On the problem of PCE's

We have the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{1}$$

which satisfies the obvious relation

$$M^{-1} = \frac{1}{2} M \tag{2}$$

We wish to find those vectors consisting of 0's and 1's such that

$$M^{\otimes N} \vec{\sigma} \tag{3}$$

is a vector with positive entries. Here $\vec{\sigma}$ is a 4^N dimensional vector of 0's and 1's. We number the entries of $\sigma_{i_1,\dots,i_N}$ (where $1 \le i_k \le 4$ for $1 \le k \le N$) consecutively by α. The entries of the positive vector are given by $2^N p_\alpha^2$. It then follows that the expression (3) satisfies

$$\left(M_{\alpha\beta}^{\otimes N} \right) \sigma_\beta = 2^N p_\alpha^2 \tag{4}$$

from which immediately follows, via (2)

$$\sigma_\alpha = \left(M_{\alpha\beta}^{\otimes N} \right) p_\beta^2 \tag{5}$$

which we analyze iteratively.

Let me show this first for $N = 1$, where it is trivial. (5) in this case reduces to

$$\begin{aligned} p_1^2 + p_2^2 + p_3^2 + p_4^2 &= 1 && \text{(6a)} \\ p_1^2 + p_2^2 - p_3^2 - p_4^2 &= \sigma_1 && \text{(6b)} \\ p_1^2 - p_2^2 + p_3^2 - p_4^2 &= \sigma_2 && \text{(6c)} \\ p_1^2 - p_2^2 - p_3^2 + p_4^2 &= \sigma_3 && \text{(6d)} \end{aligned}$$

If a real solution in the p_α exists, then the σ_α have a positive image. But if a given set of p's is real, so is any permutation. Hence we only need consider solutions up to permutations. So without loss of generality we may say that either all σ's vanish, or at least $\sigma_1 = 1$. In the latter case, by subtracting (6b) form (6a), we get

$$p_3 = p_4 = 0, \tag{7}$$

and thus, from (6c) and (6d)

$$\sigma_2 = \sigma_3 \tag{8}$$

So we have the full set of possible solutions: $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 1)$ plus all the permutations involving only $\sigma_{2,3,4}$.

Let us carry this same program through, as far as my stamina holds, for $N = 2$: there we have

$$\begin{aligned} p_{11}^2 + p_{12}^2 + p_{13}^2 + p_{14}^2 + p_{21}^2 + p_{22}^2 + p_{23}^2 + p_{24}^2 + p_{31}^2 + p_{32}^2 + p_{33}^2 & \\ + p_{34}^2 + p_{41}^2 + p_{42}^2 + p_{43}^2 + p_{44}^2 &= 1 \end{aligned} \tag{9a}$$

$$\begin{aligned} p_{11}^2 + p_{12}^2 - p_{13}^2 - p_{14}^2 + p_{21}^2 + p_{22}^2 - p_{23}^2 - p_{24}^2 + p_{31}^2 + p_{32}^2 - p_{33}^2 & \\ - p_{34}^2 + p_{41}^2 + p_{42}^2 - p_{43}^2 - p_{44}^2 &= \sigma_{12} \end{aligned} \tag{9b}$$

$$\begin{aligned} p_{11}^2 - p_{12}^2 + p_{13}^2 - p_{14}^2 + p_{21}^2 - p_{22}^2 + p_{23}^2 - p_{24}^2 + p_{31}^2 - p_{32}^2 + p_{33}^2 & \\ - p_{34}^2 + p_{41}^2 - p_{42}^2 + p_{43}^2 - p_{44}^2 &= \sigma_{13} \end{aligned} \tag{9c}$$

$$\begin{aligned} p_{11}^2 - p_{12}^2 - p_{13}^2 + p_{14}^2 + p_{21}^2 - p_{22}^2 - p_{23}^2 + p_{24}^2 + p_{31}^2 - p_{32}^2 - p_{33}^2 & \\ + p_{34}^2 + p_{41}^2 - p_{42}^2 - p_{43}^2 + p_{44}^2 &= \sigma_{14} \end{aligned} \tag{9d}$$

$$\begin{aligned} p_{11}^2 + p_{12}^2 + p_{13}^2 + p_{14}^2 + p_{21}^2 + p_{22}^2 + p_{23}^2 + p_{24}^2 - p_{31}^2 - p_{32}^2 - p_{33}^2 & \\ - p_{34}^2 - p_{41}^2 - p_{42}^2 - p_{43}^2 - p_{44}^2 &= \sigma_{21} \end{aligned} \tag{9e}$$

