

## Pauli diagonal channels constant on axes

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Received 31 October 2006, in final form 28 March 2007

Published 27 June 2007

Online at [stacks.iop.org/JPhysA/40/8171](http://stacks.iop.org/JPhysA/40/8171)

### Abstract

We define and study the properties of channels which are analogous to unital qubit channels in several ways. A full treatment can be given only when the dimension  $d = p^m$  a prime power, in which case each of the  $d + 1$  mutually unbiased bases (MUB) defines an axis. Along each axis the channel looks like a depolarizing channel, but the degree of depolarization depends on the axis. When  $d$  is not a prime power, some of our results still hold, particularly in the case of channels with one symmetry axis. We describe the convex structure of this class of channels and the subclass of entanglement breaking channels. We find new bound entangled states for  $d = 3$ . For these channels, we show that the multiplicativity conjecture for maximal output  $p$ -norm holds for  $p = 2$ . We also find channels with behaviour not exhibited by unital qubit channels, including two pairs of orthogonal bases with equal output entropy in the absence of symmetry. This provides new numerical evidence for the additivity of minimal output entropy.

PACS numbers: 03.67.-a, 03.67.Mn

### 1. Introduction

The results presented here are motivated by the desire to find channels for dimension  $d > 2$  whose properties are similar to those of the unital qubit channels, particularly with respect to optimal output purity. A channel is described by a completely positive, trace-preserving (CPT) map. The channels we construct are similar to unital qubit channels in the sense that their effect on a density matrix can be defined in terms of multipliers of components along different ‘axes’ defined in terms of mutually unbiased bases (MUB). When all multipliers are positive, these channels are very much like unital qubit channels with positive multipliers. However, when some of the multipliers are negative the new channels can exhibit behaviour not encountered for unital qubit channels.

For a fixed orthonormal basis  $\mathcal{B} = \{|\psi_k\rangle\}$ , the quantum-classical (QC) channel

$$\Psi^{\text{QC}}(\rho) = \sum_k \langle \psi_k, \rho \psi_k \rangle |\psi_k\rangle \langle \psi_k| \quad (1)$$

projects a density matrix  $\rho$  onto the corresponding diagonal matrix in this basis. A convex combination  $\sum_J t_J \Psi_J^{\text{QC}}(\rho)$  of QC channels in a collection of orthonormal bases  $\mathcal{B}_J = \{|\psi_k^J\rangle\}$  is also a channel; in fact, it is an entanglement breaking (EB) channel. We consider channels which are a linear combination of the identity map  $\mathcal{I}(\rho) = \rho$  and a convex combination of QC channels whose bases are mutually unbiased, i.e., satisfy

$$|\langle \psi_m^J, \psi_n^K \rangle|^2 = \begin{cases} \frac{1}{d} & \text{for } J \neq K \\ \delta_{mn} & \text{for } J = K. \end{cases} \quad (2)$$

Such channels can be written in the form

$$\Phi = s\mathcal{I} + \sum_L t_L \Psi_L^{\text{QC}} \quad (3)$$

with

$$s + \sum_L t_L = 1 \quad \text{and} \quad t_L \geq 0, \quad s \geq \frac{-1}{d-1}. \quad (4)$$

The first condition ensures that  $\Phi$  is trace preserving (TP), and the pair that it is completely positive (CP), as will be shown in section 2.

It is well known that  $\mathbf{C}^d$  can have at most  $d+1$  MUB and that this is always possible when  $d = p^m$  is a prime power. We are primarily interested in channels of the form (3) when such a full set of  $d+1$  MUB exist. In that case, it is natural to generalize the Bloch sphere representation so that a density matrix  $\rho$  is represented by a vector  $v_{Jj}$  as in (8) and regard each of the MUB as defining an ‘axis’. The effect of the channel (3) on a density matrix is to take  $v_{Jj} \mapsto (s + t_J)v_{Jj}$ , i.e., to multiply each  $v_{Jj}$  by the number  $\lambda_J = s + t_J$ . Since this action depends only on the axis label  $J$ , we call these channels ‘constant on axes’.

In section 2, we introduce the relevant notation and describe several equivalent ways of representing channels constant on axes. We also describe important subclasses of these channels in section 2.3 and discuss their structure as a convex set in section 2.4. More details about our approach to MUB and relevant ways of representing states and channels are given in appendix A.

In section 3, we study the entanglement-breaking (EB) subclass, emphasizing conditions on the multipliers. We also give some conditions under which the channels define bound entangled states when  $d = 3$ .

In section 4, we study channels which are linear combinations of the depolarizing channel, the projection onto the diagonal of a matrix and the completely noisy channel. These channels have one symmetry axis. They do not require MUB for their definition; however, when one has a full set of MUB they can be rewritten as channels constant on axes. We give necessary and sufficient conditions for the channels to be EB and consider their optimal output purity.

In section 5, we consider the maximal output purity of channels constant on axes, particularly the additivity conjecture for minimal entropy and the multiplicativity of the maximal  $p$ -norm. We show that for those with some negative multipliers, the optimal output purity needs not occur on the ‘longest’ axis. Indeed, one can even have two axes with different multipliers for which the corresponding MUB have equal output entropy. Numerical study of such channels gives new evidence for additivity of the minimal output entropy. In section 5.3, we conjecture that channels with non-negative multipliers achieve their maximal output purity

on axis states and explore the connection to multiplicativity. In section 5.4, we show that multiplicativity holds for  $p = 2$  for all channels constant on axes and extend this to channels constant on the ‘longest’ axis.

The paper contains a number of appendices, the first of which is primarily expository. The first two sections of appendix A describe representations of states and channels from the perspective that the  $d \times d$  matrices form a Hilbert space with the inner product  $\langle A, B \rangle = \text{Tr} A^\dagger B$ . Appendix A.3 discusses expansions in generalized Pauli matrices and their connection to MUB. Appendix A.4 gives more information about MUB; appendix A.5 considers some alternative ways of using MUB to describe channels, and appendix A.6 considers channels which are formed from conjugations on a single axis. Finally, a simple proof of the so-called computable cross norm (CCN) condition is given in appendix A.7.

The remaining appendices contain details of proofs which are omitted in the main text. Appendix B contains several proofs related to the multiplicativity conjecture. Appendix C.1 proves separability of certain state representatives which determine the EB region for channels with one symmetry axis. Appendix C.2 describes the state representative when  $d$  is prime. Appendices C.3 and C.4 use this result to obtain extreme points of the EB region for prime  $d$  as well as the PPT region for the case  $d = 3$ .

## 2. Channels constant on axes

### 2.1. Notation and generators of MUB

For any collection  $\mathcal{B}_J = \{|\psi_k^J\rangle\}$  of orthonormal bases on  $\mathbb{C}^d$ , we can define the operators

$$W_J = \sum_{k=1}^d \omega^k |\psi_k^J\rangle\langle\psi_k^J|, \quad J = 1, 2, \dots, d+1, \quad (5)$$

where  $\omega = e^{2\pi i/d}$ . It follows that

$$|\psi_n^J\rangle\langle\psi_n^J| = \frac{1}{d} \sum_{j=0}^{d-1} \bar{\omega}^{nj} W_J^j = \frac{1}{d} \left[ I + \sum_{j=1}^{d-1} \bar{\omega}^{nj} W_J^j \right]. \quad (6)$$

By construction,  $\text{Tr} W_J^m = 0$  for  $m = 1, 2, \dots, d-1$ ,  $W_J^m$  is unitary for any integer  $m$ ,  $(W_J^m)^\dagger = W_J^{-m} = W_J^{d-m}$ , and each of the operators  $W_J$  generates a cyclic group of order  $d$ . If, in addition, the bases are mutually unbiased (2) then when  $J \neq K$  and  $m$  and  $n$  are not both zero

$$\text{Tr}(W_J^m)^\dagger W_K^n = \sum_{jk} \omega^{jm-kn} |\langle\psi_j^J, \psi_k^K\rangle|^2 = \frac{1}{d} \sum_{j=1}^d \omega^{jm} \sum_{k=1}^d \bar{\omega}^{kn} = 0. \quad (7)$$

When there are  $d+1$  MUB, this gives  $d^2 - 1$  unitary operators  $\{W_J^m\}_{m=1, \dots, d-1, J=1, \dots, d+1}$  which satisfy the orthogonality condition  $\text{Tr} W_J^{d-m} W_K^n = d \delta_{JK} \delta_{mn}$  and, hence, form an orthogonal basis for the subspace of trace zero matrices in  $M_d$ . Note that this immediately implies that  $d+1$  is the maximum number of MUB for  $\mathbb{C}^d$ .

We call the unitary operators  $W_J$  the *generators* of the MUB. When we have a full set of  $d+1$  MUB, adding the identity matrix  $I$  to  $\{W_J^m\}$  gives an orthogonal basis of unitaries (OBU) for  $M_d$  with norm  $d^{1/2}$ . Therefore, any density matrix  $\rho$  can be written as

$$\rho = \frac{1}{d} \left[ I + \sum_{J=1}^{d+1} \sum_{j=1}^{d-1} v_{Jj} W_J^j \right], \quad (8)$$

with  $v_{Jj} = \text{Tr } W_J^{-j} \rho$ . This is the standard expansion of a vector in a Hilbert space using an orthogonal basis; the only novelty is that our Hilbert space is the set of  $d \times d$  matrices  $M_d$  with the Hilbert–Schmidt inner product  $\langle A, B \rangle = \text{Tr } A^\dagger B$ . Equation (8) can also be considered a generalization of the Bloch sphere representation. Both viewpoints are considered in more detail in appendix A. It is straightforward to show that

$$\Psi_K^{\text{QC}}(\rho) = \frac{1}{d} \sum_{j=0}^{d-1} W_K^j \rho W_K^{-j} = \frac{1}{d} \left[ I + \sum_{j=1}^{d-1} v_{Kj} W_K^j \right]. \quad (9)$$

This says that the effect of  $\Psi_K^{\text{QC}}(\rho)$  is simply to multiply  $v_{Jj}$  by 1 for  $J = K$  and by 0 for  $J \neq K$ . Since (9) has the Kraus operator sum form, the Kraus operators for a QC channel corresponding to the basis  $\mathcal{B}_J$  can be chosen as  $\frac{1}{\sqrt{d}} W_J^j$ ,  $j = 1, 2, \dots, d$ .

## 2.2. Equivalent representations

Given a set of MUB, a channel constant on axes can be defined in terms of the corresponding QC channels by (3) with constraints (4). The results of the previous section allow us to give some equivalent ways of writing channels constant on axes. First, observe that a map of the form (3) can be written as

$$\Phi(\rho) = a_{00}\rho + \frac{1}{d-1} \sum_{J=1}^{d+1} \sum_{j=1}^{d-1} a_J W_J^j \rho W_J^{-j}, \quad (10)$$

with  $a_{00} = s + \frac{1}{d} \sum_J t_J = \frac{1}{d}[(d-1)s + 1]$  and  $a_J = \frac{d-1}{d} t_J$ . In this form, the TP condition in (4) becomes  $a_{00} + \sum_J a_J = 1$ , and the next pair of conditions are equivalent to  $a_{00} \geq 0$  and  $a_J \geq 0$  for all  $J$ . Then (10) has the operator sum form of a CP map with Kraus operators  $\sqrt{a_{00}}I$  and  $\sqrt{a_J/(d-1)}W_J^j$ . Thus, conditions (4) suffice for  $\Phi$  to be CPT. It follows from theorem 18 in appendix A.2 that the converse is also true, i.e., a map of the form (10) is not CP unless  $a_{00} \geq 0$  and  $a_J \geq 0$  for all  $J$ .

It follows from the comment after (9) that the effect of a map of the form (3) can be expressed as

$$\Phi : \frac{1}{d} \left[ I + \sum_{J=1}^{d+1} \sum_{j=1}^{d-1} v_{Jj} W_J^j \right] \mapsto \frac{1}{d} \left[ I + \sum_{J=1}^{d+1} \lambda_J \sum_{j=1}^{d-1} v_{Jj} W_J^j \right], \quad (11)$$

so that  $v_{Jj} \mapsto \lambda_J v_{Jj}$  with  $\lambda_J := s + t_J$ . Thus, every such channel corresponds to a unique vector in  $\mathbf{R}^{d+1}$  which we write as  $[\lambda_1, \lambda_2, \dots, \lambda_{d+1}]$  with  $\lambda_J = s + t_J$ . When all of the  $\lambda_J$  are equal, the channel is depolarizing. Thus, another view of a channel constant on axes is that an input on the  $J$ th axis has the same output as a depolarizing channel with  $\lambda = \lambda_J$  in (14). This follows immediately from (11) and the fact that  $\gamma = \sum_n \mu_n |\psi_n^J\rangle\langle\psi_n^J|$  has  $v_{Lj} = 0$  for  $L \neq J$ .

**Theorem 1.** Let  $\mathbf{C}^d$  have a full set of  $d+1$  MUB and let  $[\lambda_1, \lambda_2, \dots, \lambda_{d+1}]$  be a vector in  $\mathbf{R}^{d+1}$ . Then (11) defines a CPT map if and only if

$$-\frac{1}{d-1} \leq \sum_J \lambda_J \leq 1 + d \min_K \lambda_K. \quad (12)$$

**Proof.** If one uses the TP condition in (4) to eliminate  $s$ , the two CP inequalities are equivalent to

$$\sum_{J \neq K} \lambda_J \leq 1 + (d-1)\lambda_K \quad K = 1, 2, \dots, d+1 \quad (13a)$$

**Table 1.** Comparison of axis channels classes to unital qubit classes.

	Description		Qubit		$d > 2$
	Identity	$\mathcal{I}$	$[1, 1, 1]$	$\mathcal{I}$	$[1, 1, \dots, 1]$
	Max noise	$\mathcal{N}$	$[0, 0, 0]$	$\mathcal{N}$	$[0, 0, \dots, 0]$
(a)	QC	QC	$[1, 0, 0]$	$\Psi_L^{\text{QC}}$	$[1, 0, \dots, 0]$
(b)	Phase damping		$[1, \lambda, \lambda]$	$\Psi_{L,\lambda}^{\text{PD}}$	$[1, \lambda, \dots, \lambda]$
(c)	Extreme points	$\sigma_L \rho \sigma_L$	$[1, -1, -1]$	$\Psi_L^{\text{X}}$	$\left[1, \frac{-1}{d-1}, \dots, \frac{-1}{d-1}\right]$
(d)	Extreme EB	$\sigma_j Q C \sigma_j$	$[-1, 0, 0]$	$\Psi_L^{\text{XEB}}$	$\left[\frac{-1}{d-1}, 0, \dots, 0\right]$
	Extreme EB for $d > 2$		$\left[-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$	$\Psi_L^{\text{YEB}}$	$\left[\frac{d-2}{2(d-1)}, \frac{-1}{2(d-1)}, \dots, \frac{-1}{2(d-1)}\right]$
(e)	Depolarize from $\mathcal{I}$	$\Psi_\lambda^{\text{dep}}$	$[\lambda, \lambda, \lambda]$	$\Psi_\lambda^{\text{dep}}$	$[\lambda, \lambda, \dots, \lambda]$
(f)	Max squashed	two-Pauli	$[2\lambda - 1, \lambda, \lambda]$	$\Psi_{L,\lambda}^{\text{MxSq}}$	$\left[\frac{d\lambda-1}{d-1}, \lambda, \dots, \lambda\right]$
(g)	Depolarize from $\Psi_L^{\text{X}}$		$[\lambda, -\lambda, -\lambda]$		$\left[\lambda, \frac{-\lambda}{d-1}, \dots, \frac{-\lambda}{d-1}\right]$

$$\sum_J \lambda_J \geq -\frac{1}{d-1}, \quad (13b)$$

which is clearly equivalent to (12).  $\square$

### 2.3. Subclasses

We now describe some important subclasses of channels constant on axes, in each case identifying its multipliers  $\lambda_J$  (as in (11)) and its coefficients  $a_{Lj}$  (as in (10)). This information is summarized in table 1.

