Pauli component erasing operations

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The density matrix of a system with n qubits can be expressed as

$$\rho = \frac{1}{2^n} \sum_{j_1,\ldots,j_n=0}^{3} r_{j_1,\ldots,j_n} \sigma_{j_1} \otimes \ldots \otimes \sigma_{j_n}, \qquad r_{0,\ldots,0} = 1 \tag{1}$$

where σ_{j_k} are the Pauli matrices (with σ_0 as $\mathbb{1}$) and $r_{j_1,\ldots,j_n}$ the projections of ρ onto each element of the basis of tensor products of Pauli matrices. We may refer to the $r_{j_1,\ldots,j_n}$ as the "Pauli components" of the density matrix of a system of qubits. The component $r_{0,\ldots,0}$ must be equal to one in order for ρ to satisfy $\operatorname{Tr}\rho = 1$.

Let us now define what a Pauli component erasing (PCE) operation is. A linear operation $\mathcal{E}$ is a PCE operations if it acts on a density matrix of n qubits and transforms its Pauli components as

$$r_{j_1,\ldots,j_n} \longmapsto \tau_{j_1,\ldots,j_n} r_{j_1,\ldots,j_n} \qquad \tau_{j_1,\ldots,j_n} = 0, 1, \qquad \tau_{0,\ldots,0} = 1. \tag{2}$$

In simple words, a PCE operation leaves invariant or erases the Pauli components of a density matrix of n qubits.

In this work, we are interested in studying the subset of PCE operations that are quantum channels. Let us discuss briefly the definition of a quantum channel. A quantum channel is a linear operation that acts on a density matrix ρ that is completely positive and preserves the trace of ρ. The complete positivity is a condition that captures the non-locality of quantum mechanics. One standard definition is the following: an operation $\mathcal{E}$ is completely positive if and only if $\mathcal{E}\otimes\mathbb{1}_k$ is a positive operator for every possible dimension k. In other words, $\mathcal{E} \otimes \mathbb{1}_k$ must be an operator that maps positive matrices onto positive matrices, including the entangled states between the system which $\mathcal{E}$ acts on and an arbitrary system which $\mathbb{1}_k$ acts on. In fact, the complete positivity of an operation $\mathcal{E}$ may be evaluated by checking that the maximally entangled pure state between the system and an identical copy is mapped by $\mathcal{E} \otimes \mathbb{1}$ onto a positive matrix, *i.e.* $(\mathcal{E} \otimes \mathbb{1})[|\phi^+\rangle\langle\phi^+|] \geq 0$, with $|\phi^+\rangle\langle\phi^+|$ the maximally entangled pure state state.

We're interested in characterizing the quantum PCE channels. For 1 qubit the picture of our problem is easy to understand given that the Pauli components are the components of a vector in the Bloch sphere, the so-called Bloch vector. As an example, let us consider a map that erases any single component of the Bloch vector. Geometrically, this map collapses the Bloch ball into a disk. This operation is not completely positive, and therefore is not a quantum channel. However, erasing two components yields a quantum channel.

So far, we have analytically solved the diagonalization for complete positivity of a more general class of operations than the PCE and have numerically found the all the PCE quantum channels up to 4-qubit systems. We have made several observations from our results, but we have not been able to prove them analytically and connect them all as a consequence of the mathematical characterization we desire to find of the PCE quantum channels.

Results

Complete positivity: analytical solution

Let us discuss the role of the complete positivity condition of the PCE operations in our problem. We desire to find what are the characteristics that make a PCE operation a quantum channel. In the way we defined a PCE operation in (2) it is a trace preserving operation. Therefore, the complete positivity is the only condition to be evaluated given a PCE operation.

