

Minimizing fStress and rStress by Majorizing Gauss-Newton

Jan de Leeuw

January 31, 2026

TBD

Table of contents

1 Loss Functions	3
2 Alternating Least Squares	4
2.1 Normalized Cone Regression	4
2.2 Majorization	5
References	8

Note: This is a working manuscript which will be expanded/updated frequently. All suggestions for improvement are welcome. All Rmd, tex, html, pdf, R, and C files are in the public domain. Attribution will be appreciated, but is not required. The files can be found at <https://github.com/deleeuw/rStress>

1 Loss Functions

The Multidimensional Scaling (MDS) loss function fStress is defined as

$$\sigma_f(x) := \frac{1}{2} \sum_{k=1}^K w_k (f(\delta_k) - f(d_k(x)))^2, \quad (1)$$

with f increasing and differentiable in the open interval $(0, +\infty)$. In (1) the w_k are positive *weights*, the δ_k are known *dissimilarities*. The vector x has the coordinates of n points in \mathbb{R}^p , and the $d_k(x)$ are Euclidean *distances* between pairs of these points. For each k there is a pair of indices (i, j) with $1 \leq i < j \leq n$ and a matrix A_k which is the direct sum of p copies of $(e_i - e_j)(e_i - e_j)'$, where the e_i are unit vectors (columns of the identity matrix of order n). Thus $d_k(x) = \sqrt{x' A_k x}$. Metric least squares MDS minimizes fStress over x .

fStress was introduced and studied in Groenen, De Leeuw, and Mathar (1995). No explicit algorithm to minimize (??)eq-fdef was given, but the paper has formulas for the first and second derivatives. De Leeuw (2017) uses the multivariate Faà di Bruno formula to give derivatives of fStress up to order four. These derivatives can be used in general purpose minimization methods.

An important special case of fStress is rStress (also known as powerStress), which is

$$\sigma_r(x) := \frac{1}{2} \sum_{k=1}^K w_k (\delta_k^r - d_k^r(x))^2 \quad (2)$$

Special cases of rStress are Kruskal's stress (Kruskal (1964a), Kruskal (1964b)) with $r = 1$, sstress by Takane, Young, and De Leeuw (1977) with $r = 2$, and the limiting case logarithmic stress with $r \rightarrow 0$ by Ramsay (1977). There have been various attempts to extend the majorization (or MM) method for MDS (De Leeuw (1977)) to rStress. References and links to various unpublished reports are in De Leeuw (2017). The recent smacofx package (Rusch et al. (In Press)) has R code for the `rstressMin()` function that implements the majorization method in De Leeuw, Groenen, and Mair (2016).

Minimizing of either fStress or rStress over x is a metric MDS problem. The δ_k , and consequently the $f(\delta_k)$, are K known numbers. In non-metric MDS the loss function is minimized over both x and δ , where δ is constrained to be in a subset Δ of \mathbb{R}^K . For the ordinal version of non-metric MDS, for example, we require $\delta_1 \leq \dots \leq \delta_K$. Since f is increasing, we can write fStress simply as

$$\sigma_f(x, \delta) := \frac{1}{2} \sum_{k=1}^K w_k (\delta_k - f(d_k(x)))^2, \quad (3)$$

where δ is no longer a vector of known dissimilarities, but any vector monotone with the dissimilarities. These transformed or scaled dissimilarities are often called *disparities*. Non-metric rStress is simply (3) with $f(d_k(x)) = d_k^r(x)$.

In order to exclude the trivial solution with $x = 0$ and $\delta = 0$ in addition we impose the normalization constraint

$$\eta^2(\delta) := \frac{1}{2} \sum_{k=1}^K w_k \delta_k^2 = 1. \quad (4)$$

This formulation of non-metric MDS can be generalized to $\delta \in \Delta$, where Δ is a convex cone in \mathbb{R}^K . This means we require the disparities to be in the intersection of the cone Δ and the sphere Σ defined by (4).

It is convenient to think of metric MDS as the special case in which the set Δ is the one-element set containing only the normalized dissimilarities δ . Alternatively, as in smacof, we can choose for Δ the ray of all vectors that are non-negative multiples of the dissimilarities. In other words our metric MDS treats the dissimilarities as measured on a ratio scale.

