

Is this legible?

Bisection : $|x_n - x| \leq \frac{(b-a)}{2^n}$ ← rate

§2.4 Order of convergence

$$x_n = x + O(2^{-n})$$

def: We say $x_n = x + O(\beta_n)$ if $\beta_n \rightarrow 0$ and there exists a constant C such that

$$|x_n - x| \leq C \beta_n$$

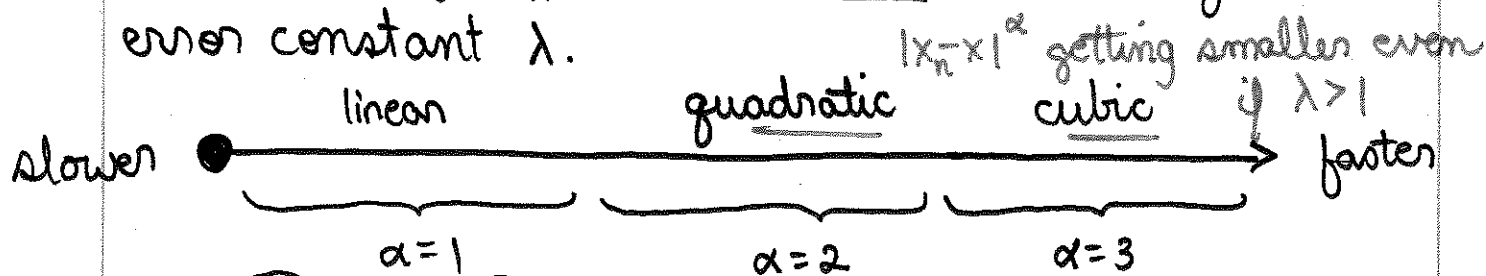
i.e., $x_n \rightarrow x$ with rate β_n .

def: Suppose $\lim_{n \rightarrow \infty} x_n = x$. If there are constants $\alpha > 0$, $\lambda > 0$, with

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - x|}{|x_n - x|^\alpha} = \lambda$$

~~$$\frac{|x_{n+1} - x|}{|x_n - x|} = 2$$~~

then we say $x_n \rightarrow x$ with order α and asymptotic error constant λ .



$$1 > \lambda > 0$$

For a fixed α , sequences with a smaller error constant converge faster.

superlinear

sublinear

e.g.) $\frac{1}{n} \rightarrow 0$ sublinearly because

$$\frac{|\frac{1}{n+1}|}{|\frac{1}{n}|^2} = \frac{n}{n+1} \rightarrow 1$$

$$\frac{\left| \frac{1}{2^{n+1}} - 0 \right|}{\left| \frac{1}{2^n} - 0 \right|} = \frac{2^n}{2^{n+1}} = \frac{1}{2} \longrightarrow \frac{1}{2}$$

linearly asymptotic error

$\sum \frac{1}{n}$ div.
 $\sum \frac{1}{2^n}$ conv.

High α and λ low is fast!

thm: Bisection method generates a sequence that converges linearly with asymptotic error constant $\frac{1}{2}$.

$$\frac{|x_{n+1} - x|}{|x_n - x|^2} \approx \frac{\frac{(b-a)}{2^{n+1}}}{\frac{(b-a)}{2^n}} = \frac{1}{2}$$

thm: Suppose $x = g(x)$, and suppose g is p times differentiable near x . Assume

$$g'(x) = g''(x) = \dots = g^{(p-1)}(x) = 0$$

but $g^{(p)}(x) \neq 0$, then $x_n = g(x_{n-1})$ converges with order p and asymptotic error $\left| \frac{g^{(p)}(x)}{p!} \right|$.

pr: Use Taylor's theorem.

Recall... to solve $f(x) = 0$

$$x_n = g(x_{n-1}) := x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

fixed point iteration!

$$g(x) = x - \frac{f(x)}{f'(x)}, \text{ say } f(x) = 0$$

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = 0$$

$g'(x)$ is zero!

$$0 = \frac{f(x)f''(x)}{[f'(x)]^2}$$

$$g''(x) = \frac{[f'(x)]^2 [f'(x)f''(x) + f(x)f'''(x)] - [f(x)f''(x)] [\dots]}{[f'(x)]^4}$$

$$= \frac{f'(x)f''(x)}{[f'(x)]^2} = \left[\frac{f''(x)}{f'(x)} \right] \neq 0$$

Since $g'(x) = 0$, but $g''(x) \neq 0$, Newton's method converges quadratically with asymptotic error

$$\rightarrow \frac{f''(x)}{2f'(x)}.$$

For large n , if $e_n = |x_n - x|$,

$$e_{n+1} \approx \lambda e_n^\alpha$$

once α is known...

$$\frac{e_{n+1}}{e_n} \approx \frac{\lambda e_n^\alpha}{\lambda e_{n-1}^\alpha} = \left(\frac{e_n}{e_{n-1}} \right)^\alpha$$

$$\frac{e_{n+1}}{e_n^\alpha} \approx \lambda$$

$$\alpha \approx \frac{\log(e_{n+1}/e_n)}{\log(e_n/e_{n-1})}$$