Midterm: Oct. 12	Oct. 5 is lab in W110
1a – 3b	Lab #2 due on Friday

Recall that the polynomial of degree n interpolating $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ is

$$P_n(x) = \sum_{k=0}^{n} L_k(x) y_k, \quad L_k(x_i) = \begin{cases} 1, & x_i = x_k \\ 0, & x_i \neq x_k \end{cases}$$

Also...

$(x - x_k)$ missing

$$L_k(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$$= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

In particular, $L_k(x)$ depend on n!

e.g.) Interpolate the data $(\overset{x_0}{0}, \overset{y_0}{1}), (\overset{x_1}{1}, \overset{y_1}{-3})$

$$L_0(x) = \frac{x-1}{0-1} \leftarrow \frac{(x-x_1)}{(x_0-x_1)}$$

$$L_1(x) = \frac{x-0}{1-0}$$

$$P_1(x) = 1 \cdot L_0(x) + -3 \cdot L_1(x)$$

What if we add data, say (2,2)? There has to be a better way to do this. (Attributed to Newton) It would convenient to know a $q(x)$ so that

$$\underbrace{P_2(x)}_{\substack{\text{interpolates at} \\ x_0=0,\ x_1=1,\ x_2=2}} = \underbrace{P_1(x)}_{\substack{\text{interpolates} \\ \text{at } x_0=0,\ x_1=1}} + q(x)$$

$\underline{x = x_0}$: $P_2(x_0) = P_1(x_0) + q(x_0) = 1$

$$= 1 + q(x_0) = 1$$

$$\boxed{q(x_0) = 0} \qquad x_0 = 0$$

$\underline{x = x_1}$: $P_2(x_1) = P_1(x_1) + q(x_1) = -3$

$$-3 + q(x_1) = -3$$

$$\boxed{q(x_1) = 0} \qquad x_1 = 1$$

$\underline{x = x_2}$: $P_2(x_2) = P_1(x_2) + q(x_2) = 2$

$$\boxed{q(x_2) = 2 - P_1(x_2)} \qquad x_2 = 2$$

In particular, q should interpolate $(0,0), (1,0), (\underline{\underline{2}}, 2 - P_1(2))$.

$$q(x) = \underbrace{\frac{x(x-1)}{2(2-1)}} \cdot (2 - P_1(2))$$

$$= \underbrace{\frac{2 - P_1(2)}{2(2-1)}}_{\text{constant}} \cdot (x - \underset{x_0}{\underline{\underline{0}}})(x - \underset{x_1}{\underline{\underline{1}}}),$$

i.e., $q(x) = a(x - x_0)(x - 1)$. Can we do this in general? (for any n)

$\underline{n=0}$: $P_0(x) = y_0$

(x_0, y_0)

$\underline{n=1}$: $P_1(x) = y_0 + a_1(x - x_0)$, $a_1 = \dfrac{y_1 - y_0}{x_1 - x_0}$

$(x_0, y_0), (x_1, y_1)$ $\quad y_1 = y_0 + a_1(x_1 - x_0)$ $\quad a_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$\underline{n=2}$:

$(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$$P_2(x) = \underbrace{y_0 + a_1(x - x_0)}_{P_1(x)} + a_2(x-x_0)(x-x_1)$$

first divided difference

$$a_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$a_2 = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_2 - x_0}$$

$$= \frac{f[x_0, x_1] - \frac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_2 - x_0}$$

$$= f[x_0, x_1, x_2]$$

second divided difference

$\underline{n=3}$:

$$P_3(x) = P_2(x) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$a_3 = f[x_0, x_1, x_2, x_3]$$

third divided difference of f

$$= \frac{f[x_0, x_1, x_2] - f[x_1, x_2, x_3]}{x_3 - x_0}$$

In general...

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$$a_k = f[x_0, x_1, x_2, \dots, x_k]$$

delete first

delete last

$$f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

e.g.) Interpolating the given data with a constant, linear, quadratic, cubic.

x	0	1	2	3
$f(x)$	4	9	15	18

$P_0(x)$

$P_1(x)$

constant: $P_0(x) = 4$

linear: $P_1(x) = P_0(x) + a_1(x-0)$

$$a_1 = f[0,1] = \frac{f(1) - f(0)}{1-0} = \frac{9-4}{1} = +5$$

$$P_1(x) = 4 - 5(x-0)$$