2 Alternating Least Squares

The technique proposed in this paper to minimize non-metric fStress or rStress is in the Alternating Least Squares (ALS) family. We start with an initial estimate $x^{(0)}$. We then minimize σ_f over δ in $\Delta \cap \Sigma$ for the current $d(x^{(0)})$. The minimizer $\delta^{(0)}$ is then used to minimize fStress over x for the current disparities, yielding $x^{(1)}$. These two steps are alternated until x and δ do not change any more. Starting in iteration $\nu = 0$ we compute

$$\delta^{(\nu)} = \operatorname{argmin}_{\delta \in \Delta \cap \Sigma} \sigma_f(x^{(\nu)}, \delta), \quad (5a)$$

$$x^{(\nu+1)} = \operatorname{argmin}_x \sigma_f(x, \delta^{(\nu)}). \quad (5b)$$

We then increase ν by one and go into the next iteration. And so on, until convergence.

2.1 Normalized Cone Regression

In metric MDS there is no need for the first step (5a), because $\delta^{(\nu)}$ is equal to the normalized dissimilarities δ for all ν . In the non-metric case step (5a) is needed, but compared to (5b) it is comparatively easy. We have to compute the least squares projection of $f(d(x))$ on the cone Δ and then normalize this projection to give it length one. If Δ is the cone of monotone vectors, as in ordinal MDS, the cone projection is *monotone regression*. The fact that projection on the intersection of Δ and Σ is the same thing as normalizing the projection on Δ is due

to De Leeuw (1975) and more generally (and more rigorously) to Bauschke, Bui, and Wang (2018). Since this step is the same as in standard MDS algorithms such as smacof (De Leeuw and Mair (2009), Mair, Groenen, and De Leeuw (2022)) we do not go into details here, and just refer to the literature.

2.2 Majorization

The second step (5b), minimizing over x for fixed current δ , is more complicated. There is no analytic solution, similar to what we have in the first step, and minimization requires an iterative process of its own. Thus, except in some very special cases, the second step requires an infinite number of “inner” iterations. Since implementing an infinite number of iterations is impossible we have to truncate the inner iteration sequence at some point. In our software we deviate from the strict ALS framework by not minimizing fStress over x for fixed δ , but by merely taking a single “inner” iteration and merely decrease fStress. If this is done judiciously we still obtain a convergent sequence of updates. This is similar to the strategy in other MDS algorithms such as smacof (De Leeuw (1977)) and alscale (Takane, Young, and De Leeuw (1977)).

In smacof the inner iteration step is a majorization step, by now more commonly known as an MM step. Briefly, we find a function $\kappa_f(x; y)$ such that

- $\kappa_f(x, y) \leq \sigma_f(x)$ for all x and y in Ξ , and
- $\kappa_f(x; y) = \sigma_f(x)$ if and only if $x = y$.

An inner iteration is of the form

$$x^{(\mu+1)} = \operatorname{argmin}_{x \in \Xi} \kappa_f(x, x^{(\mu)}). \quad (6)$$

If $x^{(\mu+1)} = x^{(\mu)}$ we declare convergence and stop. From (6),

$$\sigma_f(x^{(\mu+1)}) \leq \kappa_f(x^{(\mu+1)}, x^{(\mu)}) < \kappa_f(x^{(\mu)}, x^{(\mu)}) = \sigma_f(x^{(\mu)}). \quad (7)$$

Thus an inner majorization step upgrading x for given δ decreases fStress (unless we stop). Remember that in ALS the inner iterative process (indexed by μ) goes on within step 2 of the “outer” update (indexed by ν).

In De Leeuw, Groenen, and Mair (2016) a majorization method was proposed to minimize rStress. It majorizes d^r as a function of x , and for this it uses the specific properties of the power function. It consequently cannot be used for fStress. The technique is incorporated in the smacofx R package, and numerical experience indicates it works, but convergence can be painfully slow. Because of the generality of the function f (any differentiable increasing function) it seems impossible to develop a majorization method for fStress along the same

lines as the one for rStress. Consequently we go a somewhat different route in this paper. We do not only deviate from the strict ALS procedure by only partially minimizing loss over x for fixed d , we also deviate from the strict majorization approach by majorizing an approximation of fStress.

