Smacof at 50: A Manual
Part 2: Non-metric Smacof

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Abstract

TBD

Contents

Note: This is a working manuscript which will be expanded/updated frequently. All suggestions for improvement are welcome. All Rmd, tex, html, pdf, R, and C files are in the public domain. Attribution will be appreciated, but is not required. The code files can be found at https://github.com/deleeuw/smacofCode, the manual files at https://github.com/deleeuw/smacofManual, and the example files at https://github.com/deleeuw/smacofExamples.

1 Introduction

pick and rank

2 Loss Function

$$\sigma(X,\delta_1,\cdots,\delta_s)=\frac{\sum_{r=1}^s\omega_r\sum_{i,j}w_{ijr}(\delta_{ijr}-d_{ij}(X))^2}{\sum_{r=1}^s\omega_r\sum_{i,j}w_{ijr}d_{ij}^2(X))}$$

which must be minimized over X and over $\delta_r\in\mathcal{K}_r$, with $\mathcal{K}_r$ pointed polyhedral convex cones, defined by a partial order $\leq_r$.

Minimize of X for given δ_{ijr}.

$$\sigma_R(X,\delta_1,\cdots,\delta_s)=\sum_{r=1}^s\omega_r\sum_{i,j}w_{ijr}\delta_{ijr}^2-2\sum_{r=1}^s\omega_r\sum_{i,j}w_{ijr}\delta_{ijr}d_{ij}(X)+\sum_{r=1}^s\omega_r\sum_{i,j}w_{ijr}d_{ij}^2(X))$$

Nor use the basic smacof inequality

$$d_{ij}(X)\geq\frac{1}{d_{ij}(Y)}\text{tr }X'A_{ij}Y$$

so that

$$\sum_{r=1}^s\omega_r\sum_{i,j}w_{ijr}\delta_{ijr}d_{ij}(X)\geq\text{tr }X'B(Y)Y$$

$$B(Y):=\sum_{r=1}^s\omega_r\sum_{i,j}w_{ijr}\frac{\delta_{ijr}}{d_{ij}(Y)}A_{ij}$$

Also

$$V:=\sum_{r=1}^s\omega_r\sum_{i,j}w_{ijr}A_{ij}$$

So that

$$\sigma_R(X)\leq K-2\text{tr }X'B(Y)Y+\text{tr }X'VX$$

and the smacof update over X with tr $X'VX=1$ is the same as in smacofRR.

3 Paired Comparisons

Positive Orthant / Absolute Value / Pairwise

De Leeuw (1970) De Leeuw (2018) Hartmann (1979) Guttman (1969) Johnson (1973)

Suppose datum r says that that $(i,j) < (k,l)$. Then w_{ijr} and w_{klr} are non-zero and all other elements of W_r are zero. Thus

$$w_{ijr}(\delta_{ijr} - d_{ij})^2 + w_{klr}(\delta_{klr} - d_{kl})^2$$

Must be minimized over $\delta_{ijr} \leq \delta_{klr}$. If $d_{ij} \leq d_{kl}$ then $\hat{d}_{ijr} = d_{ij}$ and $\hat{d}_{klr} = d_{kl}$, and otherwise

$$\hat{d}_{ijr} = \hat{d}_{klr} = \frac{w_{ijr}d_{ij} + w_{klr}d_{kl}}{w_{ijr} + w_{klr}}$$

Thus

$$w_{ijr}(\delta_{ijr} - d_{ij})^2 + w_{klr}(\delta_{klr} - d_{kl})^2$$

is zero if the order of d_{ij} and d_{kl} is the same as the order in the data and

$$w_{ijr}(\frac{w_{ijr}d_{ij} + w_{klr}d_{kl}}{w_{ijr} + w_{klr}} - d_{ij})^2 + w_{klr}(\frac{w_{ijr}d_{ij} + w_{klr}d_{kl}}{w_{ijr} + w_{klr}} - d_{kl})^2$$
$$\frac{w_{ijr}w_{klr}^2}{(w_{ijr} + w_{klr})^2}(d_{ij} - d_{kl})^2 + \frac{w_{ijr}^2 w_{klr}}{(w_{ijr} + w_{klr})^2}(d_{ij} - d_{kl})^2$$
$$\frac{w_{ijr}w_{klr}}{w_{ijr} + w_{klr}}(d_{ij} - d_{kl})^2$$

B matrix

So far we have only considered the forced-choice situation in which the subject has to choose one of the pairs. If we allow for the alternative that (i,j) and (k,l) are equally similar then we have two approaches. In the primary approach we

$$\begin{bmatrix}0&0&1\\1&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}$$

4 Triads

5 pick 1/3 Method of triads

Pick the two most similar $(i, j) < (i, k)$ and $(i, j) < (j, k)$

6 order 3/3 Complete method of triads

$(i, j) < (i, k) < (j, k)$

7 Richardson –

each triple presented three times, with a different hub each time which one of the two is maximally similar to the hub stimulus. Thus the data is the single inequality $(i, k) < (j, k)$. Coombs calls this the …

8 Conditional rank Orders – Klingberg

References

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