$\boxed{\text{Assessment \#1}}$

Reminders: Assessment #3 on Friday (Practice problems, Sound?, Video)
PS #3 due Friday (look at PS #4 too)
Finals schedule should be on Lionpath

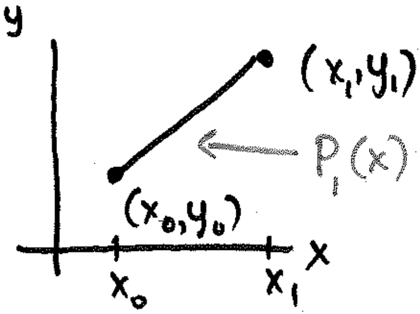
Also... midterm on 10/05. Study guide to come...

Thm: There exists a polynomial of degree at least n that interpolates data $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.

Taylor: $\underbrace{x_0 = x_1 = x_2 = \dots = x_n}_{\text{center}}, \quad y_i = f^{(i)}(x_0)$

Lagrange: $x_i \neq x_j$ for $i \neq j$, $y_i = f(x_i)$

e.g.) There exists one degree 1 polynomial interpolates $(x_0, y_0), (x_1, y_1)$.



$$P_1(x) = \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0) + y_0$$

$$= \left(\frac{x - x_0}{x_1 - x_0}\right)(y_1 - y_0) + y_0$$

$$= \left(\frac{x - x_0}{x_1 - x_0}\right) y_1 + \left(1 - \frac{x - x_0}{x_1 - x_0}\right) y_0$$

$$= \frac{x_1 - \cancel{x_0} - (x - \cancel{x_0})}{x_1 - x_0}$$

$$= \underbrace{\left(\frac{x - x_0}{x_1 - x_0}\right)}_{L_1(x)} y_1 + \underbrace{\left(\frac{x - x_1}{x_0 - x_1}\right)}_{L_0(x)} y_0$$

$$= \frac{x - x_1}{x_0 - x_1}$$

$L_1(x_1) = 1, \; L_1(x_0) = 0 \qquad L_0(x_1) = 0, \; L_0(x_0) = 1$

In particular...

$$P_1(x) = \underbrace{L_0(x)}_{\text{linear functions}} y_0 + \underbrace{L_1(x)}_{\text{degree 1}} y_1$$

In general, this is what LIP look like. For the data $(x_0, y_0), \dots, (x_n, y_n)$,

$$\underbrace{P_n(x)}_{\text{degree } n} = L_0(x)y_0 + L_1(x)y_1 + \dots + L_n(x)y_n,$$

where $L_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j. \end{cases}$

e.g.) $L_1(x_1) = 1$, $L_5(x_2) = 0$, ...

We want $L_i(x_j) = 0$ for $i \neq j$, then

$$L_i(x) \sim \underbrace{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}$$

we want all nodes other than x_i to be a root of L_i

To gaurantee $L_i(x_i) = 1$, we take

$$L_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

$$\boxed{L_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{(x-x_k)}{(x_i-x_k)}}$$

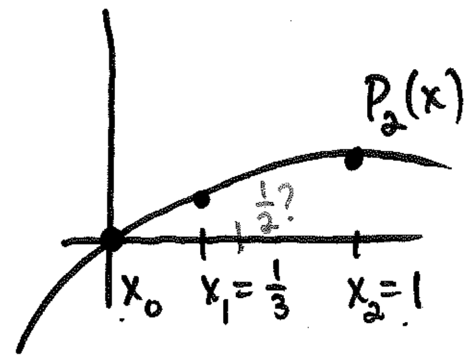
elementary Lagrangian polynomials for nodes $x_0, x_1, \dots, x_n$

Then

$$\boxed{P_n(x) = \sum_{k=0}^{n} L_k(x) y_k}$$

LIP formula

e.g.) Consider $(0,0)$, $(\frac{1}{3},\frac{1}{2})$, $(1,1)$. Suppose we want to find a quadratic interpolant.



$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$= \frac{(x-\frac{1}{3})(x-1)}{(0-\frac{1}{3})(0-1)} \longleftarrow$$

$$= \frac{1}{3}(x-\frac{1}{3})(x-1)$$

$$L_1(x) = \frac{(x-0)(x-1)}{(\frac{1}{3}-0)(\frac{1}{3}-1)} = -\frac{9}{2}(x)(x-1)$$

$$L_2(x) = \frac{x(x-\frac{1}{3})}{1(1-\frac{1}{3})} = +\frac{3}{2}x(x-\frac{1}{3})$$

$$\Rightarrow P_2(x) = \underbrace{\frac{1}{3}(x-\frac{1}{3})(x-1)}_{L_0(x)} \cdot \underbrace{0}_{y_0} - \underbrace{\frac{9}{2}x(x-1)}_{L_1(x)} \cdot \underbrace{\frac{1}{2}}_{y_1} + \frac{3}{2}x(x-\frac{1}{3}) \cdot 1$$

$$= -\frac{9}{4}x(x-1) + \frac{3}{2}x(x-\frac{1}{3})$$

The data comes from $f(x) = \sin(\frac{\pi}{2}x)$. What if we approximate $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$?

$$P_2(\frac{1}{2}) = -\frac{9}{4}(\frac{1}{2})(\frac{1}{2}-1) + \frac{3}{2}(\frac{1}{2})(\frac{1}{2}-\frac{1}{3}) = \frac{11}{16}$$

$$= 0.6875$$

This isn't bad: $\left|\frac{11}{16} - \frac{\sqrt{2}}{2}\right| \leq 0.02$.

thm: Suppose $f \in C^{n+1}$. For any $x \in [\min_{0 \le k \le n} x_k, \max_{0 \le k \le n} x_k]$, there exists a $\xi = \xi(x)$ lying in this interval s.t.

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n).$$

In particular,

$$|f(x) - P_n(x)| \le \frac{f^{(n+1)}(\xi)}{(n+1)!} \underbrace{|x-x_0||x-x_1|\cdots|x-x_n|}$$

e.g.) ~~Ex~~ $f(y) = \sin\left(\frac{\pi}{2}y\right)$, $n=2$

$$f^{(3)}(y) = -\left(\frac{\pi}{2}\right)^3 \cos\left(\frac{\pi}{2}y\right)$$

$$|f^{(3)}(y)| \le \frac{\pi^3}{8} \text{ since } \left|\cos\left(\frac{\pi}{2}y\right)\right| \le 1$$

$$\Rightarrow \left|f\left(\tfrac{1}{2}\right) - P_2\left(\tfrac{1}{2}\right)\right| \le \frac{\pi^3}{8} \cdot \frac{1}{(2+1)!} \cdot \underbrace{\left|\tfrac{1}{2}-0\right| \cdot \left|\tfrac{1}{2}-\tfrac{1}{3}\right| \cdot \left|\tfrac{1}{2}-1\right|}$$

$$= \frac{\pi^3}{48} \cdot \left(\tfrac{1}{2}\right)\left(\tfrac{1}{6}\right)\left(\tfrac{1}{2}\right) = 0.0269\ldots$$