Extrapolation and Romberg integration

Suppose a method $A(h)$ depends on a parameter $h>0$, e.g., T^h, S^h. Let $A(0)$ denote the exact value approximated by $A(h)$, e.g., $\int_a^b f(x)\,dx$. Assume there are constants $K_1, K_2, \ldots$ independent of h with

$$A(h) = A(0) + K_1 h^{p_1} + K_2 h^{p_2} + \cdots, \quad p_2 > p_1$$

Let $q<1$, and notice

$$\begin{aligned} A(qh) &= A(0) + K_1 (qh)^{p_1} + K_2 (qh)^{p_2} + \cdots \\ &= A(0) + q^{p_1} K_1 h^{p_1} + q^{p_2} K_2 h^{p_2} + \cdots \end{aligned}$$

Hence

$$A(0) - \frac{1}{q^{p_1}} A(0) = A(h) - K_1 h^{p_1} - K_2 h^{p_2} - \frac{1}{q^{p_1}} A(qh) + K_1 h^{p_1} + q^{p_2 - p_1} K_2 h^{p_2}$$

$$\left(1 - \frac{1}{q^{p_1}}\right) A(0) = A(h) - \frac{1}{q^{p_1}} A(qh) + \left(q^{p_2 - p_1} - 1\right) K_2 h^{p_2} + \cdots,$$

and so

$$A'(h) = \left\{ \frac{1}{1 - \frac{1}{q^{p_1}}} \right\} \left\{ A(h) - q^{-p_1} A(qh) \right\} + \tilde{K}_2 h^{p_2} + \cdots,$$

i.e., $A'(h)$ is an order $O(h^{p_2})$ method! All we need to know to apply Richardson extrapolation is p_1.

<u>Trapezoid rule:</u> $p_1 = 2$, $q = 2$

$$R^h = \left\{ \frac{1}{1 - \frac{1}{4}} \right\} \left\{ T^h - \frac{1}{4} T^{\frac{h}{2}} \right\} = \frac{4}{3} T^h - \frac{1}{3} T^{\frac{h}{2}}$$

Can we apply this again to R^h? Yes, as long as we know "p_1".

Luckily we can approximate p_1 for any method.

$$A(h) = A(0) + K_1 h^{p_1} + K_2 h^{p_2} + \cdots \approx A(0) + K_1 h^{p_1}$$

$$A\left(\frac{h}{2}\right) = A(0) + 2^{-p_1} K_1 h^{p_1} + 2^{-p_2} K_2 h^{p_2} + \cdots \approx A(0) + 2^{-p_1} K_1 h^{p_1}$$

Then

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$$\frac{A(2h) - A(h)}{A(h) - A\left(\frac{h}{2}\right)} \approx \frac{(2^{p_1} - 1) K_1 h^{p_1}}{(1 - 2^{-p_1}) K_1 h^{p_1}} = \frac{2^{p_1}(1 - 2^{-p_1})}{1 - 2^{-p_1}} = 2^{p_1}$$

i.e., for h small, $p_1 \approx \log_2\left(\frac{A(2h) - A(h)}{A(h) - A\left(\frac{h}{2}\right)}\right)$.

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