Is this legible?

Bisection : $|x_n - x| \le \frac{(b-a)}{2^n}$ ← rate

$x_n = x + \Theta(2^{-n})$

§2.4 Order of convergence

def: We say $x_n = x + \Theta(\beta_n)$ if $\beta_n \to 0$ and there exists a constant C such that

$$|x_n - x| \le C\beta_n,$$

i.e., $x_n \to x$ with rate β_n.

def: Suppose $\lim_{n\to\infty} x_n = x$. If there are constants $\alpha > 0$, $\lambda > 0$, with

$$\cancel{\frac{|x_{n+1}-x|}{|x_n-x|} = 2}$$

$$\lim_{n\to\infty} \frac{|x_{n+1}-x|}{|x_n-x|^\alpha} = \lambda$$

then we say $x_n \to x$ with order α and asymptotic error constant λ.

$|x_n - x|^\alpha$ getting smaller even if $\lambda > 1$

slower —— linear ($\alpha = 1$) —— quadratic ($\alpha = 2$) —— cubic ($\alpha = 3$) → faster

$1 > \lambda > 0$

For a fixed α, sequences with a smaller error constant converge faster.

sublinear

super linear

e.g.) $\frac{1}{n} \to 0$ sublinearly because $\frac{|\frac{1}{n+1}|}{|\frac{1}{n}|^1} = \frac{n}{n+1} \to 1$

$$\frac{\left|\frac{1}{2^{n+1}} - 0\right|}{\left|\frac{1}{2^n} - 0\right|^{1}} = \frac{2^n}{2^{n+1}} = \frac{1}{2} \longrightarrow \frac{1}{2}$$

linearly

asymptotic error

$\sum \frac{1}{n}$ div.

$\sum \frac{1}{2^n}$ conv.

High α and λ low is fast!

<u>thm</u>: Bisection method generates a sequence that converges linearly with asymptotic error constant $\frac{1}{2}$.

$$\frac{|x_{n+1} - x|}{|x_n - x|^1} \approx \frac{\frac{(b-a)}{2^{n+1}}}{\frac{(b-a)}{2^n}} = \frac{1}{2}$$

<u>thm</u>: Suppose $x = g(x)$, and suppose g is p times differentiable near x. Assume

$$g'(x) = g''(x) = \cdots = g^{(p-1)}(x) = 0$$

but $g^{(p)}(x) \neq 0$, then $x_n = g(x_{n-1})$ converges with order p and asymptotic error $\left|\frac{g^{(p)}(x)}{p!}\right|$.

<u>pf</u>: Use Taylor's theorem.

Recall... to solve $f(x) = 0$

fixed point iteration!

$$x_n = g(x_{n-1}) := x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$g(x) = x - \frac{f(x)}{f'(x)}, \quad \text{say } \underbrace{f(x)}_{=0} = 0$$

$$g'(x) = 1 - \frac{f'(x)f'(x) - \overbrace{f(x)}^{=0} f''(x)}{[f'(x)]^2} = 0$$

$$0 = \frac{f(x)f''(x)}{[f'(x)]^2}$$

$g'(x)$ is zero!

$$g''(x) = \frac{[f'(x)]^2[f'(x)f''(x) + \overbrace{f(x)}^{=0}f'''(x)] - [\overbrace{f(x)}^{=0}f''(x)][\cdots]}{[f'(x)]^4}$$

$$= \frac{f'(x)f''(x)}{[f'(x)]^2} = \boxed{\frac{f''(x)}{f'(x)}} \neq 0$$

Since $g'(x)=0$, but $g''(x) \neq 0$, Newton's method converges quadratically with asymptotic error

$$\longrightarrow \frac{f''(x)}{2f'(x)}.$$

For large n, if $e_n = |x_n - x|$,

$$e_{n+1} \approx \lambda e_n^\alpha$$

$$\frac{e_{n+1}}{e_n} \approx \frac{\lambda e_n^\alpha}{\lambda e_{n-1}^\alpha} = \left(\frac{e_n}{e_{n-1}}\right)^\alpha$$

$$\alpha \approx \frac{\log(e_{n+1}/e_n)}{\log(e_n/e_{n-1})}$$

once α is known...

$$\frac{e_{n+1}}{e_n^\alpha} \approx \lambda$$