$$\begin{aligned} p_{11}^2 + p_{12}^2 - p_{13}^2 - p_{14}^2 + p_{21}^2 + p_{22}^2 - p_{23}^2 - p_{24}^2 - p_{31}^2 - p_{32}^2 + p_{33}^2 & \\ + p_{34}^2 - p_{41}^2 - p_{42}^2 + p_{43}^2 + p_{44}^2 &= \sigma_{22} \end{aligned} \tag{9f}$$

$$\begin{aligned} p_{11}^2 - p_{12}^2 + p_{13}^2 - p_{14}^2 + p_{21}^2 - p_{22}^2 + p_{23}^2 - p_{24}^2 - p_{31}^2 + p_{32}^2 - p_{33}^2 & \\ + p_{34}^2 - p_{41}^2 + p_{42}^2 - p_{43}^2 + p_{44}^2 &= \sigma_{23} \end{aligned} \tag{9g}$$

$$\begin{aligned} p_{11}^2 - p_{12}^2 - p_{13}^2 + p_{14}^2 + p_{21}^2 - p_{22}^2 - p_{23}^2 + p_{24}^2 - p_{31}^2 + p_{32}^2 + p_{33}^2 & \\ - p_{34}^2 - p_{41}^2 + p_{42}^2 + p_{43}^2 - p_{44}^2 &= \sigma_{24} \end{aligned} \tag{9h}$$

$$\begin{aligned} p_{11}^2 + p_{12}^2 + p_{13}^2 + p_{14}^2 - p_{21}^2 - p_{22}^2 - p_{23}^2 - p_{24}^2 + p_{31}^2 + p_{32}^2 + p_{33}^2 & \\ + p_{34}^2 - p_{41}^2 - p_{42}^2 - p_{43}^2 - p_{44}^2 &= \sigma_{31} \end{aligned} \tag{9i}$$

$$\begin{aligned} p_{11}^2 + p_{12}^2 - p_{13}^2 - p_{14}^2 - p_{21}^2 - p_{22}^2 + p_{23}^2 + p_{24}^2 - p_{31}^2 - p_{32}^2 + p_{33}^2 & \\ + p_{34}^2 - p_{41}^2 - p_{42}^2 + p_{43}^2 + p_{44}^2 &= \sigma_{32} \end{aligned} \tag{9j}$$

$$\begin{aligned} p_{11}^2 - p_{12}^2 + p_{13}^2 - p_{14}^2 - p_{21}^2 + p_{22}^2 - p_{23}^2 + p_{24}^2 + p_{31}^2 - p_{32}^2 + p_{33}^2 & \\ - p_{34}^2 - p_{41}^2 + p_{42}^2 - p_{43}^2 + p_{44}^2 &= \sigma_{33} \end{aligned} \tag{9k}$$

$$\begin{aligned} p_{11}^2 - p_{12}^2 - p_{13}^2 + p_{14}^2 - p_{21}^2 + p_{22}^2 + p_{23}^2 - p_{24}^2 + p_{31}^2 - p_{32}^2 - p_{33}^2 & \\ + p_{34}^2 - p_{41}^2 + p_{42}^2 + p_{43}^2 - p_{44}^2 &= \sigma_{34} \end{aligned} \tag{9l}$$

$$\begin{aligned} p_{11}^2 + p_{12}^2 + p_{13}^2 + p_{14}^2 - p_{21}^2 - p_{22}^2 - p_{23}^2 - p_{24}^2 - p_{31}^2 - p_{32}^2 - p_{33}^2 & \\ - p_{34}^2 + p_{41}^2 + p_{42}^2 + p_{43}^2 + p_{44}^2 &= \sigma_{41} \end{aligned} \tag{9m}$$

$$\begin{aligned} p_{11}^2 + p_{12}^2 - p_{13}^2 - p_{14}^2 - p_{21}^2 - p_{22}^2 + p_{23}^2 + p_{24}^2 - p_{31}^2 - p_{32}^2 + p_{33}^2 & \\ + p_{34}^2 + p_{41}^2 + p_{42}^2 - p_{43}^2 - p_{44}^2 &= \sigma_{42} \end{aligned} \tag{9n}$$

$$\begin{aligned} p_{11}^2 - p_{12}^2 + p_{13}^2 - p_{14}^2 - p_{21}^2 + p_{22}^2 - p_{23}^2 + p_{24}^2 - p_{31}^2 + p_{32}^2 - p_{33}^2 & \\ + p_{34}^2 + p_{41}^2 - p_{42}^2 + p_{43}^2 - p_{44}^2 &= \sigma_{43} \end{aligned} \tag{9o}$$

$$\begin{aligned} p_{11}^2 - p_{12}^2 - p_{13}^2 + p_{14}^2 - p_{21}^2 + p_{22}^2 + p_{23}^2 - p_{24}^2 - p_{31}^2 + p_{32}^2 + p_{33}^2 & \\ - p_{34}^2 + p_{41}^2 - p_{42}^2 - p_{43}^2 + p_{44}^2 &= \sigma_{44} \end{aligned} \tag{9p}$$

