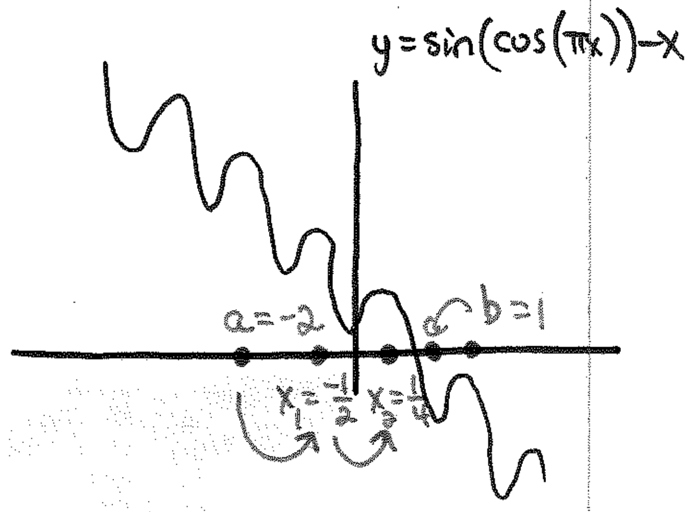
Midterm Review (Pt. 2)

Midterm FRIDAY!
PS #4 due FRI.

(2a) Apply bisection method to solve $f(x)=0$.

<u>thm</u>: (IVT) Suppose f is continuous on $[a,b]$, and if $f(a)\cdot f(b)<0$, then there exists an $x\in(a,b)$ with $f(x)=0$.