- (a) *QC channels.* Let  $\Psi_L^{\text{QC}}$  have the form (9). Then its multiplier is  $[0, \dots, 1, \dots, 0]$  and  $a_{00} = a_L = \frac{1}{d}$  and  $a_{Kj} = 0$  for  $K \neq L$ .
- (b) *Phase damping channels.* Let  $\Psi_{L,\lambda}^{\text{PD}} = \lambda \mathcal{I} + (1-\lambda) \Psi_L^{\text{QC}}$  with  $-\frac{1}{d-1} \leq \lambda \leq 1$ . Then  $\Psi_{L,\lambda}^{\text{PD}}$  has a multiplier of the form  $[\lambda, \dots, \lambda, 1, \lambda, \dots, \lambda]$  and  $a_{00} = \lambda + \frac{1-\lambda}{d}$ ,  $a_{Lj} = \frac{1-\lambda}{d}$ , and  $a_{Kj} = 0$  for  $K \neq L$ . The  $d$ -axis states which are eigenvectors of  $W_L$  are invariant and thus have pure outputs.
- (c) *Extreme phase damping channels.* Let  $\Psi_L^{\text{X}} = \Psi_{L,\lambda}^{\text{PD}}$  with  $\lambda = \frac{-1}{d-1}$  so that  $a_{00} = 0$  and  $a_{Kj} = \frac{-1}{d-1} \delta_{KL}$ . Since no axis channel (except  $\mathcal{I}$ ) can have fewer non-zero  $a_{Kj}$ , these channels are extreme points of the convex set of axis channels. Each  $\Psi_L^{\text{X}}$  has a multiplier of the form  $\left[\frac{-1}{d-1}, \dots, \frac{-1}{d-1}, 1, \frac{-1}{d-1}, \dots, \frac{-1}{d-1}\right]$  with 1 in the  $L$ th position. When  $d = 2$ , each  $\Psi_L^{\text{X}}$  is a conjugation with one of the Pauli matrices  $\sigma_L$ , and its multiplier is a permutation of  $[-1, 1, -1]$ .
- (d) *Extreme EB channels.* The channels  $\Psi_L^{\text{XEB}} \equiv \frac{-1}{d-1} \Psi_L^{\text{QC}} + \frac{1}{d-1} \mathcal{N} = \frac{1}{d} \sum_{J \neq L} \Psi_J^{\text{X}}$  have multiplier  $\left[0, \dots, 0, -\frac{1}{d-1}, 0, \dots, 0\right]$  and are extreme points of the set of EB channels. The channels we denote by  $\Psi_L^{\text{YEB}}$  have multipliers which are permutations of  $\left[\frac{d-2}{2(d-1)}, \frac{-1}{2(d-1)}, \dots, \frac{-1}{2(d-1)}\right]$ ; for  $d > 2$ , these are also extreme points of the set of EB channels, as will be shown in section 3.
- (e) *Depolarizing channels.* The channel  $\Psi_\lambda^{\text{dep}}(\rho) = \lambda \rho + (1-\lambda) \frac{1}{d} \mathcal{I}$  has multiplier  $[\lambda, \lambda, \dots, \lambda]$  and can be written as

$$\Psi_\lambda^{\text{dep}} = \lambda \mathcal{I} + (1-\lambda) \mathcal{N} = \sum_{L=1}^{d+1} \frac{1}{d+1} \Psi_{L,\zeta}^{\text{PD}}, \quad (14)$$

with  $\zeta = \frac{\lambda(d+1)-1}{d}$ . Then  $a_{00} = \lambda + \frac{1-\lambda}{d^2}$  and  $a_{Lj} = \frac{1-\lambda}{d^2}$ .

- (f) **Channels with one symmetry axis.** Channels of the form (3) with all but one of the  $t_J$  identical have multipliers  $[\lambda, \lambda, \dots, \lambda, \eta, \lambda, \dots, \lambda]$  with  $\eta = s + t_L$ . They are naturally regarded as ‘squashed’ when  $0 < \eta < \lambda$ . In general, they are symmetric with respect to ‘rotations’ about the special axis  $L$ . The boundary case with one  $t_L = 0$  has  $\eta = \frac{d\lambda-1}{d-1}$ . This is called a ‘two-Pauli’ channel in the qubit case; we call them ‘maximally squashed’. These channels can be written in several equivalent forms

$$\begin{aligned}\Psi_{L,\lambda}^{\text{MxSq}} &= \sum_{K \neq L} \frac{1}{d} \Psi_{K,\zeta}^{\text{PD}} = \zeta \mathcal{I} + \sum_{K \neq L} \frac{1}{d} \Psi_{K,\zeta}^{\text{QC}} = \lambda \mathcal{I} + \frac{1-\lambda}{d} \sum_{J \neq L} \Psi_L^{\text{X}} \\ &= \lambda \mathcal{I} + (1-\zeta) \mathcal{N} - \frac{1-\zeta}{d} \Psi_L^{\text{QC}} = \Psi_\lambda^{\text{dep}} + \frac{1-\zeta}{d} (\mathcal{N} - \Psi_L^{\text{QC}}),\end{aligned}\quad (15)$$

with  $1 \geq \lambda = \zeta + \frac{1}{d}(1-\zeta) \geq 0$ .

- (g) For qubits, the channel which takes

$$\rho \mapsto \sigma_J \Psi_\lambda^{\text{dep}} \sigma_J = \lambda \sigma_J \rho \sigma_J + (1-\lambda) \frac{1}{2} \mathbf{I} \quad J = 1, 2, 3, \quad (16)$$

can be thought of as depolarizing from conjugation with  $\sigma_J$ ; its multiplier has the form  $[-\lambda, +\lambda, -\lambda]$  (with the  $+$  sign in the  $J$ th position). For  $d > 2$ , this has no direct generalization, but one might consider channels which ‘depolarize’ from the other extreme points, e.g.,  $\lambda \Psi_L^{\text{X}} + (1-\lambda) \mathcal{N}$  which has multiplier  $[\lambda, \frac{-\lambda}{d-1}, \dots, \frac{-\lambda}{d-1}]$ . These channels are also a subclass of those with one symmetry axis.

#### 2.4. Convex structure

The set of axis channels is convex, and we have already noted that the extreme phase-damping channels  $\Psi_L^{\text{X}} = \frac{-1}{d-1} \mathcal{I} + \frac{d}{d+1} \Psi_L^{\text{QC}}$  are extreme points of this set. Adding the identity  $\mathcal{I}$  gives all the extreme points.

**Theorem 2.** When  $\mathbf{C}^d$  has  $d+1$  MUB, the set of channels constant on axes is the convex hull of the identity  $\mathcal{I}$  and the extreme phase-damping channels  $\Psi_L^{\text{X}}$ .

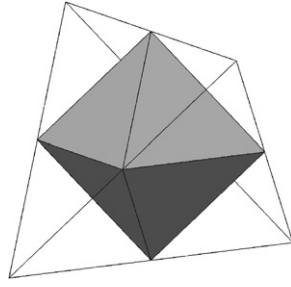
**Proof.** It suffices to observe that when  $\Phi$  is given by (3), it can be written as

$$\Phi = a_{00} \mathcal{I} + \sum_{J=1}^{d+1} a_J \Psi_J^{\text{X}} = \frac{1}{d} [(d-1)s+1] \mathcal{I} + \frac{d-1}{d} \sum_{J=1}^{d+1} t_J \Psi_J^{\text{X}}, \quad (17)$$

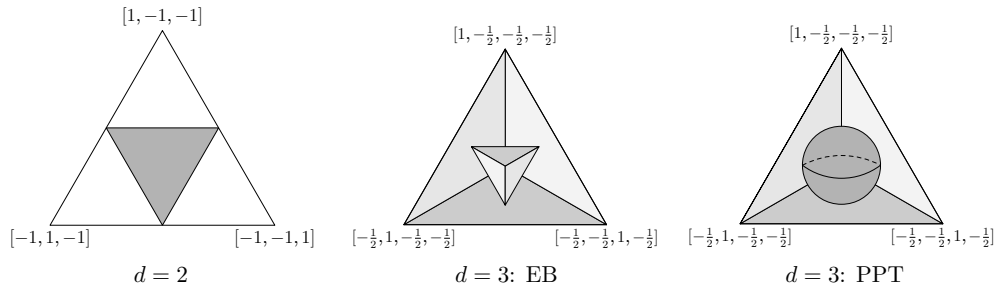
with coefficients as in (10) and (3) respectively. The TP and CP conditions (4) imply that the coefficients sum to 1 and are non-negative.  $\square$

Each of the  $d+2$  inequalities (13) defines a half space corresponding to the hyperplane defined by  $d+1$  of the extreme points in theorem 2. Then the intersection of these half spaces yields the convex set of channels constant on axes. When  $d=2$ , (13) is equivalent to  $|\lambda_j \pm \lambda_k| \leq |1 \pm \lambda_\ell|$  for  $j, k, \ell$  distinct, which are the conditions [10, 27, 35] needed to ensure that a unital qubit channel is CP.

It is now well known [10, 27, 35] that the multipliers  $[\lambda_1, \lambda_2, \lambda_3]$  for the unital qubit channels form a tetrahedron with vertices at  $[1, 1, 1]$ ,  $[1, -1, -1]$ ,  $[-1, 1, -1]$ , and  $[-1, -1, 1]$ , and that the subset of entanglement breaking (EB) channels corresponds to the octahedron obtained from the intersection of this tetrahedron with its inversion through the origin, as shown in figure 1. Removing this octahedron leaves four disjoint sets (also tetrahedrons) which can be transformed into one another by conjugation with the Pauli matrices  $\sigma_J$ . Each of these sets has multipliers with fixed signs determined by one of the maps  $\Psi_J^{\text{X}}$  and is the convex hull of this map and three QC channels. For example, the set with only



**Figure 1.** The tetrahedron of qubit channels and the octahedral EB subset.



**Figure 2.** Base of qubit and qutrit channels constant on axes showing subregions with all  $\lambda_j \leq 0$ . For qubits this is also the EB region. For qutrits, this is a tetrahedron which lies strictly within the EB region; the sphere shows the qutrit PPT channels, as discussed in section 3.3.

$\lambda_1 \geq 0$  is the convex hull of  $\Psi_1^X(\rho) = \sigma_1 \rho \sigma_1$  with multiplier  $[1, -1, -1]$  and the QC maps with multipliers  $[1, 0, 0]$ ,  $[0, -1, 0]$ ,  $[0, 0, -1]$ .

When  $d > 2$ , this picture is modified significantly. The set of CPT maps is still the convex hull of  $\mathcal{I}$  with multiplier  $[1, 1, \dots, 1]$  and  $d + 1$  channels whose multipliers are permutations of  $[1, \frac{-1}{d-1}, \dots, \frac{-1}{d-1}]$ . However, the identity is a distinguished vertex from which edges extend to give the  $d + 1$  lines of phase-damping channels between  $\mathcal{I}$  and  $\Psi_L^X$ . One also has a ‘base’ formed from the edges connecting pairs of the latter. There is only one symmetry axis. After removing the EB channels, one again has a disjoint region  $\Delta_0$  which contains the non-EB channels with all multipliers non-negative; this is the convex hull of  $\mathcal{I}$  and the  $\Psi_L^{\text{QC}}$ , as before. However, the picture with negative multipliers is far more complex.

For  $d = 3$ , the ‘base’ corresponds to the tetrahedron given by the convex hull of the four vertices  $\Psi_L^X$  with  $L = 1, 2, 3, 4$ , as shown in figure 2. The centre of each of the four faces is  $\Psi_L^{\text{XEB}} = \frac{1}{d} \sum_{j \neq L} \Psi_j^X$ . Since this is EB, the tetrahedron obtained by joining these four points (which is the set of channels with all  $\lambda_k \leq 0$ ) is a subset of the EB channels. However, it follows from theorem 4 in section 2.4 that no point on an edge connecting two  $\Psi_L^X$  is EB which means that, unlike the qubit case, removing the EB channels from the base does not leave  $d$  disjoint sets. This argument extends to all  $d > 2$ .

### 3. Entanglement breaking channels

#### 3.1. General considerations

A channel is called entanglement breaking (EB) if its action on half of an entangled state is separable. In fact, it suffices to consider its action on the maximally entangled state  $|\beta\rangle = \frac{1}{\sqrt{d}} \sum_k |e_k \otimes e_k\rangle$ . It is well known that  $\Phi \leftrightarrow \frac{1}{d} \sum_{jk} |e_j\rangle\langle e_k| \otimes \Phi(|e_j\rangle\langle e_k|) = (\mathcal{I} \otimes \Phi)|\beta\rangle\langle\beta|$

gives a one-to-one correspondence between CPT maps taking  $\mathbb{C}^d \mapsto \mathbb{C}^d$  and density matrices on  $\mathbb{C}^{d^2}$ . The latter is called the Choi–Jamiolkowski (CJ) matrix or state representative of the channel. In [18], it was shown that a channel is EB if and only if its CJ matrix is separable.

A channel of the form  $\Phi = t_L \Psi_L^{\text{QC}} + (1 - t_L)\mathcal{N}$  is clearly EB when  $t_L \geq 0$ . However, even for the larger CP range  $\frac{-1}{d-1} \leq t_L \leq 1$ , it is EB because

$$\Phi(|\psi_j^L\rangle\langle\psi_k^L|) = \delta_{jk} \sum_n \left( \frac{1}{d} (1 - t_L) + t_L \delta_{jn} \right) |\psi_n^L\rangle\langle\psi_n^L|. \quad (18)$$

This implies that the CJ matrix is diagonal in the product basis  $\{|\psi_j^L\rangle \otimes |\psi_k^L\rangle\}$  and hence separable. The CP endpoints of this line are  $\Psi_L^{\text{QC}}$  for  $t_L = 1$  and  $\Psi_L^{\text{XEB}} \equiv \frac{-1}{d-1} \Psi_L^{\text{QC}} + \frac{1}{d-1} \mathcal{N}$  for  $t_L = \frac{-1}{d-1}$ .

The positive partial transpose (PPT) condition for separability applied to the CJ matrix of a channel says that  $d(\mathcal{T} \otimes \Phi)|\beta\rangle\langle\beta| = \sum_{jk} |e_k\rangle\langle e_j| \otimes \Phi(|e_j\rangle\langle e_k|)$  is positive semi-definite. This is a necessary condition for a channel to be EB. By applying the PPT condition to the phase-damping channel  $\Psi_{L,\lambda}^{\text{PD}}$ , one can see that it is EB if and only if  $\lambda = 0$ .

It seems natural to conjecture that  $\Psi_L^{\text{QC}}$  and  $\Psi_L^{\text{XEB}}$  are the only extreme points of the convex set of EB channels constant on axes. However, this is not the case, as one can see from the following theorem which is proved in appendix C.3, where we also show that  $\Psi_L^{\text{XEB}}$  is a true extreme point for any prime  $d$ .

**Theorem 3.** *When  $d = 3$ , the channels  $\Psi_L^{\text{QC}}$ ,  $\Psi_L^{\text{XEB}}$  and  $\Psi_L^{\text{YEB}}$  are extreme points of the convex set of EB.*

For  $d = 2$ , only the first two channels give extreme points; the channel  $\Psi_L^{\text{YEB}}$  is on the boundary of EB subset, but not extreme. For  $d = 3$ , it seems natural to conjecture that set of EB channels constant on axes is the convex hull of the channels in theorem 3. However, it appears that there are regions of strict convexity for the PPT condition which yield additional extreme points for  $d \geq 3$ .

The channels  $\Psi_L^{\text{YEB}}$  are considered in section 4.2 where they are shown to be extreme points of the subset of EB channels with one symmetry axis. For  $d = 3$ , the channel  $\Psi_L^{\text{YEB}}$  has multiplier  $[\frac{+1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}]$ . For  $d = 4$ , it has multiplier  $[\frac{+1}{3}, \frac{-1}{6}, \frac{-1}{6}, \frac{-1}{6}, \frac{-1}{6}]$ ; the channel with multiplier  $[\frac{+1}{5}, \frac{-1}{5}, \frac{-1}{5}, \frac{-1}{5}, \frac{-1}{5}]$  is not CP because  $\sum_k \lambda_k = -\frac{3}{5} < -\frac{1}{3}$  violates condition (13b) and the channel with multiplier  $[\frac{+2}{9}, \frac{-2}{9}, \frac{-2}{9}, \frac{-2}{9}, \frac{-2}{9}]$  is EB but not extreme.

### 3.2. Multipliers for entanglement-breaking maps

We now consider EB criteria which can be stated in terms of the multiplier for a channel constant on axes. Any EB channel constant on axes must have  $\sum_L |\lambda_L| \leq 1$ . This is an immediate consequence of the more general requirement that  $\|\Phi\|_1 \leq 1$  for any EB channel. This is equivalent to what is sometimes called the ‘computable cross norm’ (CCN) condition or ‘rearrangement criterion’ for separability. However, as explained in appendix A.7, this condition can be applied directly to the channel without computing its CJ matrix or performing any type of rearrangement.

**Theorem 4.** *Let  $\Phi$  be a channel constant on axes. If  $\Phi$  is EB, then  $\sum_L |\lambda_L| \leq 1$ . Moreover, if all multipliers  $\lambda_k \geq 0$ , then  $\Phi$  is EB if and only if  $\sum_L \lambda_L \leq 1$ .*



**Proof of theorem 4.** The necessity follows immediately from theorem 24 and the fact that the singular values,  $\phi_s$ , of  $\Phi$  are  $|\lambda_L|$ , each with degeneracy  $d - 1$ , and 1 which is non-degenerate. Thus

$$d \geq \sum_s |\phi_s| = 1 + (d - 1) \sum_L |\lambda_L|.$$

Sufficiency follows immediately from the fact that when all  $\lambda_L \geq 0$  and  $\sum_L \lambda_L \leq 1$ , one can write

$$\Phi = \sum_L \lambda_L \Psi_L^{\text{QC}} + \left(1 - \sum_L \lambda_L\right) \mathcal{N} \quad (19)$$

as a convex combination of EB channels.  $\square$

When an EB channel is written in the form (3), we see that  $s \leq 0$ . This is an immediate corollary of theorem 4 and

$$1 \geq \sum_J |\lambda_J| \geq \sum_J \lambda_J = (d + 1)s + \sum_J t_J = ds + 1. \quad (20)$$

In [34], it was shown that a unital qubit channel is always EB when some  $\lambda_k = 0$ . It follows from (13a) that if the smallest  $\lambda_J = 0$  then  $\sum_{J \neq L} \lambda_J \leq 1$  so that the channel is EB, giving a partial extension of the qubit result. However, when some  $\lambda_k$  are negative, a channel with some  $\lambda_J = 0$  need not be EB as shown by the following example for  $d = 3$ :

$$\Phi = \frac{2}{3} \Psi_1^X + \frac{1}{3} \Psi_2^X = \left[\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}\right] \quad (21)$$

Since  $\sum_k |\lambda_k| = \frac{3}{2} > 1$ , theorem 4 implies that this channel is not EB.

