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The Sherrington-Kirkpatrick Model

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Preface

This book is an attempt to give, as much as possible, a self-contained presentation of some of the main ideas involved in the mathematical analysis of the Sherrington–Kirkpatrick model and closely related mixed p-spin models of spin glasses. Certain topics, such as the high-temperature region and phase transition, are not covered and can be found in the comprehensive manuscript of Michel Talagrand [66].

In 1975 David Sherrington and Scott Kirkpatrick introduced in [58] a model of a spin glass—a disordered magnetic alloy that exhibits unusual magnetic behavior. This model is also often interpreted as a question about a typical behavior of the optimization problem $\max_{\sigma\in\Sigma_N} H_N(\sigma)$ for a certain function $H_N(\sigma)$ on the space of *spin configurations* $\Sigma_N = \{-1,+1\}^N$. This means that the parameters of $H_N(\sigma)$ are modeled as random variables and one would like to understand the asymptotic behavior of the average $\mathbb{E}\max_{\sigma\in\Sigma_N} H_N(\sigma)$ in the *thermodynamic (infinite-volume) limit*, as the size of the system N goes to infinity. We will see in Chap. 1 that, in order to solve this problem, it is enough to compute the limit of the *free energy*,

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log \sum_{\sigma\in\Sigma_N} \exp\beta H_N(\sigma),$$

for each *inverse temperature* parameter $\beta = 1/T > 0$, and the formula for this limit was proposed by Sherrington and Kirkpatrick in [58] based on the so-called replica formalism. At the same time, they observed that their *replica symmetric solution* exhibits "unphysical behavior" at low temperature, which means that it can only be correct at high temperature.

Several years later Giorgio Parisi proposed [51, 52] another *replica symmetry breaking solution* within replica theory, now called the *Parisi ansatz*, which was consistent at any temperature $T \geq 0$ and, moreover, was in excellent agreement with computer simulations. The key feature of the celebrated Parisi ansatz was the choice of an *ultrametric parametrization* of the replica matrix in the computation of the free energy based on the replica approach. A rich theory emerged in the physics literature

during the subsequent interpretation of the Parisi solution in terms of some physical properties of the *Gibbs measure* of the model

$$G_N(\sigma) = \frac{\exp \beta H_N(\sigma)}{\sum_{\rho \in \Sigma_N} \exp \beta H_N(\rho)}.$$

In particular, in the work of Parisi [53], the *order parameter* in the ultrametric parametrization of the replica matrix was related to the distribution of the *overlap* $R_{1,2} = N^{-1} \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2$ of two spin configurations $\sigma^1, \sigma^2 \in \Sigma_N$ sampled from the Gibbs measure. The Parisi ansatz was further interpreted in terms of the geometric structure of the Gibbs measure in the work of Mézard et al. [37, 38], where it was understood, for example, that the ultrametricity of the replica matrix corresponds to the ultrametricity of the support of the Gibbs measure in the infinite-volume limit. Such reinterpretation of the Parisi solution formed a beautiful general physical theory of the model, which was described in the famous book of Mézard, Parisi, and Virasoro, "Spin Glass Theory and Beyond," [40]. In some sense, this also opened a path to a rigorous mathematical theory of the model.

Around the same time, motivated by the developments in the SK model, Bernard Derrida proposed two simplified models of spin glasses—the random energy model, REM, in [16, 17], and the generalized random energy model, GREM, in [18, 19]. The REM can be viewed as a formal limit of the family of the so-called pure p-spin models, in which the SK model corresponds to $p = 2$, and its Hamiltonian $H_N(\sigma)$ is given by an i.i.d. sequence of Gaussian random variables with variance N indexed by $\sigma \in \Sigma_N$, which is a rather classical object. The GREM combines several random energy models in a hierarchical way with the ultrametric structure built into the model from the beginning. Even though these simplified models do not shed light on the Parisi ansatz in the SK model directly, the structure of the Gibbs measures in these models was predicted to be, in some sense, identical to that of the SK model in the infinite-volume limit. For example, Derrida and Toulouse showed in [20] that the Gibbs weights in the REM have the same distribution in the thermodynamic limit as the Gibbs weights of the pure states (clusters of spin configurations) in the SK model; this latter distribution was computed earlier in [37] using the replica method. Independently, Mézard et al. [39] illustrated the connection between the REM and the SK model from a different point of view and, finally, de Dominicis and Hilhorst [15] demonstrated a similar connection between the Gibbs measure of the GREM and the global structure of the Gibbs measure in the SK model predicted by the Parisi ansatz.

The realization that the structure of the Gibbs measure in the SK model predicted by the Parisi replica theory coincides with the structure of the Gibbs measure in the GREM, which is much simpler than the SK model, turned out to be a very important step toward a deeper understanding of the Parisi ansatz. In particular, motivated by this connection with the SK model, in his seminal paper [56], David Ruelle gave an alternative explicit description of the Gibbs measure in the GREM in the thermodynamic limit in terms of a certain family of Poisson processes. As a result, one could now study the properties of these measures, nowadays called

the *Ruelle probability cascades*, using the entire arsenal of the theory of Poisson processes. Some of these properties were already described in the original paper of Ruelle [56], while other important properties, which express certain invariance features of these measures, were discovered later by Erwin Bolthausen and Alain-Sol Sznitman in [10]. We will study the Ruelle probability cascades, including their invariance properties, in Chap. 2.

Another breakthrough in the mathematical analysis of the SK model came at the end of the nineties with the discovery of the two so-called *stability properties* of the Gibbs measure in the SK model in the work of Stefano Ghirlanda and Francesco Guerra [25] and Michael Aizenman and Pierluigi Contucci [1]. It was clear that these stability properties, known as the Ghirlanda–Guerra identities and the Aizenman–Contucci stochastic stability, impose very strong constraints on the structure of the Gibbs measure, but the question was whether they lead all the way to the Ruelle probability cascades. The Aizenman–Contucci stochastic stability is identical to one part of the Bolthausen–Sznitman invariance property for the Ruelle probability cascades. The fact that the Ghirlanda–Guerra identities also hold for the Ruelle probability cascades was first proved by Michel Talagrand in [62] in the case corresponding to the REM and, soon after, by Anton Bovier and Irina Kurkova [11] in the general case corresponding to the GREM. This means that both the Aizenman–Contucci stochastic stability and the Ghirlanda–Guerra identities, which were discovered in the setting of the SK model, also appear in the setting of the Ruelle probability cascades, suggesting some connection between the two.

The first partial answer to the above question was given in an influential work of Louis-Pierre Arguin and Michael Aizenman [5] who proved that, under a technical assumption that the overlap takes finitely many values in the thermodynamic limit, the Aizenman–Contucci stochastic stability implies the ultrametricity predicted by the Parisi ansatz. Soon after, it was shown in [43] that, under the same technical assumption, the Ghirlanda–Guerra identities also imply ultrametricity (an elementary proof can be found in [47]). Another approach was proposed by Talagrand in [65]. However, since at low temperature the overlap does not necessarily take finitely many values in the thermodynamic limit, all these results were not directly applicable to the SK model. Nevertheless, they strongly suggested that the stability properties can explain the Parisi ansatz and, indeed, the fact that the Ghirlanda–Guerra identities imply ultrametricity in general, without any technical assumptions, was proved in [50]. This means that the Ghirlanda–Guerra identities characterize the Ruelle probability cascades, which confirms the prediction of the physicists that the Gibbs measure in the SK model coincides with (or can be approximated by) the Ruelle probability cascades.

Even though the proof of this result, which will be given at the end of Chap. 2, is based only on the Ghirlanda–Guerra identities, it is important to mention that the Aizenman–Contucci stochastic stability played an important role in the discovery. It started with an observation made by Talagrand in 2007 (private communication, see also [66]) who noticed that in the setting of the Ruelle probability cascades the Ghirlanda–Guerra identities are contained in the Bolthausen–Sznitman invariance.

Talagrand's observation was reversed in [49] where the Ghirlanda–Guerra identities were combined with the Aizenman–Contucci stochastic stability and expressed as one unified stability property for the Gibbs measure in the SK model, which is the exact analogue of the Bolthausen–Sznitman invariance property in the setting of the Ruelle probability cascades. From this unified stability property one can derive a new invariance property that will appear in Sect. 2.5, where it will be used to prove the ultrametricity of the Gibbs measure in the SK model predicted by the Parisi ansatz. However, this new invariance property can be obtained much more easily as a direct consequence of the Ghirlanda–Guerra identities, which means that the Ghirlanda–Guerra identities alone explain the Parisi ansatz in the SK model and, for this reason, the Aizenman–Contucci stochastic stability will not be discussed in the book, even though behind the scenes it played a very important role. In some sense, this is good news because the Aizenman–Contucci stability is a more subtle property to work with than the Ghirlanda–Guerra identities, especially in the infinite-volume limit.

Once the structure of the Gibbs measure is understood, we will be in a position to prove the celebrated Parisi formula for the free energy. This will be the main focus of Chap. 3. The proof is based on two key results in the mathematical theory of the SK model—the replica symmetry breaking interpolation bound of Guerra [27] and the cavity computation scheme of Aizenman et al. [2]. The main idea of Guerra [27] can be viewed as a very clever interpolation between the SK model and the Ruelle probability cascades, which implies, due to monotonicity, that the Parisi formula is, in fact, an upper bound on the free energy of the SK model. Following this breakthrough discovery of Guerra, Talagrand proved in his famous tour-de-force paper [64] that the Parisi formula, indeed, gives the free energy in the SK model in the thermodynamic limit. Talagrand's ingenious proof finds a way around the Parisi ansatz for the Gibbs measure, but it is quite complicated. In Chap. 3 we will describe a much more direct approach to the matching lower bound based on the Aizenman–Sims–Starr cavity computation and the fact that the Gibbs measure can be approximated by the Ruelle probability cascades. Another advantage of this approach is that it yields the Parisi formula for all mixed p-spin models, while Talagrand's proof worked only for mixed p-spin models for even $p \geq 2$. For simplicity of notation, we only consider models without the external field, but all the results hold with obvious modifications in the presence of the external field.

In Chap. 4, we will study the Gibbs measure in the mixed p-spin models in more detail and describe the joint distribution of all spins in terms of the Ruelle probability cascades. This chapter is motivated by a different family of mean-field spin glass models that includes the random K-sat and diluted p-spin models, for which the main predictions of the physicists remain open and, since we can prove these predictions (in a certain sense) in the setting of the mixed p-spin models, we use it as an illustration of what is expected in these other models.

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Contents

Chapter 1
The Free Energy and Gibbs Measure

In Sect. 1.1, we will introduce the Sherrington–Kirkpatrick model and a family of closely related mixed p-spin models and give some motivation for the problem of computing the free energy in these models. A solution of this problem in Chap. 3 will be based on a description of the structure of the Gibbs measure in the thermodynamic limit and in this chapter we will outline several connections between the free energy and Gibbs measure. At the same time, we will introduce various ideas and techniques, such as the Gaussian integration by parts, Gaussian interpolation, and Gaussian concentration, that will play essential roles in the key results of this chapter and throughout the book. In the last section, we will prove the Dovbysh–Sudakov representation for Gram-de Finetti arrays, which will allow us to define a certain analogue of the Gibbs measure in the thermodynamic limit. As a first step, we will prove the Aldous–Hoover representation for exchangeable and weakly exchangeable arrays. In Sect. 1.4, we will give a classic probabilistic proof of this result for weakly exchangeable arrays and, for a change, in the Appendix we will prove the representation for exchangeable arrays using a different approach, based on more recent ideas of Lovász and Szegedy in the framework of limits of dense graph sequences. We will describe another application of the Aldous–Hoover representations for exchangeable arrays in Chap. 4.

1.1 The Sherrington–Kirkpatrick Model

The Sherrington–Kirkpatrick model originated in physics and was introduced in 1975 as a model for a spin glass—a disordered magnetic alloy that exhibits unusual magnetic behavior. However, even in the physics literature, it is often motivated as a pure optimization problem. We will continue this tradition and consider the following scenario, called the Dean's problem. Suppose we have a group of N people indexed by the elements of $\{1,\dots,N\}$ and a collection of parameters g_{ij} for $1 \le i < j \le N$, called the interaction parameters, which describe how much people i

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and j like or dislike each other. Naturally, a positive parameter means that they like each other and a negative parameter means that they dislike each other. However unrealistic it may seem, we will assume that the feeling is mutual. We will consider different ways to divide a group into two subgroups and it will be convenient to describe them using vectors of ± 1 labels with the agreement that people with the same label belong to the same group. Therefore, vectors

$$\sigma = (\sigma_1, \ldots, \sigma_N) \in \Sigma_N = \{-1, +1\}^N$$

describe 2^N possible such partitions. For a given configuration σ, let us write $i \sim j$ whenever $\sigma_i \sigma_j = 1$ or, in other words, if i and j belong to the same subgroup, and consider the following *comfort function*:

$$c(\sigma) = \sum_{i<j} g_{ij} \sigma_i \sigma_j = \sum_{i \sim j} g_{ij} - \sum_{i \not\sim j} g_{ij}. \tag{1.1}$$

The Dean's problem is then to maximize this function over all configurations σ in Σ_N. The interpretation of this objective is clear, since maximizing the comfort function means that we would like to keep positive interactions as much as possible within the same groups and separate negative interactions into different groups. It would be interesting to try to understand how this maximum behaves in a typical situation and one natural way to give this question some meaning is to model the interaction parameters (g_{ij}) as random variables. The simplest choice is to let the interactions be independent among pairs and have the standard Gaussian distribution (we will see in Sect. 3.8 that, in some sense, the choice of the distribution is not really important). This is one common way to introduce the Sherrington–Kirkpatrick (SK) model.

As a formal definition, we will consider a Gaussian process indexed by $\sigma \in \Sigma_N$,

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} g_{ij} \sigma_i \sigma_j, \tag{1.2}$$

where the random variables g_{ij} for $1 \le i, j \le N$ are i.i.d. standard Gaussian. This process is called the *Hamiltonian* of the SK model and our goal is to study its maximum $\max_{\sigma \in \Sigma_N} H_N(\sigma)$ as the size of the system N goes to infinity or, as the physicists would say, in the *thermodynamic limit*. Notice that, compared with the comfort function above, the Hamiltonian includes N^2 terms indexed by all vectors (i, j). From a mathematical point of view this will make no difference whatsoever, but from the point of view of notation this choice will be more convenient. The normalization by $\sqrt{N}$ is, in some sense, also done for convenience of notation. With this normalization, the covariance of the Gaussian process can be written as

$$\mathbb{E} H_N(\sigma^1) H_N(\sigma^2) = \frac{1}{N} \sum_{i,j=1}^{N} \sigma_i^1 \sigma_j^1 \sigma_i^2 \sigma_j^2 = N \Big(\frac{1}{N} \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2 \Big)^2 = N R_{1,2}^2 \tag{1.3}$$

where the normalized scalar product

$$R_{1,2} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^1 \sigma_i^2 \tag{1.4}$$

is called *the overlap* of two configurations $\sigma^1, \sigma^2 \in \Sigma_N$. The fact that the covariance of the Gaussian process (1.2) is a function of the scalar product, or overlap, between points in Σ_N will have very important consequences and one could say that this property is what makes the SK model so special. For example, this means that the distribution of the maximum $\max_{\sigma\in\Sigma_N} H_N(\sigma)$ would not be affected if we replaced the index set Σ_N by its image under any orthogonal transformation on $\mathbb{R}^N$, since the scalar products and the covariance would be left unchanged. The consequences of this will gradually become clear.

Going back to the maximum of $H_N(\sigma)$, one would expect it to be of order N, on average, since this is what happens in the case of 2^N independent Gaussian random variables with variance of order N, as in Eq. (1.3). However, we are not just interested in the order of the maximum, but in the precise asymptotics

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E} \max_{\sigma\in\Sigma_N} H_N(\sigma). \tag{1.5}$$

One standard approach to this random optimization problem is to think of it as the zero-temperature case of a general family of problems at positive temperature and, instead of dealing with the maximum in Eq. (1.5) directly, first to try to compute its "smooth approximation"

$$\lim_{N\to\infty} \frac{1}{N\beta} \mathbb{E} \log \sum_{\sigma\in\Sigma_N} \exp \beta H_N(\sigma) \tag{1.6}$$

for every *inverse temperature parameter* $\beta > 0$. To connect these two quantities, let us write

$$\begin{aligned} \frac{1}{N}\mathbb{E}\max_{\sigma\in\Sigma_N} H_N(\sigma) &\le \frac{1}{N\beta}\mathbb{E}\log\sum_{\sigma\in\Sigma_N}\exp\beta H_N(\sigma) \\ &\le \frac{\log 2}{\beta} + \frac{1}{N}\mathbb{E}\max_{\sigma\in\Sigma_N} H_N(\sigma), \end{aligned} \tag{1.7}$$

where the lower bound follows by keeping only the largest term in the sum inside the logarithm and the upper bound follows by replacing each term by the largest one. This shows that Eqs. (1.5) and (1.6) differ by at most $\beta^{-1}\log 2$ and Eq. (1.6) approximates Eq. (1.5) when the inverse temperature parameter β goes to infinity. Let us denote

$$F_N(\beta) = \frac{1}{N}\mathbb{E}\log Z_N(\beta), \tag{1.8}$$

where $Z_N(\beta)$ is defined by

$$Z_N(\beta) = \sum_{\sigma\in\Sigma_N} \exp \beta H_N(\sigma). \tag{1.9}$$

The quantity $Z_N(\beta)$ is called the *partition function* and $F_N(\beta)$ is called the *free energy* of the model. We will prove in the next section that the limit

$$F(\beta) = \lim_{N\to\infty} F_N(\beta)$$

exists, which means that Eq. (1.6) is equal to $\beta^{-1}F(\beta)$. It is easy to see, by Hölder's inequality, that

$$\beta^{-1}(F_N(\beta) - \log 2) = \frac{1}{N\beta}\mathbb{E}\log\frac{1}{2^N}\sum_{\sigma\in\Sigma_N}\exp\beta H_N(\sigma)$$

is increasing in β and, therefore, so is $\beta^{-1}(F(\beta) - \log 2)$, which implies that the limit $\lim_{\beta\to\infty}\beta^{-1}F(\beta)$ exists. It then follows from Eq. (1.7) that

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\max_{\sigma\in\Sigma_N} H_N(\sigma) = \lim_{\beta\to\infty}\frac{F(\beta)}{\beta}. \tag{1.10}$$

The problem of computing Eq. (1.5) was reduced to the problem of computing the limit $F(\beta)$ of the free energy $F_N(\beta)$ at every positive temperature. The formula for $F(\beta)$ was discovered by the physicist Giorgio Parisi in 1979 and a proof of the Parisi formula will be one of the main results presented in this book.

In statistical mechanics, the notion of the free energy $F_N(\beta)$ is closely related to another notion—the *Gibbs measure* of the model—which is a random probability measure on Σ_N defined by

$$G_N(\sigma) = \frac{\exp\beta H_N(\sigma)}{Z_N(\beta)}. \tag{1.11}$$

The problem of computing the free energy in the thermodynamic limit turns out to be closely related to the problem of understanding the asymptotic structure of the Gibbs measure G_N, in a certain sense. On a purely intuitive level, it is clear that the Gibbs measure Eq. (1.11) assigns more weight to the configurations σ corresponding to the larger values of the Hamiltonian $H_N(\sigma)$ and, when the inverse temperature parameter β goes to infinity, in the limit, G_N concentrates on the optimal configurations corresponding to the maximum $\max_{\sigma\in\Sigma_N} H_N(\sigma)$. Therefore, a set where G_N carries most of its weight, in some sense, corresponds to a set where H_N takes large values and understanding the structure of this set can be helpful in the computation of the limit in Eq. (1.5), or the limit of the free energy $F_N(\beta)$. This is another way to see that the model at small but positive temperature can be viewed as an approximation of the model at zero temperature, the original optimization problem. In the remainder of the chapter, we will try to explain this connection between the free energy and Gibbs measure from several different points of view, while at the same time introducing various techniques that will be used throughout the book.

Besides the classical Sherrington–Kirkpatrick model, we will also consider its generalization, the so-called *mixed p-spin model*, corresponding to the Hamiltonian

$$H_N(\sigma) = \sum_{p\geq 1}\beta_p H_{N,p}(\sigma), \tag{1.12}$$

given by a linear combination of *pure p-spin* Hamiltonians

$$H_{N,p}(\sigma) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\dots,i_p=1}^{N} g_{i_1\dots i_p}\sigma_{i_1}\cdots\sigma_{i_p}, \tag{1.13}$$

where $(g_{i_1\dots i_p})$ are standard Gaussian random variables independent for all $p \geq 1$ and all $(i_1,\dots,i_p)$. We will assume that the coefficients (β_p) decrease fast enough, for example, $\sum_{p\geq 1} 2^p\beta_p^2 < \infty$, to ensure that the process is well defined when the sum includes infinitely many terms. The covariance of the Gaussian process $H_N(\sigma)$ can be computed similarly to Eq. (1.3) and is given by

$$\mathbb{E}H_N(\sigma^1)H_N(\sigma^2) = N\xi(R_{1,2}), \tag{1.14}$$

where

$$\xi(x) = \sum_{p\geq 1} \beta_p^2 x^p. \tag{1.15}$$

As in the Sherrington–Kirkpatrick model, the covariance is a function of the overlap Eq. (1.4). In the mixed p-spin models, we will define the free energy by

$$F_N = \frac{1}{N}\mathbb{E}\log Z_N, \tag{1.16}$$

where the partition function

$$Z_N = \sum_{\sigma\in\Sigma_N} \exp H_N(\sigma), \tag{1.17}$$

and will not explicitly write the inverse temperature parameter β as in Eq. (1.8), since it can be included in the sequence (β_p). Compared to the original SK model (1.2), no new ideas will be required to deal with the mixed p-spin model (1.12) which only includes terms for even $p \geq 2$ and $p = 1$. The case when terms for odd $p \geq 3$ are present will sometimes require extra attention, but in the end all the results will hold for all mixed p-spin models. In fact, we will see that some results will be even stronger when parameters $\beta_p \neq 0$ for all or sufficiently many $p \geq 1$.

1.2 Miscellaneous Gaussian Techniques

In this section, we will describe several Gaussian techniques, such as the Gaussian integration by parts, interpolation, and concentration, which will be used many times throughout the book. We will begin with the Gaussian integration by parts. Let g be a centered Gaussian random variable with variance v^2 and let us denote the density function of its distribution by

$$\varphi_v(x) = \frac{1}{\sqrt{2\pi}v}\exp\Big(-\frac{x^2}{2v^2}\Big). \tag{1.18}$$

Since $x\varphi_v(x) = -v^2\varphi_v'(x)$, given a continuously differentiable function $F : \mathbb{R} \to \mathbb{R}$, we can formally integrate by parts,

$$\begin{aligned}\mathbb{E}gF(g) &= \int xF(x)\varphi_v(x)\,dx = -v^2F(x)\varphi_v(x)\Big|_{-\infty}^{+\infty} + v^2\int F'(x)\varphi_v(x)\,dx \\ &= v^2\int F'(x)\varphi_v(x)\,dx = v^2\mathbb{E}F'(g),\end{aligned}$$

if the limits $\lim_{x\to\pm\infty} F(x)\varphi_v(x) = 0$ and the expectations on both sides are finite. Therefore,

$$\mathbb{E}gF(g) = \mathbb{E}g^2\mathbb{E}F'(g). \tag{1.19}$$

This computation can be generalized to Gaussian vectors. Let $g = (g_l)_{1\le l\le n}$ be a vector of jointly Gaussian random variables and, given a continuously differentiable function

$$F = F((x_l)_{1\le l\le n}) : \mathbb{R}^n \to \mathbb{R}$$

that satisfies some mild growth conditions, let us show how one can integrate $\mathbb{E}g_1F(g)$ by parts. If $v^2 = \mathbb{E}g_1^2$ then the Gaussian vector $g' = (g_l')_{1\le l\le n}$ defined by

$$g_l' = g_l - \lambda_l g_1 \text{ where } \lambda_l = v^{-2}\mathbb{E}g_1g_l, \tag{1.20}$$

is independent of g_1, since $\mathbb{E}g_1g_l' = \mathbb{E}g_1g_l - \lambda_l v^2 = 0$. If we denote $\lambda = (\lambda_l)_{1\le l\le n}$ then we can write $g = g' + g_1\lambda$. If $\mathbb{E}_1$ denotes the expectation in g_1 only then using Eq. (1.19) conditionally on g' implies that

$$\mathbb{E}_1g_1F(g) = \mathbb{E}_1g_1F(g' + g_1\lambda) = v^2\mathbb{E}_1\frac{\partial F}{\partial x}(g' + x\lambda)\Big|_{x=g_1} \tag{1.21}$$

if the limits $\lim_{x\to\pm\infty} F(g' + x\lambda)\varphi_v(x) = 0$ and both sides are finite for all g', which can be ensured by some mild growth conditions on F and its partial derivatives. If we assume that

$$g_1F(g' + g_1\lambda) \text{ and } \frac{\partial F}{\partial x}(g' + x\lambda)\Big|_{x=g_1} \tag{1.22}$$

are absolutely integrable then, integrating Eq. (1.21) in g', by Fubini's theorem,

$$\mathbb{E}g_1F(g) = v^2\mathbb{E}\frac{\partial F}{\partial x}(g' + x\lambda)\Big|_{x=g_1}. \tag{1.23}$$

Finally, if we compute the derivative,

$$\frac{\partial F}{\partial x}(g' + x\lambda)\Big|_{x=g_1} = \sum_{l\le n}\lambda_l\frac{\partial F}{\partial x_l}(g' + x\lambda)\Big|_{x=g_1} = \sum_{l\le n}\lambda_l\frac{\partial F}{\partial x_l}(g), \tag{1.24}$$

the Eq. (1.23) can be rewritten as

$$\mathbb{E}g_1F(g) = \sum_{l\le n}\mathbb{E}(g_1g_l)\mathbb{E}\frac{\partial F}{\partial x_l}(g). \tag{1.25}$$

The conditions that were used in the derivation of this formula can usually be easily verified in applications. We will encounter two types of examples, when the function $F = F((x_l)_{1\le l\le n})$ and all its partial derivatives are bounded, or, when they grow at most exponentially fast in the Euclidean norm of the vector (x_l). In both cases, all the conditions in the proof of Eq. (1.25) are, obviously, satisfied. For example, the most common application of the Gaussian integration by parts formula (1.25) will be of the following type. Suppose that we have two jointly Gaussian vectors $(x(\sigma))$ and $(y(\sigma))$ indexed by some finite set of indices $\sigma \in \Sigma$. Let G be a measure on Σ and let us define a new (random) measure on Σ by the change of density

$$G'(\sigma) = \frac{\exp y(\sigma)}{Z} G(\sigma) \text{ where } Z = \sum_{\sigma\in\Sigma} \exp(y(\sigma)) G(\sigma). \tag{1.26}$$

Let us denote by $\langle\cdot\rangle$ the average with respect to the product measure $G'^{\otimes\infty}$, which means that for any $n \ge 1$ and any function $f = f(\sigma^1, \dots, \sigma^n)$,

$$\big\langle f \big\rangle = \sum_{\sigma^1,\dots,\sigma^n\in\Sigma} f(\sigma^1, \dots, \sigma^n) G'(\sigma^1) \cdots G'(\sigma^n). \tag{1.27}$$

The following is a consequence of the Gaussian integration by parts formula in Eq. (1.25).

Lemma 1.1. *If we denote* $C(\sigma^1, \sigma^2) = \mathbb{E} x(\sigma^1) y(\sigma^2)$ *then*

$$\mathbb{E}\big\langle x(\sigma)\big\rangle = \mathbb{E}\big\langle C(\sigma^1, \sigma^1) - C(\sigma^1, \sigma^2)\big\rangle. \tag{1.28}$$

Proof. Let us consider one term in the sum

$$\mathbb{E}\big\langle x(\sigma)\big\rangle = \mathbb{E} \sum_{\sigma^1\in\Sigma} x(\sigma^1) G'(\sigma^1).$$

Let us view the function

$$F = G'(\sigma^1) = \frac{\exp y(\sigma^1)}{Z} G(\sigma^1)$$

as a function of $y(\sigma^1)$ and $(y(\sigma^2))_{\sigma^2\in\Sigma}$, which means that we view a copy of $y(\sigma^1)$ that appears in the denominator Z as a separate variable. Notice that

$$\frac{\partial F}{\partial y(\sigma^1)} = G'(\sigma^1), \ \frac{\partial F}{\partial y(\sigma^2)} = -G'(\sigma^1) G'(\sigma^2),$$

and, since each factor $G'(\sigma) \in [0, 1]$, the function F and all its derivatives are bounded and, therefore, all the conditions in the proof of the Gaussian integration by parts formula (1.25) are satisfied. Then, Eq. (1.25) implies that

$$\mathbb{E} x(\sigma^1) G'(\sigma^1) = C(\sigma^1, \sigma^1) \mathbb{E} G'(\sigma^1) - \sum_{\sigma^2\in\Sigma} C(\sigma^1, \sigma^2) \mathbb{E} G'(\sigma^1) G'(\sigma^2).$$

If we now sum this equality over $\sigma^1 \in \Sigma$, we get Eq. (1.28). □

Exercise 1.1. Given a function $\Phi = \Phi(\sigma^1, \dots, \sigma^n)$, show that

$$\mathbb{E}\big\langle \Phi x(\sigma^1)\big\rangle = \mathbb{E}\Big\langle \Phi\Big(\sum_{l=1}^{n} C(\sigma^1, \sigma^l) - nC(\sigma^1, \sigma^{n+1})\Big)\Big\rangle, \tag{1.29}$$

using a similar computation as in the proof of Lemma 1.1.

The formula (1.29) can also be used in the case when the measure G is random and independent of the Gaussian processes $(x(\sigma))$ and $(y(\sigma))$, by applying it conditionally on G. Moreover, by approximation, it can be easily extended to the case of a finite measure G on a countably infinite set Σ, under the condition that Φ is bounded and all the variances

$$\mathbb{E}x(\sigma)^2, \mathbb{E}y(\sigma)^2 \le a \tag{1.30}$$

are uniformly bounded over $\sigma \in \Sigma$. Notice that under this condition $Z < \infty$ and the measure G' in Eq. (1.26) is well defined, since, by Fubini's theorem,

$$\mathbb{E}\sum_{\sigma\in\Sigma} \exp(y(\sigma))G(\sigma) \le e^{a/2} G(\Sigma) < \infty.$$

Under these additional assumptions, we have the following.

Lemma 1.2. *If G is a finite measure on a countably infinite set Σ, Φ is a bounded function of $\sigma^1, \dots, \sigma^n$ and Eq. (1.30) holds then Eq. (1.29) still holds.*

Proof. Let G_m be the restriction of the measure G to the set of points $A_m \subseteq \Sigma$ corresponding to the largest m weights of G. Then, the formula (1.29) holds for the measure G_m and elementary estimates can be used to show that, taking a limit and using the condition (1.30), the same formula holds for the measure G. Let us write down the details for completeness. Similarly to Eq. (1.26), let us define for $\sigma \in A_m$,

$$G'_m(\sigma) = \frac{\exp y(\sigma)}{Z_m} G(\sigma) \text{ where } Z_m = \sum_{\sigma\in A_m} \exp(y(\sigma))G(\sigma). \tag{1.31}$$

Let $\langle\cdot\rangle_m$ denote the average with respect to the product measure $G_m'^{\otimes\infty}$, as in Eq. (1.27), and let us compare the left-hand side of Eq. (1.29) for these two measures,

$$\big|\mathbb{E}\big\langle \Phi x(\sigma^1)\big\rangle - \mathbb{E}\big\langle \Phi x(\sigma^1)\big\rangle_m\big| = \big|\mathbb{E}Z^{-n}Y - \mathbb{E}Z_m^{-n}Y_m\big|, \tag{1.32}$$

where we denoted

$$Y = \sum_{\sigma^1,\dots,\sigma^n\in\Sigma} \Phi(\sigma^1, \dots, \sigma^n) x(\sigma^1) \prod_{l\le n} \exp(y(\sigma^l))G(\sigma^l)$$

and

$$Y_m = \sum_{\sigma^1,\dots,\sigma^n\in A_m} \Phi(\sigma^1, \dots, \sigma^n) x(\sigma^1) \prod_{l\le n} \exp(y(\sigma^l))G(\sigma^l).$$

First of all, by Eq. (1.30) and Fubini's theorem,

$$\mathbb{E}\,|Z - Z_m| = \mathbb{E} \sum_{\sigma \in A_m^c} \exp(y(\sigma))G(\sigma) \le e^{a/2} G(A_m^c). \tag{1.33}$$

To control $\mathbb{E}|Y - Y_m|$, let us notice that, by Eq. (1.30) and Hölder's inequality,

$$\mathbb{E}\Big|\Phi(\sigma^1, \dots, \sigma^n) x(\sigma^1) \prod_{l \le n} \exp y(\sigma^l)\Big| \le L(n,a)\|\Phi\|_\infty, \tag{1.34}$$

for some constant $L(n,a)$ that depends on n and a only and, using that

$$G\Big(\Big\{\text{at least one } \sigma^1, \dots, \sigma^n \in A_m^c\Big\}\Big) \le nG(A_m^c),$$

we obtain the following bound:

$$\mathbb{E}\,|Y - Y_m| \le nL(n,a)\|\Phi\|_\infty G(A_m^c). \tag{1.35}$$

By Eqs. (1.33) and (1.35), we can choose a subsequence along which Z_m and Y_m converge to Z and Y almost surely. The fact that $\mathbb{E}Z_m^{-n} Y_m$ converges to $\mathbb{E}Z^{-n}Y$ along the same subsequence follows from the dominated convergence theorem, since if we denote by σ_0 the point in Σ corresponding to the largest weight of the measure G, then $Z_m \ge \exp(y(\sigma_0))G(\sigma_0)$ and $|Z_m^{-n} Y_m|$ can be bounded by

$$\|\Phi\|_\infty \big(\exp(y(\sigma_0))G(\sigma_0)\big)^{-n} \sum_{\sigma^1, \dots, \sigma^n} |x(\sigma^1)| \prod_{l \le n} \exp(y(\sigma^l))G(\sigma^l).$$

As in Eq. (1.34), it is easy to see that this is integrable and we proved that the quantity in Eq. (1.32) goes to zero along the above subsequence. One can treat the right-hand side of Eq. (1.29) similarly. □

Our next application of the Gaussian integration by parts will also serve as an illustration of a purely technical connection between the free energy and the Gibbs measure. The Gibbs measure appears as soon as we take a derivative of the free energy with respect to a parameter in the Hamiltonian so, in a way, they are two faces of the same coin.

Theorem 1.1 (Guerra–Toninelli). *If the mixed p-spin Hamiltonian Eq. (1.12) includes terms only for even $p \ge 2$ and $p = 1$ then the limit* $\lim_{N\to\infty} F_N$ *exists.*

Eventually, we will show that the limit exists for all mixed p-spin models, but in the case when terms for odd $p \ge 3$ are present we will prove this simultaneously with computing the limit $F(\beta)$. The proof of this theorem will also serve as the first illustration of a general idea of Gaussian interpolation, which plays a very important role throughout the book.

Proof. We will show that the sequence (NF_N) is superadditive:

$$NF_N + MF_M \le (N+M)F_{N+M} \tag{1.36}$$

for any $N, M \ge 1$, and, therefore, the limit of F_N exists by Fekete's lemma, the proof of which will be given in Lemma 1.3 below for convenience. The proof of Eq. (1.36) will be based on an interpolation between the system with $N+M$ coordinates and a product of two independent systems with N and M coordinates. Given $N, M \ge 1$, let us consider three independent mixed p-spin Hamiltonians $H_N(\rho), H_M(\tau)$, and $H_{N+M}(\sigma)$ of the type (1.12) on Σ_N, Σ_M, and $\Sigma_{N+M} = \Sigma_N \times \Sigma_M$ correspondingly. Let us represent $\sigma = (\rho, \tau) \in \Sigma_{N+M}$ and for $t \in [0,1]$ define a Hamiltonian $H_t(\sigma)$ on Σ_{N+M} by

$$H_t(\sigma) = \sqrt{t} H_{N+M}(\sigma) + \sqrt{1-t}\big(H_N(\rho) + H_M(\tau)\big).$$

Let us denote the Gibbs measure corresponding to this Hamiltonian by

$$G_t(\sigma) = \frac{\exp H_t(\sigma)}{Z_t} \text{ where } Z_t = \sum_{\sigma \in \Sigma_{N+M}} \exp H_t(\sigma),$$

and let $\langle \cdot \rangle_t$ denote the average with respect to the product measure $G_t^{\otimes \infty}$, defined similarly to Eq. (1.27). Consider a continuous function $\varphi(t) = (N+M)^{-1} \mathbb{E} \log Z_t$, which can be viewed as an interpolating free energy, since it interpolates between

$$\varphi(0) = \frac{N}{N+M} F_N + \frac{M}{N+M} F_M \text{ and } \varphi(1) = F_{N+M}.$$

Then the derivative $\varphi'(t)$ for $t \in (0,1)$ can be represented as

$$\varphi'(t) = \frac{1}{N+M} \mathbb{E} \frac{1}{Z_t} \sum_{\sigma \in \Sigma_{N+M}} \frac{\partial H_t(\sigma)}{\partial t} \exp H_t(\sigma) = \frac{1}{N+M} \mathbb{E} \Big\langle \frac{\partial H_t(\sigma)}{\partial t} \Big\rangle_t$$

and this is how an average with respect to the Gibbs measure appears. Using Eq. (1.14), it is easy to check that

$$\mathbb{E} \frac{\partial H_t(\sigma^1)}{\partial t} H_t(\sigma^2) = \frac{1}{2} \left((N+M) \xi(R_{1,2}) - N \xi(R^1_{1,2}) - M \xi(R^2_{1,2}) \right),$$

where, given $\sigma^1 = (\rho^1, \tau^1)$ and $\sigma^2 = (\rho^2, \tau^2)$, we denoted the overlaps of ρ^1, ρ^2 and τ^1, τ^2 by

$$R^1_{1,2} = \frac{1}{N} \sum_{i=1}^N \rho_i^1 \rho_i^2 \text{ and } R^2_{1,2} = \frac{1}{M} \sum_{i=1}^M \tau_i^1 \tau_i^2.$$

Since $R_{1,1} = R^1_{1,1} = R^2_{1,1} = 1$, using the Gaussian integration by parts formula (1.28),

$$\varphi'(t) = -\frac{1}{2} \mathbb{E} \Big\langle \xi(R_{1,2}) - \frac{N}{N+M} \xi(R^1_{1,2}) - \frac{M}{N+M} \xi(R^2_{1,2}) \Big\rangle_t.$$

If we write

$$R_{1,2} = \frac{N}{N+M} R^1_{1,2} + \frac{M}{N+M} R^2_{1,2}$$

and notice that the function $\xi(x)$ defined in Eq. (1.15) is convex, since we assumed that the model includes terms only for even $p \geq 2$ and $p = 1$, this proves that $\varphi'(t) \geq 0$. Therefore, $\varphi(0) \leq \varphi(1)$ and Eq. (1.36) holds. □

Lemma 1.3 (Fekete's Lemma). *Let $(x_n)_{n\geq 1}$ be a superadditive sequence, such that $x_n + x_m \leq x_{n+m}$ for all $n, m \geq 1$. Then,*

$$\lim_{n\to\infty} \frac{x_n}{n} = \sup_{n\geq 1} \frac{x_n}{n}. \tag{1.37}$$

Proof. Let us fix $k \geq 1$. The superadditivity condition implies that $a_m + na_k \leq a_{m+nk}$ for any $n, m \geq 1$. Therefore,

$$\liminf_{n\to\infty} \frac{a_{m+nk}}{m+nk} \geq \liminf_{n\to\infty} \frac{a_m + na_k}{m+nk} = \frac{a_k}{k}.$$

Since any integer can be written as $m + nk$ for some n and $m = 0, \ldots, k-1$, this implies that

$$\liminf_{n\to\infty} \frac{a_n}{n} \geq \frac{a_k}{k}.$$

Since k was arbitrary, taking supremum over $k \geq 1$ implies Eq. (1.37). □

Next, we will use the Gaussian integration by parts and Gaussian interpolation to obtain a very important Gaussian concentration inequality. Even though Gaussian concentration is a much more general phenomenon (see, e.g., [33]), we will only need it for the following free-energy-type functions. Again, consider a countable set Σ and some finite measure G on it. Let us consider the quantity

$$X = \log \sum_{\sigma\in\Sigma} \exp(g(\sigma)) G(\sigma), \tag{1.38}$$

where $(g(\sigma))_{\sigma\in\Sigma}$ is a Gaussian process such that for some constant $a > 0$,

$$\mathbb{E} g(\sigma)^2 \leq a \text{ for all } \sigma \in \Sigma. \tag{1.39}$$

The quantity (1.38) is well defined in this case, because, by Fubini's theorem,

$$\mathbb{E} \sum_{\sigma\in\Sigma} \exp(g(\sigma)) G(\sigma) \leq e^{a/2} G(\Sigma) < \infty. \tag{1.40}$$

The following concentration inequality holds.

Theorem 1.2. *If Eq. (1.39) holds then, for all $x \geq 0$,*

$$\mathbb{P}\big(|X - \mathbb{E}X| \geq x\big) \leq 2\exp\Big(-\frac{x^2}{4a}\Big), \tag{1.41}$$

which implies that $\mathbb{E}(X - \mathbb{E}X)^2 \leq 8a$.

To state one immediate application of this concentration inequality, consider the free energy F_N in the mixed p-spin model defined in Eq. (1.16). If we define $X_N = N^{-1}\log Z_N$ then $F_N = \mathbb{E}X_N$. Since, by Eq. (1.14), the variance $\mathbb{E}H_N(\sigma)^2 = N\xi(1)$ for all $\sigma \in \Sigma_N$, the Eq. (1.41) implies that

$$\mathbb{P}\big(|X_N - \mathbb{E}X_N| \geq x\big) \leq 2\exp\Big(-\frac{Nx^2}{4\xi(1)}\Big). \tag{1.42}$$

Such concentration inequalities play an indispensable role in the analysis of the Sherrington–Kirkpatrick model, for example, in the proof of the Ghirlanda–Guerra identities in Chap. 3. Another very important example of application will appear in the next section, where Theorem 1.2 will be used to prove that the Aizenman–Sims–Starr representation can be approximated by continuous functions of the distribution of the overlap array.

Proof (Theorem 1.2). It is sufficient to prove Eq. (1.41) in the case when the measure G is supported on finitely many points, since the general case follows by approximation. Indeed, if $A_m \subseteq \Sigma$ is the set of points corresponding to the m largest weights of G then, by Eq. (1.39),

$$\begin{aligned}\mathbb{E}\Big|\sum_{\sigma\in\Sigma}\exp(g(\sigma))G(\sigma) - \sum_{\sigma\in A_m}\exp(g(\sigma))G(\sigma)\Big| \\ = \mathbb{E}\sum_{\sigma\in A_m^c}\exp(g(\sigma))G(\sigma) \leq e^{a/2}G(A_m^c).\end{aligned}$$

Therefore, if similarly to Eq. (1.38) we define

$$X_m = \log\sum_{\sigma\in A_m}\exp(g(\sigma))G(\sigma),$$

then we can choose a subsequence along which X_m converges to X almost surely. The fact that $\mathbb{E}X_m$ converges to $\mathbb{E}X$ along the same subsequence follows from the dominated convergence theorem and the condition (1.39), since, if we denote by σ_0 the point corresponding to the largest weight of the measure G, we can estimate

$$g(\sigma_0) + \log G(\sigma_0) \leq X_m \leq \sum_{\sigma\in\Sigma}\exp(g(\sigma))G(\sigma).$$

Therefore, if the concentration inequality (1.41) holds for X_m then it also holds for X, by taking the limit along the above subsequence. From now on, we assume that the measure G is supported on finitely many points. Let $g_1(\sigma)$ and $g_2(\sigma)$ be two

independent copies of the process $g(\sigma)$. For $0 \le t \le 1$, we consider two interpolating Hamiltonians for $j = 1, 2$,

$$g_{j,t}(\sigma) = \sqrt{t} g_j(\sigma) + \sqrt{1-t} g(\sigma), \tag{1.43}$$

and define the corresponding analogues of Eq. (1.38) by

$$F_j = \log \sum_{\sigma \in \Sigma} \exp(g_{j,t}(\sigma)) G(\sigma). \tag{1.44}$$

We will keep the dependence of F_j on t implicit. For $0 < t < 1$, let us denote

$$d_{j,t}(\sigma) = \frac{\partial g_{j,t}(\sigma)}{\partial t} = \frac{1}{2\sqrt{t}} g_j(\sigma) - \frac{1}{2\sqrt{1-t}} g(\sigma).$$

Let us immediately observe that if we denote $C(\sigma, \rho) = \mathbb{E} g(\sigma) g(\rho)$ then

$$\mathbb{E} d_{1,t}(\sigma) g_{2,t}(\rho) = \mathbb{E} d_{2,t}(\sigma) g_{1,t}(\rho) = -\frac{1}{2} C(\sigma, \rho) \tag{1.45}$$

and

$$\mathbb{E} d_{j,t}(\sigma) g_{j,t}(\rho) = \frac{1}{2} \big(C(\sigma, \rho) - C(\sigma, \rho) \big) = 0. \tag{1.46}$$

Let us fix $\lambda \ge 0$ and consider the function $\varphi(t) = \mathbb{E} \exp \lambda (F_1 - F_2)$. Since

$$\frac{\partial F_j}{\partial t} = \exp(-F_j) \sum_{\sigma \in \Sigma} d_{j,t}(\sigma) \exp(g_{j,t}(\sigma)) G(\sigma),$$

the derivative of $\varphi(t)$ for $0 < t < 1$ can be written as

$$\begin{aligned} \varphi'(t) = {} & \lambda \mathbb{E} \exp(\lambda (F_1 - F_2) - F_1) \sum_{\sigma \in \Sigma} d_{1,t}(\sigma) \exp(g_{1,t}(\sigma)) G(\sigma) \\ & - \lambda \mathbb{E} \exp(\lambda (F_1 - F_2) - F_2) \sum_{\sigma \in \Sigma} d_{2,t}(\sigma) \exp(g_{2,t}(\sigma)) G(\sigma). \end{aligned} \tag{1.47}$$

Let us compute this derivative using the covariance formulae (1.45) and (1.46) and the Gaussian integration by parts formula (1.25). Let us fix one $\sigma \in \Sigma$ and consider the quantity from the first sum in Eq. (1.47), $\mathbb{E} d_{1,t}(\sigma) F$, where

$$F = \lambda \exp\Big(\lambda (F_1 - F_2) - F_1 + g_{1,t}(\sigma) \Big) G(\sigma).$$

By the definition (1.44), it is obvious that F, viewed as a function of the vector $(g_{j,t}(\rho))_{\rho \in \Sigma, j=1,2}$, has at most exponential growth with respect to the norm of this vector. It is easy to check that all the partial derivatives of F also have at most exponential growth and, therefore, all the conditions in the proof of the Gaussian integration by parts formula (1.25) are satisfied. Since the covariance in Eq. (1.46)

is zero, to compute $\mathbb{E}d_{1,t}(\sigma)F$, we only need to compute the partial derivatives with respect to $g_{2,t}(\rho)$:

$$\begin{aligned}\frac{\partial F}{\partial g_{2,t}(\rho)} &= -\lambda F \frac{\partial F_2}{\partial g_{2,t}(\rho)} = -\lambda F \exp(-F_2)\exp(g_{2,t}(\rho))G(\rho) \\ &= -\lambda^2 \exp\big(\lambda(F_1 - F_2) - F_1 - F_2 + g_{1,t}(\sigma) + g_{2,t}(\rho)\big)G(\sigma)G(\rho),\end{aligned}$$

and, therefore, using the covariance formula in Eq. (1.45) in the Gaussian integration by parts formula (1.25), $\mathbb{E}d_{1,t}(\sigma)F$ is equal to

$$\frac{\lambda^2}{2}\sum_{\rho\in\Sigma} C(\sigma,\rho)\mathbb{E}\exp\big(\lambda(F_1 - F_2) - F_1 - F_2 + g_{1,t}(\sigma) + g_{2,t}(\rho)\big)G(\sigma)G(\rho).$$

Summing over $\sigma \in \Sigma$ implies that the first term in Eq. (1.47) is equal to

$$\frac{\lambda^2}{2}\sum_{\sigma,\rho\in\Sigma} C(\sigma,\rho)\mathbb{E}\exp\big(\lambda(F_1 - F_2) - F_1 - F_2 + g_{1,t}(\sigma) + g_{2,t}(\rho)\big)G(\sigma)G(\rho).$$

The second term in Eq. (1.47) can be computed similarly and one can easily check that it is exactly the same. This means that the derivative in Eq. (1.47) equals

$$\lambda^2\sum_{\sigma,\rho\in\Sigma} C(\sigma,\rho)\mathbb{E}\exp\big(\lambda(F_1 - F_2) - F_1 - F_2 + g_{1,t}(\sigma) + g_{2,t}(\rho)\big)G(\sigma)G(\rho).$$

By the assumption (1.39), $|C(\sigma,\rho)| \le a$. Therefore, $\varphi'(t)$ is bounded by

$$a\lambda^2\sum_{\sigma,\rho\in\Sigma} \mathbb{E}\exp\big(\lambda(F_1 - F_2) - F_1 - F_2 + g_{1,t}(\sigma) + g_{2,t}(\rho)\big)G(\sigma)G(\rho)$$

and, recalling the definition (1.44), this bound is equal to $a\lambda^2\varphi(t)$. Therefore,

$$\big(\exp(-a\lambda^2 t)\varphi(t)\big)' = \exp(-a\lambda^2 t)\big(\varphi'(t) - a\lambda^2\varphi(t)\big) \le 0$$

and $\varphi(1) \le \varphi(0)\exp(a\lambda^2)$. By construction, $F_1 = F_2$ for $t = 0$, which means that $\varphi(0) = 1$. On the other hand, $\varphi(1) = \mathbb{E}\exp\lambda(X_1 - X_2)$ where X_1 and X_2 are two independent copies of X. Finally, by Jensen's inequality,

$$\mathbb{E}\exp\lambda(X - \mathbb{E}X) \le \mathbb{E}\exp\lambda(X_1 - X_2) \le \exp(a\lambda^2)$$

and, by Chebyshev's inequality,

$$\mathbb{P}\big(X - \mathbb{E}X \ge x\big) \le \inf_{\lambda\ge 0}\exp(a\lambda^2 - \lambda x) = \exp\big(-x^2/4a\big).$$

A similar inequality can be written for $\mathbb{E}X - X$ and, therefore, Eq. (1.41) holds. Integrating this tail bound,

$$\mathbb{E}(X-\mathbb{E}X)^2 = \int_0^\infty \mathbb{P}\big((X-\mathbb{E}X)^2 \geq x^2\big)\, d(x^2) \leq \int_0^\infty 2e^{-y/4a}\, dy = 8a,$$

finishes the proof. □

1.3 The Asymptotic Gibbs Measures

In this section, we will describe another, more conceptual, connection between the free energy and the Gibbs measure, which will also serve as a motivation for a definition of the asymptotic Gibbs measure. First, we will carry out a computation that will provide a direct link between the free energy $F_N(\beta)$ and the Gibbs measure G_N. This is the so-called *Aizenman–Sims–Starr scheme*. For simplicity, we will work with the Sherrington–Kirkpatrick model (1.2) and postpone the case of the mixed p-spin models until Chap. 3. Since the inverse temperature parameter β will always be fixed, we will make the dependence of the partition function Z_N and free energy F_N in Eq. (1.8) on β implicit. For $j \geq 0$, let us denote

$$A_j = \mathbb{E}\log Z_{j+1} - \mathbb{E}\log Z_j, \tag{1.48}$$

with the convention that $Z_0 = 1$, and rewrite the free energy as follows:

$$F_N = \frac{1}{N}\mathbb{E}\log Z_N = \frac{1}{N}\sum_{j=0}^{N-1} A_j. \tag{1.49}$$

Clearly, this representation implies that if the sequence A_N converges then its limit is also the limit of the free energy F_N. Unfortunately, we will not be able to prove that the limit of A_N exists and, in the sequel, will only use this representation to obtain a lower bound on the free energy in the limit,

$$\liminf_{N\to\infty} F_N \geq \liminf_{N\to\infty} A_N.$$

However, this representation will provide a crucial connection to the Gibbs measure if we rewrite one term, say A_N, in the following way. Let us compare the partition functions Z_N and Z_{N+1} and see what they have in common and what makes them different. If we denote $\rho = (\sigma,\varepsilon) \in \Sigma_{N+1}$ for $\sigma \in \Sigma_N$ and $\varepsilon \in \{-1,+1\}$ then we can write

$$H_{N+1}(\rho) = H'_N(\sigma) + \varepsilon z_N(\sigma) + \frac{1}{N+1} g_{(N+1)(N+1)}, \tag{1.50}$$

where

$$H'_N(\sigma) = \frac{1}{\sqrt{N+1}} \sum_{i,j=1}^{N} g_{ij}\sigma_i\sigma_j \tag{1.51}$$

and

$$z_N(\sigma) = \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N} \big(g_{i(N+1)} + g_{(N+1)i} \big) \sigma_i. \tag{1.52}$$

One the other hand, the part (1.51) of the Hamiltonian $H_{N+1}(\rho)$ is, in some sense, also a part of the Hamiltonian $H_N(\sigma)$ since, in distribution, the Gaussian process $H_N(\sigma)$ can be decomposed into a sum of two independent Gaussian processes

$$H_N(\sigma) \stackrel{d}{=} H'_N(\sigma) + y_N(\sigma), \tag{1.53}$$

where

$$y_N(\sigma) = \frac{1}{\sqrt{N(N+1)}} \sum_{i,j=1}^{N} g'_{ij} \sigma_i \sigma_j \tag{1.54}$$

for some independent array (g'_{ij}) of standard Gaussian random variables. Using the above decompositions (1.50) and (1.53), we can write

$$\mathbb{E} \log Z_{N+1} = \mathbb{E} \log \sum_{\sigma \in \Sigma_N} 2\mathrm{ch}(\beta z_N(\sigma)) \exp \beta H'_N(\sigma) \tag{1.55}$$

and

$$\mathbb{E} \log Z_N = \mathbb{E} \log \sum_{\sigma \in \Sigma_N} \exp(\beta y_N(\sigma)) \exp \beta H'_N(\sigma). \tag{1.56}$$

Finally, if we consider the Gibbs measure on Σ_N corresponding to the Hamiltonian $H'_N(\sigma)$ in Eq. (1.51),

$$G'_N(\sigma) = \frac{\exp \beta H'_N(\sigma)}{Z'_N} \text{ where } Z'_N = \sum_{\sigma \in \Sigma_N} \exp \beta H'_N(\sigma), \tag{1.57}$$

and denote the average with respect to the product measure $G'^{\otimes \infty}_N$ by $\langle \cdot \rangle'$, then Eqs. (1.55) and (1.56) can be combined to give the *Aizenman–Sims–Starr representation*,

$$A_N = \mathbb{E} \log \Big\langle 2\mathrm{ch}\big(\beta z_N(\sigma)\big) \Big\rangle' - \mathbb{E} \log \Big\langle \exp\big(\beta y_N(\sigma)\big) \Big\rangle'. \tag{1.58}$$

The method by which it was obtained is an example of what physicists might call a *cavity computation* since it was based on the comparison of the system with N coordinates to the system with an additional *cavity coordinate* ε. This representation will play a crucial role in the computation of the limit of the free energy, but it will take us a lot of work and a number of important ideas to get from Eq. (1.58) to an actual formula. At this moment, we will only use it to motivate and outline the work that will prepare us to prove many of the central predictions of the physicists in the Sherrington–Kirkpatrick model, such as the Parisi formula for the free energy.

First of all, the fact that the Gibbs measure G'_N is defined in terms of a slightly different Hamiltonian than the original Gibbs measure G_N is not so important,

because we can just as well study the Gibbs measure G'_N. What is important is that the Gaussian processes $z_N(\sigma)$ and $y_N(\sigma)$ are independent of the randomness of the measure G'_N and their covariances

$$\mathbb{E}z_N(\sigma^1)z_N(\sigma^2) = 2a_N R_{1,2} \text{ and } \mathbb{E}y_N(\sigma^1)y_N(\sigma^2) = a_N R_{1,2}^2 \tag{1.59}$$

for $a_N = N/(N+1)$ are, again, functions of the overlap $R_{1,2}$, which, of course, was a direct consequence of the covariance structure (1.3) of the Hamiltonian $H_N(\sigma)$. As we already mentioned before, this has the following key implication. Since the covariance operators (1.59) and, therefore, the distributions of the processes $z_N(\sigma)$ and $y_N(\sigma)$ are not affected by an orthogonal transformation U of the index set Σ_N, replacing the average with respect to G'_N in Eq. (1.58) by the average with respect to $G'_N \circ U^{-1}$ does not change A_N. This means that, in some sense, we only need to understand how to compute the Gibbs averages asymptotically up to orthogonal transformations. One way to encode the information about the Gibbs measure up to orthogonal transformations is as follows. For simplicity of notation, we will talk about the Gibbs measure G_N, but, of course, everything we say here also applies to the Gibbs measure G'_N. Let us consider an infinite sequence $(\sigma^l)_{l\geq 1}$ of i.i.d. random variables sampled from the measure G_N, called *replicas*, and consider the normalized Gram matrix of the overlaps

$$R = \big(R_{l,l'}\big)_{l,l'\geq 1} = \frac{1}{N}\big(\sigma^l \cdot \sigma^{l'}\big)_{l,l'\geq 1}. \tag{1.60}$$

It is easy to see that, given R, one can reconstruct the Gibbs measure G_N up to orthogonal transformations. This will be proved in a more general situation in Lemma 1.7, but in the present case it is almost obvious, because every time we observe an entry $R_{l,l'} = 1$ it means that $\sigma^l = \sigma^{l'}$. This way we can partition all replicas (σ^l) into 2^N groups and then use the law of large numbers to estimate their Gibbs weights from the frequencies of these groups in large blocks of the overlap matrix R. Of course, the overlaps only describe a relative position of points in $\mathbb{R}^N$ up to orthogonal transformations and, therefore, the Gram matrix R encodes the information about the Gibbs measure G_N up to orthogonal transformations. This information is sufficient if we want to compute the Gibbs averages $\langle\cdot\rangle$ of functions that depend on replicas (σ^l) only through their overlaps $(R_{l,l'})$. Formally, the Gibbs averages in Eq. (1.58) are not of this type, but, because of the special form of the covariances of the Gaussian processes $(z_N(\sigma))$ and $(y_N(\sigma))$, Eq. (1.58) can be approximated by functions of the overlap array R, as will be shown in the following result.

Consider an arbitrary atomic probability measure Γ on the unit sphere of a separable Hilbert space H. Let $\langle\cdot\rangle_\Gamma$ denote the average with respect to the product measure $\Gamma^{\otimes\infty}$, let (ρ^l) denote an i.i.d. sample from Γ, and let

$$Q = \big(Q_{l,l'}\big)_{l,l'\geq 1} = \big(\rho^l \cdot \rho^{l'}\big)_{l,l'\geq 1} \tag{1.61}$$

be the Gram matrix of this sample. Notice that, because the measure Γ is on the unit sphere, the scalar products $\rho^l \cdot \rho^{l'} \in [-1,1]$. Let $(z(\rho))$ and $(y(\rho))$ be two Gaussian

processes indexed by the points in the support of Γ with the covariance

$$\mathbb{E}z(\rho^1)z(\rho^2) = C_z\big(\rho^1\cdot\rho^2\big),\ \mathbb{E}y(\rho^1)y(\rho^2) = C_y\big(\rho^1\cdot\rho^2\big), \tag{1.62}$$

for some bounded continuous functions C_z, C_y on the interval $[-1,1]$. Finally, let us consider the quantity

$$\Phi(\Gamma) = \mathbb{E}\log\Big\langle 2\mathrm{ch}z(\rho)\Big\rangle_\Gamma - \mathbb{E}\log\Big\langle \exp y(\rho)\Big\rangle_\Gamma. \tag{1.63}$$

We will now show that, because of the special form of the covariance (1.62) of the Gaussian processes $(z(\rho))$ and $(y(\rho))$, the functional $\Phi(\Gamma)$ satisfies the following approximation property.