We found an analytical solution to evaluate the complete positivity. Recall that an operation $\mathcal{E}$ is completely positive if $(\mathcal{E} \otimes \mathbb{1})[|\phi^+\rangle\langle\phi^+|] \geq 0$, where $|\phi^+\rangle\langle\phi^+|$ is the maximally entangled pure state between the system and another one that is an identical copy. We showed that the diagonalization of $(\mathcal{E} \otimes \mathbb{1})[|\phi^+\rangle\langle\phi^+|] \geq 0$, with $\mathcal{E}$ an operation that acts on the Pauli components of a density matrix of n qubits as

$$r_{j_1,\ldots,j_n} \longmapsto \tau_{j_1,\ldots,j_n} r_{j_1,\ldots,j_n}, \qquad \tau_{0,\ldots,0} = 1, \tag{3}$$

yields the eigenvalues λ_i

$$\vec{\lambda} = \frac{1}{2^n} \left(\bigotimes^n a \right) \vec{\tau}, \tag{4}$$

where

$$a = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{5}$$

and the components of $\vec{\tau}$ the same ones in (3). It is very important to realize that (4) yields the diagonalization not exclusively of a PCE operation. The difference between the definition of a PCE operation in (2) and (3) is the relaxation in the value of the $\tau_{j_1,\ldots,j_n}$ (exclusively 0 or 1 for a PCE operation).

PCE quantum channels

In fig. 1 we introduce a pictorial representation for the identity map acting on a density matrix for systems of 1, 2 and 3 qubits, respectively. It will be helpful to visualize the maps we are studying making use of this tool. We'll consider any little square or cube in blank as a component erased in ρ by the map. Red squares correspond to components of Bloch vectors of one-particle reduced density matrices, blue squares to correlations between any pair of qubits, and green squares to correlations between all qubits in the system for the 3-qubits case. In the following subsections we present the PCE channels for 1 and 2 qubits, and preliminary results for 3 qubits.

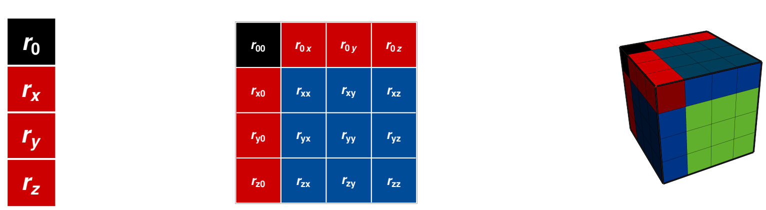


Figure 1: From left to right the identity map acting on an arbitrary density matrix of 1, 2 and 3 qubits, respectively. Red squares correspond to components of local Bloch vectors, blue squares to correlations between any pair of qubits and green squares to correlations between all qubits in the system, for the 3-qubits case.

1 qubit

In fig. 2 we present the PCE channels for 1 qubit. The first and last patterns represent identity and the total depolarizing channel. Patterns in between represent operations that map the Bloch sphere to a line on the Cartesian axes, respectively. These three channels are equivalent via permutation of the components r_i of the Bloch vector.

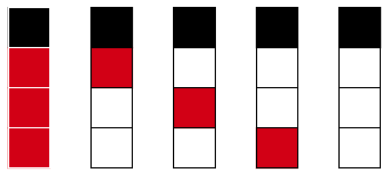


Figure 2: 1-qubit PCE channels. From left to right patterns represent the following channels: identity, mapping of the Bloch sphere to x axis, y axis, z axis, and completely depolarizing (channel that maps every state to the maximally mixed).

2 qubits

2-qubits PCE channels have been classified in equivalence classes (as shown from fig. 3 to fig. 12), such that elements in a class are connected by

1. Particle swaps: Qubits in the system can be interchanged. Therefore if a map is a PCE channel for a certain arrangement of qubits, then there are equivalent PCE channels acting on the system for all possible arrangements of qubits. In the pattern-picture this can be interpreted as transpositions.

2. Exchange of axes labels: These operations correspond to permutations of rows and/or columns and are generally are not CPTP maps. For instance the swap of rows (or columns) result in reflections of the Bloch sphere of individual particles. Aside from that, it is clear that the physics of the erasure are the same.