When all  $\lambda_k \leq 0$ , inequality (13b) implies that  $\sum_k |\lambda_k| \leq \frac{1}{d-1}$  and hence that  $\Phi$  is in the convex hull of  $\mathcal{N}$  and the set  $\{\Psi_J^{\text{XEB}}\}$ . Thus every channel with all  $\lambda_k \leq 0$  is EB. What remains is to find precise necessary and sufficient conditions for a channel with both positive and negative multipliers  $\lambda_L$  to be EB. In figure 3, in the following section, there are channels with  $\sum_J |\lambda_J| \leq 1$  which lie outside the PPT region; thus we see the condition from theorem 4 is not sufficient for EB.

Channels with exactly one symmetry axis (i.e., those for which  $d$  of the  $\lambda_J$  are equal) are studied in section 4, in which we show that  $\sum_L |\lambda_L| \leq 1$  is necessary and sufficient for  $\Phi$  to be EB. When  $d = 3$ , this implies that the channel  $\Psi_L^{\text{YEB}} = \left[\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right]$  is EB. This channel is outside the convex hull of  $\Psi_L^{\text{QC}}$  and  $\Psi_L^{\text{XEB}}$  because  $\sum_{\lambda_J < 0} \lambda_J = -\frac{3}{4} < -\frac{1}{2} = -\frac{1}{d-1}$ .

It is worth summarizing what is known about the EB subset of channels constant on axes.

- (a) If  $\Phi$  is EB, then  $\sum_J |\lambda_J| \leq 1$ .
- (b) If all  $\lambda_J \geq 0$ , and  $\sum_J |\lambda_J| \leq 1$ , then  $\Phi$  is EB.
- (c) If all  $\lambda_J \geq 0$  and some  $\lambda_J = 0$ , then  $\Phi$  is EB.
- (d) If  $\Phi$  has one symmetry axis, then  $\Phi$  is EB if and only if  $\sum_J |\lambda_J| \leq 1$ .
- (e) There are channels which satisfy  $\sum_J |\lambda_J| \leq 1$ , but are *not* EB.
- (f) If all  $\lambda_J \leq 0$ , then  $\Phi$  is EB.

We can also use the  $\lambda_J$ 's to state a necessary condition for an axis channel to be PPT.

**Theorem 5.** If a channel  $\Phi$  constant on axes satisfies the PPT condition  $(\mathcal{I} \otimes \Phi)(|\beta\rangle\langle\beta|) \geq 0$ , then  $\sum_J \lambda_J \leq 1$ .

**Proof.** First observe that for any QC channel the antisymmetric subspace is in the kernel of both  $(\mathcal{I} \otimes \Phi^{\text{QC}})(|\beta\rangle\langle\beta|)$  and  $(\mathcal{T} \otimes \Phi^{\text{QC}})(|\beta\rangle\langle\beta|)$ . To see this consider  $|v_{12}\rangle = |f \otimes g\rangle - |g \otimes f\rangle$  and write  $|f\rangle = \sum_j x_j |j\rangle$ ,  $|g\rangle = \sum_j y_j |j\rangle$  in the basis corresponding to  $\Phi^{\text{QC}}$ . In this basis

$$(\mathcal{I} \otimes \Phi^{\text{QC}})(|\beta\rangle\langle\beta|) = (\mathcal{T} \otimes \Phi^{\text{QC}})(|\beta\rangle\langle\beta|) = \sum_{kk} |kk\rangle\langle kk|,$$

and  $|v_{12}\rangle = \sum_{jk} (x_j y_k - y_j x_k) |jk\rangle$  so that

$$(\mathcal{I} \otimes \Phi^{\text{QC}})(|\beta\rangle\langle\beta|)|v_{12}\rangle = (\mathcal{T} \otimes \Phi^{\text{QC}})(|\beta\rangle\langle\beta|)|v_{12}\rangle = \frac{1}{d} \sum_{kk} |kk\rangle (x_k y_k - y_k x_k) = 0.$$

One similarly finds that the antisymmetric subspace is an eigenspace of  $(\mathcal{T} \otimes \mathcal{I})(|\beta\rangle\langle\beta|)$  with eigenvalue  $-1$ :

$$\begin{aligned} (\mathcal{T} \otimes \mathcal{I})(|\beta\rangle\langle\beta|)|v_{12}\rangle &= \frac{1}{d} \sum_{jk} |kj\rangle\langle jk| \sum_{mn} (x_m y_n - y_n x_m) |mn\rangle \\ &= \frac{1}{d} \sum_{jk} (x_j y_k - y_k x_j) |kj\rangle = -|v_{12}\rangle. \end{aligned}$$

Thus, if  $\Phi$  has the form (3) and satisfies the PPT condition, then choosing  $|v_{12}\rangle$  antisymmetric gives

$$0 \leq \langle v_{12} | (\mathcal{T} \otimes \Phi)(|\beta\rangle\langle\beta|) | v_{12} \rangle = -s, \quad (22)$$

which implies  $s \leq 0$ . Then (4) implies  $\sum_J \lambda_J = ds + 1 \leq 1$ .  $\square$

### 3.3. EB and bound entanglement when $d = 3$

We now consider some implications of the PPT and CCN conditions in more detail when  $d = 3$ . Some of the results about bound entanglement were obtained independently in [5] by a different method.

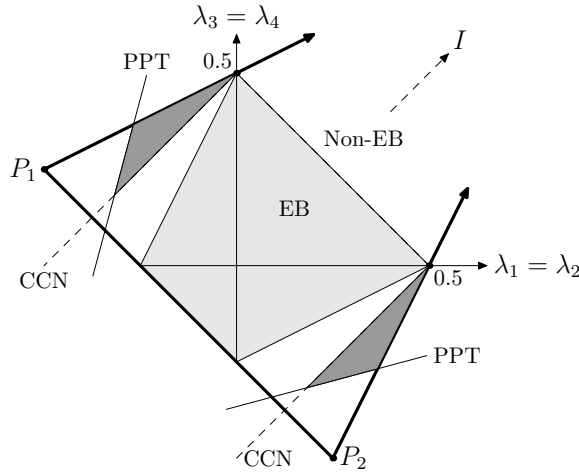
For  $d = 3$ , it is shown in appendix C.3 that the maps  $\Psi_J^{\text{XEB}}$  and  $\Psi_J^{\text{YEB}}$  are extreme points of the convex subset of EB channels. These points lie in the ‘base’ tetrahedron shown in figure 2, which also shows the tetrahedron whose vertices  $\Psi_J^{\text{XEB}}$  are the four centres of the faces. Reflecting this small tetrahedron through its centre gives the convex hull of the four  $\Psi_J^{\text{YEB}}$ . The convex hull of the eight points  $\Psi_J^{\text{XEB}}$  and  $\Psi_J^{\text{YEB}}$  is a subset of the EB channels and is inscribed in the sphere  $\sum_J |\lambda_J|^2 = \frac{1}{4}$ , which is precisely the set of PPT maps in the base tetrahedron. We conjecture that all maps in this sphere are EB; this is supported by the numerical work of Audenaert [3].

The observation about the sphere is a special case of the following theorem which is proved in appendix C.3.

**Theorem 6.** *When  $d = 3$ , a channel constant on axes is PPT if and only if it satisfies both  $\sum_J \lambda_J \leq 1$  and*

$$3 \sum_J \lambda_J^2 \leq 1 + \sum_J \lambda_J + \left( \sum_J \lambda_J \right)^2. \quad (23)$$

We can use theorem 6 to find examples of channels which are PPT but not CCN. Such channels are of some interest because they correspond to bound entangled states. We first consider  $|\lambda_J| = x$  for all  $J$ . The case all  $\lambda_J = x > 0$  is covered by theorem 4 and the case all  $\lambda_J = -x < 0$  has  $x \leq \frac{1}{8}$  and is both PPT and CCN. Permutations of  $[+x, -x, -x, -x]$



**Figure 3.** Qutrit channels with multiplier  $[\lambda_1, \lambda_1, \lambda_3, \lambda_3]$ , which is the triangle  $IP_1P_2$ .  $P_1$  and  $P_2$  correspond to the midpoints of two disjoint edges in the base tetrahedron of figure 2. Maps in the lightly shaded region are known to be EB; those in the dark regions correspond to bound entangled states.

have one symmetry axis; it is shown in section 4.2 that for these channels the PPT and CCN regions always coincide. The only remaining possibility is permutations of  $[+x, +x, -x, -x]$  for which  $\sum_j \lambda_j = 0$  and the CP condition (12) holds if and only if  $x \leq \frac{1}{3}$ . In this case, (23) becomes  $12x^2 \leq 1$ . Thus, we can conclude that channels with multiplier  $[+x, +x, -x, -x]$  are CP and bound entangled for  $\frac{1}{2\sqrt{3}} < x \leq \frac{1}{3}$ .

We now consider channels whose multipliers are permutations of  $[x, x, -y, -y]$  with  $x, y \geq 0$ . Let  $S = \sum_j \lambda_j = 2(x - y)$  and  $T = \sum_j |\lambda_j| = 2(x + y)$ . Then  $x = \frac{1}{4}(T + S)$  and  $y = \frac{1}{4}(T - S)$ . The CP conditions (12) become  $S \geq -\frac{1}{2}$  and  $2x + y = \frac{1}{4}(3T + S) \leq 1$ . The PPT condition (23) becomes

$$1 + S + S^2 \leq 6(x^2 + y^2) \leq \frac{3}{4}(S^2 + T^2)$$

or, equivalently,

$$3T^2 \leq (2 + S)^2, \quad (24)$$

which is stronger than the CCN condition  $T \leq 1$  when  $S + 2 \leq \sqrt{3}$ . Thus, we can conclude that channels of the form  $[x, x, -y, -y]$  give bound entangled states in the region

$$1 < T < \min \left\{ \frac{2 + S}{\sqrt{3}}, \frac{4 - S}{3} \right\}, \quad (25)$$

with  $\sqrt{3} - 2 < S < 1$ . In terms of  $x, y$ , this is the triangle bounded by the lines

$$2x + y = 1, \quad x + y = \frac{1}{2}, \quad \text{and} \quad y = \frac{\sqrt{3}-1}{2} + x(\sqrt{3}-2) \quad (26)$$

as shown in figure 3 with  $\lambda_1 = x, \lambda_3 = -y$ . If we drop the restriction that  $\lambda_1 = \lambda_2$  and  $\lambda_3 = \lambda_4$ , one can find additional channels with bound entangled states for any value of  $S \in (-\frac{1}{2}, 1)$ .

## 4. One symmetry axis

### 4.1. General considerations

Channels of the form (A.16) with exactly one  $t_L$  non-zero have been considered by Fukuda and Holevo [13] who wrote them in the form

**Table 2.** Extreme points of CPT maps  $b\mathcal{I} + a\Psi_1^{\text{QC}} + (1-a-b)\mathcal{N}$ .

	$(a, b)$			$d = 2$	$d = 3$
A	$(0, 1)$	$\mathcal{I}$	$[1, 1, \dots, 1]$	$[1, 1, 1]$	$[1, 1, 1, 1]$
B	$(\frac{d}{d-1}, \frac{-1}{d-1})$	$\Psi_L^{\text{X}}$	$[1, \frac{-1}{d-1}, \dots, \frac{-1}{d-1}]$	$[1, -1, -1]$	$[1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]$
E	$(\frac{-1}{d-1}, 0)$	$\Psi_L^{\text{XEB}}$	$[\frac{-1}{d-1}, 0, \dots, 0]$	$[-1, 0, 0]$	$[-\frac{1}{2}, 0, 0, 0]$

$$\Phi(a, b) = b\mathcal{I} + a\Psi^{\text{QC}} + (1-a-b)\mathcal{N}. \quad (27)$$

When  $d+1$  MUB exist, assume they are labelled so that  $\Psi^{\text{QC}}$  corresponds to  $J = 1$  so that  $\Phi$  has multiplier  $[a+b, b, \dots, b]$ . Even when a full set of MUB does not exist, (27) is a well-defined channel with the QC map in the standard basis  $|e_j\rangle$ . Moreover, we can still associate  $\Phi(a, b)$  with a multiplier in the generalized Pauli basis, as described after (A.4). Then  $\phi_s = a+b$  when  $V_s = Z^j$  for some  $j$  and  $\phi_s = b$  otherwise. These channels have exactly one symmetry axis, i.e., they satisfy the covariance condition  $\Phi(U\rho U^\dagger) = U\Phi(\rho)U^\dagger$  when  $U = \sum_j e^{i\phi_j} |e_j\rangle\langle e_j|$ .

As observed in [13] these maps are CPT when  $(a, b)$  is in the convex hull of the points  $(1, 0)$ ,  $(\frac{d}{d-1}, \frac{-1}{d-1})$ ,  $(\frac{-1}{d-1}, 0)$  as summarized in table 2. The CJ matrix for maps of the form (27) can be written as

$$\Gamma = \frac{1}{d} \sum_{jk} |e_j\rangle\langle e_k| \otimes \begin{cases} \sum_n [(a+b)\delta_{kn} + \frac{1}{d}(1-a-b)] |e_n\rangle\langle e_n| & j = k \\ b |e_j\rangle\langle e_k| & j \neq k \end{cases} \quad (28)$$

$$= \frac{1}{d^2} \left[ (1-a-b)I + bd^2 |\beta\rangle\langle\beta| + \sum_k ad |e_k \otimes e_k\rangle\langle e_k \otimes e_k| \right], \quad (29)$$

with  $|\beta\rangle = \frac{1}{\sqrt{d}} \sum_k |e_k \otimes e_k\rangle$  maximally entangled.

#### 4.2. EB channels

To find the subset of EB maps, observe that the PPT condition applied to (28) is

$$\begin{pmatrix} \frac{1}{d}(1-a-b) & b \\ b & \frac{1}{d}(1-a-b) \end{pmatrix} \geq 0$$

or, equivalently,  $\frac{1}{d}(1-a-b) \geq |b|$ , which can be written as

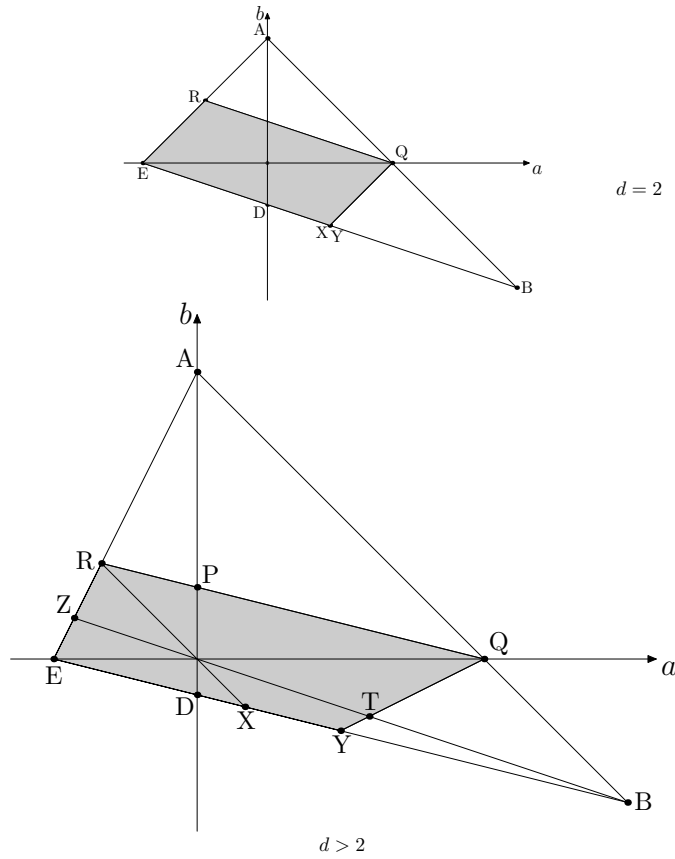
$$a + (d+1)b \leq 1 \quad b > 0 \quad (30a)$$

$$a - (d-1)b \leq 1 \quad b < 0. \quad (30b)$$

When  $a+b \geq 0$ , (30) is equivalent to the CCN condition  $\sum_L |\lambda_L| \leq 1$ . This implies that for channels of the form (28) the PPT and CCN boundaries coincide.