Theorem 1.3. *For each $\varepsilon>0$, there exists a continuous bounded function $F_\varepsilon(Q)$ of finitely many coordinates of the Gram matrix $(Q_{l,l'})$, that depends only on ε and the functions C_z, C_y in Eq. (1.62), such that*

$$\big|\Phi(\Gamma) - \big\langle F_\varepsilon(Q)\big\rangle_\Gamma\big| \le \varepsilon \tag{1.64}$$

uniformly over all possible choices of the measure Γ as above.

Proof. For simplicity of notation, we will write $\langle\cdot\rangle$ instead of $\langle\cdot\rangle_\Gamma$. We will only consider the first term in Eq. (1.63) since the second term can be treated similarly. Recall that the measure Γ is nonrandom. Given $a>0$, let us denote by

$$\log_a x = \max\big(-a, \min(\log x, a)\big),\ \mathrm{ch}_a x = \min\big(\mathrm{ch}x, \mathrm{ch}a\big),$$

the truncated versions of the logarithm and hyperbolic cosine. Using the Gaussian concentration inequality in Theorem 1.2, we will show below that for some small enough $c>0$ and for all large enough $a>0$,

$$\big|\mathbb{E}\log\big\langle \mathrm{ch}z(\rho)\big\rangle - \mathbb{E}\log_a\big\langle \mathrm{ch}_a z(\rho)\big\rangle\big| \le \exp\big(-ca^2\big). \tag{1.65}$$

First of all, by taking a sufficiently large, we can make this bound smaller than $\varepsilon/2$. Then, since $1 \le \langle \mathrm{ch}_a z(\rho)\rangle \le e^a$, approximating the logarithm by polynomials uniformly on the interval $[1, e^a]$, the quantity $\mathbb{E}\log_a\langle \mathrm{ch}_a z(\rho)\rangle$ can be approximated within $\varepsilon/2$ by some linear combinations of moments

$$\mathbb{E}\big\langle \mathrm{ch}_a z(\rho)\big\rangle^r = \mathbb{E}\Big\langle \prod_{l\le r} \mathrm{ch}_a z(\rho^l)\Big\rangle = \Big\langle \mathbb{E}\prod_{l\le r} \mathrm{ch}_a z(\rho^l)\Big\rangle. \tag{1.66}$$

Since $\mathrm{ch}_a x$ is continuous and bounded and since, by Eq. (1.62), the covariance matrix of the Gaussian vector $(z(\rho^l))_{l\le r}$ is given by $(C_z(Q_{l,l'}))_{l,l'\le r}$ for some continuous function C_z, we can write

$$\mathbb{E}\prod_{l\le r} \mathrm{ch}_a z(\rho^l) = F_r\big((Q_{l,l'})_{l,l'\le r}\big) \tag{1.67}$$

for some continuous bounded function F_r of the scalar products $(Q_{l,l'})_{l,l'\le r}$ (recall that convergence of the covariance matrix implies convergence of a Gaussian vector in distribution). Together with Eq. (1.65), this shows that we can approximate the first term in Eq. (1.63) within ε by some finite linear combination of $\langle F_r(Q)\rangle$, where each F_r is a continuous bounded function of finitely many coordinates of Q that does not depend on the choice of the measure Γ. It remains to prove Eq. (1.65). This will be done by controlling two terms separately:

$$\big|\mathbb{E}\log\langle \mathrm{ch} z(\rho)\rangle - \mathbb{E}\log_a\langle \mathrm{ch} z(\rho)\rangle\big|,\ \big|\mathbb{E}\log_a\langle \mathrm{ch} z(\rho)\rangle - \mathbb{E}\log_a\langle \mathrm{ch}_a z(\rho)\rangle\big|. \quad (1.68)$$

Let us begin with the first term. Since the support of Γ belongs to the unit sphere, by Eq. (1.62), $\mathbb{E}z(\rho)^2 = C_z(1)$ for all ρ in the support of Γ. If we write $\log\langle \mathrm{ch} z(\rho)\rangle$ as $\log\langle \mathrm{Av}_\varepsilon \exp\varepsilon z(\rho)\rangle$, where Av_ε denotes the average over $\varepsilon\in\{-1,1\}$, then the Gaussian concentration inequality in Theorem 1.2 implies that

$$\mathbb{P}\big(\big|\log\langle \mathrm{ch} z(\rho)\rangle - \mathbb{E}\log\langle \mathrm{ch} z(\rho)\rangle\big|\ge a\big) \le 2\exp\big(-ca^2\big) \quad (1.69)$$

for some constant c that depends only on $C_z(1)$. Since

$$0 \le \mathbb{E}\log\langle \mathrm{ch} z(\rho)\rangle \le \log\langle \mathbb{E}\mathrm{ch} z(\rho)\rangle = C_z(1)/2,$$

we can write the following inclusion for $a \ge C_z(1)$,

$$\Big\{\big|\log\langle \mathrm{ch} z(\rho)\rangle\big|\ge a\Big\} \subseteq \Big\{\big|\log\langle \mathrm{ch} z(\rho)\rangle - \mathbb{E}\log\langle \mathrm{ch} z(\rho)\rangle\big|\ge a/2\Big\},$$

and, therefore, the inequality (1.69) implies that for some small enough $c>0$ and for all large enough $a>0$,

$$\mathbb{P}\big(\big|\log\langle \mathrm{ch} z(\rho)\rangle\big|\ge a\big) \le \exp\big(-ca^2\big). \quad (1.70)$$

If the tail of a random variable $\xi\ge 0$ satisfies $\mathbb{P}(\xi\ge a)\le \exp(-ca^2)$ for all large enough $a>0$ then its average over the tail can be bounded as follows:

$$\begin{aligned}
\mathbb{E}\xi I(\xi\ge a) &= \mathbb{E}\int_0^\infty I\big(\xi\ge s\vee a\big)\,ds = \int_0^\infty \mathbb{P}\big(\xi\ge s\vee a\big)\,ds \\
&= a\mathbb{P}(\xi\ge a) + \int_a^\infty \mathbb{P}\big(\xi\ge s\big)\,ds \\
&\le a\exp\big(-ca^2\big) + \frac{1}{a}\int_a^\infty s\exp(-cs^2)\,ds \\
&= \Big(a+\frac{1}{2ca}\Big)\exp(-ca^2) \le \exp\Big(-\frac{ca^2}{2}\Big),
\end{aligned}$$

for large enough $a>0$. Therefore, by Eq. (1.70), the first term in Eq. (1.68) is bounded by

$$\mathbb{E}\big|\log\langle \mathrm{ch} z(\rho)\rangle\big| I\big(\big|\log\langle \mathrm{ch} z(\rho)\rangle\big|\ge a\big) \le \exp\big(-ca^2\big) \quad (1.71)$$

for some small enough $c > 0$ and for all large enough $a > 0$. Next, using that

$$\big|\log_a x - \log_a y\big| \le e^a |x-y| \text{ and } \big|\mathrm{ch}x - \mathrm{ch}_a x\big| \le \mathrm{ch}x I\big(|x| \ge a\big),$$

we can bound the second term in Eq. (1.68) by

$$e^a \mathbb{E}\big\langle \big|\mathrm{ch}z(\rho) - \mathrm{ch}_a z(\rho)\big|\big\rangle \le e^a \big\langle \mathbb{E}\mathrm{ch}z(\rho) I\big(|z(\rho)| \ge a\big)\big\rangle. \tag{1.72}$$

By Hölder's inequality, the expectation inside $\langle\cdot\rangle$ can be bounded by

$$\mathbb{E}\mathrm{ch}z(\rho) I\big(|z(\rho)| \ge a\big) \le \big(\mathbb{E}\mathrm{ch}^2 z(\rho)\big)^{1/2} \mathbb{P}\big(|z(\rho)| \ge a\big)^{1/2},$$

which is, obviously, bounded by $\exp\big(-ca^2\big)$ for some small enough $c > 0$ and for all large enough $a > 0$. Combining this with Eq. (1.71) proves Eq. (1.65). □

To represent the quantity A_N in Eq. (1.58), obtained by the Aizenman–Sims–Starr scheme, in the form (1.63), we should simply redefine the measure G'_N in Eq. (1.57) as a measure on the subset $N^{-1/2}\Sigma_N$ of the unit sphere by the map $\pi(\sigma) = N^{-1/2}\sigma$ and let $G_N^\pi = G'_N \circ \pi^{-1}$. Since the overlap $R_{1,2} = N^{-1}\sigma^1 \cdot \sigma^2$ is just the scalar product $\pi(\sigma^1) \cdot \pi(\sigma^2)$ of the rescaled configurations, the covariance formulae in Eq. (1.59) are of the type (1.62) under this definition. To ensure that F_ε in Eq. (1.64) does not depend on N, the factor $a_N = N/(N+1)$ in Eq. (1.59) can be replaced by 1 without affecting A_N much, which will be shown in Chap. 3. The bound (1.64) holds uniformly over all measures Γ and, therefore, if we first apply Eq. (1.64) conditionally on the randomness of the measure G_N^π and then average with respect to this randomness, we get

$$\Big|A_N - \mathbb{E}\Big\langle F_\varepsilon(Q)\Big\rangle_{G_N^\pi}\Big| = \Big|A_N - \mathbb{E}\Big\langle F_\varepsilon(R)\Big\rangle'\Big| \le \varepsilon. \tag{1.73}$$

Since F_ε is a continuous bounded function of finitely many scalar products, or overlaps, $\mathbb{E}\langle F_\varepsilon(R)\rangle'$ is a continuous function of the distribution of the overlap array R under the average Gibbs measure $\mathbb{E}G_N'^{\otimes\infty}$, which takes into account both sources of randomness—the randomness of the Gibbs measure G'_N coming from the random Hamiltonian and the randomness of sampling from this measure. Continuity here is with respect to the topology of weak convergence of all finite dimensional distributions of the overlap array R.

For all the above reasons, the question of understanding the structure of the Gibbs measure in the Sherrington–Kirkpatrick model is often identified with the problem of describing the distribution of the overlap matrix R in Eq. (1.60) under $\mathbb{E}G_N^{\otimes\infty}$. This point of view turns out to be very beneficial for another reason—it allows us to define and work with some asymptotic objects rather than with a fixed finite system of size 2^N. Namely, if we make the dependence of the overlap array (1.60) on N explicit, R^N, since its elements are bounded, by the selection theorem, every sequence has a subsequence along which R^N converges in distribution. Usually, it is not clear whether the entire sequence R^N converges in distribution (one such example will be given in Corollary 3.3), but if we can show that all possible limits

over subsequences have some common structure, then, often, the entire sequence will have this structure asymptotically. On the other hand, working with asymptotic distributions will allow us to use some properties of the overlap matrix that hold approximately for a system of finite size and become exact only in the thermodynamic limit. For example, in Chap. 3, we will prove that the distribution of the overlap matrix R^N satisfies the so-called Ghirlanda–Guerra identities in some approximate sense, but in the limit they become exact, resulting in far-reaching consequences.

This raises another very important question, namely, what happens to the Gibbs measure in the limit? The overlap matrix Eq. (1.60) was generated as a normalized Gram matrix R^N of a sequence of replicas (σ^l) drawn from a random measure, G_N. We consider its distribution under the average Gibbs measure $\mathbb{E}G_N^{\otimes\infty}$. Then, we consider a limit of these distributions over a subsequence and obtain some distribution of an infinite matrix R. This matrix inherits some basic properties of R^N, for example, with probability one, it will still be positive definite. Clearly, it also inherits some basic symmetry,

$$\left(R_{\pi(l),\pi(l')}\right)_{l,l'\geq 1} \stackrel{d}{=} \left(R_{l,l'}\right)_{l,l'\geq 1}, \tag{1.74}$$

for any permutation π of finitely many indices, where equality is in distribution. However, the Gibbs measure disappears in the limit, at least temporarily. It turns out that these two simple properties inherited by the limiting distribution guarantee the existence of a random measure G on the unit ball of a separable Hilbert space H such that

$$\left(R_{l,l'}\right)_{l\neq l'} \stackrel{d}{=} \left(\sigma^l \cdot \sigma^{l'}\right)_{l\neq l'}, \tag{1.75}$$

where (σ^l) is an i.i.d. sequence sampled from the measure G. A random measure $G(\omega, A)$ is a function of a generic point ω in some probability space and a Borel set A in H such that, for a fixed ω, $G(\omega,\cdot)$ is a probability measure and, for a fixed A, $G(\cdot, A)$ is a measurable function on that probability space. The fact that we can generate the asymptotic overlap array as in Eq. (1.75) is a consequence of the Dovbysh–Sudakov representation proved in Theorem 1.7 below. We will call such measures G the *asymptotic Gibbs measures*. One can think of them as the limits of the sequence G_N over subsequences if the convergence is defined by way of the overlap arrays, as above. The reason why we do not include the diagonal elements in Eq. (1.75) is because they were equal to one in Eq. (1.60), while the asymptotic Gibbs measure will not necessarily be concentrated on the unit sphere. The existence of the asymptotic Gibbs measures in the Dovbysh–Sudakov representation will play an indispensable role in the analysis of the Sherrington–Kirkpatrick model, most importantly, because their geometric nature as random measures on a Hilbert space will be very useful in the proofs of several key results. In the last two sections of this chapter, we will prove the Dovbysh–Sudakov representation (1.75). As a first step, we will prove the Aldous–Hoover representation for exchangeable arrays. This famous representation has many other applications, for example, it is used in Chap. 4 to define the asymptotic Gibbs measures in mean-field spin glass models which do not have the rotation-invariance properties of the SK model.

1.4 The Aldous–Hoover Representation

Let us consider an infinite random array $s = (s_{l,l'})_{l,l' \geq 1}$. The array s is called an *exchangeable array* if for any permutations π and ρ of finitely many indices we have equality in distribution,

$$\big(s_{\pi(l),\rho(l')}\big)_{l,l' \geq 1} \stackrel{d}{=} \big(s_{l,l'}\big)_{l,l' \geq 1}, \tag{1.76}$$

in the sense that their finite dimensional distributions are equal. Here is one natural example of an exchangeable array. Given a measurable function $\sigma : [0,1]^4 \to \mathbb{R}$ and sequences of i.i.d. random variables w, (u_l), $(v_{l'})$, $(x_{l,l'})$ that have the uniform distribution on $[0,1]$, the array

$$\big(\sigma(w, u_l, v_{l'}, x_{l,l'})\big)_{l,l' \geq 1} \tag{1.77}$$

is, obviously, exchangeable. It turns out that all exchangeable arrays are of this form.

Theorem 1.4 (Aldous–Hoover). *Any infinite exchangeable array $(s_{l,l'})_{l,l' \geq 1}$ is equal in distribution to Eq. (1.77) for some function σ.*

Another version of the Aldous–Hoover representation holds in the symmetric case. A symmetric array s is called *weakly exchangeable* if for any permutation π of finitely many indices we have equality in distribution:

$$\big(s_{\pi(l),\pi(l')}\big)_{l,l' \geq 1} \stackrel{d}{=} \big(s_{l,l'}\big)_{l,l' \geq 1}. \tag{1.78}$$

For example, the overlap matrix in Eq. (1.74) is weakly exchangeable. Because of the symmetry, we can consider only half of the array indexed by (l,l') such that $l \leq l'$ or, equivalently, consider the array indexed by sets $\{l,l'\}$ and rewrite Eq. (1.78) as

$$\big(s_{\{\pi(l),\pi(l')\}}\big)_{l,l' \geq 1} \stackrel{d}{=} \big(s_{\{l,l'\}}\big)_{l,l' \geq 1}. \tag{1.79}$$

Notice that, compared to Eq. (1.76), the diagonal elements now play somewhat different roles from the rest of the array. One natural example of a weakly exchangeable array is given by

$$s_{\{l,l\}} = g(w, u_l) \text{ and } s_{\{l,l'\}} = f\big(w, u_l, u_{l'}, x_{\{l,l'\}}\big) \text{ for } l \neq l', \tag{1.80}$$

for any measurable functions $g : [0,1]^2 \to \mathbb{R}$ and $f : [0,1]^4 \to \mathbb{R}$, which is symmetric in its middle two coordinates $u_l, u_{l'}$, and i.i.d. random variables w, (u_l), $(x_{\{l,l'\}})$ with the uniform distribution on $[0,1]$. Again, it turns out that such examples cover all possible weakly exchangeable arrays.

Theorem 1.5 (Aldous–Hoover). *Any infinite weakly exchangeable array is equal in distribution to the array (1.80) for some functions g and f, which is symmetric in its two middle coordinates.*

We will only prove Theorem 1.5 since the proof of Theorem 1.4 is very similar. (Moreover, we will prove Theorem 1.4 in the Appendix A.1 using a very different approach based on the ideas in the framework of limits of dense graph sequences.) The proof of the Dovbysh–Sudakov representation in the next section will be based on Theorem 1.4, and another application of Theorem 1.4 will be given in Sect. 4.1. As a first step, we will prove the famous de Finetti representation for exchangeable sequences. A sequence $(s_l)_{l\geq 1}$ is called *exchangeable* if for any permutation π of finitely many indices we have equality in distribution

$$\big(s_{\pi(l)}\big)_{l\geq 1} \stackrel{d}{=} \big(s_l\big)_{l\geq 1}. \tag{1.81}$$

The following holds.

Theorem 1.6 (de Finetti). *If the sequence $(s_l)_{l\geq 1}$ is exchangeable then there exists a measurable function $g : [0,1]^2 \to \mathbb{R}$ such that*

$$\big(s_l\big)_{l\geq 1} \stackrel{d}{=} \big(g(w,u_l)\big)_{l\geq 1}, \tag{1.82}$$

where w and (u_l) are i.i.d. random variables with the uniform distribution on $[0,1]$.

Before we proceed to prove the above results, let us make a couple of important observations that will be used below. Recall that a measurable space $(\Omega, \mathscr{B})$ is called a *Borel space* if there exists a one-to-one function φ from Ω onto a Borel subset $A \subseteq [0,1]$ such that both φ and φ^{-1} are measurable. For example, it is well known that all complete separable metric spaces and their Borel subsets are Borel spaces (see, e.g., Sect. 13.1 in [23]). The existence of the isomorphism φ automatically implies that if we can prove the above results in the case when the elements of the sequence (s_l) or the array $(s_{\{l,l'\}})$ take values in $[0,1]$ then the same representation results hold when the elements take values in a Borel space. Similarly, other standard results for real-valued or $[0,1]$-valued random variables are often automatically extended to Borel spaces. For example, using the quantile transform, we can generate any real-valued random variable as a function of a uniform random variable on $[0,1]$ and, therefore, any random element on a Borel space can also be generated by a function of a uniform random variable on $[0,1]$. Let us describe another typical measure theoretic argument that will be used several times below.

Lemma 1.4 (Coding Lemma). *Suppose that a random pair (X,Y) takes values in the product of a measurable space $(\Omega_1, \mathscr{B}_1)$ and a Borel space $(\Omega_2, \mathscr{B}_2)$. Then, there exists a measurable function $f : \Omega_1 \times [0,1] \to \Omega_2$ such that*

$$(X,Y) \stackrel{d}{=} \big(X, f(X,u)\big), \tag{1.83}$$

where u is a uniform random variable on $[0,1]$ independent of X.

Rather than using the Coding Lemma itself we will often use the general ideas of conditioning and generating random variables as functions of uniform random variables on $[0,1]$ that are used in its proof, without repeating the same argument.

Proof. By the definition of a Borel space, one can easily reduce the general case to the case when $\Omega_2 = [0,1]$ equipped with the Borel σ-algebra. Since in this case the regular conditional distribution $\Pr(x,B)$ of Y given X exists (see, e.g., Theorem 10.2.2 in [23]), for a fixed $x \in \Omega_1$, we can define by $F(x,y) = \Pr(x,[0,y])$ the conditional distribution function of Y given x and by

$$f(x,u) = F^{-1}(x,u) = \inf\{q \in \mathbb{Q} \cap [0,1] : u \le F(x,q)\}$$

its quantile transformation. It is easy to see that f is measurable on the product space $\Omega_1 \times [0,1]$. If u is uniform on $[0,1]$ then, for a fixed $x \in \Omega_1$, $f(x,u)$ has the distribution $\Pr(x,\cdot)$ and this finishes the proof. □

The most important way in which the exchangeability condition (1.81) will be used is to say that for any infinite subset $I \subseteq \mathbb{N}$,

$$(s_l)_{l \in I} \stackrel{d}{=} (s_l)_{l \ge 1}. \tag{1.84}$$

Let us describe one immediate consequence of this simple observation. If we let

$$\mathscr{F}_I = \sigma(s_l : l \in I) \tag{1.85}$$

be the σ-algebra generated by the random variables s_l for $l \in I$ then the following holds.

Lemma 1.5. *For any infinite subset $I \subseteq \mathbb{N}$ and $j \notin I$, the conditional expectations*

$$\mathbb{E}(f(s_j)|\mathscr{F}_I) = \mathbb{E}(f(s_j)|\mathscr{F}_{\mathbb{N}\setminus\{j\}})$$

almost surely, for any bounded measurable function $f : \mathbb{R} \to \mathbb{R}$.

Proof. First, using the property (1.84) for $I \cup \{j\}$ instead of I implies the equality in distribution,

$$\mathbb{E}(f(s_j)|\mathscr{F}_I) \stackrel{d}{=} \mathbb{E}(f(s_j)|\mathscr{F}_{\mathbb{N}\setminus\{j\}}),$$

and, therefore, the equality of the L^2-norms,

$$\|\mathbb{E}(f(s_j)|\mathscr{F}_I)\|_2 = \|\mathbb{E}(f(s_j)|\mathscr{F}_{\mathbb{N}\setminus\{j\}})\|_2. \tag{1.86}$$

On the other hand, $I \subseteq \mathbb{N}\setminus\{j\}$ and $\mathscr{F}_I \subseteq \mathscr{F}_{\mathbb{N}\setminus\{j\}}$, which implies that

$$\mathbb{E}(\mathbb{E}(f(s_j)|\mathscr{F}_I)\mathbb{E}(f(s_j)|\mathscr{F}_{\mathbb{N}\setminus\{j\}})) = \mathbb{E}(\mathbb{E}(f(s_j)|\mathscr{F}_I)^2) = \|\mathbb{E}(f(s_j)|\mathscr{F}_I)\|_2^2.$$

Combining this with Eq. (1.86) yields that

$$\|\mathbb{E}(f(s_j)|\mathscr{F}_I) - \mathbb{E}(f(s_j)|\mathscr{F}_{\mathbb{N}\setminus\{j\}})\|_2^2 = 0, \tag{1.87}$$

which finishes the proof. □

Proof (Theorem 1.6). Let us take any infinite subset $I \subseteq \mathbb{N}$ such that its complement $\mathbb{N} \setminus I$ is also infinite. By Eq. (1.84), we only need to prove the representation (1.82) for $(s_l)_{l\in I}$. First, we will show that, conditionally on $(s_l)_{l\in\mathbb{N}\setminus I}$, the random variables $(s_l)_{l\in I}$ are independent. This means that, given $n \geq 1$, any distinct $l_1, \dots, l_n \in I$, and any bounded measurable functions $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\Big(\prod_{j\leq n} f_j(s_{l_j})\Big|\mathscr{F}_{\mathbb{N}\setminus I}\Big) = \prod_{j\leq n}\mathbb{E}\big(f_j(s_{l_j})\big|\mathscr{F}_{\mathbb{N}\setminus I}\big), \tag{1.88}$$

or, in other words, for any $A \in \mathscr{F}_{\mathbb{N}\setminus I}$,

$$\mathbb{E}I_A\prod_{j\leq n} f_j(s_{l_j}) = \mathbb{E}I_A\prod_{j\leq n}\mathbb{E}\big(f_j(s_{l_j})\big|\mathscr{F}_{\mathbb{N}\setminus I}\big).$$

Since I_A and $f_1(s_{l_1}), \dots, f_{n-1}(s_{l_{n-1}})$ are $\mathscr{F}_{\mathbb{N}\setminus\{l_n\}}$-measurable, we can write

$$\begin{aligned}\mathbb{E}I_A\prod_{j\leq n} f_j(s_{l_j}) &= \mathbb{E}I_A\prod_{j\leq n-1} f_j(s_{l_j})\mathbb{E}\big(f_n(s_{l_n})\big|\mathscr{F}_{\mathbb{N}\setminus\{l_n\}}\big)\\ &= \mathbb{E}I_A\prod_{j\leq n-1} f_j(s_{l_j})\mathbb{E}\big(f_n(s_{l_n})\big|\mathscr{F}_{\mathbb{N}\setminus I}\big),\end{aligned}$$

where the second equality follows from Lemma 1.5. This implies that

$$\mathbb{E}\Big(\prod_{j\leq n} f_j(s_{l_j})\Big|\mathscr{F}_{\mathbb{N}\setminus I}\Big) = \mathbb{E}\Big(\prod_{j\leq n-1} f_j(s_{l_j})\Big|\mathscr{F}_{\mathbb{N}\setminus I}\Big)\mathbb{E}\big(f_n(s_{l_n})\big|\mathscr{F}_{\mathbb{N}\setminus I}\big)$$

and Eq. (1.88) follows by induction on n. Let us also observe that, because of Eq. (1.84), the distribution of the array $(s_l, (s_j)_{j\in\mathbb{N}\setminus I})$ does not depend on $l \in I$. This implies that, conditionally on $(s_j)_{j\in\mathbb{N}\setminus I}$, the random variables $(s_l)_{l\in I}$ are identically distributed in addition to being independent. The product space $[0,1]^\infty$ with the Borel σ-algebra corresponding to the product topology is a Borel space (recall that, equipped with the usual metric, it becomes a complete separable metric space). Therefore, in order to conclude the proof, it remains to generate $X = (s_l)_{l\in\mathbb{N}\setminus I}$ as a function of a uniform random variable w on $[0,1]$, $X = X(w)$, and then use the argument in the Coding Lemma 1.4 to generate the sequence $(s_l)_{l\in I}$ as $(f(X(w), u_l))_{l\in I}$, where $(u_l)_{l\in I}$ are i.i.d. random variables uniform on $[0,1]$. This finishes the proof. □

We are almost ready to prove the Aldous–Hoover representation in Theorem 1.5. The most important way in which the exchangeability condition (1.78) will be used is to say that, for any infinite subset $I \subseteq \mathbb{N}$,

$$\big(s_{\{l,l'\}}\big)_{l,l'\in I} \stackrel{d}{=} \big(s_{\{l,l'\}}\big)_{l,l'\geq 1}. \tag{1.89}$$

Again, one important consequence of this observation will be the following. Given $j, j' \in I$ such that $j \neq j'$, let us now define the σ-algebra

$$\mathscr{F}_I(j,j') = \sigma\big(s_{\{l,l'\}} : l,l' \in I, \{l,l'\} \neq \{j,j'\}\big). \tag{1.90}$$

In other words, this σ-algebra is generated by all elements $s_{\{l,l'\}}$ with both indices l and l' in I, excluding $s_{\{j,j'\}}$. The following analogue of Lemma 1.5 holds.

Lemma 1.6. *For any infinite subset $I \subseteq \mathbb{N}$ and any $j, j' \in I$ such that $j \neq j'$, the conditional expectations*

$$\mathbb{E}\big(f(s_{\{j,j'\}})\big|\mathscr{F}_I(j,j')\big) = \mathbb{E}\big(f(s_{\{j,j'\}})\big|\mathscr{F}_{\mathbb{N}}(j,j')\big)$$

almost surely, for any bounded measurable function $f : \mathbb{R} \to \mathbb{R}$.

Proof. The proof is almost identical to the proof of Lemma 1.5. The property (1.89) implies the equality in distribution,

$$\mathbb{E}\big(f(s_{\{j,j'\}})\big|\mathscr{F}_I(j,j')\big) \stackrel{d}{=} \mathbb{E}\big(f(s_{\{j,j'\}})\big|\mathscr{F}_{\mathbb{N}}(j,j')\big),$$

and, therefore, the equality of the L^2-norms,

$$\big\|\mathbb{E}\big(f(s_{\{j,j'\}})\big|\mathscr{F}_I(j,j')\big)\big\|_2 = \big\|\mathbb{E}\big(f(s_{\{j,j'\}})\big|\mathscr{F}_{\mathbb{N}}(j,j')\big)\big\|_2.$$

On the other hand, since $\mathscr{F}_I(j,j') \subseteq \mathscr{F}_{\mathbb{N}}(j,j')$, as in Eq. (1.87), this implies that

$$\big\|\mathbb{E}\big(f(s_{\{j,j'\}})\big|\mathscr{F}_I(j,j')\big) - \mathbb{E}\big(f(s_{\{j,j'\}})\big|\mathscr{F}_{\mathbb{N}}(j,j')\big)\big\|_2^2 = 0,$$

which finishes the proof. □

Proof (Theorem 1.5). Let us take an infinite subset $I \subseteq \mathbb{N}$ such that its complement $\mathbb{N} \setminus I$ is also infinite. By Eq. (1.89), we only need to prove the representation (1.80) for $(s_{\{l,l'\}})_{l,l' \in I}$. For each $j \in I$, let us consider the array

$$S_j = \big(s_{\{l,l'\}}\big)_{l,l' \in (\mathbb{N}\setminus I)\cup\{j\}} = \Big(s_{\{j,j\}}, \big(s_{\{j,l\}}\big)_{l \in \mathbb{N}\setminus I}, \big(s_{\{l,l'\}}\big)_{l,l' \in \mathbb{N}\setminus I}\Big). \tag{1.91}$$

It is obvious that the weak exchangeability (1.79) implies that the sequence $(S_j)_{j\in I}$ is exchangeable, since any permutation π of finitely many indices from I in the array (1.79) results in the corresponding permutation of the sequence $(S_j)_{j\in I}$. We can view each array S_j as an element of the Borel space $[0,1]^\infty$ and de Finetti's Theorem 1.6 implies that

$$\big(S_j\big)_{j\in I} \stackrel{d}{=} \big(X(w,u_j)\big)_{j\in I} \tag{1.92}$$

for some measurable function X on $[0,1]^2$ taking values in the space of such arrays. Next, we will show that, conditionally on the sequence $(S_j)_{j\in I}$, the off-diagonal elements $s_{\{j,j'\}}$ for $j, j' \in I, j \neq j'$, are independent and, moreover, the conditional distribution of $s_{\{j,j'\}}$ depends only on S_j and $S_{j'}$. This means that if we consider the σ-algebras

$$\mathscr{F} = \sigma\big((S_j)_{j\in I}\big),\ \mathscr{F}_{j,j'} = \sigma(S_j, S_{j'}) \text{ for } j, j' \in I, j \neq j', \tag{1.93}$$

then we would like to show that, for any finite set $\mathscr{C}$ of indices $\{l,l'\}$ for $l,l' \in I$ such that $l \neq l'$ and any bounded measurable functions $f_{l,l'}$ corresponding to the indices $\{l,l'\} \in \mathscr{C}$, we have

$$\mathbb{E}\Big(\prod_{\{l,l'\}\in\mathscr{C}} f_{l,l'}(s_{\{l,l'\}})\Big|\mathscr{F}\Big) = \prod_{\{l,l'\}\in\mathscr{C}} \mathbb{E}\big(f_{l,l'}(s_{\{l,l'\}})\big|\mathscr{F}_{l,l'}\big). \tag{1.94}$$

Notice that the definitions (1.90) and (1.93) imply that $\mathscr{F}_{j,j'} = \mathscr{F}_{(\mathbb{N}\setminus I)\cup\{j,j'\}}(j,j')$, since all the elements $s_{\{l,l'\}}$ with both indices in $(\mathbb{N}\setminus I)\cup\{j,j'\}$, except for $s_{\{j,j'\}}$, appear as one of the coordinates in the arrays S_j or $S_{j'}$. Therefore, by Lemma 1.6,

$$\mathbb{E}\big(f(s_{\{j,j'\}})\big|\mathscr{F}_{j,j'}\big) = \mathbb{E}\big(f(s_{\{j,j'\}})\big|\mathscr{F}_{\mathbb{N}}(j,j')\big). \tag{1.95}$$

Let us fix any $\{j,j'\} \in \mathscr{C}$, let $\mathscr{C}' = \mathscr{C}\setminus\{\{j,j'\}\}$ and consider an arbitrary set $A \in \mathscr{F}$. Since I_A and $f_{l,l'}(s_{\{l,l'\}})$ for $\{l,l'\} \in \mathscr{C}'$ are $\mathscr{F}_{\mathbb{N}}(j,j')$-measurable,

$$\begin{aligned}
&\mathbb{E} I_A \prod_{\{l,l'\}\in\mathscr{C}'} f_{l,l'}(s_{\{l,l'\}}) f_{j,j'}(s_{\{j,j'\}}) \\
&\quad= \mathbb{E}\Big(I_A \prod_{\{l,l'\}\in\mathscr{C}'} f_{l,l'}(s_{\{l,l'\}}) \mathbb{E}\big(f_{j,j'}(s_{\{j,j'\}})\big|\mathscr{F}_{\mathbb{N}}(j,j')\big)\Big) \\
&\quad= \mathbb{E}\Big(I_A \prod_{\{l,l'\}\in\mathscr{C}'} f_{l,l'}(s_{\{l,l'\}}) \mathbb{E}\big(f_{j,j'}(s_{\{j,j'\}})\big|\mathscr{F}_{j,j'}\big)\Big),
\end{aligned}$$

where the second equality follows from Eq. (1.95). Since $\mathscr{F}_{j,j'} \subseteq \mathscr{F}$, this implies that

$$\mathbb{E}\Big(\prod_{\{l,l'\}\in\mathscr{C}} f_{l,l'}(s_{\{l,l'\}})\Big|\mathscr{F}\Big) = \mathbb{E}\Big(\prod_{\{l,l'\}\in\mathscr{C}'} f_{l,l'}(s_{\{l,l'\}})\Big|\mathscr{F}\Big)\mathbb{E}\big(f_{j,j'}(s_{\{j,j'\}})\big|\mathscr{F}_{j,j'}\big)$$

and Eq. (1.94) follows by induction on the cardinality of $\mathscr{C}$. By the argument in the Coding Lemma 1.4, Eq. (1.94) implies that, conditionally on $\mathscr{F}$, we can generate

$$\big(s_{\{j,j'\}}\big)_{j\neq j'\in I} \stackrel{d}{=} \big(h(S_j,S_{j'},x_{\{j,j'\}})\big)_{j\neq j'\in I}, \tag{1.96}$$

for some measurable function h and i.i.d. uniform random variables $x_{\{j,j'\}}$ on $[0,1]$. The reason why the function h can be chosen to be the same for all $\{j,j'\}$ is because, by symmetry, the distribution of $(S_j,S_{j'},s_{\{j,j'\}})$ does not depend on $\{j,j'\}$ and the conditional distribution of $s_{\{j,j'\}}$ given S_j and $S_{j'}$ does not depend on $\{j,j'\}$. Also, the arrays $(S_j,S_{j'},s_{\{j,j'\}})$ and $(S_{j'},S_j,s_{\{j,j'\}})$ are equal in distribution and, therefore, the function h is symmetric in the coordinates S_j and $S_{j'}$. Finally, let us recall Eq. (1.92) and define the function

$$f\big(w,u_j,u_{j'},x_{\{j,j'\}}\big) = h\big(X(w,u_j),X(w,u_{j'}),x_{\{j,j'\}}\big),$$

which is, obviously, symmetric in u_j and $u_{j'}$. Then, the Eqs. (1.92) and (1.96) imply that

$$\Big(\big(S_j\big)_{j\in I}, \big(s_{\{j,j'\}}\big)_{j\neq j'\in I} \Big) \stackrel{d}{=} \Big(\big(X(w,u_j)\big)_{j\in I}, \big(f\big(w,u_j,u_{j'},x_{\{j,j'\}}\big)\big)_{j\neq j'\in I} \Big).$$

In particular, if we denote by g the first coordinate of the map X corresponding to the element $s_{\{j,j\}}$ in the array S_j in Eq. (1.91), this proves that

$$\Big(\big(s_{\{j,j\}}\big)_{j\in I}, \big(s_{\{j,j'\}}\big)_{j\neq j'\in I} \Big) \stackrel{d}{=} \Big(\big(g(w,u_j)\big)_{j\in I}, \big(f\big(w,u_j,u_{j'},x_{\{j,j'\}}\big)\big)_{j\neq j'\in I} \Big).$$

Recalling Eq. (1.89) finishes the proof of the representation (1.80). □

1.5 The Dovbysh–Sudakov Representation

Let us consider an infinite symmetric random array $R = (R_{l,l'})_{l,l'\geq 1}$, which is weakly exchangeable in the sense defined in Eq. (1.78) and positive definite with probability one, where by positive definite we will always mean nonnegative definite. Such weakly exchangeable positive definite arrays are called *Gram-de Finetti arrays*. It turns out that all such arrays are generated very similarly to the overlap matrix (1.60) in the Sherrington–Kirkpatrick model, except, in general, one can also add some nonnegative sequence on the diagonal. Let H be the Hilbert space $L^2([0,1],dv)$, where dv denotes the Lebesgue measure on $[0,1]$.

Theorem 1.7 (Dovbysh–Sudakov). *There exists a random probability measure η on $H\times\mathbb{R}^+$ such that the array $R = (R_{l,l'})_{l,l'\geq 1}$ is equal in distribution to*

$$\big(h_l\cdot h_{l'} + a_l\,\delta_{l,l'}\big)_{l,l'\geq 1}, \tag{1.97}$$

where, conditionally on η, $(h_l,a_l)_{l\geq 1}$ is a sequence of i.i.d. random variables with the distribution η and $h\cdot h'$ denotes the scalar product on H.

In particular, the marginal G of η on H can be used to generate the sequence (h_l) and the off-diagonal elements of the array R. In the case of the overlap array in Sect. 1.3, we called this G the asymptotic Gibbs measure. Let us recall that a *random measure* $\eta = \eta(\omega,A)$ is a function of a generic point $\omega\in\Omega$ for some probability space $(\Omega,\mathscr{F},\Pr)$ and a measurable set A, in this case, a Borel set in $H\times\mathbb{R}^+$, such that for a fixed ω, $\eta(\omega,\cdot)$ is a probability measure and for a fixed A, $\eta(\cdot,A)$ is a measurable function on Ω. If we denote by $\mathscr{P}(H\times\mathbb{R}^+)$ the space of all probability measures on $H\times\mathbb{R}^+$ equipped with the topology of weak convergence, then it is well known that the measurable functions on $(\Omega,\mathscr{F})$ with values in the space $\mathscr{P}(H\times\mathbb{R}^+)$ coincide with random measures in the above sense (see, e.g., Lemma 1.37 and Theorem A2.3 in [31]).

Proof (Theorem 1.7). Since the array R is positive definite, conditionally on R, we can generate a Gaussian vector g in $\mathbb{R}^{\mathbb{N}}$ with the covariance equal to R. Now, also conditionally on R, let $(g_i)_{i\geq 1}$ be independent copies of g. If, for each $i \geq 1$, we denote the coordinates of g_i by $g_{l,i}$ for $l \geq 1$, then, since the array $R = (R_{l,l'})_{l,l'\geq 1}$ is weakly exchangeable, it should be obvious that the array $(g_{l,i})_{l,i\geq 1}$ is exchangeable in the sense of Eq. (1.76). By Theorem 1.4, there exists a measurable function $\sigma : [0,1]^4 \to \mathbb{R}$ such that

$$\big(g_{l,i}\big)_{l,i\geq 1} \stackrel{d}{=} \big(\sigma(w,u_l,v_i,x_{l,i})\big)_{l,i\geq 1}, \tag{1.98}$$

where w, (u_l), (v_i), $(x_{l,i})$ are i.i.d. random variables with the uniform distribution on $[0,1]$. By the strong law of large numbers (applied conditionally on R), for any $l,l' \geq 1$,

$$\frac{1}{n}\sum_{i=1}^{n} g_{l,i}g_{l',i} \to R_{l,l'}$$

almost surely as $n \to \infty$. Similarly, by the strong law of large numbers (now applied conditionally on w and $(u_l)_{l\geq 1}$),

$$\frac{1}{n}\sum_{i=1}^{n} \sigma(w,u_l,v_i,x_{l,i})\sigma(w,u_{l'},v_i,x_{l',i}) \to \mathbb{E}'\sigma(w,u_l,v_1,x_{l,1})\sigma(w,u_{l'},v_1,x_{l',1})$$

almost surely, where $\mathbb{E}'$ denotes the expectation with respect to the random variables v_1 and $(x_{l,1})_{l\geq 1}$. Therefore, Eq. (1.98) implies that

$$\big(R_{l,l'}\big)_{l,l'\geq 1} \stackrel{d}{=} \big(\mathbb{E}'\sigma(w,u_l,v_1,x_{l,1})\sigma(w,u_{l'},v_1,x_{l',1})\big)_{l,l'\geq 1}. \tag{1.99}$$

If we denote

$$\sigma^{(1)}(w,u,v) = \int \sigma(w,u,v,x)\,dx, \;\; \sigma^{(2)}(w,u,v) = \int \sigma(w,u,v,x)^2\,dx,$$

then the off-diagonal and diagonal elements on the right-hand side of Eq. (1.99) are given by

$$\int \sigma^{(1)}(w,u_l,v)\sigma^{(1)}(w,u_{l'},v)\,dv \;\text{ and }\; \int \sigma^{(2)}(w,u_l,v)\,dv$$

correspondingly. Notice that, for almost all w and u, the function $v \to \sigma^{(1)}(w,u,v)$ is in $H = L^2([0,1],dv)$, since

$$\int \sigma^{(1)}(w,u,v)^2\,dv \leq \int \sigma^{(2)}(w,u,v)\,dv$$

and, by Eq. (1.99), the right-hand side is equal in distribution to $R_{1,1}$. Therefore, if we denote

$$h_l = \sigma^{(1)}(w,u_l,\cdot),\; a_l = \int \sigma^{(2)}(w,u_l,v)\,dv - h_l \cdot h_l,$$

then Eq. (1.99) becomes

$$\left(R_{l,l'}\right)_{l,l'\geq 1} \stackrel{d}{=} \left(h_l \cdot h_{l'} + a_l \delta_{l,l'}\right)_{l,l'\geq 1}. \tag{1.100}$$

It remains to observe that $(h_l, a_l)_{l\geq 1}$ is an i.i.d. sequence on $H \times \mathbb{R}^+$ from the random measure $\eta = \eta(w)$ given by the image of the Lebesgue measure du on $[0,1]$ by the map

$$u \to \left(\sigma^{(1)}(w,u,\cdot), \int \sigma^{(2)}(w,u,v)\,dv - \int \sigma^{(1)}(w,u,v)\sigma^{(1)}(w,u,v)\,dv\right).$$

This finishes the proof. □

Let us now suppose that there is equality instead of equality in distribution in Eq. (1.97), i.e. the array R is generated by an i.i.d. sample (h_l, a_l) from the random measure η. To complete the picture, let us show that one can reconstruct η, up to an orthogonal transformation of its marginal on H, as a measurable function of the array R itself, with values in the set $\mathscr{P}(H \times \mathbb{R}^+)$ of all probability measures on $H \times \mathbb{R}^+$ equipped with the topology of weak convergence.

Lemma 1.7. *There exists a measurable function $\eta' = \eta'(R)$ of the array $(R_{l,l'})_{l,l'\geq 1}$ with values in $\mathscr{P}(H \times \mathbb{R}^+)$ such that $\eta' = \eta \circ (U, \mathrm{id})^{-1}$ almost surely for some orthogonal operator U on H that depends on the sequence $(h_l)_{l\geq 1}$.*

Proof. Let us begin by showing that the norms $\|h_l\|$ can be reconstructed almost surely from the array R. Consider a sequence (g_l) on H such that $g_l \cdot g_{l'} = R_{l,l'}$ for all $l, l' \geq 1$. In other words, $\|g_l\|^2 = \|h_l\|^2 + a_l$ and $g_l \cdot g_{l'} = h_l \cdot h_{l'}$ for all $l \neq l'$. Without loss of generality, let us assume that $g_l = h_l + \sqrt{a_l} e_l$, where $(e_l)_{l\geq 1}$ is an orthonormal sequence orthogonal to the closed span of (h_l). If necessary, we identify H with $H \oplus H$ to choose such a sequence (e_l). Since (h_l) is an i.i.d. sequence from the marginal G of the measure η on H, with probability one, there are elements in the sequence $(h_l)_{l\geq 2}$ arbitrarily close to h_1 and, therefore, the length of the orthogonal projection of h_1 onto the closed span of $(h_l)_{l\geq 2}$ is equal to $\|h_1\|$. As a result, the length of the orthogonal projection of g_1 onto the closed span of $(g_l)_{l\geq 2}$ is also equal to $\|h_1\|$, and it is obvious that this length is a measurable function of the array $g_l \cdot g_{l'} = R_{l,l'}$. Similarly, we can reconstruct all the norms $\|h_l\|$ as measurable functions of the array R and, thus, all $a_l = R_{l,l} - \|h_l\|^2$. Therefore,

$$\left(h_l \cdot h_{l'}\right)_{l,l'\geq 1} \text{ and } (a_l)_{l\geq 1} \tag{1.101}$$

are both measurable functions of the array R. Given the matrix $(h_l \cdot h_{l'})$, we can find a sequence (x_l) in H isometric to (h_l), for example, by choosing x_l to be in the span of the first l elements of some fixed orthonormal basis. This means that all x_l are measurable functions of R and that there exists an orthogonal operator $U = U((h_l)_{l\geq 1})$ on H such that

$$x_l = U h_l \text{ for } l \geq 1. \tag{1.102}$$

Since $(h_l, a_l)_{l\geq 1}$ is an i.i.d. sequence from distribution η, by the strong law of large numbers for empirical measures (Theorem 11.4.1 in [23]),

$$\frac{1}{n}\sum_{1\leq l\leq n}\delta_{(h_l,a_l)} \to \eta \text{ weakly}$$

almost surely and, therefore, Eq. (1.102) implies that

$$\frac{1}{n}\sum_{1\leq l\leq n}\delta_{(x_l,a_l)} \to \eta\circ(U,\mathrm{id})^{-1} \text{ weakly} \tag{1.103}$$

almost surely. The left-hand side is, obviously, a measurable function of the array R in the space of all probability measures on $H\times\mathbb{R}^+$ equipped with the topology of weak convergence and, therefore, as a limit, $\eta' = \eta\circ(U,\mathrm{id})^{-1}$ is also a measurable function of R. This finishes the proof. □

Chapter 2
The Ruelle Probability Cascades

In this chapter we will describe a remarkable family of random measures on a Hilbert space, called the Ruelle probability cascades, that play a central role in the Sherrington–Kirkpatrick model, and the first three sections will be devoted to the construction of these measures and study of their properties. We will see that they satisfy certain special invariance properties, one of which, called the Ghirlanda–Guerra identities, will serve as a key link between the Ruelle probability cascades and the Gibbs measure in the Sherrington–Kirkpatrick model. This connection will be explained in the last two sections, where it will be shown that, in a certain sense, the Ghirlanda–Guerra identities completely determine a random measure up to a functional order parameter. We will see in the next chapter that, as a consequence, the asymptotic Gibbs measures in the Sherrington–Kirkpatrick and mixed p-spin models can be approximated by the Ruelle probability cascades.

2.1 Overview of Poisson Processes

In this section we will introduce and review several important properties of Poisson processes on a measurable space $(S,\mathscr{S})$. In the sequel, $(S,\mathscr{S})$ will always be some nice space, such as a Euclidean space $\mathbb{R}^n$ with the Borel σ-algebra. However, in general, it is enough to require that the diagonal $\{(s,s') : s = s'\}$ is measurable on the product space $S \times S$ which, in particular, implies that every singleton set $\{s\}$ in S is measurable as a section of the diagonal. This condition is needed to be able to write $\mathbb{P}(X = Y)$ for a pair (X,Y) of random variables defined on the product space. From now on, every time we consider a measurable space, we will assume that it satisfies this condition. Let us also notice that a product $S \times S'$ of two such spaces will also satisfy this condition since $\{(s_1,s_1') = (s_2,s_2')\} = \{s_1 = s_2\} \cap \{s_1' = s_2'\}$. Let μ and μ_n for $n \geq 1$ be some nonatomic (not having any atoms) measures on $\mathscr{S}$ such that

$$\mu = \sum_{n\geq 1} \mu_n, \ \ \mu_n(S) < \infty. \tag{2.1}$$

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For each $n \geq 1$, let N_n be a random variable with the Poisson distribution $\Pi(\mu_n(S))$ with the mean $\mu_n(S)$ and let $(X_{nl})_{l\geq 1}$ be i.i.d. random variables, also independent of N_n, with the distribution

$$p_n(B) = \frac{\mu_n(B)}{\mu_n(S)}. \tag{2.2}$$

We assume that all these random variables are independent for different $n \geq 1$. The condition that μ is nonatomic implies that $\mathbb{P}(X_{nl} = X_{mj}) = 0$ if $n \neq m$ or $l \neq j$. Let us consider random sets

$$\Pi_n = \{X_{n1}, \ldots, X_{nN_n}\} \text{ and } \Pi = \bigcup_{n\geq 1} \Pi_n. \tag{2.3}$$

The set Π will be called a *Poisson process* on S with the *mean measure* μ. Let us point out a simple observation that will be used many times below that if, first, we are given the means $\mu_n(S)$ and distributions (2.2) that were used to generate the set (2.3), then the measure μ in Eq. (2.1) can be written as

$$\mu = \sum_{n\geq 1} \mu_n(S) p_n. \tag{2.4}$$

We will show in Theorem 2.4 below that when the measure μ is σ-finite, then, in some sense, this definition of a Poisson process Π is independent of the particular representation (2.1). However, for several reasons, it is convenient to think of the above construction as the definition of a Poisson process. First of all, it allows us to avoid any discussion about what "a random set" means and, moreover, many important properties of Poisson processes follow from it rather directly. In any case, we will show in Theorem 2.5 below that all such processes satisfy the traditional definition of a Poisson process.

One important immediate consequence of the definition in Eq. (2.3) is the Mapping Theorem. Given a Poisson process Π on S with the mean measure μ and a measurable map $f : S \to S'$ into another measurable space $(S', \mathscr{S}')$, let us consider the image set

$$f(\Pi) = \bigcup_{n\geq 1} \{f(X_{n1}), \ldots, f(X_{nN_n})\}.$$

Since random variables $f(X_{nl})$ have the distribution $p_n \circ f^{-1}$ on $\mathscr{S}'$, the set $f(\Pi)$ resembles the definition (2.3) corresponding to the measure

$$\sum_{n\geq 1} \mu_n(S)\, p_n \circ f^{-1} = \sum_{n\geq 1} \mu_n \circ f^{-1} = \mu \circ f^{-1}.$$

Therefore, to conclude that $f(\Pi)$ is a Poisson process on S' we only need to require that this image measure is nonatomic.

Theorem 2.1 (Mapping Theorem). *If Π is a Poisson process on S with the mean measure μ and $f : S \to S'$ is such that $\mu \circ f^{-1}$ is nonatomic then $f(\Pi)$ is a Poisson process on S' with the mean measure $\mu \circ f^{-1}$.*

Next, let us consider a sequence (λ_m) of measures that satisfy Eq. (2.1), $\lambda_m = \sum_{n\geq 1}\lambda_{mn}$ and $\lambda_{mn}(S) < \infty$. Let $\Pi_m = \cup_{n\geq 1}\Pi_{mn}$ be a Poisson process with the mean measure λ_m defined as in Eq. (2.3) and suppose that all these Poisson processes are generated independently over $m \geq 1$. Since

$$\lambda := \sum_{m\geq 1}\lambda_m = \sum_{m\geq 1}\sum_{n\geq 1}\lambda_{mn} = \sum_{l\geq 1}\lambda_{m(l)n(l)}$$

and $\Pi := \cup_{m\geq 1}\Pi_m = \cup_{l\geq 1}\Pi_{m(l)n(l)}$ for any enumeration of the pairs (m,n) by the indices $l \in \mathbb{N}$, we immediately get the following.

Theorem 2.2 (Superposition Theorem). *If Π_m for $m \geq 1$ are independent Poisson processes with the mean measures λ_m then their superposition $\Pi = \cup_{m\geq 1}\Pi_m$ is a Poisson process with the mean measure $\lambda = \sum_{m\geq 1}\lambda_m$.*

Another important property of Poisson processes is the Marking Theorem. Consider another measurable space $(S', \mathscr{S}')$ and let $K : S \times \mathscr{S}' \to [0,1]$ be a transition function (a probability kernel), which means that for each $s \in S$, $K(s,\cdot)$ is a probability measure on $\mathscr{S}'$ and for each $A \in \mathscr{S}'$, $K(\cdot, A)$ is a measurable function on $(S, \mathscr{S})$. For each point $s \in \Pi$, let us generate a point $m(s) \in S'$ from the distribution $K(s,\cdot)$, independently for different points s. The point $m(s)$ is called a *marking* of s and a random subset of $S \times S'$,

$$\Pi^* = \big\{(s, m(s)) : s \in \Pi\big\}, \tag{2.5}$$

is called a marked Poisson process. In other words, $\Pi^* = \cup_{n\geq 1}\Pi^*_n$, where

$$\Pi^*_n = \big\{(X_{n1}, m(X_{n1})), \ldots, (X_{nN_n}, m(X_{nN_n}))\big\},$$

and each point $(X_{nl}, m(X_{nl})) \in S \times S'$ is generated according to the distribution

$$p^*_n(C) = \iint_C K(s, ds')\, p_n(ds).$$

Since p^*_n is obviously non-atomic, by definition, this means that Π^* is a Poisson process on $S \times S'$ with the mean measure

$$\sum_{n\geq 1}\mu_n(S)\, p^*_n(C) = \sum_{n\geq 1}\iint_C K(s, ds')\mu_n(ds) = \iint_C K(s, ds')\mu(ds).$$

Therefore, the following holds.

Theorem 2.3 (Marking Theorem). *The random subset Π^* in Eq. (2.5) is a Poisson process on $S \times S'$ with the mean measure*

$$\mu^*(C) = \iint_C K(s, ds')\mu(ds). \tag{2.6}$$

The above results give us several ways to generate a new Poisson process from the old one. However, in each case, the new process is generated in a way that depends

on a particular representation of its mean measure. We will now show that, when the measure μ is σ-finite, the random set Π can be generated in a way that, in some sense, does not depend on the particular representation (2.1). This point is very important, because, often, given a Poisson process with one representation of its mean measure, we study its properties using another, more convenient, representation. Suppose that S is equal to a disjoint union $\cup_{m\geq 1} S_m$ of sets such that $0 < \mu(S_m) < \infty$, in which case

$$\mu = \sum_{n\geq 1} \mu|_{S_m} \tag{2.7}$$

is another representation of the type Eq. (2.1), where $\mu|_{S_m}$ is the restriction of μ to the set S_m. The following holds.

Theorem 2.4 (Equivalence Theorem). *The process Π in Eq. (2.3) generated according to the representation (2.1) is statistically indistinguishable from a process generated using the representation (2.7).*

More precisely, we will show the following. Let us denote

$$p_{S_m}(B) = \frac{\mu|_{S_m}(B)}{\mu|_{S_m}(S)} = \frac{\mu(B\cap S_m)}{\mu(S_m)}. \tag{2.8}$$

We will show that, given the Poisson process Π in Eq. (2.3) generated according to the representation (2.1), the cardinalities $N(S_m) = |\Pi\cap S_m|$ are independent random variables with the distributions $\Pi(\mu(S_m))$ and, conditionally on $(N(S_m))_{m\geq 1}$, each set $\Pi\cap S_m$ "looks like" an i.i.d. sample of size $N(S_m)$ from the distribution p_{S_m}. This means that, if we arrange the points in $\Pi\cap S_m$ in a random order, the resulting vector has the same distribution as an i.i.d. sample from the measure p_{S_m}. Moreover, conditionally on $(N(S_m))_{m\geq 1}$, these vectors are independent over $m \geq 1$. This is exactly how one would generate a Poisson process using the representation (2.7). The proof of Theorem 2.4 will be based on the following two properties that usually appear as the definition of a Poisson process on S with the mean measure μ:

1. For any $A \in \mathscr{S}$, the cardinality $N(A) = |\Pi\cap A|$ has the Poisson distribution $\Pi(\mu(A))$ with the mean $\mu(A)$.
2. The cardinalities $N(A_1),\ldots,N(A_k)$ are independent, for any $k \geq 1$ and any disjoint sets $A_1,\ldots,A_k \in \mathscr{S}$.

When $\mu(A) = \infty$, it is understood that $\Pi\cap A$ is countably infinite. Usually, one starts with (i) and (ii) as the definition of a Poisson process and the set Π constructed in Eq. (2.3) is used to demonstrate the existence of such processes, which explains the name of the following theorem.

