On "The simplest 2D quantum walk detects chaoticity"

immediate

1 Summary

In Ref. [1], Alonso-Lobo, Carlo, and Borondo investigate a two-dimensional discrete-time quantum walk (DTQW) inside a desymmetrized Bunimovich billiard (see Fig. 1) to identify signatures of chaos.

The first indicator they examine is the level spacing distribution $P(s)$ of the eigenphases of the DTQW step, expecting it to match the predictions of random matrix theory (RMT). However, the observed $P(s)$ does not satisfactorily align with RMT predictions. Given that the system should exhibit chaotic behavior, I suspect there may be an issue related to the unfolding procedure or the presence of unaccounted symmetries. In fact, even for the rectangular billiard, the $P(s)$ distribution does not conform to a Poisson distribution, which suggests that additional factors influencing the spectral statistics need to be considered.

2D DTQW in a Bunimovich stadium

One step of the DTQW in the desymmetrized Bunimovich stadium of Fig. 1 in [1] is given by:

$$U = S_x C_2 S_y C_1, \tag{1}$$

where S_x and S_y are the conditional shift operators that move the particle through the stadium, and C_1 and C_2 are the coin operators that introduce quantum superpositions in the internal degree of freedom.

The shift operators conditionally displace the walker in the x- and y- directions, depending on the state of the internal coin degree of freedom:

$$S_x = \sum_{x=1}^{x_{\max}(y)-1} |x+1\rangle\langle x| \otimes \mathbb{1}_y \otimes |0\rangle\langle 0|_c + \sum_{x=1}^{x_{\max}(y)} |x-1\rangle\langle x| \otimes \mathbb{1}_y \otimes |1\rangle\langle 1|_c + \tag{2}$$

$$|x_{\max}(y)\rangle\langle x_{\max}(y)| \otimes \mathbb{1}_y \otimes |1\rangle\langle 0|_c + |0\rangle\langle 0| \otimes \mathbb{1}_y \otimes |0\rangle\langle 1|_c\,, \tag{3}$$

$$S_y = \sum_{y=1}^{y_{\max}(x)-1} \mathbb{1}_x \otimes |y+1\rangle\langle y| \otimes |0\rangle\langle 0|_c + \sum_{y=1}^{y_{\max}(x)} \mathbb{1}_x \otimes |y-1\rangle\langle y| \otimes |1\rangle\langle 1|_c + \tag{4}$$

$$\mathbb{1}_x \otimes |y_{\max}(x)\rangle\langle y_{\max}(x)| \otimes |1\rangle\langle 0|_c + \mathbb{1}_x \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 1|_c\,. \tag{5}$$

These operators ensure that the walker's movement is dictated by the coin state, which determines whether the displacement occurs forward or backward along each spatial direction.

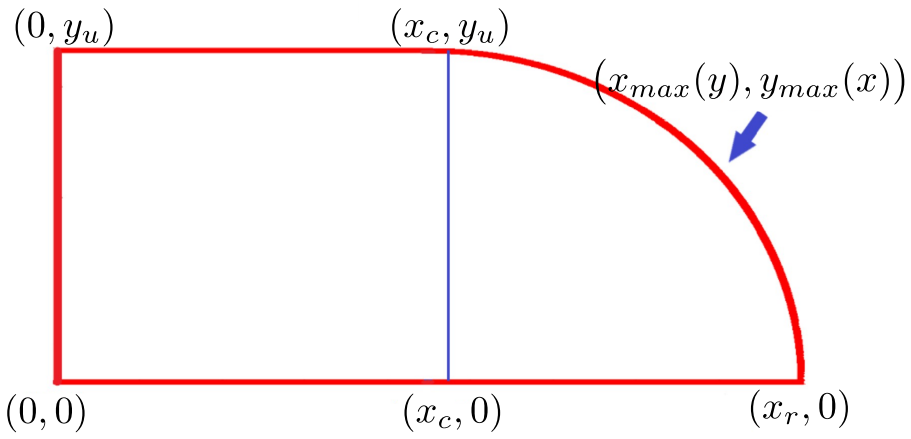

Figure 1: Desymmetrized Bunimovich stadium. Figure taken and modified from [1]. *fig:bunimovich*