In deriving the approximation and majorization we surpass the dependency of fStress on δ , because in the second ALS substep δ is just a vector of constants. We first approximate $f(d(x))$ near $d(y)$ with

$$f(d_k(x)) \approx f(d_k(y)) + \mathcal{D}f(d_k(y))(d_k(x) - d_k(y)). \quad (8)$$

Define

$$\omega_f(x; y) := \sum_{k=1}^K w_k(f(\delta_k) - f(d_k(y)) - \mathcal{D}f(d_k(y))(d_k(x) - d_k(y)))^2, \quad (9)$$

Note that in (8) the derivative is with respect to d , not with respect to x or y . This is a major difference with the majorizations in De Leeuw, Groenen, and Mair (2016).

Note that $\omega_f(x; x) = \sigma_f(x)$ for all x and if f is the identity then $\omega_f(x; y) = \sigma_f(x)$ for all x and y . If f is linear with intercept β and slope α then

$$\omega_f(x, y) = a^2 \sum_{k=1}^K w_k(\delta_k - d_k(x))^2 = a^2 \sigma_r(x), \quad (10)$$

with $r = 1$.

We now derive an alternative and more convenient expression for $\omega_f(x, y)$. Define

$$\tilde{w}_k(y) := w_k \{\mathcal{D}f(d_k(y))\}^2, \quad (11a)$$

$$\tilde{\delta}_k(y) := \frac{f(\delta_k) - f(d_k(y))}{\mathcal{D}f(d_k(y))} + d_k(y). \quad (11b)$$

Then

$$\omega_f(x; y) = \sum_{k=1}^K \tilde{w}_k(y)(\tilde{\delta}_k(y) - d_k(x))^2. \quad (12)$$

We see from (11a) that $\tilde{w}_k(y)$ is always non-negative and from (11b) that $\tilde{\delta}_k(y)$ can be negative if $d_k(y) > \delta_k$. For convex f , however, we have

$$\tilde{\delta}_k(y) = \frac{f(\delta_k) - f(d_k(y))}{\mathcal{D}f(d_k(y))} + d_k(y) \geq d_k(x) \geq 0. \quad (13)$$

For rStress equations (11a) and (11b) become

$$\tilde{w}_k(y) := r^2 w_k d_k^{2(r-1)}(y), \quad (14a)$$

$$\tilde{\delta}_k(y) := \frac{\delta_k^r + (r-1)d_k^r(y)}{rd_k^{r-1}(y)}. \quad (14b)$$

By convexity $\tilde{\delta}_k(y)$ is non-negative if $r \geq 1$. For $0 < r < 1$ we have $\tilde{\delta}_k(y) > 0$ if and only if $d_k(y) < (1-r)^r \delta_k$. If $\delta_k > d_k(y)$ then $\tilde{\delta}_k(y) > d_k(y) \geq 0$.

By writing $\omega_f(x, y)$ as in (12) we are on familiar majorization terrain. If $\delta_k(y) > 0$ we use Cauchy-Schwartz, as in standard smacof,

$$d_k(x) \geq \frac{1}{d_k(y)} \text{tr } x' A_k y, \quad (15a)$$

and if $\delta_k(y) < 0$ we use the arithmetic mean/geometric mean (AM/GM) inequality in the form

$$d_k(x) \leq \frac{1}{2} \frac{1}{d_k(y)} \{y' A_k y + x' A_k x\}, \quad (15b)$$

Use of the AM/GM inequality for majorizing terms with negative dissimilarities was pioneered by Heiser (1991).

Define the matrices

$$V(y) := \sum_{k=1}^K \tilde{w}_k(y) A_k, \quad (16a)$$

$$B(y) := \sum_{\tilde{\delta}_k(y) > 0} \tilde{w}_k \frac{\tilde{\delta}_k(y)}{d_k(y)} A_k, \quad (16b)$$

$$H(y) := \sum_{\tilde{\delta}_k(y) < 0} \tilde{w}_k \frac{\tilde{\delta}_k(y)}{d_k(y)} A_k. \quad (16c)$$