Theorem 2.5 (Existence Theorem). *The process Π in Eq. (2.3) satisfies the properties (i) and (ii).*

Proof. Given $x \geq 0$, let us denote the weights of the Poisson distribution $\Pi(x)$ with the mean x by

$$\pi_j(x) = \Pi(x)\big(\{j\}\big) = \frac{x^j}{j!}e^{-x}.$$

Consider disjoint sets $A_1, \ldots, A_k$ and let $A_0 = (\cup_{i \leq k} A_i)^c$ be the complement of their union. Fix $m_1, \ldots, m_k \geq 0$ and let $m = m_1 + \cdots + m_k$. Given any set A, denote $N_n(A) = |\Pi_n \cap A|$. With this notation, let us compute the probability of the event

$$\Omega = \big\{N_n(A_1) = m_1, \ldots, N_n(A_k) = m_k\big\}.$$

Recall that the random variables $(X_{nl})_{l \geq 1}$ are i.i.d. with the distribution p_n defined in Eq. (2.2) and, therefore, conditionally on N_n in Eq. (2.3), the cardinalities $(N_n(A_l))_{0 \leq l \leq k}$ have multinomial distribution and we can write

$$\begin{aligned}
\mathbb{P}(\Omega) &= \sum_{j \geq 0} \mathbb{P}\big(\Omega \,\big|\, N_n = m + j\big)\mathbb{P}\big(N_n = m + j\big) \\
&= \sum_{j \geq 0} \mathbb{P}\big(\big\{N_n(A_0) = j\big\} \cap \Omega \,\big|\, N_n = m + j\big)\pi_{m+j}\big(\mu_n(S)\big) \\
&= \sum_{j \geq 0} \frac{(m+j)!}{j!m_1! \cdots m_k!} p_n(A_0)^j p_n(A_1)^{m_1} \cdots p_n(A_k)^{m_k} \frac{\mu_n(S)^{m+j}}{(m+j)!} e^{-\mu_n(S)} \\
&= \sum_{j \geq 0} \frac{1}{j!m_1! \cdots m_k!} \mu_n(A_0)^j \mu_n(A_1)^{m_1} \cdots \mu_n(A_k)^{m_k} e^{-\mu_n(S)} \\
&= \frac{\mu_n(A_1)^{m_1}}{m_1!} e^{-\mu_n(A_1)} \ldots \frac{\mu_n(A_k)^{m_k}}{m_k!} e^{-\mu_n(A_k)}.
\end{aligned}$$

This means that the cardinalities $N_n(A_l)$ for $1 \leq l \leq k$ are independent random variables with the distributions $\Pi(\mu_n(A_l))$. Since all measures p_n are non-atomic, $\mathbb{P}(X_{nj} = X_{mk}) = 0$ for any $(n, j) \neq (m, k)$. Therefore, the sets Π_n are all disjoint with probability one and

$$N(A) = |\Pi \cap A| = \sum_{n \geq 1} |\Pi_n \cap A| = \sum_{n \geq 1} N_n(A).$$

First of all, the cardinalities $N(A_1), \ldots, N(A_k)$ are independent, since we showed that $N_n(A_1), \ldots, N_n(A_k)$ are independent for each $n \geq 1$, and it remains to show that $N(A)$ has the Poisson distribution with the mean $\mu(A) = \sum_{n \geq 1} \mu_n(A)$. The partial sum $S_m = \sum_{n \leq m} N_n(A)$ has the Poisson distribution with the mean $\sum_{n \leq m} \mu_n(A)$ and, since $S_m \uparrow N(A)$, for any integer $r \geq 0$, the probability $\mathbb{P}(N(A) \leq r)$ equals

$$\lim_{m \to \infty} \mathbb{P}(S_m \leq r) = \lim_{m \to \infty} \sum_{j \leq r} \pi_j\Big(\sum_{n \leq m} \mu_n(A)\Big) = \sum_{j \leq r} \pi_j\big(\mu(A)\big).$$

If $\mu(A) < \infty$, this shows that $N(A)$ has the distribution $\Pi(\mu(A))$. If $\mu(A) = \infty$ then $\mathbb{P}(N(A) \leq r) = 0$ for all $r \geq 0$, which implies that $N(A)$ is countably infinite. $\square$

Proof (Theorem 2.4). First, suppose that $\mu(S) < \infty$ and suppose that the Poisson process Π was generated as in Eq. (2.3) using some sequence of measures (μ_n). By Theorem 2.5, the cardinality $N = N(S)$ has the distribution $\Pi(\mu(S))$. Let us show that, conditionally on the event $\{N = n\}$, the set Π "looks like" an i.i.d. sample of size n from the distribution $p(\cdot) = \mu(\cdot)/\mu(S)$ or, more precisely, if we randomly assign labels $\{1,\dots,n\}$ to the points in Π, the resulting vector $(X_1,\dots,X_n)$ has the same distribution as an i.i.d. sample from p. Let us consider some measurable sets $B_1,\dots,B_n \in \mathscr{S}$ and compute

$$\mathbb{P}_n\big(X_1 \in B_1,\dots,X_n \in B_n\big) = \mathbb{P}\big(X_1 \in B_1,\dots,X_n \in B_n \,\big|\, N = n\big), \tag{2.9}$$

where we denoted by $\mathbb{P}_n$ the conditional probability given the event $\{N = n\}$. To use Theorem 2.5, we will reduce this case to the case of disjoint sets as follows. Let $A_1,\dots,A_k$ be the partition of the space S generated by the sets $B_1,\dots,B_n$, obtained by taking intersections of the sets B_l or their complements over $l \le n$. Then, for each $l \le n$, we can write

$$I(x \in B_l) = \sum_{j \le k} \delta_{lj} I(x \in A_j),$$

where $\delta_{lj} = 1$ if $B_l \supseteq A_j$ and $\delta_{lj} = 0$ otherwise. In terms of these sets, Eq. (2.9) can be rewritten as

$$\begin{aligned}\mathbb{P}_n\big(X_1 \in B_1,\dots,X_n \in B_n\big) &= \mathbb{E}_n \prod_{l \le n} \sum_{j \le k} \delta_{lj} I(X_l \in A_j) \\ &= \sum_{j_1,\dots,j_n \le k} \delta_{1j_1} \cdots \delta_{nj_n} \mathbb{P}_n\big(X_1 \in A_{j_1},\dots,X_n \in A_{j_n}\big).\end{aligned} \tag{2.10}$$

If we fix $j_1,\dots,j_n$ and for $j \le k$ let $I_j = \{l \le n : j_l = j\}$ then

$$\mathbb{P}_n\big(X_1 \in A_{j_1},\dots,X_n \in A_{j_n}\big) = \mathbb{P}_n\big(X_l \in A_j \text{ for all } j \le k, l \in I_j\big).$$

If $n_j = |I_j|$ then the last event can be expressed in words by saying that, for each $j \le k$, we observe n_j points of the random set Π in the set A_j and then assign labels in I_j to the points $\Pi \cap A_j$. By Theorem 2.5, the probability to observe n_j points in each set A_j, given that $N = n$, equals

$$\begin{aligned}\mathbb{P}_n(N(A_j) = n_j, j \le k) &= \frac{\mathbb{P}(N(A_j) = n_j, j \le k)}{\mathbb{P}(N = n)} \\ &= \prod_{j \le k} \frac{\mu(A_j)^{n_j}}{n_j!} e^{-\mu(A_j)} \Big/ \frac{\mu(S)^n}{n!} e^{-\mu(S)},\end{aligned}$$

while the probability to randomly assign the labels in I_j to the points in A_j for all $j \le k$ is equal to $\prod_{j \le k} n_j!/n!$. Therefore,

$$\mathbb{P}_n\big(X_1 \in A_{j_1},\dots,X_n \in A_{j_n}\big) = \prod_{j \le k} \Big(\frac{\mu(A_j)}{\mu(S)}\Big)^{n_j} = \prod_{l \le n} p(A_{j_l}). \tag{2.11}$$

If we plug this into Eq. (2.10), we get

$$\begin{aligned}\mathbb{P}_n\big(X_1 \in B_1, \dots, X_n \in B_n\big) &= \sum_{j_1 \le k} \delta_{1 j_1} p(A_{j_1}) \cdots \sum_{j_n \le k} \delta_{n j_n} p(A_{j_n}) \\ &= p(B_1) \cdots p(B_n).\end{aligned}$$

Therefore, conditionally on $\{N = n\}$, $X_1, \dots, X_n$ are i.i.d. with the distribution p.

Now, suppose that μ is σ-finite and Eq. (2.7) holds. Consider the random sets in Eq. (2.3) and define $N(S_m) = |\Pi \cap S_m|$. By Theorem 2.5, the random variables $(N(S_m))_{m \ge 1}$ are independent and have the Poisson distributions $\Pi(\mu(S_m))$, and we would like to show that, conditionally on $(N(S_m))_{m \ge 1}$, each set $\Pi \cap S_m$ can be generated as a sample of size $N(S_m)$ from the distribution $p_{S_m} = \mu|_{S_m}(\cdot)/\mu(S_m)$, independently over m. This can be done by considering finitely many m at a time and using exactly the same computation leading to Eq. (2.11) based on the properties proved in Theorem 2.5, with each subset $A \subseteq S_m$ producing a factor $p_{S_m}(A)$. □

2.2 The Poisson–Dirichlet Processes

In the next section we will study a special class of random measures on a Hilbert space that will be used as an approximation of the asymptotic Gibbs measures in the Sherrington–Kirkpatrick model. The key role in the construction of these measures is played by the following family of Poisson processes that we will study first. Let us consider a parameter $\zeta \in (0,1)$ and let Π be a Poisson process on $(0,\infty)$ with the mean measure

$$\mu(dx) = \zeta x^{-1-\zeta} dx. \tag{2.12}$$

Clearly, the measure μ is σ-finite, which can be seen, for example, by considering the partition $(0,\infty) = \cup_{m \ge 1} S_m$ with $S_1 = [1,\infty)$ and $S_m = [1/m, 1/(m-1))$ for $m \ge 2$, since

$$\int_1^\infty \zeta x^{-1-\zeta} dx = 1 < \infty.$$

By Theorem 2.4, each cardinality $N(S_m) = |\Pi \cap S_m|$ has the Poisson distribution with the mean $\mu(S_m)$ and, conditionally on $N(S_m)$, the points in $\Pi \cap S_m$ are i.i.d. with the distribution $\mu|_{S_m}/\mu(S_m)$. Therefore, by Wald's identity,

$$\mathbb{E} \sum_{x \in \Pi \cap S_m} x = \mathbb{E} N(S_m) \int_{S_m} \frac{x\mu(dx)}{\mu(S_m)} = \int_{S_m} x\mu(dx)$$

and, therefore,

$$\mathbb{E} \sum_{x \in \Pi} x I(x < 1) = \int_0^1 x\mu(dx) = \int_0^1 \zeta x^{-\zeta} dx = \frac{\zeta}{1-\zeta} < \infty. \tag{2.13}$$

This means that the sum $\sum_{x\in\Pi} xI(x<1)$ is finite with probability one and, since there are finitely many points in the set $\Pi\cap[1,\infty)$, the sum $\sum_{x\in\Pi} x$ is also finite with probability one. Since $\mu((0,\infty))=\infty$, Π is countably infinite, and if we let $(u_n)_{n\geq 1}$ be the decreasing enumeration of points in Π then

$$v_n = \frac{u_n}{\sum_{l\geq 1} u_l} \tag{2.14}$$

is well defined. The sequence $(v_n)_{n\geq 1}$ is a random probability distribution on $\mathbb{N}$ and it is called a *Poisson–Dirichlet process* $PD(\zeta)$ with the parameter $\zeta\in(0,1)$. It is easy to show that all the coordinates in the decreasing enumeration $(u_n)_{n\geq 1}$ are random variables by expressing them in terms of the random variables in Eq. (2.3) that generate Π. Similarly to Eq. (2.13), one can show the following property of the Poisson process Π that will be used a number of times.

Lemma 2.1. *Let Π be a Poisson process with the mean measure Eq. (2.12). Then*

$$\mathbb{E}\Big(\sum_{x\in\Pi} x\Big)^a < \infty \text{ for all } 0<a<\zeta. \tag{2.15}$$

Proof. Since $x^a\leq 1+x$ for $x>0$, by Eq. (2.13),

$$\mathbb{E}\Big(\sum_{x\in\Pi} xI(x<1)\Big)^a \leq 1+\frac{\zeta}{1-\zeta}.$$

Using that $(x+y)^a\leq x^a+y^a$ we also get, as in Eq. (2.13),

$$\mathbb{E}\Big(\sum_{x\in\Pi} xI(x\geq 1)\Big)^a \leq \mathbb{E}\sum_{x\in\Pi} x^a I(x\geq 1) = \int_1^\infty \zeta x^{-1-(\zeta-a)}dx<\infty$$

for $a<\zeta$. Combining these two inequalities proves Eq. (2.15). □

Next, we will describe what is, probably, one of the most important properties of the Poisson process with the mean measure Eq. (2.12), which is based on a special homogeneity property of this measure. Let $(S,\mathscr{S})$ be a measurable space and let

$$(X,Y):\Omega\to(0,\infty)\times S \tag{2.16}$$

be a pair of random elements defined on some probability space $(\Omega,\mathscr{F},\Pr)$ with values in the product space $(0,\infty)\times S$. Suppose that $\mathbb{E}X^\zeta<\infty$ and let us define the probability measure ν_ζ on $(S,\mathscr{S})$ by

$$\nu_\zeta(B) = \frac{\mathbb{E}X^\zeta I(Y\in B)}{\mathbb{E}X^\zeta}. \tag{2.17}$$

In other words, ν_ζ is the law of Y on $\mathscr{S}$ under the change of density $X^\zeta/\mathbb{E}X^\zeta$. Let us consider an i.i.d. sequence (X_n,Y_n) of copies of (X,Y) independent of the Poisson

process $\{u_n : n \geq 1\}$. We will think of this sequence as the markings of points in Π and, by the Marking Theorem, Theorem 2.3, the set $\{(u_n, X_n, Y_n) : n \geq 1\}$ is also a Poisson process. The following observation will play a fundamental role.

Theorem 2.6. *The Poisson process* $\{(u_n X_n, Y_n)\}$ *has the same mean measure as the process*

$$\left\{\left(\left(\mathbb{E}X^{\zeta}\right)^{1/\zeta} u_n, Y_n'\right)\right\}, \tag{2.18}$$

where (Y_n') *are i.i.d. random variables independent of* (u_n)*, with the distribution* ν_ζ.

Proof. Suppose that $\nu = \Pr \circ (X,Y)^{-1}$ is the law of (X,Y) on $(0,\infty) \times S$. Then, by the Marking Theorem 2.3, $\{(u_n, X_n, Y_n)\}$ is a Poisson process with the mean measure $\mu \otimes \nu$ on $(0,\infty) \times (0,\infty) \times S$. By the Mapping Theorem, Theorem 2.1, $\{(u_n X_n, Y_n)\}$ is a Poisson process with the mean measure given by the image of $\mu \otimes \nu$ under the map $(u,x,y) \to (ux,y)$, if this image measure has no atoms. Let us compute this measure. Given two measurable sets $A \subseteq (0,\infty)$ and $B \subseteq S$, by Fubini's theorem,

$$\mu \otimes \nu(ux \in A, y \in B) = \int \mu\big(u : uX(\omega) \in A\big) I\big(Y(\omega) \in B\big) \Pr(d\omega). \tag{2.19}$$

For any $x > 0$, we have the following homogeneity property:

$$\begin{aligned} \mu(u : ux \in A) &= \int I(ux \in A) \zeta u^{-1-\zeta} \, du \\ &= x^{\zeta} \int I(z \in A) \zeta z^{-1-\zeta} \, dz = x^{\zeta} \mu(A), \end{aligned} \tag{2.20}$$

and, by Eqs. (2.19) and (2.17),

$$\begin{aligned} \mu \otimes \nu(ux \in A, y \in B) &= \mu(A) \int X(\omega)^{\zeta} I\big(Y(\omega) \in B\big) \Pr(d\omega) \\ &= \mu(A) \mathbb{E} X^{\zeta} I\big(Y \in B\big) = \mathbb{E}X^{\zeta} \mu(A) \, \nu_\zeta(B). \end{aligned}$$

Therefore, we showed that $(\mathbb{E}X^{\zeta}\mu) \otimes \nu_\zeta$ is the mean measure of the Poisson process $\{(u_n X_n, Y_n)\}$, since this measure is obviously nonatomic. By the computation in Eq. (2.20), $\mathbb{E}X^{\zeta}\mu$ is also the mean measure of the process $\{(\mathbb{E}X^{\zeta})^{1/\zeta} u_n\}$ and, therefore, by the Marking Theorem, $(\mathbb{E}X^{\zeta}\mu) \otimes \nu_\zeta$ is the mean measure of the process (2.18). This finishes the proof. □

Let us write down one immediate consequence of Theorem 2.6. In the next section we will prove a generalization of this result in the setting of the RPC, which will yield an important representation of the Parisi formula for the free energy in Chap. 3.

Lemma 2.2. *If* $\mathbb{E}X^{\zeta} < \infty$ *then*

$$\mathbb{E} \log \sum_{n \geq 1} u_n X_n = \mathbb{E} \log \sum_{n \geq 1} u_n + \frac{1}{\zeta} \log \mathbb{E} X^{\zeta}. \tag{2.21}$$

Proof. Clearly, the equality (2.21) follows from Theorem 2.6 if we show that $\log\sum_{n\geq 1} u_n$ is integrable. By (2.15), the positive part is obviously integrable. To show that the negative part is integrable, let us bound

$$\log\sum_{n\geq 1} u_n \geq \log u_1$$

and notice that, for $t>0$, the probability $\mathbb{P}(\log u_1 \leq -t)$ can be written as

$$\mathbb{P}\big(u_1 \leq e^{-t}\big) = \mathbb{P}\big(\Pi\cap(e^{-t},\infty)=\emptyset\big) = \exp\big(-e^{\zeta t}\big),$$

since the cardinality of $\Pi\cap(e^{-t},\infty)$ has the Poisson distribution with the mean

$$\int_{e^{-t}}^{\infty} \zeta x^{-1-\zeta} dx = e^{\zeta t}.$$

This implies that the negative part is integrable, which finishes the proof. □

Let us point out a simple consequence of Eq. (2.21) that will be needed in the next section. Suppose that $Z>0$ is another random variable such that $\mathbb{E}Z^{\zeta}<\infty$ and $(Z_n)_{n\geq 1}$ are i.i.d. copies of Z, independent of all other random variables. Then,

$$\mathbb{E}\log\sum_{n\geq 1} u_n X_n Z_n = \mathbb{E}\log\sum_{n\geq 1} u_n Z_n + \frac{1}{\zeta}\log\mathbb{E}X^{\zeta}. \tag{2.22}$$

To show this, we simply apply Eq. (2.21) to both sides and use that

$$\frac{1}{\zeta}\log\mathbb{E}(XZ)^{\zeta} = \frac{1}{\zeta}\log\mathbb{E}X^{\zeta} + \frac{1}{\zeta}\log\mathbb{E}Z^{\zeta}.$$

Next, we will obtain a key invariance property of the Poisson–Dirichlet process that will be based on the following special case of Theorem 2.6. Let (g_n) be a sequence of i.i.d. standard Gaussian random variables and, given $t\in\mathbb{R}$, consider

$$X_n = \exp t\big(g_n - t\zeta/2\big) \text{ and } Y_n = g_n - t\zeta.$$

Since

$$\mathbb{E}X^{\zeta} = 1 \text{ and } \frac{X^{\zeta}}{\mathbb{E}X^{\zeta}} = \exp\Big(t\zeta g - \frac{1}{2}t^2\zeta^2\Big),$$

in this case, Eq. (2.17) becomes

$$\begin{aligned}\nu_{\zeta}(B) &= \mathbb{E}I(g-t\zeta\in B)\exp\Big(t\zeta g - \frac{1}{2}t^2\zeta^2\Big)\\ &= \frac{1}{\sqrt{2\pi}}\int I\big(x-t\zeta\in B\big)\exp\Big(-\frac{1}{2}(x-t\zeta)^2\Big)dx = \mathbb{E}I\big(g\in B\big),\end{aligned}$$

which means that $\nu_\zeta = N(0,1)$ is the standard Gaussian distribution on $\mathbb{R}$. Theorem 2.6 implies that the Poisson processes

$$\big\{\big(u_n \exp t(g_n - t\zeta/2), g_n - t\zeta\big)\big\} \text{ and } \big\{(u_n, g_n)\big\} \tag{2.23}$$

have the same mean measure. We will see below that this property will allow us to characterize the Poisson–Dirichlet process (v_n) in Eq. (2.14). Compared with (u_n), the sequence $(u_n \exp t(g_n - t\zeta/2))$ is not necessarily arranged in the decreasing order, but if we let $\pi : \mathbb{N} \to \mathbb{N}$ be a random permutation such that the sequence $(u_{\pi(n)} \exp t(g_{\pi(n)} - t\zeta/2))$ is also decreasing then Eq. (2.23) implies that

$$\big((u_{\pi(n)} \exp t(g_{\pi(n)} - t\zeta/2), g_{\pi(n)} - t\zeta)\big)_{n\geq 1} \stackrel{d}{=} \big((u_n, g_n)\big)_{n\geq 1}. \tag{2.24}$$

Let us note that $\pi : \mathbb{N} \to \mathbb{N}$ is a random bijection, since, with probability one, a process with the mean measure Eq. (2.12) has finitely many points on each interval (ε, ∞) and, therefore, by Eq. (2.23), each $u_n \exp t(g_n - t\zeta/2)$ will have a finite rank after the rearrangement. If, similarly to Eq. (2.14), we define

$$v_n^t = \frac{u_n \exp t(g_n - t\zeta/2)}{\sum_{l\geq 1} u_l \exp t(g_l - t\zeta/2)} = \frac{v_n \exp t g_n}{\sum_{l\geq 1} v_l \exp t g_l}, \tag{2.25}$$

then Eq. (2.24) implies the *Bolthausen–Sznitman invariance* property for the Poisson–Dirichlet process $PD(\zeta)$,

$$\big((v^t_{\pi(n)}, g_{\pi(n)} - t\zeta)\big)_{n\geq 1} \stackrel{d}{=} \big((v_n, g_n)\big)_{n\geq 1}. \tag{2.26}$$

We will show that this invariance property, indeed, characterizes the Poisson–Dirichlet process (v_n). However, in anticipation of a more complex situation considered in the next section and in order to make a connection to random measures on a Hilbert space, it will be instructive to associate (v_n) with a random measure G supported on some fixed orthonormal sequence (e_n) in a Hilbert space H by letting

$$G(e_n) = v_n. \tag{2.27}$$

We will also view the i.i.d. standard Gaussian sequence (g_n) as a Gaussian process $(g(e_n))$ indexed by the sequence (e_n) with the covariance matrix given by the Gram matrix

$$\mathbb{E} g(e_n) g(e_m) = e_n \cdot e_m = \delta_{nm}. \tag{2.28}$$

Of course, the geometric structure of the measure G is very simple and all the information is contained in the weights (v_n). We can think of the sequence in Eq. (2.25) as the weights of another random measure G_t obtained from G by the random change of density proportional to $\exp t g(e_n)$,

$$G_t(e_n) = \frac{G(e_n) \exp t g(e_n)}{\sum_{l\geq 1} G(e_l) \exp t g(e_l)} = v_n^t. \tag{2.29}$$

By Eq. (2.26), the weights of this new random measure after the rearrangement in the decreasing order are equal in distribution to the weights of G also arranged in the decreasing order. This does not mean that the random measures G and G_t on the Hilbert space H are equal in distribution exactly because $G(e_n)$ were arranged in the decreasing order from the beginning while $G_t(e_n)$ were not. However, notice that they are equal in distribution up to a random orthogonal transformation on H which sends $e_{\pi(n)}$ to e_n since this would again assign the nth largest weight in G_t to e_n. Let us write down what the invariance (2.26) means for these measures. Let us denote by $(\sigma^l)_{l\geq 1}$ an i.i.d. sample either from the measure G or G_t and let

$$R_{l,l'} = \sigma^l \cdot \sigma^{l'} = I(\sigma^l = \sigma^{l'}) \tag{2.30}$$

be the scalar product, or overlap, of σ^l and $\sigma^{l'}$. Again, because of the trivial geometric structure of G, the overlap simply describes whether two points are equal or not. Let us denote by $\langle\cdot\rangle$ the average with respect to the product measure $G^{\otimes\infty}$, which means that for any $n \geq 1$ and any (possibly random) function $f = f(\sigma^1, \ldots, \sigma^n)$,

$$\big\langle f \big\rangle = \sum_{j_1,\ldots,j_n\geq 1} f(e_{j_1},\ldots,e_{j_n}) G(e_{j_1})\cdots G(e_{j_n}). \tag{2.31}$$

Similarly, let us denote by $\langle\cdot\rangle_t$ the average with respect to $G_t^{\otimes\infty}$,

$$\big\langle f \big\rangle_t = \sum_{j_1,\ldots,j_n\geq 1} f(e_{j_1},\ldots,e_{j_n}) G_t(e_{j_1})\cdots G_t(e_{j_n}). \tag{2.32}$$

Then the invariance property in Eq. (2.26) implies the following.

Lemma 2.3. *For any $n \geq 1$, any integers $k_1, \ldots, k_n \geq 0$, and any bounded function f of the overlaps $R^n := (R_{l,l'})_{l,l'\leq n}$,*

$$\mathbb{E}\Big\langle f(R^n)\prod_{l\leq n}\big(g(\sigma^l) - t\zeta\big)^{k_l}\Big\rangle_t = \mathbb{E}\Big\langle f(R^n)\prod_{l\leq n} g(\sigma^l)^{k_l}\Big\rangle. \tag{2.33}$$

In fact, Eq. (2.33) is nothing but an artificial reformulation of the invariance property (2.26) in terms of the measure G. The geometric nature of G as a measure on a Hilbert space also allows us to include the information about the overlaps $(R_{l,l'})$ in Eq. (2.30) in a rather trivial way. In the next section we will increase the complexity and the corresponding result will become less trivial and much more interesting.

Proof. This should be almost obvious from the invariance property (2.26) by writing down the following formal computation:

$$\begin{aligned}
&\mathbb{E}\sum_{j_1,\ldots,j_n\geq 1} f\big((e_{j_l}\cdot e_{j_{l'}})\big)\prod_{l\leq n}(g(e_{j_l}) - t\zeta)^{k_l} G_t(e_{j_l}) \\
&\quad = \mathbb{E}\sum_{j_1,\ldots,j_n\geq 1} f\big((e_{\pi(j_l)}\cdot e_{\pi(j_{l'})})\big)\prod_{l\leq n}(g(e_{\pi(j_l)}) - t\zeta)^{k_l} G_t(e_{\pi(j_l)})
\end{aligned}$$

$$= \mathbb{E} \sum_{j_1,\dots,j_n \geq 1} f\big((e_{j_l} \cdot e_{j_{l'}})\big) \prod_{l \leq n} (g(e_{\pi(j_l)}) - t\zeta)^{k_l} G_t(e_{\pi(j_l)})$$

$$= \mathbb{E} \sum_{j_1,\dots,j_n \geq 1} f\big((e_{j_l} \cdot e_{j_{l'}})\big) \prod_{l \leq n} g(e_{j_l})^{k_l} G(e_{j_l}).$$

The first equality holds because the permutation π is a bijection, in the second equality we used that $e_j \cdot e_{j'} = e_{\pi(j)} \cdot e_{\pi(j')}$ and, finally, the last equality follows from Eq. (2.26). Of course, we implicitly used the fact that all the series are absolutely summable and that we can interchange the expected value with the sum in order to apply the invariance property (2.26) term by term. This is, indeed, the case, since f is bounded and

$$\mathbb{E} \prod_{l \leq n} |g(e_{\pi(j_l)}) - t\zeta|^{k_l} G_t(e_{\pi(j_l)}) = \mathbb{E} \prod_{l \leq n} |g(e_{j_l})|^{k_l} G(e_{j_l}) \leq c\, \mathbb{E} \prod_{l \leq n} G(e_{j_l})$$

for some constant $c = c(k_1, \dots, k_n)$, where the equality follows from Eq. (2.26) and the inequality follows by bounding the expectation in the Gaussian random variables $(g(e_l))$ using, for example, Hölder's inequality. □

Let us now set $k_1 = 1$ and $k_2 = \dots = k_n = 0$. In this case, the right-hand side of Eq. (2.33) is zero by averaging $g(\sigma^1)$ first (since the measure does not depend on the Gaussian sequence) and the left-hand side can be computed using the Gaussian integration by parts formula in Lemma 1.2. Using the fact that, by Eqs. (2.28) and (2.30), $\mathbb{E} g(\sigma^l) g(\sigma^{l'}) = R_{l,l'}$,

$$\begin{aligned} \mathbb{E}\Big\langle f(R^n)(g(\sigma^1) - t\zeta)\Big\rangle_t &= t\mathbb{E}\Big\langle f(R^n)\Big(\sum_{l=1}^n R_{1,l} - \zeta - nR_{1,n+1}\Big)\Big\rangle_t \\ &= t\mathbb{E}\Big\langle f(R^n)\Big(\sum_{l=1}^n R_{1,l} - \zeta - nR_{1,n+1}\Big)\Big\rangle, \end{aligned}$$

where in the second line we replaced the average with respect to G_t by the average with respect to G using Eq. (2.33) with all $k_l = 0$ or, in other words, using that, by Eq. (2.26), the weights $(v^t_{\pi(n)})$ and (v_n) are equal in distribution. Equating this to zero and using that $R_{1,1} = 1$, we get

$$\mathbb{E}\big\langle f(R^n) R_{1,n+1}\big\rangle = \frac{1-\zeta}{n} \mathbb{E}\big\langle f(R^n)\big\rangle + \frac{1}{n} \sum_{l=2}^n \mathbb{E}\big\langle f(R^n) R_{1,l}\big\rangle.$$

In particular, recalling Eq. (2.30), the choice of $n = 1$ and $f = 1$ gives that

$$\mathbb{E}\big\langle R_{1,2}\big\rangle = \mathbb{E}\big\langle I(\sigma^1 = \sigma^2)\big\rangle = \mathbb{E} \sum_{j \geq 1} v_j^2 = 1 - \zeta. \tag{2.34}$$

Substituting $\mathbb{E}\langle R_{1,2}\rangle$ for $1-\zeta$ in the above equation, we proved that the overlaps Eq. (2.30) of the sample from the measure G defined in Eq. (2.27) satisfy the *Ghirlanda–Guerra identities*.

Theorem 2.7. *For any $n \geq 1$ and any function f of the overlaps $R^n = (R_{l,l'})_{l,l' \leq n}$,*

$$\mathbb{E}\big\langle f R_{1,n+1}\big\rangle = \frac{1}{n}\mathbb{E}\langle f\rangle\mathbb{E}\langle R_{1,2}\rangle + \frac{1}{n}\sum_{l=2}^{n}\mathbb{E}\big\langle f R_{1,l}\big\rangle. \tag{2.35}$$

Equation (2.35) is the first appearance of the Ghirlanda–Guerra identities that will play a central role in the Sherrington–Kirkpatrick model. In this setting, they are sometimes called the Ghirlanda–Guerra identities for the Poisson–Dirichlet process $PD(\zeta)$. We will see that even in more complicated situations these identities contain almost all of the information about a random measure on a Hilbert space that satisfies them. Here, the structure of the measure is very simple, since G is supported on an orthonormal sequence (e_n), so we only need to show that the identities uniquely determine the distribution of the weights $v_n = G(e_n)$. Since the identities (2.35) themselves do not depend on the parameter ζ, we need to specify ζ via Eq. (2.34).

Theorem 2.8. *If Eqs. (2.34) and (2.35) hold then the decreasing enumeration of the weights $(v_n)_{n\geq 1}$ of the measure G is a Poisson–Dirichlet process $PD(\zeta)$.*

We will prove this by showing that the identities (2.34) and (2.35) uniquely determine all the quantities

$$S(n_1,\dots,n_k) = \mathbb{E}\sum_{j\geq 1} v_j^{n_1}\cdots\sum_{j\geq 1} v_j^{n_k}, \tag{2.36}$$

for all $k \geq 1$ and all $n_1,\dots,n_k \geq 1$. To see this, let us express Eq. (2.36) in terms of the overlaps. Let $n = n_1 + \ldots + n_k$ and let us partition the set $\{1,\dots,n\}$ into some subsets $(I_j)_{j\leq k}$ with the cardinalities $(n_j)_{j\leq k}$. For $l,l' \leq n$, let us write $l \sim l'$ if and only if l and l' belong to the same set I_j and let

$$f(R^n) = \prod_{1\leq l\sim l'\leq n} I(\sigma^l = \sigma^{l'})$$

be the indicator of the event that any two replicas with indices in the same partition set are equal. With this choice of f, it is obvious that

$$S(n_1,\dots,n_k) = \mathbb{E}\big\langle f(R^n)\big\rangle. \tag{2.37}$$

Without loss of generality, let us assume that $1 \in I_1$, in which case, similarly,

$$\mathbb{E}\big\langle f R_{1,n+1}\big\rangle = \mathbb{E}\big\langle f I\big(\sigma^1 = \sigma^{n+1}\big)\big\rangle = S(n_1+1,\dots,n_k). \tag{2.38}$$

On the other hand, Eqs. (2.34) and (2.35) imply that

$$\mathbb{E}\big\langle f R_{1,n+1}\big\rangle = \frac{1-\zeta}{n}\mathbb{E}\langle f\rangle + \frac{1}{n}\sum_{l=2}^{n}\mathbb{E}\big\langle f R_{1,l}\big\rangle.$$

Now, notice that $fR_{1,l} = f$ if $l \in I_1$ and there are $n_1 - 1$ such l between 2 and n. On the other hand, if l is one of the n_j indices in I_j for $j \geq 2$ then multiplying f by $R_{1,l}$ merges the set I_j with the set I_1 in a sense that $fR_{1,l}$ is defined just as f only with these two subsets merged into one to form a new partition of $\{1,\dots,n\}$. Combining these observations with Eqs. (2.37) and (2.38), we obtain *Talagrand's identities* for the Poisson–Dirichlet process $PD(\zeta)$,

$$S(n_1+1,\dots,n_k) = \frac{n_1-\zeta}{n} S(n_1,\dots,n_k) + \sum_{j=2}^{k} \frac{n_j}{n} S(n_2,\dots,n_j+n_1,\dots,n_k). \quad (2.39)$$

Notice that the sum of the coordinates in each term on the right-hand side has been reduced by one compared to the left-hand side. Moreover, since $\sum_j v_j = 1$, we can always omit any coordinate $n_l = 1$ in the definition of $S(n_1,\dots,n_k)$ in Eq. (2.36). Thus, using Eq. (2.39) recursively, we can express all the quantities $S(n_1,\dots,n_k)$ in Eq. (2.36) in terms of $S(2)$, which is equal to $1-\zeta$ by Eq. (2.34). Once we know all these quantities, Theorem 2.8 is intuitively clear from the following simple consideration. Given a polynomial P of one real variable, let us denote

$$m(P) = \sum_{j\geq 1} P(v_j) v_j.$$

Then, the quantities (2.36) determine all the joint moments

$$\mathbb{E} m(P_1)^{n_1} \cdots m(P_k)^{n_k}$$

for any polynomials $P_1,\dots,P_k$, which means that they determine the joint distribution of the sequence $(m(P_k))_{k\geq 1}$ for any countable collection of polynomials dense in $C([0,1], \|\cdot\|_\infty)$, for example, all polynomials with rational coefficients. Given the sequence $(m(P_k))_{k\geq 1}$, we can reconstruct $\sum_{j\geq 1} v_j I(a < v_j \leq b)$ for any $a,b \in [0,1]$ and then reconstruct the sequence (v_j), thus, showing that each v_j is measurable with respect to the σ-algebra generated by $(m(P_k))_{k\geq 1}$ and their joint distribution is determined by the quantities (2.36). Let us now make this intuition into a rigorous argument.

Proof (Theorem 2.8). Of course, we can think of the sequence of weights (v_j) of the measure G as an element of the compact product space $[0,1]^{\mathbb{N}}$ with the product topology, or even as an element of the compact subset V of this space given by

$$V = \Big\{ (x_j)_{j\geq 1} : \sum_{j\geq 1} x_j \leq 1, (x_j) \text{ is nonincreasing} \Big\}.$$

For $n \geq 2$, let us define the function $p_n : V \to \mathbb{R}$ by $p_n((x_j)) = \sum_{j\geq 1} x_j^n$. It is easy to see that all these functions are continuous on V as uniform limits of continuous functions, since

$$\Big| p_n\big((x_j)\big) - \sum_{j<m} x_j^n \Big| \leq x_m^{n-1} \sum_{j\geq m} x_j \leq x_m^{n-1} \leq m^{-(n-1)}.$$

(We excluded the value $n = 1$ because p_1 is not continuous with respect to the product topology.) Let $\mathscr{P}$ be the family of functions on V given by finite linear combinations of constants and products $p_{n_1} \cdots p_{n_k}$ for any $k \geq 1$ and $n_1, \ldots, n_k \geq 2$. Since $\mathscr{P}$ is closed under multiplication and linear combinations, it forms an algebra of continuous functions on the compact set V. If we can show that $\mathscr{P}$ separates points then, by the Stone–Weierstrass theorem, we can approximate any continuous function on V uniformly by the functions in $\mathscr{P}$. In particular, since the projections $\pi((x_j)) = x_j$ are continuous on V, this will imply that the distribution of (v_j) is determined by the quantities in Eq. (2.36) and finish the proof. It remains to show that $\mathscr{P}$ separates points. If the sequences (x_j) and (y_j) in V are different then the measures

$$\mu_x = \sum_{j \geq 1} x_j^2 \delta_{x_j} \text{ and } \mu_y = \sum_{j \geq 1} y_j^2 \delta_{y_j}$$

on $[0,1]$ are also different, since for each nonzero value x_j, we can count the multiplicity of x_j in the sequence (x_j) by $\mu_x(x_j)/x_j^2$. Therefore, we can find a polynomial P such that

$$\sum_{j \geq 1} x_j^2 P(x_j) = \int P d\mu_x \neq \int P d\mu_y = \sum_{j \geq 1} y_j^2 P(y_j),$$

and, since $\sum_{j \geq 1} x_j^2 P(x_j)$ is in $\mathscr{P}$, this proves that $\mathscr{P}$ separates points. □

2.3 The Ruelle Probability Cascades

Using the Poisson processes introduced in the previous section, in this section we will construct a class of random measures on a separable Hilbert space H that are supported on certain ultrametric sets rather than an orthonormal sequence. The points and weights of these measures will be indexed by $\mathbb{N}^r$ for some fixed $r \geq 1$. It will be very convenient to think of $\mathbb{N}^r$ as the set of leaves of a rooted tree with the vertex set

$$\mathscr{A} = \mathbb{N}^0 \cup \mathbb{N} \cup \mathbb{N}^2 \cup \ldots \cup \mathbb{N}^r, \tag{2.40}$$

where $\mathbb{N}^0 = \{\emptyset\}$, $\emptyset$ is the root of the tree, and each vertex $\alpha = (n_1, \ldots, n_p) \in \mathbb{N}^p$ for $p \leq r-1$ has children

$$\alpha n := (n_1, \ldots, n_p, n) \in \mathbb{N}^{p+1}$$

for all $n \in \mathbb{N}$. Therefore, each vertex α is connected to the root $\emptyset$ by the path

$$\emptyset \to n_1 \to (n_1, n_2) \to \cdots \to (n_1, \ldots, n_p) = \alpha.$$

We will denote all the vertices in this path by (the root is not included)

$$p(\alpha) = \big\{ n_1, (n_1, n_2), \ldots, (n_1, \ldots, n_p) \big\}. \tag{2.41}$$

The identification of the index set $\mathbb{N}^r$ with the leaves of this infinitary tree is very important, because, even though the points in the support of the random measure we will construct will be indexed by $\alpha \in \mathbb{N}^r$, the construction itself will involve random variables indexed by vertices of the entire tree. In addition to fixing some $r \geq 1$, let us fix two sequences of parameters,

$$0 < \zeta_0 < \ldots < \zeta_{r-1} < 1 \tag{2.42}$$

and

$$0 = q_0 < q_1 < \ldots < q_r < \infty. \tag{2.43}$$

For each vertex $\alpha \in \mathscr{A}$, let us denote by $|\alpha|$ its distance from the root of the tree $\emptyset$, or, equivalently, the number of coordinates in α, i.e., $\alpha \in \mathbb{N}^{|\alpha|}$. For each $\alpha \in \mathscr{A} \setminus \mathbb{N}^r$, let Π_α be a Poisson process with the mean measure Eq. (2.12) with $\zeta = \zeta_{|\alpha|}$, and let us generate these processes independently for all such α. As in the previous section, we are going to arrange all the points in Π_α in the decreasing order,

$$u_{\alpha 1} > u_{\alpha 2} > \ldots > u_{\alpha n} > \ldots, \tag{2.44}$$

and enumerate them using the children $(\alpha n)_{n \geq 1}$ of the vertex α. In other words, parent vertices $\alpha \in \mathscr{A} \setminus \mathbb{N}^r$ enumerate independent Poisson processes Π_α and child vertices $\alpha n \in \mathscr{A} \setminus \mathbb{N}^0$ enumerate individual points $u_{\alpha n}$. Of course, we will not always denote child vertices by αn, explicitly referring to their parent vertices α, so the notation u_α will also be used for individual points for $\alpha \in \mathscr{A} \setminus \mathbb{N}^0$. Given a vertex $\alpha \in \mathscr{A} \setminus \mathbb{N}^0$ and the path $p(\alpha)$ in Eq. (2.41), we define

$$w_\alpha = \prod_{\beta \in p(\alpha)} u_\beta. \tag{2.45}$$

Finally, for the leaf vertices $\alpha \in \mathbb{N}^r$ we define

$$v_\alpha = \frac{w_\alpha}{\sum_{\beta \in \mathbb{N}^r} w_\beta}. \tag{2.46}$$

The sequence $(v_\alpha)_{\alpha \in \mathbb{N}^r}$ will represent the weights of a random measure on a Hilbert space, but, of course, we will first need to show the following.

Lemma 2.4. *With probability one, the sum $\sum_{\alpha \in \mathbb{N}^r} w_\alpha < \infty$ and, hence, the sequence (2.46) is well defined.*

Proof. The case $r = 1$ was considered in the previous section, so we assume that $r \geq 2$. First of all, if for each parent vertex $\alpha \in \mathscr{A} \setminus \mathbb{N}^r$ we denote

$$U_\alpha = \sum_{n \in \mathbb{N}} u_{\alpha n} = \sum_{x \in \Pi_\alpha} x, \tag{2.47}$$

then, using that $w_{\alpha n} = w_\alpha u_{\alpha n}$ by the definition (2.45), we can write

$$\sum_{\alpha \in \mathbb{N}^r} w_\alpha = \sum_{\alpha \in \mathbb{N}^{r-1}} w_\alpha U_\alpha = \sum_{\alpha \in \mathbb{N}^{r-2}} w_\alpha \sum_{n \in \mathbb{N}} u_{\alpha n} U_{\alpha n}. \tag{2.48}$$

For a fixed $\alpha \in \mathbb{N}^{r-2}$, the random variables $(U_{\alpha n})_{n\geq 1}$ are i.i.d., since the Poisson processes $\Pi_{\alpha n}$ for $n \geq 1$ are independent and have the same mean measure Eq. (2.12) corresponding to the parameter $\zeta = \zeta_{r-1}$. By Theorem 2.6,

$$\sum_{n\in\mathbb{N}} u_{\alpha n} U_{\alpha n} \stackrel{d}{=} c \sum_{n\in\mathbb{N}} u_{\alpha n} = cU_\alpha, \tag{2.49}$$

where $c = (\mathbb{E}U_{\alpha 1}^{\zeta_{r-2}})^{1/\zeta_{r-2}}$. Equation (2.15) implies that

$$\mathbb{E}U_{\alpha 1}^{\zeta_{r-2}} = \mathbb{E}\Big(\sum_{x\in\Pi_{\alpha 1}} x\Big)^{\zeta_{r-2}} < \infty,$$

since $\zeta_{r-2} < \zeta_{r-1}$ and, moreover, this quantity is the same for all $\alpha \in \mathbb{N}^{r-2}$. Since both sides of Eq. (2.49) are i.i.d. over $\alpha \in \mathbb{N}^{r-2}$ and independent of $(w_\alpha)_{\alpha\in\mathbb{N}^{r-2}}$, Eqs. (2.48) and (2.49) imply that the sum $\sum_{\alpha\in\mathbb{N}^r} w_\alpha$ is finite with probability one if and only if the sum

$$\sum_{\alpha\in\mathbb{N}^{r-2}} w_\alpha U_\alpha = \sum_{\alpha\in\mathbb{N}^{r-2}} w_\alpha \sum_{n\in\mathbb{N}} u_{\alpha n} = \sum_{\alpha\in\mathbb{N}^{r-1}} w_\alpha$$

is finite with probability one. We can proceed by induction on r to conclude that all these sums are finite with probability one. □

Before we introduce the main object of this section, we will describe one useful generalization of Lemma 2.2 for the sequence $(v_\alpha)_{\alpha\in\mathbb{N}^r}$, which will be used in the next chapter to represent the Parisi formula for the free energy in terms of the RPC defined below. Consider i.i.d. random variables $(\omega_p)_{1\leq p\leq r}$ that have the uniform distribution on $[0,1]$ and some function

$$X_r = X_r(\omega_1, \ldots, \omega_r), \tag{2.50}$$

which satisfies $\mathbb{E}\exp\zeta_{r-1}X_r < \infty$. Let us define recursively for $0 \leq l \leq r-1$,

$$X_l = X_l(\omega_1, \ldots, \omega_l) = \frac{1}{\zeta_l}\log \mathbb{E}_l \exp\zeta_l X_{l+1}, \tag{2.51}$$

where $\mathbb{E}_l$ denotes the expectation with respect to ω_{l+1}. In particular, X_0 is non-random. By Hölder's inequality, $\exp\zeta_{r-1}X_l \leq \mathbb{E}_l \exp\zeta_{r-1}X_{l+1}$ and, therefore,

$$\mathbb{E}\exp\zeta_{r-1}X_l \leq \mathbb{E}\exp\zeta_{r-1}X_r < \infty,$$

which means that the sequence (X_l) is well defined. Now, consider a sequence $(\omega_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ of i.i.d. random variables with the uniform distribution on $[0,1]$. Then the following holds.

Theorem 2.9. *If X_0 is defined via the recursive construction (2.51) then*

$$X_0 = \mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} v_\alpha \exp X_r\big((\omega_\beta)_{\beta\in p(\alpha)}\big). \tag{2.52}$$

Proof. Let us denote $\Omega_\alpha = (\omega_\beta)_{\beta\in p(\alpha)} = (\omega_{n_1},\dots,\omega_{(n_1,\dots,n_p)})$ if $\alpha = (n_1,\dots,n_p)$. Recalling the definition (2.46), in order to prove Eq. (2.52), it is enough to show that

$$\mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} w_\alpha \exp X_r\big(\Omega_\alpha\big) = \mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} w_\alpha + X_0 \tag{2.53}$$

and that both sides are well defined. To set up the induction on r, we will consider a slightly more general situation. Let $Z > 0$ be a random variable such that $\mathbb{E}Z^{\zeta_{r-1}} < \infty$ and let $(Z_\alpha)_{\alpha\in\mathbb{N}^r}$ be a sequence of i.i.d. copies of Z independent of all other random variables. Then, we will show that

$$\mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} w_\alpha Z_\alpha \exp X_r\big(\Omega_\alpha\big) = \mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} w_\alpha Z_\alpha + X_0. \tag{2.54}$$

The case $r = 1$ was proved in Eq. (2.22) and we will proceed by induction on r. If we write $\Omega_{\alpha n} = (\Omega_\alpha, \omega_{\alpha n})$, then, similarly to Eq. (2.48),

$$\sum_{\alpha\in\mathbb{N}^r} w_\alpha Z_\alpha \exp X_r\big(\Omega_\alpha\big) = \sum_{\alpha\in\mathbb{N}^{r-1}} w_\alpha \sum_{n\ge 1} u_{\alpha n} Z_{\alpha n} \exp X_r\big(\Omega_\alpha, \omega_{\alpha n}\big). \tag{2.55}$$

Let $\mathscr{F}_{r-1}$ be the σ-algebra generated by $(w_\alpha)_{\alpha\in\mathbb{N}^{r-1}}$ and $(\Omega_\alpha)_{\alpha\in\mathbb{N}^{r-1}}$. For each $\alpha\in\mathbb{N}^{r-1}$, the random variables $(u_{\alpha n})_{n\ge1}$, $(Z_{\alpha n})_{n\ge1}$, and $(\omega_{\alpha n})_{n\ge1}$ are independent of $\mathscr{F}_{r-1}$ and, therefore, if we denote $c = (\mathbb{E}Z^{\zeta_{r-1}})^{1/\zeta_{r-1}}$ and observe that, by the recursive definition (2.51),

$$\mathbb{E}\Big(\big(Z_{\alpha n}\exp X_r(\Omega_\alpha,\omega_{\alpha n})\big)^{\zeta_{r-1}}\,\Big|\,\mathscr{F}_{r-1}\Big)^{1/\zeta_{r-1}} = c\exp X_{r-1}(\Omega_\alpha), \tag{2.56}$$

then Theorem 2.6 implies that, conditionally on $\mathscr{F}_{r-1}$,

$$\begin{aligned}\sum_{n\ge1} u_{\alpha n} Z_{\alpha n}\exp X_r\big(\Omega_\alpha,\omega_{\alpha n}\big) &\stackrel{d}{=} c\exp X_{r-1}(\Omega_\alpha)\sum_{n\ge1} u_{\alpha n}\\ &= cU_\alpha \exp X_{r-1}(\Omega_\alpha),\end{aligned} \tag{2.57}$$

where we again used the notation (2.47). Since, conditionally on $\mathscr{F}_{r-1}$, both sides of Eq. (2.57) are, obviously, independent over $\alpha\in\mathbb{N}^{r-1}$, Eq. (2.55) implies that, conditionally on $\mathscr{F}_{r-1}$,

$$\sum_{\alpha\in\mathbb{N}^r} w_\alpha Z_\alpha \exp X_r\big(\Omega_\alpha\big) \stackrel{d}{=} c\sum_{\alpha\in\mathbb{N}^{r-1}} w_\alpha U_\alpha \exp X_{r-1}(\Omega_\alpha).$$

This, of course, implies that the same equality in distribution holds unconditionally. Similarly, one can show that

$$\sum_{\alpha\in\mathbb{N}^r} w_\alpha Z_\alpha \stackrel{d}{=} c\sum_{\alpha\in\mathbb{N}^{r-1}} w_\alpha U_\alpha,$$

and, therefore, in order to prove Eq. (2.54), it is enough to show that

$$\mathbb{E}\log\sum_{\alpha\in\mathbb{N}^{r-1}} w_\alpha U_\alpha \exp X_{r-1}(\Omega_\alpha) = \mathbb{E}\log\sum_{\alpha\in\mathbb{N}^{r-1}} w_\alpha U_\alpha + X_0. \tag{2.58}$$

Since the random variables $(U_\alpha)_{\alpha\in\mathbb{N}^{r-1}}$ are i.i.d. and independent of all other random variables in Eq. (2.58) and, by (2.15),

$$\mathbb{E}U_\alpha^{\zeta_{r-2}} = \mathbb{E}\Big(\sum_{n\geq 1} u_{\alpha n}\Big)^{\zeta_{r-2}} < \infty,$$

Eq. (2.58) is of the same type as (2.54) with r replaced by $r-1$ and (Z_α) replaced by (U_α), which means that we can conclude by induction on r that Eq. (2.58) holds. This finishes the proof. □

Theorem 2.9 will sometimes be applied in the following way. We can use each random variable ω_l in (2.50) to generate N independent copies $\omega_{l,i}$ from the uniform distribution on $[0,1]$ and replace X_r by the sum $\sum_{i\leq N} X_{r,i}$ of N independent copies

$$X_{r,i} = X_r(\omega_{1,i},\dots,\omega_{r,i}).$$

By induction, Eq. (2.51) implies that each X_l is then replaced by the sum

$$\frac{1}{\zeta_l}\log\mathbb{E}_l\exp\zeta_l\sum_{i=1}^N X_{l+1,i} = \sum_{i=1}^N\frac{1}{\zeta_l}\log\mathbb{E}_l\exp\zeta_l X_{l+1,i} = \sum_{i=1}^N X_{l,i} \tag{2.59}$$

of N independent copies and, in the end, X_0 is replaced by NX_0. Similarly, if for $\alpha\in\mathscr{A}\setminus\mathbb{N}^0$ we use each random variable ω_α to generate N independent copies $\omega_{\alpha,i}$, Theorem 2.9 implies that

$$X_0 = \frac{1}{N}\mathbb{E}\log\sum_{\alpha\in\mathbb{N}^r} v_\alpha\exp\sum_{i=1}^N X_r\big((\omega_{\beta,i})_{\beta\in p(\alpha)}\big). \tag{2.60}$$

Next, we will construct a random measure on a separable Hilbert space H using the weights $(v_\alpha)_{\alpha\in\mathbb{N}^r}$ in Eq. (2.46) as follows. Let e_α for $\alpha\in\mathscr{A}\setminus\mathbb{N}^0$ be a sequence of orthonormal vectors in H. Given this sequence, we consider a set of points $h_\alpha\in H$ indexed by $\alpha\in\mathscr{A}\setminus\mathbb{N}^0$,

$$h_\alpha = \sum_{\beta\in p(\alpha)} e_\beta\big(q_{|\beta|}-q_{|\beta|-1}\big)^{1/2}, \tag{2.61}$$

where the parameters $(q_p)_{0\leq p\leq r}$ were introduced in Eq. (2.43). Finally, we define a random measure G on the Hilbert space H by

$$G(h_\alpha) = v_\alpha \text{ for } \alpha\in\mathbb{N}^r. \tag{2.62}$$

We will call the measure G the *Ruelle probability cascade* (or RPC for short) associated to the parameters (2.42) and (2.43). The dependence of the measure G on the orthonormal sequence (e_α) will be kept implicit, since, in some sense, we will be interested in this measure only up to orthogonal transformations. In fact, our main interest will be in the distribution of the scalar products of points sampled from G, which, obviously, does not depend on the choice of (e_α). From the definition (2.61), it is evident that the scalar product $h_\alpha \cdot h_\beta$ between any two points in the support of G depends only on the number of common vertices in the paths from the root $\emptyset$ to the leaves $\alpha, \beta \in \mathbb{N}^r$ that will be denoted by

$$\alpha \wedge \beta = |p(\alpha) \cap p(\beta)|. \tag{2.63}$$

With this notation, Eq. (2.61) implies that $h_\alpha \cdot h_\beta = q_{\alpha\wedge\beta}$. Now, if we take three leaves $\alpha, \beta, \gamma \in \mathbb{N}^r$ then their paths satisfy

$$\beta \wedge \gamma \geq \min\big(\alpha \wedge \beta, \alpha \wedge \gamma\big),$$

since the vertices shared by the path $p(\alpha)$ with both paths $p(\beta)$ and $p(\gamma)$ will also be shared by $p(\beta)$ and $p(\gamma)$ and, therefore,

$$h_\beta \cdot h_\gamma \geq \min\big(h_\alpha \cdot h_\beta, h_\alpha \cdot h_\gamma\big). \tag{2.64}$$

Using that $h_\alpha \cdot h_\alpha = q_r$ and $\|h_\alpha - h_\beta\|^2 = 2(q_r - h_\alpha \cdot h_\beta)$, this property can be written in terms of the distances between points as

$$\|h_\beta - h_\gamma\| \leq \max\big(\|h_\alpha - h_\beta\|, \|h_\alpha - h_\gamma)\|\big). \tag{2.65}$$

This means that the distances between points in the support of the measure G satisfy the *strong triangle* or *ultrametric inequality* or, in other words, the support of G is ultrametric in the Hilbert space H. As in the case of Poisson–Dirichlet processes, we will be interested in the invariance properties of the measure G and, in particular, the Bolthausen–Sznitman invariance property and the Ghirlanda–Guerra identities in the setting of the RPC. In the subsequent sections, we will try to understand to what extent these invariance properties determine the measure G.

In the case of the RPC, before we study the analogue of the Bolthausen–Sznitman invariance property (2.26), it will be convenient to first state the analogue of Lemma 2.3, which expresses this invariance property in terms of the distribution of scalar products, or overlaps, of the sample from the measure G under some random changes of density. Let $(\eta_\alpha)_{\alpha \in \mathscr{A} \setminus \mathbb{N}^0}$ be a sequence of i.i.d. standard Gaussian random variables and, recalling Eq. (2.61), we define a family of Gaussian random variables indexed by $(h_\alpha)_{\alpha \in \mathbb{N}^r}$,

$$g(h_\alpha) = \sum_{\beta \in p(\alpha)} \eta_\beta (q_{|\beta|} - q_{|\beta|-1})^{1/2}. \tag{2.66}$$

It is obvious that the covariance of this Gaussian process is given by

$$\mathbb{E}g(h_\alpha)g(h_\beta) = q_{\alpha\wedge\beta} = h_\alpha \cdot h_\beta, \tag{2.67}$$

which is analogous to Eq. (2.28). Given the measure G in Eq. (2.62) and $t \geq 0$, let us define a measure G_t by the random change of density,

$$G_t(h_\alpha) = \frac{G(h_\alpha)\exp tg(h_\alpha)}{\sum_{\beta\in\mathbb{N}^r} G(h_\beta)\exp tg(h_\beta)}. \tag{2.68}$$

As in the previous section, let us denote by $(\sigma^l)_{l\geq 1}$ an i.i.d. sample either from the measure G or G_t and let

$$R_{l,l'} = \sigma^l \cdot \sigma^{l'} \tag{2.69}$$

be the scalar product, or overlap, of σ^l and $\sigma^{l'}$. Let us denote by $\langle\cdot\rangle$ the average with respect to the product measure $G^{\otimes\infty}$, which means that for any $n \geq 1$ and any (possibly random) function $f = f(\sigma^1, \ldots, \sigma^n)$,

$$\big\langle f \big\rangle = \sum_{\alpha_1,\ldots,\alpha_n\in\mathbb{N}^r} f\big(h_{\alpha_1},\ldots,h_{\alpha_n}\big)\, G\big(h_{\alpha_1}\big)\cdots G\big(h_{\alpha_n}\big). \tag{2.70}$$

Similarly, let us denote by $\langle\cdot\rangle_t$ the average with respect to $G_t^{\otimes\infty}$,

$$\big\langle f \big\rangle_t = \sum_{\alpha_1,\ldots,\alpha_n\in\mathbb{N}^r} f\big(h_{\alpha_1},\ldots,h_{\alpha_n}\big)\, G_t\big(h_{\alpha_1}\big)\cdots G_t\big(h_{\alpha_n}\big). \tag{2.71}$$

Let us denote

$$b = \sum_{p=0}^{r-1} (q_{p+1} - q_p)\zeta_p. \tag{2.72}$$

Then the following generalization of Lemma 2.3 holds.

Lemma 2.5. *For any $n \geq 1$, any integers $k_1, \ldots, k_n \geq 0$ and any bounded function f of the overlaps $R^n = (R_{l,l'})_{l,l'\leq n}$,*

$$\mathbb{E}\Big\langle f(R^n)\prod_{l\leq n}\big(g(\sigma^l) - tb\big)^{k_l}\Big\rangle_t = \mathbb{E}\Big\langle f(R^n)\prod_{l\leq n} g(\sigma^l)^{k_l}\Big\rangle. \tag{2.73}$$

This coincides with Lemma 2.3 if we take $r = 1$, $q_1 = 1$, and $\zeta_0 = \zeta$, and the proof will also be based on the property of Poisson processes described in Theorem 2.6. However, before we explain this, let us observe that Eq. (2.73) implies the Ghirlanda–Guerra identities for the overlaps, almost exactly as in the previous section. Again, let us set $k_1 = 1$ and $k_2 = \ldots = k_n = 0$. The right-hand side of Eq. (2.73) is equal to zero by averaging $g(\sigma^1)$ first and the left-hand side, again, can be computed using the Gaussian integration by parts formula in Lemma 1.2 and the fact that, by Eqs. (2.67) and (2.69), $\mathbb{E}g(\sigma^l)g(\sigma^{l'}) = R_{l,l'}$,

$$\mathbb{E}\Big\langle f(R^n)\big(g(\sigma^1)-tb\big)\Big\rangle_t = t\mathbb{E}\Big\langle f(R^n)\Big(\sum_{l=1}^n R_{1,l}-b-nR_{1,n+1}\Big)\Big\rangle_t$$
$$= t\mathbb{E}\Big\langle f(R^n)\Big(\sum_{l=1}^n R_{1,l}-b-nR_{1,n+1}\Big)\Big\rangle.$$

In the second line we replaced the average with respect to G_t by the average with respect to G using Eq. (2.73) with all $k_l = 0$. Equating this to zero and using that the self-overlap $R_{1,1} = q_r$ gives

$$\mathbb{E}\big\langle f(R^n)R_{1,n+1}\big\rangle = \frac{q_r-b}{n}\mathbb{E}\big\langle f(R^n)\big\rangle + \frac{1}{n}\sum_{l=2}^n \mathbb{E}\big\langle f(R^n)R_{1,l}\big\rangle.$$

In particular, the choice of $n = 1$ and $f = 1$ gives that

$$\mathbb{E}\big\langle R_{1,2}\big\rangle = q_r - b = \sum_{p=1}^r q_p(\zeta_p - \zeta_{p-1}), \tag{2.74}$$

with the convention that $\zeta_r = 1$. Substituting $\mathbb{E}\langle R_{1,2}\rangle$ for $q_r - b$ in the above equation proves the Ghirlanda–Guerra identities,

$$\mathbb{E}\big\langle fR_{1,n+1}\big\rangle = \frac{1}{n}\mathbb{E}\big\langle f\big\rangle\mathbb{E}\big\langle R_{1,2}\big\rangle + \frac{1}{n}\sum_{l=2}^n \mathbb{E}\big\langle fR_{1,l}\big\rangle, \tag{2.75}$$

for any $n \geq 1$ and any function f of the overlaps $(R_{l,l'})_{l,l'\leq n}$. However, compared to the previous section, where the overlaps could take only two values 0 and 1, we can now do something a bit more general. Let us consider any integer $s \geq 1$ and consider a family of Gaussian random variables indexed by $(h_\alpha)_{\alpha\in\mathbb{N}^r}$,

$$g_s(h_\alpha) = \sum_{\beta\in p(\alpha)} \eta_\beta\big(q^s_{|\beta|} - q^s_{|\beta|-1}\big)^{1/2}, \tag{2.76}$$

with the covariance given by

$$\mathbb{E}g_s(h_\alpha)g_s(h_\beta) = q^s_{\alpha\wedge\beta} = \big(h_\alpha\cdot h_\beta\big)^s. \tag{2.77}$$

Working with this Gaussian family instead of Eq. (2.66) and redefining Eq. (2.72) by

$$b_s = \sum_{p=0}^{r-1}(q^s_{p+1} - q^s_p)\zeta_p,$$

one can show using exactly the same argument that the overlaps Eq. (2.69) of the sample from the measure G defined in Eq. (2.62) satisfy

$$\mathbb{E}\big\langle fR^s_{1,n+1}\big\rangle = \frac{1}{n}\mathbb{E}\big\langle f\big\rangle\mathbb{E}\big\langle R^s_{1,2}\big\rangle + \frac{1}{n}\sum_{l=2}^n \mathbb{E}\big\langle fR^s_{1,l}\big\rangle \tag{2.78}$$

for all integer $s \geq 1$, and the Eq. (2.74) now becomess

$$\mathbb{E}\langle R_{1,2}^s \rangle = \sum_{p=1}^{r} q_p^s (\zeta_p - \zeta_{p-1}). \tag{2.79}$$

The identity (2.78), obviously, holds for $s = 0$ and, since the overlap takes finitely many values $q_0, q_1, \ldots, q_r$, we can write any function of the overlap as a polynomial. This implies a more general form of the *Ghirlanda–Guerra identities.*

Theorem 2.10. *For any $n \geq 1$, any function f of the overlaps $R^n = (R_{l,l'})_{l,l' \leq n}$ and any function $\psi : \mathbb{R} \to \mathbb{R}$,*

$$\mathbb{E}\langle f\psi(R_{1,n+1}) \rangle = \frac{1}{n}\mathbb{E}\langle f \rangle \mathbb{E}\langle \psi(R_{1,2}) \rangle + \frac{1}{n}\sum_{l=2}^{n} \mathbb{E}\langle f\psi(R_{1,l}) \rangle. \tag{2.80}$$

Moreover, Eq. (2.79), which holds for all $s \geq 1$, implies that

$$\mathbb{E}\langle \psi(R_{1,2}) \rangle = \sum_{p=0}^{r} \psi(q_p)(\zeta_p - \zeta_{p-1}), \tag{2.81}$$

with the convention that $\zeta_{-1} = 0$ and $\zeta_r = 1$. In particular,

$$\mathbb{E}\langle I(R_{1,2} = q_p) \rangle = \zeta_p - \zeta_{p-1} \quad \text{for} \quad p = 0, \ldots, r. \tag{2.82}$$

This gives a new meaning to the sequences (2.42) and (2.43) as the parameters that encode the distribution of the overlap $R_{1,2}$ under the measure $\mathbb{E}G^{\otimes 2}$, which takes into account both the randomness of the measure G and sampling (σ^l) from G. The distribution function ζ corresponding to Eq. (2.82),

$$\zeta(\{q_p\}) = \zeta_p - \zeta_{p-1}, \tag{2.83}$$

is called the *functional order parameter* (f.o.p.), since we can think of ζ as one parameter that encodes all the parameters of the RPC. We will call the random measure G in Eq. (2.62) the Ruelle probability cascade corresponding to the functional order parameter ζ if the parameters in Eqs. (2.42) and (2.43) agree with Eq. (2.83). We will show in the subsequent sections that the Ghirlanda–Guerra identities, in some sense, completely determine the measure G provided that we know the distribution (2.82) of one overlap $R_{1,2}$.

