

Recall Newton's form of the interpolating polynomial

$$\begin{cases} P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) \\ \quad + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1}) \\ a_k = f[x_0, x_1, \dots, x_k] \\ f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i} \end{cases}$$

The polynomial $P_n(x)$ interpolates $(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))$.

$$a_0 = f(x_0)$$

e.g.) Consider the data $(1, 0.77), (1.3, 0.22), (1.6, 0.86), (1.9, 0.28),$

$(2.2, 0.11)$.

oth divided diff.

i	x_i	$f(x_i)$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, x_{i-2}, x_{i-1}, x_i]$
0	1.0	0.77	—	—	—
1	1.3	0.22	-1.83	—	—
2	1.6	0.86	2.13	6.61	—
3	1.9	0.28	-1.93	-6.78	—
4	2.2	0.11	-0.57	2.28	—

$f[x_{i-3}, x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-4}, x_{i-3}, x_{i-2}, x_{i-1}, x_i]$
—	—
—	—
—	—
-14.48	20.8
10.1	

$$f[x_0, x_1] = \frac{0.22 - 0.77}{x_1 - x_0} = \frac{0.22 - 0.77}{1.3 - 1.0} = -1.83$$

$$f[x_1, x_2] = \frac{0.86 - 0.22}{x_2 - x_1} = \frac{0.86 - 0.22}{1.6 - 1.3} = 2.13$$

$$f[x_0, x_1, x_2] = \frac{2.13 - (-1.83)}{x_2 - x_0} = \frac{2.13 + 1.83}{1.6 - 1.0} = 6.61$$

$$f[x_2, x_3] = \frac{0.28 - 0.86}{x_3 - x_2} = \frac{0.28 - 0.86}{1.9 - 1.6} = -1.93$$

$$f[x_3, x_4] = \frac{0.11 - 0.28}{x_4 - x_3} = \frac{0.11 - 0.28}{2.2 - 1.9} = -0.57$$

$$f[x_1, x_2, x_3] = \frac{-1.93 - 2.13}{x_3 - x_1} = \frac{-1.93 - 2.13}{1.9 - 1.3} = -6.78$$

$$f[x_2, x_3, x_4] = \frac{-0.57 - (-1.93)}{x_4 - x_2} = \frac{-0.57 + 1.93}{2.2 - 1.6} = 2.28$$

$$f[x_0, x_1, x_2, x_3] = \frac{-6.78 - 6.61}{x_3 - x_0} = \frac{-6.78 - 6.61}{1.9 - 1.0} = -14.48$$

$$f[x_1, x_2, x_3, x_4] = \frac{2.28 - (-6.78)}{x_4 - x_1} = \frac{2.28 + 6.78}{2.2 - 1.3} = 10.1$$

$$f[x_0, x_1, x_2, x_3, x_4] = \frac{10.1 - (-14.48)}{x_4 - x_0} = \frac{10.1 + 14.48}{2.2 - 1.0} = 20.8$$

The polynomial of degree 4 interpolates all the data

$$\begin{aligned}
 P_4(x) = & 0.77 + -1.83(x-1) + 6.61(x-1.3)(x-1) \\
 & a_0 + a_1(x-x_0) + a_2(x-x_1)(x-x_0) \\
 & + -14.48(x-1)(x-1.3)(x-1.6) \\
 & a_3(x-x_0)(x-x_1)(x-x_2) \\
 & + 20.8(x-1)(x-1.3)(x-1.9)(x-1.6) \\
 & a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3)
 \end{aligned}$$

$$a_4 = f[x_0, x_1, x_2, x_3, x_4]$$

{ input $x_0, x_1, \dots, x_n, f(x_0) = F_{0,0}, f(x_1) = F_{1,0}, \dots, f(x_n) = F_{n,0}$
 for $i=1, 2, \dots, n$
 for $j=1, 2, \dots, i$
 $F_{i,j} = (F_{i,j-1} - F_{i-1,j-1}) / (x_i - x_{i-j})$
 end
 end

Generates a table of divided differences for data.

$$\begin{pmatrix}
 x_0 & F_{0,0} & & & \\
 x_1 & F_{1,0} & F_{1,1} & & \\
 x_2 & F_{2,0} & F_{2,1} & F_{2,2} & \\
 x_3 & F_{3,0} & F_{3,1} & \vdots & \ddots \\
 \vdots & \vdots & \vdots & &
 \end{pmatrix}$$