.1   43 -2.015 1.0151 -0.9851 0.9851+0i 1.0075+0.0130i 1.0075-0.0130i
## 0      43 -2.015 1.0151 -0.9851 0.9851+0i 1.0075+0.0131i 1.0075-0.0131i
## 0.1    43 -2.015 1.0149 -0.9853 0.9853+0i 1.0074+0.0129i 1.0074-0.0129i
## 0.2    42 -2.014 1.0143 -0.9859 0.9859+0i 1.0070+0.0123i 1.0070-0.0123i
## 0.3    41 -2.015 1.0157 -0.9845 0.9845+0i 1.0077+0.0135i 1.0077-0.0135i
## 0.4    32 -2.015 1.0152 -0.9850 0.9850+0i 1.0075+0.0131i 1.0075-0.0131i
## 0.5    43 -2.014 1.0137 -0.9865 0.9865+0i 1.0068+0.0118i 1.0068-0.0118i
## 0.6    43 -2.014 1.0145 -0.9857 0.9857+0i 1.0071+0.0125i 1.0071-0.0125i
## 0.7    49 -2.015 1.0149 -0.9853 0.9853+0i 1.0074+0.0129i 1.0074-0.0129i
## 0.8    45 -1.985 0.9853 -1.0149 1.0149+0i 0.9926+0.0128i 0.9926-0.0128i
## 0.9    85 -2.016 1.0159 -0.9844 0.9844+0i 1.0078+0.0137i 1.0078-0.0137i
## 1      52 -1.986 0.9860 -1.0142 1.0142+0i 0.9929+0.0121i 0.9929-0.0121i
## 1.1    53 -2.014 1.0139 -0.9863 0.9863+0i 1.0069+0.0120i 1.0069-0.0120i
## 1.2    53 -1.983 0.9837 -1.0166 1.0166+0i 0.9917+0.0142i 0.9917-0.0142i
## 1.3    58 -2.016 1.0158 -0.9844 0.9844+0i 1.0078+0.0136i 1.0078-0.0136i
## 1.4    47 -2.016 1.0160 -0.9842 0.9842+0i 1.0079+0.0138i 1.0079-0.0138i
## 1.5    35 -2.015 1.0152 -0.9850 0.9850+0i 1.0075+0.0131i 1.0075-0.0131i
## 1.6    40 -2.014 1.0147 -0.9855 0.9855+0i 1.0072+0.0126i 1.0072-0.0126i
## 1.7    60 -1.985 0.9857 -1.0145 1.0145+0i 0.9927+0.0125i 0.9927-0.0125i
## 1.8    59 -1.983 0.9836 -1.0167 1.0167+0i 0.9917+0.0143i 0.9917-0.0143i
## 1.9    45 -1.984 0.9838 -1.0165 1.0165+0i 0.9918+0.0141i 0.9918-0.0141i
## 2     119 -1.986 0.9864 -1.0138 1.0138+0i 0.9931+0.0118i 0.9931-0.0118i
## 2.1    44 -2.016 1.0162 -0.9840 0.9840+0i 1.0080+0.0140i 1.0080-0.0140i
## 2.2    53 -1.985 0.9854 -1.0148 1.0148+0i 0.9926+0.0127i 0.9926-0.0127i
## 2.3    39 -1.985 0.9853 -1.0149 1.0149+0i 0.9925+0.0128i 0.9925-0.0128i
## 2.4    42 -1.984 0.9847 -1.0155 1.0155+0i 0.9922+0.0133i 0.9922-0.0133i
## 2.5    41 -1.985 0.9856 -1.0146 1.0146+0i 0.9927+0.0125i 0.9927-0.0125i
## 2.6    40 -1.985 0.9848 -1.0155 1.0155+0i 0.9923+0.0133i 0.9923-0.0133i
## 2.7    84 -2.016 1.0163 -0.9840 0.9840+0i 1.0080+0.0140i 1.0080-0.0140i
## 2.8    83 -2.015 1.0151 -0.9851 0.9851+0i 1.0074+0.0130i 1.0074-0.0130i
## 2.9    86 -1.984 0.9845 -1.0158 1.0158+0i 0.9921+0.0135i 0.9921-0.0135i
## 3      51 -1.986 0.9865 -1.0137 1.0137+0i 0.9932+0.0117i 0.9932-0.0117i
##          sigma      rate
## -3   2.011e-12 7.772e-07
## -2.9 1.792e-12 2.965e-07
## -2.8 1.576e-12 7.471e-07
## -2.7 1.366e-12 7.090e-07
## -2.6 1.165e-12 3.299e-06
## -2.5 3.719e-12 4.242e-07
## -2.4 3.039e-12 6.586e-07
## -2.3 2.418e-12 1.565e-06
## -2.2 1.865e-12 1.267e-06
```

