Pauli component erasing operations

June 14, 2021

Contents

Carlos' ideas

1 CP from eigenvalues

Choi matrix's eigenvalues of a 1-qubit Pauli Channel are

$$\lambda_\mu = 1 + \sum_{j=1}^{3} a_j^\mu \tau_j \qquad \mu = 0,1,2,3; \qquad a_j^\mu = \left\{ \begin{array}{rl} 1, & \mu = j \vee \mu = 0 \\ -1, & \mu \neq j \end{array} \right. . \tag{1}$$

Therefore, the inequalities needed to be sattisfied for a 1-qubit unital Pauli channel in order to be CP may be expressed in the following compact expression:

$$\sum_{j=0}^{3} a_j^\mu \tau_j \geq -1, \qquad \mu = 1,2,3; \qquad a_j^\mu = \left\{ \begin{array}{rl} 1, & \mu = j \\ -1, & \mu \neq j \end{array} \right. . \tag{2}$$

Now, Choi matrix's eigenvalues of an n-qubit Pauli Channel are

$$\lambda_{\mu_1,\ldots,\mu_n} = 1 + \sum_{\substack{j_1,\ldots,j_n=0 \\ (j_1,\ldots,j_n\neq 0)}}^{3} a_{j_1}^{\mu_1} \ldots a_{j_n}^{\mu_n} \tau_{j_1,\ldots,j_n} \qquad \mu_l = 0,1,2,3,$$
$$a_{j_l}^{\mu_l} = \left\{ \begin{array}{rl} 1, & \mu_l = j_l \vee j_l = 0 \vee \mu_1,\ldots,\mu_n = 0 \\ -1, & \mu_l \neq j_l \end{array} \right. . \tag{3}$$

Therefore, the inequalities needed to be sattisfied for the CP are

$$\sum_{\substack{j_1,\ldots,j_n=0 \\ (j_1,\ldots,j_n\neq 0)}}^{3} a_{j_1}^{\mu_1} \ldots a_{j_n}^{\mu_n} \tau_{j_1,\ldots,j_n} \geq -1, \qquad \mu_l = 0,1,2,3; \qquad a_{j_l}^{\mu_l} = \left\{ \begin{array}{rl} 1, & \mu_l = j_l \vee j_l = 0 \\ -1, & \mu_l \neq j_l \end{array} \right. , \tag{4}$$

where we used that $\lambda_{0,\ldots,0}$ is always non-negative.

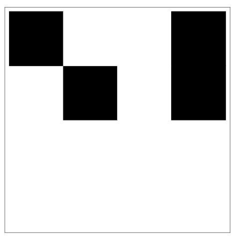
1.1 Another observation from this

Choi's matrix of 2 qubits is diagonal in Pauli product basis (Kraus operators). Does this follow in the general case????

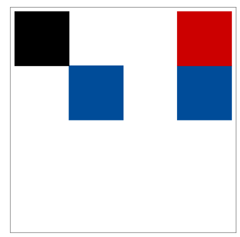
2 General characterisation of PCE quantum channels

To avoid confusion

CP: Igual el canal totalmente despolarizante y el estado totalmente mixto tendrían el mismo diagrama. Quiza el borde se puede ser de diferente color JA: Me gustaría discutir antes si igual vale la pena hacer la distinción. Dado que al final la distinción sólo es una ayuda que se me ocurrió para explicar gráficamente el procedimiento de la concatenación. The figures we've been using (columns, boards, cubes) may represent both PCE operations (dynamics) and density matrices (kinematics). We will make a distinction between both representations with colors. B&W represent PCE operations, and colored figures represent density matrices in Pauli products basis.



(a) PCE operation that erases all Pauli components except for $r_{0,0}$, $r_{1,1}$, $r_{1,3}$, and $r_{0,3}$.



(b) Density matrix in Pauli basis with all components equal to zero except for $r_{0,0}$, $r_{1,1}$, $r_{1,3}$, and $r_{0,3}$.