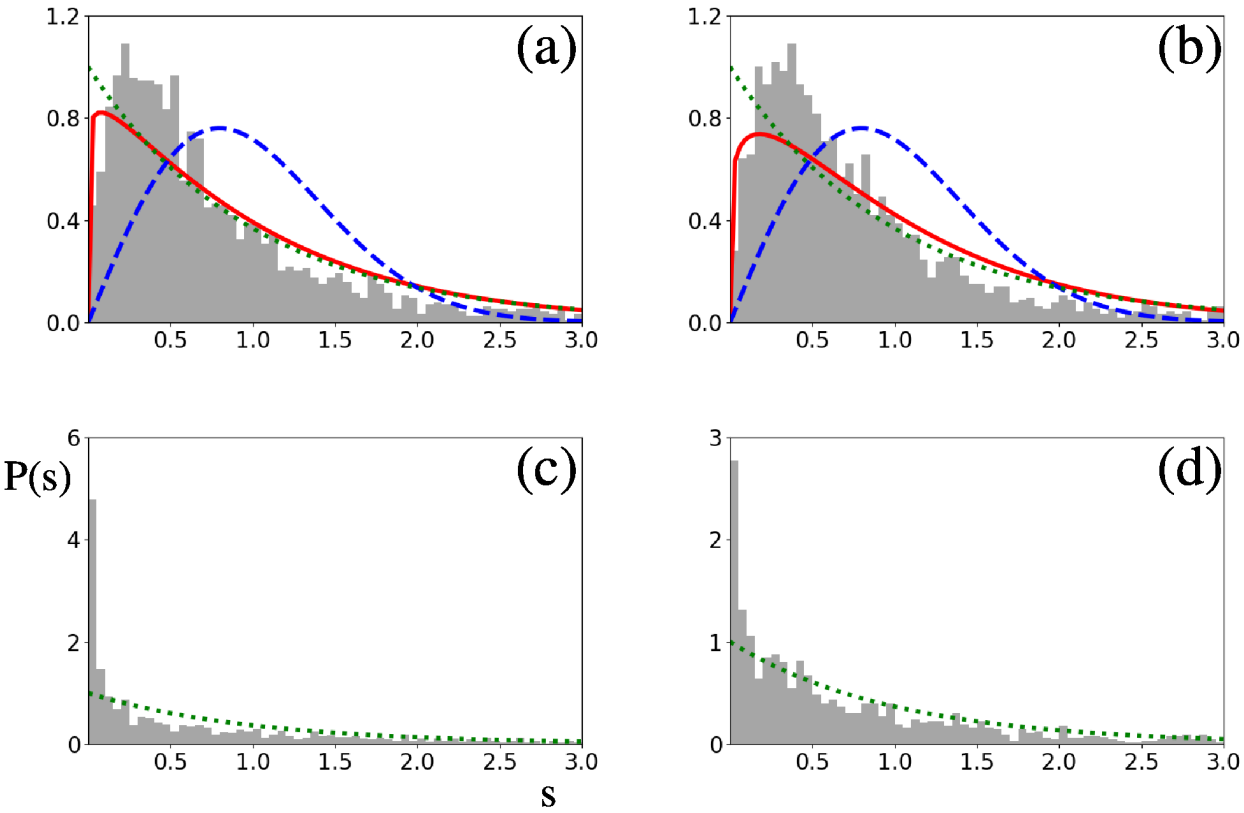

Figure 2: Unfolded level spacing distributions $P(s)$. In (a) and (b) the Bunimovich stadium with coin angles $\alpha = \beta = \pi/4$, and $\alpha = \pi/4, \beta = \pi/3$, respectively. In (c) and (d) the rectangular stadium with coin angles $\alpha = \beta = \pi/4$, and $\alpha = \pi/4, \beta = \pi/3$, respectively. Green and blue dashed lines represent Poisson distribution and Wigner surmise for GOE ensemble, respectively, and red solid line is the best fit for the Brody distribution. Figure taken from [1]. *fig:level:spacing:dist*

The coin operators introduce quantum coherence by applying a unitary rotation in the internal spin space at each step:

$$C_1 = \mathbb{1}_x \otimes \mathbb{1}_y \otimes \begin{pmatrix} \cos\alpha & \sin\alpha \\ -e^{i\pi/4}\sin\alpha & e^{i\pi/4}\cos\alpha \end{pmatrix}, \quad C_2 = \mathbb{1}_x \otimes \mathbb{1}_y \otimes \begin{pmatrix} \cos\beta & \sin\beta \\ -e^{i\pi/4}\sin\beta & e^{i\pi/4}\cos\beta \end{pmatrix}. \tag{6}$$

These unitary transformations mix the two internal states of the walker, playing a crucial role in generating interference effects that distinguish quantum walks from their classical counterparts.

This formulation defines a discrete-time quantum walk (DTQW) in which the quantum state of the walker evolves through a sequence of conditional shifts and internal coin rotations. The interplay between the shift and coin operations determines the statistical properties of the eigenphases of the evolution operator U, which are key to analyzing the spectral signatures of chaos in the system.