```
## -2.1 1.386e-12 1.049e-06
## -2   3.757e-12 8.881e-07
## -1.9 2.524e-12 1.903e-07
## -1.8 1.581e-12 1.680e-06
## -1.7 3.463e-12 7.068e-07
## -1.6 1.771e-12 9.286e-07
## -1.5 2.967e-12 4.478e-07
## -1.4 1.051e-12 4.571e-06
## -1.3 1.017e-12 4.491e-06
## -1.2 1.975e-12 2.627e-07
## -1.1 2.647e-12 9.601e-07
## -1   0.000e+00 0.000e+00
## -0.9 1.120e-12 3.090e-06
## -0.8 1.347e-12 3.468e-06
## -0.7 1.487e-12 3.688e-06
## -0.6 1.272e-12 3.345e-06
## -0.5 3.495e-12 1.005e-06
## -0.4 1.865e-12 9.152e-07
## -0.3 1.859e-12 2.733e-07
## -0.2 2.006e-12 1.088e-06
## -0.1 1.882e-12 6.678e-07
## 0    1.896e-12 8.007e-07
## 0.1  1.745e-12 1.805e-06
## 0.2  1.344e-12 3.512e-06
## 0.3  2.356e-12 2.338e-06
## 0.4  1.967e-12 1.201e-06
## 0.5  1.041e-12 2.957e-06
## 0.6  1.465e-12 3.658e-06
## 0.7  1.741e-12 4.082e-06
## 0.8  1.762e-12 3.065e-07
## 0.9  2.492e-12 6.282e-07
## 1    1.302e-12 2.043e-06
## 1.1  1.156e-12 3.157e-06
## 1.2  3.351e-12 3.294e-07
## 1.3  2.441e-12 1.050e-06
## 1.4  2.657e-12 1.254e-06
## 1.5  1.965e-12 9.237e-07
## 1.6  1.566e-12 3.809e-06
## 1.7  1.535e-12 1.593e-06
## 1.8  3.461e-12 2.367e-07
## 1.9  3.229e-12 6.267e-07
## 2    1.114e-12 1.199e-06
## 2.1  2.850e-12 1.200e-06
## 2.2  1.716e-12 7.107e-07
## 2.3  1.786e-12 4.215e-07
## 2.4  2.263e-12 8.494e-07
## 2.5  1.585e-12 1.770e-06
## 2.6  2.213e-12 6.911e-07
## 2.7  2.886e-12 1.770e-06
## 2.8  1.851e-12 4.613e-07
## 2.9  2.508e-12 4.813e-07
## 3    1.056e-12 4.585e-06
```

3.4 Example 4: $f(x) = (x-1)(x^2-4x+6)$

The cubic $f(x) = (x-1)(x^2-4x+6)$, with (p, q, r) equal to -5, 10, -6, has a single real root and two conjugate complex roots.

$(\alpha, \beta, \gamma) = (-4, 6, -1)$



3.4.1 Newton