Hypothesis

Let us denote $\mathsf{PCE_n}$ the set of all PCE quantum channels for n qubits.

Hypothesis 1 *There exists a set $\Gamma_n \subset \mathsf{PCE_n}$ that is sufficient (and necessary?) to generate the rest of elements in $\mathsf{PCE_n}$ as concatenations of different elements in Γ_n.*

Let us consider $\mathcal{E}_j \in \Gamma_n$ ($j = 1, \dots, 4^n - 1$), and $\Lambda \in \mathsf{PCE_n}$. All Λ are a composition of $\mathcal{E}_j$,

$$\underbrace{\mathcal{E}_{j_{2n}} \dots \mathcal{E}_{j_1}}_{k \text{ elements}} = \Lambda, \qquad j_l \neq j_{l+1}, \qquad k = 1, \dots, 2n. \tag{5}$$

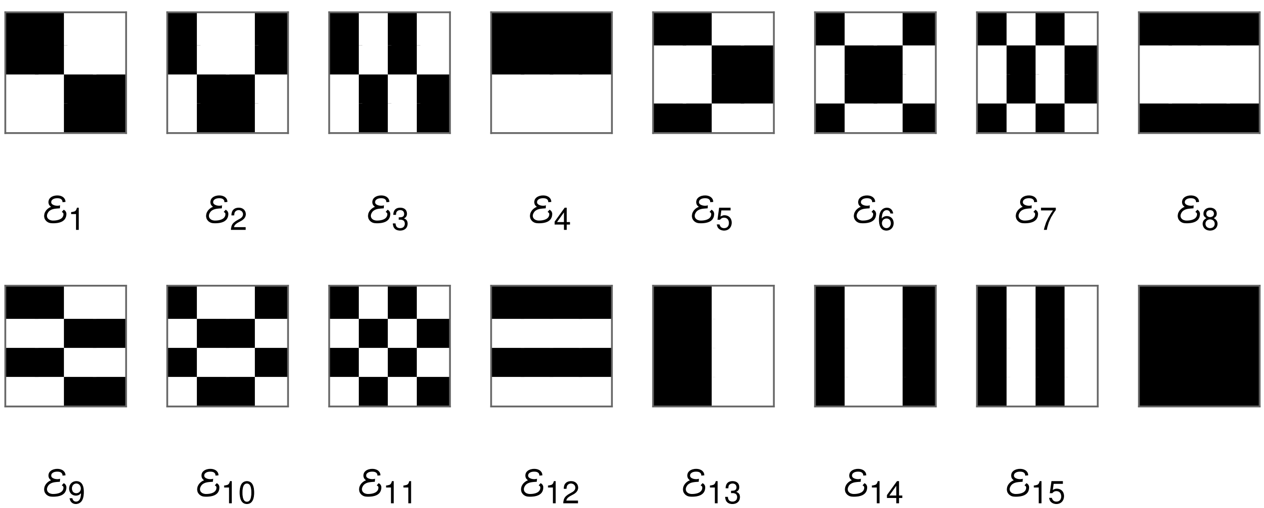
A way to find Γ

PCE quantum channels in Γ_n may be found as the quantum channels defined by the the coefficients a_j^μ of the τ_i in the eigenvalues expression of Choi matrix, with $a_k^k = 0$ (instead of -1). Let us show the 2-qubits case as an example to illustrate this. The eigenvalues of Choi matrix of 2-qubits Pauli