At this moment, however, let us return to Lemma 2.5 and explain the main idea behind it, namely, the *Bolthausen–Sznitman invariance* property in the setting of the RPC, which appeared in Eq. (2.26) in the case of the Poisson–Dirichlet process. Another generalization of this property will be proved in Theorem 4.4 in Chap. 4.

Theorem 2.11. *There exists a random bijection $\pi : \mathbb{N}^r \to \mathbb{N}^r$, which preserves the number of common vertices (2.63), $\pi(\alpha) \wedge \pi(\beta) = \alpha \wedge \beta$, such that*

$$\Big(\big(G_t(h_{\pi(\alpha)}), g(h_{\pi(\alpha)}) - tb\big)\Big)_{\alpha\in\mathbb{N}^r} \stackrel{d}{=} \Big(\big(G(h_\alpha), g(h_\alpha)\big)\Big)_{\alpha\in\mathbb{N}^r}. \tag{2.84}$$

First, let us notice that Theorem 2.11 implies Lemma 2.5.

Proof (Lemma 2.5). The proof that Eq. (2.84) implies Lemma 2.5 is identical to the proof of Lemma 2.3 from Eq. (2.26), using that the permutation π is a bijection on $\mathbb{N}^r$ and that $h_\alpha \cdot h_\beta = q_{\alpha\wedge\beta} = q_{\pi(\alpha)\wedge\pi(\beta)} = h_{\pi(\alpha)} \cdot h_{\pi(\beta)}$. □

Proof (Theorem 2.11). We will need a slightly more general version of Eq. (2.23), which states that if (g_n) is an i.i.d. sequence of Gaussian random variables with the variance δ then the Poisson processes

$$\big\{(u_n \exp t(g_n - t\delta\zeta/2), g_n - t\delta\zeta)\big\} \text{ and } \big\{(u_n, g_n)\big\} \tag{2.85}$$

have the same mean measure. Of course, this follows from Eq. (2.23) by rescaling, or one can simply repeat the same proof. For $\alpha \in \mathscr{A} \setminus \mathbb{N}^0$, let

$$g_\alpha = \eta_\alpha \big(q_{|\alpha|} - q_{|\alpha|-1}\big)^{1/2} \tag{2.86}$$

be a sequence of independent Gaussian random variables with the variances

$$\delta_{|\alpha|-1} := q_{|\alpha|} - q_{|\alpha|-1} \tag{2.87}$$

that depend on the distance $|\alpha|$ of the vertex α from the root of the tree. Since for a fixed $\alpha \in \mathscr{A} \setminus \mathbb{N}^r$, $\Pi_\alpha = \{u_{\alpha n} : n \geq 1\}$ is a Poisson process with the mean measure Eq. (2.12) with $\zeta = \zeta_{|\alpha|}$, Eq. (2.85) implies that the Poisson processes

$$\Big\{\big(u_{\alpha n} \exp t(g_{\alpha n} - t\delta_{|\alpha|}\zeta_{|\alpha|}/2), g_{\alpha n} - t\delta_{|\alpha|}\zeta_{|\alpha|}\big)\Big\} \text{ and } \big\{(u_{\alpha n}, g_{\alpha n})\big\} \tag{2.88}$$

have the same mean measure. Similarly to Eq. (2.24), if we define a random bijection $\pi_\alpha : \mathbb{N} \to \mathbb{N}$ that sorts the first process in the decreasing order of the first coordinate then the sequences $S_\alpha = \big((u_{\alpha n}, g_{\alpha n})\big)_{n\geq 1}$ and

$$S'_\alpha = \Big(\big(u_{\alpha\pi_\alpha(n)} \exp t(g_{\alpha\pi_\alpha(n)} - t\delta_{|\alpha|}\zeta_{|\alpha|}/2), g_{\alpha\pi_\alpha(n)} - t\delta_{|\alpha|}\zeta_{|\alpha|}\big)\Big)_{n\geq 1} \tag{2.89}$$

are equal in distribution, $S'_\alpha \stackrel{d}{=} S_\alpha$. So far, we have simply restated the Bolthausen–Sznitman invariance property for each Poisson process Π_α and made the dependence of all the variables on α explicit. Since the processes Π_α are independent of each other, the permutations π_α and sequences S_α, S'_α are also independent for all $\alpha \in \mathscr{A} \setminus \mathbb{N}^r$; everything is defined locally. Therefore,

$$(S'_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^r} \stackrel{d}{=} (S_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^r}. \tag{2.90}$$

Now, we are going to combine these sequences in a way that implies Eq. (2.84). The right-hand side of Eq. (2.84) can be expressed in terms of the random variables

$$w_\alpha = \prod_{\beta \in p(\alpha)} u_\beta \text{ and } g(h_\alpha) = \sum_{\beta \in p(\alpha)} g_\beta \tag{2.91}$$

for $\alpha \in \mathbb{N}^r$ and, in some sense, all we need to do is to replace the pairs (u_β, g_β) by the corresponding pairs on the left-hand side of Eq. (2.90). However, since local permutations do not preserve the global structure of the tree, we have to take care when we define the global permutation π to ensure that $\pi(\alpha) \wedge \pi(\beta) = \alpha \wedge \beta$. To achieve this, we will combine local permutations (π_α) into a global permutation $\pi : \mathscr{A} \to \mathscr{A}$ of all vertices of the tree as follows:

(i) The permutation π fixes the root of the tree, $\pi(\emptyset) = \emptyset$.
(ii) Recursively, given $\pi(\alpha)$, we define $\pi(\alpha n) = \pi(\alpha)\pi_{\pi(\alpha)}(n)$.

In other words, if α is mapped into $\pi(\alpha)$, then its children αn are mapped into children of $\pi(\alpha)$, i.e. the permutation π preserves the parent–child relationship. In particular, it preserves paths, $p(\pi(\alpha)) = \pi(p(\alpha))$, where by $\pi(p(\alpha))$ we mean the permutation applied to each vertex in the path $p(\alpha)$. Moreover, the permutation π restricted to the vertices at a given distance from the root, $|\alpha| = p$, is a random bijection of $\mathbb{N}^p$ into itself, which easily follows from (ii) by induction on p. Let us show that the recursive nature of the permutation π has the following consequence,

$$(S'_{\pi(\alpha)})_{\alpha \in \mathscr{A} \setminus \mathbb{N}^r} \stackrel{d}{=} (S_\alpha)_{\alpha \in \mathscr{A} \setminus \mathbb{N}^r}. \tag{2.92}$$

This also follows easily from the definition (i) and (ii) above by showing that

$$(S'_{\pi(\alpha)})_{|\alpha| \le p} \stackrel{d}{=} (S_\alpha)_{|\alpha| \le p},$$

by induction on $p \le r-1$. By (i) and Eq. (2.90), this holds for $p = 0$. To make an induction step, let us denote by $\mathscr{F}_p$ the σ-algebra generated by the processes Π_α for $|\alpha| \le p$ and let us notice that, by (ii), the permutation π on the vertices $\mathbb{N}^{p+1}$ is defined completely in terms of these processes. Therefore, conditionally on $\mathscr{F}_p$, the permutation π restricted to the vertices at the distance $|\alpha| = p+1$ from the root is a nonrandom bijection of $\mathbb{N}^{p+1}$ into itself. On the other hand, it is obvious that the sequences (S_α) and (S'_α) for $|\alpha| = p+1$ are i.i.d. and independent of $\mathscr{F}_p$. Together with Eq. (2.90) these observations imply that, conditionally on $\mathscr{F}_p$,

$$(S'_{\pi(\alpha)})_{|\alpha|=p+1} \stackrel{d}{=} (S_\alpha)_{|\alpha|=p+1},$$

since on the left-hand side, we simply reordered an i.i.d. sequence by some nonrandom permutation. Since the distribution of these sequences does not depend on the condition $\mathscr{F}_p$, this completes the proof of the induction step. Let us notice that the definition of S'_α in Eq. (2.89) and (ii) imply that

$$S'_{\pi(\alpha)} = \Big(\big(u_{\pi(\alpha n)} \exp t(g_{\pi(\alpha n)} - t\delta_{|\alpha|}\zeta_{|\alpha|}/2), g_{\pi(\alpha n)} - t\delta_{|\alpha|}\zeta_{|\alpha|} \big) \Big)_{n \ge 1},$$

and Eq. (2.92) can be expressed in terms of individual coordinates by saying that

$$\Big(\big(u_{\pi(\alpha)} \exp t(g_{\pi(\alpha)} - t\delta_{|\alpha|-1}\zeta_{|\alpha|-1}/2), g_{\pi(\alpha)} - t\delta_{|\alpha|-1}\zeta_{|\alpha|-1} \big) \Big)_{\alpha \in \mathscr{A} \setminus \mathbb{N}^0} \tag{2.93}$$

and $((u_\alpha, g_\alpha))_{\alpha \in \mathscr{A} \setminus \mathbb{N}^0}$ are equal in distribution. This is the analogue of Eq. (2.24) in the setting of the Poisson–Dirichlet processes. Now, we are going to match pairs (u_β, g_β) in Eq. (2.91) with the corresponding pairs in (2.93). Since the permutation π preserves paths, $p(\pi(\alpha)) = \pi(p(\alpha))$, for each leaf vertex $\alpha \in \mathbb{N}^r$,

$$\sum_{\beta \in p(\alpha)} (g_{\pi(\beta)} - t\delta_{|\beta|-1}\zeta_{|\beta|-1}) = \sum_{\beta \in p(\pi(\alpha))} (g_\beta - t\delta_{|\beta|-1}\zeta_{|\beta|-1}),$$

$$\prod_{\beta \in p(\alpha)} u_{\pi(\beta)} \exp t(g_{\pi(\beta)} - t\delta_{|\beta|-1}\zeta_{|\beta|-1}/2) = \prod_{\beta \in p(\pi(\alpha))} u_\beta \exp t(g_\beta - t\delta_{|\beta|-1}\zeta_{|\beta|-1}/2),$$

which are equal to $g(h_{\pi(\alpha)}) - tb$ and $w_{\pi(\alpha)} \exp t(g(h_{\pi(\alpha)}) - tb/2)$ correspondingly, by Eqs. (2.66), (2.72), (2.86), and (2.87). Then, equality in distribution in Eq. (2.93) implies

$$\Big(\big(w_{\pi(\alpha)} \exp t(g(h_{\pi(\alpha)}) - tb/2), g(h_{\pi(\alpha)}) - tb \big) \Big)_{\alpha \in \mathbb{N}^r} \stackrel{d}{=} \big((w_\alpha, g(h_\alpha)) \big)_{\alpha \in \mathbb{N}^r}. \tag{2.94}$$

Now, combining the definitions in Eqs. (2.46), (2.62), and (2.68), we can write

$$\frac{w_{\pi(\alpha)} \exp t(g(h_{\pi(\alpha)}) - tb/2)}{\sum_{\beta \in \mathbb{N}^r} w_{\pi(\beta)} \exp t(g(h_{\pi(\beta)}) - tb/2)} = \frac{v_{\pi(\alpha)} \exp t g(h_{\pi(\alpha)})}{\sum_{\beta \in \mathbb{N}^r} v_{\pi(\beta)} \exp t g(h_{\pi(\beta)})} = G_t(h_{\pi(\alpha)}),$$

and from this it is clear that Eq. (2.94) implies Eq. (2.84). The number of common vertices (2.63) is preserved, $\pi(\alpha) \wedge \pi(\beta) = \alpha \wedge \beta$, because the permutation π preserves the parent–child relationship. □

Before we move on to study the implications of the invariance properties, let us describe another interesting property of the RPC, which represents one step in the construction of a discrete *Bolthausen–Sznitman coalescent*. It is also based on the property of Poisson processes proved in Theorem 2.6. This will not be used in the sequel. Suppose that we observe the measure G, defined in Eq. (2.62), up to orthogonal transformations and without reference to the index set $\mathbb{N}^r$. This means that we observe the sequence of weights of this measure arranged in the decreasing order, say $(v(l))_{l \geq 1}$, and if $(h(l))_{l \geq 1}$ is the sequence of points in the support of the measure G corresponding to these weights, then we observe the Gram matrix of their scalar products $Q = (h(l) \cdot h(l'))_{l,l' \geq 1}$. The following holds.

Theorem 2.12. *The sequence $(v(l))_{l \geq 1}$ is the Poisson–Dirichlet process $PD(\zeta_{r-1})$ independent of the Gram matrix Q, which can be generated by*

$$\Big(X_l \cdot X_{l'} + \delta_{l,l'}(q_r - q_{r-1}) \Big)_{l,l' \geq 1}, \tag{2.95}$$

where $(X_l)_{l \geq 1}$ is an i.i.d. sequence from some random measure G' on H.

Let us begin with the following observation. Given Eq. (2.42), consider a sequence of parameters,

$$0 < \frac{\zeta_0}{\zeta_{r-1}} < \ldots < \frac{\zeta_{r-2}}{\zeta_{r-1}} < 1, \tag{2.96}$$

and, recalling Eq. (2.45), consider a sequence of random variables indexed by $\alpha \in \mathbb{N}^{r-1}$,

$$\lambda_\alpha = \frac{w_\alpha^{\zeta_{r-1}}}{\sum_{\beta \in \mathbb{N}^{r-1}} w_\beta^{\zeta_{r-1}}}. \tag{2.97}$$

As a part of the next result, we will see that this sequence is well defined.

Lemma 2.6. *The sequence $(\lambda_\alpha)_{\alpha \in \mathbb{N}^{r-1}}$ in Eq. (2.97) is a sequence of weights of the Ruelle probability cascade corresponding to the parameters (2.96).*

Proof. If Π is a Poisson process with the mean measure in Eq. (2.12) then the Poisson process

$$\left\{x^{\zeta'} : x \in \Pi\right\} \text{ has the mean measure } \frac{\zeta}{\zeta'} x^{-1-\zeta/\zeta'} dx \tag{2.98}$$

on $(0,\infty)$. This follows from the Mapping Theorem 2.1, since, given a set $A \subseteq (0,\infty)$,

$$\mu\left(x : x^{\zeta'} \in A\right) = \int I\left(x^{\zeta'} \in A\right) \zeta x^{-1-\zeta} dx = \int_A \frac{\zeta}{\zeta'} x^{-1-\zeta/\zeta'} dx.$$

Notice that, when $\zeta < \zeta'$, this mean measure is of the same type as Eq. (2.12), with ζ in (2.12) now replaced by ζ/ζ'. For $\alpha \in \mathbb{N}^{r-1}$, let us write

$$w_\alpha^{\zeta_{r-1}} = \prod_{\beta \in p(\alpha)} u_\beta^{\zeta_{r-1}}$$

and observe that, by Eq. (2.98), these random variables are defined in exactly the same way as Eq. (2.45), only in terms of the Poisson processes with the mean measures $(\zeta_p/\zeta_{r-1}) x^{-1-\zeta_p/\zeta_{r-1}} dx$ instead of $\zeta_p x^{-1-\zeta_p} dx$ for $p \leq r-2$. Comparing Eq. (2.97) with Eq. (2.46) finishes the proof. □

Proof (Theorem 2.12). Until the very end of the proof, let us fix all the random variables w_α in Eq. (2.45) for $\alpha \in \mathbb{N}^{r-1}$ and consider the random measure G in Eq. (2.62) conditionally on these random variables. Recall the definition (2.44) and, for each $\alpha \in \mathbb{N}^{r-1}$, consider the set

$$\Pi_\alpha^+ = \left\{(w_\alpha u_{\alpha n}, \alpha) : u_{\alpha n} \in \Pi_\alpha\right\} \subseteq (0,\infty) \times \mathbb{N}^{r-1}.$$

Even though for a fixed $\alpha \in \mathbb{N}^{r-1}$ all the second coordinates are equal to α, we would like to think of Π_α^+ as a random set on the product space $(0,\infty) \times \mathbb{N}^{r-1}$. By the definition (2.45), the quantity $w_\alpha u_{\alpha n} = w_{\alpha n}$ is simply a coordinate of the process

w indexed by the leaf vertex αn, separated into a fixed nonrandom factor w_α and a random factor $u_{\alpha n}$. If we denote by μ the mean measure of Π_α given by (2.12) with $\zeta = \zeta_{r-1}$ then, as in the proof of Theorem 2.6, by Eq. (2.20) and the Marking Theorem, Π_α^+ is a Poisson process with the mean measure $(w_\alpha)^{\zeta_{r-1}}\mu \otimes \delta_\alpha$, where δ_α is the Dirac measure at α. By the Superposition Theorem 2.2, their union

$$\Pi^+ = \bigcup_{\alpha\in\mathbb{N}^{r-1}} \Pi_\alpha^+ = \big\{(w_\alpha u_{\alpha n}, \alpha) : \alpha \in \mathbb{N}^{r-1}, n \in \mathbb{N}\big\} \tag{2.99}$$

is a Poisson process with the mean measure

$$\chi = \sum_{\alpha\in\mathbb{N}^{r-1}} w_\alpha^{\zeta_{r-1}} \mu \otimes \delta_\alpha. \tag{2.100}$$

By Lemma 2.6, with probability one, $\lambda := \sum_{\alpha\in\mathbb{N}^{r-1}} w_\alpha^{\zeta_{r-1}} < \infty$, and if we recall Eq. (2.97) and define a probability measure Λ on $\mathbb{N}^{r-1}$ by $\Lambda(\alpha) = \lambda_\alpha$ then the measure χ in Eq. (2.100) can be rewritten as

$$\chi = \sum_{\alpha\in\mathbb{N}^{r-1}} \lambda\mu \otimes \lambda_\alpha \delta_\alpha = \lambda\mu \otimes \Lambda. \tag{2.101}$$

Let us reindex the Poisson process in Eq. (2.99),

$$\Pi^+ = \big\{(w(l), \alpha(l)) : l \geq 1\big\}, \tag{2.102}$$

so that the first coordinates $(w(l))_{l\geq 1}$ are now arranged in the decreasing order. Then, the weights $(v(l))_{l\geq 1}$ of the measure G arranged in the decreasing order are precisely

$$v(l) = \frac{w(l)}{\sum_{j\geq 1} w(j)}. \tag{2.103}$$

Moreover, by the definition (2.61), for any two parent vertices $\alpha, \beta \in \mathbb{N}^{r-1}$,

$$h_{\alpha n} \cdot h_{\beta m} = h_\alpha \cdot h_\beta + (q_r - q_{r-1}) I(\alpha n = \beta m),$$

and, since $\alpha(l) \in \mathbb{N}^{r-1}$ in Eq. (2.102) represents the parent vertex corresponding to the weight $v(l)$, we also have

$$Q = \big(h(l) \cdot h(l')\big)_{l,l'\geq 1} = \big(h_{\alpha(l)} \cdot h_{\alpha(l')} + \delta_{l,l'}(q_r - q_{r-1})\big)_{l,l'\geq 1}. \tag{2.104}$$

By the Marking Theorem 2.3, the representation (2.101) of the mean measure of the Poisson process (2.102) implies that we can think of $(\alpha(l))_{l\geq 1}$ as independent markings, with the distribution Λ, of the points of the Poisson process $\{w(l) : l \geq 1\}$ with the mean measure $\lambda\mu$. First of all, since μ is the measure Eq. (2.12) with $\zeta = \zeta_{r-1}$, by the computation (2.20), the process $\{\lambda^{-1/\zeta_{r-1}} w(l) : l \geq 1\}$ has the mean measure μ and, therefore, by the definition (2.14), the weights $(v(l))$ in Eq. (2.103) have the Poisson–Dirichlet distribution $PD(\zeta_{r-1})$. Moreover, it is clear that the matrix Q in Eq. (2.104) is independent of the weights $(v(l))$, since the markings $(\alpha(l))$ in

Eq. (2.102) are independent of $(w(l))$. Finally, if we define

$$G'(h_\alpha) = \Lambda(\alpha) = \lambda_\alpha \text{ for } \alpha \in \mathbb{N}^{r-1},$$

then we can think of $(h_{\alpha(l)})_{l\geq 1}$ as an i.i.d. sample from G'. So far, all of the above was proved conditionally on $(w_\alpha)_{\alpha\in\mathbb{N}^{r-1}}$, but, since the distribution $PD(\zeta_{r-1})$ of the weights $(v(l))$ does not depend on the conditioning, they are independent of the matrix Q and of the sequence $(w_\alpha)_{\alpha\in\mathbb{N}^{r-1}}$. This finishes the proof. □

2.4 A Characterization by Invariance

The Ghirlanda–Guerra identities (2.80) were first discovered in the setting of the Sherrington–Kirkpatrick and mixed p-spin models and in the next chapter we will show that the asymptotic Gibbs measures in these models satisfy these identities. A natural question arises, what can one say about a random measure on a Hilbert space that satisfies the Ghirlanda–Guerra identities? We will answer this question in the remainder of the chapter and show that if the Ghirlanda–Guerra identities hold then the measure G is determined up to an orthogonal transformation by the functional order parameter, the distribution of the overlap $R_{1,2}$ under $\mathbb{E}G^{\otimes 2}$.

Any transformation $G \circ U^{-1}$ of the measure G by an orthogonal operator U on a Hilbert space H, obviously, does not affect the distribution of the Gram matrix $R = (\sigma^l \cdot \sigma^{l'})$ of an i.i.d. sample (σ^l) from this measure and, since by their nature the Ghirlanda–Guerra identities express a property of the distribution of the matrix R under $\mathbb{E}G^{\otimes\infty}$, they cannot distinguish two measures that differ by an orthogonal transformation. On the other hand, we showed in Lemma 1.7 that one can reconstruct the measure G from the Gram matrix R up to orthogonal transformations, which means that our ultimate goal would be to uniquely determine the distribution of the array R in terms of the distribution of one element $R_{1,2}$. This goal also fits well with the discussion in Sect. 1.3, where we explained that the question of understanding the structure of the Gibbs measure G_N in the SK model can be identified with the problem of describing the distribution of the overlap matrix R under $\mathbb{E}G_N^{\otimes\infty}$. Now, if we assume that the Ghirlanda–Guerra identities hold, what is the connection between the distribution of the overlap matrix R and the RPC studied in the previous section? In the case when the distribution of one element $R_{1,2}$ is discrete, as in Eq. (2.82), and if, indeed, it uniquely determines the distribution of the entire matrix R then this matrix can be generated by the Ruelle probability cascade with the corresponding parameters, by Theorem 2.10. In the case when the distribution of $R_{1,2}$ is not discrete, we will show later in this section that the Gram matrix can be approximated by discrete matrices generated by the RPC.

Let G be a random measure on a Hilbert space H and let us denote by $\langle\cdot\rangle$ the average with respect to $G^{\otimes\infty}$. As before, let (σ^l) be an i.i.d. sample from the measure G and let

$$R = \big(R_{l,l'}\big)_{l,l'\geq 1} = \big(\sigma^l \cdot \sigma^{l'}\big)_{l,l'\geq 1} \tag{2.105}$$

be the Gram matrix, also called the overlap matrix, of the sequence (σ^l). Suppose that, for any $n \geq 1$ and any bounded measurable functions $f = f((R_{l,l'})_{l\neq l'\leq n})$ and $\psi : \mathbb{R} \to \mathbb{R}$, the Ghirlanda–Guerra identities hold:

$$\mathbb{E}\big\langle f\psi(R_{1,n+1})\big\rangle = \frac{1}{n}\mathbb{E}\big\langle f\big\rangle\mathbb{E}\big\langle\psi(R_{1,2})\big\rangle + \frac{1}{n}\sum_{l=2}^{n}\mathbb{E}\big\langle f\psi(R_{1,l})\big\rangle. \tag{2.106}$$

It is important to point out that i.i.d. replicas (σ^l) play interchangeable roles and the index 1 in $R_{1,n+1}$ can be replaced by any index $1 \leq j \leq n$, in which case the sum on the right-hand side will be over $1 \leq l \leq n$ such that $l \neq j$. This obvious observation will be used implicitly many times below. Let us also emphasize that in Eq. (2.106) we consider functions f that depend only on the off-diagonal elements $R_{l,l'}$ for $l \neq l'$ and do not depend on the self-overlaps $R_{l,l}$. In the case of the RPC, the self-overlaps were constant by construction, so this distinction would not make any difference. In other cases, we sometimes do not know a priori that the self-overlaps are constant and the Ghirlanda–Guerra identities will involve only the overlaps $R_{l,l'}$ for $l \neq l'$. We will show in Theorem 2.15 below that, a posteriori, the Ghirlanda–Guerra identities imply that the self-overlaps are constant, so the distinction turns out to be temporary. Let us denote by ζ the distribution of the overlap $R_{1,2}$ under the measure $\mathbb{E}G^{\otimes 2}$,

$$\zeta(A) = \mathbb{E}\big\langle I\big(R_{1,2} \in A\big)\big\rangle \tag{2.107}$$

for any measurable set A on $\mathbb{R}$. In all applications, random measures G will have bounded support in H so, without loss of generality, we will assume that

$$G(\sigma : \|\sigma\| \leq 1) = 1 \tag{2.108}$$

with probability one, in which case the overlaps $R_{l,l'} \in [-1, 1]$. The main result of this section is the following.

Theorem 2.13. *Suppose that the Ghirlanda–Guerra identities (2.106) hold. Then the distribution of the entire overlap matrix $R = (R_{l,l'})_{l,l'\geq 1}$ under $\mathbb{E}G^{\otimes\infty}$ is uniquely determined by ζ in Eq. (2.107).*

The main difficulty in the proof of Theorem 2.13 will be to show that the structure of the support of the measure G is similar to the structure of the RPC, namely, that it is tree-like or ultrametric, as in Eq. (2.64). In the setting of the Sherrington–Kirkpatrick model this is known as the *Parisi ultrametricity conjecture.*

Theorem 2.14. *Suppose that the Ghirlanda–Guerra identities (2.106) hold. Then, the overlap matrix* $(R_{l,l'})_{l,l'\geq 1}$ *is ultrametric,*

$$\mathbb{E}\big\langle I\big(R_{1,2}\geq \min(R_{1,3},R_{2,3})\big)\big\rangle = 1. \tag{2.109}$$

Again, the replicas play interchangeable roles, so indices $1,2$, and 3 can be replaced by indices l_1,l_2, and l_3. We will postpone the proof of this theorem until the next section and focus on the proof of Theorem 2.13 under the assumption that Eq. (2.109) holds. However, we will begin with two observations that do not require the use of ultrametricity. The first observation shows that the Ghirlanda–Guerra identities determine the self-overlaps $R_{l,l}$ and they are, indeed, constant.

Theorem 2.15. *Suppose that the Ghirlanda–Guerra identities (2.106) hold. If* q^* *is the largest point in the support of* ζ*, then, with probability one,* $G(\|\sigma\|^2 = q^*) = 1$.

Because of this observation, the diagonal elements are nonrandom, $R_{l,l} = q^*$, and, if we wish, we can include them in the statement of the Ghirlanda–Guerra identities. The proof of Theorem 2.15 will be based on one elementary calculation.

Lemma 2.7. *Consider a measurable set* $A \subseteq \mathbb{R}$. *With probability one over the choice of G:*

(a) If $\zeta(A) > 0$ *then for G-almost all* σ^1, $G(\sigma^2 : \sigma^1\cdot\sigma^2 \in A) > 0$.
(b) If $\zeta(A) = 0$ *then for G-almost all* σ^1, $G(\sigma^2 : \sigma^1\cdot\sigma^2 \in A) = 0$.

Proof. (a) Suppose that $a = \zeta(A^c) < 1$. First of all, using the Ghirlanda–Guerra identities (2.106), one can write,

$$\begin{aligned}\mathbb{E}\big\langle I\big(R_{1,l}\in A^c, 2\leq l\leq n+1\big)\big\rangle &= \mathbb{E}\big\langle I\big(R_{1,l}\in A^c, 2\leq l\leq n\big) I\big(R_{1,n+1}\in A^c\big)\big\rangle \\ &= \frac{n-1+a}{n}\,\mathbb{E}\big\langle I\big(R_{1,l}\in A^c, 2\leq l\leq n\big)\big\rangle.\end{aligned}$$

Repeating the same computation, one can show by induction on n that this equals

$$\frac{(n-1+a)\cdots(1+a)a}{n!} = \frac{a(1+a)}{n}\Big(1+\frac{a}{2}\Big)\cdots\Big(1+\frac{a}{n-1}\Big).$$

Using the inequality $1+x\leq e^x$, it is now easy to see that

$$\mathbb{E}\big\langle I\big(R_{1,l}\in A^c, 2\leq l\leq n+1\big)\big\rangle \leq \frac{a(1+a)}{n}e^{a\log n} = \frac{a(1+a)}{n^{1-a}}.$$

If we rewrite the left-hand side using Fubini's theorem then, since $a<1$, letting $n\to\infty$ implies that

$$\lim_{n\to\infty}\mathbb{E}\int G(\sigma^2 : \sigma^1\cdot\sigma^2\in A^c)^n\, dG(\sigma^1) = 0.$$

This would lead to contradiction if we assume that $G(\sigma^2 : \sigma^1 \cdot \sigma^2 \in A^c) = 1$ with positive probability over the choice of G and the choice of σ^1, which proves part (a). Part (b) simply follows by Fubini's theorem. □

Proof (Theorem 2.15). Since q^* is the largest point in the support of ζ,

$$\zeta\big((q^*,\infty)\big) = 0 \text{ and } \zeta\big([q^* - n^{-1}, q^*]\big) > 0 \text{ for all } n \geq 1.$$

Using Lemma 2.7, we get that with probability one, for G-almost all σ^1,

$$G\big(\sigma^2 : \sigma^1 \cdot \sigma^2 \leq q^*\big) = 1 \tag{2.110}$$

and, for all $n \geq 1$,

$$G\big(\sigma^2 : \sigma^1 \cdot \sigma^2 \geq q^* - n^{-1}\big) > 0. \tag{2.111}$$

The equality in Eq. (2.110) implies that $G(\|\sigma\|^2 \leq q^*) = 1$. Otherwise, there exists $\sigma \in H$ with $\|\sigma\|^2 > q^*$ such that $G(B_\varepsilon(\sigma)) > 0$ for any $\varepsilon > 0$, where $B_\varepsilon(\sigma)$ is the ball of radius ε centered at σ. Taking $\varepsilon > 0$ small enough, so that $\sigma^1 \cdot \sigma^2 > q^*$ for all $\sigma^1, \sigma^2 \in B_\varepsilon(\sigma)$ contradicts Eq. (2.110).

Next, let us show that $G(\|\sigma\|^2 < q^*) = 0$. Otherwise, $G(\|\sigma\|^2 < q^* - \varepsilon) > 0$ for some small enough $\varepsilon > 0$, while for all $\sigma^1 \in \{\|\sigma\|^2 < q^* - \varepsilon\}$ and $\sigma^2 \in \{\|\sigma\|^2 \leq q^*\}$ we have

$$\sigma^1 \cdot \sigma^2 < \sqrt{q^*(q^* - \varepsilon)} < q^* - n^{-1}$$

for some large enough $n \geq 1$. Since we already proved that $G(\|\sigma\|^2 \leq q^*) = 1$, this contradicts the fact that Eq. (2.111) holds for all $n \geq 1$. □

In the setting of the RPC, the overlap $R_{1,2}$ could take only non-negative values, by construction. Our second observation shows that this is a necessary condition for the Ghirlanda–Guerra identities to hold, which is known as the *Talagrand positivity principle*.

Theorem 2.16. *Suppose that the Ghirlanda–Guerra identities (2.106) hold. Then the overlap is non-negative, $\zeta([0,\infty)) = 1$.*

Proof. Given a set A, consider the event $A_n = \{R_{l,l'} \in A : l \neq l' \leq n\}$ and notice that

$$I_{A_{n+1}} = I_{A_n} \prod_{l \leq n} \big(1 - I(R_{l,n+1} \notin A)\big) \geq I_{A_n} - \sum_{l \leq n} I_{A_n} I(R_{l,n+1} \notin A). \tag{2.112}$$

For each $l \leq n$, the Ghirlanda–Guerra identities (2.106) imply that

$$\mathbb{E}\big\langle I_{A_n} I(R_{l,n+1} \notin A)\big\rangle = \frac{\zeta(A^c)}{n} \mathbb{E}\big\langle I_{A_n}\big\rangle,$$

and together with Eq. (2.112) this gives

$$\mathbb{E}\big\langle I_{A_{n+1}}\big\rangle \geq \mathbb{E}\big\langle I_{A_n}\big\rangle - \zeta(A^c)\mathbb{E}\big\langle I_{A_n}\big\rangle = \zeta(A)\mathbb{E}\big\langle I_{A_n}\big\rangle \geq \zeta(A)^n,$$

by induction on n. Therefore, if $\zeta(A) > 0$, for any $n \geq 1$, with positive probability over the choice of G, one can find n points $\sigma^1, \ldots, \sigma^n$ in the support of G such that their overlaps $R_{l,l'} \in A$. If $A = (-\infty, -\varepsilon]$ for some $\varepsilon > 0$, this would imply that

$$0 \leq \Big\| \sum_{l \leq n} \sigma^l \Big\|^2 = \sum_{l,l' \leq n} R_{l,l'} \leq nq^* - n(n-1)\varepsilon < 0$$

for large n, and we can conclude that $\zeta((-\infty, -\varepsilon]) = 0$ for all $\varepsilon > 0$. □

Proof (of Theorem 2.13 assuming Theorem 2.14). By Theorem 2.15, the diagonal elements $R_{l,l}$ are determined by ζ and we only need to consider the off-diagonal elements. By Talagrand's positivity principle in Theorem 2.16, we can assume that all the overlaps are non-negative. Let us begin with the discrete case and suppose for a moment that the overlaps take only finitely many values

$$0 = q_0 < q_1 < \ldots < q_{r-1} < q_r = 1, \tag{2.113}$$

with probabilities

$$\zeta(\{q_p\}) = \mathbb{E}\big\langle I(R_{1,2} = q_p)\big\rangle = \delta_p \tag{2.114}$$

for $p = 0, \ldots, r$ and some $\delta_p \geq 0$ such that $\sum_{p=0}^r \delta_p = 1$. Some δ_p here can be equal to zero, which is the reason we set $q_r = 1$ in Eq. (2.113), because we are free to include it. In this case, we only need to show how to compute the probability of a particular configuration of finitely many overlaps,

$$\mathbb{E}\big\langle I\big(R_{l,l'} = q_{l,l'} : l \neq l' \leq n+1\big)\big\rangle, \tag{2.115}$$

for any $n \geq 1$ and any $q_{l,l'} \in \{q_0, \ldots, q_r\}$. Let us find the largest elements among $q_{l,l'}$ for $l \neq l'$ and, without loss of generality, suppose that $q_{1,n+1}$ is one of them. Because of the ultrametricity property Eq. (2.109), we only have to consider $(q_{l,l'})$ that have this property, since, otherwise, Eq. (2.115) is equal to zero. In particular, since $q_{1,n+1}$ is the largest, for $2 \leq l \leq n$,

$$q_{1,l} \geq \min\big(q_{1,n+1}, q_{l,n+1}\big) = q_{l,n+1} \quad \text{and} \quad q_{l,n+1} \geq \min\big(q_{1,n+1}, q_{1,l}\big) = q_{1,l},$$

which implies that $q_{1,l} = q_{l,n+1}$. Hence, if the overlap $R_{1,n+1} = q_{1,n+1}$, then, for all $2 \leq l \leq n$, $R_{1,l} = q_{1,l}$ automatically implies that $R_{l,n+1} = q_{1,l}$ and Eq. (2.115) equals

$$\mathbb{E}\big\langle I\big(R_{l,l'} = q_{l,l'} : l, l' \leq n\big) I\big(R_{1,n+1} = q_{1,n+1}\big)\big\rangle. \tag{2.116}$$

By the Ghirlanda–Guerra identities (2.106) and (2.114), this is equal to

$$\begin{aligned} &\frac{1}{n}\zeta\big(\{q_{1,n+1}\}\big)\mathbb{E}\big\langle I\big(R_{l,l'} = q_{l,l'} : l, l' \leq n\big)\big\rangle \\ &+\frac{1}{n}\sum_{l=2}^{n} I\big(q_{1,l} = q_{1,n+1}\big)\mathbb{E}\big\langle I\big(R_{l,l'} = q_{l,l'} : l, l' \leq n\big)\big\rangle. \end{aligned} \tag{2.117}$$

We can continue this computation recursively over n and, in the end, Eq. (2.115) will be expressed completely in terms of the distribution of one overlap Eq. (2.114). The general case follows by approximation.

Given $r \geq 1$, let us consider a sequence of points as in Eq. (2.113) and a function $\kappa(q)$ on $[0,1]$ such that

$$\kappa(q) = q_p \text{ if } q_p \leq q < q_{p+1} \tag{2.118}$$

for $p = 0, \ldots, r-1$ and $\kappa(q_r) = q_r = 1$. Notice that, for $0 \leq q \leq 1$,

$$|\kappa(q) - q| \leq \Delta_r := \max_{1 \leq p \leq r} (q_p - q_{p-1}). \tag{2.119}$$

We will show that the discrete approximation

$$R^\kappa = \big(\kappa(R_{l,l'})\big)_{l,l' \geq 1} \tag{2.120}$$

of the matrix $R = (R_{l,l'})$ satisfies all the properties needed to apply the first part of the proof and, therefore, its distribution is uniquely determined by ζ. First of all, since the function κ is non-decreasing, Eq. (2.109) obviously implies that R^κ is also ultrametric. Moreover, Eq. (2.109) implies that, for any q, the inequality $q \leq R_{l,l'}$ defines an equivalence relation $l \sim l'$, since $q \leq R_{l_1,l_2}$ and $q \leq R_{l_1,l_3}$ imply that $q \leq R_{l_2,l_3}$. Therefore, the array $(I(q \leq R_{l,l'}))_{l,l' \geq 1}$ is non-negative definite, since it is block-diagonal with blocks consisting of all elements equal to one. This implies that R^κ is non-negative definite, since it can be written as a convex combination of such arrays,

$$\kappa\big(R_{l,l'}\big) = \sum_{p=1}^{r} (q_p - q_{p-1}) I\big(q_p \leq R_{l,l'}\big). \tag{2.121}$$

It is obvious that R^κ is weakly exchangeable under $\mathbb{E}G^{\otimes\infty}$, as in Eq. (1.78), and the Dovbysh–Sudakov representation, Theorem 1.7, yields that (we use here that the diagonal elements of R^κ are equal to $\kappa(q^*)$)

$$R^\kappa \stackrel{d}{=} Q^\kappa = \Big(\rho^l \cdot \rho^{l'} + \delta_{l,l'}\big(\kappa(q^*) - \|\rho^l\|^2\big)\Big)_{l,l' \geq 1}, \tag{2.122}$$

where (ρ^l) is an i.i.d. sample from some random measure G' on ℓ^2. Let us denote by $\langle\cdot\rangle'$ the average with respect to $G'^{\otimes\infty}$ and, for any function f of the overlaps $(R_{l,l'})_{l \neq l'}$, let us denote $f_\kappa\big((R_{l,l'})\big) = f\big((\kappa(R_{l,l'}))\big)$. Since

$$\mathbb{E}\big\langle f\big((Q^\kappa_{l,l'})\big)\big\rangle' = \mathbb{E}\big\langle f\big((\kappa(R_{l,l'}))\big)\big\rangle = \mathbb{E}\big\langle f_\kappa\big((R_{l,l'})\big)\big\rangle,$$

the Ghirlanda–Guerra identities (2.106) for the measure G imply that the measure G' also satisfies these identities. The discrete case considered in the first part of the proof implies that the distribution of Q^κ, and R^κ, is uniquely determined by the distribution of one element $\kappa(R_{1,2})$, which is given by the image measure $\zeta \circ \kappa^{-1}$.

If we choose sequences (2.113) in such a way that $\lim_{r\to\infty}\Delta_r = 0$ then, by Eq. (2.119), R^κ converges to R almost surely and in distribution, which means that the distribution of R is also uniquely determined by ζ. □

In Theorem 2.13, the distribution ζ of one overlap in Eq. (2.107) was determined by the random measure G. The question is, given a probability distribution ζ on $[0,1]$, can we find a random measure on a separable Hilbert space H, which satisfies the Ghirlanda–Guerra identities and such that the distribution of one overlap is equal to ζ? We already know that if ζ is concentrated on finitely many points in $[0,1]$ and $\zeta(\{0\}) > 0$ then ζ is a functional order parameter of some Ruelle probability cascade G as in Eq. (2.83) and, by Theorem 2.10, G satisfies the Ghirlanda–Guerra identities (the reason why we require that $\zeta(\{0\}) > 0$ is because in our construction of the RPC the overlap distribution in Eq. (2.82) always had a positive weight ζ_0 at 0). As a consequence, we can show the following.

Theorem 2.17. *For every probability distribution ζ on $[0,1]$ there exists a random measure G on a separable Hilbert space H, which satisfies the Ghirlanda–Guerra identities (2.106) and such that Eq. (2.107) holds. Moreover, the distribution of $(R_{l,l'})_{l\neq l'}$ in Eq. (2.105) is continuous with respect to the weak convergence of ζ.*

By Theorem 2.15, the diagonal elements $R_{l,l}$ are, obviously, not continuous with respect to the weak convergence of ζ, since it is easy to see that the largest point in the support of ζ is not continuous with respect to this convergence.

Proof. Suppose that the distributions ζ^n are such that the corresponding random measures G^n exist, and suppose that $\zeta^n \to \zeta^*$ weakly. Let us show that, in this case, the distribution of $(R_{l,l'})_{l\neq l'}$ under $\mathbb{E}(G^n)^{\otimes\infty}$ converges. Let us consider a limit of these distributions over some subsequence and let us choose a further subsubsequence along which the entire array R (including the diagonal elements) converges in distribution to some array R^*. The array R^* is a Gram–de Finetti array, namely, positive definite and weakly exchangeable, as in Eq. (1.74), since it obviously inherits these properties from R. Therefore, by the Dovbysh–Sudakov representation in Theorem 1.7, we can assume that the off-diagonal elements of R^* are generated by

$$\big(R^*_{l,l'}\big)_{l\neq l'} = \big(\sigma^l\cdot\sigma^{l'}\big)_{l\neq l'},$$

where (σ^l) is an i.i.d. sample from some random measure G^* on the unit ball of a separable Hilbert space. Since we assumed that the measures G^n satisfy the Ghirlanda–Guerra identities, obviously, the measure G^* also satisfies the Ghirlanda–Guerra identities. Then, by Theorem 2.13, the distribution of $(R^*_{l,l'})_{l\neq l'}$ is uniquely determined by the distribution ζ^* of one overlap $R^*_{1,2}$ and, therefore, all limits over subsequences coincide and the distribution of $(R_{l,l'})_{l\neq l'}$ under $\mathbb{E}(G^n)^{\otimes\infty}$ converges. Since the functional order parameters of the RPC form a dense subset of all probability distributions on $[0,1]$, we demonstrated the existence of a random measure G^* for any distribution ζ^* on $[0,1]$, as well as the continuity of the distribution of $(R_{l,l'})_{l\neq l'}$ with respect to weak convergence of ζ. □

Remark 2.1. The above proof explains the fundamental connection between RPC and the overlap matrix (2.105) generated by a random measure G that satisfies the Ghirlanda–Guerra identities, which we will now recapitulate for future reference. Given the distribution ζ of the overlap $R_{1,2}$ under $\mathbb{E}G^{\otimes 2}$ in Eq. (2.107), let ζ^n be any sequence of probability distributions such that $\zeta^n \to \zeta$ weakly and such that each ζ^n is concentrated on finitely many points in $[0,1]$ and $\zeta^n(\{0\}) > 0$. Let G^n be the Ruelle probability cascade corresponding to the functional order parameter ζ^n. Then, by Theorem 2.17, the overlap array $(R_{l,l'})_{l\neq l'}$ generated by G^n converges in distribution to the overlap array generated by G. As for the diagonal elements $R_{l,l}$, we have some freedom of choosing the largest point in the support of each ζ^n without affecting the weak convergence $\zeta^n \to \zeta$. For example, we can choose it to be equal to the largest point q^* in the support of ζ, in which case $R_{l,l} = q^*$ and the entire array $(R_{l,l'})$ converges in distribution. However, as we will see in the proof of the Parisi formula, sometimes it will be convenient to make a different choice, for example, $R_{l,l} = 1$, which can always be achieved by assigning a small weight $\zeta^n(\{1\})$ to the point 1, if necessary. Such approximation of the overlap array (2.105) by Gram matrices of i.i.d samples from the RPC, under the Ghirlanda–Guerra identities, will play a central role in the analysis of the Sherrington–Kirkpatrick model and mixed p-spin models.

2.5 Invariance and Ultrametricity

In this section we will prove Theorem 2.14 and show that the Ghirlanda–Guerra identities imply ultrametricity of the overlaps. The main idea behind the proof is that, due to the Ghirlanda–Guerra identities, the distribution of the overlaps $(R_{l,l'})$ is invariant under a large family of changes of density and, as we will show, this invariance property contains a lot of information about the geometric structure of the random measure G. The invariance property can be stated as follows. Given $n \geq 1$, we consider n bounded measurable functions $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$ and let

$$F(\sigma, \sigma^1, \dots, \sigma^n) = f_1(\sigma \cdot \sigma^1) + \dots + f_n(\sigma \cdot \sigma^n). \tag{2.123}$$

For $1 \leq l \leq n$, we define

$$F_l(\sigma, \sigma^1, \dots, \sigma^n) = F(\sigma, \sigma^1, \dots, \sigma^n) - f_l(\sigma \cdot \sigma^l) + \mathbb{E}\big\langle f_l(R_{1,2})\big\rangle, \tag{2.124}$$

and, for $l \geq n+1$, we define

$$F_l(\sigma, \sigma^1, \dots, \sigma^n) = F(\sigma, \sigma^1, \dots, \sigma^n). \tag{2.125}$$

The definition (2.125) for $l \geq n+1$ will not be used in the statement, but will appear in the proof of the following invariance property.

Theorem 2.18. *Suppose the Ghirlanda–Guerra identities (2.106) hold and let* Φ *be a bounded measurable function of* $R^n = (R_{l,l'})_{l,l' \le n}$. *Then,*

$$\mathbb{E}\langle \Phi \rangle = \mathbb{E}\Big\langle \frac{\Phi \exp \sum_{l=1}^n F_l(\sigma^l, \sigma^1, \dots, \sigma^n)}{\langle \exp F(\sigma, \sigma^1, \dots, \sigma^n) \rangle_-^n} \Big\rangle, \tag{2.126}$$

where the average $\langle \cdot \rangle_-$ *with respect to* G *in the denominator is in* σ *only for fixed* $\sigma^1, \dots, \sigma^n$, *and the outside average* $\langle \cdot \rangle$ *of the ratio is in* $\sigma^1, \dots, \sigma^n$.

When $n = 1$, it is understood that Φ is a constant. Notice that one can easily recover the original Ghirlanda–Guerra identities from Eq. (2.126) by taking $f_1 = t\psi$ and $f_2 = \dots = f_n = 0$ and computing the derivative at $t = 0$. The motivation behind Theorem 2.18 is not obvious, but, with some effort, one can obtain Eq. (2.126) from the Bolthausen–Sznitman invariance property in Theorem 2.11. Using the Ghirlanda–Guerra identities, the proof of Eq. (2.126) is rather straightforward.

Proof (Theorem 2.18). Without loss of generality, let us assume that $|\Phi| \le 1$ and suppose that $|f_l| \le L$ for all $l \le n$ for some large enough L. For $t \ge 0$, let us define

$$\varphi(t) = \mathbb{E}\Big\langle \frac{\Phi \exp \sum_{l=1}^n t F_l(\sigma^l, \sigma^1, \dots, \sigma^n)}{\langle \exp t F(\sigma, \sigma^1, \dots, \sigma^n) \rangle_-^n} \Big\rangle. \tag{2.127}$$

We will show that the Ghirlanda–Guerra identities (2.106) imply that the function $\varphi(t)$ is constant for all $t \ge 0$, proving the statement of the theorem, $\varphi(0) = \varphi(1)$. If for $k \ge 1$ we denote

$$D_{n+k} = \sum_{l=1}^{n+k-1} F_l(\sigma^l, \sigma^1, \dots, \sigma^n) - (n+k-1) F_{n+k}(\sigma^{n+k}, \sigma^1, \dots, \sigma^n),$$

then, recalling Eq. (2.125) and using that the average $\langle \cdot \rangle_-$ is in σ only, one can easily check, by induction on k, that

$$\varphi^{(k)}(t) = \mathbb{E}\Big\langle \frac{\Phi D_{n+1} \cdots D_{n+k} \exp \sum_{l=1}^{n+k} t F_l(\sigma^l, \sigma^1, \dots, \sigma^n)}{\langle \exp t F(\sigma, \sigma^1, \dots, \sigma^n) \rangle_-^{n+k}} \Big\rangle.$$

Next, we will show that $\varphi^{(k)}(0) = 0$. If we introduce the notation

$$\Phi' = \Phi D_{n+1} \cdots D_{n+k-1},$$

then Φ' is a function of the overlaps $(R_{l,l'})_{l,l' \le n+k-1}$ and $\varphi^{(k)}(0)$ equals

$$\begin{aligned} &\mathbb{E}\Big\langle \Phi' \Big(\sum_{l=1}^{n+k-1} F_l(\sigma^l, \sigma^1, \dots, \sigma^n) - (n+k-1) F_{n+k}(\sigma^{n+k}, \sigma^1, \dots, \sigma^n) \Big) \Big\rangle \\ &\quad = \sum_{j=1}^n \mathbb{E}\Big\langle \Phi' \Big(\sum_{l \ne j, l=1}^{n+k-1} f_j(R_{j,l}) + \mathbb{E}\langle f_j(R_{1,2}) \rangle - (n+k-1) f_j(R_{j,n+k}) \Big) \Big\rangle = 0, \end{aligned}$$

by the Ghirlanda–Guerra identities (2.106) applied to each term j. Furthermore, since $|\Phi| \le 1$, $|F_l| \le Ln$, and $|D_{n+k}| \le 2L(n+k-1)n$, we can bound

$$\begin{aligned}\big|\varphi^{(k)}(t)\big| &\le \Big(\prod_{l=1}^{k} 2L(n+l-1)n\Big)\,\mathbb{E}\Big\langle \frac{\exp\sum_{l=1}^{n+k} tF_l(\sigma^l,\sigma^1,\dots,\sigma^n)}{\langle \exp tF(\sigma,\sigma^1,\dots,\sigma^n)\rangle_-^{n+k}}\Big\rangle \\ &= \Big(\prod_{l=1}^{k} 2L(n+l-1)n\Big)\,\mathbb{E}\Big\langle \frac{\exp\sum_{l=1}^{n} tF_l(\sigma^l,\sigma^1,\dots,\sigma^n)}{\langle \exp tF(\sigma,\sigma^1,\dots,\sigma^n)\rangle_-^{n}}\Big\rangle,\end{aligned}$$

where the equality follows from the fact that the denominator depends only on the first n coordinates and, recalling Eq. (2.125), the average of the numerator in σ^l for each $n < l \le n+k$ will cancel exactly one factor in the denominator. Moreover, if we consider an arbitrary $T > 0$, using that $|F_l| \le Ln$, the last ratio can be bounded by $\exp(2LTn^2)$ for $0 \le t \le T$ and, therefore,

$$\max_{0\le t\le T}\big|\varphi^{(k)}(t)\big| \le \exp(2LTn^2)\frac{(n+k-1)!}{(n-1)!}(2Ln)^k.$$

Since we proved above that $\varphi^{(k)}(0) = 0$ for all $k \ge 1$, using Taylor's expansion, we can write

$$\big|\varphi(t)-\varphi(0)\big| \le \max_{0\le s\le t}\frac{|\varphi^{(k)}(s)|}{k!}t^k \le \exp(2LTn^2)\frac{(n+k-1)!}{k!\,(n-1)!}(2Lnt)^k.$$

Letting $k \to \infty$ proves that $\varphi(t) = \varphi(0)$ for $0 \le t < (2Ln)^{-1}$. This implies that for any $t_0 < (2Ln)^{-1}$ we have $\varphi^{(k)}(t_0) = 0$ for all $k \ge 1$ and, again, by Taylor's expansion for $t_0 \le t \le T$,

$$\begin{aligned}\big|\varphi(t)-\varphi(t_0)\big| &\le \max_{t_0\le s\le t}\frac{|\varphi^{(k)}(s)|}{k!}(t-t_0)^k \\ &\le \exp(2LTn^2)\frac{(n+k-1)!}{k!\,(n-1)!}\big(2Ln(t-t_0)\big)^k.\end{aligned}$$

Letting $k \to \infty$ proves that $\varphi(t) = \varphi(0)$ for $0 \le t < 2(2Ln)^{-1}$. We can proceed in the same fashion to prove this equality for all $t < T$ and note that T was arbitrary. □

Let us write down a formal generalization of Theorem 2.18 on which the proof of ultrametricity will be based. Consider a finite index set $\mathscr{A}$. Given $n \ge 1$ and $\sigma^1,\dots,\sigma^n \in H$, let $(B_\alpha)_{\alpha\in\mathscr{A}}$ be some partition of the Hilbert space H such that, for each $\alpha \in \mathscr{A}$, the indicator $I_{B_\alpha}(\sigma) = I(\sigma \in B_\alpha)$ is a measurable function of $R^n = (R_{l,l'})_{l,l'\le n}$ and $(\sigma\cdot\sigma^l)_{l\le n}$. In other words, the sets in the partition are expressed in terms of some conditions on the scalar products between $\sigma,\sigma^1,\dots,\sigma^n$. Let

$$W_\alpha = W_\alpha(\sigma^1,\dots,\sigma^n) = G(B_\alpha) \tag{2.128}$$

be the weights of the sets in this partition with respect to the measure G. Let us define a map T by

$$W=(W_\alpha)_{\alpha\in\mathscr{A}}\to T(W)=\Big(\frac{\langle I_{B_\alpha}(\sigma)\exp F(\sigma,\sigma^1,\dots,\sigma^n)\rangle_-}{\langle\exp F(\sigma,\sigma^1,\dots,\sigma^n)\rangle_-}\Big)_{\alpha\in\mathscr{A}}. \tag{2.129}$$

Then the following holds.

Theorem 2.19. *Suppose that the Ghirlanda–Guerra identities (2.106) hold. Then, for any bounded measurable function* $\varphi:\mathbb{R}^{n\times n}\times\mathbb{R}^{|\mathscr{A}|}\to\mathbb{R}$,

$$\mathbb{E}\big\langle\varphi(R^n,W)\big\rangle=\mathbb{E}\Big\langle\frac{\varphi(R^n,T(W))\exp\sum_{l=1}^n F_l(\sigma^l,\sigma^1,\dots,\sigma^n)}{\langle\exp F(\sigma,\sigma^1,\dots,\sigma^n)\rangle_-^n}\Big\rangle. \tag{2.130}$$

Proof. Let $n_\alpha\ge 0$ be some integers for $\alpha\in\mathscr{A}$ and let $m=n+\sum_{\alpha\in\mathscr{A}}n_\alpha$. Let $(S_\alpha)_{\alpha\in\mathscr{A}}$ be any partition of $\{n+1,\dots,m\}$ such that the cardinalities $|S_\alpha|=n_\alpha$. Consider a continuous function $\Phi=\Phi(R^n)$ of the overlaps of n replicas and let

$$\Phi'=\Phi(R^n)\prod_{\alpha\in\mathscr{A}}\varphi_\alpha,\ \text{ where }\ \varphi_\alpha=I\big(\sigma^l\in B_\alpha,\forall l\in S_\alpha\big).$$

We will apply Theorem 2.18 to the function Φ', but since it now depends on m coordinates, we have to choose m bounded measurable functions $f_1,\dots,f_m$ in the definition (2.123). We will choose the first n functions to be arbitrary and we let $f_{n+1}=\dots=f_m=0$. First of all, integrating out the coordinates $(\sigma^l)_{l>n}$, the left-hand side of Eq. (2.126) can be written as

$$\mathbb{E}\big\langle\Phi'\big\rangle=\mathbb{E}\Big\langle\Phi(R^n)\prod_{\alpha\in\mathscr{A}}\varphi_\alpha\Big\rangle=\mathbb{E}\Big\langle\Phi(R^n)\prod_{\alpha\in\mathscr{A}}W_\alpha^{n_\alpha}(\sigma^1,\dots,\sigma^n)\Big\rangle, \tag{2.131}$$

where W_α's were defined in Eq. (2.128). Let us now compute the right-hand side of Eq. (2.126). Since $f_{n+1}=\dots=f_m=0$, the coordinates $\sigma^{n+1},\dots,\sigma^m$ are not present in all the functions defined in Eqs. (2.123)–(2.125) and we will continue to write them as functions of $\sigma,\sigma^1,\dots,\sigma^n$ only. Then, it is easy to see that the denominator on the right-hand side of Eq. (2.126) is equal to $\langle\exp F(\sigma,\sigma^1,\dots,\sigma^n)\rangle_-^m$ and the sum in the numerator equals

$$\sum_{l=1}^n F_l(\sigma^l,\sigma^1,\dots,\sigma^n)+\sum_{l=n+1}^m F(\sigma^l,\sigma^1,\dots,\sigma^n).$$

Since the function Φ and the denominator do not depend on $(\sigma^l)_{l>n}$, integrating the numerator in the coordinate σ^l for $l\in S_\alpha$ produces a factor

$$\big\langle I_{B_\alpha}(\sigma)\exp F(\sigma,\sigma^1,\dots,\sigma^n)\big\rangle_-.$$

For each $\alpha\in\mathscr{A}$, we have $|S_\alpha|=n_\alpha$ such coordinates and, therefore, the right-hand side of Eq. (2.126) is equal to

$$\mathbb{E}\Big\langle \frac{\Phi(R^n)\exp\sum_{l=1}^n F_l(\sigma^l,\sigma^1,\dots,\sigma^n)}{\langle \exp F(\sigma,\sigma^1,\dots,\sigma^n)\rangle_-^n} \prod_{\alpha\in\mathscr{A}} \Big(\frac{\langle I_{B_\alpha}\exp F(\sigma,\sigma^1,\dots,\sigma^n)\rangle_-}{\langle \exp F(\sigma,\sigma^1,\dots,\sigma^n)\rangle_-} \Big)^{n_\alpha} \Big\rangle.$$

Comparing this with Eq. (2.131) and recalling the notation (2.129) proves Eq. (2.130) for

$$\varphi(R^n,W) = \Phi(R^n)\prod_{\alpha\in\mathscr{A}} W_\alpha^{n_\alpha}.$$

The general case then follows by approximation. First, we can approximate a continuous function ϕ on $[0,1]^{|\mathscr{A}|}$ by polynomials to obtain Eq. (2.130) for products $\Phi(R^n)\phi(W)$. This, of course, implies the result for continuous functions $\varphi(R^n,W)$ and then for arbitrary bounded measurable functions. □

Let us recall that, by Theorem 2.15, the measure G is concentrated on the sphere $\|\sigma\|^2 = q^*$ and from now on all σ's will be on this sphere. Because of this, whenever we write that the scalar product $\sigma^1\cdot\sigma^2 \geq q$ is larger than some number, this means that the points σ^1,σ^2 are within some distance from each other and, vice versa, if we write that the scalar product $\sigma^1\cdot\sigma^2 \leq q$ is smaller than some number, this means that the points σ^1,σ^2 are separated by a certain distance (this relationship will be very helpful in visualizing the geometric picture since everything will be written in terms of scalar products). Consider a symmetric non-negative definite matrix

$$A = \big(a_{l,l'}\big)_{l,l'\leq n} \tag{2.132}$$

such that $a_{l,l} = q^*$ for $l \leq n$. Given $\varepsilon > 0$, we will write $x \approx a$ to denote that $a-\varepsilon < x < a+\varepsilon$ and $R^n \approx A$ to denote that $R_{l,l'} \approx a_{l,l'}$ for all $l \neq l' \leq n$ and, for simplicity of notation, we will keep the dependence of $\approx$ on ε implicit. Below, the matrix A will be used to describe a set of constraints such that the overlaps in R^n can take values close to A,

$$\mathbb{E}\big\langle I\big(R^n \approx A\big)\big\rangle > 0, \tag{2.133}$$

for a given $\varepsilon > 0$. Let us consider the quantity

$$a_n^* = \max\big(a_{1,n},\dots,a_{n-1,n}\big), \tag{2.134}$$

which describes the constraint on the overlap corresponding to the closest replica among $\sigma^1,\dots,\sigma^{n-1}$ to the last replica σ^n. We will only need to consider the case when $a_n^* < q^*$, because, otherwise, the closest replica essentially coincides with σ^n. The following is a key step in the proof of Theorem 2.14.