```
##      itel alpha beta gamma root1    root2    root3     sigma      rate
## -3      9    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 2.597e-08
## -2.9    7    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 0.000e+00
## -2.8    8    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 2.999e-07
## -2.7    9    -4    6    -1  1+0i 2+1.414i 2-1.414i 1.578e-30 4.239e-08
## -2.6    9    -4    6    -1  1+0i 2+1.414i 2-1.414i 3.944e-31 8.531e-08
## -2.5    9    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 0.000e+00
## -2.4    8    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 6.671e-09
## -2.3    9    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 2.507e-08
## -2.2    9    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 2.684e-05
## -2.1    9    -4    6    -1  1+0i 2+1.414i 2-1.414i 3.944e-31 4.768e-08
## -2      9    -4    6    -1  1+0i 2+1.414i 2-1.414i 1.972e-30 6.448e-08
## -1.9    9    -4    6    -1  1+0i 2+1.414i 2-1.414i 1.972e-30 2.892e-07
## -1.8    9    -4    6    -1  1+0i 2+1.414i 2-1.414i 3.944e-31 1.016e-07
## -1.7    8    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 5.814e-08
## -1.6    8    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 9.123e-08
## -1.5    7    -4    6    -1  1+0i 2+1.414i 2-1.414i 3.944e-31 1.102e-07
## -1.4    7    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 0.000e+00
## -1.3    6    -4    6    -1  1+0i 2+1.414i 2-1.414i 3.944e-31 7.067e-09
## -1.2    5    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 3.820e-06
## -1.1    5    -4    6    -1  1+0i 2+1.414i 2-1.414i 1.578e-30 2.641e-08
## -1      1    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 0.000e+00
## -0.9    5    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 0.000e+00
```

```
## -0.8    5    -4    6    -1  1+0i 2+1.414i 2-1.414i 1.578e-30 1.161e-06
## -0.7    6    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 9.309e-06
## -0.6    6    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 1.198e-07
## -0.5    6    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 2.327e-08
## -0.4    6    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 4.796e-08
## -0.3    6    -4    6    -1  1+0i 2+1.414i 2-1.414i 1.972e-30 1.212e-07
## -0.2    6    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 2.198e-07
## -0.1    6    -4    6    -1  1+0i 2+1.414i 2-1.414i 3.944e-31 2.790e-07
## 0       6    -4    6    -1  1+0i 2+1.414i 2-1.414i 3.944e-31 2.581e-07
## 0.1     6    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 2.321e-07
## 0.2     6    -4    6    -1  1+0i 2+1.414i 2-1.414i 1.972e-30 3.389e-07
## 0.3     6    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 5.642e-07
## 0.4     6    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 1.451e-07
## 0.5     6    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 1.340e-08
## 0.6     7    -4    6    -1  1+0i 2+1.414i 2-1.414i 1.578e-30 1.013e-07
## 0.7     8    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 2.912e-07
## 0.8     9    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 6.686e-07
## 0.9    11    -4    6    -1  1+0i 2+1.414i 2-1.414i 1.972e-30 5.340e-07
## 1      14    -4    6    -1  1+0i 2+1.414i 2-1.414i 3.944e-31 2.789e-08
## 1.1    17    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 0.000e+00
## 1.2    10    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 2.652e-08
## 1.3    13    -4    6    -1  1+0i 2+1.414i 2-1.414i 3.944e-31 1.153e-07
## 1.4    15    -4    6    -1  1+0i 2+1.414i 2-1.414i 1.578e-30 6.084e-07
## 1.5    14    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 3.285e-06
## 1.6    13    -4    6    -1  1+0i 2+1.414i 2-1.414i 3.944e-31 1.009e-07
## 1.7    12    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 1.780e-07
## 1.8    11    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 1.039e-06
## 1.9    10    -4    6    -1  1+0i 2+1.414i 2-1.414i 3.944e-31 6.279e-07
## 2      10    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 0.000e+00
## 2.1    11    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 1.328e-06
## 2.2    15    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 0.000e+00
## 2.3    18    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 5.724e-08
## 2.4    13    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 4.997e-07
## 2.5    11    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 3.212e-08
## 2.6    13    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 2.953e-08
## 2.7    17    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 0.000e+00
## 2.8    12    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 7.905e-09
## 2.9    19    -4    6    -1  1+0i 2+1.414i 2-1.414i 0.000e+00 0.000e+00
## 3      18    -4    6    -1  1+0i 2+1.414i 2-1.414i 3.944e-31 1.028e-07
```

3.4.2 ALS