Channel are

$$
\begin{aligned}
\lambda_1 &= \tau_{0,0} + \tau_{0,1} - \tau_{0,2} - \tau_{0,3} + \tau_{1,0} + \tau_{1,1} - \tau_{1,2} - \tau_{1,3} - \tau_{2,0} - \tau_{2,1} + \tau_{2,2} + \tau_{2,3} - \tau_{3,0} - \tau_{3,1} + \tau_{3,2} + \tau_{3,3}\\
\lambda_2 &= \tau_{0,0} - \tau_{0,1} - \tau_{0,2} + \tau_{0,3} + \tau_{1,0} - \tau_{1,1} - \tau_{1,2} + \tau_{1,3} - \tau_{2,0} + \tau_{2,1} + \tau_{2,2} - \tau_{2,3} - \tau_{3,0} + \tau_{3,1} + \tau_{3,2} - \tau_{3,3}\\
\lambda_3 &= \tau_{0,0} - \tau_{0,1} + \tau_{0,2} - \tau_{0,3} + \tau_{1,0} - \tau_{1,1} + \tau_{1,2} - \tau_{1,3} - \tau_{2,0} + \tau_{2,1} - \tau_{2,2} + \tau_{2,3} - \tau_{3,0} + \tau_{3,1} - \tau_{3,2} + \tau_{3,3}\\
\lambda_4 &= \tau_{0,0} + \tau_{0,1} + \tau_{0,2} + \tau_{0,3} + \tau_{1,0} + \tau_{1,1} + \tau_{1,2} + \tau_{1,3} - \tau_{2,0} - \tau_{2,1} - \tau_{2,2} - \tau_{2,3} - \tau_{3,0} - \tau_{3,1} - \tau_{3,2} - \tau_{3,3}\\
\lambda_5 &= \tau_{0,0} + \tau_{0,1} - \tau_{0,2} - \tau_{0,3} - \tau_{1,0} - \tau_{1,1} + \tau_{1,2} + \tau_{1,3} - \tau_{2,0} - \tau_{2,1} + \tau_{2,2} + \tau_{2,3} + \tau_{3,0} + \tau_{3,1} - \tau_{3,2} - \tau_{3,3}\\
\lambda_6 &= \tau_{0,0} - \tau_{0,1} - \tau_{0,2} + \tau_{0,3} - \tau_{1,0} + \tau_{1,1} + \tau_{1,2} - \tau_{1,3} - \tau_{2,0} + \tau_{2,1} + \tau_{2,2} - \tau_{2,3} + \tau_{3,0} - \tau_{3,1} - \tau_{3,2} + \tau_{3,3}\\
\lambda_7 &= \tau_{0,0} - \tau_{0,1} + \tau_{0,2} - \tau_{0,3} - \tau_{1,0} + \tau_{1,1} - \tau_{1,2} + \tau_{1,3} - \tau_{2,0} + \tau_{2,1} - \tau_{2,2} + \tau_{2,3} + \tau_{3,0} - \tau_{3,1} + \tau_{3,2} - \tau_{3,3}\\
\lambda_8 &= \tau_{0,0} + \tau_{0,1} + \tau_{0,2} + \tau_{0,3} - \tau_{1,0} - \tau_{1,1} - \tau_{1,2} - \tau_{1,3} - \tau_{2,0} - \tau_{2,1} - \tau_{2,2} - \tau_{2,3} + \tau_{3,0} + \tau_{3,1} + \tau_{3,2} + \tau_{3,3}\\
\lambda_9 &= \tau_{0,0} + \tau_{0,1} - \tau_{0,2} - \tau_{0,3} - \tau_{1,0} - \tau_{1,1} + \tau_{1,2} + \tau_{1,3} + \tau_{2,0} + \tau_{2,1} - \tau_{2,2} - \tau_{2,3} - \tau_{3,0} - \tau_{3,1} + \tau_{3,2} + \tau_{3,3}\\
\lambda_{10} &= \tau_{0,0} - \tau_{0,1} - \tau_{0,2} + \tau_{0,3} - \tau_{1,0} + \tau_{1,1} + \tau_{1,2} - \tau_{1,3} + \tau_{2,0} - \tau_{2,1} - \tau_{2,2} + \tau_{2,3} - \tau_{3,0} + \tau_{3,1} + \tau_{3,2} - \tau_{3,3}\\
\lambda_{11} &= \tau_{0,0} - \tau_{0,1} + \tau_{0,2} - \tau_{0,3} - \tau_{1,0} + \tau_{1,1} - \tau_{1,2} + \tau_{1,3} + \tau_{2,0} - \tau_{2,1} + \tau_{2,2} - \tau_{2,3} - \tau_{3,0} + \tau_{3,1} - \tau_{3,2} + \tau_{3,3}\\
\lambda_{12} &= \tau_{0,0} + \tau_{0,1} + \tau_{0,2} + \tau_{0,3} - \tau_{1,0} - \tau_{1,1} - \tau_{1,2} - \tau_{1,3} + \tau_{2,0} + \tau_{2,1} + \tau_{2,2} + \tau_{2,3} - \tau_{3,0} - \tau_{3,1} - \tau_{3,2} - \tau_{3,3}\\
\lambda_{13} &= \tau_{0,0} + \tau_{0,1} - \tau_{0,2} - \tau_{0,3} + \tau_{1,0} + \tau_{1,1} - \tau_{1,2} - \tau_{1,3} + \tau_{2,0} + \tau_{2,1} - \tau_{2,2} - \tau_{2,3} + \tau_{3,0} + \tau_{3,1} - \tau_{3,2} - \tau_{3,3}\\
\lambda_{14} &= \tau_{0,0} - \tau_{0,1} - \tau_{0,2} + \tau_{0,3} + \tau_{1,0} - \tau_{1,1} - \tau_{1,2} + \tau_{1,3} + \tau_{2,0} - \tau_{2,1} - \tau_{2,2} + \tau_{2,3} + \tau_{3,0} - \tau_{3,1} - \tau_{3,2} + \tau_{3,3}\\
\lambda_{15} &= \tau_{0,0} - \tau_{0,1} + \tau_{0,2} - \tau_{0,3} + \tau_{1,0} - \tau_{1,1} + \tau_{1,2} - \tau_{1,3} + \tau_{2,0} - \tau_{2,1} + \tau_{2,2} - \tau_{2,3} + \tau_{3,0} - \tau_{3,1} + \tau_{3,2} - \tau_{3,3}\\
\lambda_{16} &= \tau_{0,0} + \tau_{0,1} + \tau_{0,2} + \tau_{0,3} + \tau_{1,0} + \tau_{1,1} + \tau_{1,2} + \tau_{1,3} + \tau_{2,0} + \tau_{2,1} + \tau_{2,2} + \tau_{2,3} + \tau_{3,0} + \tau_{3,1} + \tau_{3,2} + \tau_{3,3}.
\end{aligned}
\tag{6}
$$