Theorem 2.20. *Suppose that the Ghirlanda–Guerra identities (2.106) hold. Given $\varepsilon > 0$, if the matrix A satisfies Eq. (2.133) and $a_n^* + \varepsilon < q^*$ then*

$$\mathbb{E}\big\langle I\big(R^n \approx A, R_{l,n+1} \approx a_{l,n} \text{ for } l \leq n-1, R_{n,n+1} < a_n^* + \varepsilon\big)\big\rangle > 0. \tag{2.135}$$

Remark 2.2. This result will be used in the following way. Suppose that $a_n^* < q^*$ and the matrix A is in the support of the distribution of R^n under $\mathbb{E}G^{\otimes\infty}$, which means that Eq. (2.133) holds for all $\varepsilon > 0$. Since $a_n^* + \varepsilon < q^*$ for small $\varepsilon > 0$, Eq. (2.135) holds for all $\varepsilon > 0$. Therefore, the support of the distribution of R^{n+1} under $\mathbb{E}G^{\otimes\infty}$ intersects the event in Eq. (2.135) for every $\varepsilon > 0$ and, hence, it contains a point in the set

$$\left\{R^{n+1} : R^n = A, R_{l,n+1} = a_{l,n} \text{ for } l \le n-1, R_{n,n+1} \le a_n^*\right\}, \tag{2.136}$$

since the support is compact.

The motivation for this property becomes clear if we recall how the support of the RPC was constructed in Eq. (2.61). If we take n points in the support indexed by $\alpha_1, \dots, \alpha_n \in \mathbb{N}^r$, the point closest to h_{α_n} will correspond to the largest value of $\alpha_l \wedge \alpha_n$ defined in Eq. (2.63), because $h_{\alpha_l} \cdot h_{\alpha_n} = q_{\alpha_l \wedge \alpha_n}$. Suppose that $\alpha_1 \wedge \alpha_n$ is one of the largest values. Obviously, we can always find another index $\alpha_{n+1} \in \mathbb{N}^r$ such that

$$\alpha_n \wedge \alpha_{n+1} = \alpha_1 \wedge \alpha_n \text{ and } \alpha_l \wedge \alpha_{n+1} = \alpha_l \wedge \alpha_n \text{ for } l = 1, \dots, n-1.$$

This means that the points $h_{\alpha_{n+1}}$ and h_{α_n} are at exactly the same distances from all the other points and, moreover, they are not too close to each other, since $h_{\alpha_{n+1}}$ is at the same distance from h_{α_n} as the closest of the other points, in this case, h_{α_1}. One can think of $h_{\alpha_{n+1}}$ as a duplicate of h_{α_n} and, in some sense, it is a non-trivial duplicate, since they are not too close to each other. This is exactly the property expressed in Eq. (2.136), only with positive probability rather than always. In the previous section we explained that, under the ultrametricity assumption, the overlap array can be approximated in distribution by an array generated by the RPC and, since we are trying to prove ultrametricity, we should expect that the property (2.136) holds. On the other hand, it is very easy to see that the possibility of "duplicating a point" characterizes ultrametricity in the sense that this property can hold only if the support of the measure G is ultrametric. Let us explain this first assuming the validity of Theorem 2.20.

Proof (Theorem 2.14). The proof is by contradiction. Suppose that Eq. (2.109) is violated, in which case there exist $a < b \le c < q^*$ such that the matrix

$$\begin{pmatrix} q^* & a & b \\ a & q^* & c \\ b & c & q^* \end{pmatrix} \tag{2.137}$$

is in the support of the distribution of R^3 under $\mathbb{E}G^{\otimes\infty}$, so it satisfies Eq. (2.133) for every $\varepsilon > 0$. In this case, Theorem 2.20 implies the following. Given any $n_1, n_2, n_3 \ge 1$ and $n = n_1 + n_2 + n_3$, we can find a matrix A in the support of the distribution of R^n under $\mathbb{E}G^{\otimes\infty}$ such that for some partition of indices $\{1, \dots, n\} = I_1 \cup I_2 \cup I_3$ with $|I_j| = n_j$, we have $j \in I_j$ for $j \le 3$ and

(a) $a_{l,l'} \le c$ for all $l \ne l' \le n$
(b) $a_{l,l'} = a$ if $l \in I_1, l' \in I_2$, $a_{l,l'} = b$ if $l \in I_1, l' \in I_3$ and $a_{l,l'} = c$ if $l \in I_2, l' \in I_3$

This can be proved by induction on n_1, n_2, n_3. First of all, by the choice of the matrix (2.137), this holds for $n_1 = n_2 = n_3 = 1$. Assuming that the claim holds for some n_1, n_2, and n_3 with the matrix A, let us show how one can increase any of the n_j's by one. For example, let us assume for simplicity of notation that $n \in I_3$ and show that the claim holds with $n_3 + 1$. Since $a_n^* \le c < q^*$, we can use the remark below Theorem 2.20 to find a matrix A' in the support of the distribution of R^{n+1} under $\mathbb{E}G^{\otimes\infty}$ that belongs to the set (2.136). Hence,

$$a'_{l,l'} \le c \text{ for all } l \ne l' \le n+1 \text{ and } a'_{l,n+1} = a_{l,n} \text{ for } l \le n-1,$$

so, in particular,

$$a'_{l,n+1} = b \text{ for } l \in I_1 \text{ and } a'_{l,n+1} = c \text{ for } l \in I_2,$$

which means that A' satisfies the conditions (a), (b) with I_3 replaced by $I_3 \cup \{n+1\}$. In a similar fashion, one can increase the cardinality of I_1 and I_2 which completes the induction. Now, let $n_1 = n_2 = n_3 = m$, find the matrix A as above and find $\sigma^1, \dots, \sigma^n$ on the sphere of radius $\sqrt{q^*}$ such that $R_{l,l'} = \sigma^l \cdot \sigma^{l'} = a_{l,l'}$ for all $l, l' \le n$. Let $\bar\sigma^j$ be the barycenter of the set $\{\sigma^l : l \in I_j\}$. The condition (a) implies that

$$\|\bar\sigma^j\|^2 = \frac{1}{m^2}\sum_{l\in I_j}\|\sigma^l\|^2 + \frac{1}{m^2}\sum_{l\ne l'\in I_j} R_{l,l'} \le \frac{mq^* + m(m-1)c}{m^2},$$

and the condition (b) implies that

$$\bar\sigma^1 \cdot \bar\sigma^2 = a, \bar\sigma^1 \cdot \bar\sigma^3 = b \text{ and } \bar\sigma^2 \cdot \bar\sigma^3 = c. \tag{2.138}$$

Therefore,

$$\|\bar\sigma^2 - \bar\sigma^3\|^2 = \|\bar\sigma^2\|^2 + \|\bar\sigma^3\|^2 - 2\bar\sigma^2 \cdot \bar\sigma^3 \le \frac{2(q^* - c)}{m}$$

and $0 < b - a = \bar\sigma^1 \cdot \bar\sigma^3 - \bar\sigma^1 \cdot \bar\sigma^2 \le Km^{-1/2}$. We arrive at contradiction by letting $m \to \infty$, which finishes the proof. □

It remains to prove Theorem 2.20. As we mentioned above, the proof will be based on the invariance property in the form of Theorem 2.19. We will see that, in some sense, the invariance places strong constraints on the support of the measure G, which force the distribution of the overlaps to satisfy the duplication property (2.135).

Proof (Theorem 2.20). We will prove Eq. (2.135) by contradiction, so suppose that the left-hand side is equal to zero. We will apply Theorem 2.19 with $\mathscr{A} = \{1,2\}$ and the partition

$$B_1 = \{\sigma : \sigma \cdot \sigma^n \geq a_n^* + \varepsilon\}, B_2 = B_1^c.$$

Since we assume that $a_n^* + \varepsilon < q^*$, the set B_1 contains a small neighborhood of σ^n on the sphere of radius $\sqrt{q^*}$ and, on the event $\{R^n \approx A\}$, its complement B_2 contains small neighborhoods of $\sigma^1, \ldots, \sigma^{n-1}$, since $R_{l,n} < a_{l,n} + \varepsilon \leq a_n^* + \varepsilon$. Therefore, for $\sigma^1, \ldots, \sigma^n$ in the support of G, on the event $\{R^n \approx A\}$, the weights

$$W_1 = G(B_1) \text{ and } W_2 = G(B_2) = 1 - W_1$$

are strictly positive. Then, Eq. (2.133) implies that we can find $0 < p < p' < 1$ and small $\delta > 0$ such that

$$\mathbb{E}\Big\langle I\big(R^n \approx A, W_1 \in (p,p')\big)\Big\rangle \geq \delta. \tag{2.139}$$

Let us apply Theorem 2.19 with the above partition, the choice of

$$\varphi(R^n, W) = I\big(R^n \approx A, W_1 \in (p,p')\big) \tag{2.140}$$

and the choice of functions $f_1 = \ldots = f_{n-1} = 0$ and $f_n(x) = tI(x \geq a_n^* + \varepsilon)$ for $t \in \mathbb{R}$. The sum in the numerator on the right-hand side of Eq. (2.130) will become

$$\begin{aligned}\sum_{l=1}^{n} F_l(\sigma^l, \sigma^1, \ldots, \sigma^n) &= \sum_{l=1}^{n-1} tI\big(R_{l,n} \geq a_n^* + \varepsilon\big) + t\,\mathbb{E}\big\langle I\big(R_{1,2} \geq a_n^* + \varepsilon\big)\big\rangle \\ &= t\,\mathbb{E}\big\langle I\big(R_{1,2} \geq a_n^* + \varepsilon\big)\big\rangle =: t\gamma,\end{aligned}$$

since, again, on the event $\{R^n \approx A\}$, the overlaps $R_{l,n} < a_{l,n} + \varepsilon \leq a_n^* + \varepsilon$ for all $l \leq n-1$ and the denominator will become

$$\begin{aligned}\big\langle \exp F(\sigma, \sigma^1, \ldots, \sigma^n)\big\rangle_- &= \big\langle \exp tI\big(\sigma \cdot \sigma^n \geq a_n^* + \varepsilon\big)\big\rangle_- \\ &= G(B_1)e^t + G(B_2) = W_1 e^t + 1 - W_1. \end{aligned} \tag{2.141}$$

If we denote $W = (W_1, W_2)$ and $\Delta_t(W) = W_1 e^t + 1 - W_1$ then the map $T(W)$ in the Eq. (2.129) becomes, in this case,

$$T_t(W) = \Big(\frac{W_1 e^t}{\Delta_t(W)}, \frac{1 - W_1}{\Delta_t(W)}\Big). \tag{2.142}$$

Since $\Delta_t(W) \geq 1$ for $t \geq 0$, Eqs. (2.130) and (2.139) imply

$$\begin{aligned}\delta &\leq \mathbb{E}\Big\langle \frac{I(R^n \approx A, (T_t(W))_1 \in (p,p'))\, e^{t\gamma}}{\Delta_t(W)^n}\Big\rangle \\ &\leq \mathbb{E}\Big\langle I\big(R^n \approx A, (T_t(W))_1 \in (p,p')\big)\, e^{t\gamma}\Big\rangle. \end{aligned} \tag{2.143}$$

In the average $\langle\cdot\rangle$ on the right-hand side let us fix $\sigma^1,\dots,\sigma^{n-1}$ and consider the average with respect to σ^n first. Clearly, on the event $\{R^n \approx A\}$, such average will be taken over the set

$$\Omega(\sigma^1,\dots,\sigma^{n-1}) = \big\{\sigma : \sigma\cdot\sigma^l \approx a_{l,n} \text{ for } l \le n-1\big\}. \tag{2.144}$$

Let us make the following crucial observation about the diameter of this set on the support of G. Suppose that with positive probability over the choice of the measure G and replicas $\sigma^1,\dots,\sigma^{n-1}$ from G satisfying the constraints in A, i.e., $R_{l,l'} \approx a_{l,l'}$ for $l,l' \le n-1$, we can find two points σ' and σ'' in the support of G that belong to the set $\Omega(\sigma^1,\dots,\sigma^{n-1})$ and such that $\sigma'\cdot\sigma'' < a_n^* + \varepsilon$. This would then imply Eq. (2.135), since for (σ^n,σ^{n+1}) in a small neighborhood of (σ',σ''), the vector $(\sigma^1,\dots,\sigma^n,\sigma^{n+1})$ would belong to the event

$$\big\{R^n \approx A, R_{l,n+1} \approx a_{l,n} \text{ for } l \le n-1, R_{n,n+1} < a_n^* + \varepsilon\big\}$$

on the left-hand side of Eq. (2.135). Since we assume that the left-hand side of (2.135) is equal to zero, we must have that, for almost all choices of the measure G and replicas $\sigma^1,\dots,\sigma^{n-1}$ satisfying the constraints in A, any two points σ',σ'' in the support of G that belong to the set $\Omega(\sigma^1,\dots,\sigma^{n-1})$ satisfy $\sigma'\cdot\sigma'' \ge a_n^* + \varepsilon$. Now, let us also recall that in Eq. (2.143) we are averaging over σ^n that satisfy the condition $(T_t(W))_1 \in (p,p')$. This means that if we fix any such σ' in the support of G that satisfies this condition and belongs to the set Eq. (2.144) then the Gibbs average in σ^n will be taken over its neighborhood

$$B_1 = B_1(\sigma') = \big\{\sigma'' : \sigma'\cdot\sigma'' \ge a_n^* + \varepsilon\big\}$$

of measure $W_1 = W_1(\sigma') = G(B_1(\sigma'))$ that satisfies $(T_t(W))_1 \in (p,p')$. It is easy to check that the inverse of the map in Eq. (2.142) is $T_t^{-1} = T_{-t}$ and, using this for $(T_t(W))_1 \in (p,p')$, implies that

$$W_1(\sigma') \in \Big\{\frac{qe^{-t}}{qe^{-t}+1-q} : q \in (p,p')\Big\}$$

and, therefore, $W_1(\sigma') \le (1-p')^{-1}e^{-t}$. This means that the average on the right-hand side of Eq. (2.143) over σ^n for fixed $\sigma^1,\dots,\sigma^{n-1}$ is bounded by $(1-p')^{-1}e^{-t}e^{t\gamma}$ and, thus, for $t \ge 0$,

$$0 < \delta \le \mathbb{E}\Big\langle I\big(R^n \approx A, (T_t(W))_1 \in (p,p')\big)e^{t\gamma}\Big\rangle \le (1-p')^{-1}e^{-t(1-\gamma)}.$$

Since A satisfies Eq. (2.133), $1-\gamma = \mathbb{E}\langle I(R_{1,2} < a_n^* + \varepsilon)\rangle > 0$, and letting $t \to +\infty$ we arrive at contradiction. □

Chapter 3
The Parisi Formula

The main goal of this chapter is to prove the celebrated Parisi formula for the free energy in the mixed p-spin models. The Ruelle probability cascades studied in the previous chapter will be used in a number of different ways, but their most important role will be as an approximation of the asymptotic Gibbs measures that generate the overlap matrix in the thermodynamic limit. As we explained in Sect. 2.4, a link between the Gibbs measure and the Ruelle probability cascades can be established using the Ghirlanda–Guerra identities and in this chapter we will show how these identities arise in the setting of the mixed p-spin models. In addition, the proof of the Parisi formula will be based on several other essential ideas, such as the Talagrand positivity principle, the Guerra replica symmetry breaking interpolation, and the Aizenman–Sims–Starr scheme.

3.1 Overview of the Parisi Formula

In this chapter we will study the mixed p-spin model introduced in Eq. (1.12). Let us remind for convenience that its Hamiltonian

$$H_N(\sigma) = \sum_{p\geq 1} \beta_p H_{N,p}(\sigma) \tag{3.1}$$

is defined by a linear combination of pure p-spin Hamiltonians

$$H_{N,p}(\sigma) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} g_{i_1\ldots i_p}\sigma_{i_1}\cdots\sigma_{i_p}, \tag{3.2}$$

where the random variables $(g_{i_1\ldots i_p})$ are independent for all $p \geq 1$ and all $(i_1,\ldots,i_p)$ and have the standard Gaussian distribution. Thus, the Hamiltonian $H_N(\sigma)$ in Eq. (3.1) is a Gaussian process with the covariance

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$$\mathbb{E}H_N(\sigma^1)H_N(\sigma^2) = N\xi(R_{1,2}), \tag{3.3}$$

where the function

$$\xi(x) = \sum_{p\geq 1} \beta_p^2 x^p. \tag{3.4}$$

Let us introduce a related function that will be used throughout this chapter,

$$\theta(x) = x\xi'(x) - \xi(x) = \sum_{p\geq 1} (p-1)\beta_p^2 x^p. \tag{3.5}$$

We remind that the coefficients (β_p) are assumed to decrease fast enough to ensure that the process is well defined when the sum includes infinitely many terms. Let us begin by formulating the Parisi formula for the free energy in the mixed p-spin model that will be proved in this chapter. We will first give the standard formulation and then describe an equivalent representation in terms of the Ruelle probability cascades. Because of this connection, we will use some of the notation from the previous chapter. As in Eqs. (2.42) and (2.43), given $r \geq 1$, we consider two sequences of parameters,

$$0 < \zeta_0 < \ldots < \zeta_{r-1} < 1 \tag{3.6}$$

and

$$0 = q_0 < q_1 < \ldots < q_r = 1, \tag{3.7}$$

only now we set $q_r = 1$. We mentioned in Eq. (2.83) that these two sequences can be identified with the distribution function given by

$$\zeta\big(\{q_p\}\big) = \zeta_p - \zeta_{p-1}, \tag{3.8}$$

for $p = 0, \ldots, r$, with the convention that $\zeta_{-1} = 0, \zeta_r = 1$. The distribution function ζ is called the *functional order parameter* (f.o.p.). Let us consider i.i.d. standard Gaussian random variables $(\eta_p)_{0\leq p\leq r}$ and define

$$X_r = \log 2\mathrm{ch}\Big(\eta_0\xi'(0)^{1/2} + \sum_{1\leq p\leq r} \eta_p\big(\xi'(q_p) - \xi'(q_{p-1})\big)^{1/2}\Big). \tag{3.9}$$

Notice that $\xi'(0) \neq 0$ if and only if $\beta_1 \neq 0$ in Eq. (3.1). Recursively over $0 \leq l \leq r-1$, we define

$$X_l = \frac{1}{\zeta_l}\log \mathbb{E}_l \exp \zeta_l X_{l+1}, \tag{3.10}$$

where $\mathbb{E}_l$ denotes the expectation with respect to η_{l+1} only. Notice that X_0 is a function of η_0. Finally, we define the so-called *Parisi functional* by

$$\mathscr{P}(\zeta) = \mathbb{E}X_0 - \frac{1}{2}\sum_{0\leq p\leq r-1} \zeta_p\big(\theta(q_{p+1}) - \theta(q_p)\big). \tag{3.11}$$

The main goal of this chapter is to prove the following formula for the free energy in the mixed p-spin models.

Theorem 3.1 (The Parisi Formula). *The limit of the free energy equals*

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log\sum_{\sigma\in\Sigma_N}\exp H_N(\sigma)=\inf_{\zeta}\mathscr{P}(\zeta). \tag{3.12}$$

The infimum is taken over all distribution functions of the type (3.8) or, in other words, over all $r\geq 1$ and sequences (3.6) and (3.7). We remind that, as in Eq. (1.16), we did not include the inverse temperature parameter β in the free energy in Eq. (3.12), since we can think of it as a factor in the parameters (β_p) in Eq. (3.1). It is customary to relax the definition (3.6) to allow non-strict inequalities $0\leq\zeta_0$ and $\zeta_{r-1}\leq 1$ and, if $\zeta_0=0$, to interpret Eq. (3.10) for $l=0$ as $X_0=\mathbb{E}_0X_1$ which corresponds to the limit $\zeta_0\downarrow 0$. Including these cases, obviously, does not affect the infimum of $\mathscr{P}(\zeta)$ on the right-hand side of Eq. (3.12), so we can assume Eq. (3.6). Next, we will explain a formal connection of the Parisi formula to the Ruelle probability cascades by showing how the functional $\mathscr{P}(\zeta)$ in Eq. (3.11) can be represented in terms of the cascade corresponding to the parameters (3.6) and (3.7).

The recursive construction (3.10) should look familiar, since we have seen a similar construction in Eq. (2.51) and Theorem 2.9. Clearly, the uniform random variables $(\omega_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ in Theorem 2.9 can be used to generate any other random variables and here we will use them to generate the following Gaussian sequences. As in Eq. (2.76), let $(\eta_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ be a sequence of i.i.d. standard Gaussian random variables indexed by all non-root vertices of the tree $\mathscr{A}$ in Eq. (2.40) and let η_0 be another independent standard Gaussian random variable. Given a nondecreasing function $\psi:[0,\infty)\to[0,\infty)$, consider a family of Gaussian random variables

$$g_\psi(h_\alpha)=\eta_0\psi(0)^{1/2}+\sum_{\beta\in p(\alpha)}\eta_\beta\big(\psi(q_{|\beta|})-\psi(q_{|\beta|-1})\big)^{1/2}, \tag{3.13}$$

indexed by the sequence of vectors $(h_\alpha)_{\alpha\in\mathbb{N}^r}$ defined in Eq. (2.61). Recall that $p(\alpha)$ denotes the path (2.41) from the root $\emptyset$ of the tree $\mathscr{A}$ to the vertex α and $|\beta|$ is the distance of the vertex β from the root. Similarly to Eq. (2.67), the covariance of the Gaussian process (3.13) is given by

$$\mathbb{E}g_\psi(h_\alpha)g_\psi(h_\beta)=\psi\big(h_\alpha\cdot h_\beta\big). \tag{3.14}$$

Below, we will use this process for $\psi=\xi'$ and $\psi=\theta$, and the first term $\eta_0\psi(0)^{1/2}$ in Eq. (3.13) was specifically included to cover the case when $\xi'(0)\neq 0$. Let us recall the sequence $(v_\alpha)_{\alpha\in\mathbb{N}^r}$ defined in Eq. (2.46), which describes the weights of the Ruelle probability cascade in Eq. (2.62) corresponding to the parameters (3.6) and (3.7). Theorem 2.9 implies the following representation.

Lemma 3.1. *The Parisi functional $\mathscr{P}(\zeta)$ in Eq. (3.11) can be represented as*

$$\mathscr{P}(\zeta) = \mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} v_\alpha 2\operatorname{ch} g_{\xi'}(h_\alpha) - \mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} v_\alpha \exp g_\theta(h_\alpha). \tag{3.15}$$

Proof. We will apply Theorem 2.9 separately to the first and second term in Eq. (3.15). For the first term, let us fix η_0 in Eqs. (3.9) and (3.13) and, for a moment, think of the function $X_r = X_r(\eta_1,\dots,\eta_r)$ as a function of $\eta_1,\dots,\eta_r$ only. Then, the recursive constructions (3.10) and (2.51) are identical and, using Theorem 2.9 conditionally on η_0 and then averaging in η_0, imply that

$$\begin{aligned}\mathbb{E}X_0 &= \mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} v_\alpha \exp\log 2\operatorname{ch} g_{\xi'}(h_\alpha)\\ &= \mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} v_\alpha 2\operatorname{ch} g_{\xi'}(h_\alpha).\end{aligned}$$

For the second term in Eq. (3.15), since $\theta(0)=0$, we need to show that the recursive construction (2.51), applied now to the function

$$X_r = X_r(\eta_1,\dots,\eta_r) = \sum_{1\le p\le r} \eta_p\big(\theta(q_p)-\theta(q_{p-1})\big)^{1/2},$$

will produce the second term in Eq. (3.11). Equation (2.51) for $l = r-1$ implies that (notice that $\mathbb{E}_{r-1}$ is now the expectation with respect to η_r only)

$$\begin{aligned}X_{r-1}(\eta_1,\dots,\eta_{r-1}) = &\sum_{1\le p\le r-1} \eta_p\big(\theta(q_p)-\theta(q_{p-1})\big)^{1/2}\\ &+\frac{1}{2}\zeta_{r-1}\big(\theta(q_r)-\theta(q_{r-1})\big).\end{aligned}$$

The second term is constant and we observe that a constant added to Eq. (2.50), $X_r + c$, propagates through Eq. (2.51) and we end up with $X_0 + c$. Therefore, we can proceed by induction on r to conclude that the second terms in Eqs. (3.15) and (3.11) are equal. □

The right-hand side of Eq. (3.15) is reminiscent of the Aizenman–Sims–Starr representation in Eq. (1.58), especially, if we write Eq. (3.15) as

$$\mathscr{P}(\zeta) = \mathbb{E}\log\big\langle 2\operatorname{ch} g_{\xi'}(\sigma)\big\rangle - \mathbb{E}\log\big\langle \exp g_\theta(\sigma)\big\rangle, \tag{3.16}$$

where $\langle\cdot\rangle$ is the average with respect to the random measure G defined in Eq. (2.62) by $G(h_\alpha) = v_\alpha$. For simplicity of notation, in Chap. 1 we considered the Aizenman–Sims–Starr scheme only for the classical Sherrington–Kirkpatrick model, for which $\xi(x) = \beta^2x^2$, $\xi'(x) = 2\beta^2 x$, and $\theta(x) = \beta^2x^2$. If we compare the covariance of the Gaussian process (3.13) in Eq. (3.14) for $\psi = \xi'$ and $\psi = \theta$ with the covariance (1.59) of the Gaussian processes $(z_N(\sigma))$ and $(y_N(\sigma))$ that appear in the Aizenman–Sims–Starr representation, the similarity between Eqs. (3.16) and (1.58)

is quite complete. Of course, as we mentioned a number of times before, this is not a coincidence, but rather a consequence of the main result of the last chapter. Once we prove that the Gibbs measure in the mixed p-spin model asymptotically satisfies the Ghirlanda–Guerra identities, Theorem 2.13 and the discussion after its proof will imply that, in some sense, the Gibbs measure looks like the Ruelle probability cascades.

3.2 The Ghirlanda–Guerra Identities

In this section, we will see how the Ghirlanda–Guerra identities that appeared in the last chapter in the setting of the Ruelle probability cascades arise in the setting of the mixed p-spin models. As we mentioned before, these identities were first discovered by Ghirlanda and Guerra in the setting of the mixed p-spin models, where, as we will see below, they follow from some general principle of the concentration of the Hamiltonian that holds under mild assumptions and can be proved in a wide variety of situations. Originally, this concentration principle was proved on average over the parameters (β_p) in Eq. (3.1) in the mixed p-spin model, but later on it was realized that the core idea is widely applicable if we utilize the mixed p-spin Hamiltonian in the role of a *perturbation* Hamiltonian. Namely, for all $p \geq 1$, let us consider

$$g_p(\sigma) = \frac{1}{N^{p/2}} \sum_{i_1,\dots,i_p=1}^{N} g'_{i_1\dots i_p} \sigma_{i_1} \dots \sigma_{i_p}, \tag{3.17}$$

where the coefficients $(g'_{i_1\dots i_p})$ are again i.i.d. standard Gaussian random variables independent of all the random variables in Eq. (3.1), and define

$$g(\sigma) = \sum_{p\geq 1} 2^{-p} x_p g_p(\sigma) \tag{3.18}$$

for some parameters $(x_p)_{p\geq 1}$ that will always belong to the interval $x_p \in [0,3]$ for all $p \geq 1$. For simplicity of notation, we will keep the dependence of $g(\sigma)$ on N implicit. This Gaussian process is of the same nature as the mixed p-spin Hamiltonian Eq. (3.1) except for a different normalization in Eq. (3.17), which implies that

$$\mathbb{E} g(\sigma^1) g(\sigma^2) = \sum_{p\geq 1} 4^{-p} x_p^2 R_{1,2}^p. \tag{3.19}$$

In other words, $g(\sigma)$ is of a smaller (constant) order than $H_N(\sigma)$ because of the additional factor $N^{-1/2}$. Given an arbitrary Hamiltonian $H(\sigma)$ on Σ_N, either random or nonrandom, let us consider a model with the perturbed Hamiltonian

$$H^{\text{pert}}(\sigma) = H(\sigma) + s g(\sigma), \tag{3.20}$$

for some parameter $s \geq 0$. Of course, the main example we have in mind will be given by the mixed p-spin Hamiltonian $H(\sigma) = H_N(\sigma)$ in Eq. (3.1), but the case of nonrandom Hamiltonians $H(\sigma)$ will also have important applications, specifically, in the Talagrand positivity principle in Sect. 3.3. What is the advantage of adding the perturbation term (3.18) to the original Hamiltonian of the model? The answer to this question lies in the fact that, under certain conditions, this perturbation term allows us to regularize the Gibbs measure and force it to satisfy useful properties without affecting our mail goal—the computation of the free energy. Using Eq. (3.19) and the independence of $g(\sigma)$ and $H(\sigma)$, it is easy to see that

$$\begin{aligned} \frac{1}{N}\mathbb{E}\log\sum_{\sigma\in\Sigma_N}\exp H(\sigma) &\leq \frac{1}{N}\mathbb{E}\log\sum_{\sigma\in\Sigma_N}\exp\big(H(\sigma)+sg(\sigma)\big) \\ &\leq \frac{1}{N}\mathbb{E}\log\sum_{\sigma\in\Sigma_N}\exp H(\sigma)+\frac{s^2}{2N}\sum_{p\geq 1}4^{-p}x_p^2. \end{aligned} \tag{3.21}$$

Both inequalities follow from Jensen's inequality applied either to the sum or the expectation with respect to $g(\sigma)$ conditionally on $H(\sigma)$. This implies that if we let $s = s_N$ in Eq. (3.20) depend on N in such a way that

$$\lim_{N\to\infty} N^{-1}s_N^2 = 0, \tag{3.22}$$

then the limit of the free energy is unchanged by the perturbation term $sg(\sigma)$. On the other hand, if $s = s_N$ is not too small, then the perturbation term has a non-trivial influence on the Gibbs measure of the model, as we shall see below. This is one of the most influential ideas in spin glasses. Let us formulate a condition under which the perturbation Hamiltonian Eq. (3.18) will force the Gibbs measure to satisfy the Ghirlanda–Guerra identities and explain exactly in what sense these identities hold. Let

$$G_N(\sigma) = \frac{\exp H^{\text{pert}}(\sigma)}{Z_N} \text{ where } Z_N = \sum_{\sigma\in\Sigma_N}\exp H^{\text{pert}}(\sigma) \tag{3.23}$$

be the Gibbs measure corresponding to the perturbed Hamiltonian Eq. (3.20) and, as usual, let $\langle\cdot\rangle$ denote the average with respect to $G_N^{\otimes\infty}$. Consider a function

$$\varphi = \log Z_N = \log\sum_{\sigma\in\Sigma_N}\exp\big(H(\sigma)+sg(\sigma)\big), \tag{3.24}$$

that will be viewed as a random function $\varphi = \varphi\big((x_p)\big)$ of the parameters (x_p), and suppose that

$$\sup\Big\{\mathbb{E}|\varphi-\mathbb{E}\varphi| : 0\leq x_p\leq 3, p\geq 1\Big\} \leq v_N(s) \tag{3.25}$$

for some function $v_N(s)$ that describes how well $\varphi((x_p))$ is concentrated around its expected value uniformly over all possible choices of the parameters (x_p) from the

interval $[0,3]$. Our main condition about the model will be expressed in terms of this concentration function, namely, that there exists a sequence $s = s_N$ such that

$$\lim_{N\to\infty} s_N = \infty \text{ and } \lim_{N\to\infty} s_N^{-2} v_N(s_N) = 0. \tag{3.26}$$

Under this assumption, the perturbation term will force the Gibbs measure to satisfy the Ghirlanda–Guerra identities, which, of course, will be useful only if the sequence s_N also satisfies Eq. (3.22) and the perturbation term does not affect the limit of the free energy. For any $n \geq 2, p \geq 1$ and any function f of the overlaps $R^n = (R_{l,l'})_{l,l'\leq n}$ of n replicas, let us define

$$\Delta(f,n,p) = \Big| \mathbb{E}\langle f R_{1,n+1}^p \rangle - \frac{1}{n} \mathbb{E}\langle f \rangle \mathbb{E}\langle R_{1,2}^p \rangle - \frac{1}{n} \sum_{l=2}^{n} \mathbb{E}\langle f R_{1,l}^p \rangle \Big|. \tag{3.27}$$

This is the familiar expression that appeared in Eq. (2.80) in the setting of the Ruelle probability cascades, with $\psi(x) = x^p$. Here, each quantity (3.27) depends on the parameters $(x_p)_{p\geq 1}$ in the perturbation Hamiltonian Eq. (3.18) and the Ghirlanda–Guerra identities hold asymptotically on average over these parameters, in the following sense. If we think of $(x_p)_{p\geq 1}$ as a sequence of i.i.d. random variables with the uniform distribution on $[1,2]$ and denote by $\mathbb{E}_x$ the expectation with respect to such sequence, then the following holds.

Theorem 3.2 (The Ghirlanda–Guerra Identities). *If $s = s_N$ in Eqs. (3.20) and (3.26) holds then*

$$\lim_{N\to\infty} \mathbb{E}_x \Delta(f,n,p) = 0 \tag{3.28}$$

for any $p \geq 1, n \geq 2$ and any bounded measurable function $f = f((R_{l,l'})_{l,l'\leq n})$.

The fact that these identities hold on average will be sufficient, because Eq. (3.28) will allow us to choose the parameters (x_p^N) in the perturbation Hamiltonian that depend on N such that, asymptotically, the Ghirlanda–Guerra identities hold with this specific choice of perturbation rather than on average. There will be other considerations involved, so we will make this choice later, but it should be emphasized that in the end the asymptotic Gibbs measures constructed via the Dovbysh–Sudakov representation will satisfy these identities exactly and not on average. Before we prove Theorem 3.2, let us give two examples of the models for which one can choose the sequence s_N such that the conditions (3.22) and (3.26) are satisfied. Both examples will be based on the Gaussian concentration inequality in Theorem 1.2.

Example 3.1 (Nonrandom Hamiltonian). If $H(\sigma)$ is nonrandom, then we can use Theorem 1.2 with the measure $G(\sigma) = \exp H(\sigma)$ on Σ_N. Since, by Eq. (3.19),

$$\mathbb{E} g(\sigma)^2 = \sum_{p\geq 1} 4^{-p} x_p^2 \leq 3$$

if all $0 \le x_p \le 3$, Theorem 1.2 implies that $\mathbb{E}(\varphi - \mathbb{E}\varphi)^2 \le 24s^2$ and Eq. (3.25) holds with $v_N(s) = 5s$. Therefore, both Eqs. (3.22) and (3.26) hold if we can take $s_N = N^\gamma$ for any $0 < \gamma < 1/2$.

Example 3.2 (Mixed p-spin Hamiltonian). If $H(\sigma) = H_N(\sigma)$ is the mixed p-spin Hamiltonian in Eq. (3.1) then we can use Theorem 1.2 with the counting measure G on Σ_N. By Eqs. (3.3) and (3.19),

$$\mathbb{E}\big(H_N(\sigma) + sg(\sigma)\big)^2 \le N\xi(1) + 3s^2$$

if all $0 \le x_p \le 3$ and Theorem 1.2 implies that $\mathbb{E}(\varphi - \mathbb{E}\varphi)^2 \le 8(N\xi(1) + 3s^2)$. Hence, Eq. (3.25) holds with $v_N(s) = 5(N\xi(1) + s^2)^{1/2}$ and both Eqs. (3.22) and (3.26) hold if we can take $s_N = N^\gamma$ for any $1/4 < \gamma < 1/2$.

In addition to the concentration inequality (3.25), the proof of Theorem 3.2 will utilize some simple properties of convex functions. The following lemma quantifies the fact that if two convex functions are close to each other then their derivatives are also close.

Lemma 3.2. *If $\varphi(x)$ and $\phi(x)$ are two differentiable convex functions then, for any $y > 0$,*

$$|\varphi'(x) - \phi'(x)| \le \phi'(x+y) - \phi'(x-y) + \frac{\delta}{y}, \tag{3.29}$$

where $\delta = |\varphi(x+y) - \phi(x+y)| + |\varphi(x-y) - \phi(x-y)| + |\varphi(x) - \phi(x)|$.

Proof. The convexity of ϕ implies that, for any $y > 0$,

$$\phi(x+y) - \phi(x) \le y\phi'(x+y) \text{ and } \phi'(x-y) \le \phi'(x).$$

Therefore,

$$\phi(x+y) - \phi(x) \le y\big(\phi'(x) + \phi'(x+y) - \phi'(x-y)\big).$$

The convexity of φ and the definition of δ imply

$$\begin{aligned} y\varphi'(x) &\le \varphi(x+y) - \varphi(x) \le \phi(x+y) - \phi(x) + \delta \\ &\le y\big(\phi'(x) + \phi'(x+y) - \phi'(x-y)\big) + \delta. \end{aligned}$$

Similarly, one can show that

$$y\varphi'(x) \ge y\big(\phi'(x) - \phi'(x+y) + \phi'(x-y)\big) - \delta$$

and combining these two inequalities finishes the proof. □

The main idea behind the Ghirlanda–Guerra identities is that, for each $p \ge 1$, the presence of the term $g_p(\sigma)$ in the perturbation Hamiltonian Eq. (3.18) provides some information about the pth moment of the overlaps, which can be deduced from the following concentration inequality mentioned at the beginning of this section.

Theorem 3.3. *For any $p \geq 1$, if $s > 0$ is such that $s^{-2} v_N(s) \leq 4^{-p}$ then*

$$\int_1^2 \mathbb{E}\big\langle \big| g_p(\sigma) - \mathbb{E}\big\langle g_p(\sigma)\big\rangle \big| \big\rangle dx_p \leq 2 + 48\sqrt{v_N(s)}. \tag{3.30}$$

Of course, the condition (3.26) will ensure that the assumption $s^{-2} v_N(s) \leq 4^{-p}$ is satisfied for $s = s_N$ and N large enough. We will see below that the Ghirlanda–Guerra identities are obtained simply by testing the concentration of $g_p(\sigma)$ on an arbitrary function of the overlaps.

Proof. Given the function φ in Eq. (3.24), we define $\phi = \mathbb{E}\varphi$. Let us fix $p \geq 1$ and denote $s_p = s2^{-p}$. Throughout the proof, we will think of φ and ϕ as the functions of $x = x_p$ only and work with one term,

$$s2^{-p} x_p g_p(\sigma) = x s_p g_p(\sigma),$$

in the perturbation Hamiltonian Eq. (3.18). First, let us observe that

$$\varphi'(x) = s_p \big\langle g_p(\sigma)\big\rangle \text{ and } \phi'(x) = s_p \mathbb{E}\big\langle g_p(\sigma)\big\rangle. \tag{3.31}$$

Since the covariance $\mathbb{E} g_p(\sigma^1) g_p(\sigma^2) = R_{1,2}^p$, the Gaussian integration by parts in Lemma 1.1 implies that

$$\phi'(x) = s_p \mathbb{E}\big\langle g_p(\sigma)\big\rangle = x s_p^2 \big(1 - \mathbb{E}\big\langle R_{1,2}^p\big\rangle\big) \in \big[0, 2x s_p^2\big]. \tag{3.32}$$

Differentiating the derivative $\phi'(x)$ in Eq. (3.31), it is easy to see that

$$\phi''(x) = s_p^2 \mathbb{E}\big(\big\langle g_p(\sigma)^2\big\rangle - \big\langle g_p(\sigma)\big\rangle^2\big) = s_p^2 \mathbb{E}\big\langle \big(g_p(\sigma) - \big\langle g_p(\sigma)\big\rangle\big)^2\big\rangle,$$

and integrating this over the interval $1 \leq x \leq 2$ and using Eq. (3.32) imply

$$s_p^2 \int_1^2 \mathbb{E}\big\langle \big(g_p(\sigma) - \big\langle g_p(\sigma)\big\rangle\big)^2\big\rangle dx = \phi'(2) - \phi'(1) \leq 4 s_p^2.$$

If we cancel s_p^2 on both sides then Jensen's inequality implies that

$$\int_1^2 \mathbb{E}\big\langle \big| g_p(\sigma) - \big\langle g_p(\sigma)\big\rangle \big| \big\rangle dx \leq 2. \tag{3.33}$$

To prove Eq. (3.30), it remains to approximate $\langle g_p(\sigma)\rangle$ by $\mathbb{E}\langle g_p(\sigma)\rangle$ and this is where the convexity plays its role. Since $\varphi(x)$ and $\phi(x)$ are convex and differentiable, we can apply the inequality (3.29). We will consider $1 \leq x \leq 2$ and $0 \leq y \leq 1$, in which case $0 \leq x - y, x, x + y \leq 3$ and we can use the definition (3.25) to conclude that δ in Eq. (3.29) satisfies $\mathbb{E}\delta \leq 3 v_N(s)$. Averaging the inequality (3.29) implies

$$\mathbb{E}|\varphi'(x) - \phi'(x)| \leq \phi'(x+y) - \phi'(x-y) + \frac{3 v_N(s)}{y}. \tag{3.34}$$

By Eq. (3.32), $|\phi'(x)| \le 6s_p^2$ for all $0 \le x \le 3$ and, by the mean value theorem,

$$\int_1^2 (\phi'(x+y) - \phi'(x-y))\,dx = \phi(2+y) - \phi(2-y) - \phi(1+y) + \phi(1-y) \le 24 y s_p^2.$$

Therefore, integrating Eq. (3.34) and recalling Eq. (3.31), we get

$$\int_1^2 \mathbb{E}\big|\langle g_p(\sigma)\rangle - \mathbb{E}\langle g_p(\sigma)\rangle\big|\,dx \le 24\Big(y s_p + \frac{v_N(s)}{y s_p}\Big). \tag{3.35}$$

The minimum of the right-hand side over $y > 0$ is equal to $48 v_N(s)^{1/2}$ and is achieved at $y = v_N(s)^{1/2}/s_p$. Throughout the argument we assumed that $0 \le y \le 1$ and this is guaranteed by the condition $s^{-2} v_N(s) \le 4^{-p}$. Combining Eq. (3.35) with this optimal choice of y and Eq. (3.33) implies Eq. (3.30). □

Proof (Theorem 3.2). Let us fix $n \ge 2$ and consider a bounded function $f = f(R^n)$ of the overlaps of n replicas. Without loss of generality, we can assume that $|f| \le 1$. Then,

$$\big|\mathbb{E}\langle f g_p(\sigma^1)\rangle - \mathbb{E}\langle f\rangle \mathbb{E}\langle g_p(\sigma)\rangle\big| \le \mathbb{E}\big\langle \big|g_p(\sigma) - \mathbb{E}\langle g_p(\sigma)\rangle\big|\big\rangle. \tag{3.36}$$

We can think of the left-hand side as a way to test the concentration of the process $(g_p(\sigma))$ on some function of the overlaps, $f(R^n)$. Using the Gaussian integration by parts formula in Lemma 1.1 and recalling the factor $s2^{-p}x_p$ in front of $g_p(\sigma)$ in Eq. (3.20),

$$\mathbb{E}\langle g_p(\sigma)\rangle = s2^{-p}x_p\big(1 - \mathbb{E}\langle R_{1,2}^p\rangle\big),$$

and, by the formula (1.29) in the Exercise 1.1 below Lemma 1.1,

$$\mathbb{E}\langle f g_p(\sigma^1)\rangle = s2^{-p}x_p \mathbb{E}\Big\langle f\Big(\sum_{l=1}^n R_{1,l}^p - n R_{1,n+1}^p\Big)\Big\rangle.$$

Therefore, since the self-overlap $R_{1,1} = 1$, the left-hand side of Eq. (3.36) equals $s2^{-p}x_p n\Delta(f,n,p)$, where $\Delta(f,n,p)$ was defined in Eq. (3.27). Integrating the inequality (3.36) over $1 \le x_p \le 2$ and using Theorem 3.3 imply

$$s2^{-p} n \int_1^2 \Delta(f,n,p)\,dx_p \le 2 + 48\sqrt{v_N(s)}, \tag{3.37}$$

if $s^{-2} v_N(s) \le 4^{-p}$. If we divide both sides by $s2^{-p}n$ and then average over all (x_p) on the interval $[1,2]$, using Fubini's theorem, we get

$$\mathbb{E}_x \Delta(f,n,p) \le \frac{2^p}{n}\Big(\frac{2}{s} + 48\frac{\sqrt{v_N(s)}}{s}\Big), \tag{3.38}$$

if $s^{-2}v_N(s) \le 4^{-p}$. Using this bound for $s = s_N$ that satisfies the condition (3.26) implies the Ghirlanda–Guerra identities in Eq. (3.28). □

3.3 The Talagrand Positivity Principle

Primarily, the Ghirlanda–Guerra identities will be used in their exact form, once we pass to the limit, to characterize the asymptotic Gibbs measures. However, we will need one corollary of these identities—the positivity of the overlap—for systems of finite size, when N is large but finite. In the case when the Ghirlanda–Guerra identities are exact, this was proved in Talagrand's positivity principle in Theorem 2.16. We will need an analogue of this result in some approximate sense when the Ghirlanda–Guerra identities also hold approximately on average $\mathbb{E}_x$ over the parameters (x_p) in the perturbation Hamiltonian Eq. (3.18), as in Theorem 3.2. We will be able to prove such result uniformly over all Hamiltonians $H(\sigma)$ in Eq. (3.20); so, in this section, $H(\sigma)$ is an arbitrary nonrandom Hamiltonian on Σ_N and G_N is the Gibbs measure (3.23) corresponding to the perturbed Hamiltonian in Eq. (3.20), $H^{\text{pert}}(\sigma) = H(\sigma) + sg(\sigma)$. The following holds.

Theorem 3.4 (Talagrand's Positivity Principle). *If $s = s_N$ in Eq. (3.20) is such that $\lim_{N\to\infty} s_N = \infty$ then, for any $\varepsilon > 0$,*

$$\lim_{N\to\infty} \sup_{H(\sigma)} \mathbb{E}_x \mathbb{E} G_N^{\otimes 2}\big(R_{1,2} \le -\varepsilon\big) = 0. \tag{3.39}$$

Notice that here we only need the first part of the assumption (3.26). The general idea of the proof will be the same as in Theorem 2.16 and we only need to adapt it slightly to take into account that the Ghirlanda–Guerra identities hold in some approximate sense. First of all, let us recall that in the Example 3.1 below Theorem 3.2 we showed that, whenever $H(\sigma)$ is nonrandom, one can take $v_N(s) = 5s$ in Eq. (3.25). In particular, Eq. (3.38) implies that

$$\mathbb{E}_x \Delta(f, n, p) \le \frac{2^p}{n}\Big(\frac{2}{s} + 48\sqrt{\frac{5}{s}}\Big) \tag{3.40}$$

if $5/s \le 4^{-p}$. It is crucial that this bound does not depend on $H(\sigma)$. Similarly to Eq. (3.27), let us define

$$\Delta(f, n, \psi) = \Big|\mathbb{E}\big\langle f\psi(R_{1,n+1})\big\rangle - \frac{1}{n}\mathbb{E}\big\langle f\big\rangle \mathbb{E}\big\langle \psi(R_{1,2})\big\rangle - \frac{1}{n}\sum_{l=2}^{n} \mathbb{E}\big\langle f\psi(R_{1,l})\big\rangle\Big|$$

for any continuous function $\psi : [-1, 1] \to \mathbb{R}$. If we approximate ψ uniformly by polynomials and use the bound (3.40) for $s = s_N$ such that $s_N \to \infty$, we can see that

$$\lim_{N\to\infty} \sup_{H(\sigma)} \mathbb{E}_x \Delta(f,n,\psi) = 0. \tag{3.41}$$

This observation will be the main tool in the proof of Theorem 3.4. The rest of the proof will mimic the argument of Theorem 2.16 and, at the same time, approximate indicators of sets by continuous functions.

Proof (Theorem 3.4). Given $\varepsilon > 0$, let us fix any integer $n > 1 + 2\varepsilon^{-1}$ and, for $2 \le k \le n$, define

$$a_k = -\varepsilon + \frac{k-2}{n-2} \times \frac{\varepsilon}{2}.$$

Notice that $-\varepsilon = a_2 < \ldots < a_n = -\varepsilon/2$. For $2 \le k \le n$, let us define the events

$$A_k = \Big\{ (\sigma^l)_{l\le k} : R_{l,l'} \le a_k, 1 \le l \ne l' \le k \Big\}.$$

In particular, $A_2 = \{R_{1,2} \le -\varepsilon\}$ and $A_n = \emptyset$, since our choice of $n > 1 + 2\varepsilon^{-1}$ implies that for $(\sigma^1, \ldots, \sigma^n) \in A_n$,

$$0 \le N^{-1} \Big\| \sum_{l\le n} \sigma^l \Big\|^2 = n + \sum_{l\ne l'} R_{l,l'} \le n - n(n-1)\frac{\varepsilon}{2} < 0.$$

Since $a_k < a_{k+1}$, for $2 \le k \le n-1$, we have the following relation between two consecutive events:

$$\begin{aligned} I_{A_{k+1}} &\ge I_{A_k} \prod_{l\le k} I\big(R_{l,k+1} \le a_{k+1}\big) = I_{A_k} \prod_{l\le k} \big(1 - I\big(R_{l,k+1} > a_{k+1}\big)\big) \\ &\ge I_{A_k} - \sum_{l\le k} I_{A_k} I\big(R_{l,k+1} > a_{k+1}\big). \end{aligned}$$

If we let $\psi_k(x)$ be a continuous function equal to 0 for $x \le a_k$, 1 for $x \ge a_{k+1}$, and linear on the interval $[a_k, a_{k+1}]$ then $I(x > a_{k+1}) \le \psi_k(x)$ and, therefore,

$$I_{A_{k+1}} \ge I_{A_k} - \sum_{l\le k} I_{A_k} \psi_k(R_{l,k+1}).$$

Integrating this inequality and using the symmetry between replicas, we get

$$\begin{aligned} \mathbb{E}\big\langle I_{A_{k+1}} \big\rangle &\ge \mathbb{E}\big\langle I_{A_k} \big\rangle - \sum_{l\le k} \mathbb{E}\big\langle I_{A_k} \psi_k(R_{l,k+1}) \big\rangle \\ &= \mathbb{E}\big\langle I_{A_k} \big\rangle - k\mathbb{E}\big\langle I_{A_k} \psi_k(R_{1,k+1}) \big\rangle. \end{aligned} \tag{3.42}$$

Let us now consider Eq. (3.41) for $\Delta_k := \Delta(I_{A_k}, k, \psi_k)$. Observe that $I_{A_k} \psi_k(R_{1,l}) = 0$ for $l = 2, \ldots, k$, since $R_{1,l} < a_k$ on the event A_k, while $\psi_k(x) = 0$ for $x \le a_k$. Therefore,

$$\Delta_k = \Big| \mathbb{E}\big\langle I_{A_k} \psi_k(R_{1,k+1}) \big\rangle - \frac{1}{k} \mathbb{E}\big\langle I_{A_k} \big\rangle \mathbb{E}\big\langle \psi_k(R_{1,2}) \big\rangle \Big|,$$

and Eq. (3.42) implies that

$$\mathbb{E}\langle I_{A_{k+1}}\rangle \geq \mathbb{E}\langle I_{A_k}\rangle - \mathbb{E}\langle I_{A_k}\rangle \mathbb{E}\langle \psi_k(R_{1,2})\rangle - \Delta_k. \tag{3.43}$$

Let us denote $a = \mathbb{E}\langle I_{A_2}\rangle$ and $\Delta = \max\{\Delta_k : 2 \leq k \leq n-1\}$. With this notation, the inequality $1 - \psi_k(x) \geq I(x \leq -\varepsilon)$ implies that $1 - \mathbb{E}\langle \psi_k(R_{1,2})\rangle \geq a$ and Eq. (3.43) implies that $\mathbb{E}\langle I_{A_{k+1}}\rangle \geq a\mathbb{E}\langle I_{A_k}\rangle - \Delta$. By induction on k, one can easily show that $\mathbb{E}\langle I_{A_k}\rangle \geq a^{k-2} - (k-2)\Delta$ and, recalling the fact that $A_n = \emptyset$, we get

$$0 = \mathbb{E}\langle I_{A_n}\rangle \geq a^{n-2} - (n-2)\Delta.$$

Solving this inequality for a and averaging over the parameters (x_p) give

$$\mathbb{E}_x a = \mathbb{E}_x \mathbb{E} G_N^{\otimes 2}\big(R_{1,2} \leq -\varepsilon\big) \leq \big((n-2)\mathbb{E}_x\Delta\big)^{1/(n-2)}.$$

It remains to bound the maximum, Δ, by the sum and apply Eq. (3.41) to each Δ_k. Since Eq. (3.41) holds uniformly over all nonrandom Hamiltonians $H(\sigma)$ on Σ_N, this finishes the proof. □

3.4 The Guerra RSB Interpolation

In this section we will describe Guerra's *replica symmetry breaking (RSB) interpolation* method which shows that the Parisi formula (3.12) gives an upper bound on the free energy in the thermodynamic limit.

Theorem 3.5 (Guerra's RSB Bound). *For any f.o.p. ζ in Eq. (3.8),*

$$\limsup_{N\to\infty} \frac{1}{N}\mathbb{E}\log \sum_{\sigma\in\Sigma_N} \exp H_N(\sigma) \leq \mathscr{P}(\zeta). \tag{3.44}$$

Therefore, we can minimize the right-hand side over ζ. The representation (3.15) of the Parisi functional $\mathscr{P}(\zeta)$ in terms of the Ruelle probability cascades will play a key role in the proof. With the notation in Eq. (3.15), for $1 \leq i \leq N$, let $(g_{\xi',i}(h_\alpha))$ and $(g_{\theta,i}(h_\alpha))$ be independent copies of the Gaussian processes $(g_{\xi'}(h_\alpha))$ and $(g_\theta(h_\alpha))$. For $0 \leq t \leq 1$, consider the Hamiltonian

$$H_{N,t}(\sigma,\alpha) = \sqrt{t}H_N(\sigma) + \sqrt{1-t}\sum_{i=1}^N g_{\xi',i}(h_\alpha)\sigma_i + \sqrt{t}\sum_{i=1}^N g_{\theta,i}(h_\alpha), \tag{3.45}$$

indexed by $\Sigma_N \times \mathbb{N}^r$ or, more precisely, by vectors (σ, h_α) such that σ belongs to the support of the Gibbs measure G_N and h_α belongs to the support of the measure G in Eq. (2.62). This Hamiltonian is the essence of the Guerra RSB interpolation method. To this Hamiltonian one can associate the free energy

$$\varphi(t) = \frac{1}{N}\mathbb{E}\log\sum_{\sigma,\alpha} v_\alpha \exp\bigl(H_{N,t}(\sigma,\alpha) + s_N g(\sigma)\bigr), \tag{3.46}$$

where $(g(\sigma))$ is the p-spin perturbation Hamiltonian defined in Eq. (3.18) and, in this section, the expectation in Eq. (3.46) also includes the average $\mathbb{E}_x$ with respect to the sequence (x_p) of i.i.d. uniform random variables on $[1,2]$ that appear in the definition of $(g(\sigma))$ in Eq. (3.18). We choose any sequence (s_N) such that

$$\lim_{N\to\infty} N^{-1}s_N^2 = 0 \text{ and } \lim_{N\to\infty} s_N = \infty. \tag{3.47}$$

The first condition in Eq. (3.47) coincides with Eq. (3.22) and, by a computation similar to Eq. (3.21), ensures that the perturbation term $s_N g(\sigma)$ in Eq. (3.46) does not affect $\varphi(t)$ in the limit, uniformly over $t \in [0,1]$. The second condition in Eq. (3.47) will ensure the validity of Talagrand's positivity principle in Theorem 3.4, which is the only reason why the perturbation term was included in Eq. (3.46). We will show that the derivative $\varphi'(t)$ is essentially non-positive and the inequality (3.44) will follow by comparing the values $\varphi(1)$ and $\varphi(0)$. Let us define the Gibbs measure on $\Sigma_N \times \mathbb{N}^r$,

$$\Gamma_t(\sigma,\alpha) = \frac{v_\alpha \exp\bigl(H_{N,t}(\sigma,\alpha) + s_N g(\sigma)\bigr)}{Z_t}, \tag{3.48}$$

associated with the perturbed Hamiltonian Eq. (3.45), where Z_t is the corresponding partition function, the sum in the logarithm in Eq. (3.46).