```
##     itel alpha   beta  gamma    root1        root2        root3     sigma
## 0    345 -4.00  6.000 -1.000  1.000+0i 2.0000+1.414i  2.000-1.414i 1.257e-19
## 0.1  345 -4.00  6.000 -1.000  1.000+0i 2.0000+1.414i  2.000-1.414i 1.247e-19
## 0.2  345 -4.00  6.000 -1.000  1.000+0i 2.0000+1.414i  2.000-1.414i 1.214e-19
## 0.3  344 -4.00  6.000 -1.000  1.000+0i 2.0000+1.414i  2.000-1.414i 1.273e-19
## 0.4  343 -4.00  6.000 -1.000  1.000+0i 2.0000+1.414i  2.000-1.414i 1.161e-19
## 0.5  332 -4.00  6.000 -1.000  1.000+0i 2.0000+1.414i  2.000-1.414i 1.176e-19
## 0.6  373 -4.00  6.000 -1.000  1.000+0i 2.0000+1.414i  2.000-1.414i 1.257e-19
## 0.7  532 -4.00  6.000 -1.000  1.000+0i 2.0000+1.414i  2.000-1.414i 1.165e-19
## 0.8  958 -4.00  6.000 -1.000  1.000+0i 2.0000+1.414i  2.000-1.414i 1.164e-19
## 0.9  979 -4.00  6.000 -1.000  1.000+0i 2.0000+1.414i  2.000-1.414i 1.222e-19
```

```
## 1    344 -4.00  6.000 -1.000  1.000+0i 2.0000+1.414i  2.000-1.414i 1.253e-19
## 1.1   25  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 1.2   25  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 1.3   24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 1.4   24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 1.5   24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 1.6   24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 1.7   24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 1.8   24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 1.9   24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 2     24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 2.1   24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 2.2   24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 2.3   24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 2.4   24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 2.5   24  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 2.6   23  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 2.7   23  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 2.8   23  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 2.9   22  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
## 3     22  2.61 -1.572  3.114 -3.114+0i 0.5046+0.000i -3.114+0.000i 6.404e+01
##       rate  trate
## 0   0.9434 0.9434
## 0.1 0.9434 0.9434
## 0.2 0.9433 0.9434
## 0.3 0.9434 0.9434
## 0.4 0.9435 0.9434
## 0.5 0.9434 0.9434
## 0.6 0.9434 0.9434
## 0.7 0.9434 0.9434
## 0.8 0.9434 0.9434
## 0.9 0.9434 0.9434
## 1   0.9433 0.9434
## 1.1 0.3735 0.3735
## 1.2 0.3735 0.3735
## 1.3 0.3735 0.3735
## 1.4 0.3735 0.3735
## 1.5 0.3735 0.3735
## 1.6 0.3735 0.3735
## 1.7 0.3735 0.3735
## 1.8 0.3735 0.3735
## 1.9 0.3735 0.3735
## 2   0.3735 0.3735
## 2.1 0.3735 0.3735
## 2.2 0.3735 0.3735
## 2.3 0.3735 0.3735
## 2.4 0.3735 0.3735
## 2.5 0.3735 0.3735
## 2.6 0.3735 0.3735
## 2.7 0.3735 0.3735
## 2.8 0.3735 0.3735
## 2.9 0.3735 0.3735
## 3   0.3735 0.3735
```

3.4.3 newtonLS

```
##      itel  alpha    beta  gamma     root1          root2         root3
## -3     59  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## -2.9   62 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -2.8   24 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -2.7   42 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -2.6   23  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## -2.5   81  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## -2.4   19  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## -2.3   18  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## -2.2   13  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## -2.1   21  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## -2     43  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## -1.9   24 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -1.8   20 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -1.7   20  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## -1.6   23 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -1.5   32  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## -1.4   12 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -1.3   32 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -1.2   14 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -1.1    7 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -1      1 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -0.9    6 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -0.8    7 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -0.7    8 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -0.6    8 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -0.5    9 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -0.4    9 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -0.3    9 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -0.2   10 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## -0.1   10 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## 0      10 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## 0.1    11 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## 0.2    12 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## 0.3    14 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## 0.4    12  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## 0.5     8  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 0.6     6  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## 0.7     5  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## 0.8     5  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## 0.9     5  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## 1       5  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## 1.1     4  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## 1.2     5  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## 1.3     6  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## 1.4     7  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## 1.5    39 -4.000  6.0000 -1.000  1.000+0i  2.0000+1.414i  2.000-1.414i
## 1.6    13  1.785  0.7636  1.073 -1.073+0i -0.7113+0.000i -1.073+0.000i
## 1.7    15  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 1.8    11  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 1.9     7  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 2       6  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
```