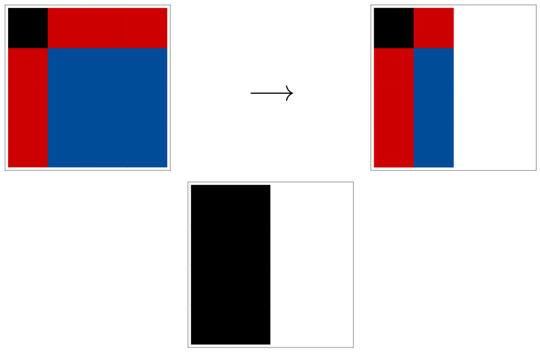
Now, taking all $\tau_{i,j}$ with coefficients -1 equal to zero one is led to 16 elements of PCE_2 (one for each eigenvalue):



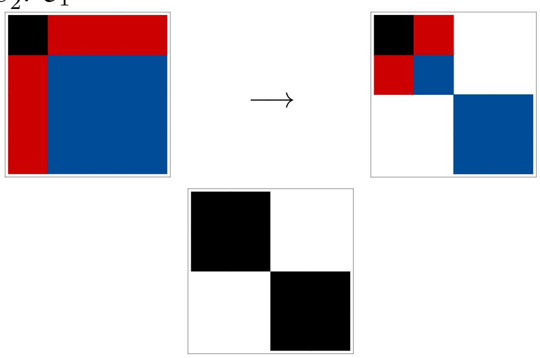
The first 15 elements are the elements of Γ_2.