$\sin(\cos(\pi x))-x=0$

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input a, b
for n=1,2,3,...
  x_n = 1/2(a+b)
  if f(x_n)·f(a)<0
     b = x_n
  else
     a = x_n
  end
end
```

gaurantees $x\in[a,x_n]$ or $x\in[x_n,b]$

(2b) Apply fixed-point iteration to solve $g(x)=x$.

<u>thm</u>: Suppose for some interval $[a,b]$ we have the following conditions.

(i) $g(\cancel{x}\,y)\in[a,b]$ for $\cancel{x}\,y\in[a,b]$

(ii) $|g'(\cancel{x}\,y)|<1$ for $\cancel{x}\,y\in[a,b]$.

Then there exists a point $x\in[a,b]$ s.t. $g(x)=x$.

Moreover if $x_0\in[a,b]$, then the sequence generated by $x_n=g(x_{n-1})$, $n\geq1$ converges to x.

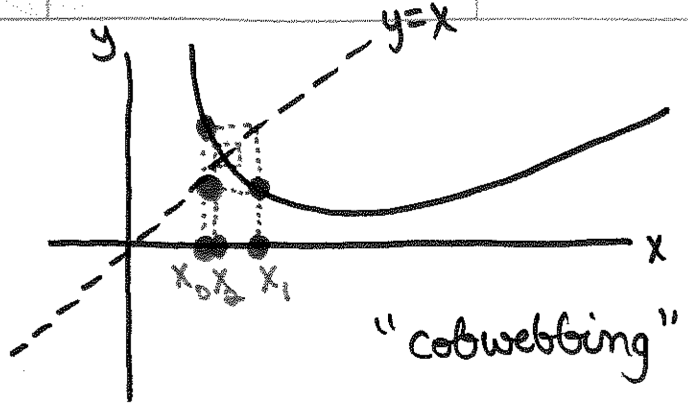
input x_0

for $n=1,2,3,\ldots$

$x_n = g(x_0)$

$x_0 = x_n$

end



"cobwebbing"

$$\underbrace{y = \frac{1}{2}x + \frac{3}{x}}_{g(x)}$$

Verify $\{x_n\}$ converges for any $x_0 \in [2,5]$.

(i) $y \in [2,5]$, is $g(y) \in \underbrace{[2,5]}$?

$g(2) = 2.5$

$g(5) = 3.1$

$g(\sqrt{6}) = \underbrace{\frac{\sqrt{6}}{2}} + \underbrace{\frac{3}{\sqrt{6}}} \leq 4$

$g(x) = \frac{1}{2}x + \frac{3}{x}$

$g'(x) = \frac{1}{2} - \frac{3}{x^2}$

$= \frac{x^2 - 6}{2x^2} = 0$

$x = 0, \pm\sqrt{6}$

$g''(y) = \frac{6}{x^3} = 0$

$x = 0$

(ii) $|g'(y)| < 1$

$g'(2) = -\frac{1}{4} \leftarrow$

$g'(5) = 0.38 \leftarrow$

$|g'(y)| < 1$

(2c) Apply Newton's method to solve $f(x) = 0$.

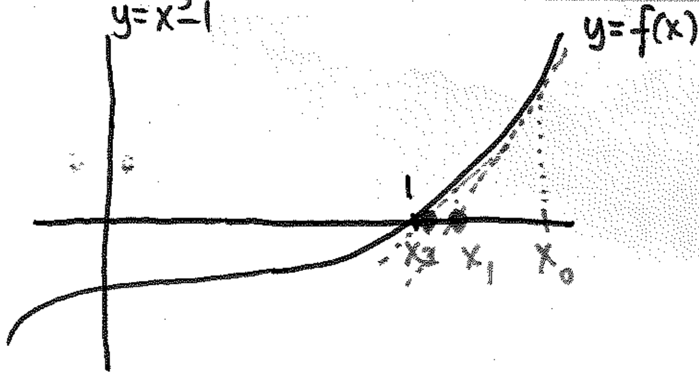
<u>thm</u>: Suppose x is a simple root of f, i.e., $f(x) = 0$ but $f'(x) \neq 0$. Then there exists a $\delta > 0$ so that the sequence

$$\boxed{x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}}, \quad n \geq 1$$

RHS $g(x)$

converge for any $x_0 \in [x-\delta, x+\delta]$.

input x_0
for $n=1,2,3,\ldots$
  $x_n = x_0 - \frac{f(x_0)}{f'(x_0)}$
  $x_0 = x_n$
end



$$y = f'(x_0)(x-x_0) + f(x_0)$$

↑ $y=0$ is x-int.

(2d) Summarize in pseudocode or graphically how they converge.

(2e) Compute <u>order of convergence</u> and <u>asymptotic error constants</u>. (higher α, faster) (small λ, faster)

<u>def</u>: A sequence converges with order α if there exists a $\lambda < \infty$ with

$$\lim_{n\to\infty} \frac{|x_{n+1}-x|}{|x_n - x|^{\alpha}} = \lambda.$$

Linear $\begin{cases} \text{Sub linear } (\lambda=1) \\ 0<\lambda<1 \\ \text{Super linear } (\lambda=0) \end{cases}$

Quadratic $\alpha=2$

Cubic $\alpha=3$

e.g.) $x_n = 2^{-7\cdot 2^n}$, $\lim_{n\to\infty} x_n = 0$

$$\frac{|2^{-7\cdot 2^{n+1}}|}{|2^{-7\cdot 2^n}|} = \frac{(2^{-7\cdot 2^n})^2}{2^{-7\cdot 2^n}} = 2^{-7\cdot 2^n} \xrightarrow[n\to\infty]{} 0$$

x_n converges at least super linearly

$$\frac{2^{-7\cdot 2^{n+1}}}{(2^{-7\cdot 2^n})^2} = \frac{(2^{-7\cdot 2^n})^2}{(2^{-7\cdot 2^n})^2} = 1 \xrightarrow[n\to\infty]{} 1$$

x_n converges quadratically

(2g) Accelerate by applying Aitken's Δ^2-method or Steffensen's.

apply to any sequence

only for fixed-point iterations

Δ^2-method: $\{x_n\}$

$$\hat{x}_n = x_n - \frac{(\Delta x_n)^2}{\Delta^2 x_n}, \quad \Delta x_n = x_{n+1} - x_n$$

Steffensens:

$x_n = g(x_{n-1})$

input x_0
for $n = 1, 2, 3, \ldots$
  $x_1 = g(x_0)$
  $x_2 = g(x_1)$
  $$\hat{x}_n = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0} \leftarrow (x_2 - x_1) - (x_1 - x_0)$$
  $x_0 = \hat{x}_n$
end