```
## 2.1     6  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 2.2     6  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 2.3     6  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 2.4     6  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 2.5     5  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 2.6     5  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 2.7     5  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 2.8     5  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 2.9     5  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
## 3       4  2.610 -1.5715  3.114 -3.114+0i  0.5046+0.000i -3.114+0.000i
##          sigma      rate
## -3   8.093e+01 3.104e-07
## -2.9 0.000e+00 5.087e-07
## -2.8 0.000e+00 7.126e-06
## -2.7 0.000e+00 6.972e-07
## -2.6 8.093e+01 3.658e-07
## -2.5 8.093e+01 2.383e-06
## -2.4 8.093e+01 1.027e-08
## -2.3 8.093e+01 1.459e-08
## -2.2 8.093e+01 6.772e-07
## -2.1 8.093e+01 9.314e-09
## -2   8.093e+01 1.551e-06
## -1.9 0.000e+00 0.000e+00
## -1.8 0.000e+00 0.000e+00
## -1.7 8.093e+01 1.400e-07
## -1.6 0.000e+00 5.501e-07
## -1.5 8.093e+01 3.607e-07
## -1.4 0.000e+00 6.637e-08
## -1.3 0.000e+00 7.077e-06
## -1.2 1.578e-30 2.909e-07
## -1.1 0.000e+00 3.303e-06
## -1   0.000e+00 0.000e+00
## -0.9 3.944e-31 5.073e-07
## -0.8 0.000e+00 2.377e-07
## -0.7 0.000e+00 0.000e+00
## -0.6 0.000e+00 2.321e-06
## -0.5 0.000e+00 0.000e+00
## -0.4 0.000e+00 4.477e-07
## -0.3 0.000e+00 7.525e-06
## -0.2 0.000e+00 2.020e-06
## -0.1 0.000e+00 0.000e+00
## 0    0.000e+00 0.000e+00
## 0.1  0.000e+00 2.205e-08
## 0.2  3.944e-31 1.559e-07
## 0.3  0.000e+00 9.685e-06
## 0.4  8.093e+01 1.202e-06
## 0.5  6.404e+01 1.159e-07
## 0.6  8.093e+01 2.474e-08
## 0.7  8.093e+01 4.154e-08
## 0.8  8.093e+01 4.078e-07
## 0.9  8.093e+01 2.882e-08
## 1    8.093e+01 2.181e-07
## 1.1  8.093e+01 2.902e-07
## 1.2  8.093e+01 4.402e-08
```

```
## 1.3  8.093e+01 2.608e-07
## 1.4  8.093e+01 2.325e-08
## 1.5  3.944e-31 4.661e-07
## 1.6  8.093e+01 8.012e-09
## 1.7  6.404e+01 2.435e-08
## 1.8  6.404e+01 1.420e-07
## 1.9  6.404e+01 1.436e-08
## 2    6.404e+01 1.937e-06
## 2.1  6.404e+01 1.060e-07
## 2.2  6.404e+01 2.126e-08
## 2.3  6.404e+01 8.930e-08
## 2.4  6.404e+01 4.467e-07
## 2.5  6.404e+01 3.567e-06
## 2.6  6.404e+01 7.975e-07
## 2.7  6.404e+01 2.076e-07
## 2.8  6.404e+01 3.600e-08
## 2.9  6.404e+01 8.447e-08
## 3    6.404e+01 2.877e-06
```

4 Appendix: Code

4.1 lsFunc.R

```r
# the least squares loss function
# with its first and second derivatives

lsFunc <- function(x, y) {
  alpha <- x[1]
  beta <- x[2]
  gamma <- x[3]
  p <- y[1]
  q <- y[2]
  r <- y[3]
  f <- 0.0
  f <- f + (alpha + gamma - p) ^ 2
  f <- f + (beta + (alpha * gamma) - q) ^ 2
  f <- f + ((beta * gamma) - r) ^ 2
  return(f / 2)
}

numDer <- function(x, y) {
  g <- grad(lsFunc, x, y = y)
  h <- hessian(lsFunc, x, y = y)
  return(list(grad = g, hess = h))
}

anaDer <- function(x, y) {
  alpha <- x[1]
  beta <- x[2]
  gamma <- x[3]
  p <- y[1]
  q <- y[2]
```