For the sake of completeness let us show how to construct all elements in PCE_2 from Γ_2. Recall that PCE_2 can be ordered in equivalence classes with PCE quantum channels that are connected via particle swaps and local permutations of basis.

- **8 components:**
  - C_1^8: $\mathcal{E}_{13}$

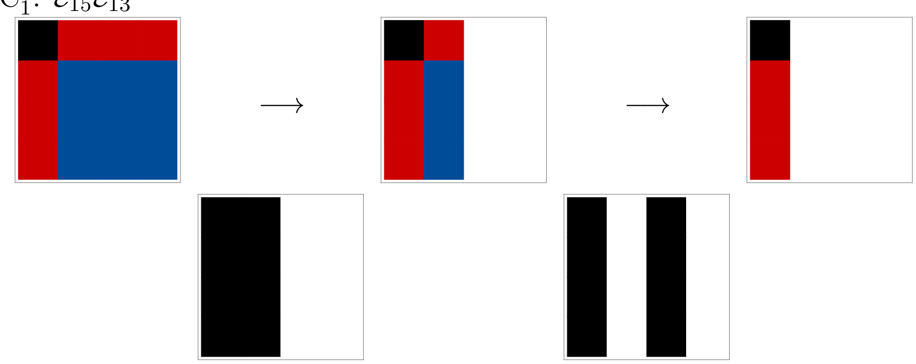


- C_2^8: $\mathcal{E}_1$

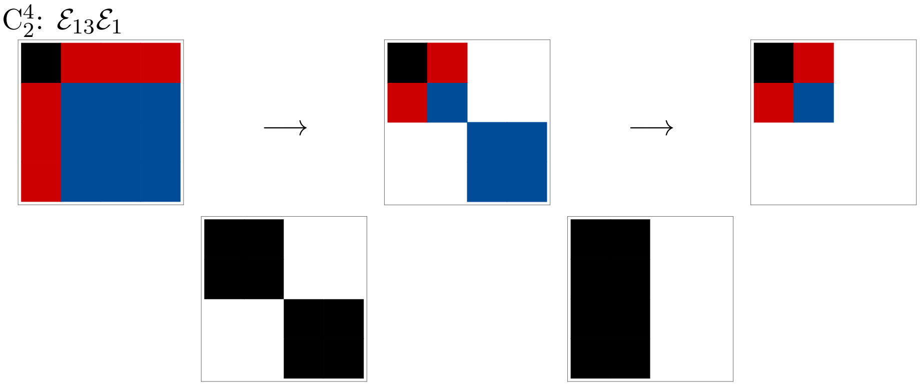


- **4 components:**

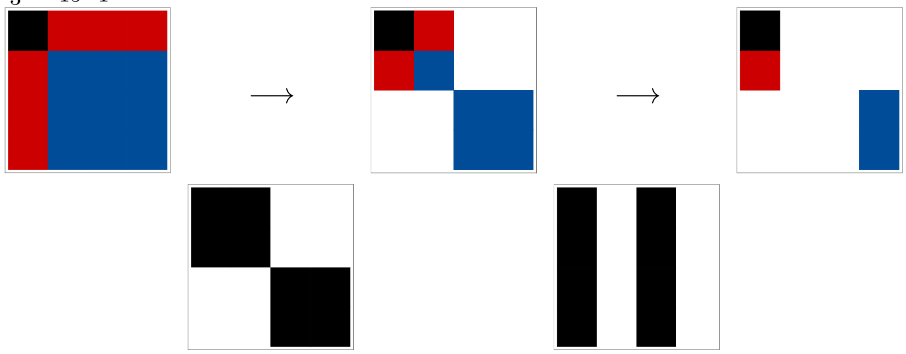
  - C_1^4: $\mathcal{E}_{15}\mathcal{E}_{13}$