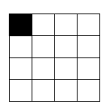


Figure 3: C^1

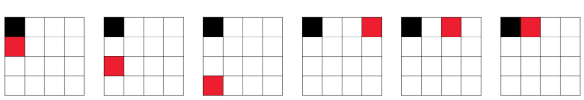


Figure 4: C^2_1

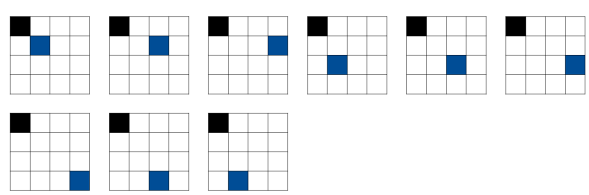


Figure 5: C^2_2

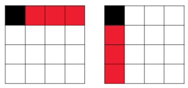


Figure 6: C_1^4

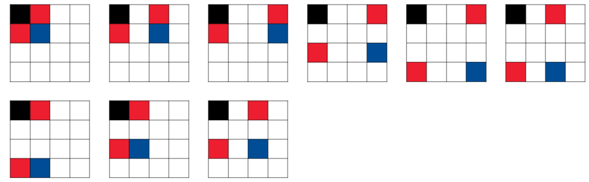


Figure 7: C_2^4

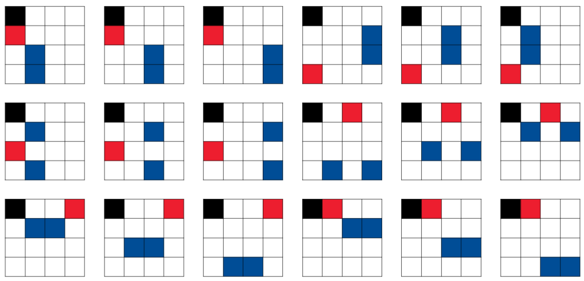


Figure 8: C_3^4

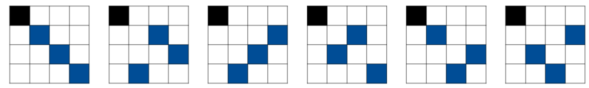


Figure 9: C_4^4

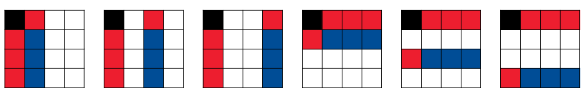


Figure 10: C_1^8

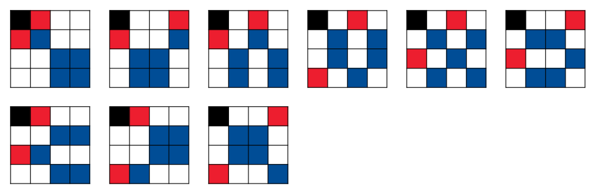


Figure 11: C_2^8

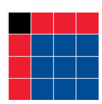
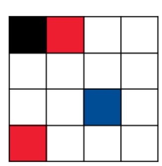


Figure 12: C^{16}

In general, our results exhibit the following features:

- (*Power-of-2 rule*) Only a power-of-2 number of Pauli components in ρ may be left invariant by a PCE channel. However, not only the number of Pauli components in ρ to leave invariant determines complete positivity since not all maps that leave 2^k Pauli components invariant in ρ are quantum channels. For example, the pattern below corresponds to a map that is a potential element of C_2^4 (Fig. **??**). Nevertheless, complete positivity is not satisfied and then the map is not a quantum channel.



- *(Mirroring)* The number of PCE quantum channels as a function of the number of Pauli components left invariant shows to be symmetric with respect to a particular value. The number of PCE operations and quantum channels, as a function of the number of Pauli components left invariant, for 2 qubits are shown in table 1.

	Number of Pauli components left invariant															
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
# of PCE operations	1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1
# of PCE qtm channels	1	15	0	35	0	0	0	15	0	0	0	0	0	0	0	1

Table 1: Tabla

Let us focus only on the quantum channels:

	Number of r_{ij} inv.				
	1	**2**	**4**	**8**	**16**
# of PCE qtm channels	1	15	35	15	1

We denote $f(k)$ the number of PCE quantum channels as a function of the number k of Pauli components left invariant. The number k must be a power of 2. In the case of 2 qubits, $f(k)$ is symmetric with respect to $k = 4$. Our numerical results show that $f(k)$ is symmetric with respect to $k = 2^n$, in general. When considering only a power-of-2 Pauli components invariant, table 1 shows that this symmetry is not a property of the PCE operations, but only of the quantum channels.

