Gaussian quadrature

All of our previous quadrature formulas have fallen into the category of Newton-Cotes formulas (i.e., "interpolate and integrate"). We measured error in integration according to error in interpolation.

def: The precision n of a quadrature formula is the largest positive integer such that the formula is exact for x^k, $k = 0, 1, \ldots, n$.

e.g.) The Trapezoid Rule has precision $n = 1$.

$$\int_a^b dx = b - a \qquad T_{[a,b]}(1) = \frac{b-a}{2}\{1+1\} = b-a$$

$$\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2) = \frac{1}{2}(b-a)(b+a) \qquad T_{[a,b]}(x) = \frac{b-a}{2}\{a+b\} = \frac{1}{2}(b-a)(b+a)$$

e.g.) Simpson's Rule has precision $n = 3$. (This is interesting because Simpson's rule resulted from interpolating with a quadratic polynomial.)

In Gaussian quadrature, coefficients and points of evaluation in $[a,b]$ (i.e., nodes) are chosen to optimize precision. That is, coefficients c_i and nodes x_i in the formula

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \overbrace{c_i}^{c_1, c_2, \ldots, c_n \text{ (n parameters)}} f(\underbrace{x_i}_{x_1, x_2, \ldots, x_n \text{ (n parameters)}})$$

are chosen so the formula is exact for as many polynomials as possible. With $2n$ parameters to choose, we should be able to develop a rule with precision $2n-1$ (polynomials of degree $2n-1$ have $2n$ coefficients).

$\boxed{n=1}$ We choose c_1, x_1 so that

$$\int_{-1}^{1} x\,dx = c_1 x_1 \qquad\qquad \int_{-1}^{1} dx = c_1$$

$$\tfrac{1}{2}\left[1-(-1)^2\right] = 2x_1 \qquad\qquad 2 = c_1$$

$$0 = x_1$$

Then a quadrature rule on $[-1,1]$ with precision 1 is

$$\int_{-1}^{1} f(x)\,dx = 2f(0),$$

which is exactly the midpoint (or centered-rectangle) rule.

$\boxed{n=2}$ We can find c_1, c_2, x_1, x_2 so that the rule is exact for up to degree 3!
For the ease of it, we find the rule on $[-1,1]$.

$$\int_{-1}^{1} y^2\,dx = c_1x_1^2 + c_2x_2^2 \qquad \int_{-1}^{1} x\,dy = c_1x_1 + c_2x_2 \qquad \int_{-1}^{1} dx = c_1 + c_2$$

$$\frac{2}{3} = c_1x_1^2 + c_2x_2^2 \qquad 0 = c_1x_1 + c_2x_2 \qquad 2 = c_1 + c_2$$

and

$$\int_{-1}^{1} x^3\,dy = c_1x_1^3 + c_2x_2^3 = 0,$$

which gives 4 (~~linearly~~ independent?) equations with 4 unknowns. With a little difficulty,

$$c_1 = 1,\quad c_2 = 1,\quad x_1 = \frac{-\sqrt{3}}{3},\quad x_2 = \frac{\sqrt{3}}{3},$$

i.e., $\displaystyle\int_{-1}^{1} f(x)\,dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$ is the two-point Gauss quadrature rule on $[-1,1]$.

1. What is the Gauss quadrature rule on arbitrary $[a,b]$?
2. How bad is this rule for functions other than polynomials?
3. Are there other options for finding the correct nodes?
4. If we aren't interpolating, why should the formula be exact for polynomials?

We'll start with 1. Note if we make the change of variables $x = a\left(\frac{1-t}{2}\right) + b\left(\frac{1+t}{2}\right)$,

$$\int_a^b f(x)\,dx = \int_{-1}^{1} g(t)\left(\frac{b-a}{2}\right)dt, \quad t = \frac{2x-a-b}{b-a}, \quad g(t) = f\left(\tfrac{1}{2}\left[a+b+(b-a)t\right]\right)$$

and so $$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{2}\right) f\left(\frac{a+b+(b-a)\frac{-\sqrt{3}}{3}}{2}\right) + \left(\frac{b-a}{2}\right) f\left(\frac{a+b+(b-a)\frac{\sqrt{3}}{3}}{2}\right).$$

We'll spend the next few classes discussing 3.