Reminders: PS #2 due Friday
Witkowski 110 on Friday

§2.5 Accelerating convergence

def: A sequence $\{x_n\}$ converges linearly to x if there exists $\lambda \in (0,1)$ such that

$$\lim_{n\to\infty} \frac{|x_{n+1}-x|}{|x_n - x|^1} = \lambda .$$

For really large n,

$$\frac{x_{n+1}-x}{x_n - x} \approx \lambda$$

Then

$$\frac{x_{n+2}-x}{x_{n+1}-x} \approx \frac{x_{n+1}-x}{x_n - x}$$

Solving for x...

$$(x_{n+2}-x)(x_n - x) \approx (x_{n+1}-x)^2$$

$$x_{n+2}x_n - x x_{n+2} - x x_n + x^2 \approx x_{n+1}^2 - 2x x_{n+1} + x^2$$

$$(2x_{n+1} - x_{n+2} - x_n)x \approx x_{n+1}^2 - x_{n+2}x_n$$

$$x \approx \frac{x_{n+1}^2 - x_{n+2}x_n}{2x_{n+1} - x_{n+2} - x_n}$$

Aitken's Δ^2-method

thm: If $x_n \to x$ linearly, then $\{\hat{x}_n\}$ defined by

$$\hat{x}_n = \frac{x_{n+1}^2 - x_{n+2}x_n}{2x_{n+1} - x_{n+2} - x_n}$$

converges to x faster than x_n, i.e.,

$$\lim_{n\to\infty} \frac{|\hat{x}_n - x|}{|x_n - x|} = 0$$

$\leftarrow$ abs. error for $\hat{x}_n$

$|\hat{x}_n - x|$ decays faster

Typically we use the notation $\Delta x_n = x_{n+1} - x_n$,

$$\boxed{\hat{x}_n = x_n - \frac{(\Delta x_n)^2}{\Delta^2 x_n}}$$

$(\Delta x_n)^2 = (x_{n+1} - x_n)^2$

$\Delta^2 x_n = \Delta(\Delta x_n)$

$= \Delta(x_{n+1} - x_n)$

$= [x_{n+2} - x_{n+1}] - [x_{n+1} - x_n]$

$= x_{n+2} - 2x_{n+1} + x_n$

To generate $\hat{x}_n$, you need to know x_n, x_{n+1}, x_{n+2}. To apply ~~this~~ to a fixed-point iteration, we stagger our application of Δ^2-method.

$x = 10^{-x}$

e.g.) Consider $x_n = 10^{-x_{n-1}}$, $x_0 = 0.5$.

$$\left.\begin{aligned} x_0 &= 0.5 \\ x_1 &= 10^{-0.5} \\ x_2 &= 10^{-x_1} \end{aligned}\right\} \hat{x}^{(0)} = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0}$$

$\cancel{x_3 = 10^{-x_2}}$

$$\left.\begin{aligned} x_0 &= \hat{x}^{(0)} \\ x_1 &= 10^{-x_0} \\ x_2 &= 10^{-x_1} \end{aligned}\right\} \hat{x}^{(1)} = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0}$$

$$\left.\begin{aligned} x_0 &= \hat{x}^{(1)} \\ &\vdots \end{aligned}\right\} \hat{x}^{(2)} = \ldots$$

thm: For an appropriate x_0, the sequence $\{\hat{x}^{(j)}\}_j$ converges quadratically to a solution of $x = g(x)$.