Proof (Theorem 3.5). Let us begin with the computation of the derivative $\varphi'(t)$. If we denote by $\langle\cdot\rangle_t$ the average with respect to the Gibbs measure $\Gamma_t^{\otimes\infty}$ then

$$\varphi'(t) = \frac{1}{N}\mathbb{E}\Bigl\langle \frac{\partial H_{N,t}(\sigma,\alpha)}{\partial t}\Bigr\rangle_t,$$

for $t \in (0,1)$. Let us recall the definition of θ in Eq. (3.5) and for $a,b \in \mathbb{R}$, let us denote

$$C(a,b) = \xi(a) - a\xi'(b) + \theta(b) = \xi(a) - \xi(b) - \xi'(a)(a-b).$$

Given two configurations $(\sigma^1,\alpha^1),(\sigma^2,\alpha^2) \in \Sigma_N \times \mathbb{N}^r$ and using the covariance formulae in Eqs. (3.3) and (3.14) for $\psi = \xi'$ and $\psi = \theta$, it is easy to check that

$$\frac{1}{N}\mathbb{E}\frac{\partial H_{N,t}(\sigma^1,\alpha^1)}{\partial t} H_{N,t}(\sigma^2,\alpha^2) = C\bigl(R_{1,2}, h_{\alpha^1}\cdot h_{\alpha^2}\bigr).$$

Since $R_{1,1} = 1$, $h_{\alpha^1}\cdot h_{\alpha^1} = q_r = 1$ by Eq. (3.7) and $C(1,1) = 0$, the Gaussian integration by parts formula in Lemma 1.1 (see also Lemma 1.2) implies that

$$\varphi'(t) = -\mathbb{E}\bigl\langle C\bigl(R_{1,2}, h_{\alpha^1}\cdot h_{\alpha^2}\bigr)\bigr\rangle_t. \tag{3.49}$$

If the p-spin Hamiltonian Eq. (3.1) does not include the terms for odd $p \geq 3$ then the function ξ is convex on the entire real line, which implies that $C(a,b) \geq 0$ for all $a,b \in \mathbb{R}$. Therefore, $\varphi'(t) \leq 0$ and, in this case, we do not need the perturbation term $s_N g(\sigma)$. However, if some terms for odd $p \geq 3$ are present then we can only claim that the function ξ is convex on $[0,\infty)$ and $C(a,b) \geq 0$ for $a,b \geq 0$. The scalar product $h_{\alpha^1} \cdot h_{\alpha^2} \geq 0$ by the definition of the Ruelle probability cascades. However, a priori, the overlap $R_{1,2}$ can take values in $[-1,1]$ and Eq. (3.49) only implies

$$\varphi'(t) \leq \|C\|_\infty \mathbb{E}\big\langle I\big(R_{1,2} \leq -\varepsilon\big)\big\rangle_t + \delta(\varepsilon), \tag{3.50}$$

where $\delta(\varepsilon) = \max\{-C(a,b) : a \geq -\varepsilon, b \geq 0\} \leq L\varepsilon$, for some constant $L > 0$ that depends on ξ only. To control the first term in Eq. (3.50), we will apply Talagrand's positivity principle in Theorem 3.4. Since the event $\{R_{1,2} \leq -\varepsilon\}$ does not depend on the indices (α^l), we can replace the Gibbs average $\langle\cdot\rangle_t$ in Eq. (3.50) with respect to the Gibbs measure Γ_t by the average with respect to its marginal G_N on Σ_N. By the definition (3.48), it is clear that if we define a new Hamiltonian $H(\sigma)$ by

$$\exp H(\sigma) = \sum_{\alpha \in \mathbb{N}^r} v_\alpha \exp H_{N,t}(\sigma,\alpha) \tag{3.51}$$

then the marginal G_N on Σ_N is equal to

$$G_N(\sigma) = \frac{\exp(H(\sigma) + s_N g(\sigma))}{Z_N} \text{ where } Z_N = \sum_{\sigma \in \Sigma_N} \exp\big(H(\sigma) + s_N g(\sigma)\big).$$

Therefore, we can bound the first term in Eq. (3.50) using that

$$\mathbb{E}\big\langle I\big(R_{1,2} \leq -\varepsilon\big)\big\rangle_t = \mathbb{E} G_N^{\otimes 2}\big(R_{1,2} \leq -\varepsilon\big) \leq \sup_{H(\sigma)} \mathbb{E}_x \mathbb{E}_g G_N^{\otimes 2}\big(R_{1,2} \leq -\varepsilon\big),$$

where $\mathbb{E}_g$ and $\mathbb{E}_x$ denote the expectations with respect to the Gaussian processes $(g_p(\sigma))$ and uniform random variables (x_p) on $[1,2]$ that appear in the definition of the perturbation Hamiltonian $(g(\sigma))$ in Eq. (3.18), and where we replaced the average with respect to the random Hamiltonian $H(\sigma)$ in Eq. (3.51) by the supremum over all nonrandom Hamiltonians on Σ_N. Notice that this bound is uniform over t and, therefore, Talagrand's positivity principle in Theorem 3.4 and Eq. (3.50) implies that

$$\limsup_{N\to\infty} \sup_{0<t<1} \varphi'(t) \leq L\varepsilon.$$

Since $\varphi(t)$ is continuous on $[0,1]$, integrating the derivative and letting $\varepsilon \downarrow 0$,

$$\limsup_{N\to\infty} \varphi(1) \leq \limsup_{N\to\infty} \varphi(0). \tag{3.52}$$

Let us observe that the perturbation term $s_N g(\sigma)$ had played its role and, since, by the first condition in Eq. (3.47), asymptotically it does not affect $\varphi(0)$ and $\varphi(1)$, without loss of generality, we can now omit it in the definition of $\varphi(0)$ and $\varphi(1)$. Then, it is easy to see that

$$\varphi(0) = \frac{1}{N}\mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} v_\alpha \sum_{\sigma\in\Sigma_N} \exp\sum_{i=1}^N g_{\xi',i}(h_\alpha)\sigma_i$$
$$= \frac{1}{N}\mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} v_\alpha \exp\sum_{i=1}^N \log 2\mathrm{ch}\, g_{\xi',i}(h_\alpha)$$

and

$$\varphi(1) = \frac{1}{N}\mathbb{E}\log \sum_{\sigma\in\Sigma_N} \exp H_N(\sigma) + \frac{1}{N}\mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} v_\alpha \exp\sum_{i=1}^N g_{\theta,i}(h_\alpha).$$

To finish the proof, it remains to appeal to Theorem 2.9 and the corollary of Theorem 2.9 in Eq. (2.60), which imply that

$$\varphi(0) = \mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} v_\alpha 2\mathrm{ch}\, g_{\xi'}(h_\alpha)$$

and

$$\varphi(1) = \frac{1}{N}\mathbb{E}\log \sum_{\sigma\in\Sigma_N} \exp H_N(\sigma) + \mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} v_\alpha \exp g_\theta(h_\alpha).$$

To finish the proof, it remains to compare this to the representation of the Parisi functional $\mathscr{P}(\zeta)$ in Lemma 3.1 and use the inequality (3.52). □

3.5 The Aizenman–Sims–Starr Scheme

In the last section we proved an upper bound in the Parisi formula and in the next section we will prove the matching lower bound based on the Aizenman–Sims–Starr scheme. We already described the Aizenman–Sims–Starr scheme in the Sherrington–Kirkpatrick model in Sect. 1.3, which began with the telescopic sum (1.49) and ended with the representation (1.58). An analogous computation can be carried out in the setting of the mixed p-spin models, but this time we will consider the free energy

$$F_N = \frac{1}{N}\mathbb{E}\log Z_N \tag{3.53}$$

where

$$Z_N = \sum_{\sigma\in\Sigma_N} \exp\big(H_N(\sigma) + s_N g_N(\sigma)\big), \tag{3.54}$$

corresponding to the Hamiltonian which includes the perturbation term $(g_N(\sigma))$ defined in Eq. (3.18), whose dependence on N is now made explicit. In this section, we are not averaging with respect to the sequence (x_p) of parameters that appear in the definition of $(g_N(\sigma))$, so we assume that this sequence is fixed. Of course,

one can write the Aizenman–Sims–Starr scheme without the perturbation term, but it will be needed in the next section to ensure the validity of the Ghirlanda–Guerra identities. From now on, we will take

$$s_N = N^\gamma, \tag{3.55}$$

for any $1/4 < \gamma < 1/2$, as in the Example 3.2 below Theorem 3.2. With this choice, the perturbation term does not affect the free energy F_N in the limit by Eq. (3.21) but will ensure that the asymptotic Gibbs measures satisfy the Ghirlanda–Guerra identities. Using Eq. (1.49), we begin with the relation

$$\liminf_{N\to\infty} F_N \geq \liminf_{N\to\infty} A_N, \tag{3.56}$$

where

$$A_N = \mathbb{E}\log Z_{N+1} - \mathbb{E}\log Z_N. \tag{3.57}$$

As in Sect. 1.3, we can compare the Hamiltonians in the partition functions Z_N and Z_{N+1} and separate them into their "common part" and the remaining "individual parts." Let us denote $\rho = (\sigma, \varepsilon) \in \Sigma_{N+1}$ for $\sigma \in \Sigma_N$ and $\varepsilon \in \{-1, +1\}$. If we compare

$$H_{N,p}(\sigma) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} g_{i_1\ldots i_p}\sigma_{i_1}\cdots\sigma_{i_p},$$

$$H_{N+1,p}(\rho) = \frac{1}{(N+1)^{(p-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N+1} g_{i_1\ldots i_p}\rho_{i_1}\cdots\rho_{i_p},$$

then the common part of $H_N(\sigma)$ and $H_{N+1}(\rho)$ will be given by the Hamiltonian

$$H'_N(\sigma) = \sum_{p\geq 1} \beta_p H'_{N,p}(\sigma), \tag{3.58}$$

where $H'_{N,p}(\sigma)$ are differently scaled pure p-spin Hamiltonians,

$$H'_{N,p}(\sigma) = \frac{1}{(N+1)^{(p-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} g_{i_1,\ldots,i_p}\sigma_{i_1}\cdots\sigma_{i_p}. \tag{3.59}$$

One can now carefully write down the remaining parts of the Hamiltonians $H_{N+1}(\rho)$ and $H_N(\sigma)$ and study their covariance structure, as we did in Chap. 1 for the Sherrington–Kirkpatrick model. This explicit calculation is not very difficult and is a good exercise to understand how one would initially extend the Aizenman–Sims–Starr scheme for the Sherrington–Kirkpatrick model to the case of the mixed p-spin models. However, from the point of view of notation, this approach is a bit messy, especially, because we also have to consider the perturbation Hamiltonian in Eq. (3.53). We will use the interpolation method instead. Let $(z_N(\sigma))$ and $(y_N(\sigma))$ be two Gaussian processes indexed by $\Sigma_N = \{-1, 1\}^N$ with the covariance

$$\mathbb{E}z_N(\sigma^1)z_N(\sigma^2) = \xi'(R_{1,2}),\ \mathbb{E}y_N(\sigma^1)y_N(\sigma^2) = \theta(R_{1,2}), \tag{3.60}$$

independent of each other and all other random variables. Such processes exist and can be constructed explicitly, similarly to the perturbation Hamiltonian Eq. (3.18) with the covariance (3.19), since, by Eqs. (3.4) and (3.5), the Taylor series of the functions $\xi'(x)$ and $\theta(x)$ have positive coefficients. Let us denote by

$$G'_N(\sigma) = \frac{\exp\big(H'_N(\sigma) + s_N g_N(\sigma)\big)}{Z'_N}, \tag{3.61}$$

where

$$Z'_N = \sum_{\sigma\in\Sigma_N} \exp\big(H'_N(\sigma) + s_N g_N(\sigma)\big), \tag{3.62}$$

the Gibbs measure corresponding to the perturbed Hamiltonian Eq. (3.58), and let $\langle\cdot\rangle'$ be the average with respect to $G'^{\otimes\infty}_N$. Then the following analogue of the Aizenman–Sims–Starr representation (1.58) holds in the setting of the mixed p-spin models.

Theorem 3.6 (Aizenman–Sims–Starr Representation). *The quantity A_N in Eq. (3.57) satisfies*

$$A_N = \mathbb{E}\log\Big\langle 2\operatorname{ch} z_N(\sigma)\Big\rangle' - \mathbb{E}\log\Big\langle \exp y_N(\sigma)\Big\rangle' + o(1) \tag{3.63}$$

uniformly over all $x_p \in [1,2]$ for all $p \geq 1$ in the perturbation term (3.18).

Proof. In each term of A_N in Eq. (3.57) we would like to replace the original Hamiltonian by a very close one, and we will achieve this using the Gaussian interpolation between the two. For the second term, we will use the interpolating Hamiltonian on Σ_N, for $0 \leq t \leq 1$,

$$H_t(\sigma) = \sqrt{t}H_N(\sigma) + \sqrt{1-t}\,\big(H'_N(\sigma) + y_N(\sigma)\big) + s_N g_N(\sigma). \tag{3.64}$$

The perturbation term will be left untouched by this interpolation. For the first term, recalling the notation $\rho = (\sigma,\varepsilon) \in \Sigma_{N+1}$ for $\sigma \in \Sigma_N$ and $\varepsilon \in \{-1,+1\}$, we will use the Hamiltonian on Σ_{N+1}, for $0 \leq t \leq 1$,

$$\begin{aligned} H_t(\rho) = {} & \sqrt{t}H_{N+1}(\rho) + \sqrt{1-t}\big(H'_N(\sigma) + \varepsilon z_N(\sigma)\big) \\ & + \sqrt{t}s_{N+1}g_{N+1}(\rho) + \sqrt{1-t}\,s_N g_N(\sigma). \end{aligned} \tag{3.65}$$

We assume that the perturbation Hamiltonians $g_N(\sigma)$ and $g_{N+1}(\rho)$ are defined in terms of independent Gaussian sequences and in this interpolation we also replace the perturbation term in the partition function Z_{N+1} by the one in Z_N. Let us consider the functions

$$\varphi_1(t) = \mathbb{E}\log\sum_{\sigma\in\Sigma_N}\exp H_t(\sigma),\ \varphi_2(t) = \mathbb{E}\log\sum_{\rho\in\Sigma_{N+1}}\exp H_t(\rho).$$

By estimating the derivatives of these functions, we will show that

$$\varphi_1(1)+\varphi_2(1)=\varphi_1(0)+\varphi_2(0)+o(1), \tag{3.66}$$

and one can easily check that this coincides with Eq. (3.63). First, if we denote by $\langle\cdot\rangle_t$ the average with respect to the Gibbs measure on Σ_N corresponding to the Hamiltonian $H_t(\sigma)$ and, for $0<t<1$, denote

$$C_1(\sigma^1,\sigma^2)=\mathbb{E}\frac{\partial H_t(\sigma^1)}{\partial t}H_t(\sigma^2),$$

then, by the Gaussian integration by parts formula in Lemma 1.1,

$$\varphi_1'(t)=\mathbb{E}\Big\langle\frac{\partial H_t(\sigma)}{\partial t}\Big\rangle_t=\mathbb{E}\Big\langle C_1(\sigma^1,\sigma^1)-C_1(\sigma^1,\sigma^2)\Big\rangle_t. \tag{3.67}$$

First of all, using Eqs. (3.3) and (3.60), it is easy to see that

$$C_1(\sigma^1,\sigma^2)=\frac{1}{2}\Big(N\xi(R_{1,2})-\theta(R_{1,2})-\mathbb{E}H_N'(\sigma^1)H_N'(\sigma^2)\Big). \tag{3.68}$$

It is also easy to see that the covariance of the Hamiltonian $H_N'(\sigma)$ in Eq. (3.58) satisfies

$$\begin{aligned}\mathbb{E}H_N'(\sigma^1)H_N'(\sigma^2)&=N\sum_{p\ge1}\Big(\frac{N}{N+1}\Big)^{p-1}\beta_p^2R_{1,2}^p\\&=N\xi(R_{1,2})-\theta(R_{1,2})+O(N^{-1}),\end{aligned} \tag{3.69}$$

if we recall the definition of θ in Eq. (3.5) and use that

$$\Big(\frac{N}{N+1}\Big)^{p-1}=1-\frac{p-1}{N}+O(N^{-1}).$$

Combining Eqs. (3.68) and (3.69) implies that $C_1(\sigma^1,\sigma^2)=O(N^{-1})$ and integrating the derivative $\varphi_1'(t)$ in Eq. (3.67) implies that $\varphi_1(1)=\varphi_1(0)+o(1)$. The argument in the case of the function $\varphi_2(t)$ is identical and we only need to show how one can control

$$C_2(\rho^1,\rho^2)=\mathbb{E}\frac{\partial H_t(\rho^1)}{\partial t}H_t(\rho^2) \tag{3.70}$$

for $\rho^1=(\sigma^1,\varepsilon^1),\rho^2=(\sigma^2,\varepsilon^2)\in\Sigma_{N+1}$. The computation is more involved in this case but still very straightforward and, similarly to Eq. (3.69), we will need to rewrite the covariance of each term in Eq. (3.65). Let us begin with the non-perturbation terms in the first line in Eq. (3.65). We have already computed the covariance of the Hamiltonian $H_N'(\sigma)$ in Eq. (3.69). By Eq. (3.60),

$$\mathbb{E}\varepsilon^1z_N(\sigma^1)\varepsilon^2z_N(\sigma^2)=\varepsilon^1\varepsilon^2\xi'(R_{1,2}).$$

Finally, if we write the overlap of ρ^1 and ρ^2 as

$$R_{1,2}^+ = \frac{1}{N+1}\sum_{i=1}^{N+1}\rho_i^1\rho_i^2 = \frac{1}{N+1}\sum_{i=1}^{N}\sigma_i^1\sigma_i^2 + \frac{1}{N+1}\varepsilon^1\varepsilon^2$$
$$= R_{1,2} - \frac{1}{N+1}R_{1,2} + \frac{1}{N+1}\varepsilon^1\varepsilon^2, \tag{3.71}$$

then, by Eq. (3.3) and Taylor's expansion, the covariance of the $H_{N+1}(\rho)$ satisfies

$$\mathbb{E}H_{N+1}(\rho^1)H_{N+1}(\rho^2) = (N+1)\xi(R_{1,2}^+)$$
$$= (N+1)\xi(R_{1,2}) - R_{1,2}\xi'(R_{1,2}) + \varepsilon^1\varepsilon^2\xi'(R_{1,2}) + O\big(N^{-1}\big).$$

Using all these covariance formulae, it is now easy to see that the contribution on the right-hand side of Eq. (3.70) of all non-perturbation terms from the first line in Eq. (3.65) will be (up to a factor $1/2$)

$$(N+1)\xi(R_{1,2}) - R_{1,2}\xi'(R_{1,2}) + \varepsilon^1\varepsilon^2\xi'(R_{1,2})$$
$$-N\xi(R_{1,2}) + \theta(R_{1,2}) - \varepsilon^1\varepsilon^2\xi'(R_{1,2}) + O(N^{-1}) = O(N^{-1}).$$

To compute the contribution of all the perturbation terms from the second line in Eq. (3.65), let us denote by $f(R_{1,2})$ the covariance function of the process $(g_N(\sigma))$ in Eq. (3.19). Then, the perturbation terms will contribute (again, up to a factor $1/2$)

$$s_{N+1}^2 f(R_{1,2}^+) - s_N^2 f(R_{1,2}) = (N+1)^{2\gamma} f(R_{1,2}^+) - N^{2\gamma} f(R_{1,2}),$$

by Eq. (3.55). Again, using Eq. (3.71) and the fact that $2\gamma < 1$, it is easy to see that this quantity is of order $O(N^{-(1-2\gamma)}) = o(1)$ and $C_2(\rho^1,\rho^2) = o(1)$. By the definition of the covariance function $f(R_{1,2})$ in Eq. (3.19), we can control f and its derivative uniformly over all choices of the parameters $x_p \in [1,2]$, so $o(1)$ is uniform in (x_p). Integrating the derivative $\varphi_2'(t)$ implies that $\varphi_2(1) = \varphi_2(0) + o(1)$ and Eq. (3.66) holds. This finishes the proof. □

3.6 Proof of the Parisi Formula

In this section we will finish the proof of the Parisi formula in Theorem 3.1. The upper bound was proved in Theorem 3.5 using Guerra's RSB interpolation and in this section we will prove the corresponding lower bound using the Aizenman–Sims–Starr scheme and the characterization of the distributions of the overlap arrays that satisfy the Ghirlanda–Guerra identities proved in Sect. 2.4. We will begin with the following variation of Eqs. (3.56) and (3.63). Let

$$F_N = \frac{1}{N}\mathbb{E}\log\sum_{\sigma\in\Sigma_N}\exp H_N(\sigma)$$

denote the free energy corresponding to the original mixed p-spin Hamiltonian Eq. (3.1) without the perturbation term. Let $x = (x_p)$ be the sequence of the parameters in the definition (3.18) of the perturbation Hamiltonian and (slightly abusing notation) let

$$A_N = A_N(x) = \mathbb{E}\log\Big\langle 2\operatorname{ch} z_N(\sigma)\Big\rangle' - \mathbb{E}\log\Big\langle \exp y_N(\sigma)\Big\rangle' \tag{3.72}$$

denote the representation on the right-hand side of Eq. (3.63), which depends on the sequence $x = (x_p)$. Then, Eqs. (3.56) and (3.63) imply that

$$\liminf_{N\to\infty} F_N \geq \liminf_{N\to\infty} \mathbb{E}_x A_N, \tag{3.73}$$

where $\mathbb{E}_x$ is the average with respect to $x = (x_p)$ viewed as a sequence of i.i.d. random variables that have the uniform distribution on $[1,2]$. Indeed, this inequality holds because, by Eq. (3.21), the difference between the free energy with and without the perturbation term can be controlled uniformly over (x_p) and the error in Eq. (3.63) is also uniform in (x_p). Recall that the average $\langle\cdot\rangle'$ in Eq. (3.72) is with respect to the Gibbs measure G'_N in Eq. (3.61), and also recall the choice of s_N made in Eq. (3.55). The Example 3.2 below Theorem 3.2 applies almost verbatim to the Hamiltonian H'_N in Eq. (3.58) and, therefore, the measure G'_N satisfies the Ghirlanda–Guerra identities in Theorem 3.2. This means that if, for any $n \geq 2, p \geq 1$ and any function f of the overlaps $R^n = (R_{l,l'})_{l,l'\leq n}$ of n replicas, we now define

$$\Delta(f,n,p) = \Big|\mathbb{E}\big\langle f R_{1,n+1}^p\big\rangle' - \frac{1}{n}\mathbb{E}\big\langle f\big\rangle'\mathbb{E}\big\langle R_{1,2}^p\big\rangle' - \frac{1}{n}\sum_{l=2}^{n}\mathbb{E}\big\langle f R_{1,l}^p\big\rangle'\Big|$$

then

$$\lim_{N\to\infty} \mathbb{E}_x \Delta(f,n,p) = 0. \tag{3.74}$$

At this moment, the Aizenman–Sims–Starr representation (3.73) and the Ghirlanda–Guerra identities (3.74) are written on average over (x_p). However, to make sense of these properties in the limit, we will need to replace the average by a specific nonrandom choice of the parameters $x^N = (x_p^N)$ that may vary with N. This can be done as follows. First of all, to pass to the limit, we will temporarily encode the Ghirlanda–Guerra identities for arbitrary functions of the overlaps in terms of some countable family of functions. Let us consider a collection

$$\mathscr{F} = \Big\{(f,n,p) : p \geq 1, n \geq 2, f \text{ is a monomial of } (R_{l,l'})_{l,l'\leq n}\Big\}.$$

Since $\mathscr{F}$ is countable, we can enumerate it, $((f_j,n_j,p_j))_{j\geq 1}$, and define a function

$$\Delta_N = \Delta_N(x) = \sum_{j\geq 1} 2^{-j}\Delta(f_j,n_j,p_j), \tag{3.75}$$

which depends on the sequence $x=(x_p)_{p\ge 1}$. Since $|f|\le 1$ for each monomial f, it is easy to see that $|\Delta(f,n,p)|\le 2$ and, therefore, Eq. (3.74) implies that $\lim_{N\to\infty}\mathbb{E}_x\Delta_N = 0$. We will need the following simple lemma.

Lemma 3.3. *We can find $x=(x_p)_{p\ge 1}$ such that*

$$\Delta_N(x)\le 2(c\mathbb{E}_x\Delta_N)^{1/2} \text{ and } A_N(x)\le \mathbb{E}_x A_N + 2(c\mathbb{E}_x\Delta_N)^{1/2}, \tag{3.76}$$

where c is a constant that depends only on the function ξ.

Proof. If we denote by $\mathbb{E}_z$ and $\mathbb{E}_y$ the expectations with respect to $(z_N(\sigma))$ and $(y_N(\sigma))$ then Eq. (3.60) and Jensen's inequality imply

$$0\le \mathbb{E}\log\big\langle \operatorname{ch} z_N(\sigma)\big\rangle' \le \mathbb{E}\log\big\langle \mathbb{E}_z \operatorname{ch} z_N(\sigma)\big\rangle' = \frac{\xi'(1)}{2},$$
$$0\le \mathbb{E}\log\big\langle \exp y_N(\sigma)\big\rangle' \le \mathbb{E}\log\big\langle \mathbb{E}_y \exp y_N(\sigma)\big\rangle' = \frac{\theta(1)}{2}.$$

Therefore, $-c\le A_N(x)\le c$ for $c=\xi'(1)+\theta(1)$. Given $\varepsilon>0$, consider the event

$$\Omega=\Big\{x=(x_p)_{p\ge 1} : A_N(x)\le \mathbb{E}_x A_N+\varepsilon\Big\}.$$

Then, if $\mathbb{P}_x$ denotes the probability with respect to the i.i.d. sequence $(x_p)_{p\ge 1}$ with the uniform distribution on $[1,2]$,

$$\mathbb{E}_x A_N \ge \big(\mathbb{E}_x A_N+\varepsilon\big)\mathbb{P}_x(\Omega^c) - c\,\mathbb{P}_x(\Omega)$$

and, therefore,

$$\mathbb{P}_x(\Omega)\ge \frac{\varepsilon}{\mathbb{E}_x A_N+\varepsilon+c} > \frac{\varepsilon}{3c}$$

for $\varepsilon<c$. On the other hand, Chebyshev's inequality implies

$$\mathbb{P}_x\big(\Delta_N\le\varepsilon\big)\ge 1-\frac{\mathbb{E}_x\Delta_N}{\varepsilon},$$

and $\Omega\cap\{\Delta_N\le\varepsilon\}\ne\emptyset$ if $\varepsilon/3c>\mathbb{E}_x\Delta_N/\varepsilon$. Taking $\varepsilon=2(c\mathbb{E}_x\Delta_N)^{1/2}$ implies that we can find a point x that satisfies both inequalities in Eq. (3.76). □

For each N, let us choose a sequence $x^N=(x_p^N)_{p\ge 1}$ that satisfies Eq. (3.76). Since $\mathbb{E}_x\Delta_N\to 0$, we get

$$\lim_{N\to\infty}\Delta_N(x^N)=0, \tag{3.77}$$

and Eq. (3.73) can be replaced by

$$\liminf_{N\to\infty} F_N \ge \liminf_{N\to\infty} A_N(x^N). \tag{3.78}$$

This shows that the Ghirlanda–Guerra identities and the Aizenman–Sims–Starr representation hold simultaneously for the Gibbs measure G'_N defined with the choice of the parameters x^N in the perturbation term $s_N g_N(\sigma)$. Of course, our main concern now is to understand what happens to the Aizenman–Sims–Starr representation

$$A_N(x^N) = \mathbb{E}\log\Big\langle 2\mathrm{ch}\, z_N(\sigma)\Big\rangle' - \mathbb{E}\log\Big\langle \exp y_N(\sigma)\Big\rangle' \tag{3.79}$$

in the limit and to explain how it is related to the Parisi functional in Eq. (3.16),

$$\mathscr{P}(\zeta) = \mathbb{E}\log\Big\langle 2\mathrm{ch}\, g_{\xi'}(\sigma)\Big\rangle - \mathbb{E}\log\Big\langle \exp g_\theta(\sigma)\Big\rangle, \tag{3.80}$$

expressed in terms of the Ruelle probability cascades. This is where the Ghirlanda–Guerra identities will play their most important role, since they will allow us to appeal to Theorem 2.13 in Chap. 2 to characterize the asymptotic distribution of the overlaps and the asymptotic Gibbs measures. Let $(\sigma^l)_{l\geq 1}$ be an i.i.d. sequence of replicas from the Gibbs measure G'_N and let

$$R^N = \big(R^N_{l,l'}\big)_{l,l'\geq 1} = N^{-1}\big(\sigma^l\cdot\sigma^{l'}\big)_{l,l'\geq 1}$$

be the array of overlaps of this sample. Consider a subsequence (N_k) along which the lower limit on the right-hand side of Eq. (3.78) is achieved and, at the same time, the distribution of R^N under $\mathbb{E}G'^{\otimes\infty}_N$ converges weakly to the distribution of some infinite array $R = (R_{l,l'})$ in the sense of convergence of all their finite dimensional distributions. The array R is a Gram-de Finetti array, namely, positive definite and weakly exchangeable, as in Eq. (1.74), since it obviously inherits these properties from R^N. Therefore, by the Dovbysh–Sudakov representation in Theorem 1.7, we can assume that this limiting array is generated by

$$\big(R_{l,l'}\big)_{l,l'\geq 1} = \Big(\rho^l\cdot\rho^{l'} + \delta_{l,l'}\big(1-\|\rho^l\|^2\big)\Big)_{l,l'\geq 1}, \tag{3.81}$$

where (ρ^l) is an i.i.d. sample from some random measure G on the unit ball of a separable Hilbert space. The diagonal elements are equal to one, since $R^N_{l,l} = 1$ for all N. If we denote by $\langle\cdot\rangle$ the average with respect to G then, by Eqs. (3.77) and (3.75), the measure G satisfies the Ghirlanda–Guerra identities:

$$\mathbb{E}\big\langle f R^p_{1,n+1}\big\rangle = \frac{1}{n}\mathbb{E}\big\langle f\big\rangle\mathbb{E}\big\langle R^p_{1,2}\big\rangle + \frac{1}{n}\sum_{l=2}^{n}\mathbb{E}\big\langle f R^p_{1,l}\big\rangle, \tag{3.82}$$

for all $p\geq 1, n\geq 2$ and all monomials f of $(R_{l,l'})_{l,l'\leq n}$. Approximating bounded functions of the overlaps by polynomials (in the L^1 sense) implies that

$$\mathbb{E}\big\langle f\psi(R_{1,n+1})\big\rangle = \frac{1}{n}\mathbb{E}\big\langle f\big\rangle\mathbb{E}\big\langle\psi(R_{1,2})\big\rangle + \frac{1}{n}\sum_{l=2}^{n}\mathbb{E}\big\langle f\psi(R_{1,l})\big\rangle \tag{3.83}$$

for all bounded measurable functions f and ψ. This means that we can apply the results of Sect. 2.4. Let us clarify a minor point first. Since the diagonal elements $R_{l,l}$ are equal to one, they play no role in Eq. (3.83) and the identities really express a property of the off-diagonal elements, just like in Eq. (2.106) in Sect. 2.4. This also means that the results in Sect. 2.4 apply to the Gram matrix $(\rho^l \cdot \rho^{l'})_{l,l' \geq 1}$, as in Eq. (2.105), which coincides with the right-hand side of Eq. (3.81) on the off-diagonal elements. If q^* is the largest point in the support of the distribution ζ of one overlap $R_{1,2}$ under $\mathbb{E}G^{\otimes 2}$ then, by Theorem 2.15, the measure G is concentrated on the sphere of radius $\sqrt{q^*}$ and all $\|\rho^l\|^2 = q^*$ in Eq. (3.81). Therefore, the main difference with Sect. 2.4 is that now, by construction, the diagonal elements are equal to 1 instead of q^*. This is a minor point and we only raise it to clarify the difference between the overlap matrix (3.81) and the overlap matrix in Sect. 2.4.

The most important consequence of the Ghirlanda–Guerra identities (3.83) is due to Theorem 2.13 and the discussion in the Remark 2.1 at the end of Sect. 2.4. Namely, we can approximate the asymptotic overlap matrix (3.81) in distribution by the Gram matrices of i.i.d. samples from the Ruelle probability cascades, as follows. Given the distribution ζ of $R_{1,2}$ under $\mathbb{E}G^{\otimes 2}$, choose any sequence of distributions ζ^n such that $\zeta^n \to \zeta$ weakly and each ζ^n is concentrated on finitely many points in $[0,1]$. Let us make sure that $\zeta^n(\{0\}) > 0$ and $\zeta^n(\{1\}) > 0$ for all n, by assigning some small weights to the points 0 and 1, if necessary. Let G^n be the Ruelle probability cascade, the random measure in Eq. (2.62), corresponding to the f.o.p. ζ^n, which means that the overlap distribution under $\mathbb{E}G^{n\otimes 2}$ in Eq. (2.82) is equal to ζ^n. Then, Theorem 2.17 implies that the Gram matrices Q^n of i.i.d. samples from G^n converge in distribution to the overlap matrix R in Eq. (3.81). Since we made sure that $\zeta^n(\{1\}) > 0$, the measure G^n concentrates on the sphere of radius one and the diagonal elements are also equal to one, $Q^n_{l,l} = 1$, so the convergence applies to the entire array and not only the off-diagonal elements. Let us denote by $\mathscr{P}(\zeta^n)$ the Parisi functional corresponding to ζ^n, which can be expressed in terms of the Ruelle probability cascade G^n as in Eq. (3.80) or Eq. (3.16). Then, to finish the proof of the Parisi formula, it is enough to show that

$$\lim_{k\to\infty} A_{N_k} = \lim_{n\to\infty} \mathscr{P}(\zeta^n), \tag{3.84}$$

since the right-hand side is bounded from below by $\inf_\zeta \mathscr{P}(\zeta)$. Recall that (N_k) is a subsequence along which the lower limit in Eq. (3.78) was achieved. To prove Eq. (3.84), it remains to apply Theorem 1.3 with the choice of $C_z = \xi'$ and $C_y = \theta$ in Eq. (1.62). First of all, if we compare the covariance formulae in Eq. (1.62) with Eq. (3.14) for $\psi = \xi'$ and $\psi = \theta$ and compare Eq. (1.63) with Eq. (3.80), it is obvious that $\mathscr{P}(\zeta^n) = \mathbb{E}\Phi(G^n)$. The bound (1.64) holds uniformly over all measures Γ on the unit sphere of a Hilbert space and, therefore, if we first apply Eq. (1.64) conditionally on the randomness of the measure G^n and then average with respect to this randomness, we get

$$\left| \mathscr{P}(\zeta^n) - \mathbb{E}\left\langle F_\varepsilon(Q) \right\rangle_{G^n} \right| \leq \varepsilon. \tag{3.85}$$

Even though we explained this below Theorem 1.3, let us explain one more time that, in order to represent A_N in Eq. (3.79) in the form (1.63), we should simply redefine the measure G'_N as a measure on the subset $N^{-1/2}\Sigma_N$ of a unit sphere by the map $\pi(\sigma) = N^{-1/2}\sigma$ and let $G_N^\pi = G'_N \circ \pi^{-1}$. Since the overlap

$$R_{1,2} = N^{-1}\sigma^1 \cdot \sigma^2 = \pi(\sigma^1) \cdot \pi(\sigma^2)$$

is just a scalar product of the rescaled configurations, the covariance formulae in Eq. (1.62) with the choice of $C_z = \xi'$ and $C_y = \theta$ agree with Eq. (3.60) under this definition and, therefore, $A_N = \mathbb{E}\Phi(G_N^\pi)$. Again, by Eq. (1.64),

$$\left| A_N - \mathbb{E}\left\langle F_\varepsilon(Q) \right\rangle_{G_N^\pi} \right| \le \varepsilon. \tag{3.86}$$

The Gram matrix Q generated by the i.i.d. samples from the measures G_N^π and G^n was denoted above by R^N and Q^n correspondingly and, since, by construction, R^{N_k} and Q^n converge in distribution to the same overlap matrix R in Eq. (3.81),

$$\lim_{k\to\infty} \mathbb{E}\left\langle F_\varepsilon(Q) \right\rangle_{G_{N_k}^\pi} = \lim_{n\to\infty} \mathbb{E}\left\langle F_\varepsilon(Q) \right\rangle_{G^n}.$$

Equations (3.85) and (3.86) imply Eq. (3.84) and this finishes the proof of the lower bound in the Parisi formula. The upper bound was proved by the Guerra RSB interpolation and, therefore, the proof of the Parisi formula is complete. □

3.7 Generic Mixed p-Spin Models

In this section, we will show that one can say more in the case of the so-called *generic mixed p-spin models*, when the Hamiltonian Eq. (3.1) contains sufficiently many pure p-spin terms (3.2), so that the following condition is satisfied:

(G) Linear span of constants and power functions x^p corresponding to $\beta_p \neq 0$ in Eq. (3.1) is dense in $(C[-1,1], \|\cdot\|_\infty)$.

We will see that each pure p-spin term contains some information about the pth moment of the overlap and, as a result, the condition (G) allows us to characterize the Gibbs measure in the thermodynamic limit without the help of the perturbation term in Eq. (3.18), so the whole picture becomes more pure, in some sense. More specifically, if G_N is the Gibbs measure corresponding to the Hamiltonian Eq. (3.1) of a generic mixed p-spin model, then we will show that:

(i) The distribution of the overlap $R_{1,2}$ under $\mathbb{E}G_N^{\otimes 2}$ converges to some limit ζ^*.
(ii) The distribution of the entire overlap array $(R_{l,l'})_{l,l'\ge 1}$ under $\mathbb{E}G_N^{\otimes\infty}$ converges.
(iii) The limit of the free energy is given by $\lim_{N\to\infty} F_N = \mathscr{P}(\zeta^*)$.

It will also be shown that the support of ζ^* belongs to the interval $[0,1]$ and, when ζ^* is not of the form (3.6)–(3.8), we define

$$\mathscr{P}(\zeta^*) = \lim_{n\to\infty} \mathscr{P}(\zeta^n) \tag{3.87}$$

for any sequence of distributions (ζ^n) of that form such that $\zeta^n \to \zeta^*$ weakly.

Lemma 3.4. *The limit Eq. (3.87) exists and does not depend on the sequence* (ζ^n).

Proof. Let G^n be the Ruelle probability cascade corresponding to the f.o.p. ζ^n. Then, as in Eq. (3.85), by Theorem 1.3,

$$\Big| \mathscr{P}(\zeta^n) - \mathbb{E}\Big\langle F_\varepsilon(Q) \Big\rangle_{G^n} \Big| \le \varepsilon.$$

If we can show that the convergence $\zeta^n \to \zeta^*$ implies that the distribution of the overlap array Q under $\mathbb{E}(G^n)^{\otimes\infty}$ converges to the distribution of some array Q^* then $\mathbb{E}\langle F_\varepsilon(Q)\rangle_{G_n}$ converges to $\mathbb{E}F_\varepsilon(Q^*)$ and the assertion of the lemma follows. The diagonal elements $Q_{l,l} = 1$, because $\zeta^n(\{1\}) > 0$ and G^n is supported on the sphere of radius one, and the off-diagonal elements $(Q_{l,l'})_{l\neq l'}$ converge in distribution by Theorem 2.17. □

The properties (i)–(iii) above will be corollaries of the Parisi formula for the free energy. First, we will show that the Parisi formula is differentiable with respect to each inverse temperature parameter β_p in the definition of the Hamiltonian Eq. (3.1) and express the derivatives in terms of the asymptotic distributions of the overlap $R_{1,2}$. Under the condition (G), for generic mixed p-spin models, this will imply property (i). Then, as a consequence of differentiability, we will prove a strong version of the Ghirlanda–Guerra identities for the original Gibbs measure, which in contrast with Theorem 3.2 does not involve any perturbation. This will allow us to deduce the property (ii). Moreover, the validity of the Ghirlanda–Guerra identities implies that, in the thermodynamic limit, the overlap array $(R_{l,l'})_{l,l'\ge 1}$ can be approximated in distribution by the overlap arrays Q^n generated by the Ruelle probability cascades G^n corresponding to the sequence ζ^n in Eq. (3.87). This is, of course, a consequence of the results of Sect. 2.4. Finally, the property (iii) will be proved using the Aizenman–Sims–Starr representation.

Let us make the dependence of the Parisi functional in Eq. (3.11) on the sequence of parameters (β_p) in the definition of the mixed p-spin Hamiltonian in Eq. (3.1) explicit and write $\mathscr{P}(\zeta) = \mathscr{P}(\zeta,(\beta_p))$. Similarly, let us denote the Parisi formula by

$$\mathscr{P} = \mathscr{P}\big((\beta_p)\big) = \inf_{\zeta} \mathscr{P}\big(\zeta,(\beta_p)\big). \tag{3.88}$$

Let $\mathscr{M}$ be the set of all limits over subsequences of the distribution of the overlap $R_{1,2}$ under $\mathbb{E}G_N^{\otimes 2}$, where G_N is the Gibbs measure corresponding to the Hamiltonian Eq. (3.1). Then the following holds (we do not assume the condition (G) here).

Theorem 3.7. *For each $p \geq 1$, the Parisi formula $\mathscr{P}\big((\beta_p)\big)$ is differentiable with respect to β_p and*

$$\frac{\partial \mathscr{P}}{\partial \beta_p} = \beta_p\Big(1 - \int q^p \, dm(q)\Big) \tag{3.89}$$

for all limits $m \in \mathscr{M}$ of the overlap distribution under $\mathbb{E}G_N^{\otimes 2}$.

Corollary 3.1. *Under the condition (G), property (i) holds.*

Proof. If $\beta_p \neq 0$, then Eq. (3.89) implies that all the limits $m \in \mathscr{M}$ have the same pth moment. Therefore, the condition (G), obviously, implies property (i). □

The proof of Theorem 3.7 will be based on two basic observations. First of all, by Hölder's inequality, the free energy

$$F_N = \frac{1}{N}\mathbb{E}\log \sum_{\sigma\in\Sigma_N} \exp \sum_{p\geq 1} \beta_p H_{N,p}(\sigma) \tag{3.90}$$

is convex in each parameter β_p and, in fact, in the entire sequence (β_p). This means that the Parisi formula $\mathscr{P}\big((\beta_p)\big)$ in Eq. (3.12) is also convex in (β_p), which is not at all obvious from its definition. Convexity will play the most important role, but we will also need the following technical fact about the Parisi functional $\mathscr{P}(\zeta)$.

Lemma 3.5. *For each $p \geq 1$, there exist constants c_p, d_p that depend only on p such that*

$$\Big|\frac{\partial^2 \mathscr{P}(\zeta)}{\partial \beta_p^2}\Big| \leq d_p + c_p \beta_p^2 \tag{3.91}$$

for all f.o.ps ζ in Eq. (3.8).

Proof. The second derivative in β_p of the second term in Eq. (3.11) can be computed explicitly,

$$(p-1)\sum_{0\leq l\leq r-1} \zeta_l (q_{l+1}^p - q_l^p),$$

and it is bounded by $p-1$. To compute the second derivative of the first term in Eq. (3.11), we will use its representation in terms of the Ruelle probability cascades in Eq. (3.15),

$$\mathbb{E}X_0 = \mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} \sum_{\varepsilon=\pm 1} v_\alpha \exp \varepsilon g_{\xi'}(h_\alpha). \tag{3.92}$$

Let us recall the definition of the Gaussian process $g_{\xi'}(h_\alpha)$ in Eq. (3.13). To make the dependence of $g_{\xi'}(h_\alpha)$ on β_p explicit, let us redefine it by

$$g_{\xi'}(h_\alpha) = \sum_{p\geq 1} \sqrt{p}\beta_p g_{p-1}(h_\alpha), \tag{3.93}$$

where each $(g_p(h_\alpha))$ is a Gaussian process defined as in Eq. (3.13) for $\psi(x) = x^p$, with the covariance $\mathbb{E}g_p(h_\alpha)g_p(h_\beta) = (h_\alpha \cdot h_\beta)^p$, and these processes are independent for all $p \geq 0$. To write the derivatives of Eq. (3.92) in a compact form, let us consider a Gibbs measure on $H \times \{-1,+1\}$,

$$G(h_\alpha, \varepsilon) = \frac{v_\alpha \exp \varepsilon g_{\xi'}(h_\alpha)}{Z},$$

where, as usual, Z is the partition function,

$$Z = \sum_{\alpha \in \mathbb{N}^r} \sum_{\varepsilon = \pm 1} v_\alpha \exp \varepsilon g_{\xi'}(h_\alpha),$$

and let us denote by $\langle \cdot \rangle$ the average with respect to $G^{\otimes \infty}$. We will denote a generic element of $H \times \{-1,+1\}$ by (σ, ε) and denote the overlap $\sigma^l \cdot \sigma^{l'} = R_{l,l'}$. Then, using Eq. (3.93), the derivative of Eq. (3.92) with respect to β_p can be written as

$$\frac{\partial}{\partial \beta_p} \mathbb{E}X_0 = \sqrt{p}\, \mathbb{E}\big\langle \varepsilon g_{p-1}(\sigma) \big\rangle. \tag{3.94}$$

Since the covariance $\mathbb{E}g_{p-1}(\sigma^1)g_{\xi'}(\sigma^2) = \sqrt{p}\beta_p R_{1,2}^{p-1}$, the Gaussian integration by parts formula in Lemma 1.1 (see also Lemma 1.2) implies that this derivative is equal to

$$\frac{\partial}{\partial \beta_p} \mathbb{E}X_0 = p\beta_p \big(1 - \mathbb{E}\big\langle \varepsilon^1 \varepsilon^2 R_{1,2}^{p-1} \big\rangle\big).$$

Similarly to Eq. (3.94), the derivative of $\mathbb{E}\big\langle \varepsilon^1 \varepsilon^2 R_{1,2}^{p-1} \big\rangle$ in β_p can be written as

$$\begin{aligned} &\sqrt{p}\, \mathbb{E}\Big\langle \varepsilon^1 \varepsilon^2 R_{1,2}^{p-1} \Big(\varepsilon^1 g_{p-1}(\sigma^1) + \varepsilon^2 g_{p-1}(\sigma^2) - 2\varepsilon^3 g_{p-1}(\sigma^3) \Big) \Big\rangle \\ &\quad = 2p\beta_p \mathbb{E}\Big\langle \varepsilon^1 \varepsilon^2 R_{1,2}^{p-1} \Big(\varepsilon^1 \varepsilon^2 R_{1,2}^{p-1} - 4\varepsilon^1 \varepsilon^3 R_{1,3}^{p-1} + 3\varepsilon^3 \varepsilon^4 R_{3,4}^{p-1} \Big) \Big\rangle, \end{aligned}$$

where the equality follows by the Gaussian integration by parts formula in Lemma 1.2 and where we combined some terms by the symmetry between replicas. This is bounded in absolute value by $16p\beta_p$, which finishes the proof. □

Proof (Theorem 3.7). For simplicity of notation, we will fix $p \geq 1$, fix all parameters $(\beta_l)_{l \neq p}$ and think of $\mathscr{P}(\zeta, (\beta_p))$ and $\mathscr{P}((\beta_p))$ as functions $\mathscr{P}(\zeta, \beta)$ and $\mathscr{P}(\beta)$ of one variable $\beta = \beta_p$. First, we will prove that the function $\mathscr{P}(\beta)$ is differentiable. By Eq. (3.88), for a fixed β_p, we can find f.o.ps ζ^n such that

$$\mathscr{P}(\zeta^n, \beta_p) \leq \mathscr{P}(\beta_p) + \frac{1}{n}. \tag{3.95}$$

To prove that a convex function $\mathscr{P}(\beta)$ is differentiable at $\beta = \beta_p$, we need to show that its subdifferential $\partial \mathscr{P}(\beta_p)$ consists of a single point. Let $a \in \partial \mathscr{P}(\beta_p)$. Using

Eq. (3.95), the convexity of $\mathscr{P}(\beta)$ and the fact that $\mathscr{P}(\beta) \le \mathscr{P}(\zeta^n, \beta)$ for all β, we get that for all $y > 0$,

$$a \le \frac{\mathscr{P}(\beta_p + y) - \mathscr{P}(\beta_p)}{y} \le \frac{\mathscr{P}(\zeta^n, \beta_p + y) - \mathscr{P}(\zeta^n, \beta_p) + n^{-1}}{y}$$

and

$$a \ge \frac{\mathscr{P}(\beta_p) - \mathscr{P}(\beta_p - y)}{y} \ge \frac{\mathscr{P}(\zeta^n, \beta_p) - \mathscr{P}(\zeta^n, \beta_p - y) - n^{-1}}{y}.$$

By Eq. (3.91), the second derivative of $\mathscr{P}(\zeta^n, \beta)$ can be controlled in a small neighborhood of β_p uniformly over all $n \ge 1$ and, using Taylor's expansion, the above inequalities imply

$$\frac{\partial \mathscr{P}}{\partial \beta}(\zeta^n, \beta_p) - L_p y - \frac{n^{-1}}{y} \le a \le \frac{\partial \mathscr{P}}{\partial \beta}(\zeta^n, \beta_p) + L_p y + \frac{n^{-1}}{y}.$$

Setting $y = n^{-1/2}$ and letting $n \to \infty$ prove that $\partial \mathscr{P}(\beta_p)$ consists of a unique point, which proves that $\mathscr{P}(\beta)$ is differentiable. It remains to show Eq. (3.89). This will follow from the Parisi formula, $\lim_{N\to\infty} F_N(\beta) = \mathscr{P}(\beta)$, where we made the dependence of the free energy on β explicit. The convexity of $F_N(\beta)$ implies that

$$\frac{F_N(\beta_p) - F_N(\beta_p - y)}{y} \le F_N'(\beta_p) \le \frac{F_N(\beta_p + y) - F_N(\beta_p)}{y}$$

for all $y > 0$. Taking the limit $N \to \infty$ first, then letting $y \downarrow 0$, and using that the function $\mathscr{P}(\beta)$ is differentiable prove that

$$\lim_{N\to\infty} F_N'(\beta) = \mathscr{P}'(\beta). \tag{3.96}$$

Finally, since the derivative of the free energy in Eq. (3.90) in $\beta = \beta_p$ is equal to

$$F_N'(\beta) = \frac{1}{N}\mathbb{E}\Big\langle H_{N,p}(\sigma)\Big\rangle, \tag{3.97}$$

the Gaussian integration by parts formula in Lemma 1.1 implies that

$$F_N'(\beta) = \beta\Big(1 - \mathbb{E}\langle R_{1,2}^p\rangle\Big). \tag{3.98}$$

If we recall that $\mathscr{M}$ is the set of all limits of the distribution of the overlap $R_{1,2}$ under $\mathbb{E}G_N^{\otimes 2}$ over subsequences, Eqs. (3.96) and (3.98) imply that

$$\mathscr{P}'(\beta_p) = \lim_{N\to\infty} F_N'(\beta_p) = \beta_p\Big(1 - \int q^p \, dm(q)\Big), \tag{3.99}$$

for any $m \in \mathscr{M}$. This finishes the proof. □

Next, we will show how the differentiability of the Parisi formula implies a strong version of the Ghirlanda–Guerra identities. As in Sect. 3.2, the Ghirlanda–Guerra identities will be a consequence of a general principle of the concentration of the Hamiltonian under the Gibbs measure.

Theorem 3.8. *For any $p \geq 1$,*

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E}\Big\langle \Big| H_{N,p}(\sigma) - \mathbb{E}\big\langle H_{N,p}(\sigma)\big\rangle \Big| \Big\rangle = 0. \tag{3.100}$$

Similarly to the argument in Theorem 3.2, Eq. (3.100) implies the Ghirlanda–Guerra identities via the Gaussian integration by parts. Consider a bounded function $f = f((R_{l,l'})_{l,l'\leq n})$ and, without loss of generality, suppose that $|f| \leq 1$. Then, as in Eq. (3.36),

$$\frac{1}{N}\Big| \mathbb{E}\big\langle f H_{N,p}(\sigma)\big\rangle - \mathbb{E}\big\langle f\big\rangle \mathbb{E}\big\langle H_{N,p}(\sigma)\big\rangle \Big| \leq \frac{1}{N} \mathbb{E}\Big\langle \Big| H_{N,p}(\sigma) - \mathbb{E}\big\langle H_{N,p}(\sigma)\big\rangle \Big| \Big\rangle.$$

Using the Gaussian integration by parts as in the proof of Theorem 3.2, one can check that the left-hand side equals

$$\beta_p \Big| n\mathbb{E}\big\langle f R_{1,n+1}^p\big\rangle - \mathbb{E}\big\langle f\big\rangle \mathbb{E}\big\langle R_{1,2}^p\big\rangle - \sum_{l=2}^{n} \mathbb{E}\big\langle f R_{1,l}^p\big\rangle \Big|$$

and, since the right-hand side goes to zero by Eq. (3.100), we obtain the following strong version of the Ghirlanda–Guerra identities.

Corollary 3.2. *For any $p \geq 1$ such that $\beta_p \neq 0$,*

$$\lim_{N\to\infty} \Big| \mathbb{E}\big\langle f R_{1,n+1}^p\big\rangle - \frac{1}{n}\mathbb{E}\big\langle f\big\rangle \mathbb{E}\big\langle R_{1,2}^p\big\rangle - \frac{1}{n}\sum_{l=2}^{n} \mathbb{E}\big\langle f R_{1,l}^p\big\rangle \Big| = 0. \tag{3.101}$$

By comparison with Theorem 3.2 this result does not use any perturbation, but we get the identity for the pth moment of the overlap only if $\beta_p \neq 0$ and the pure p-spin term is present in the Hamiltonian, which allows us to deduce the Ghirlanda–Guerra identities in the limit for general functions only if our model includes sufficiently many pure p-spin terms. As a consequence, we get the following.

Corollary 3.3. *Under the condition (G), property (ii) holds.*

Proof. By Corollary 3.1, the distribution of the overlap $R_{1,2}$ converges to ζ^*. The Corollary 3.2 and condition (G) imply that all asymptotic distributions of $(R_{l,l'})_{l,l'\geq 1}$ over subsequences satisfy the Ghirlanda–Guerra identities. Since, by Theorem 2.13, the distribution of one overlap uniquely determines the distribution of the entire overlap array, all the limits over subsequences coincide, which finishes the proof. Notice that, since the Ghirlanda–Guerra identities hold, ζ^* is supported on $[0,1]$ by Talagrand's positivity principle. □

Proof (Theorem 3.8). As in the proof of Theorem 3.7, for simplicity of notation, let us fix $p \geq 1$, fix all parameters $(\beta_l)_{l \neq p}$ and think of all quantities that depend on (β_p) as functions of one variable $\beta = \beta_p$. Let us write $H(\sigma) = H_{N,p}(\sigma)$. First, we will show that

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E} \big| H(\sigma) - \langle H(\sigma) \rangle \big| = 0. \tag{3.102}$$

Let $\langle \cdot \rangle_\beta$ denote the Gibbs average corresponding to the Hamiltonian Eq. (3.1) where β_p has been replaced by β. Consider $\beta_p' > \beta_p$ and let $\delta = \beta_p' - \beta_p$. We start with the following obvious equality,

$$\begin{aligned} \delta \mathbb{E} \big| H(\sigma^1) - H(\sigma^2) \big|_{\beta_p} &= \int_{\beta_p}^{\beta_p'} \mathbb{E} \big| H(\sigma^1) - H(\sigma^2) \big|_\beta \, d\beta \\ &\quad - \int_{\beta_p}^{\beta_p'} \int_{\beta_p}^{x} \frac{\partial}{\partial \beta} \mathbb{E} \big| H(\sigma^1) - H(\sigma^2) \big|_\beta \, d\beta \, dx. \end{aligned} \tag{3.103}$$

The derivative of $\mathbb{E} \big| H(\sigma^1) - H(\sigma^2) \big|_\beta$ in the last integral is equal to

$$\mathbb{E} \big| H(\sigma^1) - H(\sigma^2) \big| \big(H(\sigma^1) + H(\sigma^2) - 2H(\sigma^3) \big) \big|_\beta$$

and, using Hölder's inequality, it can be bounded in absolute value by

$$2 \mathbb{E} \big| \big(H(\sigma^1) - H(\sigma^2) \big)^2 \big|_\beta \leq 8 \mathbb{E} \big| \big(H(\sigma) - \langle H(\sigma) \rangle_\beta \big)^2 \big|_\beta.$$

Using this estimate of the derivative in Eq. (3.103), we obtain the following sequence of obvious inequalities,

$$\begin{aligned} \mathbb{E} \big| H(\sigma^1) - H(\sigma^2) \big|_{\beta_p} &\leq \frac{1}{\delta} \int_{\beta_p}^{\beta_p'} \mathbb{E} \big| H(\sigma^1) - H(\sigma^2) \big|_\beta \, d\beta \\ &\quad + \frac{8}{\delta} \int_{\beta_p}^{\beta_p'} \int_{\beta_p}^{x} \mathbb{E} \big| \big(H(\sigma) - \langle H(\sigma) \rangle_\beta \big)^2 \big|_\beta \, d\beta \, dx \\ &\leq \frac{2}{\delta} \int_{\beta_p}^{\beta_p'} \mathbb{E} \big| H(\sigma) - \langle H(\sigma) \rangle_\beta \big|_\beta \, d\beta \\ &\quad + 8 \int_{\beta_p}^{\beta_p'} \mathbb{E} \big| \big(H(\sigma) - \langle H(\sigma) \rangle_\beta \big)^2 \big|_\beta \, d\beta \\ &\leq 2 \Big(\frac{1}{\delta} \int_{\beta_p}^{\beta_p'} \mathbb{E} \big| \big(H(\sigma) - \langle H(\sigma) \rangle_\beta \big)^2 \big|_\beta \, d\beta \Big)^{1/2} \\ &\quad + 8 \int_{\beta_p}^{\beta_p'} \mathbb{E} \big| \big(H(\sigma) - \langle H(\sigma) \rangle_\beta \big)^2 \big|_\beta \, d\beta. \end{aligned}$$

Therefore, if we introduce the notation

$$\varepsilon_N = \frac{1}{N} \int_{\beta_p}^{\beta_p'} \mathbb{E} \big| \big(H(\sigma) - \langle H(\sigma) \rangle_\beta \big)^2 \big|_\beta \, d\beta,$$

we have shown that

$$\frac{1}{N}\mathbb{E}\big\langle\big|H(\sigma^1)-H(\sigma^2)\big|\big\rangle \le 2\sqrt{\frac{\varepsilon_N}{N\delta}}+8\varepsilon_N. \tag{3.104}$$

Starting with the formula for $F_N'(\beta)$ in Eq. (3.97), it is easy to see that

$$F_N''(\beta)=\frac{1}{N}\mathbb{E}\big\langle\big(H(\sigma)-\langle H(\sigma)\rangle_\beta\big)^2\big\rangle_\beta$$

and, therefore, we can write

$$\varepsilon_N=\int_{\beta_p}^{\beta_p'}F_N''(\beta)\,d\beta=F_N'(\beta_p')-F_N'(\beta_p).$$

Since $F_N(\beta)$ is convex, this implies that for any $y>0$,

$$\varepsilon_N\le\frac{F_N(\beta_p'+y)-F_N(\beta_p')}{y}-\frac{F_N(\beta_p)-F_N(\beta_p-y)}{y},$$

and Eq. (3.104) together with the Parisi formula implies

$$\limsup_{N\to\infty}\frac{1}{8N}\mathbb{E}\big\langle\big|H(\sigma^1)-H(\sigma^2)\big|\big\rangle\le\frac{\mathscr{P}(\beta_p'+y)-\mathscr{P}(\beta_p')}{y}-\frac{\mathscr{P}(\beta_p)-\mathscr{P}(\beta_p-y)}{y}.$$

Since the left-hand side does not depend on β_p' and y, letting $\beta_p'\to\beta_p$ first, then letting $y\downarrow 0$, and using that, by Theorem 3.7, $\mathscr{P}(\beta)$ is differentiable prove Eq. (3.102). To prove Eq. (3.100), it remains to show that

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\big|\big\langle H(\sigma)\big\rangle-\mathbb{E}\big\langle H(\sigma)\big\rangle\big|=0. \tag{3.105}$$

Let us denote $\phi_N(\beta)=N^{-1}\log Z_N$, so that the free energy $F_N(\beta)=\mathbb{E}\phi_N(\beta)$. Since

$$\frac{1}{N}\mathbb{E}\big|\big\langle H(\sigma)\big\rangle-\mathbb{E}\big\langle H(\sigma)\big\rangle\big|=\mathbb{E}\big|\phi_N'(\beta)-F_N'(\beta)\big|, \tag{3.106}$$

our goal is to demonstrate that these derivative are close in the limit. Again, the convexity plays a major role. Given $y>0$, let us denote

$$\begin{aligned}\delta_N(y)=\;&|\phi_N(\beta-y)-F_N(\beta-y)|+|\phi_N(\beta)-F_N(\beta)|\\&+|\phi_N(\beta+y)-F_N(\beta+y)|.\end{aligned}$$

The convexity of $\phi_N(\beta)$ implies that we can bound $\phi_N'(\beta)-F_N'(\beta)$ from above by

$$\frac{\phi_N(\beta+y)-\phi_N(\beta)}{y}-F_N'(\beta)\le\Big|\frac{F_N(\beta+y)-F_N(\beta)}{y}-F_N'(\beta)\Big|+\frac{\delta_N(y)}{y}$$

and, similarly, we can bound $\phi_N'(\beta) - F_N'(\beta)$ from below by

$$\frac{\phi_N(\beta) - \phi_N(\beta - y)}{y} - F_N'(\beta) \geq -\left|\frac{F_N(\beta) - F_N(\beta - y)}{y} - F_N'(\beta)\right| - \frac{\delta_N(y)}{y}.$$

Combining these two estimates and taking expectations give

$$\begin{aligned}\mathbb{E}|\phi_N'(\beta) - F_N'(\beta)| \leq &\left|\frac{F_N(\beta + y) - F_N(\beta)}{y} - F_N'(\beta)\right| \\ &+ \left|\frac{F_N(\beta) - F_N(\beta - y)}{y} - F_N'(\beta)\right| + \frac{\mathbb{E}\delta_N(y)}{y}. \end{aligned} \tag{3.107}$$

Using the Gaussian concentration inequality in Theorem 1.2, it is easy to check that $\mathbb{E}\delta_N(y) \leq LN^{-1/2}$ for some constant $L > 0$ that depends only on the sequence of parameters (β_p), as long as y stays bounded. Letting $N \to \infty$ in Eq. (3.107) and using the Parisi formula and Eq. (3.96) imply that

$$\begin{aligned}\limsup_{N\to\infty} \mathbb{E}\left|\phi_N'(\beta) - F_N'(\beta)\right| \leq &\left|\frac{\mathscr{P}(\beta + y) - \mathscr{P}(\beta)}{y} - \mathscr{P}'(\beta)\right| \\ &+ \left|\frac{\mathscr{P}(\beta) - \mathscr{P}(\beta - y)}{y} - \mathscr{P}'(\beta)\right|.\end{aligned}$$

Letting $y \downarrow 0$ and using the differentiability of the Parisi formula prove Eq. (3.105). Together with Eq. (3.102), this finishes the proof. □

It remains to explain why the property (iii) holds for generic mixed p-spin models. To show this, we will simply combine the Aizenman–Sims–Starr scheme with the techniques developed in this section in the proof of the properties (i) and (ii). As in Sect. 3.5, we begin with Eq. (1.49) and only now we will use that

$$\lim_{N\to\infty} F_N = \lim_{N\to\infty} A_N \tag{3.108}$$

if the limit on the right-hand side exists, where

$$A_N = \mathbb{E}\log Z_{N+1} - \mathbb{E}\log Z_N.$$

Of course, in the absence of the perturbation, the proof of the Aizenman–Sims–Starr representation in Theorem 3.6 only simplifies and Eq. (3.63) becomes

$$A_N = \mathbb{E}\log\Big\langle 2\operatorname{ch} z_N(\sigma)\Big\rangle' - \mathbb{E}\log\Big\langle \exp y_N(\sigma)\Big\rangle' + o(1), \tag{3.109}$$

where $\langle\cdot\rangle'$ now denotes the average with respect to the Gibbs measure

$$G_N'(\sigma) = \frac{\exp H_N'(\sigma)}{Z_N'}, \text{ where } Z_N' = \sum_{\sigma\in\Sigma_N} \exp H_N'(\sigma), \tag{3.110}$$

corresponding to the Hamiltonian $H'_N(\sigma)$ in Eq. (3.58). The proof of the property (iii) will then easily follow from the following.