Note that all three matrices (more precisely matrix-valued functions) are positive semi-definite. We can use these matrices for a majorization of $\omega_f(x; y)$ at y is

$$\omega_f(x; y) = C + \frac{1}{2} x' (V(y) + H(y)) x - x' B(y) x \leq \kappa_f(x; y), \quad (17)$$

with

$$\kappa_f(x; y) := C + \frac{1}{2} x' (V(y) + H(y)) x - x' B(y) y, \quad (18)$$

where C is a constant that depends on y but not on x .

$$\sigma_f(x^+) \approx \omega_f(x^+; x) \leq \kappa_f(x^+; x) < \kappa_f(x; x) = \sigma_f(x).$$

References

- Bauschke, Heinz H., Minh N. Bui, and Xianfu Wang. 2018. “Projecting onto the Intersection of a Cone and a Sphere.” *SIAM Journal on Optimization* 28: 2158–88.
- De Leeuw, Jan. 1975. “A Normalized Cone Regression Approach to Alternating Least Squares Algorithms.” Department of Data Theory FSW/RUL. <https://jansweb.netlify.app/publication/deleeuw-u-75-a/deleeuw-u-75-a.pdf>.
- . 1977. “Applications of Convex Analysis to Multidimensional Scaling.” In *Recent Developments in Statistics*, edited by J. R. Barra, F. Brodeau, G. Romier, and B. Van Cutsem, 133–45. Amsterdam, The Netherlands: North Holland Publishing Company.
- . 2017. “Higher Partials of fStress. Who Needs Them ?” 2017. <https://jansweb.netlify.app/publication/deleeuw-e-17-r/deleeuw-e-17-r.pdf>.
- De Leeuw, Jan, Patrick Groenen, and Patrick Mair. 2016. “Minimizing rStress Using Majorization.” 2016. <https://jansweb.netlify.app/publication/deleeuw-groenen-mair-e-16-a/deleeuw-groenen-mair-e-16-a.pdf>.
- De Leeuw, Jan, and Patrick Mair. 2009. “Multidimensional Scaling Using Majorization: SMACOF in R.” *Journal of Statistical Software* 31 (3): 1–30. <https://www.jstatsoft.org/article/view/v031i03>.
- Groenen, Patrick J. F., Jan De Leeuw, and Rudolf Mathar. 1995. “Least Squares Multidimensional Scaling with Transformed Distances.” In *From Data to Knowledge: Theoretical and Practical Aspects of Classification, Data Analysis and Knowledge Organization*, edited by W. Gaul and D. Pfeifer. Berlin, Germany: Springer Verlag. <https://jansweb.netlify.app/publication/groenen-deleeuw-mathar-c-95/groenen-deleeuw-mathar-c-95.pdf>.
- Heiser, W. J. 1991. “A Generalized Majorization Method for Least Squares Multidimensional Scaling of Pseudodistances that May Be Negative.” *Psychometrika* 56 (1): 7–27.
- Kruskal, Joseph B. 1964a. “Multidimensional Scaling by Optimizing Goodness of Fit to a Nonmetric Hypothesis.” *Psychometrika* 29: 1–27.
- . 1964b. “Nonmetric Multidimensional Scaling: a Numerical Method.” *Psychometrika* 29: 115–29.
- Mair, Patrick, Patrick J. F. Groenen, and Jan De Leeuw. 2022. “More on Multidimensional Scaling in R: smacof Version 2.” *Journal of Statistical Software* 102 (10): 1–47. <https://www.jstatsoft.org/article/view/v102i10>.
- Ramsay, James O. 1977. “Maximum Likelihood Estimation in Multidimensional Scaling.” *Psychometrika* 42: 241–66.
- Rusch, Thomas, Jan De Leeuw, Patrick Mair, and Kurt Hornik. In Press. “Flexible Multidimensional Scaling with the r Package Smacofx.” *Journal of Statistical Software*, In Press. <https://jansweb.netlify.app/publication/rusch-deleeuw-mair-hornik-a-25/rusch-deleeuw-mair-hornik-a-25.pdf>.
- Takane, Yoshio, Forrest W. Young, and Jan De Leeuw. 1977. “Nonmetric Individual Differences in Multidimensional Scaling: An Alternating Least Squares Method with Optimal

Scaling Features.” *Psychometrika* 42: 7–67.