Error for composite trapezoid rule:

Recall on one interval $[x_0, x_1]$, the trapezoid rule is given by

$$\int_{x_0}^{x_1} f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + f(x_1)\right],$$

where $x_1 - x_0 = h$. If we partition an interval $[a,b] = [x_0,x_1] \cup \ldots \cup [x_{n-1}, x_n]$ with $h = x_i - x_{i-1}$ for $i = 1,2,\ldots,n$, we get the composite trapezoid rule

$$\begin{aligned}
\int_a^b f(x)\,dx &= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx \\
&\approx \sum_{i=1}^{n} \frac{h}{2}\left[f(x_{i-1}) + f(x_i)\right] \\
&\approx \frac{h}{2}\left[f(a) + f(b) + \sum_{i=1}^{n-1} f(x_i)\right] := T^h_{[a,b]}(f)
\end{aligned}$$

Recall we used Lagrange interpolation on each subinterval, i.e.,

$$f(x) = P_i(x) + \frac{f^{(2)}(\xi_i)}{2!}(x - x_{i-1})(x - x_i)$$

for $x \in [x_{i-1}, x_i]$, where $\xi_i \in [x_{i-1}, x_i]$. On each subinterval,

$$\int_{x_{i-1}}^{x_i} f(x)\,dx - \int_{x_{i-1}}^{x_i} P_i(x)\,dx = \frac{1}{2}\int_{x_{i-1}}^{x_i} f^{(2)}(\xi_i)(x - x_{i-1})(x - x_i)\,dx.$$

Notice $\omega_i(x) = (x - x_{i-1})(x - x_i) \leq 0$ on $[x_{i-1}, x_i]$. Moreover

$$\int_{x_{i-1}}^{x_i} (x - x_{i-1})(x - x_i)\,dx$$

$$= \frac{1}{2}(x - x_{i-1})^2(x - x_i)\Big|_{x_{i-1}}^{x_i} - \frac{1}{6}(x - x_{i-1})^3\Big|_{x_{i-1}}^{x_i} = \frac{-1}{6}(x_i - x_{i-1})^3$$

If $f \in C^2[a,b]$, then $f^{(2)}$ attains maximum M_i and minimum m_i on $[x_{i-1}, x_i]$. In particular,

$$m_i \cdot W_i \leq \int_{x_{i-1}}^{x_i} f^{(2)}(\xi_i(x))\, \omega_i(x)\, dx \leq M_i \cdot W_i,$$

where $W_i = \int_{x_{i-1}}^{x_i} \omega_i(x)\, dx$. In particular,

$$m_i \leq \frac{1}{W_i} \int_{x_{i-1}}^{x_i} f^{(2)}(\xi_i(x))\, \omega_i(x)\, dx \leq M_i.$$

By IVT, there exists a point $\zeta_i \in [x_{i-1}, x_i]$ with $W_i f^{(2)}(\zeta_i) = \int_{x_{i-1}}^{x_i} f^{(2)}(\xi_i)\omega_i(x)\,dx$.

Hence

$$\frac{1}{2}\int_{x_{i-1}}^{x_i} f^{(2)}(\xi_i)\, \omega_i(x)\, dx = \frac{1}{2} f^{(2)}(\zeta_i) \int_{x_{i-1}}^{x_i} \omega_i(x)\,dx = \frac{-1}{12} h^3 f(\zeta_i).$$

This is only one interval!

$$\left| \int_a^b f(x)\,dx - T^h_{[a,b]}(f) \right| = \left| \sum_{i=1}^{n} \frac{1}{12} h^3 f^{(2)}(\zeta_i) \right|$$

$$\leq \sum_{i=1}^{n} \frac{1}{12} h^3 \left| f^{(2)}(\zeta_i) \right|$$

If $f^2 \in C[a,b]$, then $\sum_{i=1}^n |f^{(2)}(\zeta_i)| \leq nM$, $M = \max_{x\in[a,b]} |f^{(2)}(x)|$, and so

$$\left| \int_a^b f(x)\,dx - T^h_{[a,b]}(f) \right| \leq M \frac{n}{12} h^3 = M \frac{(b-a)}{12} h^2.$$

How is n related to h?

$$b - a = \sum_{i=1}^{n} x_i - x_{i-1} = \sum_{i=1}^{n} h = nh$$

We say $T^{h}_{[a,b]}(f)$ is $O(h^2)$ since

$$T^{h}_{[a,b]}(f) = \int_a^b f(x)\,dx + O(h^2).$$

~~e.g.) We saw for~~ $\cancel{f(x) = x^2\sqrt{x^3+1},\ [a,b]=[-1,1],\ h=0.5,}$

$$\cancel{T^{0.5}_{[-1,1]}(f) = \frac{0.5}{2}\left[0 + \sqrt{2} + 2\left\{\frac{1}{4}\sqrt{\frac{7}{8}} + 0 + \frac{1}{4}\sqrt{\frac{9}{8}}\right\}\right]}$$

$$= \cancel{0.60306270540\ldots}$$

~~Since~~ $\cancel{\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big|_0^2 = \frac{2}{9}\left\{\sqrt{8}-0\right\}} = 0.62853936105\ldots$ ~~Notice~~

$$\cancel{f'(x) = 2x\sqrt{x^3+1} + \frac{3x^4}{2\sqrt{x^3+1}}}$$

$$\cancel{f''(x) = 2\sqrt{x^3+1} + \frac{9x^3}{\sqrt{x^3+1}} - \frac{9x^6}{4(x^3+1)^{3/2}}}$$

e.g.) Consider $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\exp\left[\frac{-x^2}{2}\right]dx$, $h=0.5$.

$$T^{0.5}_{[-1,1]}(f) = \frac{0.5}{2}\left[f(-1) + f(1) + 2f(-\tfrac{1}{2}) + 2f(0) + 2f(\tfrac{1}{2})\right]$$

$$= \frac{0.25}{\sqrt{2\pi}}\left[e^{\frac{-1}{2}} + e^{\frac{-1}{2}} + 2e^{\frac{-1}{8}} + 2 + 2e^{\frac{-1}{8}}\right]$$

$$= 0.672521829 22\ldots$$

Hence,

$$\left|\int_{-1}^{1}\frac{\exp\left[\frac{-x^2}{2}\right]}{\sqrt{2\pi}}dx - T^{0.5}_{[-1,1]}(f)\right| \le \frac{(1+1)}{12}\max_{x\in[-1,1]}|f^{(2)}(x)|\cdot(0.5)^2 \le \frac{1}{24\sqrt{2\pi}} = 0.016622595\ldots$$

where $f'(x) = \frac{-x}{\sqrt{2\pi}}\exp\left[\frac{-x^2}{2}\right]$, $f''(x) = -f(x) + x^2 f(x) = (1-x^2)f(x)$.