- *(1 : 1 correspondence)* There might be a 1 : 1 correspondence between PCE quantum channels that leave 2^k and 2^{2n-k} Pauli components invariant. That is to say in the 2-qubits case, PCE channels in Figs. 4 and 11 correspond to one PCE channel in Figs. 10 and 11. In fact, we've deduced from our numerical results the correspondence between PCE channels that leave 2 and the half the total Pauli components. For 2 qubits, given the set $\{j_1, j_2\}$ of indices of the Pauli component r_{j_1,j_2} left invariant, other than $r_{0,0}$, of every member in figs. 10 and 10 one can one can find a single partner in figs. 10 and 11 via

$$\Phi_{j_1,j_2} = \mathcal{E}_{j_1} \otimes \mathcal{E}_{j_2} + (\mathbb{1} - \mathcal{E}_{j_1}) \otimes (\mathbb{1} - \mathcal{E}_{j_2}), \tag{6}$$

where $\mathcal{E}_{j_l}$ are the first four 1-qubit PCE channels in fig. 2, the identity and the 3 PCE operations that map the Bloch sphere to a line on every Cartesian axis. In general, a PCE quantum channel that leaves invariant $r_{0,\dots,0}$ and $r_{j_1,\dots,j_n}$ has a single PCE partner that leaves invariant half of the total Pauli components of the form

$$\Phi_{j_1,\dots,j_n} = \sum_{i=0}^{i\leq n/2} \sum_{j\in\mathcal{S}_{2i}^n} \bigotimes_{j_i=1}^{n} \Omega_{j_i}, \tag{7}$$

with $\mathcal{S}_{2i}^n$ the set of all subsets with $2i$ elements from $\{1, 2, \dots, n\}$ and

$$\Omega_{j_i} = \begin{cases} \mathcal{E}_{j_i}, & j_i \notin j \\ \mathbb{1} - \mathcal{E}_{j_i}, & j_i \in j. \end{cases} \tag{8}$$

We elaborate the 3-qubits and 4-qubits case. For 3 qubits, every PCE quantum channel that leaves invariant $r_{0,0,0}$ and r_{j_1,j_2,j_3} has a PCE partner that leaves 32 Pauli components that can be written in the form

$$\begin{aligned} \Phi_{j_1,j_2,j_3} &= \sum_{i=0}^{i\leq 3/2} \sum_{j\in\mathcal{S}_{2i}^3} \bigotimes_{j_i=1}^{3} \Omega j_i \\ &= \mathcal{E}_{j_1} \otimes \mathcal{E}_{j_2} \otimes \mathcal{E}_{j_3} + (\mathbb{1} - \mathcal{E}_{j_1}) \otimes (\mathbb{1} - \mathcal{E}_{j_2}) \otimes \mathcal{E}_{j_3} \\ &\quad + (\mathbb{1} - \mathcal{E}_{j_1}) \otimes \mathcal{E}_{j_2} \otimes (\mathbb{1} - \mathcal{E}_{j_3}) + \mathcal{E}_{j_1} \otimes (\mathbb{1} - \mathcal{E}_{j_2}) \otimes (\mathbb{1} - \mathcal{E}_{j_3}). \end{aligned} \tag{9}$$

For 4 qubits, every PCE quantum channel that leaves invariant $r_{0,0,0,0}$ and r_{j_1,j_2,j_3,j_4} has a PCE partner that leaves 128 Pauli components that can be written in the form

