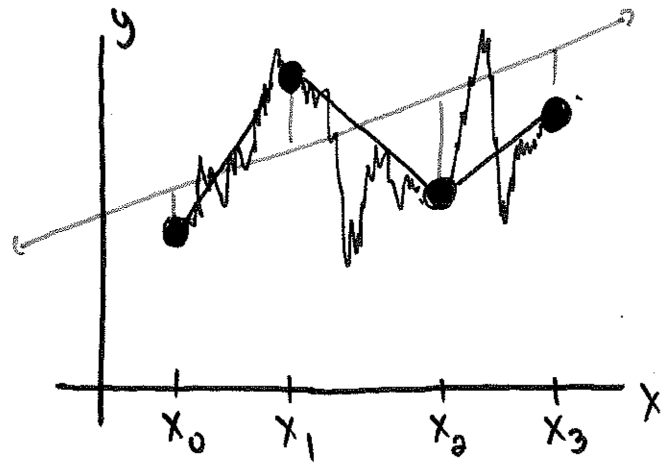
Polynomial interpolation:



- Approximating the data
  LS, minimizing error
- Interpolating
  Taylor polynomial

def: Given $(n+1)$-distinct points $x_0, x_1, x_2, \ldots, x_n$ and $(n+1)$-distinct values $y_0, y_1, \ldots, y_n$, to interpolate the data is to find a function P s.t. $P(x_i) = y_i$, $0 \le i \le n$.

We'll try interpolating data (functions) with polynomials! Polynomials have all derivatives, easy to integrate!

thm: (Weierstrass Approx. Theorem) If f is continuous on $[a,b]$, then for any $\varepsilon > 0$ there will be a polynomial P such that $|f(x) - P(x)| < \varepsilon$ for $x \in [a,b]$.

↳ Mathematically polynomials are dense.

How do we find this polynomial?

e.g.) Taylor polynomials! The nth Taylor polynomial for f centered at $x = a$ is

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In short...

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

e.g.) Consider $f(x)=\sqrt{x}$, $a=4$, $n=3$.

$$\begin{aligned}
f(x) &= \sqrt{x} & f(a) &= 2 \\
f'(x) &= \tfrac{1}{2}x^{-1/2} & f'(a) &= \tfrac{1}{4} \\
f''(x) &= \tfrac{-1}{4}x^{-3/2} & f''(a) &= \tfrac{-1}{4}(4)^{-3/2} = \frac{-1}{4\cdot 2^3} = \frac{-1}{32} \\
f'''(x) &= \tfrac{3}{8}x^{-5/2} & f'''(a) &= \tfrac{3}{8}(4)^{-5/2} = \frac{3}{8\cdot 2^5} = \frac{3}{256}
\end{aligned}$$

$$P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3.$$

thm: Suppose $f^{(n+1)}(y)$ exists & is continuous. Let K satisfy $|f^{(n+1)}(y)| \le K$ for any y between x and a. Then

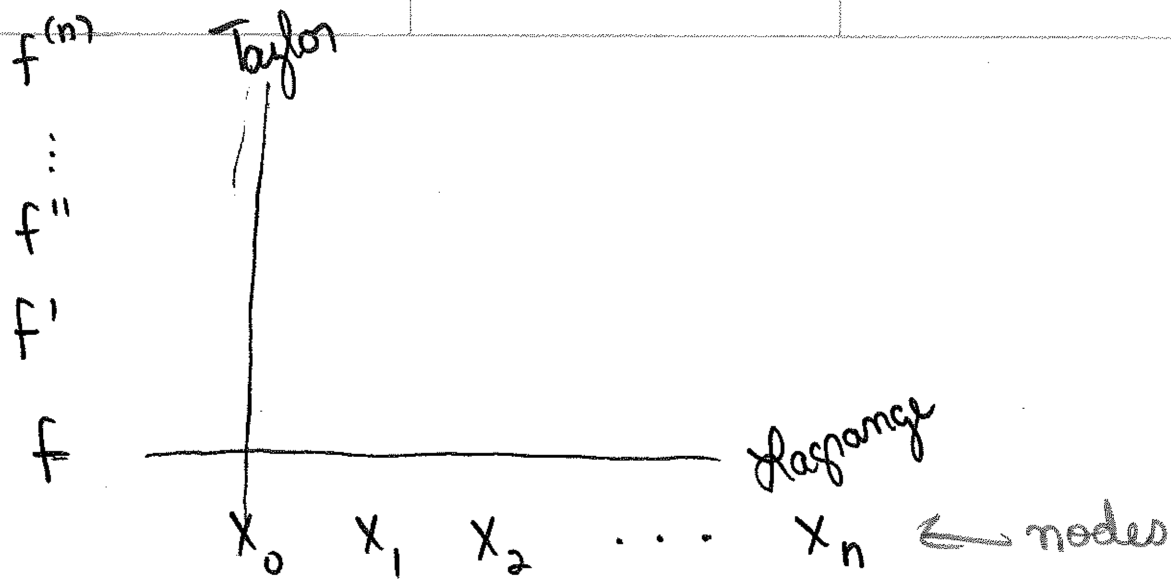
$$|P_n(x) - f(x)| \le K\,\frac{|x-a|^{n+1} \leftarrow \text{inc. error}}{(n+1)! \leftarrow \text{dec. error}}$$

e.g.) $P_3(9) = 3.104$, $f(9) = 3$ $\longrightarrow$ $|3.104 - 3| = 0.104$

Note $f^{(4)}(x) = \frac{-15}{16}x^{-7/2}$. For $y \in [4,9]$, $|f^{(4)}(y)| \le \frac{15}{3328}$ ≈ 0.004507

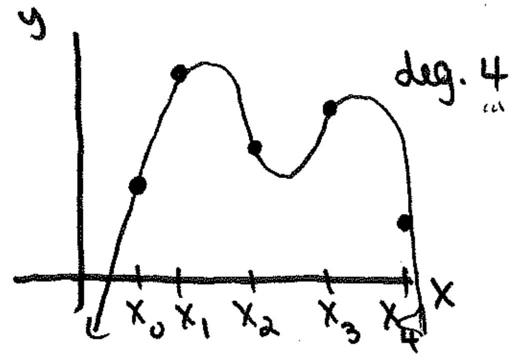
$$|P_3(9) - f(9)| \le 0.004507 \cdot \frac{|9-4|^4}{(3+1)!} \approx 0.117$$

The nth Taylor polynomial agrees with $f(a), f'(a), \ldots, f^{(n)}(a)$.



Given $(n+1)$ distinct points, the Lagrange interpolating polynomial interpolate a function f at each point.

For $(n+1)$ distinct points, the LIP will have degree n.



<u>thm</u>: Given $(n+1)$ distinct data points, say (x_0, y_0), $(x_1, y_1), \ldots, (x_n, y_n)$, there is a degree n polynomial P with $P(x_i) = y_i$. $\Big\}$ think $y_i = f(x_i)$