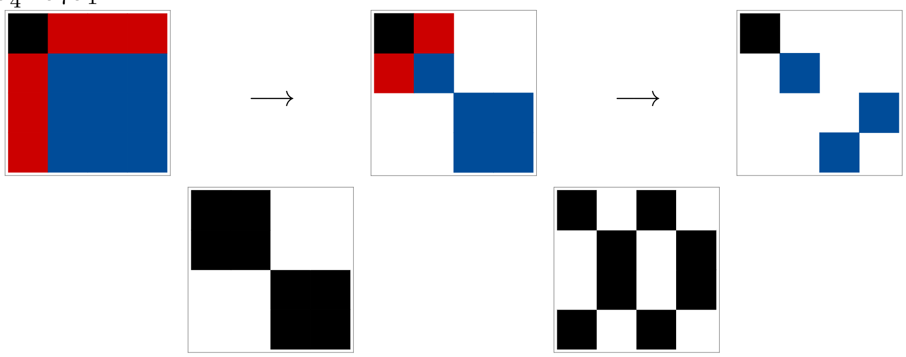
  - C_2^4: $\mathcal{E}_{13}\mathcal{E}_1$



- C^4_3: $\mathcal{E}_{15}\mathcal{E}_1$

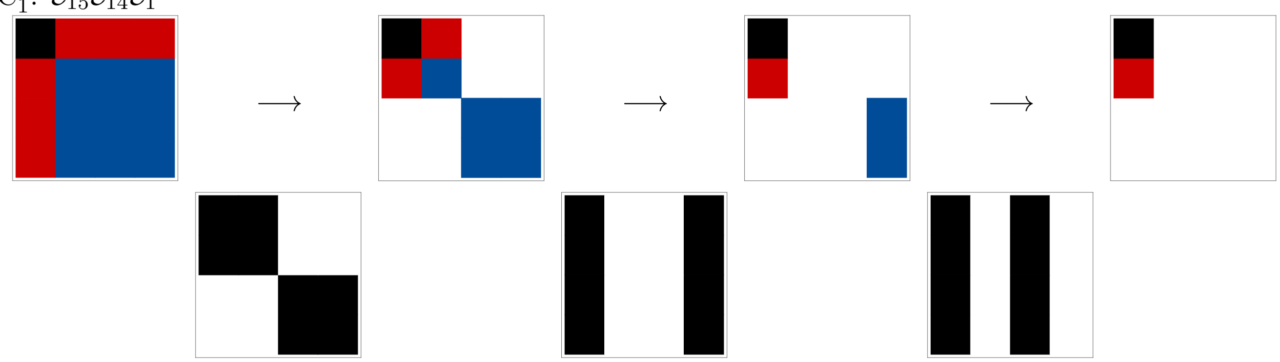


- C^4_4: $\mathcal{E}_7\mathcal{E}_1$

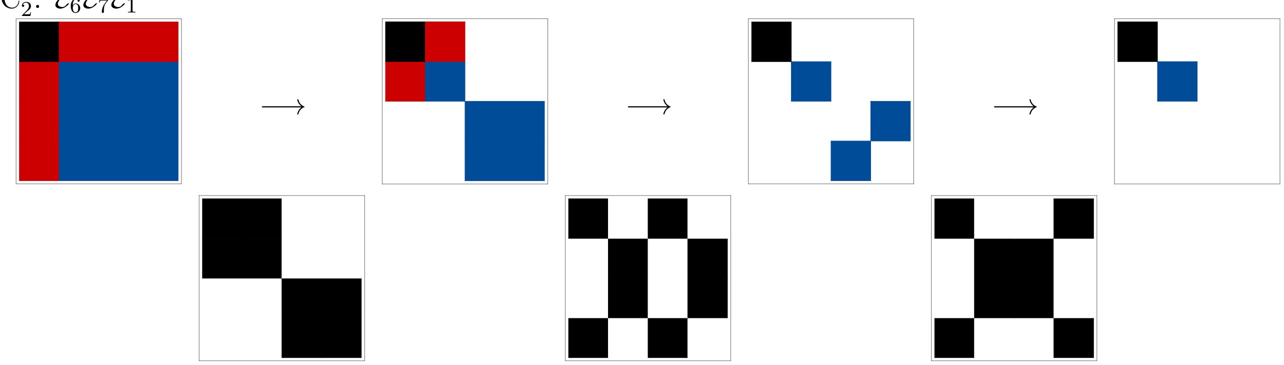


- **2 components:**

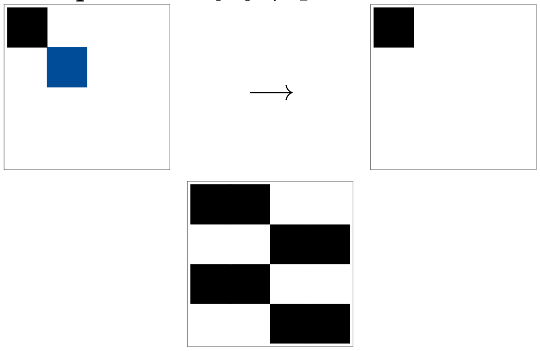
  - C^2_1: $\mathcal{E}_{15}\mathcal{E}_{14}\mathcal{E}_1$



  - C^2_2: $\mathcal{E}_6\mathcal{E}_7\mathcal{E}_1$



- **1 component:** $\mathcal{E}_9\mathcal{E}_6\mathcal{E}_7\mathcal{E}_1$



Hypothesis 2 *Only n elements in Γ_n are sufficient to generate the remaining elements.*

2.1 Numerical Results

Hypothesis 1 has strong numerical support. The search for PCE quantum channels has been done in two different ways:

1. Using the inequalities in (4) to test CP of all PCE operations (even the ones that do not follow the power-of-2 rule).

2. Using hypoteshis 1.

Both methods have analyzed the cases of 1, 2, 3 and partially 4 qubits. Both of them get the same results.

Qubits	Number of components invariant								
	1	2	4	8	16	32	64	128	256
1	1	3	1						
2	1	15	35	15	1				
3	1	63	651	1395	651	1			
4	1	255	10795	97155?	????	97155	10795	255	1

Tested with both methods

Tested only with inequalities

Tested only with concatenation hypothesis

3 Things to do and questions to answer

1. This characterization may explain the power-of-2 rule. It is quite obvious (in the figures representation) that the concatenation always erases half of the number of non-zero components. A proof for that is missing.