$$
\begin{aligned}
\Phi_{j_1,j_2,j_3,j_4} =& \sum_{i=0}^{i\leq 2} \sum_{j\in \mathcal{S}^4_{2i}} \bigotimes_{j_i=1}^{4} \Omega j_i \\
=& \,\mathcal{E}_{j_1}\otimes\mathcal{E}_{j_2}\otimes\mathcal{E}_{j_3}\otimes\mathcal{E}_{j_4} + (\mathbb{1}-\mathcal{E}_{j_1})\otimes(\mathbb{1}-\mathcal{E}_{j_2})\otimes\mathcal{E}_{j_3}\otimes\mathcal{E}_{j_4} \\
&+ (\mathbb{1}-\mathcal{E}_{j_1})\otimes\mathcal{E}_{j_2}\otimes(\mathbb{1}-\mathcal{E}_{j_3})\otimes\mathcal{E}_{j_4} + (\mathbb{1}-\mathcal{E}_{j_1})\otimes\mathcal{E}_{j_2}\otimes\mathcal{E}_{j_3}\otimes(\mathbb{1}-\mathcal{E}_{j_4}) \\
&+ \mathcal{E}_{j_1}\otimes(\mathbb{1}-\mathcal{E}_{j_2})\otimes(\mathbb{1}-\mathcal{E}_{j_3})\otimes\mathcal{E}_{j_4} + \mathcal{E}_{j_1}\otimes(\mathbb{1}-\mathcal{E}_{j_2})\otimes\mathcal{E}_{j_3}\otimes(\mathbb{1}-\mathcal{E}_{j_4}) \\
&+ \mathcal{E}_{j_1}\otimes\mathcal{E}_{j_2}\otimes(\mathbb{1}-\mathcal{E}_{j_3})\otimes(\mathbb{1}-\mathcal{E}_{j_4}) \\
&+ (\mathbb{1}-\mathcal{E}_{j_1})\otimes(\mathbb{1}-\mathcal{E}_{j_2})\otimes(\mathbb{1}-\mathcal{E}_{j_3})\otimes(\mathbb{1}-\mathcal{E}_{j_4}).
\end{aligned}
\tag{10}
$$

- There exists a set of PCE generators and every other PCE quantum channel can be written as concatenation of the generators. The identity plus the PCE channels that leave invariant half the total components constitute the set of generators. For example, the 2-qubits generators are

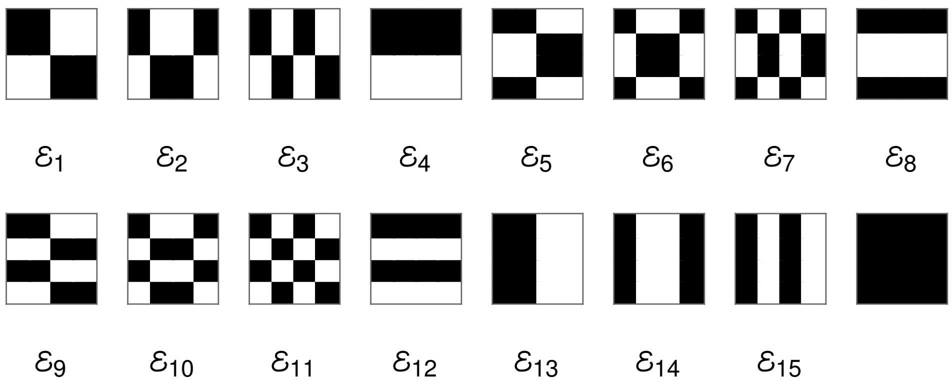
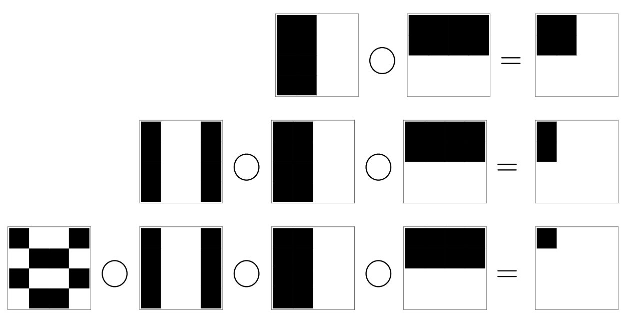


Figure 13: Generators of 2 qubits.

and concatenation with the patterns is understood as the intersection of black squares in the patterns of the generators. For example:

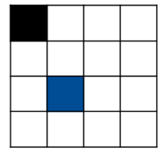


This shows the idea and that is possible to construct from the generators in Fig. 13 all PCE channels in Figs. from 3 to 9

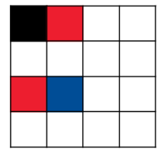
JA: Considero que los siguientes items meten ruido y no aportan

- Empirical observations of results for 2 qubits led us to some rules that patterns showed from fig. 3 to fig. 12 obey:

  1. If a component with indices ij is left invariant, then two options are allowed: *a)* both components with indices $i0$ and $0j$ are left invariant too, or *b)* both components with indices $i0$ and $0j$ are erased. Let us take a look at the next example.



