Romberg integration

Recall the composite trapezoid rule on $[a,b]$ with step size h

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a)+f(b)+2\sum_{i=1}^{n-1}(a+ih)\right].$$

This is a consequence of interpolating f with a linear polynomial on $[x_{i-1},x_i]$ for $i=1,2,\ldots,n$, integrating each interpolant, and using the fact

$$\int_a^b f(x) = \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)\,dx.$$

If quadratic interpolants are used on $[x_{i-1},x_i]$ (with the third node the midpoint], we get Simpson's rule

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(a)+f(b)+2\sum_{i=1}^{n-1} f(a+ih)+4\sum_{i=1}^{n-1} f\left(a+i\frac{h}{2}\right)\right]$$

Since these rules come from polynomial interpolation, we're just a step away from error formulas. Last time we used a "weighted MVT for integrals."

<u>thm</u>: For $F,G\in C[a,b]$, if $G\geq 0$, there exists $c\in(a,b)$ with

$$\int_a^b F(t)G(t)\,dt = F(c)\int_a^b G(t)\,dt.$$

Hence,

$$\begin{aligned}\int_a^b f(x)\,dx - T^h_{[a,b]}(f) &= \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\frac{f''(\xi_i)}{2}(x-x_{i-1})(x-x_i)\,dx\\ &= \frac{1}{2}\sum_{i=1}^{n} f''(\zeta_i)\int_{x_{i-1}}^{x_i}(x-x_{i-1})(x-x_i)\,dx\\ &= \frac{h^3}{12}\sum_{i=1}^{n} f''(\zeta_i) = \frac{(b-a)}{12}f''(\mu)\,h^2,\quad \mu\in(a,b)\end{aligned}$$

and so

$$\left|\int_a^b f(x)\,dx - T^h_{[a,b]}(f)\right| \leq \frac{(b-a)}{12}M_2h^2,\quad M_2=\max_{[a,b]}|f''|$$

If you apply the same techniques to composite

Simpson's,

$$\left|\int_a^b f(x)\,dx - S^h_{[a,b]}(f)\right| \le \frac{(b-a)}{180} M_4 h^4, \quad M^4 = \max_{[a,b]} |f^{(4)}|.$$

e.g.) Determine a step size h so that $\int_1^2 x\ln x\,dx$ is accurate to within 10^{-4} using $T^h_{[1,2]}(x\ln x)$.

$f(x) = x\ln x$

$f'(x) = \ln x + 1$

$f''(x) = \frac{1}{x}$

$f'''(x) = \frac{-1}{x^2}$

$f''''(x) = \frac{2}{x^3}$

$$\left|\int_a^b x\ln x\,dx - T^h_{[1,2]}(x\ln x)\right| \le \frac{h^2}{12} M_2 (b-a)$$

$$\frac{h^2}{12}(1)(2-1) = \frac{h^2}{12} < 10^{-4}$$

$$h < \sqrt{12 \cdot 10^{-4}} \approx 0.03464$$

$\longrightarrow$ Take $h \le 0.34 \times 10^{-1}$.

We say $T^h_{[a,b]}(f)$ is an $O(h^2)$ method (and $S^h_{[a,b]}(f)$ is an $O(h^4)$ method), specifically because

$$T^h_{[a,b]}(f) = \int_a^b f(x)\,dx + K h^2,$$

where $K = (b-a)f''(\mu)/12$. If we double our subintervals (or take half our step size),

$$T^{h/2}_{[a,b]}(f) = \int_a^b f(x)\,dx + K\left(\frac{h}{2}\right)^2$$

$$= \int_a^b f(x)\,dx + \frac{K}{4} h^2.$$

Notice

$$4T^{h/2}_{[a,b]}(f) - T^h_{[a,b]}(f) = 3\int_a^b f(x)\,dx + Kh^2 - Kh^2 \ldots$$

does this mean

$$R^{T,h}_{[a,b]}(f) = \frac{4}{3} T^{\frac{h}{2}}_{[a,b]}(f) - \frac{1}{3} T^h_{[a,b]}(f)$$

is exact??? No, but it does improve the accuracy. The constants K depended on the partition of $[a,b]$. However, with more thorough analysis one can show

$$T^{h}_{[a,b]}(f) = \int_a^b f(x)\,dx + K_1 h^2 + K_2 h^4 + K_3 h^6 + \dots$$

where $\{K_i\}$ are independent of the partition (in particular K_i depends only on $f^{(2i-1)}$ at a and b). The point is,

$$R^{T,h}_{[a,b]}(f) = \int_a^b f(x)\,dx + O(h^4).$$

e.g.) determine $R^{T,h}_{[1,2]}(x\ln x)$ with $h = 0.5$.

We need $T^{h}_{[1,2]}(x\ln x)$ with $h = 0.5$ and $h = 0.25$.

$$T^{0.5}_{[1,2]}(x\ln x) = \frac{0.5}{2}\left[1\cdot\ln(1) + 2\ln(2) + 2\left(\tfrac{3}{2}\right)\ln\left(\tfrac{3}{2}\right)\right]$$

$$= 0.25\left[2.602689685\right] = 0.65067242\dots$$

$$T^{0.25}_{[1,2]}(x\ln x) = \frac{0.25}{2}\left[1\cdot\ln(1) + 2\ln(2) + 2\left(\tfrac{5}{4}\right)\ln\left(\tfrac{5}{4}\right) + 2\left(\tfrac{3}{2}\right)\ln\left(\tfrac{3}{2}\right) + 2\left(\tfrac{7}{4}\right)\ln\left(\tfrac{7}{4}\right)\right]$$

$$= 0.125\left[5.11920382\right] = 0.639900477\dots$$

Applying the previously given formula,

$$R^{T,0.5}_{[1,2]}(x\ln x) = \frac{4}{3}(0.6399004\dots) - \frac{1}{3}(0.650672\dots)$$

$$= 0.63631\dots$$

The actual value: $0.636294\dots$

We may continue applying this idea. Indeed

$$R^{h}(f) = \int_a^b f(x)\,dx + K_1 h^4 + K_2 h^6 + K_3 h^8 + \dots$$

$$R^{\frac{h}{2}}(f) = \int_a^b f(x)\,dx + K_1\left(\frac{h}{2}\right)^4 + K_2\left(\frac{h}{2}\right)^6 + K_3\left(\frac{h}{2}\right)^8 + \dots$$

and so

$$R^{h,2}(f) = \frac{16}{15} R^{\frac{h}{2},1}(f) - \frac{1}{15} R^{h,1}(f)$$

is an $O(h^6)$ method. We can keep going and develop an $O(h^{2k})$ method for any k.

[redacted]