Midterm Review

Midterm this Fri!

(1a) How do machines store numbers?

e.g.) Consider 3-digit rounding.

$$x = \underbrace{0.d_1 d_2 \ldots d_k}_{\text{mantissa}} \times 10^{\overset{\text{exponent}}{n}}$$

$$\sum_{n=0}^{4} 10^{-n} = 1 + 0.1 + 0.01 + 0.001 + \underbrace{0.0001}_{0.1 \times 10^{-3}}$$

$$= 1.1 + 0.01 + 0.001 + 0.0001$$

$$= \underbrace{1.11 + 0.001} + 0.0001$$

$$= 1.11 + 0.0001$$

$$x^* = 1.11 = 0.111 \times 10^1$$

Exact: $1.1111 = x$

Approx: $1.11 = x^*$

(1b) Measuring & predicting abs. and rel. error.

$$|x - x^*| = |1.1111 - 1.11|$$

$$= 0.0011$$

absolute: $|x - x^*|$

relative: $\dfrac{|x - x^*|}{|x|}$ ←

$$\frac{|x - x^*|}{|x|} = \frac{0.0011}{1.1111} = 0.0009\ldots$$

0.005

If $x = 0.d_1 d_2 d_3 \ldots \times 10^n$, then in 3-digit arithmetic,

$$x^* = \begin{cases} 0.d_1 d_2 d_3 \times 10^n, & d_4 < 5 \\ 0.d_1 d_2 d_3 \times 10^n + 1 \times 10^{n-3}, & d_4 \geq 5 \end{cases}$$

k

$$|x - x^*| = |0.d_4 d_5 \ldots \times 10^{n-3}| \leq \boxed{0.5 \times 10^{n-3}} \quad \text{if } d_4 < 5$$

$$|x - x^*| = |0.d_1 d_2 \ldots \times 10^n - 0.d_1 d_2 d_3 \times 10^n - 1 \times 10^{n-3}| \quad \text{if } d_4 \geq 5$$

k

$$= |0.d_4 d_5 \ldots \times 10^{n-3} - 1 \times 10^{n-3}| \leq \boxed{0.5 \times 10^{n-3}}$$

3-digit

$x = 0.d_1 d_2 \ldots \times 10^n, \quad d_1 \geq 1$

$|x| \geq 0.1 \times 10^n$

$$\frac{|x-x^*|}{|x|} \leq \frac{0.5 \times 10^{n-3}}{|x|} \leq \frac{0.5 \times 10^{n-3}}{0.1 \times 10^n} \leq 5 \times 10^{-3}$$

$$= \boxed{0.5 \times 10^{-2}} \leftarrow \text{eps}$$

(1c) Give examples or predict error-producing cancelation.

e.g.) $f(x) = 1 - \sqrt{1+x^2}$

Cancelation occurs when two numbers of nearly equal magnitude but opp. sign are added

3-digit

$x = 0.1$:

$f(0.1) = 1 - \sqrt{1 + 0.01}$ $\quad \sqrt{1.01} = 1.00498\ldots$

$= 1 - \sqrt{1.01}$

$= 1 - 1 = 0$

$x = 1$:

$f(1) = -0.41$

$$\frac{|f(0.1) - 0|}{|f(0.1)|} = \frac{|f(0.1)|}{|f(0.1)|} = 1$$

$$\frac{|f(1) + 0.41|}{|f(1)|} = 0.001017\ldots$$

(1d) Rearrange calculations to minimize possible cancelation issues.

$$f(x) = (1 - \sqrt{1+x^2})\left(\frac{1 + \sqrt{1+x^2}}{1 + \sqrt{1+x^2}}\right) = \frac{-x^2}{1 + \sqrt{1+x^2}} := g(x)$$

$$g(0.1) = \frac{-0.1^2}{1 + \underbrace{\sqrt{1+0.1^2}}}$$

$$= \frac{-0.01}{1+1} = -0.005$$

Vieta's: $(x - x_1)(x - x_2)$

$x^2 - x_2 x - x_1 x + \underbrace{x_1 x_2}$

$$h(x) = 1 - \frac{1}{(x^4 + 1)} = \frac{x^4}{x^4 + 1}$$

$h(0.1) = 0$

(1e) Characterize algorithms and their stability with regards error-propogation.

linear prop. is okay! stable

exp prop is bad! unstable

$n!,\ n^n,\ n!^{n!}\ldots$

$b_{n+1} = 4b_n - 1, \quad b_0 = \frac{1}{3}$

$= 4[4b_{n-1} - 1] - 1 = 4^2 b_{n-1} - (4+1)$

$= 4^2[4b_{n-2} - 1] - (4+1)$

$= 4^3 b_{n-2} - (4^2+4+1)$

$\vdots$

$$\boxed{b_{n+1} = 4^{n+1} b_0 - \sum_{k=0}^{n} 4^k}$$

$c_{n+1} = \frac{1}{4}(c_n + 1), \quad c_0 = 0$

$\vdots$

$c_{n+1} = \frac{1}{4^{n+1}} c_0 + \frac{1}{3}$

$$|c_{n+1} - c^*_{n+1}| = \underbrace{\frac{1}{4^{n+1}}}_{\text{neg. exp.}} |c_0 - c^*_0|$$

~~errored~~ contaminated $b_0 = b^*_0$

$b^*_{n+1} = 4^{n+1} b^*_0 - \sum_{k=0}^{n} 4^k$

$$|b_{n+1} - b^*_{n+1}| = |4^{n+1} b_0 - \sum_{k=0}^{n} 4^k - 4^{n+1} b^*_0 + \sum_{k=0}^{n} 4^k|$$

$$= \underbrace{4^{n+1}}_{\text{exp.}} |b_0 - b^*_0|$$