2. Analytic proof for (4).

3. Analytic proof that (5) is necessary and sufficient to find all elements in $\mathsf{PCE_n}$.

4. Analytic proof that taking $-1 \rightarrow 0$ for a_k^k leads to CP. I suggest using particle swaps and local basis permutation in order to reduce the problem to n inequalities.

5. Find a way to count PCE channels.

4 Progress

4.1 An idea to prove equation (3)

A density matrix of 1 qubit in Pauli matrices basis is written as

$$\rho = \frac{1}{2}\sum_{i=0}^{3} r_i \sigma_i, \tag{7}$$

where σ_i are a 2×2 identity plus Pauli matrices. For 2 qubits, in the general case, the density matrix cannot be written as

$$\rho = \frac{1}{4}\left(\sum_{i=0}^{3} r_i \sigma_i \otimes \sum_{j=0}^{3} r_j \sigma_j\right) \tag{8}$$

because a 2-qubits state is not separable, in general. Therefore, the density matrix is written as

$$\rho = \frac{1}{4}\sum_{i,j=0}^{3} r_{i,j}\sigma_{i,j}. \tag{9}$$

In some sense, expanding (8)

$$\frac{1}{4}\left(\sum_{i=0}^{3} r_i \sigma_i \otimes \sum_{j=0}^{3} r_j \sigma_j\right) = \frac{1}{4}\left(\mathbb{1}_4 \otimes \mathbb{1}_4 + \mathbb{1} \otimes \sum_{j=1}^{3} r_j \sigma_j + \sum_{i=1}^{3} r_i \sigma_i \otimes \mathbb{1} + \sum_{i=1}^{3} r_i \sigma_i \otimes \sum_{j=1}^{3} r_j \sigma_j\right) \tag{10}$$

makes us a hint realize that, in the general case, the components of 2-qubits density matrix in tensor products of Pauli matrices basis are

$$r_i r_j \to r_{i.j}, \qquad r_i \to r_{i,0}, \qquad r_j \to r_{0,j}. \tag{11}$$

