simple

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Majorization

$$\sigma(X)=\sum w_k(\delta_k-d_k(X))^2$$

Now $d_k(X)=d_k(\tilde X)+(d_k(X)-d_k(\tilde X))$. Thus

$$\sigma(X)=\sum w_k((\delta_k-d_k(\tilde X))-(d_k(X)-d_k(\tilde X)))^2\leq\sigma(\tilde X)-2w_\star\sum\frac{w_k}{w_\star}(\delta_k-d_k(\tilde X))(d_k(X)-d_k(\tilde X))+w_\star\sum(d_k(X)-$$

$$\hat d_k(\tilde X)=\frac{w_k}{w_\star}\delta_k+(1-\frac{w_k}{w_\star})d_k(\tilde X)$$

So decrease

$$\sum(d_k(X)-\hat d_k(\tilde X))^2$$

Note this can also be applied if some weights are zero.

$$\frac{1}{n}\overline{B}(\tilde X)\tilde X=\frac{1}{n}\sum\frac{\frac{w_k}{w_\star}\delta_k+(1-\frac{w_k}{w_\star})d_k(\tilde X)}{d_k(\tilde X)}\tilde X=(nw_\star)^{-1}\{B(\tilde X)\tilde X+(nw_\star I-V)\tilde X\}$$

Alt:

$$\eta^2(X)=\text{tr }X'VX=\sum w_{ij}d_{ij}^2(X)\leq w_\star\sum d_{ij}^2(X)=nw_\star\text{ tr }X'X$$

$$\eta^2(\tilde X+(X-\tilde X))\leq\eta^2(\tilde X)+2\text{ tr }\tilde X'V(X-\tilde X)+nw_\star\text{ tr }(X-\tilde X)'(X-\tilde X)$$

$$\mathcal{D}\omega=V\tilde X+nw_\star(X-\tilde X)$$

$$B(\tilde X)\tilde X=V\tilde X+nw_\star(X-\tilde X)$$

$$X=\tilde X+\frac{1}{nw_\star}\{B(\tilde X)-V\}\tilde X=\tilde X-\frac{1}{nw_\star}\nabla\sigma(\tilde X)$$

while for $X = V^+B(\tilde{X})\tilde{X}$
$$X = \tilde{X} - V^+\nabla\sigma(\tilde{X})$$

Multiplication

$Y = VX$ can be computed as
$$Y = \sum_{i<j} w_{ij}A_{ij}X = \sum_{i<j} w_{ij}(e_i - e_j)(x_i - x_j)'$$
$$y_{ks} = e_k'Ye_s = \sum_{i<j} w_{ij}(\delta^{ik} - \delta^{jk})(x_{is} - x_{js})$$

$$y_{ks} = \sum_{j=1}^{n} w_{kj}(x_{ks} - x_{js})$$

```r
set.seed(12345)
w <- as.matrix(dist(matrix(rnorm(12), 6, 2)))
b <- -w
diag(b) <- -rowSums(b)
x <- matrix(rnorm(12), 6, 2)
y1 <- b %*% x
y2 <- matrix(0, 6, 2)
for (s in 1:2) {
  y2[, s] <- rowSums(w * outer(x[, s], x[, s], "-"))
}
print(cbind(y1, y2))
```

```
       [,1]       [,2]      [,3]       [,4]
1  3.138992   9.004121  3.138992   9.004121
2  3.846538   3.573911  3.846538   3.573911
3 -4.495247   6.877840 -4.495247   6.877840
4  8.135394  14.644141  8.135394  14.644141
5 -5.290701  -2.517587 -5.290701  -2.517587
6 -5.334975 -31.582426 -5.334975 -31.582426
```