2 Distribution of Level Spacing Ratios $P(r)$

Results from [1] for the level spacing distribution $P(s)$ of the eigenphases of a DTQW step in the Bunimovich and rectangular billiards are shown in Fig. 2. The authors argue that, although their statistics do not match the RMT prediction for the Bunimovich billiard nor the Poisson statistics for the rectangular billiard, the difference between both statistics distinguishes the chaotic and integrable cases.

From my perspective, one should be able to match the RMT prediction for the Bunimovich billiard and Poisson statistics for the rectangular one. If not, there may be an issue with desymmetrization or the unfolding of eigenphases. Since no obvious symmetry exists in the case of different coins ($C_1 \neq C_2$), I initially suspected that the unfolding procedure was the problem. A way to circumvent this is to compute the level spacing ratio distribution $P(r)$, for which the RMT prediction has been derived analytically [2].

Figure 3 shows the distribution $P(r)$ for 80% of the eigenphases, taken from the middle of the spectrum. It closely matches the Wigner surmise for the GOE, clearly supporting the BGS conjecture. However, when computing the histogram of $P(r)$ for the case of equal coins ($C_1 = C_2$), corresponding to Fig. 2(a), a large

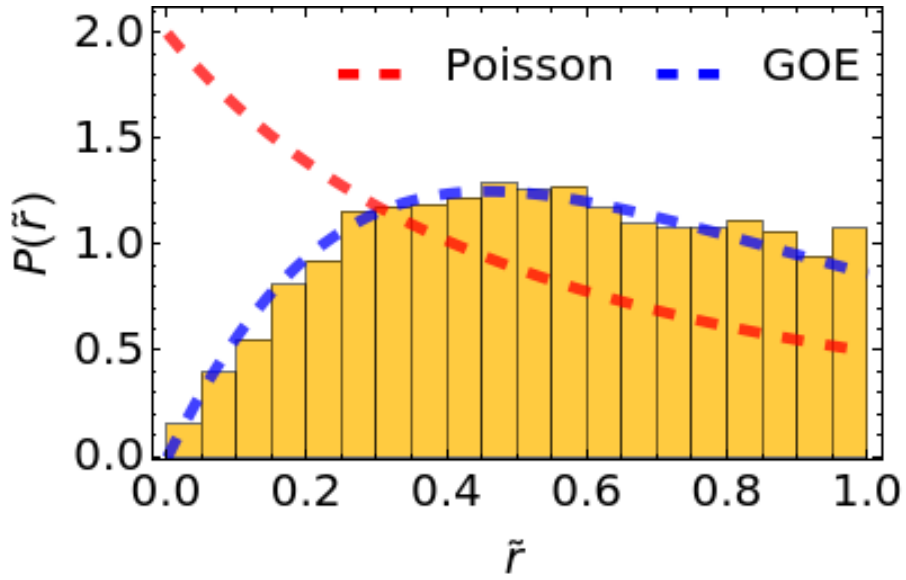

Figure 3: Monedas de la v2 del arxiv [1]. Dimensiones del estadio: $\alpha = \pi/4$, $\beta = \pi/3$. $x_c = 50$, $x_r = 2x_c$ y $y_u = x_c$. *fig:1*

number of occurrences of $\tilde{r} = 0$ were found, preventing the construction of a reliable histogram of $P(r)$. Since the unfolding should not be an issue in this case, I am inclined to think that an additional, unaccounted-for symmetry—besides the spatial one—is present. However, this symmetry is not obvious to me, and I have not yet been able to resolve the discrepancy with the Wigner surmise.

Notably, the left-most bin in Fig. 2(a) and (c) is systematically taller than in Fig. 2(b) and (d), further supporting the hypothesis of a missing symmetry when $C_1 = C_2$.

3 Open questions

- Is there a symmetry that prevents the spectral statistics from matching Poisson and RMT predictions?
- Could it be a quasi-symmetry?
- If such a symmetry exists, how can it be identified?
- The paper discusses other static signatures of chaos, such as the morphology of the eigenfunctions of a DTQW step. Would it be interesting to investigate dynamical signatures, such as the spectral form factor, survival probability, and Loschmidt echo?
- While the original Bunimovich stadium problem is formulated as a single-particle system, in the DTQW formulation, one may consider the entanglement between position and spin degrees of freedom. Would it be worthwhile to investigate if it detects the integrable-chaos transition?

4 2D DTQW with a 4-degrees coin

5 The Shift Operator and Boundary Conditions

In the context of a 2D Discrete-Time Quantum Walk (DTQW) confined within a Bunimovich stadium, the Hilbert space is defined by $\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_C$, where $\mathcal{H}_P$ represents the lattice position spanned by $|x, y\rangle$, and $\mathcal{H}_C$ is the four-dimensional coin space spanned by the basis states $\{|\uparrow\rangle, |\downarrow\rangle, |\leftarrow\rangle, |\rightarrow\rangle\}$.