Now we start distinguishing cases. Since $\sigma_{11} = 1$, the assumption $\sigma_{12} = 1$ is different from $\sigma_{22} = 1$, but there are in fact only those 2 which are essentially different. Consider then first $\sigma_{12} = 1$. This leads to

$$p_{13} = p_{14} = p_{23} = p_{24} = p_{33} = p_{34} = p_{43} = p_{44} = 0 \tag{10}$$

This reduces (9) to

$$\begin{aligned}
p_{11}^2 + p_{12}^2 + p_{21}^2 + p_{22}^2 + p_{31}^2 + p_{32}^2 + p_{41}^2 + p_{42}^2 &= 1 && \text{(11a)}\\
p_{11}^2 + p_{12}^2 + p_{21}^2 + p_{22}^2 + p_{31}^2 + p_{32}^2 + p_{41}^2 + p_{42}^2 &= \sigma_{12} && \text{(11b)}\\
p_{11}^2 - p_{12}^2 + p_{21}^2 - p_{22}^2 + p_{31}^2 - p_{32}^2 + p_{41}^2 - p_{42}^2 &= \sigma_{13} && \text{(11c)}\\
p_{11}^2 - p_{12}^2 + p_{21}^2 - p_{22}^2 + p_{31}^2 - p_{32}^2 + p_{41}^2 - p_{42}^2 &= \sigma_{14} && \text{(11d)}\\
p_{11}^2 + p_{12}^2 + p_{21}^2 + p_{22}^2 - p_{31}^2 - p_{32}^2 - p_{41}^2 - p_{42}^2 &= \sigma_{21} && \text{(11e)}\\
p_{11}^2 + p_{12}^2 + p_{21}^2 + p_{22}^2 - p_{31}^2 - p_{32}^2 - p_{41}^2 - p_{42}^2 &= \sigma_{22} && \text{(11f)}\\
p_{11}^2 - p_{12}^2 + p_{21}^2 - p_{22}^2 - p_{31}^2 + p_{32}^2 - p_{41}^2 + p_{42}^2 &= \sigma_{23} && \text{(11g)}\\
p_{11}^2 - p_{12}^2 + p_{21}^2 - p_{22}^2 - p_{31}^2 + p_{32}^2 - p_{41}^2 + p_{42}^2 &= \sigma_{24} && \text{(11h)}\\
p_{11}^2 + p_{12}^2 - p_{21}^2 - p_{22}^2 + p_{31}^2 + p_{32}^2 - p_{41}^2 - p_{42}^2 &= \sigma_{31} && \text{(11i)}\\
p_{11}^2 + p_{12}^2 - p_{21}^2 - p_{22}^2 - p_{31}^2 - p_{32}^2 - p_{41}^2 - p_{42}^2 &= \sigma_{32} && \text{(11j)}\\
p_{11}^2 - p_{12}^2 - p_{21}^2 + p_{22}^2 + p_{31}^2 - p_{32}^2 - p_{41}^2 + p_{42}^2 &= \sigma_{33} && \text{(11k)}\\
p_{11}^2 - p_{12}^2 - p_{21}^2 + p_{22}^2 + p_{31}^2 - p_{32}^2 - p_{41}^2 + p_{42}^2 &= \sigma_{34} && \text{(11l)}\\
p_{11}^2 + p_{12}^2 - p_{21}^2 - p_{22}^2 - p_{31}^2 - p_{32}^2 + p_{41}^2 + p_{42}^2 &= \sigma_{41} && \text{(11m)}\\
p_{11}^2 + p_{12}^2 - p_{21}^2 - p_{22}^2 - p_{31}^2 - p_{32}^2 + p_{41}^2 + p_{42}^2 &= \sigma_{42} && \text{(11n)}\\
p_{11}^2 - p_{12}^2 - p_{21}^2 + p_{22}^2 - p_{31}^2 + p_{32}^2 + p_{41}^2 - p_{42}^2 &= \sigma_{43} && \text{(11o)}\\
p_{11}^2 - p_{12}^2 - p_{21}^2 + p_{22}^2 - p_{31}^2 + p_{32}^2 + p_{41}^2 - p_{42}^2 &= \sigma_{44} && \text{(11p)}
\end{aligned}$$

which shows that

$$\begin{aligned}
\sigma_{13} &= \sigma_{14}, \quad \sigma_{21} = \sigma_{22}, \quad \sigma_{23} = \sigma_{24} && \text{(12)}\\
\sigma_{31} &= \sigma_{32}, \quad \sigma_{33} = \sigma_{34}, \quad \sigma_{41} = \sigma_{42}, \quad \sigma_{43} = \sigma_{44} && \text{(13)}
\end{aligned}$$