When  $d+1$  MUB exist, one can write the maps with  $a = -b = -\frac{1}{d}$  or  $\frac{1}{d(d-1)}$  as  $\sum_{J \neq L} \frac{1}{d} \Psi_J^{\text{QC}}$  and  $\sum_{J \neq L} \frac{1}{d} \Psi_J^{\text{XEB}}$ , respectively, which implies that they are EB. Showing that they are EB for arbitrary  $d$  is harder<sup>3</sup>. It is natural to conjecture that these are also extreme

<sup>3</sup> This problem was mentioned at a talk in Torun in June, 2006. Shortly after this talk, Myrheim [30] and Horodecki [20] independently gave explicit constructions for separability of the CJ matrix for  $b = \frac{1}{d}$ . However, the separability of the CJ matrix for  $b = -\frac{1}{d(d-1)}$  was settled only by observing that  $X$  is on the line  $EY$  in figure 4 and  $Y$  is separable.



See Tables 2, 3 and 4 for coordinates of marked points.

$$\begin{aligned}
 AB: a + b &= 1 & AE: a(d-1) - b &= -1 & BE: a + b(d+1) &= -\frac{1}{d-1} \\
 RQ: a + b(d+1) &= 1 & YQ: a - (d-1)b &= 1 \\
 RX: a + b &= 0 & ZB: a + db &= 0
 \end{aligned}$$

**Figure 4.** Maps of the form (27) which are CPT are in the convex hull of ABE. Maps in the convex hull of ERQY are EB. For  $d = 2$ , the points  $X$  and  $Y$  coincide.

points of the convex subset of EB maps of this type, in which case the convex hull of ERQX in figure 4 would give the EB channels. This is false, however. The next result says that all channels with one symmetry axis which satisfy the PPT condition (or, equivalently, the CCN condition) are EB; this corresponds to the convex hull of ERQY as shown in figure 4.

**Theorem 7.** A channel of the form (27) is EB if and only if it satisfies (30).

**Proof.** Since the set of EB channels is convex, it suffices to show that each of the channels corresponding to the points E, R, Q, Y in table 3 and figure 4 is EB. The point Q corresponds to  $\Phi^{\text{QC}}$  which is EB and E has a separable CJ matrix because  $b = 0$ . Decompositions showing that the CJ matrices for R and Y are separable and given in appendix C.1.  $\square$

It is well known that the depolarizing channel,  $\Psi_\lambda^{\text{dep}}$ , is EB for  $\lambda \leq \frac{1}{d+1}$ , which is consistent with theorem 4. If one ‘depolarizes’ from an extreme point other than the identity, the resulting channel  $\Phi = \lambda \Psi_L^X + (1 - \lambda)\mathcal{N}$  has one symmetry axis. We can then use

**Table 3.** Extreme points for subset of EB channels.

	$(a, b)$			$d = 2$	$d = 3$
$Q$	$(1, 0)$	$\Psi_L^{\text{QC}}$	$[1, 0, \dots, 0]$	$[1, 0, 0]$	$[1, 0, 0, 0]$
$E$	$\left(\frac{-1}{d-1}, 0\right)$	$\Psi_L^{\text{XEB}}$	$\left[\frac{-1}{d-1}, 0, \dots, 0\right]$	$[-1, 0, 0]$	$\left[\frac{-1}{2}, 0, 0, 0\right]$
$R$	$\left(\frac{-1}{d}, \frac{1}{d}\right)$	$\sum_{K \neq L} \frac{1}{d} \Psi_L^{\text{QC}}$	$\left[0, \frac{1}{d}, \dots, \frac{1}{d}\right]$	$\left[0, \frac{1}{2}, \frac{1}{2}\right]$	$\left[0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$
$Y$	$\left(\frac{1}{2}, -\frac{1}{2(d-1)}\right)$	$\Psi_L^{\text{YEB}}$	$\left[\frac{d-2}{2(d-1)}, \frac{-1}{2(d-1)}, \dots, \frac{-1}{2(d-1)}\right]$	$\left[0, -\frac{1}{2}, -\frac{1}{2}\right]$	$\left[\frac{1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}\right]$

**Table 4.** Other interesting points.

	$(a, b)$			$d = 2$	$d = 3$
$X$	$\left(\frac{1}{d(d-1)}, \frac{-1}{d(d-1)}\right)$	$\sum_{K \neq L} \frac{1}{d} \Psi_L^{\text{XEB}}$	$\left[0, \frac{-1}{d(d-1)}, \dots, \frac{-1}{d(d-1)}\right]$	$\left[0, -\frac{1}{2}, -\frac{1}{2}\right]$	$\left[0, \frac{-1}{6}, \frac{-1}{6}, \frac{-1}{6}\right]$
$\mathcal{N}$	$(0, 0)$		$[0, 0, \dots, 0]$	$[0, 0, 0]$	$[0, 0, 0, 0]$
$P$	$\left(0, \frac{1}{d+1}\right)$	$\Psi_{\frac{1}{d+1}}^{\text{dep}}$		$\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$	$\left[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right]$
$D$	$\left(0, \frac{-1}{d^2-1}\right)$	$\Psi_{\frac{-1}{d^2-1}}^{\text{dep}}$		$\left[-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right]$	$\left[-\frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}\right]$
$T$	$\left(\frac{d}{2d-1}, \frac{-1}{2d-1}\right)$			$\left[\frac{1}{3}, \frac{-1}{3}, \frac{-1}{3}\right]$	$\left[\frac{2}{5}, \frac{-1}{5}, \frac{-1}{5}, \frac{-1}{5}\right]$
$Z$	$\left(\frac{-d}{d^2-d+1}, \frac{1}{d^2-d+1}\right)$			$\left[\frac{-1}{3}, \frac{1}{3}, \frac{1}{3}\right]$	$\left[\frac{-2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right]$

theorem 7 to conclude that the channel is EB when  $\lambda \leq \frac{d-1}{2d-1}$ , for which the limiting case has multiplier  $\left[\frac{d-1}{2d-1}, \frac{-1}{2d-1}, \dots, \frac{-1}{2d-1}\right]$ . Note that the CP range is  $\frac{-d+1}{d^2-d+1} \leq \lambda \leq 1$  which has multiplier  $\left[\frac{-d+1}{d^2-d+1}, \frac{1}{d^2-d+1}, \dots, \frac{1}{d^2-d+1}\right]$  at the boundary. For  $d = 3$ , the EB portion of the line segment  $\lambda \Psi_1^X + (1 - \lambda)\mathcal{N}$  is bounded by the channels with multipliers  $\left[\frac{2}{5}, \frac{-1}{5}, \frac{-1}{5}, \frac{-1}{5}\right]$  and  $\left[\frac{-2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right]$ .

#### 4.3. Multiplicativity

Although this topic is more fully studied in the following section, where complete definitions are given, it is worth making some observations here. We use ‘multiplicative’ to mean that (35) holds with  $\Omega$  arbitrary. One can apply Fukuda’s lemma [11] to show that

$$\Phi = \Psi_y^{\text{dep}} \circ \Psi_{L,x}^{\text{PD}} = xy\mathcal{I} + (1-x)y\Psi_L^{\text{QC}} + (1-y)\mathcal{N} \quad (31)$$

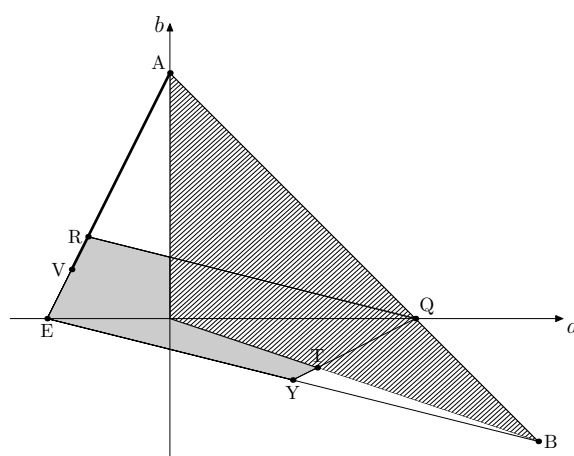
is multiplicative for  $\frac{-1}{d-1} \leq x \leq 1$  and  $\frac{-1}{d^2-1} \leq y \leq 1$ . The relations  $b = xy$  and  $a = (1-x)y$  imply that

$$\frac{-1}{d^2-1} \leq a+b \leq 1 \quad \text{and} \quad \frac{-1}{d-1} \leq \frac{b}{a+b} \leq 1. \quad (32)$$

This gives the following result.

**Theorem 8.** A map of the form (27) is multiplicative if either of the following sets of conditions hold:

- (i)  $a > 0$  and  $a + bd \geq 0$ , or
- (ii)  $a < 0$  and  $-b - \frac{1}{d^2-1} \leq a \leq -bd$ .



The second set of conditions (ii) corresponds to a very small region entirely contained within the set of EB channels.

Proving multiplicativity for the triangle YTB presents a different challenge.

### 5.1. General considerations

$$S_{\min}(\Phi \otimes \Omega) = S_{\min}(\Phi) + S_{\min}(\Omega). \quad (33)$$

Another measure of the optimal output purity of a channel is the maximal output  $p$ -norm defined as

$$v_p(\Phi) = \sup_{\gamma} \|\Phi(\gamma)\|_p = \sup_{\gamma} (\text{Tr}[\Phi(\gamma)]^p)^{1/p}. \quad (34)$$

$$\nu_p(\Phi \otimes \Omega) = \nu_p(\Phi) \nu_p(\Omega) \quad (35)$$

at least for  $1 \leq p \leq 2$ . Moreover, it was shown in [2] that if (35) holds for  $p \in (1, 1 + \epsilon)$  for some  $\epsilon > 0$ , then (33) holds.

Recently, Fukuda [12] showed that if (33) or (35) holds for all unital channels, then it holds for arbitrary  $\Phi$ . Our original motivation for studying channels constant on axes was to find a class to which one could extend King's proof [22] of these conjectures in the case of unital qubit channels. Instead, we have merely gained additional insight into the reasons his argument does not work when  $d > 2$ . We can however prove multiplicativity for channels constant on axes in the important case  $p = 2$ , as is shown in section 5.4.

We conjecture that for channels constant on axes, the maximal output  $p$ -norm and minimal output entropy are both achieved with an axis state.

**Conjecture 9.** *Let  $\Phi$  be a channel of the form (3). Then the maximal output  $p$ -norm and minimal output entropy can be achieved with an axis state, i.e., for each  $p$  one can find  $L$  (which may depend upon  $p$ ) such that  $v_p(\Phi) = \|\Phi(|\psi_n^L\rangle\langle\psi_n^L|)\|_p = \|\Psi_{\lambda_L}^{\text{dep}}\|_p$ , and  $S_{\min}(\Phi) = S(\Phi(|\psi_n^L\rangle\langle\psi_n^L|)) = S_{\min}(\Psi_{\lambda_L}^{\text{dep}})$  for some  $L$ .*

When some of the  $\lambda_J < 0$ , the axis  $L$  can depend upon  $p$  and our evidence for this conjecture is only numerical, as described in the following section. When all  $\lambda_J \geq 0$ , the analogy with unital qubit channels suggests that  $L$  satisfies  $\sup_J \lambda_J = \lambda_L$ , and it is easy to see that if  $\lambda_J < \lambda_L$  then  $\|\Phi(|\psi_J^n\rangle\langle\psi_J^n|)\|_p \leq \|\Phi(|\psi_L^n\rangle\langle\psi_L^n|)\|_p$  for all  $p$ . However, we have not been able to exclude the possibility that  $v_p(\Phi)$  is attained on a superposition of axis states.

Conjecture 9 is known for certain classes of axis channels, such as QC channels and depolarizing channels, and it is shown to hold for all axis channels in the special cases  $p = 2$  and  $p = \infty$ , as discussed in appendix B.

The following result, which is a special case of lemma 28 proved in appendix B, is consistent with this conjecture.

**Theorem 10.** *Let  $\Phi$  be a channel constant on axes and  $\gamma(t)$  a differentiable one-parameter family of pure states with  $\gamma(0) = |\psi_n^J\rangle\langle\psi_n^J|$  an axis state. Then for all  $p$ ,  $\|\Phi[\gamma(t)]\|_p$  has a critical point at  $t = 0$ . Moreover,  $S(\Phi[\gamma(t)])$  also has a critical point at  $t = 0$ . If  $\Phi_1$  and  $\Phi_2$  are both constant on axes, this extends to the channel  $\Phi_1 \otimes \Phi_2$  with input  $\gamma_{12}(t)$ , where  $\gamma_{12}(0) = |\psi_n^J \otimes \psi_m^K\rangle\langle\psi_n^J \otimes \psi_m^K|$ .*

Whenever  $v_p(\Phi)$  is achieved with an axis state, one can use Fukuda's lemma [11] to show that if (35) holds for  $\Phi$ , it can be extended to  $\Phi \circ M_J$ , where  $M_J$  is a convex combination of conjugations on a single axis as in (A.20) (see appendix A.6). The key point is that  $M_J(|\psi_n^J\rangle\langle\psi_n^J|) = |\psi_n^J\rangle\langle\psi_n^J|$ .

**Theorem 11.** *Let  $\Phi$  be a channel constant on axes and let  $M_J$  be given by (A.20). If  $v_p(\Phi) = \|\Phi(|\psi_n^J\rangle\langle\psi_n^J|)\|_p$  and (35) holds for  $v_p(\Phi \otimes \Omega)$ , then it also holds for  $\Phi \circ M_J$ .*

## 5.2. Numerical study of new behaviour

For a channel constant on axes, the output of any axis state is

$$\Phi(|\psi_n^L\rangle\langle\psi_n^L|) = (1 - \lambda_L) \frac{1}{d} \mathbf{I} + \lambda_L |\psi_n^L\rangle\langle\psi_n^L|, \quad (36)$$

which has eigenvalues  $\frac{1}{d}[1 + (d-1)\lambda_L]$  and  $\frac{1}{d}(1 - \lambda_L)$  with degeneracy  $d-1$ . This implies  $v_p(\Phi) \geq v_p(\Psi_{\lambda_L}^{\text{dep}})$  when  $\lambda_L \geq -\frac{1}{d^2-1}$ . When all  $\lambda_K \geq 0$ , we conjecture that this is optimal, i.e.,  $v_p$  is achieved with an axis state corresponding to the largest  $\lambda_K$  and provide some evidence



in this direction. We also show that channels with negative multipliers can have fundamentally different behaviour.

In particular, it can happen that  $0 < -\lambda_{L_1} = |\lambda_{L_1}| < \lambda_{L_2}$  but  $\|\Phi(|\psi_n^{L_1}\rangle\langle\psi_n^{L_1}|)\|_p > \|\Phi(|\psi_n^{L_2}\rangle\langle\psi_n^{L_2}|)\|_p$ . For example, consider the channel

$$\Phi = a\Psi_1^X + (1-a)\Psi_2^X \text{ with multiplier } \left[ \frac{3a-1}{2}, \frac{2-3a}{2}, -\frac{1}{2}, -\frac{1}{2} \right]. \quad (37)$$

For  $a = \frac{2}{3}$ , this becomes  $[+0.5, 0, -0.5, -0.5]$  which has larger output  $p$ -norms when  $1 < p < 2$  for inputs along the  $-0.5$  axes than for those along the  $+0.5$  axis. This behaviour persists for  $[0.6, -0.1, -0.5, -0.5]$  when  $1 < p < 1.2$ . Moreover, for  $0 < \lambda_1 < 0.65$ , the minimal output entropy of the channel with multiplier  $[\lambda_1, 0.5 - \lambda_1, -0.5, -0.5]$  is  $S_{\min}(\Phi) = S[\Phi(|\psi_n^3\rangle\langle\psi_n^3|)] = 1$ , but inputs on the ‘long’ axis have  $S[\Phi(|\psi_n^1\rangle\langle\psi_n^1|)] > 1$ .

Numerical studies of the minimal output entropy of channels with multiplier  $[\lambda_1, 0.5 - \lambda_1, -0.5, -0.5]$  have been carried out for  $\lambda_1$  near the crossing point  $\lambda_1 = 0.659$ . For a single use of the channel,  $S_{\min}(\Phi)$  is always achieved with an axis state and satisfies  $S_{\min}(\Phi) = 1$  for  $\lambda_1 < 0.659$ . For the product, one finds  $S_{\min}(\Phi \otimes \Phi) = 2$  is always achieved with a product of axis states. Moreover, near the crossing the maximally entangled state  $|\beta\rangle = \frac{1}{\sqrt{d}} \sum_n |\psi_n^1\rangle \otimes |\psi_n^3\rangle$  has entropy  $S[(\Phi \otimes \Phi)(|\beta\rangle\langle\beta|)] = 2.74041$ .