To confine the walker within the Bunimovich geometry Ω, the Shift operator S acts conditionally depending on the validity of the target site. Let $\mathbf{x} = (x, y)$ denote the lattice position and $\mathbf{e}_c$ the unit displacement vector associated with coin state $|c\rangle$. The action of S on a generic basis state $|\mathbf{x}, c\rangle$ is defined as:

eq:boundary_shift

$$S|\mathbf{x}, c\rangle = \begin{cases} |\mathbf{x} + \mathbf{e}_c, c\rangle & \text{if } (\mathbf{x} + \mathbf{e}_c) \in \Omega \quad \text{(Bulk propagation)} \\ |\mathbf{x}, \bar{c}\rangle & \text{if } (\mathbf{x} + \mathbf{e}_c) \notin \Omega \quad \text{(Boundary reflection)} \end{cases} \tag{7}$$

where $\bar{c}$ denotes the reversed direction index (e.g., if $c = \rightarrow$, then $\bar{c} = \leftarrow$). This "flip-flop" boundary condition ensures global unitarity by mapping any trajectory that would exit the billiard back into the valid Hilbert space $\mathcal{H}_\Omega$.

5.1 C_{2v}-Symmetric Coin Families

To investigate quantum chaos in the stadium, the coin operator C must respect the geometric symmetries of the domain. For the Bunimovich stadium, the relevant symmetry group is C_{2v} (invariance under reflections σ_x, σ_y and π-rotation).

Below are three one-parameter families of coins $C(\theta)$ that satisfy $[C, \mathcal{R}_{C_{2v}}] = 0$, ensuring that any breaking of symmetry in the probability distribution arises solely from the chaotic dynamics of the boundary, not the coin bias.

5.2 The Weighted Grover Family (Anisotropic Diffusion)

This family generalizes the standard Grover diffusion coin. It is constructed as a reflection over a symmetric superposition state $|s(\theta)\rangle$.

$$C_G(\theta) = 2|s(\theta)\rangle\langle s(\theta)| - I_4 \tag{8}$$

where the superposition state allows for tuning the coupling between horizontal and vertical degrees of freedom:

$$|s(\theta)\rangle = \cos\theta \left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} \right) + \sin\theta \left(\frac{|\leftarrow\rangle + |\rightarrow\rangle}{\sqrt{2}} \right) \tag{9}$$

- **Control Parameter:** θ acts as a tunability parameter for anisotropy. At $\theta = \pi/4$, we recover the isotropic Grover coin (maximum diffusion).
- **Transport Analysis:** Varying θ allows one to study how the classical chaotic map (the stadium billiard) reacts to walkers with different "masses" or drift velocities along the longitudinal vs. transverse axes.

5.2.1 The Block-Diagonal Rotation Family (Separable Dynamics)

This family consists of independent rotations for the vertical and horizontal subspaces. It does not mix the x and y coin states directly; mixing occurs *only* via boundary reflections in the curved sections of the stadium.

$$C_B(\phi) = \begin{pmatrix} \cos\phi & i\sin\phi & 0 & 0 \\ i\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & i\sin\phi \\ 0 & 0 & i\sin\phi & \cos\phi \end{pmatrix} \tag{10}$$

- **Geometric Probe:** Since this coin creates no entanglement between x and y directions in the bulk, it is the ideal choice for isolating the "geometric chaos" generated purely by the semicircular boundaries of the stadium.
- **Chirality:** The parameter ϕ controls the chirality or phase accumulation, useful for studying time-reversal symmetry breaking effects.

5.2.2 The Generalized Hadamard Family (H_4)

A parameterization of the 4×4 Hadamard matrix that maintains unitarity and symmetry, often used to study interference effects.

$$C_H(\xi) = \frac{1}{\sqrt{2}} \begin{pmatrix} R(\xi) & R(\xi) \\ R(\xi) & -R(\xi) \end{pmatrix}, \quad \text{where } R(\xi) = \begin{pmatrix} \cos\xi & \sin\xi \\ \sin\xi & -\cos\xi \end{pmatrix} \tag{11}$$

- **Interference Contrast:** This coin maximizes quantum interference. In chaotic systems, it is useful for analyzing the transition from constructive interference (localization) to uniform distribution (ergodicity) as the stadium deformation increases.

References

[1] C. Alonso-Lobo, Gabriel G. Carlo, and F. Borondo. The simplest 2D quantum walk detects chaoticity, January 2025. arXiv:2501.13900.

[2] Y. Y. Atas, E. Bogomolny, O. Giraud, and G. Roux. Distribution of the Ratio of Consecutive Level Spacings in Random Matrix Ensembles. *Physical Review Letters*, 110(8):084101, February 2013.