The component with indices 21 in the pattern is left invariant, then the only options allowed are: *a)* components with indices 01 and 20 are erased, as in this pattern above, or *b)* components with indices 01 and 20 are left invariant too, as in the following pattern

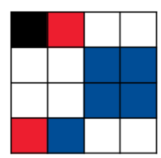
,

which is in fact another PCE channel. This rule can be derived using the complete positive condition over the following pattern:

1	a_1	0	0
0	0	0	0
a_2	b	0	0
0	0	0	0

(11)

2. Considering only components in the correlation matrix (blue squares): if a component with indices ij is left invariant and the previous rule is obeyed, then, in the correlation matrix, remaining components on row i and column j are erased, and the rest of the components are left invariant by a PCE channel. Let's go back to the last pattern in the previous example. This rule ensures that components in the correlation matrix outside of row 2 and column 1 may be left invariant by a PCE channel, i.e.

,

which is another PCE channel.

Actually, this two simple rules allow us to relate different equivalence classes. In the previous example we were able to connect an element of C_2^2 to one of C_3^4, to another one of C_2^8.

- *(Rainbow hypothesis)* PCE channels that leave 2^k components invariant and 2^{2n-k} have a 1:1 correspondence. That is to say, two numbers of the same color in fig. **??** correspond to PCE channels that have a 1:1 correspondence.

- The action of a PCE channel on every subsystem must be another PCE channel. This can be seen from the patterns in fig. 3 to fig. 12. Recall that red squares represent components of local Bloch vectors and blue squares represent correlations shared between qubits. Then, it may be noted in the first column and first row of every pattern there is a 1-qubit PCE quantum channel of the form of fig. 2.

3 qubits

Recall that for 3 qubits system we have numerically analyzed only maps that leave invariant 1, 2, 3, 4, and 64 components in ρ. Therefore, results presented in this section are preliminary. Maps that are PCE channels follow the same features of 2-qubits system. Not all 3-qubits PCE channels will be shown as in the previous section, but only one element of every equivalence class found. Elements in equivalence classes are connected via particle swaps and permutation of individual components.

1-invariant-component maps

The only map that leaves invariant 1 component in ρ is the completely depolarizing channel, and it is trivially a PCE channel.

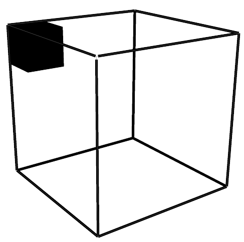


Figure 14: Completely depolarizing channel for 3 qubits.

2-invariant-components maps

All 63 maps that leave invariant 2 components in ρ are PCE channels. It's important to mention that in the 2-qubits case all 2-invariant-components maps are PCE channels, too. We can distinguish 3 equivalence classes: quantum channels that leave invariant

1. any component of a local Bloch vector,
2. any correlation between any pair of qubits,
3. any correlation between all qubits in the system.

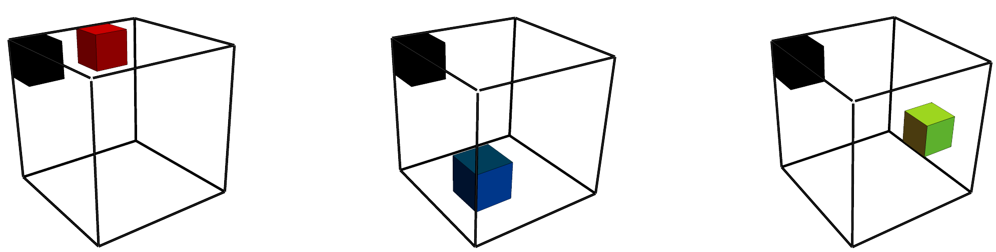


Figure 15: 3-qubits 2-components-invariant PCE channels. One element of each of the 3 equivalence classes (from left to right): leaves invariant one component of any local Bloch vector, one correlation between any pair or qubits, and one correlation between all qubits in the system.