```r
  r <- y[3]
  da <- (gamma + alpha - p) + gamma * (beta + (alpha * gamma) - q)
  db <- (beta + (alpha * gamma) - q) + gamma * ((beta * gamma) - r)
  dc <-
    (gamma + alpha - p) + alpha * (beta + (alpha * gamma) - q) + beta * ((beta * gamma) - r)
  g <- c(da, db, dc)
  daa <- 1 + gamma ^ 2
  dab <- gamma
  dac <- 1 + (beta - q) + 2 * alpha * gamma
  dbb <- 1 + gamma ^ 2
  dbc <- alpha - r + 2 * beta * gamma
  dcc <- 1 + alpha ^ 2 + beta ^ 2
  h <- matrix(c(daa, dab, dac, dab, dbb, dbc, dac, dbc, dcc), 3, 3)
  return(list(grad = g, hess = h))
}

convRate <- function(hess) {
  a <- hess[3, 3]
  b <- hess[3,][-3]
  c <- hess[1:2, 1:2]
  return(sum(b * solve(c, b)) / a)
}
```

4.2 findRoots.R

```r
library(polynom)

polyRoots <- function(y) {
  p <- polynomial(rev(c(1, y)))
  roots <- solve(p)
  return(roots)
}

findRoots <- function(x) {
  alpha <- x[1]
  beta <- x[2]
  gamma <- x[3]
  d <- sqrt(as.complex((alpha ^ 2) - (4 * beta)))
  roots <- c(-gamma, (-alpha + d) / 2, (-alpha - d) / 2)
  return(roots)
}
```

4.3 newton.R

```r
# newton method to solve the equations
# for alpha, beta, gamma

jacobian <- function(x) {
  alpha <- x[1]
  beta <- x[2]
  gamma <- x[3]
```

```r
  jc <- matrix(c(1, 0, 1, gamma, 1, alpha, 0, gamma, beta), 3, 3, byrow = TRUE)
  return(jc)
}

newton <- function(y,
                   gamma,
                   itmax = 100,
                   eps = 1e-10,
                   verbose = TRUE) {
  p <- y[1]
  q <- y[2]
  r <- y[3]
  itel <- 1
  xold <- c(abFromC(gamma, y), gamma)
  alpha <- xold[1]
  beta <- xold[2]
  gamma <- xold[3]
  f1 <- alpha + gamma - p
  f2 <- beta + (alpha * gamma) - q
  f3 <- (beta * gamma) - r
  fold <- c(f1, f2, f3)
  sold <- lsFunc(xold, y)
  cold <- Inf
  repeat {
    jacginv <- ginv(jacobian(xold))
    delta <- jacginv %*% fold
    xnew <- xold - delta
    cnew <- sqrt(sum(delta ^ 2))
    rate <- cnew / cold
    alpha <- xnew[1]
    beta <- xnew[2]
    gamma <- xnew[3]
    f1 <- alpha + gamma - p
    f2 <- beta + (alpha * gamma) - q
    f3 <- (beta * gamma) - r
    fnew <- c(f1, f2, f3)
    snew <- lsFunc(xnew, y)
    if (verbose) {
      cat(
        "itel ",
        formatC(itel, width = 4, format = "d"),
        "fold ",
        formatC(sold, digits = 15, format = "f"),
        "fnew ",
        formatC(snew, digits = 15, format = "f"),
        "cnew ",
        formatC(cnew, digits = 15, format = "f"),
        "rate ",
        formatC(rate, digits = 15, format = "f"),
        "\n"
      )
    }
    if ((itel == itmax) || (cnew < eps)) {
```