It follows that n-quits density matrix is

$$\rho = \frac{1}{2^n}\sum_{j_1,\ldots,j_n=0}^{3} r_{j_1,\ldots,j_n}\sigma_{j_1} \otimes \ldots \otimes \sigma_{j_n}, \tag{12}$$

therefore,

$$r_{j_1}\ldots r_{j_k}\ldots r_{j_n} \to r_{j_1,\ldots,j_n}, \qquad r_{j_k} \to r_{0,\ldots,j_k,\ldots,0}, \tag{13}$$

where

$$r_{j_1}\ldots r_{j_k}\ldots r_{j_n} \neq r_{j_1,\ldots,j_n}, \qquad r_{j_k} \neq r_{0,\ldots,j_k,\ldots,0}, \tag{14}$$

Now, the state $\rho_{\mathcal{E}}$ isomorphic to the Choi matrix $D_{\mathcal{E}}$ (is this written ok?) of a 1-qubit PCE channel $\mathcal{E}$ is

$$\rho_{\mathcal{E}} = \sum_{i=0}^{3} \lambda_i \left|\sigma_i\right\rangle\!\left\langle\sigma_i\right|, \tag{15}$$

where $|\sigma_i\rangle$ are the vectorized σ_i and λ_i are (this has been shown analytically)

$$\begin{aligned} \lambda_0 &= \frac{1}{4}(1+\tau_1+\tau_2+\tau_3), & \lambda_1 &= \frac{1}{4}(1+\tau_1-\tau_2-\tau_3), \\ \lambda_2 &= \frac{1}{4}(1-\tau_1+\tau_2-\tau_3), & \lambda_3 &= \frac{1}{4}(1-\tau_1-\tau_2+\tau_3). \end{aligned} \tag{16}$$

I claim that, in the same sense of (13), the eigenvalues of $\rho_{\mathcal{E}}$ of n-qubits PCE quantum channel $\mathcal{E}$ are of the form

$$\lambda_{j_1}\ldots\lambda_{j_k}\ldots\lambda_{j_n} \to \lambda_{j_1,\ldots,j_n}, \qquad \lambda_{j_k} \to \lambda_{0,\ldots,j_k,\ldots,0}, \tag{17}$$

where λ_k are the eigenvalues (16).

5 Number of PCE channels by the ratio of components left invariant

- 1/2 of total components invariant:

$$4^n - 1 \tag{18}$$

- 1/4 of total components invariant:

$$\frac{\binom{4^n-1}{2}}{3}, \tag{19}$$

- 1/8 of total components invariant:

$$\frac{\binom{4^n-1}{3} - \binom{4^n-1}{2}/3}{28} \tag{20}$$

- 1/16 of total components invariant:

$$\frac{\binom{4^n-1}{4} - 35\left(\frac{\binom{4^n-1}{3}-\binom{4^n-1}{2}/3}{28}\right)}{840} \tag{21}$$

1						**1**	**3**	**1**					
2					**1**	**15**	**35**	**15**	**1**				
3				**1**	**63**	**651**	**1,395**	**651**	**63**	**1**			
4			**1**	**255**	**10,795**	97,155	200,787	**97,155**	**10,795**	**255**	**1**		
5		1	1,023	174,251	6,347,715	53,743,987		53,743,987	6,347,715	174,251	**1,023**	**1**	
6	1	4,095	2,794,155	408,345,795	13,910,980,083				13,910,980,083	408,345,795	2,794,155	**4,095**	**1**

David's ideas

Alejo's ideas