Lemma 3.6. *Suppose the condition (G) holds. Then, under $\mathbb{E}G'^{\otimes\infty}_N$, the distribution of the overlap $R_{1,2}$ converges to ζ^*, the distribution of the overlap array $(R_{l,l'})_{l,l'\geq 1}$ converges, and the Ghirlanda–Guerra identities hold in the limit.*

In other words, all the results that we proved in this section under the measure $\mathbb{E}G_N^{\otimes\infty}$ also hold under the measure $\mathbb{E}G'^{\otimes\infty}_N$ with the same limit ζ^*.

Corollary 3.4. *Under the condition (G), property (iii) holds.*

Proof. If we choose a sequence (ζ^n) as in Eq. (3.87) then, using Lemma 3.6, one can repeat the proof of Eq. (3.84) in the previous section to show that

$$\lim_{N\to\infty} A_N = \lim_{n\to\infty} \mathscr{P}(\zeta^n) = \mathscr{P}(\zeta^*). \tag{3.111}$$

Recalling Eq. (3.108) finishes the proof. □

Proof (Lemma 3.6). Let us denote the free energy corresponding to the Hamiltonian $H'_N(\sigma)$ by $\hat{F}_N = N^{-1}\mathbb{E}\log Z'_N$. We have seen in the proof of the Aizenman–Sims–Starr representation [recall Eqs. (3.60) and (3.69)[that the Hamiltonian $H_N(\sigma)$ is very close in distribution to the Hamiltonian $H'_N(\sigma) + y_N(\sigma)$ and, since $y_N(\sigma)$ is of constant order, the computation similar to Eq. (3.21) shows that $|F_N - \hat{F}_N| = O(N^{-1})$. This means that $\lim_{N\to\infty}\hat{F}_N = \lim_{N\to\infty} F_N = \mathscr{P}$. Then, exactly the same argument that was used to prove Eq. (3.96) also gives

$$\lim_{N\to\infty} \frac{\partial \hat{F}_N}{\partial \beta_p} = \frac{\partial \mathscr{P}}{\partial \beta_p}. \tag{3.112}$$

Let us observe that the covariance of one term $H'_{N,p}(\sigma)$ in the definition of the Hamiltonian $H'_N(\sigma)$ in Eq. (3.58) satisfies

$$\mathbb{E}H'_{N,p}(\sigma^1)H'_{N,p}(\sigma^2) = \Big(\frac{N}{N+1}\Big)^{p-1} N R_{1,2}^p. \tag{3.113}$$

Therefore, as in Eqs. (3.97) and (3.98), we can write

$$\frac{\partial \hat{F}_N}{\partial \beta_p} = \frac{1}{N}\mathbb{E}\Big\langle H'_{N,p}(\sigma)\Big\rangle' = \Big(\frac{N}{N+1}\Big)^{p-1}\beta_p\Big(1 - \mathbb{E}\langle R_{1,2}^p\rangle'\Big). \tag{3.114}$$

Since, by Eqs. (3.96) and (3.112), the derivatives $\partial F_N/\partial\beta_p$ and $\partial\hat{F}_N/\partial\beta_p$ converge to the same limit, comparing Eqs. (3.97) and (3.114) implies that $\mathbb{E}\langle R_{1,2}^p\rangle'$ and $\mathbb{E}\langle R_{1,2}^p\rangle$ converge to the same limit if $\beta_p \neq 0$. If the condition (G) holds, this proves the first assertion that the distribution of the overlap $R_{1,2}$ under $\mathbb{E}G'^{\otimes 2}_N$ also converges to ζ^*.

Next, since $\lim_{N\to\infty} \hat{F}_N = \mathscr{P}$, the proof of Theorem 3.8 can be repeated verbatim to show that for any $p \geq 1$,

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E}\Big\langle \Big| H'_{N,p}(\sigma) - \mathbb{E}\big\langle H'_{N,p}(\sigma)\big\rangle' \Big| \Big\rangle' = 0. \tag{3.115}$$

Then, using Eq. (3.113) and property (G), the argument in Corollaries 3.2 and 3.3 yields that the distribution of the overlap array $(R_{l,l'})_{l,l'\geq 1}$ under $\mathbb{E}G_N'^{\otimes\infty}$ converges and satisfies the Ghirlanda–Guerra identities in the limit. □

3.8 Universality in the Disorder

In this section we will consider the following question. Suppose that instead of the original Hamiltonian Eq. (1.2) in the Sherrington–Kirkpatrick model we consider a Hamiltonian

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} x_{ij}\sigma_i\sigma_j, \tag{3.116}$$

where the random variables x_{ij} for $1 \leq i, j \leq N$ are i.i.d., but not necessarily standard Gaussian. Does the free energy

$$F_N = \frac{1}{N} \mathbb{E} \log \sum_{\sigma\in\Sigma_N} \exp\beta H_N(\sigma) \tag{3.117}$$

depend on the distribution of (x_{ij}) in the thermodynamic limit? We will prove that the answer is no, under the assumption that

$$\mathbb{E}x_{11} = 0, \mathbb{E}x_{11}^2 = 1 \text{ and } \mathbb{E}|x_{11}|^3 < \infty. \tag{3.118}$$

In other words, if the third moment is finite, the limit of the free energy depends only on the variance $\mathbb{E}x_{11}^2$, which is normalized to be equal to one. The proof will be based on a familiar interpolation technique and an approximate integration by parts formula. For $0 \leq t \leq 1$, let us consider the following interpolating Hamiltonian,

$$H_{N,t}(\sigma) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} \big(\sqrt{t}x_{ij} + \sqrt{1-t}g_{ij}\big)\sigma_i\sigma_j, \tag{3.119}$$

and the corresponding interpolating free energy

$$F_N(t) = \frac{1}{N} \mathbb{E} \log \sum_{\sigma\in\Sigma_N} \exp\beta H_{N,t}(\sigma). \tag{3.120}$$

Since, collectively, the random variables (x_{ij}) are called the *disorder* of the model, the following result expresses the "universality in the disorder."

Theorem 3.9 (Carmona-Hu). *Under the assumption (3.118),*

$$\big|F_N(1) - F_N(0)\big| \le \frac{3\beta^3 \mathbb{E}|x_{11}|^3}{\sqrt{N}}. \tag{3.121}$$

A similar result can be proved in the setting of the mixed p-spin models as well, but, for simplicity of notation, we will only write down the proof for the Sherrington–Kirkpatrick model. The bound (3.121) will be obtained by estimating the derivative $F_N'(t)$, using the following approximate integration by parts formula, which can be viewed as an analogue of the Gaussian integration by parts formula (1.19).

Lemma 3.7. *Suppose that x is a random variable that satisfies $\mathbb{E}x = 0$, $\mathbb{E}|x|^3 < \infty$. If $F : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and $\|F''\|_\infty < \infty$ then*

$$\big|\mathbb{E}xF(x) - \mathbb{E}x^2\mathbb{E}F'(x)\big| \le \frac{3}{2}\|F''\|_\infty \mathbb{E}|x|^3. \tag{3.122}$$

Proof. Since $\mathbb{E}x = 0$, we can rewrite the left-hand side as

$$\mathbb{E}xF(x) - \mathbb{E}x^2\mathbb{E}F'(x) = \mathbb{E}x\big(F(x) - F(0) - xF'(0)\big) - \mathbb{E}x^2\mathbb{E}\big(F'(x) - F'(0)\big).$$

Using Taylor's formula, this can be bounded in absolute value by

$$\frac{1}{2}\|F''\|_\infty \mathbb{E}|x|^3 + \|F''\|_\infty \mathbb{E}x^2\mathbb{E}|x|$$

and Eq. (3.122) follows by Hölder's inequality, since $\mathbb{E}x^2\mathbb{E}|x| \le \mathbb{E}|x|^3$. □

Proof (Theorem 3.9). For $0 \le t \le 1$, consider the Gibbs measure corresponding to the interpolating Hamiltonian Eq. (3.119),

$$G_{N,t}(\sigma) = \frac{\exp\beta H_{N,t}(\sigma)}{Z_{N,t}} \text{ where } Z_{N,t} = \sum_{\sigma\in\Sigma_N} \exp\beta H_{N,t}(\sigma), \tag{3.123}$$

and let us denote by $\langle\cdot\rangle_t$ the average with respect to $G_{N,t}^{\otimes\infty}$. Then, for $0 < t < 1$, the derivative of the interpolating free energy Eq. (3.120) can be written as

$$F_N'(t) = \frac{\beta}{2\sqrt{t}N^{3/2}} \sum_{i,j=1}^N \mathbb{E}x_{ij}\langle\sigma_i\sigma_j\rangle_t - \frac{\beta}{2\sqrt{1-t}N^{3/2}} \sum_{i,j=1}^N \mathbb{E}g_{ij}\langle\sigma_i\sigma_j\rangle_t. \tag{3.124}$$

Let us fix the indices $1 \le i, j \le N$ and denote $F = \langle \sigma_i \sigma_j \rangle_t$. It is easy to see that

$$\frac{\partial F}{\partial x_{ij}} = \frac{\beta\sqrt{t}}{\sqrt{N}}\Big(1 - \big\langle \sigma_i^1 \sigma_j^1 \sigma_i^2 \sigma_j^2 \big\rangle_t\Big),$$
$$\frac{\partial^2 F}{\partial x_{ij}^2} = -\frac{\beta^2 t}{N}\Big(\big\langle \sigma_i^1 \sigma_j^1 \big\rangle_t + \big\langle \sigma_i^2 \sigma_j^2 \big\rangle_t - 2\big\langle \sigma_i^1 \sigma_j^1 \sigma_i^2 \sigma_j^2 \sigma_i^3 \sigma_j^3 \big\rangle_t\Big).$$

The second derivative is, obviously, bounded in absolute value by $4\beta^2 t N^{-1}$ and, therefore, recalling that $\mathbb{E}x_{ij}^2 = 1$ by Eq. (3.118), the approximate integration by parts formula (3.122) implies that

$$\Big|\frac{\beta}{2\sqrt{t}N^{3/2}}\mathbb{E}x_{ij}\langle \sigma_i \sigma_j \rangle_t - \frac{\beta^2}{2N^2}\Big(1 - \mathbb{E}\big\langle \sigma_i^1 \sigma_j^1 \sigma_i^2 \sigma_j^2 \big\rangle_t\Big)\Big| \le \frac{3\beta^3\sqrt{t}\mathbb{E}|x_{ij}|^3}{N^{5/2}}.$$

Similarly, the exact Gaussian integration by parts formula (1.19) implies that

$$\frac{\beta}{2\sqrt{1-t}N^{3/2}}\mathbb{E}g_{ij}\langle \sigma_i \sigma_j \rangle_t = \frac{\beta^2}{2N^2}\Big(1 - \mathbb{E}\big\langle \sigma_i^1 \sigma_j^1 \sigma_i^2 \sigma_j^2 \big\rangle_t\Big)$$

and, therefore, the derivative Eq. (3.124) can be controlled by

$$|F_N'(t)| \le \sum_{i,j=1}^N \frac{3\beta^3\sqrt{t}\mathbb{E}|x_{ij}|^3}{N^{5/2}} \le \frac{3\beta^3\mathbb{E}|x_{11}|^3}{\sqrt{N}}.$$

Using the fact that the function $F_N(t)$ is continuous on $[0,1]$ and integrating the derivative prove Eq. (3.121). □

For example, in the context of the Dean's problem in Sect. 1.1, we can choose the interaction parameters g_{ij} to be independent random signs ± 1, which describe whether two people simply like or dislike each other. Theorem 3.9 implies that, in this case, we can still use the Parisi formula to express the asymptotic behavior of the Dean's problem of optimizing the comfort function (1.1).

Chapter 4
Toward a Generalized Parisi Ansatz

In the analysis of the Sherrington–Kirkpatrick and mixed p-spin models, a key role is played by the fact that the Hamiltonian of these models in Eqs. (1.2) and (1.12) is a Gaussian process with the covariance given by a function of the overlap of spin configurations in $\{-1,+1\}^N$. The distribution of such processes is invariant under orthogonal transformations and, as a result, the computation of the free energy can be reduced to the description of asymptotic distributions of the overlaps, which, in some sense, encode the Gibbs measure up to orthogonal transformations. However, for other random Hamiltonians on $\{-1,+1\}^N$, understanding the distribution of the overlaps is not sufficient and one would like to study the asymptotic distributions of all coordinates, or spins, of the configurations sampled from the Gibbs measure. In certain models, it is expected that the structure of these asymptotic distributions can be described by some particular realizations of the Ruelle probability cascades on a separable Hilbert space, but, in most cases, these predictions remain an open problem. In this chapter, we will describe an approach that, in some sense, proves these predictions in the setting of the mixed p-spin models. Unfortunately, again, the special Gaussian nature of the Hamiltonian will play a crucial role, but at least, we will obtain new information beyond the distribution of the overlaps.

4.1 Asymptotic Spin Distributions

Let us consider any random Hamiltonian $H_N(\sigma)$ on $\Sigma_N = \{-1,+1\}^N$ whose distribution is invariant under permutations of the coordinates of σ, which means that if for any permutation π of $\{1,\ldots,N\}$ we define $\sigma_\pi = (\sigma_{\pi(1)},\ldots,\sigma_{\pi(N)})$ then

$$\big(H_N(\sigma_\pi)\big)_{\sigma\in\Sigma_N} \stackrel{d}{=} \big(H_N(\sigma)\big)_{\sigma\in\Sigma_N}. \tag{4.1}$$

This condition is called the *symmetry between sites* and it simply means that the individual coordinates of the configurations σ play interchangeable roles in

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the model. Spin glass models corresponding to such Hamiltonians are usually called *mean-field* models on Σ_N. Let G_N be the Gibbs measure corresponding to the Hamiltonian H_N and let $(\sigma^l)_{l\geq 1}$ be an i.i.d. sequence of replicas sampled from G_N. The individual coordinates of a vector $\sigma \in \Sigma_N$ are called *spins*. Let

$$S^N = \big(\sigma_i^l\big)_{1\leq i\leq N, 1\leq l} \tag{4.2}$$

be the array of all spins of all replicas. In certain mean-field models, for example, in the so-called diluted models, in order to compute the limit of the free energy, one would like to understand the asymptotic distributions of all spins in S^N rather than, for example, the distribution of the overlaps, which was sufficient in the SK model. Notice that the overlaps $(R_{l,l'})$ are, obviously, functions of the array of spins, but S^N contains much more information than $(R_{l,l'})$ and it is easy to see that one can reconstruct the Gibbs measure G_N from S^N exactly and not only up to orthogonal transformations. Let us denote by μ_N the distribution of S^N under $\mathbb{E}G_N^{\otimes\infty}$, which means that for any choice of signs $a_i^l \in \{-1,+1\}$ and for any $n \geq 1$,

$$\mu_N\big(\{\sigma_i^l = a_i^l : i \leq N, l \leq n\}\big) = \mathbb{E}G_N^{\otimes n}\big(\{\sigma_i^l = a_i^l : i \leq N, l \leq n\}\big). \tag{4.3}$$

Let us think of μ_N as a distribution on the set of $\infty\times\infty$ matrices $(\sigma_i^l)_{i,l\geq 1}$ simply by setting $\sigma_i^l = 0$ for $i > N$ and let us denote by $\mathscr{M}$ the family of all possible limits of (μ_N) over subsequences, as usual, in the sense of the convergence of their finite-dimensional marginals. Our goal will be to study the family $\mathscr{M}$ of limiting distributions of all spins and, as the first step, we will explain how one can define an asymptotic analogue of the Gibbs measure G_N for these measures.

By the symmetry between sites in Eq. (4.1), the distribution μ_N of the array S^N is invariant under permutations of the coordinate indices $1 \leq i \leq N$ and, clearly, it is always invariant under permutations of the replica indices $l \geq 1$. Obviously, these symmetries will be inherited by the limiting distributions and, therefore, if we consider a random array $(s_i^l)_{i,l\geq 1}$ with the distribution $\mu \in \mathscr{M}$, it will be exchangeable in the sense of the definition (1.76), that is, for any permutations π and ρ of finitely many indices,

$$\big(s_{\pi(i)}^{\rho(l)}\big)_{i,l\geq 1} \stackrel{d}{=} \big(s_i^l\big)_{i,l\geq 1}. \tag{4.4}$$

Since all the elements s_i^l take values ± 1, the Aldous–Hoover representation in Theorem 1.4 (or Theorem A.1) implies that there exists a measurable function $\sigma : [0,1]^4 \to \{-1,+1\}$ such that μ coincides with the distribution of the array

$$\big(\sigma(w, u_l, v_i, x_{l,i})\big)_{i,l\geq 1}, \tag{4.5}$$

where the random variables w, (u_l), (v_i), $(x_{l,i})$ are i.i.d. with the uniform distribution on $[0,1]$. For each element, the last coordinate $x_{l,i}$ is a dummy variable that corresponds to flipping a biased coin to generate a Bernoulli random variable with the expected value

$$\bar{\sigma}(w, u_l, v_i) = \mathbb{E}_x \sigma(w, u_l, v_i, x_{l,i}), \tag{4.6}$$

where $\mathbb{E}_x$ denotes the expectation with respect to the last coordinate only. This means that the function $\bar{\sigma} : [0,1]^3 \to [-1,1]$ encodes the distribution μ of the array (s_i^l). This function $\bar{\sigma}$ can be viewed as an asymptotic analogue of the Gibbs measure that can be used to generate the array of all spins. By comparison, the Dovbysh–Sudakov representation guaranteed the existence of a random measure on a Hilbert space that can be used to generate the array of overlaps in the limit. If we denote by du or dv the Lebesgue measure on $[0,1]$ then one can also associate $\bar{\sigma}$ with a random measure on $L^2([0,1],dv)$ defined by

$$G = G_w = du \circ (u \to \bar{\sigma}(w,u,\cdot))^{-1}. \tag{4.7}$$

In other words, the coordinate w corresponds to the randomness of this measure and, for a fixed w, the measure G_w is the image of the Lebesgue measure du by the map $u \in [0,1] \to \bar{\sigma}(w,u,\cdot) \in L^2([0,1],dv)$. An i.i.d. sequence of replicas from the measure G_w is given by $(\bar{\sigma}(w,u_l,\cdot))_{l\geq 1}$, which means that, in the limit, vectors in Σ_N are replaced by functions in $L^2([0,1],dv)$ intersected with the unit ball of L^∞, since we know that $|\bar{\sigma}| \leq 1$. Given a "configuration" $\bar{\sigma}(w,u_l,\cdot)$, the spins can be generated by plugging an i.i.d. sequence (v_i) into this function and then flipping a biased coin as above.

Another point of view on the function $\bar{\sigma}$ as a random measure Eq. (4.7) is related to an equivalent way to encode the distribution of the array of spins S^N in terms of the so-called *multi-overlaps*, which is more common in physics literature. Given a sequence (σ^l) of replicas from the Gibbs measure G_N, the quantity

$$R_{l_1,\dots,l_n} = \frac{1}{N}\sum_{i=1}^N \sigma_i^{l_1}\cdots\sigma_i^{l_n} \tag{4.8}$$

is called the multi-overlap of spin configurations $\sigma^{l_1},\dots,\sigma^{l_n}$. It is easy to see that the array of all multi-overlaps encodes the same information asymptotically as the array of all spins in Eq. (4.2). The reason behind this is that, asymptotically, the joint moments of multi-overlaps can be expressed in terms of the joint moments of spins and vice versa. Instead of writing general formulae, for simplicity of notation, let us explain this point with one example. If we write

$$\begin{aligned} R_{1,2}R_{1,3,4}^2 &= \frac{1}{N}\sum_{i=1}^N \sigma_i^1\sigma_i^2 \times \frac{1}{N}\sum_{i=1}^N \sigma_i^1\sigma_i^3\sigma_i^4 \times \frac{1}{N}\sum_{i=1}^N \sigma_i^1\sigma_i^3\sigma_i^4 \\ &= \frac{1}{N^3}\sum_{i_1,i_2,i_3=1}^N (\sigma_{i_1}^1\sigma_{i_1}^2)(\sigma_{i_2}^1\sigma_{i_2}^3\sigma_{i_2}^4)(\sigma_{i_3}^1\sigma_{i_3}^3\sigma_{i_3}^4) \end{aligned}$$

then the contribution of the terms for which at least two of the indices i_1,i_2,i_3 are equal will be negligible in the limit, since the number of such terms is of order N^2. The symmetry between sites in Eq. (4.1) implies that

$$\lim_{N\to\infty} \mathbb{E}\langle R_{1,2}R_{1,3,4}^2\rangle = \lim_{N\to\infty} \mathbb{E}\big\langle(\sigma_1^1\sigma_1^2)(\sigma_2^1\sigma_2^3\sigma_2^4)(\sigma_3^1\sigma_3^3\sigma_3^4)\big\rangle$$

along any subsequence (N_k) for which either of these limits exists. If $\mu \in \mathscr{M}$ is the limit of the distribution of spins μ_N along some subsequence then the asymptotic representation (4.5) implies that the right-hand side can be written as $\mathbb{E}R^{\infty}_{1,2}(R^{\infty}_{1,3,4})^2$ with the notation

$$R^{\infty}_{l_1,\dots,l_n} = \mathbb{E}_v\, \bar{\sigma}(w,u_{l_1},v)\cdots\bar{\sigma}(w,u_{l_n},v), \tag{4.9}$$

where $\mathbb{E}_v$ is the expectation in the last coordinate v only. The general case is very similar and all the joint moments of the multi-overlaps in Eq. (4.8) under $G_N^{\otimes\infty}$ converge to the joints moments of the array (4.9) along the same subsequence. From the point of view of multi-overlaps, the interpretation of the function $\bar{\sigma}$ as a random measure Eq. (4.7) is also very natural, since the average of the product of spin configurations in Eq. (4.8) becomes the average of the product of the asymptotic configurations in Eq. (4.9), given by functions in the intersection of $L^2([0,1],dv)$ with the unit ball of L^{∞}. Notice that, in the limit, the overlaps $(R^{\infty}_{l,l'})$ are defined by the scalar products of replicas $(\bar{\sigma}(w,u_l,\cdot))_{l\geq 1}$ from the random measure G_w in Eq. (4.7) on the Hilbert space $L^2([0,1],dv)$, which means that, in this particular case, we obtain the Dovbysh–Sudakov representation directly from the Aldous–Hoover representation (4.5). The existence of the asymptotic Gibbs measure Eq. (4.7) which encodes the asymptotic distribution of all spins $\mu \in \mathscr{M}$ can be very useful in the study of mean-field models on Σ_N and, in the case of the mixed p-spin models, we will approximate this measure explicitly in terms of the Ruelle probability cascades.

4.2 Regularizing Perturbations

From now on we will consider the mixed p-spin model with the Hamiltonian $H_N(\sigma)$ in Eq. (3.1). As we have seen in the previous chapters, the study of the asymptotic distributions of the overlaps was motivated by their role in the computation of the free energy in the thermodynamic limit. The main tool that allowed us to describe these asymptotic distributions was the Ghirlanda–Guerra identities, which connected the asymptotic Gibbs measures with the Ruelle probability cascades. The Ghirlanda–Guerra identities were proved by adding a small perturbation term to the original Hamiltonian, small enough not to affect the limit of the free energy but large enough to force the Gibbs measure to satisfy these identities. In the same spirit, if computing the free energy is our primary goal then we are free to add any perturbation to the original Hamiltonian if it helps us study the asymptotic distributions of spins, as long as the perturbation is small enough not to affect the free energy. In this section, we will introduce one such perturbation which implies a very useful property of convergence of spin distributions. In the next section we will see that, due to this property, the asymptotic Gibbs measures satisfy an invariance principle that will be used later to describe the structure of these measures. Besides these new properties, we will still need the Ghirlanda–Guerra identities to describe the asymptotic overlap distributions in terms of the Ruelle probability cascades and,

for this purpose, we should include the perturbation Hamiltonian Eq. (3.17) as in Sect. 3.2, which forces the Gibbs measure to satisfy these identities. However, for simplicity of notation, we will omit the perturbation term Eq. (3.17) in this chapter and will focus on the new perturbation and its properties. It is very easy to see that when we include both perturbations, they do not affect each other's influence on the Gibbs measure and still induce their corresponding properties. Consider any increasing sequence (c_N) such that

$$\lim_{N\to\infty} c_N = +\infty \text{ and } \lim_{N\to\infty} (c_{N+1} - c_N) = 0. \tag{4.10}$$

The last property obviously implies that $c_N = o(N)$. For $k \geq 1$, let $(z_{N,k}(\sigma))$ and $(y_{N,k}(\sigma))$ be independent copies of the Gaussian processes defined in Eq. (3.60) and consider the perturbation Hamiltonian

$$h_N(\sigma) = \sum_{k\leq\pi(c_N)} \log \mathrm{ch}\, z_{N,k}(\sigma) + \sum_{k\leq\pi'(c_N)} y_{N,k}(\sigma), \tag{4.11}$$

where $\pi(c_N)$ and $\pi'(c_N)$ are independent Poisson random variables with the mean c_N. Using a computation similar to Eq. (3.21), one can easily check that adding this perturbation term to the original Hamiltonian $H_N(\sigma)$, certainly, does not affect the limit of the free energy, since $c_N = o(N)$. The Gaussian processes defined in Eq. (3.60) appeared in the Aizenman–Sims–Starr scheme, which was based on a certain cavity computation, and the perturbation Hamiltonian Eq. (4.11) will be used here to introduce some useful stability properties into similar cavity computations. One of the main consequences of including the perturbation term Eq. (4.11) into our model will be the following property of convergence of the spins distributions in (4.3). Of course, from now on, the Gibbs measure G_N and the spin distributions μ_N correspond to the perturbed Hamiltonian $H_N(\sigma) + h_N(\sigma)$.

Theorem 4.1. *If μ_N in Eq. (4.3) converges to μ over some subsequence $(N_k)_{k\geq 1}$ then it also converges to μ over the subsequence $(N_k + n)_{k\geq 1}$ for any $n \geq 1$.*

First, we will prove a simple technical result. For a moment, let $\langle\cdot\rangle$ denote the average with respect to the Gibbs measure corresponding to the sum of an arbitrary Hamiltonian on Σ_N and a perturbation term Eq. (4.11), and let $\langle\cdot\rangle'$ denote the average with respect to the Gibbs measure corresponding to the sum of the same arbitrary Hamiltonian and a perturbation term as in Eq. (4.11) only with the number of terms replaced by $\pi(c_N) + n$ instead of $\pi(c_N)$ in the first sum and $\pi'(c_N) + m$ instead of $\pi'(c_N)$ in the second sum, for any finite $m, n \geq 0$. Then the following holds.

Lemma 4.1. *For any bounded function h of finitely many coordinates (spins) in the array (4.2),*

$$\lim_{N\to\infty} \left| \mathbb{E}\langle h\rangle' - \mathbb{E}\langle h\rangle \right| = 0. \tag{4.12}$$

Proof. Without loss of generality, we can assume that $|h| \leq 1$. If we now denote by $\pi(\lambda, k) = \lambda^k e^{-\lambda}/k!$ the weights of the Poisson distribution with the mean λ and

denote by $\langle\cdot\rangle_{i,j}$ the Gibbs average $\langle\cdot\rangle$ conditionally on the event that $\pi(c_N)=i$ and $\pi'(c_N)=j$ then

$$\mathbb{E}\langle h\rangle = \sum_{i,j\geq 0} \pi(c_N,i)\pi(c_N,j)\mathbb{E}\langle h\rangle_{i,j},$$

and, similarly,

$$\begin{aligned}\mathbb{E}\langle h\rangle' &= \sum_{i,j\geq 0} \pi(c_N,i)\pi(c_N,j)\mathbb{E}\langle h\rangle_{i+n,j+m} \\ &= \sum_{i\geq n,j\geq m} \pi(c_N,i-n)\pi(c_N,j-m)\mathbb{E}\langle h\rangle_{i,j}.\end{aligned}$$

Therefore, the difference $|\mathbb{E}\langle h\rangle' - \mathbb{E}\langle h\rangle|$ can be bounded by

$$\begin{aligned}&\sum_{i\geq n,j\geq m} \big|\pi(c_N,i-n)\pi(c_N,j-m)-\pi(c_N,i)\pi(c_N,j)\big| \\ &\quad + \sum_{i<n}\pi(c_N,i) + \sum_{j<m}\pi(c_N,j) \\ &\leq \sum_{i\geq n}\big|\pi(c_N,i-n)-\pi(c_N,i)\big| + \sum_{j\geq m}\big|\pi(c_N,j-m)-\pi(c_N,j)\big| \\ &\quad + \sum_{i<n}\pi(c_N,i) + \sum_{j<m}\pi(c_N,j).\end{aligned}$$

The last two sums, obviously, go to zero as $N\to\infty$. One can show that the first and second sums also go to zero, as follows. By Chebyshev's inequality, the Poisson distribution with the mean c_N is concentrated inside the interval

$$c_N - \sqrt{c_N\log c_N} \leq i \leq c_N + \sqrt{c_N\log c_N} \tag{4.13}$$

with probability at least $1-(\log c_N)^{-1}$. If we write

$$\big|\pi(c_N,i-n)-\pi(c_N,i)\big| = \pi(c_N,i)\Big|1-\frac{i!}{(i-n)!}(c_N)^{-n}\Big| \tag{4.14}$$

then it remains to observe that

$$\frac{i!}{(i-n)!}(c_N)^{-n} = \frac{i(i-1)\dots(i-n+1)}{(c_N)^n} \to 1$$

as $N\to\infty$ uniformly over i in Eq. (4.13). This finishes the proof. □

Remark 4.1. The same result also holds if we replace $\pi(c_N)$ and $\pi'(c_N)$ by Poisson random variables $\pi(c_N+n)$ and $\pi'(c_N+m)$ with the means c_N+n and c_N+m, correspondingly. In this case, one can write

$$\mathbb{E}\langle h\rangle' = \sum_{i,j\geq 0}\pi(c_N+n,i)\pi(c_N+m,j)\mathbb{E}\langle h\rangle_{i,j}$$

and, instead of Eq. (4.14), use

$$\left|\pi(c_N+n,i)-\pi(c_N,i)\right| = \pi(c_N,i)\left|1-\left(1+\frac{n}{c_N}\right)^i e^{-n}\right|$$

and notice that, again, the last factor goes to zero uniformly over the interval Eq. (4.13). Of course, this also means that the same result holds in the mixed cases when, for example, $\pi(c_N)$ is replaced by $\pi(c_N+n)$ and $\pi'(c_N)$ is replaced by $\pi'(c_N)+m$.

Proof (Theorem 4.1). We will show that the joint moments of spins converge to the same limit over the subsequences that differ by a finite shift n. Let

$$h=\prod_{l\le m} h_l \text{ where } h_l=\prod_{i\in C_l}\sigma_i^l,$$

for some finite sets of spin coordinates C_l. Let $\langle\cdot\rangle_N$ denote the average with respect to the Gibbs measure G_N on Σ_N corresponding to the Hamiltonian $H_N(\sigma)+h_N(\sigma)$. We will show that $\lim_{N\to\infty}|\mathbb{E}\langle h\rangle_{N+n}-\mathbb{E}\langle h\rangle_N|=0$. Let us rewrite $\mathbb{E}\langle h\rangle_{N+n}$ using the so-called cavity computation, by treating the last n coordinates as the cavity coordinates. Let us write $\rho=(\sigma,\varepsilon)\in\Sigma_{N+n}$, where $\sigma=(\sigma_1,\dots,\sigma_N)$ are the first N coordinates and $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)$ are the last n (cavity) coordinates, and rewrite $H_{N+n}(\rho)$ as

$$H_{N+n}(\sigma)+\sum_{i\le n}\varepsilon_i z_i(\sigma)+\delta(\rho), \tag{4.15}$$

where we define (slightly abusing notation)

$$H_{N+n}(\sigma):=\sum_{p\ge1}\frac{\beta_p}{(N+n)^{(p-1)/2}}\sum_{i_1,\dots,i_p=1}^{N} g_{i_1\dots i_p}\sigma_{i_1}\dots\sigma_{i_p}, \tag{4.16}$$

each term $\varepsilon_i z_i(\sigma)$ consists of all the terms in $H_{N+n}(\rho)$ with only one factor ε_i from ε present, and the last term $\delta(\rho)$ is the sum of all the terms with at least two factors in ε. We have seen the decomposition (4.15) in the case of $n=1$ in Sect. 1.3 for the SK model. It is easy to check that

$$\mathbb{E}z_i(\sigma^1)z_i(\sigma^2)=\xi'(R_{1,2})+o(1),$$

uniformly over all $\sigma^1,\sigma^2\in\Sigma_N$, and the covariance of $\delta(\rho)$ is also of order $o(1)$, uniformly over ρ^1,ρ^2. Using the Gaussian interpolation method as in the proof of Theorem 3.6, one can replace (and we also redefine) the Hamiltonian $H_{N+n}(\rho)$ in the Gibbs average $\langle\cdot\rangle_{N+n}$ by

$$H_{N+n}(\rho)=H_{N+n}(\sigma)+\sum_{i\le n}\varepsilon_i z_i(\sigma), \tag{4.17}$$

where the Gaussian processes $z_i(\sigma)$ are independent of Eq. (4.16) and of each other and have the covariance exactly

$$\mathbb{E} z_i(\sigma^1) z_i(\sigma^2) = \xi'(R_{1,2}), \tag{4.18}$$

without the error term $o(1)$. Since $|c_{N+n} - c_N| \to 0$ by the assumption (4.10), the remark after the proof of Lemma 4.1 implies that we can replace the random variables $\pi(c_{N+n})$ and $\pi'(c_{N+n})$ in the perturbation Hamiltonian $h_{N+n}(\rho)$ by $\pi(c_N)$ and $\pi'(c_N)$ without affecting $\mathbb{E}\langle h\rangle_{N+n}$ asymptotically. After that, one can use Gaussian interpolation as in the proof of Theorem 3.6 and the fact that $c_N = o(N)$ to show that the random variables $z_{N+n,k}(\rho)$ and $y_{N+n,k}(\rho)$ in the perturbation Hamiltonian $h_{N+n}(\rho)$ can be replaced by $z_{N,k}(\sigma)$ and $y_{N,k}(\sigma)$. In other words, since perturbation is of a smaller order, we can replace $h_{N+n}(\rho)$ in $\langle\cdot\rangle_{N+n}$ by the same perturbation perm $h_N(\sigma)$ as in $\langle\cdot\rangle_N$ without affecting $\mathbb{E}\langle h\rangle_{N+n}$ asymptotically. Therefore, from now on, we can assume that the average $\langle\cdot\rangle_{N+n}$ corresponds to the Hamiltonian

$$H_{N+n}(\sigma) + \sum_{i\le n} \varepsilon_i z_i(\sigma) + h_N(\sigma). \tag{4.19}$$

If we denote by $\langle\cdot\rangle'_N$ the average on Σ_N with respect to the Gibbs measure corresponding to the Hamiltonian

$$H'_N(\sigma) = H_{N+n}(\sigma) + h_N(\sigma) \tag{4.20}$$

then each factor in

$$\langle h\rangle_{N+n} = \prod_{l\le m} \langle h_l\rangle_{N+n} = \prod_{l\le m} \Big\langle \prod_{i\in C_l} \rho_i \Big\rangle_{N+n} = \prod_{l\le m} \Big\langle \prod_{i\in C_l} \sigma_i \Big\rangle_{N+n}$$

(in the last equality we used that, for large N, all the sets C_l will include only the first N coordinates) can be written as

$$\langle h_l\rangle_{N+n} = \frac{\big\langle \prod_{i\in C_l}\sigma_i \operatorname{Av}_\varepsilon \exp\sum_{i\le n}\varepsilon_i z_i(\sigma)\big\rangle'_N}{\big\langle \operatorname{Av}_\varepsilon \exp\sum_{i\le n}\varepsilon_i z_i(\sigma)\big\rangle'_N} = \Big\langle \prod_{i\in C_l}\sigma_i \Big\rangle''_N,$$

where we denoted by $\langle\cdot\rangle''_N$ the average on Σ_N with respect to the Gibbs measure corresponding to the Hamiltonian

$$H''_N(\sigma) = H'_N(\sigma) + \sum_{i\le n} \log \operatorname{ch} z_i(\sigma) = H_{N+n}(\sigma) + h'_N(\sigma),$$

where

$$h'_N(\sigma) = h_N(\sigma) + \sum_{i\le n} \log \operatorname{ch} z_i(\sigma).$$

Since the processes $z_i(\sigma)$ are independent copies of $z_{N,k}(\sigma)$, we can redefine

$$h'_N(\sigma) = \sum_{k\le\pi(c_N)+n} \log \mathrm{ch}\, z_{N,k}(\sigma) + \sum_{k\le\pi'(c_N)} y_{N,k}(\sigma), \tag{4.21}$$

and we showed that $\mathbb{E}\langle h\rangle_{N+n} = \mathbb{E}\langle h\rangle''_N$. Let us now consider $\mathbb{E}\langle h\rangle_N$. It is easy to check that, in distribution, the Hamiltonian $H_N(\sigma)$ can be related to the Hamiltonian $H_{N+n}(\sigma)$ in Eq. (4.16) by

$$H_N(\sigma) = H_{N+n}(\sigma) + \sum_{i\le n} y_i(\sigma), \tag{4.22}$$

where $(y_i(\rho))$ are independent Gaussian processes with the covariance

$$\mathbb{E} y_i(\sigma^1) y_i(\sigma^2) = \theta(R_{1,2}) + o(1).$$

Again, using the Gaussian interpolation as in the proof of Theorem 3.6, one can show that redefining the covariance of $y_i(\sigma)$ to be exactly $\theta(R_{1,2})$ will not affect $\mathbb{E}\langle h\rangle_N$ asymptotically. Since such $y_i(\sigma)$ are independent copies of $y_{N,k}(\sigma)$, we can assume that the average $\langle\cdot\rangle_N$ is defined in terms of the Hamiltonian $H_{N+n}(\sigma) + h''_N(\sigma)$ where

$$h''_N(\sigma) = \sum_{k\le\pi(c_N)} \log \mathrm{ch}\, z_{N,k}(\sigma) + \sum_{k\le\pi'(c_N)+n} y_{N,k}(\sigma). \tag{4.23}$$

Lemma 4.1 then implies that both perturbation terms (4.21) and (4.23) can be replaced by the original perturbation term (4.11) without affecting $\mathbb{E}\langle h\rangle''_N$ and $\mathbb{E}\langle h\rangle_N$ asymptotically and this finishes the proof. □

4.3 Invariance via Cavity Method

Consider a subsequential limit $\mu \in \mathscr{M}$ of the spin distributions μ_N defined in Eq. (4.3) and let $\bar{\sigma}(w,u,v)$ be the function in Eq. (4.6) which encodes this distribution. We have seen that, along the same subsequence, the multi-overlaps (4.8) converge in distribution to the asymptotic multi-overlaps (4.9). Suppose that the overlaps $(R^\infty_{l,l'})$ satisfy the Ghirlanda–Guerra identities (3.83). This can be achieved as in Sect. 3.2 by adding a perturbation Hamiltonian Eq. (3.17), which ensures the validity of the Ghirlanda–Guerra identities on average, and then choosing specific parameters in the perturbation as in the proof of the Parisi formula. One can expect this approach to yield the Ghirlanda–Guerra identities in many other spin glass models, since the proof of these identities in Theorem 3.2 used only mild assumptions on the concentration of the free energy. If the overlaps $(R^\infty_{l,l'})$ satisfy the Ghirlanda–Guerra identities, we know they can be approximated in distribution by Gram matrices $Q^r = (Q^r_{l,l'})$ of i.i.d. samples from the Ruelle probability cascades G^r on the unit sphere of a Hilbert space, as $r \to \infty$. This was explained in the Remark 2.1 at the end of Sect. 2.4. A question then arises, can we approximate all multi-overlaps, or all spins, in terms of these Ruelle probability cascades? Or, can we describe the

function $\bar{\sigma}(w,u,v)$, in some sense, in terms of the sequence G^r? We will see that in the mixed p-spin models the answer is yes, due to the following invariance property induced by the regularizing perturbation Eq. (4.11). Let $g_{\xi'}$ be a Gaussian process defined in Lemma 3.1 with the covariance (3.14) with $\psi = \xi'$ and the parameters corresponding to the Ruelle probability cascade G^r. For simplicity of notation, we will keep the dependence of $g_{\xi'}$ on r implicit and will keep track of r by denoting the average with respect to $(G^r)^{\otimes\infty}$ by $\langle\cdot\rangle_r$. Let $g_{\xi',i}$ be independent copies of $g_{\xi'}$ for $i \geq 1$. Let us consider integers $n,m \geq 1$ and, for each $l \leq m$, consider a set $C_l \subseteq \{1,\dots,n\}$. Integer m will represent a number of replicas and C_l will be some set of spin coordinates corresponding to the replica index l. If $(s_i^l)_{i,l\geq 1}$ is the array of spins with the asymptotic distribution μ, then, by Eq. (4.5), we can write the joint moment of spins in terms of the function $\bar{\sigma}(w,u,v)$ as

$$\mathbb{E}\prod_{l\leq m}\prod_{i\in C_l} s_i^l = \mathbb{E}\prod_{l\leq m}\prod_{i\in C_l}\bar{\sigma}(w,u_l,v_i). \tag{4.24}$$

We will prove that the joint moments of spins can be approximated using the above Ruelle probability cascades G^r as follows.

Theorem 4.2. *For any $n,m \geq 1$ and any sets $C_l \subseteq \{1,\dots,n\}$ for $l \leq m$,*

$$\mathbb{E}\prod_{l\leq m}\prod_{i\in C_l} s_i^l = \lim_{r\to\infty}\mathbb{E}\frac{\big\langle\prod_{l\leq m}\prod_{i\in C_l}\operatorname{th} g_{\xi',i}(\sigma^l)\prod_{i\leq n}\operatorname{ch} g_{\xi',i}(\sigma^l)\big\rangle_r}{\big\langle\prod_{i\leq n}\operatorname{ch} g_{\xi',i}(\sigma)\big\rangle_r^m}. \tag{4.25}$$

We will give a detailed interpretation of this invariance property in the next section and, at this moment, notice that, intuitively, the right-hand side depends only on the distribution of the overlaps, since the covariance of the Gaussian processes $g_{\xi',i}$ is a function of the overlap. Indeed, it will be shown in the proof that the right-hand side is a continuous function of the distribution of the overlap matrix $(R^\infty_{l,l'})$. This means that, in some sense, in the mixed p-spin models the asymptotic distribution of all spins, or all multi-overlaps, is determined by the distribution of the overlaps.

Proof (Theorem 4.2). Let us consider a product of spins

$$h = \prod_{l\leq m} h_l, \text{ where } h_l = \prod_{i\in C_l}\sigma_i^l,$$

for some finite sets of spin coordinates $C_l \subseteq \{1,\dots,n\}$. By Theorem 4.1, the left-hand side of Eq. (4.25), which was the limit of $\mathbb{E}\langle h\rangle_N$ over some subsequence (N_k), is also the limit of $\mathbb{E}\langle h\rangle_{N+n}$ over the same subsequence. Let us rewrite $\mathbb{E}\langle h\rangle_{N+n}$ using the same cavity computation as in the proof of Theorem 4.1, only now the spins with the coordinates $i \leq n$ will play the roles of cavity coordinates instead of the spins with the coordinates $N+1 \leq i \leq N+n$. Let us write $\rho = (\varepsilon,\sigma) \in \Sigma_{N+n}$, where we denote the first n (cavity) coordinates by $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n)$ and the last N coordinates by $\sigma = (\sigma_1,\dots,\sigma_N)$. First of all, using exactly the same computation leading to Eq. (4.19) in the proof of Theorem 4.1, without affecting $\mathbb{E}\langle h\rangle_{N+n}$

asymptotically, we can replace the Hamiltonian $H_{N+n}(\rho)+h_{N+n}(\rho)$ in the average $\langle\cdot\rangle_{N+n}$ by the Hamiltonian

$$H_{N+n}(\sigma)+\sum_{i\le n}\varepsilon_i z_i(\sigma)+h_N(\sigma), \tag{4.26}$$

where $H_{N+n}(\sigma)$ was defined in Eq. (4.16) and $z_i(\sigma)$ are independent copies of the Gaussian process with the covariance (4.18). Again, as in Eq. (4.20), if we denote by $\langle\cdot\rangle'_N$ the average on Σ_N with respect to the Gibbs measure corresponding to the Hamiltonian

$$H'_N(\sigma)=H_{N+n}(\sigma)+h_N(\sigma), \tag{4.27}$$

then, using Eq. (4.26), each factor in

$$\langle h\rangle_{N+n}=\prod_{l\le m}\langle h_l\rangle_{N+n}=\prod_{l\le m}\Big\langle\prod_{i\in C_l}\rho_i\Big\rangle_{N+n}=\prod_{l\le m}\Big\langle\prod_{i\in C_l}\varepsilon_i\Big\rangle_{N+n}$$

(now, the first n coordinates are the cavity coordinates ε) can be written as

$$\begin{aligned}\Big\langle\prod_{i\in C_l}\varepsilon_i\Big\rangle_{N+n}&=\frac{\big\langle \mathrm{Av}_\varepsilon\prod_{i\in C_l}\varepsilon_i\exp\sum_{i\le n}\varepsilon_i z_i(\sigma)\big\rangle'_N}{\big\langle \mathrm{Av}_\varepsilon\exp\sum_{i\le n}\varepsilon_i z_i(\sigma)\big\rangle'_N}\\&=\frac{\big\langle\prod_{i\in C_l}\mathrm{th}\,z_i(\sigma)\prod_{i\le n}\mathrm{ch}\,z_i(\sigma)\big\rangle'_N}{\big\langle\prod_{i\le n}\mathrm{ch}\,z_i(\sigma)\big\rangle'_N}.\end{aligned}$$

Therefore, this cavity computation allows us to rewrite the joint moment of spins as

$$\mathbb{E}\langle h\rangle_{N+n}=\mathbb{E}\frac{\big\langle\prod_{l\le m}\prod_{i\in C_l}\mathrm{th}\,z_i(\sigma^l)\prod_{i\le n}\mathrm{ch}\,z_i(\sigma^l)\big\rangle'_N}{\big\langle\prod_{i\le n}\mathrm{ch}\,z_i(\sigma)\big\rangle'^{\,m}_N}, \tag{4.28}$$

which looks very similar to the right-hand side of Eq. (4.25), especially, if we notice that the covariance of both processes z_i and $g_{\xi',i}$ is given by the same function $\xi'(R_{1,2})$ of the overlap, albeit on different spaces. Another important fact that was observed in the proof of Theorem 4.1 is that, because of the relationship (4.22) and Lemma 4.1, the spin distributions under the Gibbs averages $\langle\cdot\rangle_N$ and $\langle\cdot\rangle'_N$ corresponding to the Hamiltonians $H_N(\sigma)+h_N(\sigma)$ and $H_{N+n}(\sigma)+h_N(\sigma)$ converge to the same limit over any subsequence and, in particular, the overlap matrix $(R_{l,l'})$ converges in distribution to the same limit. Let us denote

$$X(\sigma)=\sum_{i\le n}\log\mathrm{ch}\,z_i(\sigma),\ Y=\log\big\langle\exp X(\sigma)\big\rangle'_N,$$

and let Z denote the numerator on the right-hand side of Eq. (4.28). With this notation, the right-hand side of Eq. (4.28) is equal to $\mathbb{E}Z\exp(-mY)$. The rest of the argument resembles the proof of Theorem 1.3. Similarly to Eq. (1.70), the Gaussian concentration inequality in Theorem 1.2 implies that, for small enough $c>0$ and all large enough $a>0$,

$$\mathbb{P}(|Y|\ge a)=\mathbb{P}\big(\big|\log\big\langle\exp X(\sigma)\big\rangle'_N\big|\ge a\big)\le\exp(-ca^2). \tag{4.29}$$

Let $(x)_a = \max(-a, \min(x,a))$ and notice that

$$\left|\exp(-mx) - \exp(-m(x)_a)\right| \le \max\left(e^{-ma}, \exp(-mx)\right) I\left(|x| > a\right).$$

Obviously, the numerator in Eq. (4.28) satisfies $\mathbb{E}Z^2 \le L$ for some large enough $L > 0$ that depends on n, m and the function ξ', and Hölder's inequality and Eq. (4.29) imply that

$$\begin{aligned}&\left|\mathbb{E}Z\exp(-mY) - \mathbb{E}Z\exp(-m(Y)_a)\right| \\ &\quad \le \mathbb{E}|Z| \max\left(e^{-ma}, \exp(-mY)\right) I\left(|Y| > a\right) \le \exp(-ca^2),\end{aligned} \tag{4.30}$$

for some small enough $c > 0$ and large enough $a > 0$. Next, let us denote

$$\exp_a x = \max\left(e^{-a}, \min(e^x, e^a)\right)$$

and let $Y' = \log\langle \exp_a X(\sigma)\rangle'_N$. It is easy to check that for all $x, y \in \mathbb{R}$,

$$\left|\exp(-m(x)_a) - \exp(-m(y)_a)\right| \le m e^{(m+1)a} \left|\exp x - \exp y\right|,$$

and this implies that

$$\left|\exp(-m(Y)_a) - \exp(-m(Y')_a)\right| \le m e^{(m+1)a} \left\langle \left|\exp X(\sigma) - \exp_a X(\sigma)\right|\right\rangle'_N.$$

Since for all $x \in \mathbb{R}$,

$$|\exp x - \exp_a x| \le \max(e^{-a}, \exp x) I\left(|x| \ge a\right),$$

we obtain the following bound:

$$\begin{aligned}&\left|\exp(-m(Y)_a) - \exp(-m(Y')_a)\right| \\ &\quad \le m e^{(m+1)a} \left\langle \max\left(e^{-a}, \exp X(\sigma)\right) I\left(|X(\sigma)| \ge a\right)\right\rangle'_N.\end{aligned}$$

It is easy to see that, for a fixed $\sigma \in \Sigma_N$, $\mathbb{P}(|X(\sigma)| \ge a) \le \exp(-ca^2)$ for small enough $c > 0$ and large enough $a > 0$, and, using Hölder's inequality and the fact that $X(\sigma)$ is independent of the randomness in $\langle\cdot\rangle'_N$,

$$\begin{aligned}&\left|\mathbb{E}Z\exp(-m(Y)_a) - \mathbb{E}Z\exp(-m(Y')_a)\right| \\ &\quad \le m e^{(m+1)a} \left(\mathbb{E}Z^2\right)^{1/2} \left(\mathbb{E}\left\langle \max\left(e^{-4a}, \exp 4X(\sigma)\right)\right\rangle'_N\right)^{1/4} \\ &\qquad \times \left(\mathbb{E}\left\langle \mathbb{P}\left(|X(\sigma)| \ge a\right)\right\rangle'_N\right)^{1/4} \le \exp(-ca^2),\end{aligned}$$

for small enough $c > 0$ and large enough $a > 0$. Combining this with Eq. (4.30), we proved that the right-hand side of Eq. (4.28), $\mathbb{E}Z\exp(-mY)$, can be approximated by

$$\left|\mathbb{E}Z\exp(-mY) - \mathbb{E}Z\exp(-m(Y')_a)\right| \le \exp(-ca^2), \tag{4.31}$$

for small enough $c > 0$ and large enough $a > 0$. Since $\langle \exp_a X(\sigma)\rangle'_N$ takes values in the interval $[e^{-a}, e^a]$, we have $(Y')_a = Y'$, and the quantity

$$\exp(-m(Y')_a) = \big\langle \exp_a X(\sigma)\big\rangle_N'^{-m}$$

can be approximated uniformly by polynomials of $\langle \exp_a X(\sigma)\rangle'_N$. Therefore, Eq. (4.31) implies that the right-hand side of Eq. (4.28), $\mathbb{E}Z\exp(-mY)$, can be approximated by a linear combination of terms

$$\mathbb{E}Z\big\langle \exp_a X(\sigma)\big\rangle_N'^{s}. \tag{4.32}$$

The argument leading to this approximation did not use explicitly the definition of the Gibbs measure in $\langle\cdot\rangle'_N$, which means that the right-hand side of Eq. (4.25) can be approximated by exactly the same linear combination in terms of the average $\langle\cdot\rangle_r$ and the processes $g_{\xi',i}(\sigma)$. Therefore, to finish the proof, it is enough to show that Eq. (4.32) is a continuous function of the distribution of the overlap matrix $(R_{l,l'})$, since our choice of the Ruelle probability cascades G^r was exactly such that the overlap matrices generated by these cascades converge in distribution to the same limit as the overlap matrices under the Gibbs average $\langle\cdot\rangle_N$, which, as we mentioned above, is the same under the Gibbs average $\langle\cdot\rangle'_N$. Using replicas, we can rewrite (4.32) as

$$\mathbb{E}\Big\langle \mathbb{E}_z \prod_{j\le s} \exp_a X(\sigma^{m+j}) \prod_{l\le m}\prod_{i\in C_l} \mathrm{th}\, z_i(\sigma^l) \prod_{i\le n} \mathrm{ch}\, z_i(\sigma^l)\Big\rangle'_N,$$

where $\mathbb{E}_z$ is the expectation with respect to $(z_i(\sigma))$, which can be taken inside the Gibbs average, because these processes are independent of the randomness in $\langle\cdot\rangle'_N$. Since the covariance of the Gaussian processes $(z_i(\sigma))$ is given by a continuous function $\xi'(R_{1,2})$ of the overlap, the expression inside the Gibbs average is a continuous function of the overlaps on $m+s$ replicas. This finishes the proof. □

4.4 Structure of Spin Distributions

Let us fix $r \ge 1$. For simplicity of notation, let us denote by $G = G^r$ and $\langle\cdot\rangle = \langle\cdot\rangle_r$ the Ruelle probability cascade and its average in Eq. (4.25) and consider the quantity

$$\mathbb{E}\frac{\big\langle \prod_{l\le m}\prod_{i\in C_l} \mathrm{th}\, g_{\xi',i}(\sigma^l) \prod_{i\le n} \mathrm{ch}\, g_{\xi',i}(\sigma^l)\big\rangle}{\big\langle \prod_{i\le n} \mathrm{ch}\, g_{\xi',i}(\sigma)\big\rangle^m}. \tag{4.33}$$

We can think of the hyperbolic cosines as a change of density of the measure G, as follows. If we denote

$$X(\sigma) = \sum_{i\le n} \log \mathrm{ch}\, g_{\xi',i}(\sigma),$$

define the random measure G' by

$$G'(\sigma) = \frac{G(\sigma)\exp X(\sigma)}{\langle \exp X(\sigma)\rangle}, \tag{4.34}$$

and let $\langle\cdot\rangle'$ be the average with respect to $G'^{\otimes\infty}$ then Eq. (4.33) can be rewritten as

$$\mathbb{E}\Big\langle \prod_{l\le m}\prod_{i\in C_l} \mathrm{th}\, g_{\xi',i}(\sigma^l)\Big\rangle'. \tag{4.35}$$

Formally, the change of density in Eq. (4.34) depends on n. However, we will show that, due to some invariance property of the Ruelle probability cascades, for all $n \ge 1$,

$$\mathbb{E}\Big\langle \prod_{l\le m}\prod_{i\in C_l} \mathrm{th}\, g_{\xi',i}(\sigma^l)\Big\rangle' = \mathbb{E}\Big\langle \prod_{l\le m}\prod_{i\in C_l} \mathrm{th}\, g'_{\xi',i}(\sigma^l)\Big\rangle, \tag{4.36}$$

where, for $i \ge 1$, $g'_{\xi',i}(\sigma)$ are independent copies of some random process $g'_{\xi'}(\sigma)$, which are also independent of the randomness in the average $\langle\cdot\rangle$ with respect to the original Ruelle probability cascade G in Eq. (4.33). The process $g'_{\xi'}(\sigma)$ will be described below. Once we prove Eq. (4.36), Eq. (4.25) can be interpreted as follows. If the asymptotic distribution of one overlap $R^\infty_{1,2}$ is approximated by the functional order parameter of the Ruelle probability cascade G, as at the beginning of Sect. 4.3, then the array of all spins (s^l_i) can be approximated in distribution by the following procedure. First, we generate the array

$$\bar\sigma_{i,l} = \mathrm{th}\, g'_{\xi',i}(\sigma^l) \text{ for } i,l \ge 1, \tag{4.37}$$

where (σ^l) is an i.i.d. sequence of replicas from G, and then we flip independent biased ± 1 coins with the means $\bar\sigma_{i,l}$. Equations (4.25) and (4.36) imply that the joint moments of spins generated in this way will be approximately equal to the corresponding joint moments of spins in (s^l_i). This is how the invariance property (4.25) of the asymptotic spin distributions in $\mathscr{M}$, induced by the regularizing perturbation (4.11), allows us to describe the distribution of spins in terms of the Ruelle probability cascades.