```
      break
    }
    itel <- itel + 1
    xold <- xnew
    fold <- fnew
    cold <- cnew
  }
  return(list(x = xnew, itel = itel, f = fnew, sigma = snew, rate = rate))
}

newtonRunner <- function(y) {
  gamma0 <- (1:61) / 10 - 3.1
  alpha <- seq(-3, 3, length = 61)
  beta <- seq(-3, 3, length = 61)
  gamma <- seq(-3, 3, length = 61)
  root1 <- seq(-3, 3, length = 61)
  root2 <- seq(-3, 3, length = 61)
  root3 <- seq(-3, 3, length = 61)
  itel <- seq(-3, 3, length = 61)
  sigma <- seq(-3, 3, length = 61)
  rate <- seq(-3, 3, length = 61)
  for (i in 1:61) {
    xtmp <-
      newton(
        y,
        gamma = gamma0[i],
        itmax = 10000,
        verbose = FALSE,
        eps = 1e-10
      )
    roots <- findRoots(xtmp$x)
    alpha[i] <- (xtmp$x)[1]
    beta[i] <- (xtmp$x)[2]
    gamma[i] <- (xtmp$x)[3]
    itel[i] <- xtmp$itel
    sigma[i] <- xtmp$sigma
    rate[i] <- xtmp$rate
    root1[i] <- roots[1]
    root2[i] <- roots[2]
    root3[i] <- roots[3]
  }
  dtmp <- data.frame(
    itel = itel,
    alpha = alpha,
    beta = beta,
    gamma = gamma,
    root1 = root1,
    root2 = root2,
    root3 = root3,
    sigma = sigma,
    rate = rate,
    row.names = gamma0
  )
```

```r
  print(dtmp, digits = 4)
}
```

4.4 alsSolve.R

```r
# ALS method to solve the equations
# for alpha, beta, gamma

abFromC <- function(gamma, y) {
  p <- y[1]
  q <- y[2]
  r <- y[3]
  u <- matrix(c(1, gamma, 0, 0, 1, gamma), 3, 2)
  v <- c(p - gamma, q, r)
  return(qr.solve(u, v))
}

cFromAb <- function(alpha, beta, y) {
  p <- y[1]
  q <- y[2]
  r <- y[3]
  s <- 1 + (alpha ^ 2) + (beta ^ 2)
  t <- (p - alpha) + alpha * (q - beta) + beta * r
  return(t / s)
}


alsSolve <- function(y,
                     gamma,
                     itmax = 100000,
                     eps = 1e-15,
                     verbose = FALSE) {
  p <- y[1]
  q <- y[2]
  r <- y[3]
  xold <- c(abFromC(gamma, y), gamma)
  fold <- lsFunc(xold, y)
  itel <- 1
  cold <- Inf
  repeat {
    alpha <- xold[1]
    beta <- xold[2]
    gamma <- cFromAb(alpha, beta, y)
    xnew <- c(abFromC(gamma, y), gamma)
    fnew <- lsFunc(xold, y)
    cnew <- sqrt(sum(xold - xnew) ^ 2)
    rate <- cnew / cold
    if (verbose) {
      cat(
        "itel ",
        formatC(itel, digits = 3, format = "d"),
        "fold ",
```

```
        formatC(fold, digits = 10, format = "f"),
        "fnew ",
        formatC(fnew, digits = 10, format = "f"),
        "chng ",
        formatC(cnew, digits = 8, format = "f"),
        "rate ",
        formatC(rate, digits = 8, format = "f"),
        "\n"
      )
    }
    if ((itel == itmax) || (cnew < eps)) {
      break
    }
    itel <- itel + 1
    fold <- fnew
    xold <- xnew
    cold <- cnew
  }
  trate <- convRate(anaDer(xnew, y)$hess)
  return(list(
    x = xnew,
    f = fnew,
    rate = rate,
    trate = trate,
    itel = itel
  ))
}

alsRunner <- function(y) {
  gamma0 <- (1:61) / 10 - 3.1
  alpha <- seq(-3, 3, length = 61)
  beta <- seq(-3, 3, length = 61)
  gamma <- seq(-3, 3, length = 61)
  root1 <- seq(-3, 3, length = 61)
  root2 <- seq(-3, 3, length = 61)
  root3 <- seq(-3, 3, length = 61)
  itel <- seq(-3, 3, length = 61)
  rate <- seq(-3, 3, length = 61)
  trate <- seq(-3, 3, length = 61)
  sigma <- seq(-3, 3, length = 61)
  for (i in 1:61) {
    xtmp <-
      alsSolve(
        y,
        gamma = gamma0[i],
        itmax = 10000,
        verbose = FALSE,
        eps = 1e-10
      )
    roots <- findRoots(xtmp$x)
    itel[i] <- xtmp$itel
    rate[i] <- xtmp$rate
    trate[i] <- xtmp$trate
```