### 5.3. Non-negative multipliers

King’s approach to the unital qubit channels is to reduce the problem to multiplicativity of ‘two-Pauli’ channels  $\Psi_{L,x}^{\text{MxSq}}$  by considering channels of the form  $[\lambda_1, \lambda_2, x]$  with  $|\lambda_j| \leq x$ . This subclass of channels has extreme points with multipliers

$$\begin{array}{lll} [x, x, x], & [-x, -x, x], & [2x - 1, x, x], \\ [x, 2x - 1, x] & [1 - 2x, -x, x] & [-x, 1 - 2x, x] \end{array} \quad (38)$$

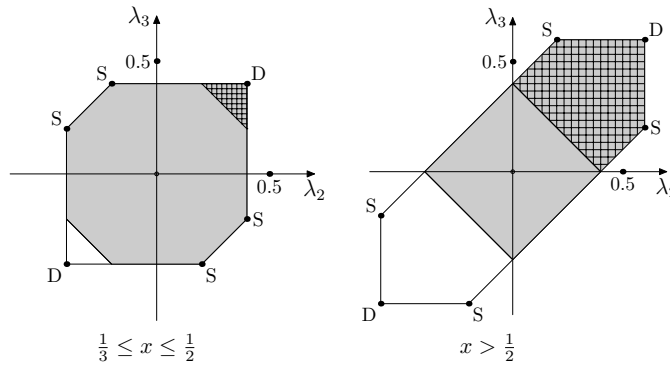
for  $x > \frac{1}{3}$ . Here,  $\Psi_{J,x}^{\text{MxSq}}$  has multiplier  $[2x - 1, x, x]$ . If one can show that  $\nu_p(\Psi_{J,x}^{\text{MxSq}}) = \nu_p(\Psi_x^{\text{dep}})$  and that  $\nu_p(\Psi_{J,x}^{\text{MxSq}} \otimes \Omega)$  satisfies (35), then multiplicativity follows from lemma 25, first using  $B = \nu_p(\Psi_x^{\text{dep}})$  and then using  $B = \nu_p(\Psi_x^{\text{dep}})\nu_p(\Omega)$ . King’s argument exploits the fact that changing  $\lambda_j \rightarrow -\lambda_j$  for  $j = 1, 2$  is equivalent to a unitary conjugation with  $\sigma_z$ . This property does not extend to channels constant on axes. However, we can make an analogous reduction on the subset of channels with non-negative multipliers under the assumption that conjecture 9 holds for these channels.

For qubits, the subset of channels with multiplier  $[\lambda_1, \lambda_2, x]$  with  $0 \leq \lambda_j \leq x$  has extreme points

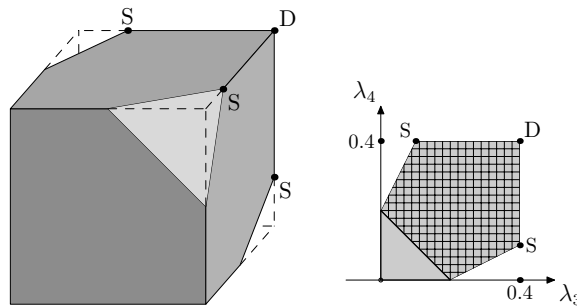
$$\begin{array}{llll} [0, 0, x], & [0, x, x], & [x, 0, x], & [x, x, x] \quad x \leq \frac{1}{2} \\ [0, 0, x], & [0, 1 - x, x], & [1 - x, 0, x], & [2x - 1, x, x], \quad [x, 2x - 1, x], \quad [x, x, x] \quad x > \frac{1}{2} \end{array}$$

as shown in figure 6. In both cases, the first three channels are EB and the last the depolarizing channel. The difference between the two situations is that the latter includes channels of the form  $\Psi_{J,x}^{\text{MxSq}}$  but the former does not. For  $d > 2$ , the convex set of channels which are not EB and satisfy  $0 \leq \lambda_j \leq \lambda_{d+1} = x$  has analogous extreme points as shown in figure 7. We observe here only that any channel  $\Phi_x$  in this set is a convex combination of

- (a)  $\Psi_x^{\text{EB}}$  and  $\Psi_x^{\text{dep}}$  when  $0 < x \leq \frac{1}{d}$ , and
- (b)  $\Psi_x^{\text{EB}}$ ,  $\Psi_x^{\text{dep}}$  and  $\{\Psi_{J,x}^{\text{MxSq}} : J = 1, 2, \dots, d\}$  when  $\frac{1}{d} < x \leq 1$ ,



**Figure 6.** Qubit channels with  $|\lambda_j| \leq x = \lambda_1$ , with grey shading indicating the EB region and cross-hatch the subset of the non-EB channels with all  $\lambda_j \geq 0$ . The evident symmetry for  $\lambda_j \mapsto -\lambda_j$  is lost for  $d > 2$  but the picture for  $\lambda_j \geq 0$  is similar, as shown in figure 7.



**Figure 7.** Qutrit channels with  $0 \leq \lambda_j \leq \lambda_1 = 0.4$ . The 3-dimensional view is a cube with three corners removed. The vertices of the front corner are the squashed channel  $S = [0.4, 0.4, 0.4, 0.1]$  along with  $[0.4, 0.4, 0.2, 0]$  and  $[0.4, 0.2, 0.4, 0]$ . The two-dimensional view shows the face of the cube with  $\lambda_2 = 0.4$ .

where  $\Psi_x^{\text{EB}}$  denotes some EB channel whose multiplier satisfies  $0 \leq \lambda_J \leq x$ . The extreme EB channels of this type will be permutations of  $[0, \dots, 0, 1 - \kappa x, x, \dots, x]$  with  $\kappa$  chosen so that  $0 < 1 - \kappa x \leq x$ .

The following consequence of lemma 25 shows that if conjecture 9 holds for EB channels with non-negative multipliers, then we can reduce the general situation to the maximally squashed channels.

**Theorem 12.** Let  $\Phi$  be a channel of the form (11) with  $\lambda_J \geq 0$  for all  $J$ . Choose  $L^*$  so that  $\sup_J \lambda_J = \lambda_{L^*}^*$ . Then

(a) if conjecture 9 holds for  $\Phi_{J, \lambda_{L^*}^*}^{\text{MxSq}}$  and any EB channel with  $0 \leq \lambda_J \leq \lambda_{L^*}^*$ , then

$$\nu_p(\Phi) = \|\Phi(|\psi_n^{L^*}\rangle\langle\psi_n^{L^*}|)\|_p = \nu_p(\Psi_{\lambda_{L^*}^*}^{\text{dep}}); \quad (39)$$

(b) if  $p \geq 1$ , the hypotheses in (a) above hold, and (35) holds for  $\nu_p(\Phi_{J, \lambda_{L^*}^*}^{\text{MxSq}} \otimes \Omega)$ , then (35) holds for  $\nu_p(\Phi \otimes \Omega)$ .

**Proof.** A channel satisfying the hypothesis above is a convex combination of  $\Psi_x^{\text{dep}}$  and EB channels with multipliers that are permutations of  $[0, \dots, 0, 1 - \kappa x, x, \dots, x]$ . By the remark in section 4.3 such EB channels have the same maximal output  $p$ -norm as  $\nu_p(\Psi_x^{\text{dep}})$ . Then

part (a) follows from lemma 25 with  $B = \nu_p(\Psi_x^{\text{dep}})$  and the fact that the bound can be attained with an axis state. Both depolarizing channels [24] and EB channels [25] are multiplicative for all  $p$ . Therefore, part (b) also follows from the lemma 25 with  $B = \nu_p(\Psi_x^{\text{dep}})\nu_p(\Omega)$ .  $\square$

In the case of qubits, conjecture 9 is known to hold, so theorem 12 gives a new proof of multiplicativity for channels whose multiplier  $[\lambda_1, \lambda_2, \lambda_3]$  satisfies  $0 \leq \lambda_j \leq \frac{1}{2}$  and  $\sum_j \lambda_j > 1$ . One can then conjugate with  $\sigma_j$  and combine with known results about EB channels to prove that (35) holds for any unital qubit channel with all  $|\lambda_j| \leq \frac{1}{2}$ . This last step does not extend to  $d > 2$ . If the maximal  $\lambda_j$  is greater than  $\frac{1}{2}$  for a qubit channel, this gives a new reduction of multiplicativity to maximally squashed channels. Both cases for qubits can be seen in figure 6.

Recall  $\Phi_{L,x}^{\text{MxSq}} = \sum_{K \neq L} \Psi_{L,\zeta}^{\text{PD}}$  with  $\zeta$  as in (15) and that intuitively one would expect that  $\nu_p(\Phi_{L,x}^{\text{MxSq}})$  is achieved with a state which is 0 on the ‘short’ axis  $L$ , i.e., for which  $u_{Lj} = 0$  in (A.15). For such states, King’s proof [24] of the multiplicativity of the depolarizing channel carries over. However, the channel  $\Phi_{L,x}^{\text{MxSq}}$  has symmetry around the axis  $L$ . This does not allow one to assume that  $w_{Lj} = 0$ . To overcome this problem, King [22] uses the convex decomposition  $\Phi_{3,x}^{\text{MxSq}} = \frac{3x-1}{2x} \Psi_{2,x}^{\text{PD}} + \frac{1-x}{2x} \sigma_1 \Psi_{2,x}^{\text{MxSq}} \sigma_1$  or, equivalently,

$$[x, x, 2x-1] = \frac{3x-1}{2x} [x, 1, x] + \frac{1-x}{2x} [x, 1-2x, -x] \quad (40)$$

after making a rotation so that  $\rho = \frac{1}{2}(I + w_1\sigma + w_3\sigma_3)$ , i.e.,  $w_2 = 0$ . Then both channels on the right have  $\nu_p(\Psi) = \nu_p(\Psi_x^{\text{dep}})$ . However, the channel  $[x, 1-2x, -x]$  with negative multipliers has no direct analogue in the  $d > 2$  case. Therefore, King’s argument does not generalize. Amosov [1] has given a new proof of additivity (33) for unital qubit channels. Because his argument is based on King’s decomposition (40), it does not readily generalize to  $d > 2$ .

Proving (35) for channels with positive multipliers seems to require a new approach to the multiplicativity for channels  $\Phi_{L,x}^{\text{MxSq}}$ . However, we have shown that if conjecture 9 holds, then multiplicativity for channels with positive multipliers can be reduced to the maximally squashed case. Channels with some negative multipliers present a different challenge.

#### 5.4. Results for $p = 2$

In the case  $p = 2$ , we can prove more, including multiplicativity for all channels constant on axes. The results of this section are based on the following theorem which was proved by Fukuda and Holevo in [13] in the case of generalized Pauli matrices; inequality (41) was obtained independently by Nathanson in [26]. For completeness, we present a proof in appendix B.4 (essentially identical to that in [13]) that covers the more general class of maps that are diagonal in an orthogonal basis of unitaries (OBU), as described in appendix A.

**Theorem 13** (Fukuda-Holevo-Nathanson). *Let  $\Phi$  be a CPT map which is diagonal when represented in an OBU, and let  $\phi_s$  denote its diagonal elements. Then*

$$\nu_2(\Phi) \leq \left( \frac{1}{d} [1 + (d-1) \sup_s |\phi_s|^2] \right)^{1/2}. \quad (41)$$

Moreover, if the bound (41) is attained, then

$$\nu_2(\Phi \otimes \Omega) = \nu_2(\Phi) \nu_2(\Omega) \quad (42)$$

for any CPT map  $\Omega$ .

This result implies that all channels constant on axes satisfy the multiplicativity conjecture for  $p = 2$ .

**Theorem 14.** *Let  $d$  be a prime power and  $\Phi : M_d \mapsto M_d$  be a CPT map constant on axes. Then (35) holds for  $p = 2$ , i.e.,  $v_2(\Phi \otimes \Omega) = v_2(\Phi)v_2(\Omega)$  with  $\Omega$  an arbitrary CPT map.*

**Proof.** By theorem 13, it suffices to show that

$$[v_2(\Phi)]^2 = \frac{1}{d}[1 + (d-1)\lambda^2], \quad (43)$$

where  $\lambda = \sup_L |\lambda_L| = \sup_L |s + t_L|$ . For channels constant on axes, it is straightforward to verify that (43) is attained with any axis state  $|\psi_n^{L*}\rangle\langle\psi_n^{L*}|$  for which  $|\lambda_{L*}| = \lambda$ .  $\square$

One can extend this slightly to cover channels of the form (A.16), which includes example 4 in [13].

**Theorem 15.** *Let  $\Phi$  be a channel on  $M_d$  of the form (A.16) with  $\kappa$  MUB. Then (35) holds for  $p = 2$  with  $\Omega$  being an arbitrary CPT map.*

For channels in the OBU  $\{U_{Lj}\}$  defined in theorem 20, one can relax the requirement that the channel is ‘constant’ on the longest axis.

**Theorem 16.** *Let  $\Psi$  be a channel constant on axes,  $\lambda_{L_0} \geq |\lambda_L| \forall L$ , and  $M_J$  a channel of the form (A.20) with  $J = L_0$ . If either  $\Phi = xM_J + (1-x)\Psi$ , with  $0 \leq x \leq 1$ , or  $\Phi = \Psi \circ M_J$ , then (35) holds for  $p = 2$  with  $\Omega$  being an arbitrary CPT map.*

**Proof.** Both  $\Psi$  and  $M_J$  are diagonal in the orthogonal unitary basis  $\{U_{Lj}\}$  with multipliers  $\lambda_L$  and  $\mu_{Lj}$  respectively. In both cases, one can verify that  $\Phi$  is also diagonal with  $\phi_{L_0j} = \lambda^*$  independent of  $j$  and  $\phi_{L_0j} \geq |\phi_{Lj}|$ . (In the first case,  $\lambda^* = x + (1-x)\lambda_{L_0}$ ; in the second  $\lambda^* = \lambda_{L_0}$ .) Therefore,  $\sup_s |\phi_s| = \lambda^*$  and any axis state  $|\psi_n^{L_0}\rangle\langle\psi_n^{L_0}|$  saturates the bound (41). The result then follows from theorem 13.  $\square$

The channel  $\Phi$  is constant on the ‘longest’ axis in the sense that the multiplier  $\phi_{L_0j}$  is independent of  $j$  on this axis. But it is the constraint  $a_{L_0j}$  independent of  $j$  that has been relaxed. Maps of the form above with  $\Psi$  a depolarizing channel were studied in [8] and shown to satisfy (35) for all  $p$ .

## 6. Bloch sphere picture

The Bloch sphere picture has proved so useful for  $d = 2$  that there have been numerous attempts to extend it to higher dimensions, and (A.1) can be regarded as such an extension. Moreover, conditions (A.2) extend the standard criterion on the components of the vector representing a density matrix. The fact that the vector in (A.1) are complex rather than real is an inessential consequence of our decision to focus on OBU rather than bases with Hermitian elements. (Replacing  $W_J^j$  and  $W_J^{-j}$  by  $\frac{1}{2}(W_J^j + W_J^{-j})$  and  $\frac{i}{2}(W_J^j - W_J^{-j})$ , replaces  $v_{Jj}$  and  $v_{J,d-j}$  by  $\text{Re } v_{Jj}$  and  $\text{Im } v_{Jj}$  respectively.) The essential problem is that (A.2) is a necessary, but not sufficient condition for a matrix of the form (A.1) to yield a positive semi-definite matrix. Finding simple sufficient conditions for positivity, or even purity, is the real roadblock.

For qubits, all vectors on the surface of the unit ball correspond to pure states and its image under a CP is an ellipsoid contained in this ball. As shown in [10, 27, 35] not every ellipsoid corresponds to a CP map, but those that do define a unique CP map with positive multipliers. However, the role of negative multipliers is lost completely. The Bloch sphere

picture does not show rotations (unless composed with another map) and does not show the effect of, e.g., a bit flip even when composed with another map.

The channels presented here do allow a partial generalization of the Bloch sphere picture in the sense of axes, with a multiplier effect similar to that of unital qubit channels in the case of positive multipliers. The inadequacy of this picture in the case of negative multipliers arises already for qubits. However, it is obscured by the unitary equivalence of maps composed with conjugation by a Pauli matrix  $\sigma_k$ . For channels constant on axes, this simple map is replaced by  $\Psi_J^X$ , which is the *average* of conjugations with powers of the axis generators  $W_J^\ell$ , and the picture for negative multipliers breaks down completely.

## Acknowledgments

It is a pleasure to acknowledge helpful conversations with many people, including some with J Emerson and D Gottesman about MUB, with D Bruss, J Myrheim, M Horodecki, P Horodecki and others in Gdansk about separability criteria, and with K Audenaert who did numerical tests of separability. Much of this work was done when MBR was visiting the Perimeter Institute in Waterloo and MN was at Kenyon College. Partially supported by the National Science Foundation under grants DMS-0314228 and DMS-0604900 and by the National Security Agency and Advanced Research and Development Activity under Army Research Office contract number DAAD19-02-1-0065.

## Appendix A. Convex combinations of unitary conjugations

### A.1. Orthogonal bases of unitary operators

An *orthogonal basis of unitaries* (OBU) for  $M_d$  is a set of  $d^2$  unitary matrices  $\{V_0, V_1, \dots, V_{d^2-1}\}$  with  $V_0 = I$  satisfying  $\text{Tr } V_s^\dagger V_t = d\delta_{st}$ . Since  $M_d$  becomes a Hilbert space when equipped with the inner product  $\langle A, B \rangle = \text{Tr } A^\dagger B$ , one can expand an element of  $M_d$  in this basis. In particular, any density matrix,  $\rho$ , which is a positive semi-definite operator with  $\text{Tr } \rho = 1$  can be written in the form

$$\rho = \frac{1}{d} \sum_{s=0}^{d^2-1} v_s V_s = \frac{1}{d} \left[ I + \sum_{s>0} v_s V_s \right], \quad (\text{A.1})$$

with  $v_s = \text{Tr } V_s^\dagger \rho$ . It follows easily [13, 26] that

$$|v_s| \leq 1 \quad \text{and} \quad \sum_{s>0} |v_s|^2 \leq d - 1, \quad (\text{A.2})$$

with equality in the latter if and only if  $\rho$  is pure. Although (A.1) can be regarded as a generalization of the Bloch sphere representation to  $d > 2$ , the conditions in (A.2) are necessary but not sufficient for an expression of the form (A.1) to define a positive semi-definite operator.