This reduces (11) to

$$\begin{aligned}
p_{11}^2 + p_{12}^2 + p_{21}^2 + p_{22}^2 + p_{31}^2 + p_{32}^2 + p_{41}^2 + p_{42}^2 &= 1 && \text{(14a)}\\
p_{11}^2 - p_{12}^2 + p_{21}^2 - p_{22}^2 + p_{31}^2 - p_{32}^2 + p_{41}^2 - p_{42}^2 &= \sigma_{13} && \text{(14b)}\\
p_{11}^2 + p_{12}^2 + p_{21}^2 + p_{22}^2 - p_{31}^2 - p_{32}^2 - p_{41}^2 - p_{42}^2 &= \sigma_{21} && \text{(14c)}\\
p_{11}^2 - p_{12}^2 + p_{21}^2 - p_{22}^2 - p_{31}^2 + p_{32}^2 - p_{41}^2 + p_{42}^2 &= \sigma_{23} && \text{(14d)}\\
p_{11}^2 + p_{12}^2 - p_{21}^2 - p_{22}^2 + p_{31}^2 + p_{32}^2 - p_{41}^2 - p_{42}^2 &= \sigma_{31} && \text{(14e)}\\
p_{11}^2 - p_{12}^2 - p_{21}^2 + p_{22}^2 + p_{31}^2 - p_{32}^2 - p_{41}^2 + p_{42}^2 &= \sigma_{33} && \text{(14f)}\\
p_{11}^2 + p_{12}^2 - p_{21}^2 - p_{22}^2 - p_{31}^2 - p_{32}^2 + p_{41}^2 + p_{42}^2 &= \sigma_{41} && \text{(14g)}\\
p_{11}^2 - p_{12}^2 - p_{21}^2 + p_{22}^2 - p_{31}^2 + p_{32}^2 + p_{41}^2 - p_{42}^2 &= \sigma_{43} && \text{(14h)}
\end{aligned}$$

If we now assume, for instance, $\sigma_{21} = 1$, we get

$$p_{31} = p_{32} = p_{41} = p_{42} = 0 \tag{15}$$

and (14) reduces to

$$\begin{aligned}
p_{11}^2+p_{12}^2+p_{21}^2+p_{22}^2 &= 1 && \text{(16a)}\\
p_{11}^2-p_{12}^2+p_{21}^2-p_{22}^2 &= \sigma_{13} && \text{(16b)}\\
p_{11}^2+p_{12}^2+p_{21}^2+p_{22}^2 &= \sigma_{21} && \text{(16c)}\\
p_{11}^2-p_{12}^2+p_{21}^2-p_{22}^2 &= \sigma_{23} && \text{(16d)}\\
p_{11}^2+p_{12}^2-p_{21}^2-p_{22}^2 &= \sigma_{31} && \text{(16e)}\\
p_{11}^2-p_{12}^2-p_{21}^2+p_{22}^2 &= \sigma_{33} && \text{(16f)}\\
p_{11}^2+p_{12}^2-p_{21}^2-p_{22}^2 &= \sigma_{41} && \text{(16g)}\\
p_{11}^2-p_{12}^2-p_{21}^2+p_{22}^2 &= \sigma_{43} && \text{(16h)}
\end{aligned}$$

which leads to

$$\sigma_{13}=\sigma_{23}, \quad \sigma_{31}=\sigma_{41}, \quad \sigma_{33}=\sigma_{43} \tag{17}$$

which reduces (16) to

$$\begin{aligned}
p_{11}^2+p_{12}^2+p_{21}^2+p_{22}^2 &= 1 && \text{(18a)}\\
p_{11}^2-p_{12}^2+p_{21}^2-p_{22}^2 &= \sigma_{13} && \text{(18b)}\\
p_{11}^2+p_{12}^2-p_{21}^2-p_{22}^2 &= \sigma_{31} && \text{(18c)}\\
p_{11}^2-p_{12}^2-p_{21}^2+p_{22}^2 &= \sigma_{33} && \text{(18d)}
\end{aligned}$$

There, however, we already know the solutions: either $\sigma_{13}=\sigma_{31}=\sigma_{33}=0$, or else only one is 1, or else all are 1. So we have finished the branch $\sigma_{12}=\sigma_{21}=1$. One should now proceed with other possibilities. For instance $\sigma_{12}=\sigma_{23}=1$, which is inequivalent. Or else the entire branch in which the initial σ that is 1 shares no index with σ_{11}. So we may ask what happens if we start with σ_{22}.

Clearly, this is in some need of formalization and/or computerization: even running this through by hand for $N=2$ is beyond me. $N=3$, and even more $N=17$, are wholly beyond this brute force approach. But I do believe it shows some structure which might usefully be employed in the general case.