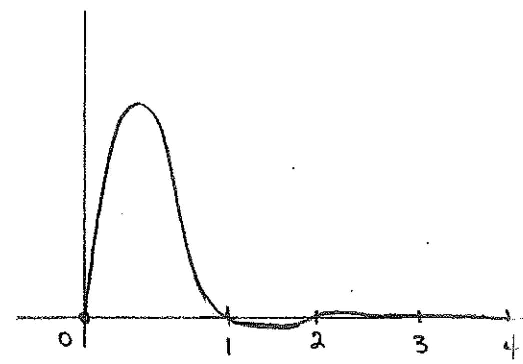
Adaptive quadrature

Composite integration techniques can ~~fail~~ be ~~misleading~~ more work than necessary when integrating a function on an interval where the variation of the function is ~~unpredictable~~. both large and small.

e.g.) Consider integrating $f(x) = 5e^{-5x}\sin(x)$ on $[0,4]$.

Should we use a small step size or a large step size?



- On $[0,1]$, it makes sense to sample the function more.
- On $[1,4]$ (even more so on $[2,4]$) it makes sense to use a large step size and sample the function less.

Which is a more accurate approximation to $\int_0^4 f(x)\,dx$?

$$\underbrace{T^{0.5}_{[0,2]}(f)}_{\text{9 evaluations of } f} \qquad \underbrace{T^{0.25}_{[0,1]}(f) + T^{1}_{[1,4]}(f)}_{\text{8 evaluations of } f} \qquad \underbrace{T^{0.25}_{[0,1]}(f) + T^{1}_{[1,2]}(f) + T^{2}_{[2,4]}(f)}_{\text{7 evaluations of } f}$$

$$\int_0^4 5e^{-5x}\sin(x)\,dx = \frac{5}{26}e^{-20}\left(1 - 5\sin(4) - 5\cos(4)\right) \approx 0.19230769406\ldots$$

$T^{0.5}_{[0,4]}(f) = 0.114046687045633$

$T^{0.25}_{[0,1]}(f) + T^{1}_{[1,4]}(f) = 0.182845273127360$

$T^{0.25}_{[0,1]}(f) + T^{1}_{[1,2]}(f) + T^{2}_{[2,4]}(f) = 0.183154660878357$

For regions where the values of the function ~~oscillate wildly~~ vary a lot, a smaller step size is needed. When the values don't vary too much, a larger step size can be used. Can we derive a scheme where the step size is adapted on various portions of the interval of integration?

Suppose we wish to approximate $\int_a^b f(x)dx$ with the Trapezoid rule with tolerance $\varepsilon>0$. When $b-a=h$ (e.g., $n=1$),

$$\int_a^b f(x)\,dx = T^h_{[a,b]}(f) - \frac{h^2}{12}\overset{(b-a)=h}{} f''(\xi) = T^h_{[a,b]}(f) - \frac{h^3}{12} f''(\xi) \qquad (*)$$

~~Similarly when b-a~~ If we apply the Trapezoid rule with step size $\frac{h}{2}$,

$$\int_a^b f(x)dx = \underbrace{T^{h/2}(f)} - \frac{\left(\frac{h}{2}\right)^2}{12}\left(2\frac{h}{2}\right) f''(\xi'), \quad \xi' \in (a,b)$$

$$= T^h_{[a,\frac{a+b}{2}]}(f) + T^h_{[\frac{a+b}{2},b]}(f) - \frac{1}{4}\frac{h^3}{12} f''(\xi'). \qquad (**)$$

Assume $f''(\xi) \approx f''(\xi')$ (will discuss the validity of this later). Then

$$T_{[a,\frac{a+b}{2}]}(f) + T_{[\frac{a+b}{2},b]}(f) - \frac{h^3}{48} f''(\xi') = T_{[a,b]}(f) - \frac{h^3}{12} f''(\xi)$$

$$\frac{h^3}{12} f''(\xi) \approx \frac{4}{3}\left\{ T_{[a,b]}(f) - T_{[a,\frac{a+b}{2}]}(f) - T_{[\frac{a+b}{2},b]}(f) \right\}$$

Using this in $(**)$,

$$\left| \int_a^b f(x) - T_{[a,\frac{a+b}{2}]}(f) - T_{[\frac{a+b}{2},b]}(f) \right| \approx \frac{1}{3} \underbrace{\left| T_{[a,b]}(f) - T_{[a,\frac{a+b}{2}]}(f) - T_{[\frac{a+b}{2},b]}(f) \right|}_{(***)},$$

i.e., $T_{[a,\frac{a+b}{2}]}(f) + T_{[\frac{a+b}{2},b]}(f)$ approximates about 3 times better than $T_{[a,b]}(f)$,

and so if $(***) < 3\varepsilon$, we will have $\int_a^b f(x)dx$ within ε.