Let  $\{|\psi_n\rangle\}$  be an orthonormal basis for  $\mathbb{C}^d$ . Then the span of  $\{|\psi_n\rangle\langle\psi_n|\}$  is a  $d$ -dimensional subspace of the  $d^2$ -dimensional space  $M_d$ . Now suppose that  $\{V_s\}$  is an OBU for  $M_d$  such that  $\{V_0, V_1, \dots, V_{d-1}\}$  span the same subspace. Then the projections  $\gamma_n = |\psi_n\rangle\langle\psi_n|$  and the operators  $\frac{1}{\sqrt{d}} V_s$  with  $s = 0, 1, \dots, d-1$  give two orthonormal bases for this subspace. Hence they are related by a unitary transformation, i.e.,

$$\gamma_n = \frac{1}{d} \sum_{s=0}^{d-1} x_{ns} V_s \quad \text{and} \quad V_s = \sum_{n=1}^d \overline{x_{ns}} \gamma_n, \quad (\text{A.3})$$

with  $\frac{1}{\sqrt{d}}(x_{ns})$  being unitary. Since the  $\gamma_n$  commute, so do the  $V_s$ . In fact, the vectors  $|\psi_n\rangle$  are simultaneous eigenvectors of these  $V_s$  with (A.3) the spectral decomposition. This (or the purity condition (A.2)) implies that  $|x_{ns}| = 1$  for all  $n, s$ .

We will consider two special cases of an OBU in detail: those associated with the generalized Pauli matrices introduced in appendix A.3 and those associated with generators of mutually unbiased bases (MUB) introduced in appendix A.4. In both cases, each matrix  $V_s$  will be labelled by a pair of indices, so that  $s \sim (j, k)$  or  $s \sim (J, j)$ . Despite its two indices,  $v_{jk}$  gives coefficients in a basis and is best regarded as a column vector after some ordering of the indices rather than as a matrix.

## A.2. Representations of linear operators on $M_d$

When a linear operator  $\Phi : M_d \mapsto M_d$  is represented by the  $d^2 \times d^2$  matrix  $T^\Phi$  with elements

$$T_{st} = \frac{1}{d} \text{Tr } V_s^\dagger \Phi(V_t), \quad (\text{A.4})$$

its action on  $\rho$  corresponds to  $v_s \mapsto \sum_t T_{st} v_t$ . When  $T_{st} = \delta_{st} \phi_t$  is a diagonal matrix, the channel is called *diagonal* and its action on  $\rho$  reduces to  $v_s \mapsto \phi_s v_s$ , i.e., it acts like a multiplier on the vector representing  $\rho$ .

If the unitary requirement is temporarily dropped and  $V_s \sim |e_j\rangle\langle e_k|$  in the standard basis for  $\mathbb{C}^d$ , then  $T_{(i,k),(j,\ell)} = \text{Tr}[|e_i\rangle\langle e_k|V(|e_j\rangle\langle e_\ell|)]$  has the same entries as the Choi–Jamiolkowski state representative but a very *different* ordering! It is important that the pair  $(i, k)$  labels rows and  $(j, \ell)$  columns in order to correctly describe the action of  $\Phi$  by matrix multiplication. The conversion from this ordering to the Choi–Jamiolkowski state form is sometimes called the ‘canonical shuffle’ [32].

We are primarily interested in maps of the form

$$\Phi(\rho) = \sum_s a_s V_s \rho V_s^\dagger, \quad (\text{A.5})$$

with  $a_s \geq 0$  and  $\sum_s a_s = 1$ . Then  $\Phi$  is a unital completely positive, trace-preserving (CPT) map or unital quantum channel and  $T_{0s} = T_{s0} = \delta_{0s}$ .

**Theorem 17.** *Let  $\{V_s\}$  be an OBU satisfying a commutation relation of the form*

$$V_s V_t V_s^\dagger V_t^\dagger = \xi_{st} I. \quad (\text{A.6})$$

*Then  $|\xi_{st}| = 1$  and a channel of the form (A.5) is diagonal with multiplier  $\phi_s = \sum_u \xi_{su} a_u$ .*

A channel of the form (A.5) can be represented by a diagonal matrix even when the commutation condition does not hold. However, when only one  $a_s$  is non-zero, i.e.,  $\Phi(\rho) = V_u \rho V_u^\dagger$  for some fixed  $u$ , the channel is diagonal if and only if (A.6) holds.

The next result may seem obvious; however, if the  $V_s$  are not mutually orthogonal, one can have a map of the form (A.5) which is CP even though some  $a_J$  are negative. An example is the qubit channel

$$\Phi(\rho) = V \rho V^\dagger = \rho + \sigma_x \rho \sigma_x - V^\dagger \rho V, \quad (\text{A.7})$$

where  $V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ .

**Theorem 18.** *Let  $\{V_s\}$  be an OBU and  $\Phi$  a map of the form (A.5). Then  $\Phi$  is CP if and only if all  $a_J \geq 0$ .*

**Proof.** The key point is that when  $\Phi(\rho) = U\rho U^\dagger$  with  $U$  unitary, its CJ matrix is the projection  $|U\rangle\langle U|$ , where we employ a slight abuse of notation in which  $|U\rangle$  denotes the  $d^2 \times 1$  vectors obtained by ‘stacking’ the columns of  $U$ . When  $\Phi$  has the form (A.5), its CJ matrix can be written as

$$\Gamma_\Phi = \frac{1}{d} \sum_s a_s |V_s\rangle\langle V_s|. \quad (\text{A.8})$$

Moreover, when the  $V_s$  are mutually orthogonal, the corresponding  $|V_s\rangle$  are also orthogonal and are therefore eigenvectors of the CJ matrix with eigenvalue  $a_s$ . Thus,  $\Gamma_\Phi$  is positive semi-definite if and only if all  $a_s \geq 0$ .  $\square$

### A.3. Generalized Pauli matrices

In  $d$  dimensions, one can define the generalized Pauli matrices  $X$  and  $Z$  by their action on a fixed orthonormal basis  $\mathbf{C}^d$ :

$$X|e_k\rangle = |e_{k+1}\rangle \quad \text{and} \quad Z|e_k\rangle = \omega^k |e_k\rangle, \quad (\text{A.9})$$

with  $\omega = e^{i2\pi/d}$  and addition mod  $d$  in the subscript. They are unitary and satisfy the commutation relation  $ZX = \omega XZ$ . Thus, the set of generalized Pauli operators  $\mathcal{P} = \{X^j Z^k : j, k = 0, 1, \dots, d-1\}$  forms an orthogonal unitary basis for  $M_d$ . We are interested in channels which have the form (A.5) in this basis, i.e., for which

$$\Phi(\rho) = \sum_{jk} a_{jk} X^j Z^k \rho (X^j Z^k)^\dagger, \quad (\text{A.10})$$

with  $a_{jk} \geq 0$  and  $\sum_{jk} a_{jk} = 1$ . In view of theorem 17, the matrix representing  $\Phi$  is a diagonal matrix; however, the diagonal elements will not in general be real.

It is evident that  $Z$  has the same properties as one of the  $W_J$  in section 2.1. In addition,  $X$  and many other members of  $\mathcal{P}$  are unitarily equivalent to  $Z$  and share these properties. Whenever  $W = X^j Z^k$  with either  $j$  or  $k$  relatively prime to  $d$ , then  $W$  generates a cyclic group of order  $d$ . We want to exploit this group structure to relabel the matrices  $X^j Z^k$  and associate them with ‘axes’ whenever possible. For this purpose we do not need to distinguish between, e.g.,  $(X^j Z^k)^2$  and  $X^{2j} Z^{2k}$  although  $(X^j Z^k)^2 = \omega^{jk} X^{2j} Z^{2k}$ . With this notion of equivalence, we find that if  $W_1 = X^{j_1} Z^{k_1}$  and  $W_2 = X^{j_2} Z^{k_2}$  with  $\gcd(j_1, k_1, d) = \gcd(j_2, k_2, d) = 1$ , then they generate cyclic groups  $\mathcal{W}_1$  and  $\mathcal{W}_2$  which are either equal or have no common element other than  $I$ .

Thus, when  $d$  is prime, the set of generalized Pauli operators  $\mathcal{P}$  can be partitioned into the identity  $I$  and  $d+1$  disjoint sets of the form  $\{W^j : j = 1, 2, \dots, d-1\}$ . Let  $W_K, K = 1, 2, \dots, d+1$  denote some fixed choice of generators, and note that  $W_L W_K = \omega^t W_K W_L$ , where  $\omega = e^{2\pi i/d}$  and  $t$  is an integer which depends on  $L$  and  $K$ . (One specific choice, used in appendix C.2, is  $W_J = XZ^J$  for  $J = 1, 2, \dots, d$  and  $W_{d+1} = Z$ .) The eigenvectors  $|\psi_n^J\rangle, |\psi_n^J|$  of these  $W_J$  form a set of  $d+1$  MUB. In view of (6), the eigenstates of  $W_J$  can be regarded as generalizations of the qubit states  $\frac{1}{2}[I \pm \sigma_j]$  at the ends of the three axes of the Bloch sphere. Thus, it is natural to call them *axis states*.

When  $d = d_1 d_2$  is not prime, then  $M_d \simeq M_{d_1} \otimes M_{d_2}$  and one can form another OBU from tensor products of generalized Pauli matrices in dimensions  $d_1$  and  $d_2$ . However, only when  $d = p^m$  is a prime power it is known that one can make a similar division into MUB.

One might ask why we did not consider maps of the form (A.10) with  $a_s \sim a_{jk}$  independent of  $k$ :

$$\Phi(\rho) = \frac{1}{d} \sum_j a_j \sum_k X^j Z^k \rho Z^{-k} X^{-j} = \sum_j a_j X^j \Psi^{\text{QC}}(\rho) X^{-j}. \quad (\text{A.11})$$



We see that such a channel is a convex combination of EB channels and, hence, also EB. Therefore, this choice would not yield a particularly interesting new class of channels.

#### A.4. Mutually unbiased bases

A pair of orthonormal bases  $\{\vartheta_n\}$  and  $\{\psi_n\}$  is called mutually unbiased if  $\langle \vartheta_m, \psi_n \rangle = \frac{1}{d}$ . A set of mutually unbiased bases (MUB) for  $\mathbb{C}^d$  is a collection of orthonormal bases  $\{|\psi_n^J\rangle\}$ ,  $J = 1, 2, \dots, \kappa$  which are pairwise mutually unbiased as in (2). Our treatment of MUB is based on an association with generators  $W_J$  as in (5).

**Theorem 19.** *A collection of orthonormal bases  $\{|\psi_k^J\rangle\}$  is pairwise mutually unbiased if and only if the associated generators satisfy the orthogonality condition  $\text{Tr } W_J^{d-m} W_L^n = d \delta_{JL} \delta_{mn}$ .*

**Proof.** One implication was shown in (7). The other follows immediately from

$$\begin{aligned} |\langle \psi_m^J, \psi_n^L \rangle|^2 &= \text{Tr}(|\psi_m^J\rangle\langle\psi_m^J|)(|\psi_n^L\rangle\langle\psi_n^L|) \\ &= \frac{1}{d^2} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} \bar{\omega}^{mj+kn} \text{Tr } W_J^j W_L^k = \frac{1}{d^2} \text{Tr } I = \frac{1}{d}, \end{aligned} \quad (\text{A.12})$$

since  $\text{Tr } W_J^j W_L^k = \delta_{j0} \delta_{k0}$  when  $J \neq L$ .  $\square$

As observed after (7), one can have at most  $d + 1$  MUB for  $\mathbb{C}^d$ . It follows immediately from theorem 19 that the existence of a maximal set of  $d + 1$  MUB is equivalent to the existence of  $d + 1$  unitary  $W_J$  whose powers generate an OBU. Moreover, this is equivalent to the existence of  $d + 1$  mutually orthogonal unitary  $W_J$  with non-degenerate eigenvalues  $\omega^k$  with  $\omega = e^{2\pi i/d}$ .

The question of whether or not a maximal set of MUB exists when  $d$  is a composite of different primes is a difficult open problem. However, it is known that  $d + 1$  MUB exist when  $d = p^m$  is a prime power [4, 29, 40]. One method of constructing MUB is based on partitioning tensor products of Pauli matrices [4]. (See also [21].)

**Theorem 20.** *When  $d = p^m$  is a prime power, one can decompose the OBU formed by taking tensor products of generalized Pauli matrices into the identity and  $d + 1$  disjoint subsets of  $d - 1$  elements which we denote by  $U_{Lj}$  with  $L = 1, 2, \dots, d + 1$  and  $j = 1, 2, \dots, d - 1$ . Moreover, for each  $L$ ,  $\{I, U_{Lj} : j = 1, 2, \dots, d - 1\}$  forms a maximal Abelian subgroup of the Pauli group and the simultaneous eigenvectors of the  $U_{Lj}$  generate the MUB  $\{|\psi_n^L\rangle\}$ .*

In this setting, the Abelian subgroups which define the MUB are not cyclic. Although one can still use the MUB to define generators  $W_J$ , they need not be equivalent to generalized Pauli matrices. When  $d$  is prime, the  $W_J$  can be chosen to be generalized Pauli matrices and, hence, satisfy a commutation relation.

**Question 21.** *Can the generators  $W_J$  for a fixed maximal set of MUB always be chosen so that they satisfy  $W_J W_L = \xi_{JL} W_L W_J$  for some complex numbers  $\xi_{JL}$  with  $|\xi_{JL}| = 1$ ?*

Limited testing when  $d = 4$  suggests that the answer is negative. However,  $W_J$  is not unique; it depends on the ordering of the basis.

Even when  $d$  is not a prime power, one can find at least three cyclic subgroups  $\mathcal{W}_L$  with  $\mathcal{W}_K \cap \mathcal{W}_L = I$  when  $K \neq L$ . One can choose as generators  $X, Z, XZ$ , and define a set of three associated MUB.



**Question 22.** When  $3 < \kappa < d + 1$ , one can always extend  $\{W_L^j\}$  to an orthogonal basis for  $M_d$ . Can this be done so that the additional elements are also unitary?

When  $\kappa = 3$ , the generators  $W_L$  can be chosen to be generalized Pauli matrices, as in appendix A.3. Can generators of an MUB always be chosen to be either generalized Pauli matrices or tensor products of generalized Pauli matrices? If they are always one or the other, the answer to the previous question is positive.

#### A.5. Channels based on MUB

Channels constant on axes are special cases of channels of the form (A.5) with  $V_s \sim W_L^j$  and  $a_s \simeq a_{Lj} = \frac{1}{d-1}a_L$  with  $a_L$  as in (10). One could also consider  $V_s \sim U_{Lj}$  as defined in terms of generalized Pauli matrices in theorem 20; however, a channel of the form (A.5) with  $V_s \sim W_L^j$  needs not have this form with  $V_s \sim U_{Lj}$ . If the coefficients depend on  $j$  as well as  $L$ , the conversion could lead to cross terms of the form  $U_{Lj}\rho U_{Lk}$  with  $j \neq k$ . In fact, a channel of the form (A.5) with  $V_s \sim U_{Lj}$  is always diagonal, but one with  $V_s \sim W_L^j$  need not be. However, for channels constant on axes, both are diagonal.

For each fixed  $L$ ,  $\{W_L^j\}_{j=1,2,\dots,d-1}$  and  $\{U_{Lj}\}_{j=1,2,\dots,d-1}$  span the same subspace of  $M_d$ , and many of the relations in section 2 can be written using  $U_{Lj}$ . In particular,

$$\Psi_L^{\text{QC}}(\rho) = \frac{1}{d} \sum_j U_{Lj} \rho U_{Lj}^\dagger \quad (\text{A.13})$$

and

$$|\psi_n^L\rangle\langle\psi_n^L| = \frac{1}{d} \left[ I + \sum_{j=1}^{d-1} u_{Lj} U_{Lj} \right], \quad (\text{A.14})$$

with  $|u_{Lj}| = 1$  and  $\sum_j |u_{Lj}|^2 = d - 1$ . We can also rewrite (A.1) as

$$\rho = \frac{1}{d} \left[ I + \sum_{J=1}^{d+1} \sum_{j=1}^{d-1} u_{Jj} U_{Jj} \right], \quad (\text{A.15})$$

with  $u_{Jj} = \text{Tr } U_{Jj}^\dagger \rho$ .