In order to prove Eq. (4.36), we will need a generalization of the invariance property (2.84) in the setting of the Ruelle probability cascades. The central idea behind Eq. (2.84) was the invariance property of the Poisson process in Theorem 2.6 and its consequence in Eq. (2.23), which was based on a simple fact that a Gaussian random variable g under a change of density proportional to $\exp tg$ has Gaussian distribution with different parameters. Theorem 2.6 will still be our basic tool here, but we will consider more general changes of density of the Ruelle probability cascades. From now on we will assume that the random measure G is defined in Eq. (2.62) and use the notation in Sect. 2.3. First, let us recall the sequence $(u_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ in Eq. (2.44). Given a parent vertex $\alpha \in \mathscr{A}\setminus\mathbb{N}^r$, the sequence $(u_{\alpha n})_{n\ge1}$ is the decreasing rearrangement of a Poisson process Π_α with the mean measure

in Eq. (2.12) corresponding to the parameter $\zeta_{|\alpha|}$ in Eq. (2.42). Recall that we denote the children of the vertex α by αn and $|\alpha|$ is the distance of α from the root of the tree $\emptyset$. Next, we are going to construct a random process $(z_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ via some general iterative procedure, where each z_α will be a random element on some measurable space $(\mathscr{X},\mathscr{B})$. Let ν_0 be a probability measure on $(\mathscr{X},\mathscr{B})$ and, for $1\le p\le r-1$, let

$$\nu_p : \mathscr{X}^p\times\mathscr{B}\to[0,1] \tag{4.38}$$

be a transition function, which means that for each element x of the product space, $\mathscr{X}^p$, $\nu_p(x,\cdot)$ is a probability measure on the σ-algebra $\mathscr{B}$ and for each $B\in\mathscr{B}$, $\nu_p(\cdot,B)$ is a measurable function with respect to the product σ-algebra $\mathscr{B}^{\otimes p}$ on $\mathscr{X}^p$. Given these transition functions, we generate $(z_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ recursively from the root of the tree, as follows. The sequence $(z_\alpha)_{|\alpha|=1}$ is i.i.d. with the distribution ν_0 and, for $1\le p\le r-1$, given $(z_\alpha)_{|\alpha|\le p}$, for each $\alpha\in\mathbb{N}^p$, the sequence $(z_{\alpha n})_{n\ge1}$ is i.i.d. with the distribution $\nu_p(Z_\alpha,\cdot)$, where we used the notation

$$Z_\alpha=(z_\beta)_{\beta\in p(\alpha)}\in\mathscr{X}^p \tag{4.39}$$

and where $p(\alpha)$ is the path from the root of the tree to the vertex α, defined in Eq. (2.41). Moreover, the processes $(z_{\alpha n})_{n\ge1}$ are generated independently of each other for $\alpha\in\mathbb{N}^p$. We also assume that the process (z_α) is independent of (u_α). Our basic object will be this pair of processes $(u_\alpha),(z_\alpha)$ and we will study their distribution under certain changes of density.

Let us consider a function $X_r:\mathscr{X}^r\to\mathbb{R}$ and, recursively over $0\le p\le r-1$, define functions $X_p:\mathscr{X}^p\to\mathbb{R}$ by

$$X_p(x)=\frac{1}{\zeta_p}\log\int_{\mathscr{X}}\exp\zeta_pX_{p+1}(x,x_{p+1})\,\nu_p(x,dx_{p+1}), \tag{4.40}$$

for $x\in\mathscr{X}^p$. In particular, X_0 is a constant. At the moment, we simply assume that X_r is such that all these functions are well defined for all $0\le p\le r-1$ and all $x\in\mathscr{X}^p$. Let us define a function

$$W_p(x,x_{p+1})=\exp\bigl(X_{p+1}(x,x_{p+1})-X_p(x)\bigr) \tag{4.41}$$

for $x\in\mathscr{X}^p$ and $x_{p+1}\in\mathscr{X}$. Let us point out that, by the definition (4.40),

$$\int_{\mathscr{X}}W_p^{\zeta_p}(x,x_{p+1})\,\nu_p(x,dx_{p+1})=1 \tag{4.42}$$

and, therefore, for each $x\in\mathscr{X}^p$, we can think of $W_p^{\zeta_p}(x,\cdot)$ as a change of density under the transition probability $\nu_p(x,\cdot)$ and define a new transition function

$$\nu_p'(x,B)=\int_B W_p^{\zeta_p}(x,x_{p+1})\,\nu_p(x,dx_{p+1}). \tag{4.43}$$

For $p=0$, ν'_0 is just a probability on $(\mathscr{X},\mathscr{B})$. Let $(z'_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ be a random process defined iteratively from the root of the tree exactly as (z_α) above, using the transition functions $\nu'_p(x,B)$ instead of $\nu_p(x,B)$. Given the random arrays $(u_\alpha),(z_\alpha)$ above, for each parent vertex $\alpha\in\mathscr{A}\setminus\mathbb{N}^r$, let us define

$$u'_{\alpha n}=u_{\alpha n}W_{|\alpha|}(Z_\alpha,z_{\alpha n}). \tag{4.44}$$

As in Sect. 2.3, we will say that a bijection $\pi:\mathscr{A}\to\mathscr{A}$ of the vertices of the tree $\mathscr{A}$ preserves the parent-child relationship if children αn of α are mapped into children of $\pi(\alpha)$, $\pi(\alpha n)=(\pi(\alpha),k)$ for some $k\in\mathbb{N}$. We will prove the following generalization of the invariance property (2.93).

Theorem 4.3. *There exists a random bijection $\pi:\mathscr{A}\to\mathscr{A}$ of the vertices of the tree $\mathscr{A}$, which preserves the parent-child relationship, such that*

$$\big(u'_{\pi(\alpha)}\big)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}\overset{d}{=}(u_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0},\ \big(z_{\pi(\alpha)}\big)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}\overset{d}{=}(z'_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0} \tag{4.45}$$

and these two processes are independent of each other.

Notice that the process (u'_α) in Eq. (4.44), formally, depends on the process (z_α), but after permutation they become independent. One can recover the invariance property (2.93) by letting the transition functions $\nu_p(x,B)=\nu_p(B)$ be Gaussian distributions on the real line independent of x, which means that the random variables z_α are all independent of each other, and taking $X_r(x)$ to be a linear function. In this case, all X_p will be affine, one can compute the transition functions ν'_p in Eq. (4.43) explicitly and they will again be Gaussian shifted by some constants, as in Eq. (2.93).

Proof (Theorem 4.3). Let $\mathscr{F}_0$ be the trivial σ-algebra and, for $0\le p\le r$, let $\mathscr{F}_p$ be the σ-algebra generated by the random processes u_α and z_α for $|\alpha|\le p$. We will construct the permutation π recursively starting from the root of the tree, $\pi(\emptyset)=\emptyset$, in such a way that π restricted to $\mathbb{N}^p$ is an $\mathscr{F}_p$-measurable bijection of $\mathbb{N}^p$ into itself. Simultaneously, we will prove by induction on p that

$$\big(u'_{\pi(\alpha)}\big)_{|\alpha|\le p}\overset{d}{=}(u_\alpha)_{|\alpha|\le p},\ \big(z_{\pi(\alpha)}\big)_{|\alpha|\le p}\overset{d}{=}(z'_\alpha)_{|\alpha|\le p}, \tag{4.46}$$

and these two sequences are independent of each other. Suppose that π has been constructed for all $|\alpha|\le p$ and Eq. (4.46) holds. Let us fix $\alpha\in\mathbb{N}^p$ and consider the sequence

$$\Big(\big(u'_{\pi(\alpha)n},z_{\pi(\alpha)n}\big)\Big)_{n\ge1}=\Big(\big(u_{\pi(\alpha)n}W_{|\alpha|}(Z_{\pi(\alpha)},z_{\pi(\alpha)n}),z_{\pi(\alpha)n}\big)\Big)_{n\ge1}, \tag{4.47}$$

conditionally on $\mathscr{F}_p$. Then, $\pi(\alpha)$ and $Z_{\pi(\alpha)}$ are fixed and, by construction, random elements $(z_{\pi(\alpha)n})_{n\ge1}$ are i.i.d. and have the distribution $\nu_p(Z_{\pi(\alpha)},\cdot)$. By Eq. (4.42), the conditional expectation

$$\mathbb{E}\big(W_p^{\zeta_p}(Z_{\pi(\alpha)},z_{\pi(\alpha)n})\big|Z_{\pi(\alpha)}\big)=1,$$

and, therefore, Theorem 2.6 implies the following. If $\pi_{\pi(\alpha)}(n)$ is the random bijection of $\mathbb{N}$ that arranges the weights $(u'_{\pi(\alpha)n})$ in the decreasing order then, conditionally on $\mathscr{F}_p$,

$$\Big(\big(u'_{\pi(\alpha)\pi_{\pi(\alpha)}(n)}, z_{\pi(\alpha)\pi_{\pi(\alpha)}(n)} \big) \Big)_{n\geq 1} \stackrel{d}{=} \Big(\big(u_{\alpha n}, z'_{\alpha n} \big) \Big)_{n\geq 1}, \tag{4.48}$$

where $(u_{\alpha n})_{n\geq 1}$ is a decreasing rearrangement of a Poisson process with the mean measure in Eq. (2.12) with the parameter $\zeta = \zeta_p$ and, in agreement with Eq. (2.17), $(z'_{\alpha n})_{n\geq 1}$ are independent markings with the distribution $\nu'_p(Z_{\pi(\alpha)}, \cdot)$ defined in Eq. (4.43), which is the distribution of the random elements $z_{\pi(\alpha)n}$ under the change of density $W_p^{\zeta_p}(Z_{\pi(\alpha)}, z_{\pi(\alpha)n})$. If we define π for $\alpha n \in \mathbb{N}^{p+1}$ by

$$\pi(\alpha n) = \pi(\alpha)\pi_{\pi(\alpha)}(n), \tag{4.49}$$

then, conditionally on $\mathscr{F}_p$, for each $\alpha \in \mathbb{N}^p$,

$$\Big(\big(u'_{\pi(\alpha n)}, z_{\pi(\alpha n)} \big) \Big)_{n\geq 1} \stackrel{d}{=} \Big(\big(u_{\alpha n}, z'_{\alpha n} \big) \Big)_{n\geq 1}. \tag{4.50}$$

First of all, by the definition (4.47), it is obvious that, conditionally on $\mathscr{F}_p$, the sequences $(u'_{\pi(\alpha n)})_{n\geq 1}$ are independent for $\alpha \in \mathbb{N}^p$. By Eq. (4.50), their distribution does not depend on the condition, which implies that these sequences are independent of $\mathscr{F}_p$ and independent of each other for $\alpha \in \mathbb{N}^p$. This proves that

$$\big(u'_{\pi(\alpha)} \big)_{|\alpha|\leq p+1} \stackrel{d}{=} (u_\alpha)_{|\alpha|\leq p+1}. \tag{4.51}$$

By the definition (4.47), it is also obvious that, conditionally on $\mathscr{F}_p$, the sequences $(z_{\pi(\alpha n)})_{n\geq 1}$ are independent for $\alpha \in \mathbb{N}^p$. We also showed that, conditionally on $\mathscr{F}_p$, each sequence $(z_{\pi(\alpha n)})_{n\geq 1}$ is generated according to the distribution $\nu'_p(Z_{\pi(\alpha)}, \cdot)$ and, by the induction assumption (4.46), this implies that

$$\big(z_{\pi(\alpha)} \big)_{|\alpha|\leq p+1} \stackrel{d}{=} (z'_\alpha)_{|\alpha|\leq p+1}, \tag{4.52}$$

because this is exactly how the sequence (z'_α) is generated. Since the markings $(z'_{\alpha n})_{n\geq 1}$ in Eqs. (4.48) and (4.50) are independent of $(u_{\alpha n})_{n\geq 1}$, the sequences (4.51) and (4.52) are again independent of each other. Finally, notice that the definition (4.49) implies that π is now an $\mathscr{F}_{p+1}$-measurable bijection of $\mathbb{N}^{p+1}$ into itself. This proves Eq. (4.46) for $p+1$ and the induction step is complete. Moreover, by Eq. (4.49), π preserves the parent-child relationship. This finishes the proof. □

From Eq. (4.45) we obtain the following. Let us recall the definition of the sequence (w_α) in Eq. (2.45) and, recalling Eq. (4.44), define $w'_\alpha = \prod_{\beta\in p(\alpha)} u'_\beta$. Then, for any leaf vertex $\alpha \in \mathbb{N}^r$, Eqs. (4.41) and (4.44) imply that $w'_\alpha = w_\alpha \exp(X_r(Z_\alpha) - X_0)$ and, given a permutation π as in Eq. (4.45), Theorem 4.3 implies that

$$\begin{aligned}\big(w_{\pi(\alpha)}\exp(X_r(Z_{\pi(\alpha)})-X_0)\big)_{\alpha\in\mathbb{N}^r} &\stackrel{d}{=} (w_\alpha)_{\alpha\in\mathbb{N}^r},\\ \big(z_{\pi(\alpha)}\big)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0} &\stackrel{d}{=} (z'_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0},\end{aligned} \tag{4.53}$$

and the two processes are independent of each other. Of course, here we used that, by the definition (4.39) and the fact that π preserves parent-child relationship,

$$Z_{\pi(\alpha)} = (z_\beta)_{\beta\in p(\pi(\alpha))} = (z_{\pi(\beta)})_{\beta\in p(\alpha)}.$$

Let us recall the definition of the sequence (v_α) in Eq. (2.46) and the definition of the Ruelle probability cascade, the random measure G in Eq. (2.62) defined by $G(h_\alpha) = v_\alpha$. By analogy with Eq. (2.68), let us define a random measure

$$G'(h_\alpha) = \frac{G(h_\alpha)\exp X_r(Z_\alpha)}{\sum_{\beta\in\mathbb{N}^r} G(h_\beta)\exp X_r(Z_\alpha)} \tag{4.54}$$

by the change of density proportional to $\exp X_r(Z_\alpha)$. Since X_0 is a constant, Eq. (4.53) implies the following generalization of Eq. (2.84).

Theorem 4.4. *There exists a random bijection $\pi : \mathscr{A} \to \mathscr{A}$ of the vertices of the tree $\mathscr{A}$, which preserves the parent-child relationship, such that*

$$\big(G'(h_{\pi(\alpha)})\big)_{\alpha\in\mathbb{N}^r} \stackrel{d}{=} \big(G(h_\alpha)\big)_{\alpha\in\mathbb{N}^r},\ (z_{\pi(\alpha)})_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0} \stackrel{d}{=} (z'_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0} \tag{4.55}$$

and these processes are independent of each other.

This means that, up to a random bijection, the change of density in Eq. (4.54) does not affect the distribution of the weights $G(h_\alpha)$ and, after the rearrangement, the random process (z_α) is generated recursively by the sequence of new transition functions Eq. (4.43). When we apply this result to Gibbs averages, the random bijection π disappears, since we sum over all indices $\alpha \in \mathbb{N}^r$. For example, given a bounded function $F = F(Z_\alpha)$, we can write

$$\begin{aligned}\mathbb{E}\sum_{\alpha\in\mathbb{N}^r} F(Z_\alpha)G'(h_\alpha) &= \mathbb{E}\sum_{\alpha\in\mathbb{N}^r} F(Z_{\pi(\alpha)})G'(h_{\pi(\alpha)})\\ &= \mathbb{E}\sum_{\alpha\in\mathbb{N}^r} F(Z'_\alpha)G(h_\alpha),\end{aligned}$$

and the case of multivariate functions of several replicas is similar. Of course, we have seen this type of argument in Lemma 2.3. Therefore, to conclude that Theorem 4.4 implies Eq. (4.36), we only need to observe that the change of density in Eq. (4.34) is of the same kind as Eq. (4.54). First, consider the case $n = 1$. If we recall Eq. (3.13) then

$$g_{\xi'}(h_\alpha) = \eta_0\xi'(0)^{1/2} + \sum_{\beta\in p(\alpha)} \eta_\beta\big(\xi'(q_{|\beta|}) - \xi'(q_{|\beta|-1})\big)^{1/2}, \tag{4.56}$$

where $\eta_0, (\eta_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ are i.i.d. standard Gaussian random variables. This means that, in this case, Eq. (4.34) corresponds to Eq. (4.54) with the choice of

$$X_r\big(\eta_0, (\eta_\beta)_{\beta\in p(\alpha)}\big) = \log \mathrm{ch}\, g_{\xi'}(h_\alpha). \tag{4.57}$$

If we fix η_0 and identify $(\eta_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ with the process $(z_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ in the above construction then we can apply Theorem 4.4 conditionally on η_0 to obtain Eq. (4.36) with

$$g'_{\xi'}(h_\alpha) = \eta_0 \xi'(0)^{1/2} + \sum_{\beta\in p(\alpha)} \eta'_\beta \big(\xi'(q_{|\beta|}) - \xi'(q_{|\beta|-1})\big)^{1/2}, \tag{4.58}$$

where the random variables $(\eta'_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ are generated recursively according to some new transition functions (4.43) that now also depend on η_0. Of course, these transition functions cannot be computed in a closed form because, from the very first step, the recursive construction (4.40) involves computing integrals of the type $\mathbb{E}\mathrm{ch}^\zeta(a+b\eta)$ for a standard Gaussian random variable η, but in principle, they can be computed. It should be obvious that, with the choice of the function X_r in Eq. (4.57), all the functions in the recursive construction (4.40) are well defined. In the general case, $n \geq 1$, Eq. (4.34) corresponds to Eq. (4.54) with the choice of

$$X_r = \sum_{i\leq n} \log \mathrm{ch}\, g_{\xi',i}(h_\alpha),$$

for independent copies $g_{\xi',i}$ of the process (4.58). Of course, in this case, each z_α can be identified with the vector $(\eta_{\alpha,i})_{i\leq n}$ of independent copies of η_α, and the space $\mathscr{X}$ on which z_α are defined is now $\mathbb{R}^n$. It is easy to check that independent copies of $(\eta_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ will become independent copies of $(\eta'_\alpha)_{\alpha\in\mathscr{A}\setminus\mathbb{N}^0}$ under the new transition functions (4.43), because, in this case, all the changes of density in Eq. (4.42) will be written as a product of densities in terms of different coordinates for $i \leq n$. One can see this by induction in the recursive construction (4.40), similarly to the computation in Eq. (2.59). This means that the processes $g'_{\xi',i}$ will be independent copies of Eq. (4.58) and, therefore, Eq. (4.36) still holds, which completes the description of the asymptotic spin distributions in the mixed p-spin models in terms of the Ruelle probability cascades.

A general conjecture in physics states that in many mean-field models, such as the random K-sat and random p-spin models, one should be able to approximate the asymptotic distributions of spins by generating the array of means $\bar{\sigma}_{i,l}$ exactly the same way as in Eq. (4.37); only the particular process $g'_{\xi'}(h_\alpha)$, which appears in the setting of the mixed p-spin models, is replaced by a more general process of the type $X_r(Z_\alpha)$, for general functions of $(Z_\alpha)_{\alpha\in\mathbb{N}^r}$ generated recursively as in Eq. (4.39) using some sequences of transition functions (4.38). In the mixed p-spin models, the invariance property (4.25), which allowed us to express the joint moments of all spins in terms of the distribution of the overlaps, was the consequence of the very special Gaussian nature of the Hamiltonian in these models. However, in other models, there is no such obvious relation between the spins and the overlaps and some new ideas are needed.

Appendix

A.1 The Aldous–Hoover Representation II

In this section, we will prove the Aldous–Hoover representation for exchangeable arrays stated in Theorem 1.4 using a different approach based on the ideas of Lovász and Szegedy in the framework of limits of dense graph sequences. For convenience, let us recall that an infinite random array $s = (s_{l,l'})_{l,l' \geq 1}$ is called an *exchangeable array* if for any permutations π and ρ of finitely many indices we have equality in distribution

$$\big(s_{\pi(l),\rho(l')}\big)_{l,l' \geq 1} \stackrel{d}{=} \big(s_{l,l'}\big)_{l,l' \geq 1}. \tag{A.1}$$

We will prove the following.

Theorem A.1 (Aldous–Hoover). *For any infinite exchangeable array* $(s_{l,l'})_{l,l' \geq 1}$,

$$\big(s_{l,l'}\big)_{l,l' \geq 1} \stackrel{d}{=} \big(\sigma(w, u_l, v_{l'}, x_{l,l'})\big)_{l,l' \geq 1} \tag{A.2}$$

for some measurable function $\sigma : [0,1]^4 \to \mathbb{R}$ *and i.i.d. random variables* w, (u_l), $(v_{l'})$, *and* $(x_{l,l'})$ *that have the uniform distribution on* $[0,1]$.

As was explained in Sect. 1.4, we only need to consider the case when the entries $s_{l,l'}$ take values in $[0,1]$. In fact, on first reading, one can even suppose that the entries take two values 0 and 1, since this case already illustrates all the main ideas, while simplifying the argument. Since two such arrays are equal in distribution if and only if all the joint moments of their entries are equal, we would like to show that for some function σ, for any $n \geq 1$ and any integer $k_{l,l'} \geq 0$,

$$\begin{aligned} \mathbb{E} \prod_{l,l' \leq n} s_{l,l'}^{k_{l,l'}} &= \mathbb{E} \prod_{l,l' \leq n} \sigma(w, u_l, v_{l'}, x_{l,l'})^{k_{l,l'}} \\ &= \mathbb{E} \prod_{l,l' \leq n} \sigma^{(k_{l,l'})}(w, u_l, v_{l'}), \end{aligned} \tag{A.3}$$

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where we introduced the notation

$$\sigma^{(k)}(w,u,v) = \int_0^1 \sigma(w,u,v,x)^k \, dx. \tag{A.4}$$

When $s_{l,l'} \in \{0,1\}$, it is enough to consider only $k = 0,1$. We will construct such function σ from the array $(s_{l,l'})$ itself, using the exchangeability condition (A.1) in the following way. We know that the joint distribution of the entries of $(s_{l,l'})$ is not affected by permutations and

$$\mathbb{E} \prod_{l,l' \le n} s_{l,l'}^{k_{l,l'}} = \mathbb{E} \prod_{l,l' \le n} s_{\pi(l),\rho(l')}^{k_{l,l'}}.$$

Then, this equality also holds for random permutations and we can take π and ρ to be independent uniform random permutations of $\{1,\dots,N\}$ for $N \ge n$ keeping all other indices fixed. When N is much larger than n, the random indices $\pi(l)$ and $\rho(l)$ for $l \le n$ behave almost like independent uniform random variables on $\{1,\dots,N\}$. More precisely, if we redefine $\pi(l)$ and $\rho(l)$ for $l \le n$ to be independent and uniform on $\{1,\dots,N\}$ then, conditionally on the event

$$\big\{\pi(l) \ne \pi(l'), \rho(l) \ne \rho(l') \text{ for } 1 \le l \ne l' \le n\big\},$$

they will have the same distribution as the first n coordinates of a random permutation. Since, for large N, this event has overwhelming probability,

$$\mathbb{E} \prod_{l,l' \le n} s_{l,l'}^{k_{l,l'}} = \lim_{N\to\infty} \mathbb{E} \prod_{l,l' \le n} s_{\pi(l),\rho(l')}^{k_{l,l'}}.$$

This already resembles Eq. (A.3) if we represent the random variables $\pi(l)$ and $\rho(l')$ as functions of i.i.d. random variables u_l and $v_{l'}$, uniform on $[0,1]$. If we generate the random array $s = s(w)$ as a function of a uniform random variable w on $[0,1]$ and define

$$\sigma_N(w,u,v) = s_{l,l'}(w) \text{ when } u \in \Big[\frac{l-1}{N}, \frac{l}{N}\Big), v \in \Big[\frac{l'-1}{N}, \frac{l'}{N}\Big) \tag{A.5}$$

for $l,l' \in \{1,\dots,N\}$, then we can rewrite the equation above as

$$\mathbb{E} \prod_{l,l' \le n} s_{l,l'}^{k_{l,l'}} = \lim_{N\to\infty} \mathbb{E} \prod_{l,l' \le n} \sigma_N(w,u_l,v_{l'})^{k_{l,l'}}. \tag{A.6}$$

Exchangeability had played its role. Now, a difficult question arises, how to extract a function σ from a (sub)sequence of functions σ_N, so that the limit (A.6) can be written as Eq. (A.3). The key idea is to think of $\sigma_N(w,\cdot,\cdot)$ as a (random) function on $[0,1]^2$ and to define a distance between such functions that is weak enough to allow us to find a converging subsequence but, at the same time, strong enough to imply the convergence of the joint moments in Eq. (A.6). Another subtle point is that the limiting function σ depends on one more coordinate $x \in [0,1]$, so we will, in fact,

be defining simultaneous convergence of the powers $\sigma_N(w,\cdot,\cdot)^k$ to some functions $\sigma^{(k)}(w,\cdot,\cdot)$ for all $k \geq 1$ that will then be related to one function σ via the Eq. (A.4). To motivate the definition of a distance, let us see how we can control the difference between the joint moments

$$\mathbb{E}\prod_{l,l'\leq n}\sigma_N(w,u_l,v_{l'})^{k_{l,l'}} - \mathbb{E}\prod_{l,l'\leq n}\sigma^{(k_{l,l'})}(w,u_l,v_{l'}) \tag{A.7}$$

in terms of some distance between the functions σ_N^k and $\sigma^{(k)}$. If we consider a pair $T=(T_1,T_2)$ of measurable functions $T_j : [0,1]^2 \to [0,1]$ such that for all $w \in [0,1]$,

$$T_1(w,\cdot), T_2(w,\cdot) : [0,1] \to [0,1]$$

are (Lebesgue) measure-preserving transformations, then, conditionally on w, the sequences $(T_1(w,u_l))_{l\geq 1}$, $(T_2(w,v_l))_{l\geq 1}$ are i.i.d. with the uniform distribution on $[0,1]$ and, therefore, the difference in Eq. (A.7) equals

$$\mathbb{E}\prod_{l,l'\leq n}\sigma_N(w,T_1(w,u_l),T_2(w,v_{l'}))^{k_{l,l'}} - \mathbb{E}\prod_{l,l'\leq n}\sigma^{(k_{l,l'})}(w,u_l,v_{l'}).$$

In other words, from the point of view of the joint moments, we have a freedom to redefine how the function σ_N depends on the uniform random variables u and v on $[0,1]$. For simplicity of notation, given a function $f : [0,1]^3 \to \mathbb{R}$, let us write

$$f_T(w,u,v) = f\big(w,T_1(w,u),T_2(w,v)\big). \tag{A.8}$$

The next idea is crucial. If we arrange all pairs (l,l') for $1 \leq l,l' \leq n$ in some linear order $\leq$ and denote

$$f_{l,l'} = \prod_{(j,j')<(l,l')}\sigma_{N,T}(w,u_j,v_{j'})^{k_{j,j'}}\prod_{(j,j')>(l,l')}\sigma^{(k_{j,j'})}(w,u_j,v_{j'})$$

then the above difference can be written as a telescopic sum:

$$\sum_{l,l'}\mathbb{E}\big(\sigma_{N,T}(w,u_l,v_{l'})^{k_{l,l'}} - \sigma^{(k_{l,l'})}(w,u_l,v_{l'})\big)f_{l,l'}. \tag{A.9}$$

Let us first integrate one term in u_l and $v_{l'}$, conditionally on all the other random variables. Notice that each factor in $f_{l,l'}$ depends only on u_l, or $v_{l'}$, or neither, but not on both, and all factors are bounded in absolute value by 1. Therefore, if we define a distance between two functions $\sigma_1,\sigma_2 : [0,1]^2 \to \mathbb{R}$ by

$$\|\sigma_1-\sigma_2\|_\square = \sup_{\|f\|_\infty,\|g\|_\infty\leq 1}\left|\iint_{[0,1]^2}\big(\sigma_1(u,v)-\sigma_2(u,v)\big)f(u)g(v)\,dudv\right|, \tag{A.10}$$

then the sum in Eq. (A.9) can be bounded in absolute value by

$$\sum_{l,l'\leq n}\mathbb{E}\big\|\sigma_{N,T}(w,\cdot,\cdot)^{k_{l,l'}} - \sigma^{(k_{l,l'})}(w,\cdot,\cdot)\big\|_\square. \tag{A.11}$$

Since the difference in Eq. (A.7) does not depend on T, we can minimize the bound (A.11) over T, which means that, in order to prove Eq. (A.3) and the Aldous–Hoover representation, it is enough to show the following.

Theorem A.2 (Lovász–Szegedy). *For any sequence of functions $\sigma_N : [0,1]^3 \to [0,1]$, there exists a function $\sigma : [0,1]^4 \to [0,1]$ such that*

$$\liminf_{N\to\infty} \inf_T \sum_{k\geq 1} 2^{-k} \mathbb{E} \left\| \sigma_{N,T}(w,\cdot,\cdot)^k - \sigma^{(k)}(w,\cdot,\cdot) \right\|_\square = 0, \tag{A.12}$$

where the functions $\sigma^{(k)}$ are defined in Eq. (A.4).

The distance $\|\cdot\|_\square$ is called the *cut norm* or *rectangle norm*. It also equals to

$$\|\sigma_1 - \sigma_2\|_\square = \sup_{A_1,A_2\subseteq[0,1]} \left| \iint_{A_1\times A_2} \big(\sigma_1(u,v) - \sigma_2(u,v)\big)\, du dv \right|, \tag{A.13}$$

which is easy to see by approximating the functions $f(u)$, $g(v)$ in Eq. (A.10) by the symmetric convex combinations of indicators of measurable sets in $[0,1]$. The distance (A.13) possesses good properties that will allow us to extract a limiting object σ along some subsequence of (σ_N) in the sense of Eq. (A.12). Its main property will be expressed in the regularity lemma below which will be based on the following.

Lemma A.1. *Let H be a Hilbert space and let $(H_n)_{n\geq 1}$ be a sequence of arbitrary subsets $H_n \subseteq H$. For any $h \in H$ and $\varepsilon > 0$, there exist $n \leq 1+\varepsilon^{-2}$ and $h_i \in H_i, \lambda_i \in \mathbb{R}$ for $i \leq n$ such that for all $h' \in H_{n+1}$,*

$$\Big|\big\langle h', h - \sum_{i\leq n} \lambda_i h_i \big\rangle\Big| \leq \varepsilon \big\|h'\big\| \big\|h\big\|. \tag{A.14}$$

Proof. Given $h \in H$ and $n \geq 1$, let us define

$$\delta_n = \inf\Big\{ \|h - h^*\|^2 : h^* = \sum_{i\leq n} \lambda_i h_i, h_i \in H_i, \lambda_i \in \mathbb{R} \Big\}.$$

Since $\|h\|^2 \geq \delta_1 \geq \delta_2 \geq \ldots \geq 0$, this implies that

$$\delta_n < \delta_{n+1} + \varepsilon^2 \|h\|^2$$

for some $n \leq 1+\varepsilon^{-2}$, because, otherwise, for $\varepsilon^{-2} < n_0 \leq 1+\varepsilon^{-2}$,

$$\|h\|^2 \geq \delta_1 \geq \sum_{1\leq i\leq n_0} (\delta_i - \delta_{i+1}) \geq n_0 \varepsilon^2 \|h\|^2 > \|h\|^2.$$

For such n, let us choose $h^* = \sum_{i\leq n} \lambda_i h_i$ such that $\|h-h^*\|^2 \leq \delta_{n+1} + \varepsilon^2\|h\|^2$. Then, by the definition of δ_{n+1}, for any $h' \in H_{n+1}$ and $t \in \mathbb{R}$,

$$\|h - h^*\|^2 \leq \|h - h^* - th'\|^2 + \varepsilon^2 \|h\|^2,$$

and it is easy to check that this can hold only if Eq. (A.14) holds. □

Lemma A.1 applied to the Hilbert space $H = L^2([0,1]^2, dudv)$ will yield a very useful approximation property with respect to the norm $\|\cdot\|_\square$. We will say that $\mathscr{D}$ is a *finite product* σ*-algebra* on $[0,1]^2$ if there exist two finite measurable partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ of $[0,1]$ such that $\mathscr{D}$ is generated by the rectangles $P_1 \times P_2$ such that $P_1 \in \mathscr{P}_1$ and $P_2 \in \mathscr{P}_2$. Let us define the size

$$|\mathscr{D}| = \max(|\mathscr{P}_1|, |\mathscr{P}_2|) \tag{A.15}$$

of the partition $\mathscr{D}$ by the largest of the cardinalities of the partitions $\mathscr{P}_1$, $\mathscr{P}_2$. Given $h \in H$, let us denote by $h_\mathscr{D}$ the conditional expectation $\mathbb{E}(h|\mathscr{D})$ of h given $\mathscr{D}$ with respect to the Lebesgue measure,

$$h_\mathscr{D}(u,v) = \frac{1}{|P_1||P_2|} \iint_{P_1 \times P_2} h(x,y)\,dxdy, \tag{A.16}$$

for $(u,v) \in P_1 \times P_2$ whenever $|P_1||P_2| > 0$; otherwise, $h_\mathscr{D}(u,v) = 0$. The following lemma describes a property of the rectangle norm $\|\cdot\|_\square$ which will be the key to proving Eq. (A.12). Let $\|\cdot\|$ denote the L^2-norm on H.

Lemma A.2 (Regularity Lemma). *For any $\varepsilon > 0$ and any function $h \in H$, there exists a finite product σ-algebra $\mathscr{D}$ on $[0,1]^2$ such that*

$$\|h - h_\mathscr{D}\|_\square \le 2\varepsilon\|h\| \tag{A.17}$$

and the size $|\mathscr{D}| \le 2^{1+1/\varepsilon^2}$.

This means that any h can be approximated within $\varepsilon\|h\|$ in the cut norm by a simple step function $h_\mathscr{D}$ with the number of steps $|\mathscr{D}|$ controlled uniformly over all h. Of course, $\mathscr{D}$ itself can depend on h, but its size (A.15) is bounded independently of h. To appreciate this result, let us notice that we can, of course, approximate any $h \in H$ by a step function even in the stronger L^2-norm, but the number of steps, in general, will not be independent of h.

Proof (Lemma A.2). Given a rectangle $A = A_1 \times A_2 \subseteq [0,1]^2$, let us denote by $\mathscr{D}(A)$ a finite product σ-algebra generated by the rectangles $A_1 \times [0,1]$ and $[0,1] \times A_2$. Let

$$H_1 = \big\{h \in H : h \text{ is } \mathscr{D}(A)\text{-measurable for some rectangle } A\big\}.$$

Here, the choice of a rectangle A can depend on h. Let $H_n := H_1$ for all $n \ge 2$. By Lemma A.1, for any $h \in H$, we can find $n \le 1 + \varepsilon^{-2}$ and a linear combination

$$h^* = \sum_{i \le n} \lambda_i h_i$$

with some $h_i \in H_i, \lambda_i \in \mathbb{R}$ for $i \le n$, such that for all $h' \in H_{n+1}$,

$$\big|\langle h', h - h^*\rangle\big| \le \varepsilon\|h'\|\|h\|.$$

Since, for any rectangle $A \subseteq [0,1]^2$, its indicator $h' = I_A$ belongs to $H_{n+1} = H_1$, the definition (A.13) implies that

$$\|h - h^*\|_\square = \sup_A \left|\langle I_A, h - h^* \rangle\right| \leq \varepsilon \|h\|.$$

If $A^i = A^i_1 \times A^i_2$ is the rectangle corresponding to $h_i \in H_i = H_1$, then the linear combination h^* is, obviously, measurable with respect to the σ-algebra $\mathscr{D}$ generated by $\{A^i : i \leq n\}$. If for $j = 1,2$, $\mathscr{P}_j$ is the partition of $[0,1]$ generated by the sets $(A^i_j)_{i \leq n}$ then $\mathscr{D}$ is obviously a finite product σ-algebra generated by the partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ and its size $|\mathscr{D}|$, defined in Eq. (A.15), is bounded by 2^n. Let us now consider two functions $f = f(u)$ and $g = g(v)$ such that $\|f\|_\infty, \|g\|_\infty \leq 1$. Since Eq. (A.16) implies that $(fg)_\mathscr{D} = f_\mathscr{D} g_\mathscr{D}$ and since both $h_\mathscr{D}$ and h^* are $\mathscr{D}$-measurable,

$$\begin{aligned}\left|\iint (h^* - h_\mathscr{D}) f g \, du dv\right| &= \left|\iint (h^* - h_\mathscr{D}) f_\mathscr{D} g_\mathscr{D} \, du dv\right| \\ &= \left|\iint (h^* - h) f_\mathscr{D} g_\mathscr{D} \, du dv\right| \leq \|h^* - h\|_\square \leq \varepsilon \|h\|.\end{aligned}$$

Therefore, by Eq. (A.10), $\|h^* - h_\mathscr{D}\|_\square \leq \varepsilon \|h\|$ and $\|h - h_\mathscr{D}\|_\square \leq 2\varepsilon \|h\|$. □

It is worth writing down a simple observation at the end of the above proof as a separate statement.

Lemma A.3. *If h^* is measurable with respect to some finite product σ-algebra $\mathscr{D}$ then $\|h - h_\mathscr{D}\|_\square \leq 2\|h - h^*\|_\square$.*

Suppose that, using the regularity lemma, we found a finite product σ-algebra $\mathscr{D}$ such that $\|h - h_\mathscr{D}\|_\square \leq \varepsilon$ and $\mathscr{D}$ is generated by some finite partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ of $[0,1]$. Since in Eq. (A.12) we have the freedom of applying a measure-preserving transformation to the coordinates u and v, before we proceed to prove Theorem A.2, let us first observe that we can "transfer" measurable sets in the partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ into proper intervals by some measure-preserving transformation without losing control of $\|h - h_\mathscr{D}\|_\square$.

Lemma A.4. *Suppose we are given two partitions of $[0,1]$ into measurable sets $\{P_1, \ldots, P_k\}$ and intervals $\{I_1, \ldots, I_k\}$ such that their measures $|P_j| = |I_j|$ for $j \leq k$. Then, there exists a measure-preserving transformation T such that $T(I_j) \subseteq P_j$ for $j \leq k$ and T is one-to-one outside a set of measure zero.*

Proof. Since this is a very basic exercise in probability, we will only give a brief sketch. For each $j \leq k$, the restriction of the uniform distribution on $[0,1]$ to P_j can be realized by the quantile transformation on the interval I_j of the same measure. Since this restriction is a continuous measure, the quantile will be one-to-one. It is not necessarily into P_j, but the measure of points mapped outside of P_j will, obviously, be zero and we can redefine the transformation to map them into a fixed point in P_j. Define T on I_j by this redefined quantile transformation. □

Let T_1, T_2 be two such measure-preserving transformations that transfer the partitions $\mathscr{P}_1, \mathscr{P}_2$ that generate $\mathscr{D}$ in Lemma A.2 into interval partitions $\mathscr{I}_1, \mathscr{I}_2$. Let $\mathscr{F}$ be the finite product σ-algebra generated by $\mathscr{I}_1$ and $\mathscr{I}_2$. Similarly to Eq. (A.8), for a function $f : [0,1]^2 \to \mathbb{R}$ let us define

$$f_T(u,v) = f(T_1(u), T_2(v)). \tag{A.18}$$

Notice that $h_{\mathscr{D},T} := (h_{\mathscr{D}})_T$ is now measurable with respect to $\mathscr{F}$, since its value on the rectangle $I_1 \times I_2$ is given by the value of $h_{\mathscr{D}}$ on the rectangle $P_1 \times P_2$ such that $T_1(I_1) \subseteq P_1$ and $T_2(I_2) \subseteq P_2$, i.e.

$$h_{\mathscr{D},T}(u,v) = \frac{1}{|P_1||P_2|} \iint_{P_1 \times P_2} h(x,y)\,dxdy$$

for $(u,v) \in I_1 \times I_2$. This means that $h_{\mathscr{D},T}$ is a step function with steps given by "geometric" and not just measurable rectangles, which will be helpful in the proof of Theorem A.2 below. On the other hand, $h_{\mathscr{D},T}$ approximates h_T just as well as $h_{\mathscr{D}}$ approximated h.

Lemma A.5. *If the measure-preserving transformations T_1 and T_2 are one-to-one outside a set of measure zero then*

$$\|h_T - f_T\|_\square = \|h - f\|_\square. \tag{A.19}$$

Proof. By the definition (A.18), for any measurable sets $A_1, A_2 \subseteq [0,1]$,

$$\iint\limits_{A_1 \times A_2} (h_T - f_T)\,dudv = \iint\limits_{A_1 \times A_2} \big(h(T_1(u), T_2(v)) - f(T_1(u), T_2(v))\big)\,dudv.$$

Since the map T_1 is one-to-one outside a set of measure zero, the sets $\{u \in A_1\}$ and $\{T_1(u) \in T_1(A_1)\}$ differ by a set of measure zero and in the above integral we can replace A_1 by $T_1^{-1}(T_1(A_1))$ and, similarly, A_2 by $T_2^{-1}(T_2(A_2))$. Then, making the change of variables, we get

$$\iint\limits_{A_1 \times A_2} (h_T - f_T)\,dudv = \iint\limits_{T_1(A_1) \times T_2(A_2)} \big(h(u,v) - f(u,v)\big)\,dudv.$$

By the definition (A.13), this implies $\|h_T - f_T\|_\square \le \|h - f\|_\square$. To obtain the opposite inequality in Eq. (A.19), we start with the sets $T_1^{-1}(A_1)$ and $T_2^{-1}(A_2)$ to get

$$\iint\limits_{T_1^{-1}(A_1) \times T_2^{-1}(A_2)} (h_T - f_T)\,dudv = \iint\limits_{A_1 \times A_2} \big(h(u,v) - f(u,v)\big)\,dudv.$$

By the definition (A.13), this implies $\|h_T - f_T\|_\square \ge \|h - f\|_\square$. □

On first reading, one can assume that functions σ, σ_N in the proof below take values in $\{0,1\}$, in which case we only need to consider $k = 1$.

Proof (Theorem A.2). First, consider a function $\sigma : [0,1]^2 \to [0,1]$. Using the regularity Lemma A.2, for each $m \geq 1$ and each $k \leq m$, let us find a finite product σ-algebra $\mathscr{D}_{mk}$ such that

$$\left\|\sigma^k - (\sigma^k)_{\mathscr{D}_{mk}}\right\|_{\square} \leq \frac{1}{2m},$$

where σ^k is the kth power of σ. For each $m \geq 1$, let $\mathscr{D}_m = \vee_{n \leq m} \vee_{k \leq n} \mathscr{D}_{nk}$. This definition implies that

$$\mathscr{D}_1 \subseteq \mathscr{D}_2 \subseteq \ldots \subseteq \mathscr{D}_m \subseteq \ldots \tag{A.20}$$

and, since $\mathscr{D}_m$ is a refinement of $\mathscr{D}_{mk}$ for $k \leq m$, Lemma A.3 implies that

$$\left\|\sigma^k - (\sigma^k)_{\mathscr{D}_m}\right\|_{\square} \leq \frac{1}{m} \text{ for } k \leq m.$$

Finally, for each $m \geq 1$, let us utilize Lemma A.4 and find a pair of measure-preserving transformations $T_m = (T_{m1}, T_{m2})$ that transfer $\mathscr{D}_m$ into a finite product σ-algebra $\mathscr{F}_m$ generated by some partitions into intervals $\mathscr{I}_1^m$ and $\mathscr{I}_2^m$. By Eq. (A.20), partitions generating $(\mathscr{D}_m)$ are being refined as m increases and when we define (T_m) that transfer sets in these partitions into intervals we can define (T_m) recursively in a way that preserves the inclusion relation. In other words, if $\mathscr{D}_m$ is generated by the partitions $\mathscr{P}_1^m$ and $\mathscr{P}_2^m$ and if an element, say $P \in \mathscr{P}_1^m$, is broken into a disjoint union of sets at the next step $m+1$,

$$P = P_1 \cup \ldots \cup P_r \text{ for some } P_1, \ldots, P_r \in \mathscr{P}_1^{m+1},$$

then, if P is transferred into an interval I by the transformation T_{m1}, $T_{m1}(I) \subseteq P$, then $P_1, \ldots, P_r$ are transferred into some intervals $I_1, \ldots, I_r$ by the transformation $T_{(m+1)1}$ such that I is the disjoint union of $I_1, \ldots, I_r$. The inclusion structure of the partitions generating Eq. (A.20) should be preserved after we transfer them into intervals. In particular,

$$\mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \ldots \subseteq \mathscr{F}_m \subseteq \ldots . \tag{A.21}$$

Let us recall that each function $(\sigma^k)_{\mathscr{D}_m, T_m}$ is now measurable with respect to $\mathscr{F}_m$ and, by Eq. (A.19), we will have

$$\left\|(\sigma^k)_{T_m} - (\sigma^k)_{\mathscr{D}_m, T_m}\right\|_{\square} \leq \frac{1}{m} \text{ for } k \leq m.$$

Notice that, for each $k \geq 1$, the sequence $((\sigma^k)_{\mathscr{D}_m}, \mathscr{D}_m)_{m \geq 1}$ is a martingale and, since the transformations (T_m) preserve the inclusion structure and the measure of the elements of the partitions, the sequence $((\sigma^k)_{\mathscr{D}_m, T_m}, \mathscr{F}_m)_{m \geq 1}$ is also a martingale. To simplify the notation, let us now denote

$$\sigma_{mk} = (\sigma^k)_{\mathscr{D}_m, T_m}. \tag{A.22}$$

To summarize, given a function $\sigma : [0,1]^2 \to [0,1]$, we constructed a sequence of pairs of measure-preserving transformations (T_m), a sequence $(\mathscr{F}_m)$ of finite product σ-algebras generated by partitions into intervals and, for each $k \geq 1$, a martingale sequence $(\sigma_{mk}, \mathscr{F}_m)_{m \geq 1}$ such that

$$\left\| (\sigma^k)_{T_m} - \sigma_{mk} \right\|_{\square} \leq \frac{1}{m} \text{ for } k \leq m. \tag{A.23}$$

In particular, this implies that

$$\inf_T \sum_{1 \leq k \leq m} 2^{-k} \left\| (\sigma^k)_T - \sigma_{mk} \right\|_{\square} \leq \frac{1}{m}. \tag{A.24}$$

Another important feature of the sequence (σ_{mk}) is that, for any fixed m and any $u, v \in [0,1]$, the sequence $\sigma_{mk}(u,v)$ for $k \geq 1$ is a sequence of moments of some probability distribution on $[0,1]$. This is because for any $(u,v) \in P_1 \times P_2$ in some measurable rectangle in $\mathscr{D}_m$,

$$(\sigma^k)_{\mathscr{D}_m}(u,v) = \frac{1}{|P_1||P_2|} \iint_{P_1 \times P_2} \sigma^k(x,y)\, dxdy \tag{A.25}$$

is the sequence of moments of σ viewed as a random variable on $P_1 \times P_2$ with the uniform distribution, and σ_{mk} are obtained from $(\sigma^k)_{\mathscr{D}_m}$ for all $k \geq 1$ by the same change of variables (A.22). Let us observe a simple fact that will be used below that, if for each $N \geq 1$, $(\mu_k^N)_{k \geq 0}$ is a sequence of moments of some probability distribution on $[0,1]$ and it converges to $(\mu_k)_{k \geq 0}$ as $N \to \infty$ then, by the selection theorem, the limiting sequence is also a sequence of moments of some probability distribution on $[0,1]$. Finally, we come to the main argument. Given a function $\sigma_N : [0,1]^3 \to [0,1]$, for each fixed $w \in [0,1]$, we can think of $\sigma_N(w,\cdot,\cdot)$ as a function on $[0,1]^2$ and, as above, find a sequence of functions $\sigma_{N,mk}(w,\cdot,\cdot)$ such that

$$\inf_T \sum_{1 \leq k \leq m} 2^{-k} \left\| \sigma_{N,T}^k(w,\cdot,\cdot) - \sigma_{N,mk}(w,\cdot,\cdot) \right\|_{\square} \leq \frac{1}{m}. \tag{A.26}$$

Of course, the sequence $(\mathscr{F}_{N,m}(w))$ of corresponding finite product σ-algebras will now depend on w but, otherwise, $(\sigma_{N,mk}(w,\cdot,\cdot))$ will satisfy the same properties as (σ_{mk}) for each $N \geq 1$ and all $w \in [0,1]$. (Remark: There is a minor issue here of whether the functions $\sigma_{N,mk}$ are jointly measurable. However, this issue can be avoided if we first approximate σ_N arbitrarily well in the L^2-norm by a step function on $[0,1]^3$ and then apply the above construction to this discretized approximation.) Even though the sequence $(\mathscr{F}_{N,m}(w))$ depends on w, the size of each σ-algebra $\mathscr{F}_{N,m}(w)$, in the sense of the definition (A.15), is bounded by a function of m only, which was the key point of the regularity Lemma A.2. Therefore, the σ-algebra $\mathscr{F}_{N,m}(w)$ can be described by a finite collection of random variables (functions of w) that encode the number of intervals in the partitions $\mathscr{I}_1, \mathscr{I}_2$ generating this finite product σ-algebra and the Lebesgue measures of all these intervals. The functions $\sigma_{N,mk}(w,\cdot,\cdot)$ can be similarly described by random variables given by their values on the rectangles that generate $\mathscr{F}_{N,m}(w)$. We can find a subsequence (N_j) along

which all these random variables, for all $k, m \geq 1$, converge almost surely and, since for each $m \geq 1$ the number of intervals in the partitions stays bounded, the limiting random variables can be used in a natural way to define a sequence of finite product σ-algebras $(\mathscr{F}_m(w))$ and functions $(\sigma_{mk}(w,\cdot,\cdot))$. It is clear from this construction that, for all $m, k \geq 1$,

$$\lim_{j\to\infty} \left\| \sigma_{N_j,mk}(w,\cdot,\cdot) - \sigma_{mk}(w,\cdot,\cdot) \right\|_{\square} = 0$$

almost surely (in fact, here we could use a stronger L^2-norm) and Eq. (A.26) implies that

$$\limsup_{j\to\infty} \inf_{T} \sum_{1\leq k\leq m} 2^{-k} \left\| \sigma^k_{N_j,T}(w,\cdot,\cdot) - \sigma_{mk}(w,\cdot,\cdot) \right\|_{\square} \leq \frac{1}{m} \tag{A.27}$$

almost surely. It is also clear from the construction that the limiting sequence σ_{mk} will satisfy the same properties as $\sigma_{N,mk}$ for almost all w. First of all, for each $k \geq 1$, the sequence

$$\left(\sigma_{mk}(w,\cdot,\cdot), \mathscr{F}_m(w)\right)_{m\geq 1}$$

is again a martingale, since, in the limit, a value of $\sigma_{mk}(w,\cdot,\cdot)$ on any rectangle in the σ-algebra $\mathscr{F}_m(w)$ will still be obtained by averaging the values of $\sigma_{(m+1)k}(w,\cdot,\cdot)$ over sub-rectangles from the σ-algebra $\mathscr{F}_{m+1}(w)$. Moreover, for each $m \geq 1$ and almost all $w, u, v \in [0,1]$, the sequence $(\sigma_{mk}(w,u,v))_{k\geq 1}$ is a sequence of moments of a probability distribution on $[0,1]$, since such sequences are closed under taking limits, as we mentioned above. By the martingale convergence theorem, for all such w, the limits

$$\sigma^{(k)}(w,\cdot,\cdot) = \lim_{m\to\infty} \sigma_{mk}(w,\cdot,\cdot) \tag{A.28}$$

exist almost surely and in $L^2([0,1]^2, dudv)$ for all $k \geq 1$ and, therefore,

$$\lim_{m\to\infty} \sum_{1\leq k\leq m} 2^{-k} \left\| \sigma_{mk}(w,\cdot,\cdot) - \sigma^{(k)}(w,\cdot,\cdot) \right\|_{\square} = 0.$$

Combining with Eq. (A.27), we obtain that

$$\lim_{j\to\infty} \inf_{T} \sum_{k\geq 1} 2^{-k} \| \sigma^k_{N_j,T}(w,\cdot,\cdot) - \sigma^{(k)}(w,\cdot,\cdot) \|_{\square} = 0. \tag{A.29}$$

This holds almost surely and, hence, on average over w. Finally, by Eq. (A.28), for almost all (w,u,v), the sequence $(\sigma^{(k)}(w,u,v))_{k\geq 1}$ is again a sequence of moments of a probability distribution on $[0,1]$, and we can redefine these limits on the set of measure zero to have this property for all (w,u,v). This precisely means that there exists a function $\sigma : [0,1]^4 \to [0,1]$ such that Eq. (A.4) holds, which finishes the proof. (Remark: Again, there is a minor issue of measurability of the function σ on $[0,1]^4$. First of all, all moments $\sigma^{(k)}$ are measurable on $[0,1]^3$ as the limits of measurable functions, and there is a standard way to approximate a distribution

on $[0,1]$ and its quantile in terms of finitely many moments at a time (see, e.g., Hausdorff moment problem in Sect. 7.3 in [24]). It is easy to check that, using such approximation procedure, one can define σ in terms of the sequence $\sigma^{(k)}$ in a measurable way.) □

Bibliography

Notes and Comments

Chapter 1. As we mentioned in the preface, the Sherrington–Kirkpatrick model was introduced by Sherrington and Kirkpatrick [58] in 1975 and the formula for the free energy was invented in a celebrated work of Parisi [51, 52].

The concentration (self-averaging) of the free energy type functionals in Theorem 1.2 is just one example of a general Gaussian concentration phenomenon (see, e.g., [33]). The concentration of the free energy in the SK model given in Eq. (1.42) was first observed by Pastur and Shcherbina in [55]. Guerra and Toninelli [26] proved the existence of the limit of the free energy in the Sherrington–Kirkpatrick model in Theorem 1.1. Aizenman et al. [2] invented the scheme leading to the representation (1.58) in Sect. 1.3. The proof of Theorem 1.3 is from Panchenko [48], but it was shown to me earlier by Talagrand. The Aldous–Hoover representation in Sect. 1.4 was proved by Aldous [3] and Hoover [29] (see also [4, 30]). A very nice modern overview of the topic of exchangeability can be found in Austin [7], and the proof of the Aldous–Hoover representation for weakly exchangeable arrays in Sect. 1.4 essentially follows [7]. Dovbysh and Sudakov [22] proved the representation result for Gram-de Finetti arrays that appears in Sect. 1.5 and another proof can be found in Panchenko [44], where some of the ideas are taken from Hestir [28]. The short and elegant proof we give here was discovered by Tim Austin in [8]. The relevance of these representation results in the analysis of the Sherrington-Kirkpatrick model was first brought to light by Arguin and Aizenman [5].

Chapter 2. The Ruelle probability cascades were introduced by Ruelle in [56] and, as the title of [56] suggests, their definition was motivated by Derrida's random energy model (REM) [16, 17] and the generalized random energy model (GREM) [18, 19], which are two simplified mean-field models of spin glasses. As we mentioned in the preface, the random measures in Sect. 2.2 describe to the asymptotic Gibbs measures of the REM (this is explained very well in Sect. 1.2 in [62]) and the random measures in Sect. 2.3 (the Ruelle probability cascades) describe to the asymptotic Gibbs measures of the GREM, which was rigorously proved by Bovier

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and Kurkova [11]. An overview of the Poisson processes in Sect. 2.1 follows the classical text by Kingman [32]. Theorem 2.6, which plays a key role in the proof of all the invariance properties in this chapter, as well as Theorem 2.12, was proved in the seminal work of Bolthausen and Sznitman [10]. Talagrand [62] first discovered the identities (2.39), which express the Ghirlanda–Guerra identities in the setting of the Poisson–Dirichlet processes. Following a similar idea, the Ghirlanda–Guerra identities in the setting of the Ruelle probability cascades were proved by Bovier and Kurkova [11]. The current approach to both results in Theorems 2.8 and 2.10 is due to Talagrand [66]. The material in Sect. 2.4—the result in Theorem 2.13 that the Ghirlanda–Guerra identities and ultrametricity together uniquely determine the distribution of the overlap array—was well known following the proof of the Ghirlanda–Guerra identities in [25] and can be found, for example, in Baffioni and Rosati [9], Bovier and Kurkova [11], or Talagrand [66]. Theorem 2.15 is from Panchenko [43]. The fact that the Ghirlanda–Guerra identities imply ultrametricity of the overlaps, Theorem 2.14, was proved in Panchenko [50]. A partial result, which jump-started most of the research in this direction, was proved in a paper of Arguin and Aizenman [5], where, instead of the Ghirlanda–Guerra identities, the authors used the stochastic stability property discovered by Aizenman and Contucci [1]. Several other partial results, based on the Ghirlanda–Guerra identities, can be found in Panchenko [43, 47], Parisi and Talagrand [54] and Talagrand [65].

Chapter 3. The Parisi formula in Theorem 3.1 was first proved in the setting of the mixed even p-spin models in the famous work of Talagrand [64], following the breakthrough invention of the replica symmetry breaking interpolation by Guerra in [27]. The proof of the general case presented in Chap. 3, which also covers the case of odd p-spin interactions, follows the argument in Panchenko [48]; however, besides the main result in [50], all other ideas of the proof were well known to many people; see, e.g., Arguin and Chatterjee [6]. Ghirlanda and Guerra [25] proved the fundamental identities described in Sect. 3.2. Even though we do not use it here, a related property, called the stochastic stability, was discovered by Aizenman and Contucci around the same time in [1] and played an important role in the development of the area, for example, in the work of Arguin and Aizenman [5] that we mentioned above. Various proofs of the stochastic stability can be found in Contucci and Giardina [14] and Talagrand [65]. A stability property which unifies the Ghirlanda–Guerra identities and Aizenman–Contucci stochastic stability was proved in Panchenko [49] and, in some sense, it is an analogue of the Bolthausen–Sznitman invariance property (2.84) in the setting of the Ruelle probability cascades. The positivity principle, which appears in Theorems 2.16 under the exact Ghirlanda–Guerra identities and in Theorem 3.4 under the approximate Ghirlanda–Guerra identities, was originally proved by Talagrand in [62]. The current proof is new and much simplified, and it was also pointed out to me by Talagrand. As we mentioned above, the replica symmetry breaking interpolation in Sect. 3.4 was first invented by Guerra in [27], but the original proof did not utilize the Ruelle probability cascades. The general idea of the current proof in terms of the Ruelle probability cascades, which allows to simplify the computations, appears in Aizenman et al. [2]. The role of the positivity principle in the Guerra replica

symmetry breaking interpolation, in the case when odd p-spin terms are present, was observed by Talagrand in [60], and the presentation here follows Panchenko [41]. We already mentioned that the cavity computation scheme in Sect. 3.5 was discovered by Aizenman, Sims, and Starr in [2]. The differentiability of the Parisi formula in Theorem 3.7 was proved by Talagrand in [61, 63], and the proof we give here is a simplification of the argument in Panchenko [42]. Strong version of the Ghirlanda–Guerra identities for the mixed p-spin models in Theorem 3.8 is from Panchenko [45]. Arguin and Chatterjee [6] proved a similar strong version of the Aizenman–Contucci stochastic stability. An application of the strong version of the Ghirlanda–Guerra identities to the chaos problem can be found in Chen and Panchenko [13]. Carmona and Hu [12] proved universality in the disorder result in Sect. 3.8, generalizing an earlier result of Talagrand [59] in the case of the Bernoulli disorder.

Chapter 4. This chapter covers about half of the material in Panchenko [46]. The other half, which concerns diluted models, is not considered here and is the main motivation to study the asymptotic distributions of all spins. In [46] one can also find the details of how the generalized Parisi ansatz is related to the computation of the free energy in some family of diluted models.

Appendix. The proof of the Aldous–Hoover representation for exchangeable arrays presented in Sect. A.1 follows the ideas of Lovász and Szegedy [34, 35, 36] in the framework of limits of dense graph sequences. A connection between the Aldous–Hoover representation and the Lovász–Szegedy representation was first explained by Diaconis and Janson in [21].

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