3-invariants-component maps

No PCE channels were found in the set of all maps that leave 3 components invariant in ρ.

4-invariant-components maps

There are 39,711 maps that leave invariant 4 components in ρ, 651 are PCE channels and may be classified in 10 equivalence classes. One arbitrary element in each class is shown in fig. 16. We understand how to infer 5 out of the 10 equivalence classes from 1 and 2-qubits PCE channels, as it will be discussed.

On top of fig. 16 the first 4 elements (from left to right) correspond to PCE channels that are separable in a 1 or 2 qutbis PCE channel acting on any subsystem and a completely depolarzing channel acting on the rest of the system. The fifth and last element on top of the figure may be understood invoking an extension for 3 qubits of one of the empirical rules presented for 2 qubits. The extensions of those rules will be discussed at the end of this section. PCE channels belonging to equivalence classes of bottom elements in the figure are such that we still cannot infer them with our previous results.

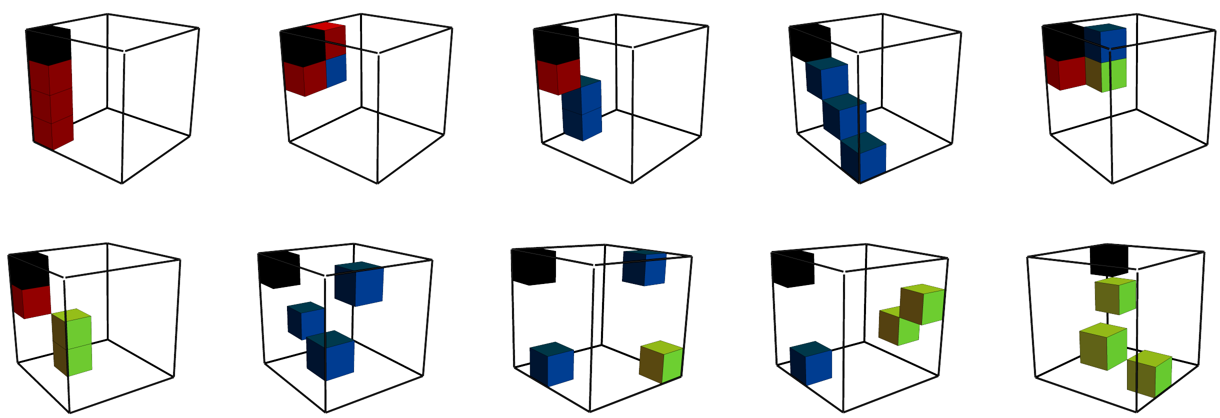


Figure 16: 3-qubits 4-components-invariant PCE channels. One element of every of the 10 equivalence classes.

The empirical rules already presented for 2 qubits can be formulated for 3 qubits as follows:

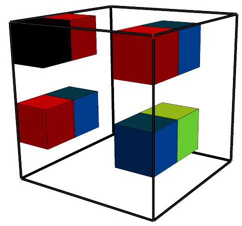
- A component with indices ijk (all different from zero) is left invariant by a PCE channel if and only if one of the cases is followed:

  1. Components $ij0$, $0jk$, $i0k$, $i00$, $0j0$, and $00k$ are erased.
  2. Components $i00$ and $0jk$ are also left invariant.
  3. Components $0j0$ and $i0k$ are also left invariant.
  4. Components $00k$ and $ij0$ are also left invariant.
  5. Components $ij0$, $0jk$, $i0k$, $i00$, $0j0$, and $00k$ are also left invariant.

  It can be noted that the element in the right upper corner of fig. 16 is understood as one of the cases 2-4 of this item.

- Considering only components that correspond to correlations between 3 qubits in the system (green cubes): if a little cube with indices ijk (all different from zero) is left invariant and any of the cases of the previous rule is obeyed, then the remaining components on columns i, j, and k are erased and the rest of components in the correlation tensor are left invariant by a PCE channel.

Let us consider an example to make use of both empirical rules. If we begin with an element of the same equivalence class as the second element on top of fig. 16, then we make use of one of the cases 2-4 of the first empirical rule for 3 qubits and get the PCE channel below.