We are primarily interested in channels of the form (3) when a full set of  $d + 1$  MUB exists. However, even when only  $\kappa < d + 1$  MUB exist, one can generalize (3) to

$$\Phi = s\mathcal{I} + \sum_{L=1}^{\kappa} t_L \Psi_L^{\text{QC}} + u\mathcal{N}, \quad (\text{A.16})$$

with the CPT conditions given by  $s + \sum_L t_L + u = 1$ ,  $s + \frac{1}{d} \sum_L t_L + \frac{1}{d^2} u \geq 0$ , and  $\frac{1}{d} t_L + \frac{1}{d^2} u \geq 0$ .

When  $d + 1$  MUB exist, the completely noisy channel  $\mathcal{N} : \rho \mapsto (\text{Tr } \rho) \frac{1}{d} I$  satisfies

$$\mathcal{N} = \frac{1}{d} \sum_L \Psi_L^{\text{QC}} - \frac{1}{d} \mathcal{I}, \quad (\text{A.17})$$

which allows one to reduce (A.16) to (3) by letting  $s \rightarrow s - \frac{1}{d}u$  and  $t_L = t_L + \frac{1}{d}u$ ; in both forms one has  $\lambda_L = \tilde{s} + \tilde{t}_L$ . Even when  $\kappa < d + 1$ , one can associate a multiplier with the channel (A.16) by completing the orthogonal basis  $W_j^k$ . In this case

$$\phi_m = \begin{cases} s + t_L & m \sim (L, j) \\ s & \text{otherwise.} \end{cases} \quad (\text{A.18})$$

### A.6. Conjugations on a single axis

We denote conjugation with a single unitary matrix  $U$  by  $\Gamma_U$  so that  $\Gamma_U(\rho) \equiv U\rho U^\dagger$ . When  $U = U_{Jj}$  is an element of the OBU  $\{U_{Lk}\}$ , the channel  $\Gamma_{U_{Jj}}$  is diagonal in this basis with multiplier  $\phi_s \sim \phi_{Lk}$  satisfying  $|\phi_{Lk}| = 1$  and  $\phi_{Jk} = 1$  for all  $k$ . However, the map  $\Gamma_{W_J^j}(\rho) = W_J^j \rho W_J^{-j}$  is *not* diagonal unless the commutation condition (A.6) holds as in question 21. When  $d$  is prime,  $U_{Jj} = W_J^j$  and we can say a bit more:

$$\Gamma_{U_{Jj}}(|\psi_m^L\rangle\langle\psi_m^L|) = \Gamma_{W_J^j}(|\psi_m^L\rangle\langle\psi_m^L|) = |\psi_n^L\rangle\langle\psi_n^L|, \quad (\text{A.19})$$

where  $n$  is a function of  $L$  and  $m$ . Thus,  $\Gamma_{W_J^j}$  permutes axis states when  $d$  is prime.

It is useful to consider the special case of (A.5) in which the unitary conjugations involve only a single axis  $J$ . The channels

$$M_J(\rho) = \sum_j c_j \Gamma_{U_{Jj}}(\rho) = \sum_j c_j U_{Jj} \rho U_{Jj}^\dagger \quad (\text{A.20})$$

are diagonal with multiplier satisfying  $\phi_{Jj} = 1$  and  $|\phi_{Lk}| \leq 1$  for  $L \neq J$ . If  $\Psi$  is a channel constant on axes, then a channel of the form  $\Phi = \Psi \circ M_J$  still has a constant multiplier  $\lambda_J$  on the axis  $J$  but has multipliers  $|\phi_{Lk}| \leq \lambda_L$  on the other axes. A channel of the form  $\Phi = x M_J + (1-x)\Psi$ , with  $0 \leq x \leq 1$  has a constant multiplier  $x + (1-x)\lambda_J$  on the axis  $J$ , but has multipliers  $|\phi_{Lk}| \leq x + (1-x)\lambda_L$  on the other axes. Relaxing the requirement that the coefficients  $a_{Lj}$  are constant on one axis  $J$  yields channels whose multipliers are constant only on that axis.

### A.7. EB condition on the $L_1$ norm of a channel

We give a simple proof of the so-called ‘computable cross-norm’ condition for separability. This says that a bipartite density matrix  $\Gamma_\Phi$  is separable if and only if  $\sum_j \mu_j \leq 1$  when  $\mu_j$  are the singular values after the canonical reshuffling of the elements so that  $\Gamma_\Phi$  is the CJ matrix of a CP map. The conventions that  $\text{Tr } \Gamma_\Phi = 1$  and  $\Phi_\Gamma$  satisfies the trace-preserving condition  $\text{Tr } \Phi_\Gamma(\rho) = \text{Tr } \rho$  are not consistent unless the reshuffling is accompanied by multiplication by  $d$ . Thus, theorem below gives an upper bound of  $d$  rather than 1.

**Theorem 23.** *Let  $\Phi$  be any EB channel. Then  $\|\Phi\|_1 \leq d$ .*

**Proof.** It was shown in [18] that a channel is EB if and only if it can be written in the form  $\Phi(\rho) = \sum_k R_k \text{Tr } \rho E_k$ , where each  $R_k$  is a density matrix and  $\{E_k\}$  forms a POVM, i.e., each  $E_k \geq 0$  and  $\sum_k E_k = I$ . Then, with respect to the Hilbert–Schmidt inner product,  $\Phi$  can be written as

$$\Phi = \sum_k |R_k\rangle\langle E_k|, \quad (\text{A.21})$$

with the columns of  $R_k$  and  $E_k$  ‘stacked’ as in (A.8).

Any positive semi-definite matrix satisfies  $\text{Tr } E_k^2 \leq (\text{Tr } E_k)^2$ . Therefore, it follows from the triangle inequality that

$$\begin{aligned} \|\Phi\|_1 &\leq \sum_k \| |R_k\rangle\langle E_k| \|_1 = \sum_k (\text{Tr } E_k^2)^{1/2} (\text{Tr } R_k^2)^{1/2} \\ &\leq \sum_k \text{Tr } E_k = \text{Tr } I = d. \end{aligned} \quad (\text{A.22})$$

□

An immediate corollary applies to diagonal channels with  $\Phi(\sum_s a_s V_s) = \sum_s \phi_s a_s V_s$ .

**Theorem 24.** *Let  $\Phi$  be a channel which is diagonal in an OBU. If  $\Phi$  is EB, then  $\sum_s |\phi_s| \leq d$ .*

## Appendix B. Some multiplicativity proofs

### B.1. Convex combinations of channels

The following elementary lemma is needed in section 5.3.

**Lemma 25.** *Let  $\Lambda_j$  be any set of channels for which  $v_p(\Lambda_j) \leq B$  for all  $j$ , and let  $\Lambda$  be a convex combination of the  $\Lambda_j$ . Then  $v_p(\Lambda) \leq B$ . Moreover, if  $\|\Lambda(\rho)\|_p = B$  for some  $\rho$ , then  $v_p(\Lambda) = B$ .*

As an illustration, we consider an application to channels with one symmetry axis.

**Theorem 26.** *Let  $\Phi(a_*, b_*)$  be a channel of the form (27) with  $b_* > 0$  and  $a_* < 0$ .*

- (i) *If  $v_p[\Phi(a_*, b_*)] = v_p[\Phi(0, b_*)]$ , then  $v_p[\Phi(a, b_*)] = v_p[\Phi(0, b_*)]$  for all  $a \in (a_*, 0)$ .*
- (ii) *If (a) holds and (35) holds with  $\Phi = \Phi(a_*, b_*)$ , then it also holds for  $\Phi = \Phi(a, b_*)$  with  $a \in (a_*, 0)$ .*

**Proof.** The proof follows immediately from lemma 25 with  $B = v_p[\Phi(0, b_*)]$  in part (a) and  $B = v_p[\Phi(0, b_*)]v_p(\Omega)$  in part (b).  $\square$

### B.2. Conjecture 9 when $p = 2$ and $p = \infty$

Conjecture 9 posits that the maximal output  $p$ -norm of a channel constant on axes is achieved on an axis state. We show this to be true in the special cases  $p = 2$  and  $p = \infty$ .

**Theorem 27.** *Let  $\Psi$  be a channel constant on axes with  $|\lambda_L| \leq |\lambda_{L^*}|$  for all  $L$ . Then*

$$v_2(\Psi) = \|\Psi(|\psi_0^{L^*}\rangle\langle\psi_0^{L^*}|)\|_2 \quad (\text{B.1})$$

$$v_\infty(\Psi) = \|\Psi(|\psi_0^{L^*}\rangle\langle\psi_0^{L^*}|)\|_\infty. \quad (\text{B.2})$$

In the case  $p = 2$ , this uses the fact that the axis state saturates inequality (41) in section 5.4. When the multipliers are all non-negative, we could prove the  $p = \infty$  case using theorem 16 from [26]. However, a more general proof using singular values works for all axis channels and, indeed, it seems likely that one could generalize it to all  $p > 2$ . We present the proof for channels with  $|\lambda_{L^*}| \geq \frac{1}{d}$ ; a similar argument works when  $|\lambda_{L^*}| < \frac{1}{d-1}(1 - \lambda_{L^*})$ .

For a density matrix  $\rho$ , let  $\{y_i\}$  be the singular values of  $\Psi(\rho)$  with  $y_0 = \|\Psi(\rho)\|_\infty$ . If  $y_0 > |\lambda_{L^*}|$ , then

$$\|\Psi(\rho)\|_2^2 = \sum_{i=0}^{d-1} y_i^2 = y_0^2 + \sum_{i=1}^{d-1} y_i^2 \quad (\text{B.3})$$

$$\geq y_0^2 + \frac{1}{d-1} \left( \sum_{i=1}^{d-1} y_i \right)^2 = y_0^2 + \frac{1}{d-1} (1 - y_0)^2 \quad (\text{B.4})$$

$$> \lambda_{L^*}^2 + \frac{1}{d-1} (1 - \lambda_{L^*})^2 = \|\Psi(|\psi_0^{L^*}\rangle\langle\psi_0^{L^*}|)\|_2^2, \quad (\text{B.5})$$

which contradicts (B.1).

Therefore,  $\|\Psi(\rho)\|_\infty \leq \|\Psi(|\psi_0^{L^*}\rangle\langle\psi_0^{L^*}|)\|_\infty$  for all  $\rho$ .

### B.3. Critical points

The following result emerged from our study of channels constant on axes. We present it here in full generality.

**Lemma 28.** *Let  $\Phi$  be a positivity-preserving linear map on  $M_d$  and  $\{|\psi_n\rangle\}$  an orthonormal basis for  $\mathbb{C}^d$  such that  $\text{span}\{|\psi_n\rangle\langle\psi_n|\}_{n=1,2,\dots,d}$  is an invariant subspace of both  $\Phi$  and  $\widehat{\Phi}$ . Let  $\gamma(t)$  be a differentiable one-parameter family of pure states with  $\gamma(0) = |\psi_m\rangle\langle\psi_m|$  for some  $m$ . Then for all  $p \geq 1$ , the functions  $\|\Phi[\gamma(t)]\|_p$  and  $S(\Phi[\gamma(t)])$  each have a critical point at  $t = 0$ . Moreover, if  $\Phi_1$  and  $\Phi_2$  are positivity-preserving linear maps on  $M_{d_1}$  and  $M_{d_2}$  and each satisfies the same hypotheses, then the result holds for any differentiable family  $\gamma_{12}(t)$  of pure states on  $\mathbb{C}^{d_1 d_2}$  with  $\gamma_{12}(0) = |\psi_n^1 \otimes \psi_m^2\rangle\langle\psi_n^1 \otimes \psi_m^2|$ .*

**Proof.** First, observe that since  $\gamma(t)$  is a pure for all  $t$ ,  $\text{Tr } \gamma(t) = \text{Tr } [\gamma(t)]^2 = 1$  is constant, which implies  $\text{Tr } \gamma'(t) = 0$ . Then, writing  $\gamma(t) = |\chi(t)\rangle\langle\chi(t)|$ , we see that

$$\gamma'(t) = |\chi(t)\rangle\langle\chi'(t)| + |\chi'(t)\rangle\langle\chi(t)|.$$

Thus

$$0 = \text{Tr } \gamma'(0) = \langle\chi'(0), \psi_m\rangle + \langle\psi_m, \chi'(0)\rangle = 2\text{Re}\langle\psi_m, \chi'(0)\rangle. \quad (\text{B.6})$$

Now let  $f(t) = \text{Tr}(\Phi[\gamma(t)])^p$  and observe that

$$f'(t) = p \text{Tr}(\Phi[\gamma(t)]^{p-1} \Phi[\gamma'(t)]) = p \text{Tr } \widehat{\Phi}[(\Phi[\gamma(t)])^{p-1}] \gamma'(t). \quad (\text{B.7})$$

The invariance condition on  $\Phi$  implies that  $\Phi[\gamma(0)] = \sum_n \mu_n |\psi_n\rangle\langle\psi_n|$ , and the fact that it is positivity preserving implies that  $\mu_n \geq 0$ . Then it follows from the orthogonality of the  $\psi_n$  that  $\Phi[\gamma(0)]^{p-1} = \sum_n \mu_n^{p-1} |\psi_n\rangle\langle\psi_n|$ . Then using the invariance of  $\widehat{\Phi}$ , we can find  $\tilde{\mu}_n$  such that

$$\begin{aligned} f'(0) &= p \text{Tr} \left( \sum_n \tilde{\mu}_n |\psi_n\rangle\langle\psi_n| \right) \gamma'(0) \\ &= p \text{Tr} \left( \sum_n \tilde{\mu}_n |\psi_n\rangle\langle\psi_n| \right) (|\psi_m\rangle\langle\chi'(0)| + |\chi'(0)\rangle\langle\psi_m|) \\ &= 2p \tilde{\mu}_m (\langle\psi_m, \chi'(0)\rangle + \langle\chi'(0), \psi_m\rangle) = 0 \end{aligned} \quad (\text{B.8})$$

by (B.6). A similar argument holds for  $f(t) = S(\Phi[\gamma(t)])$ . Note that  $\Phi$  positivity preserving is needed only to ensure that  $(\Phi[\gamma(t)])^{p-1}$  and  $\log(\Phi[\gamma(t)])^{p-1}$  are well defined and differentiable. In the case of a tensor product, it suffices to observe that the channel  $\Phi_1 \otimes \Phi_2$  and the products  $\{|\psi_m^1 \otimes \psi_n^2\rangle\langle\psi_m^1 \otimes \psi_n^2|\}$  satisfy the hypotheses of the lemma for  $d = d_1 d_2$ .  $\square$

Note that because  $M_d$  has dimension  $d^2$  the hypothesis that the  $d$ -dimensional space  $\text{span}\{|\psi_n\rangle\langle\psi_n|\}$  is invariant is far from trivial. Special cases are given below.

- (a)  $\Phi$  is a channel constant on axes and  $\{|\psi_n^j\rangle\}$  is one of the MUB.
- (b)  $\Phi$  is a Pauli diagonal channel, and  $\{|\psi_n\rangle\}$  are the common eigenvectors of a commuting subset of  $d$  generalized Pauli matrices. When  $d$  is not prime, the subgroup need not be cyclic. For example, for  $d = 4$ , the lemma applies to the simultaneous eigenvectors of the set  $\{I, X^2, Z^2, X^2 Z^2\}$ .
- (c) Let  $\Phi$  be a tensor product of unital qubit channels and  $|\beta_n\rangle = (I \otimes \sigma_n)|\beta_0\rangle$  be the four maximally entangled states (with  $|\beta_0\rangle = |00\rangle + |11\rangle$ ). Then

$$\text{span}\{|\beta_n\rangle\langle\beta_n|\} = \text{span}\{I \otimes I, \sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z\},$$

and this is an invariant subspace of  $\Phi$ . So for a product of unital qubit channels, we have a critical point at the maximally entangled states. Even when the inputs are not optimal, this critical point can be a relative maximum; see the example in figure 6 of [15].

- (d) This lemma does not apply directly to the Werner–Holevo channel [39]  $\mathcal{W}(\rho) = \frac{1}{d-1}(I - \rho^T)$  because  $\mathcal{W} = \widehat{\mathcal{W}}$  maps a basis to its complex conjugate, i.e.,

$$\mathcal{W}(\text{span}\{|\psi_n\rangle\langle\psi_n|\}) = \text{span}\{|\overline{\psi}_n\rangle\langle\overline{\psi}_n|\}. \quad (\text{B.9})$$

For any pure input,  $(\mathcal{W}(|\psi\rangle\langle\psi|))^p = \frac{1}{(d-1)^p} \mathcal{W}(|\psi\rangle\langle\psi|)$ , and  $\widehat{\mathcal{W}}(|\overline{\psi}_n\rangle\langle\overline{\psi}_n|) \in \text{span}\{|\psi_n\rangle\langle\psi_n|\}$  since  $\widehat{\mathcal{W}} = \mathcal{W}$ . Therefore, we have all that is needed for the proof of the statement

$$\widehat{\mathcal{W}}(\mathcal{W}(|\psi_n\rangle\langle\psi_n|))^p \in \text{span}\{|\psi_n\rangle\langle\psi_n|\}, \quad (\text{B.10})$$

so that the conclusion still holds.

