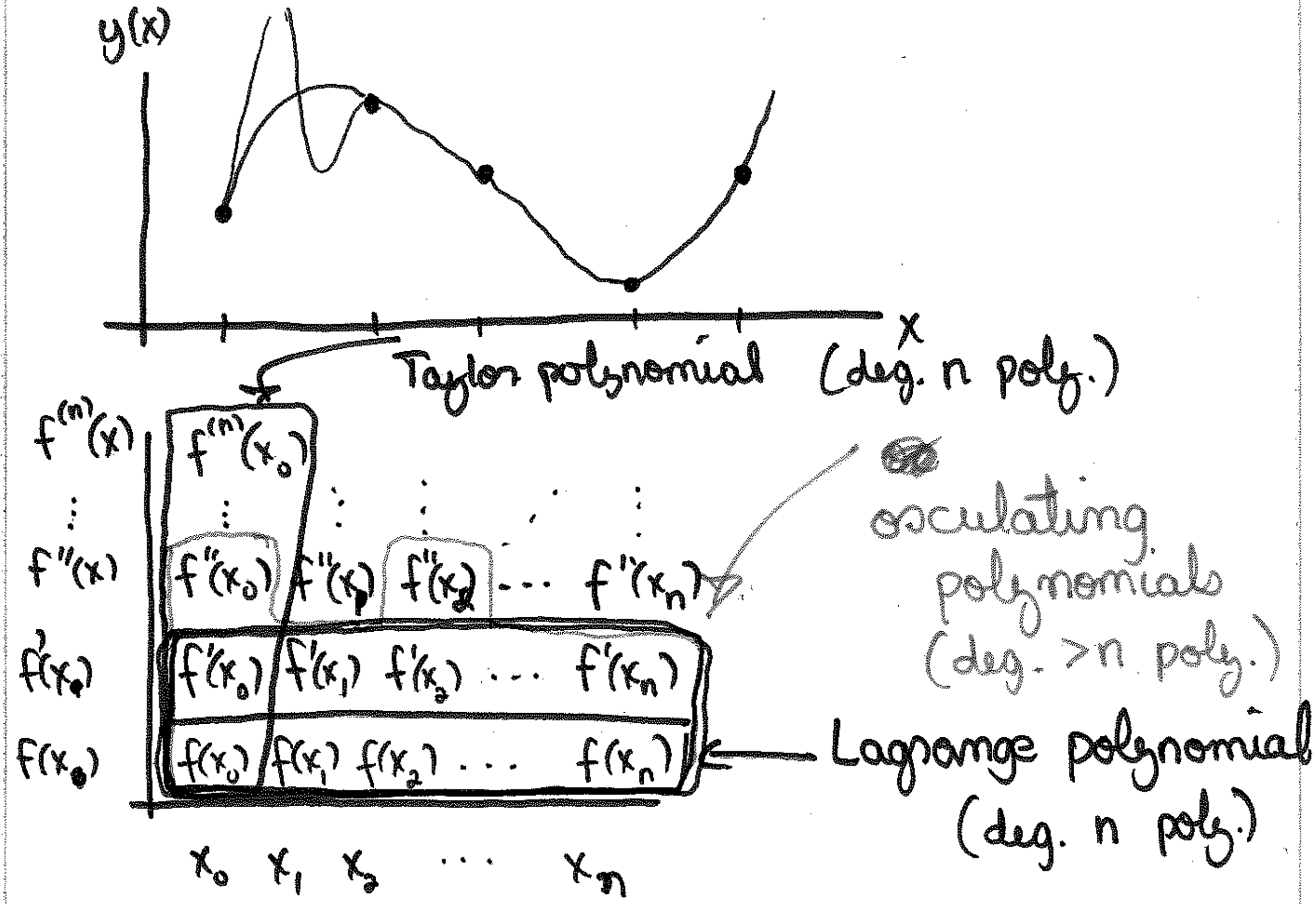


<u>Hermite polynomial</u>: The Hermite polynomial for a function $f$ relative to $x_0, x_1, \dots, x_n$ agrees with the function values $(y_i = f(x_i))$ at each node <u>and</u> its ~~agrees with~~ derivative agrees with the derivative of the function.

<u>thm</u>: If $f \in C^1[a,b]$, $x_0, x_1, \dots, x_n$ distinct, the unique Hermite polynomial $H$ of degree at most $2n+1$ and $H(x_i) = f(x_i)$, $H'(x_i) = f'(x_i)$ is given by

$$H(x) = \sum_{k=0}^{n} y_k H_k(x) + \sum_{k=0}^{n} y'_k \hat{H}_k(x),$$

where $y_k = f(x_k)$, $y'_k = f'(x_k)$,

$$H_k(x) = \{1 - 2(x - x_k) L'_k(x)\} L_k^2(x)$$

$$\hat{H}_k(x) = (x - x_k) L_k^2(x)$$

$$L_k = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

Why do these work??

$$H_i(x_i) = \{1 - 2 \cdot 0\, L'_i(x_i)\} L_i^2(x_i) = 1$$

$$H_i(x_j) = \{1 - 2(x_j - x_i) L'_i(x_j)\} \underbrace{L_i^2(x_j)} = 0$$

$$\hat{H}_i(x_j) = (x_j - x_i) \underbrace{L_i^2(x_j)} = 0 \quad \text{for } j$$

$$H(x_i) = \underbrace{\sum_{k=0}^{n} y_k H_k(x_i)}_{y_i} + \underbrace{\sum_{k=0}^{n} y'_k \hat{H}_k(x_i)}_{0}$$

↳ i.e., $H$ interpolates the data.

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$$H'(x) = \sum_{k=0}^{n} y_k H'_k(x) + \sum_{k=0}^{n} y'_k \hat{H}'_k(x)$$

We want... $H'_k(x_i) = 0$ for all $i$

$$\hat{H}'_k(x_i) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

$$H_k'(x) = \left\{ -2L_k'(x) - 2\underbrace{(x-x_k)}_{}L_k''(x) \right\} \underbrace{L_k^2(x)}_{}$$
$$+ \left\{ 1 - 2\underbrace{(x-x_k)}_{}L_k'(x) \right\} 2\underbrace{L_k(x)}_{}L_k'(x)$$

$H_k'(x_i) = 0 \leftarrow H_k'$ involves $L_k$ in each term

$$H_k'(x_k) = -2L_k'(x_k)\underbrace{L_k^2(x_k)}_{1} + 2\underbrace{L_k(x_k)}_{1}L_k'(x_k) = 0$$

↳ $H_k'(x)$ is zero at all nodes!

$$\hat{H}_k'(x) = \frac{d}{dx}\left\{ (x-x_k)L_k^2(x) \right\}$$
$$= \underbrace{L_k^2(x)}_{} + (x-x_k)\,2\underbrace{L_k(x)}_{}L_k'(x)$$

$\hat{H}_k'(x_i) = 0 \leftarrow \hat{H}_k'$ involves $L_k$ in each term

$$\hat{H}_k'(x_k) = L_k^2(x_k) + (x_k-x_k)\,2L_k(x_k)L_k'(x_k)$$
$$= 1$$

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$H(x_i) = y_i\ ,\quad H'(x_i) = y_i'$