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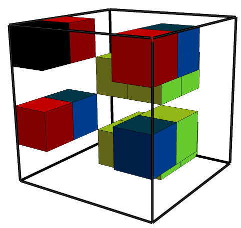
Finally, we make use of the second rule to left invariant only components corresponding to correlations between all qubits in the system (green cubes) and get another PCE channel (pattern below).



What we have done is start with a PCE channel that leaves 4 invariant components, make our way trough a 8-invariant-components PCE channel and finally arrive at a 16-invariant-components PCE channel.

In principle, the rainbow hypothesis ensures the correspondence between the channels we have found and channels that leave invariant 16, 32 and 64 components. The empirical rules for 3 qubits provide one way to *travel* from a k-components-invariant quantum channel to a $(2n-k)$-components-invariant channel. It is sufficient to found one element in an equivalence class because the rest of the elements are found by particle swaps and permutation of individual components.

Are this channels a subset of Pauli diagonal channels constant on axes?

We explored if PCE channels are contained within the set of Pauli diagonal channels constant on axes [**?**]. Let us call them $\mathcal{R}$. For 2 qubits, we concluded that Kraus rank 4 PCE channels are the only candidates to be in $\mathcal{R}$, in fact we are convinced that all of them are in, but are still working in the proof. To see this let us analyze the action of maps in $\mathcal{R}$ on arbitrary density matrices

$$\rho \longmapsto \frac{1}{d}\left[\mathbb{1} + \sum_{J=1}^{d+1} \lambda_J \sum_{j=1}^{d-1} v_{Jj} W_J^j\right], \tag{12}$$

where λ_J are the eigenvalues of a given map, and operators W_J are unitaries defined as

$$W_J = \sum_{k=1}^{d} \omega^k \left|\psi_k^J\middle\rangle\middle\langle\psi_k^J\right|, \tag{13}$$

where sets of vectors $\left\{\left|\psi_k^J\right\rangle\right\}_k$ with $J = 1, \ldots, 1+d$ are mutually unbiased basis (MUB) over $\mathbb{C}^d$, and $\omega = e^{2\pi i/d}$. There are d^2-1 unitaries generated with the powers of the W_Js, i.e. $\{W_J^m\}_{m=1,\ldots,d,J=1,\ldots,d+1}$. They, together the identity matrix, form a basis in the space of $d \times d$ matrices.

To examine if PCE channels are contained in $\mathcal{R}$ let us consider the rank of both maps and discuss how only a subset of PCE channels could be in $\mathcal{R}$. Our results show that PCE channels have a power-of-2 rank. On the other hand, from (12) it can be seen that the rank of a map in $\mathcal{R}$ is 1 plus the number of all λ_J different from zero. Notice that eigenvalues λ_J have multiplicity $d-1$, for each J. Consequently, the allowed matrix ranks of maps in $\mathcal{R}$ are $1 + k(d-1)$, where $k = 1, 2, \ldots, d+1$. It follows that a *necessary* condition for maps in $\mathcal{R}$ and PCE channels to have the same rank is that

$$2^j = 1 + k(2^{2n} - 1), \qquad j = 0, 1, \ldots, 2n \tag{14}$$

is always true for some $k \in \mathbb{Z}^+$, and n the number of qubits. If we take $n = 2$ and $j = 3$ (2 qubits, 2^3 components invariant)

$$k = \frac{2^3 - 1}{2^{2(2)} - 1} = \frac{7}{15}. \tag{15}$$

Therefore the rank of a PCE channel may not be the same of any map in $\mathcal{R}$. Then, there are PCE channels that cannot be in $\mathcal{R}$, like all 8-components-invariant PCE channels, as shown in the previous example.

To-do

In order to fully understand our results and generalize this kind of maps for n qubits we propose the following:

1. Investigate the Kraus operator representation of this quantum channels.
2. Investigate the Schmidt spectrum of the Choi matrix.
3. Investigate the Jamiolkowski isomorphism to find an equivalence between CP and the empirical rules listed previously.
4. Use our current results to propose an efficient way to do numerical analysis to find 3-qubit quantum channels that leave 8 components invariant.