```
    alpha[i] <- (xtmp$x)[1]
    beta[i] <- (xtmp$x)[2]
    gamma[i] <- (xtmp$x)[3]
    root1[i] <- roots[1]
    root2[i] <- roots[2]
    root3[i] <- roots[3]
    sigma[i] <- xtmp$f
  }
  dtmp <- data.frame(
    itel = itel,
    alpha = alpha,
    beta = beta,
    gamma = gamma,
    root1 = root1,
    root2 = root2,
    root3 = root3,
    sigma = sigma,
    rate = rate,
    trate = trate,
    row.names = gamma0
  )
  print(dtmp, digits = 4)
}
```

4.5 newtonLS.R

```
source("findRoots.R")

abFromC <- function(gamma, y) {
  p <- y[1]
  q <- y[2]
  r <- y[3]
  u <- matrix(c(1, gamma, 0, 0, 1, gamma), 3, 2)
  v <- c(p - gamma, q, r)
  return(qr.solve(u, v))
}

newtonLS <-
  function(y,
           gamma,
           itmax = 100,
           eps = 1e-10,
           verbose = FALSE) {
    itel <- 1
    xold <- c(abFromC(gamma, y), gamma)
    sold <- lsFunc(xold, y)
    cold <- Inf
    repeat {
      h <- anaDer(xold, y)
      grad <- h$grad
      hess <- h$hess
      diff <- drop(ginv(hess) %*% grad)
```

```
      xnew <- xold - diff
      snew <- lsFunc(xnew, y)
      cnew <- sqrt(sum(diff ^ 2))
      rate <- cnew / cold
      if (verbose)
      {
        cat(
          "itel ", formatC(itel, width = 5, format = "d"),
          "sold ", formatC(sold, digits = 15, width = 20, format = "f"),
          "snew ", formatC(snew, digits = 15, width = 20, format = "f"),
          "cnew ", formatC(cnew, digits = 15, width = 20, format = "f"),
          "rate ", formatC(rate, digits = 15, width = 20, format = "f"),
          "\n")
      }
      if ((itel == itmax) || (cnew < eps)) {
        break
      }
      xold <- xnew
      sold <- snew
      cold <- cnew
      itel <- itel + 1
    }
    return(list(x = xnew, s = snew, itel = itel, rate = rate))
  }

nlsRunner <- function(y) {
  gamma0 <- (1:61) / 10 - 3.1
  alpha <- seq(-3, 3, length = 61)
  beta <- seq(-3, 3, length = 61)
  gamma <- seq(-3, 3, length = 61)
  root1 <- seq(-3, 3, length = 61)
  root2 <- seq(-3, 3, length = 61)
  root3 <- seq(-3, 3, length = 61)
  itel <- seq(-3, 3, length = 61)
  rate <- seq(-3, 3, length = 61)
  sigma <- seq(-3, 3, length = 61)
  for (i in 1:61) {
    xtmp <-
      newtonLS(
        y,
        gamma = gamma0[i],
        itmax = 10000,
        verbose = FALSE,
        eps = 1e-10
      )
    roots <- findRoots(xtmp$x)
    itel[i] <- xtmp$itel
    alpha[i] <- (xtmp$x)[1]
    beta[i] <- (xtmp$x)[2]
    gamma[i] <- (xtmp$x)[3]
    root1[i] <- roots[1]
    root2[i] <- roots[2]
    root3[i] <- roots[3]
```

```
    rate[i] <- xtmp$rate
    sigma[i] <- xtmp$s
  }
  dtmp <- data.frame(
    itel = itel,
    alpha = alpha,
    beta = beta,
    gamma = gamma,
    root1 = root1,
    root2 = root2,
    root3 = root3,
    sigma = sigma,
    rate = rate,
    row.names = gamma0
  )
  print(dtmp, digits = 4)
}
```

References

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