For a single use of  $\Phi$ , this affirms that any pure state is a critical point of the  $p$ -norm, which is clear since all pure state outputs have the same spectrum. For the product  $\mathcal{W} \otimes \mathcal{W}$ , this shows that any maximally entangled state is a critical point of both the output  $p$ -norm and entropy.

#### B.4. Proof of theorem 13

Using the notation of appendix A.1 with  $\{V_s\}$  the OBU, observe that any  $\rho_{12}$  can be written as  $\rho_{12} = \frac{1}{d} \sum_s V_s \otimes A_s$  with  $A_s = \text{Tr}_1(V_s^\dagger \otimes I)\rho_{12}$ . Then  $\rho_2 = \frac{1}{d} A_0$  and

$$(\Phi \otimes \Omega)(\rho_{12}) = \frac{1}{d} \sum_s \phi_s V_s \otimes \Omega(A_s). \quad (\text{B.11})$$

Then defining  $|\phi_{\max}| = \sup_{s>0} |\phi_s|$ , one finds

$$\begin{aligned} \|(\Phi \otimes \Omega)(\rho_{12})\|_2^2 &= \text{Tr}(\Phi \otimes \Omega)(\rho_{12})^\dagger (\Phi \otimes \Omega)(\rho_{12}) \\ &= \frac{1}{d} \left[ \text{Tr}|\Omega(A_0)|^2 + \sum_{s>0} |\phi_s|^2 \text{Tr}|\Omega(A_s)|^2 \right] \\ &\leq \frac{1}{d} \left( \text{Tr}|\Omega(\rho_2)|^2 (1 - |\phi_{\max}|^2) + |\phi_{\max}|^2 \sum_s \text{Tr}|\Omega(A_s)|^2 \right) \\ &\leq \frac{1}{d} ([v_2(\Omega)]^2 (1 - |\phi_{\max}|^2) + |\phi_{\max}|^2 d \|(\mathcal{I}_1 \otimes \Omega)(\rho_{12})\|_2^2) \\ &\leq \frac{1}{d} ([v_2(\Omega)]^2 (1 - |\phi_{\max}|^2) + |\phi_{\max}|^2 d [v_2(\mathcal{I} \otimes \Omega)]^2) \\ &= \frac{1}{d} (1 + (d-1)|\phi_{\max}|^2) [v_2(\Omega)]^2, \end{aligned} \quad (\text{B.12})$$

where we used the fact [2] that  $v_2(\mathcal{I} \otimes \Omega) = v_2(\Omega)$ . When  $\dim \mathcal{H}_2 = 1$ , (41) follows. Moreover, if the upper bound (41) is attained, then (B.13) implies that

$$\|(\Phi \otimes \Omega)(\rho_{12})\|_2^2 \leq [v_2(\Phi)]^2 [v_2(\Omega)]^2, \quad (\text{B.13})$$

and this bound can always be attained by using a tensor product input.

## Appendix C. Separability of some CJ matrices

### C.1. Extreme points with one symmetry axis

To prove theorem 7 in section 4.2, we need to establish that the points  $R$  and  $Y$  in figure 4 correspond to channels with separable CJ matrices.

For the point  $R = (-\frac{1}{d}, \frac{1}{d})$ , we will use a construction due to Horodecki [20] which extends an argument in [19]. One can verify that the CJ matrix (29) can be written as

$$\begin{aligned}\Gamma &= \frac{1}{d^2} \left( I + \sum_{j \neq k} |e_j \otimes e_j\rangle \langle e_k \otimes e_k| \right) \\ &= \frac{1}{d^2} \frac{1}{m^{d-1}} \sum_{x_2, \dots, x_d} |\phi_{x_2, \dots, x_d} \otimes \bar{\phi}_{x_2, \dots, x_d}\rangle \langle \phi_{x_2, \dots, x_d} \otimes \bar{\phi}_{x_2, \dots, x_d}|, \end{aligned} \quad (\text{C.1})$$

where  $m \geq 3$  is an integer,  $x_1 = 1$  and each of the  $d - 1$  remaining  $x_j$  is chosen from among the  $m$ th roots of unity  $e^{2\pi i n/m}$ , the sum runs over all possible choices of  $x_2, \dots, x_d$ , and

$$|\phi_{x_2, \dots, x_d}\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^d x_j |e_j\rangle.$$

The point  $Y = (\frac{1}{2}, \frac{-1}{2(d-1)})$  corresponds to the channel  $\Psi_J^{\text{YEB}}$ , for which the CJ matrix (29) can be written as

$$\begin{aligned}\Gamma &= \frac{1}{2d(d-1)} \left( \sum_{j \neq k} |e_j \otimes e_k\rangle \langle e_j \otimes e_k| + (d-1) \right. \\ &\quad \times \sum_k |e_k \otimes e_k\rangle \langle e_k \otimes e_k| - \sum_{j \neq k} |e_j \otimes e_j\rangle \langle e_k \otimes e_k| \left. \right) \\ &= \frac{1}{2d(d-1)} \sum_{j < k} \gamma_{jk}, \end{aligned} \quad (\text{C.2})$$

where  $\gamma_{jk}$  is given by

$$|e_j \otimes e_k\rangle \langle e_j \otimes e_k| + |e_k \otimes e_j\rangle \langle e_k \otimes e_j| + (|e_j \otimes e_j\rangle - |e_k \otimes e_k\rangle)(\langle e_j \otimes e_j| - \langle e_k \otimes e_k|).$$

Each  $\gamma_{jk}$  corresponds to a qubit density matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

which is separable because it satisfies the PPT condition. Thus, (C.2) is a convex combination of separable matrices.

### C.2. State representatives for $d$ prime

To further characterize the EB maps, we need more information about the CJ matrix. We first consider only the case of prime  $d$ , for which the generators  $W_J$  are generalized Pauli matrices. There is no loss of generality in assuming that  $W_J = XZ^J$  for  $J = 1, 2, \dots, d$  and that  $W_{d+1} = Z$ . For  $J \neq d+1$ :

$$W_J^m (|e_j\rangle \langle e_k|) W_J^{-m} = \omega^{mJ(j-k)} |e_{j+m}\rangle \langle e_{k+m}|,$$

so that

$$\Psi_J^X(|e_j\rangle\langle e_k|) = \frac{1}{d-1} \sum_{m \neq 0} \omega^{mJ(j-k)} |e_{j+m}\rangle\langle e_{k+m}|. \quad (\text{C.3})$$

Note that this implies that for  $J \neq d+1$ , the CJ matrix  $\Gamma_{\Psi_J^X}$  has the coefficient of  $|e_j \otimes e_j\rangle\langle e_k \otimes e_k|$  equal to zero for all  $j, k$ , which means that the maximally entangled state  $|\beta\rangle$  is in its kernel. The same is true for the CJ matrix of the CP map  $\widehat{\Phi}_{d+1} = \sum_{J \neq d+1} a_J \Psi_J^X$  with  $a_J > 0$ . When  $d = 3$ , we can write its CJ matrix explicitly as

$$\widehat{\Gamma}_{d+1} = \frac{1}{6} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & \bar{z} & z & 0 & 0 \\ 0 & 0 & \alpha & z & 0 & 0 & 0 & \bar{z} & 0 \\ 0 & 0 & \bar{z} & \alpha & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & \alpha & \bar{z} & 0 & 0 \\ 0 & \bar{z} & 0 & 0 & 0 & z & \alpha & 0 & 0 \\ 0 & 0 & z & \bar{z} & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{C.4})$$

where  $z = a_1\omega + a_2\omega^2 + a_3$  and  $\alpha = a_1 + a_2 + a_3$  which is 1 when  $\widehat{\Phi}_{d+1}$  is TP.

To obtain the general CJ matrix, observe that we can use (17) to write any channel constant on axes as

$$\begin{aligned} \Phi &= a_{00}\mathcal{I} + a_{d+1}\Psi_{d+1}^X + \sum_{J=1}^d a_J \Psi_J^X \\ &= \frac{1}{d}[(d-1)s + 1 - t_{d+1}]\mathcal{I} + t_{d+1}\Phi_{d+1}^{\text{QC}} + \widehat{\Phi}_{d+1}, \end{aligned} \quad (\text{C.5})$$

with  $a_{00}, a_J, s, t_J$  related as following (10). Then

$$\Gamma_\Phi = \frac{1}{d}[(d-1)s + 1 - t_{d+1}][|\beta\rangle\langle\beta| + \frac{t_{d+1}}{d} \sum_j |e_j \otimes e_j\rangle\langle e_j \otimes e_j| + \widehat{\Gamma}_{d+1}]. \quad (\text{C.6})$$

We now give the nonzero elements of  $\Gamma_\Phi$  with the conventions that indices with different letters are always unequal. The first two come from the first two terms in (C.6) and the next two from  $\widehat{\Phi}_{d+1}$  and (C.3).

Term:

$$|e_j \otimes e_j\rangle\langle e_j \otimes e_j|$$

$$|e_j \otimes e_j\rangle\langle e_k \otimes e_k|$$

$$|e_j \otimes e_k\rangle\langle e_j \otimes e_k|$$

$$|e_j \otimes e_{j+m}\rangle\langle e_k \otimes e_{k+m}|$$

Coefficient:

$$\frac{1}{d^2}[(d-1)s - (d-1)t_{d+1}] = \frac{1}{d^2}[1 + (d-1)\lambda_{d+1}]$$

$$\frac{1}{d^2}[1 + ds - \lambda_{d+1}] = \frac{1}{d^2} \left[ \sum_{J=1}^d \lambda_J \right]$$

$$\frac{1}{d(d-1)}(1 - a_{00} - a_{d+1}) = \frac{1}{d^2}(1 - \lambda_{d+1})$$

$$\frac{1}{d^2} \sum_{J \neq d+1} \omega^{mJ(j-k)} t_J = \frac{1}{d^2} \sum_{J \neq d+1} \omega^{mJ(j-k)} \lambda_J.$$

We write the CJ matrix explicitly in the case  $d = 3$ :

$$\frac{1}{9} \begin{pmatrix} 1+2\lambda_4 & 0 & 0 & 0 & u & 0 & 0 & 0 & u \\ 0 & 1-\lambda_4 & 0 & 0 & 0 & z & \bar{z} & 0 & 0 \\ 0 & 0 & 1-\lambda_4 & \bar{z} & 0 & 0 & 0 & z & 0 \\ 0 & 0 & z & 1-\lambda_4 & 0 & 0 & 0 & \bar{z} & 0 \\ u & 0 & 0 & 0 & 1+2\lambda_4 & 0 & 0 & 0 & u \\ 0 & \bar{z} & 0 & 0 & 0 & 1-\lambda_4 & z & 0 & 0 \\ 0 & z & 0 & 0 & 0 & \bar{z} & 1-\lambda_4 & 0 & 0 \\ 0 & 0 & \bar{z} & z & 0 & 0 & 0 & 1-\lambda_4 & 0 \\ u & 0 & 0 & 0 & u & 0 & 0 & 0 & 1+2\lambda_4 \end{pmatrix}, \quad (\text{C.7})$$

where  $u = \lambda_1 + \lambda_2 + \lambda_3$  and  $z = \lambda_1\omega + \lambda_2\omega^2 + \lambda_3$ .

### C.3. Implications and proofs for $d = 3$

**Proof of theorem 6.** Applying the partial transpose to (C.7) gives a matrix which can be permuted to give three similar  $3 \times 3$  blocks so that the PPT condition is equivalent to

$$\frac{1}{9} \begin{pmatrix} 1+2\lambda_1 & \bar{z} & \bar{z} \\ z & 1-\lambda_1 & u \\ z & u & 1-\lambda_1 \end{pmatrix} \geq 0. \quad (\text{C.8})$$

Conjugating this with the unitary matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

gives the equivalent condition

$$\frac{1}{9} \begin{pmatrix} 1+2\lambda_1 & \sqrt{2}\bar{z} & 0 \\ \sqrt{2}z & 1-\lambda_1+u & 0 \\ 0 & 0 & 1-\lambda_1-u \end{pmatrix} \geq 0. \quad (\text{C.9})$$

This gives a pair of necessary and sufficient conditions for PPT

$$1 - \lambda_1 - u \geq 0 \quad \text{and} \quad (1 + 2\lambda_1)(1 - \lambda_1) \geq 2|z|^2.$$

The first is equivalent to  $\sum_J \lambda_J \leq 1$  which was shown in theorem 5 to be necessary for all  $d$ ; the second is (23).  $\square$

On the base tetrahedron,  $\sum_J \lambda_J = -\frac{1}{2}$  and the second condition reduces to  $\sum_J \lambda_J^2 \leq \frac{1}{4}$ ; this is the equation of the inscribed sphere which just touches the faces of the base tetrahedron, as shown in figure 2.

**C.3.1. Extreme points.** Next consider a channel of the form  $\hat{\Phi}_K = \sum_{J \neq K} a_J \Psi_J^X$  which is a convex combination of  $d$  of the  $\Psi_J^X$ . When  $d = 3$  and  $K = d + 1$ , the CJ matrix of  $\hat{\Phi}_K$  is given by (C.4). It is easy to see that the PPT condition for separability can never be satisfied unless  $z = 0$ . But this will happen if and only if all  $a_J = \frac{1}{d}$ . This corresponds to a channel  $\Psi_J^{\text{XEB}}$  in the centre of one of the faces of the ‘base’ tetrahedron shown in figure 2. Therefore,  $\Psi_J^{\text{XEB}}$  is an extreme point of the convex hull of EB maps for  $d = 3$ .



The point  $\Psi_J^{\text{YEB}}$  lies on the line segment  $t\Psi_J^{\text{XEB}} + (1-t)\Psi_J^{\text{X}}$ , which is the segment BE in figure 4. It follows from theorem 7 that these channels are EB if and only if  $\frac{1}{2} \leq t \leq 1$ , i.e., that  $\Psi_J^{\text{XEB}}$  and  $\Psi_J^{\text{YEB}}$  are the extreme points of the EB channels restricted to this line segment. In the case  $d = 3$ ,  $\Psi_J^{\text{YEB}} = [\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}]$  lies on the boundary of the sphere  $\sum_J \lambda_J^2 \leq \frac{1}{4}$  which encloses the set of PPT maps in the base tetrahedron. Since this set is strictly convex,  $\Psi_J^{\text{YEB}}$  must be an extreme point of the subset of EB maps when  $d = 3$ .

Note that there are channels in the base tetrahedron which satisfy the CCN condition  $\sum_J |\lambda_J| \leq 1$  but are not PPT. For example,  $\frac{1}{6}\Psi_1^{\text{X}} + \frac{2}{6}\Psi_2^{\text{X}} + \frac{3}{6}\Psi_3^{\text{X}}$  gives a channel with multiplier  $[-\frac{1}{4}, 0, +\frac{1}{4}, -\frac{1}{2}]$ . This channel has  $|\lambda_J|^2 = \frac{3}{8} > \frac{1}{4}$  and lies on a face of the base tetrahedron with unequal  $a_J$ .

#### C.4. Some observations for $d > 3$ :

The form of the CJ matrices in appendix C.2 generalizes to prime  $d > 3$ , in particular the pattern of non-zero elements in the matrix. Indeed, the only elements with non-zero coefficients must have the form  $|e_j\rangle\langle e_k| \otimes |e_{j+m}\rangle\langle e_{k+m}|$ . When  $j = k$  or  $m = 0$ , the coefficients were given explicitly after (C.6).

Applying the PPT condition to a channel of the form  $\hat{\Phi}_K = \sum_{J \neq K} a_J \Psi_J^{\text{X}}$  with  $K = d+1$  will yield  $d-1$  equations of the form  $\sum_{J=1}^d \omega^{n_J} a_J = 0$ , with each of the  $d$  roots of unity occurring exactly once. We note that setting all  $a_J = \frac{1}{d}$  yields a solution to these equations. Moreover, combining the  $d-1$  PPT equations with the normalization condition  $\sum_J a_J = 1$ , gives  $d$  equations for the  $d$  numbers  $a_J$ . The coefficient matrix for the  $a_J$  can be written in the form  $x_{jk} = \omega^{jk}$ . This is a unitary matrix which implies that  $a_J = \frac{1}{d} \forall J$  is the only solution. Thus, as for  $d = 3$ , the only EB maps on the ‘faces’ of the base are the  $\Psi_J^{\text{XEB}}$ , which are thus true extreme points of the EB subset of channels constant on axes.

We remark that the positivity of the submatrix of  $\Gamma_\Phi$

$$\begin{pmatrix} 1 + (d-1)\lambda_K & u & u & \dots & u \\ u & 1 + (d-1)\lambda_K & u & \dots & u \\ \vdots & & & \ddots & \vdots \\ u & \dots & & u & 1 + (d-1)\lambda_K \end{pmatrix}$$

yields inequalities (12). In fact, one eigenvalue is

$$(d-1)u + 1 + (d-1)\lambda_K = 1 + (d-1) \sum_J \lambda_J,$$

and the requirement that this is  $\geq 0$  gives (13b). Considering a  $2 \times 2$  submatrix gives  $(1 + (d-1)\lambda_K)^2 \geq u^2$  which implies (13a).

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