Legendre polynomials (on $[-1,1]$)

Recall our quadrature rule with precision 3 on $[-1,1]$:

$$\int_{-1}^{1} f(x)\,dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

e.g.) $\int_{-1}^{1} x^3 - 2x^2 + 1\,dx \overset{\text{(exact)}}{=} \frac{1}{4}x^4 - \frac{2}{3}x^3 + x\Big|_{-1}^{1} = \left(\frac{1}{4} - \frac{2}{3} + 1\right) - \left(\frac{1}{4} + \frac{2}{3} - 1\right) = \frac{2}{3}$

(rule) $\longrightarrow \;= \left(\frac{-\sqrt{3}}{3}\right)^3 - 2\left(\frac{-\sqrt{3}}{3}\right)^2 + 1 + \left(\frac{\sqrt{3}}{3}\right)^3 - 2\left(\frac{\sqrt{3}}{3}\right)^2 + 1 = -\frac{4}{3} + 2 = \frac{2}{3}$

We computed the coefficients c_1, c_2 and nodes x_1, x_2 using an "undetermined coefficients"-like method. However, there are easier ways via Legendre polynomials.

def: The Legendre polynomials on $[-1,1]$ are a set of polynomial functions $\{\pi_0(x), \pi_1(x), \dots, \pi_n(x), \dots\}$ ~~$\{P_0(x), P_1(x), \dots, P_n(x), \dots\}$~~ satisfying the following conditions.

(i) $\pi_k(x)$ ~~$P_k(x)$~~ is a polynomial of degree k. (independence)

(ii) The lead coefficient of $\pi_k(x)$ ~~$P_k(x)$~~ is 1. (monic)

(iii) $\int_{-1}^{1} \pi_i(x)\pi_j(x)$ ~~$P_i(x)P_j(x)$~~ $dx = \begin{cases} \rho_i & \text{if } i=j, \\ 0 & \text{if } i \neq j. \end{cases}$ (orthogonal on $[-1,1]$ with respect to weight $\omega(x)=1$)

lem: If $P(x)$ is a polynomial of degree n, then $\int_{-1}^{1} P(x)\pi_n(x)$ ~~$P_n(x)$~~ $dx = 0$.

pf: This follows from the fact that $\{\pi_0, \pi_1, \dots, \pi_{n-1}\}$ forms a basis for the vector space of polynomials on $[-1,1]$ with degree n-1 or less, i.e., there exist coefficients α_k so that $P(x) = \sum_{i=0}^{n-1} \alpha_i \pi_i(x)$. Then

$$\int_{-1}^{1} P(x)P_n(x)\,dx = \sum_{i=0}^{n-1} \alpha_i \int_{-1}^{1} \pi_i(x)\pi_n(x)\,dx = 0.$$

The only degree 0 satisfying (ii) is $\pi_0(x) = 1$. We can determine $\pi_1(x)$ using (ii) and (iii).

$$\int_{-1}^{1} \pi_0(x)[x+c]\,dx = \int_{-1}^{1} x\,dx + \int_{-1}^{1} c\,dx = 2c = 0 \longrightarrow \pi_1(x) = x.$$

We can determine the polynomials of higher degree using the Gram-Schmidt orthogonalization process (Monday/Wednesday next week). For now, the next few polynomials are

$$\pi_2(x) = x^2 - \frac{1}{3}, \quad \pi_3(x) = x^3 - \frac{3}{5}x, \quad \pi_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35},$$

$$\pi_5(x) = x^5 - \frac{70}{63}x^3 + \frac{5}{21}x, \quad \pi_6(x) = x^6 - \frac{315}{231}x^4 + \frac{105}{231}x^2 - \frac{5}{231}, \ \ldots$$

e.g.) The roots of $\pi_3(x)$ are $x = 0, \sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}$, and a Gaussian quadrature rule with precision 5 is

$$\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right).$$

How do we determine the coefficients?

<u>thm:</u> Let $x_1, x_2, \ldots, x_n$ denote the roots of $\pi_n(x)$, i.e., $\pi_n(x_i) = 0$, $i = 1, 2, \ldots, n$. define the numbers c_i for $i = 1, 2, \ldots, n$ by

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}\,dx.$$

(elementary Lagrange polynomials relative to nodes $x_1, x_2, \ldots, x_n$)

Then the quadrature rule $\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} c_i f(x_i)$ has precision $2n-1$.

e.g.) The roots of $\pi_3(x)$ are $x_1 = -\sqrt{\frac{3}{5}}$, $x_2 = 0$, $x_3 = \sqrt{\frac{3}{5}}$. The coefficient c_1 is

$$c_1 = \int_{-1}^{1} \frac{(x-0)(x-\sqrt{\frac{3}{5}})}{(-\sqrt{\frac{3}{5}}-0)(-\sqrt{\frac{3}{5}}-\sqrt{\frac{3}{5}})}\,dx = \frac{1}{2(\sqrt{\frac{3}{5}})^2} \int_{-1}^{1} x(x-\sqrt{\tfrac{3}{5}})\,dx = \frac{5}{6}\left[\frac{1}{3}x^3 - \frac{1}{2}\sqrt{\frac{3}{5}}x^2\right] = \frac{5}{9}$$

The other coefficients can be determined likewise. Actually, since

$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

is a rule with precision 3, all we need is to evaluate L_2, L_3 in a couple places.

$$L_2(x) = \frac{\left(x+\sqrt{\frac{3}{5}}\right)\left(x-\sqrt{\frac{3}{5}}\right)}{-\left(\sqrt{\frac{3}{5}}\right)^2} = -\frac{5}{3}\left(x+\sqrt{\frac{3}{5}}\right)\left(x-\sqrt{\frac{3}{5}}\right) \longrightarrow c_2 = L_2\left(\frac{-\sqrt{3}}{3}\right) + L_2\left(\frac{\sqrt{3}}{3}\right)$$

$$L_3(x) = \frac{\left(x+\sqrt{\frac{3}{5}}\right)(x-0)}{2\sqrt{\frac{3}{5}}\cdot\sqrt{\frac{3}{5}}} = \frac{5}{6}\left(x+\sqrt{\frac{3}{5}}\right)x \longrightarrow c_3 = L_3\left(\frac{-\sqrt{3}}{3}\right) + L_3\left(\frac{-\sqrt{3}}{3}\right)$$

This is a convenient approach given that the Legendre polynomials are generally generated recursively, i.e.,

$$\begin{cases} \pi_{k+1}(x) = x\,\pi_k(x) - (4-k^{-2})^{-1}\pi_{k-1}(x), & k>1 \\ \pi_1(x) = x, \quad \pi_0(x) = 1 \end{cases}$$

We'll talk more about this later.