Group structure of PCEs

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For a PCE quantum channel If $\tau_{pq} = \tau_{p'q'} = 1$, with $p, q \neq p', q'$, then $\tau_{p+p',q+q'} = 1$. Our proof goes as follows. With the expressions for τ_{pq} in terms of $\lambda_{\mu\nu}$ and the sum of all $\lambda_{\mu\nu}$ we derive another expression that must hold for a particular set of indices μ, ν. With this, the algebraic property is computed straightforwardly. This very important property reveals the semigroup structure of PCE quantum channels.

We will begin from the expression for the Choi's eigenvalues $\lambda_{\mu\nu}$ with binary indices, that is $\mu, \nu \in \{0, 1\}$. This form of the eigenvalues read

$$\lambda_{\mu\nu} = \frac{1}{2} \sum_{m,n=0}^{1} (-1)^{\mu m + \nu n} \tau_{mn} \pmod 2, \tag{1}$$

where sums are modulo 2. From here on we will use implicitly the modulo 2 arithmetic and drop the (mod 2) notation. To invert this expression let us multiply each side by $(-1)^{-\mu p - \nu q}$ and sum over μ and ν,

$$\sum_{\mu,\nu=0}^{1} (-1)^{-\mu p - \nu q} \lambda_{\mu\nu} = \frac{1}{2} \sum_{m,n,\mu,\nu=0}^{1} (-1)^{\mu(m-p)m + \nu(n-q)} \tau_{mn}. \tag{2}$$

We notice that

$$\sum_{j=0}^{1} (-1)^{j(k-l)} = 1 + (-1)^{k-l} = 2\delta_{kl} \tag{3}$$

because $k - l = 0$ when $k = l$ and $k - l = 1$ when $k \neq l$. There is no other possible value because we are using modulo 2 arithmetic. Using this property, we resume to eq. (2),

$$\sum_{\mu,\nu=0}^{1} (-1)^{-\mu p - \nu q} \lambda_{\mu\nu} = \frac{1}{2} \sum_{m,n,\mu,\nu=0}^{1} (2\delta_{mp})(2\delta_{nq}) \tau_{mn} \tag{4a}$$

$$\tau_{pq} = \frac{1}{2} \sum_{\mu,\nu=0}^{1} (-1)^{-\mu p - \nu q} \lambda_{\mu\nu}. \tag{4b}$$

Now, we subtract $\sum_{\mu,\nu} \lambda_{\mu\nu} = 2$ from $2 = \sum_{\mu,\nu} (-1)^{-\mu p - \nu q} \lambda_{\mu\nu}$ (setting $\tau_{pq} = 1$ in (4b)),

$$\sum_{\mu,\nu=0}^{1} \lambda_{\mu\nu} \left[(-1)^{-\mu p - \nu q} - 1 \right] = 0. \tag{5}$$

Recall that $\lambda_{\mu\nu} \geq 0$ for the corresponding map to be completely positive, and $(-1)^{-\mu p - \nu q} - 1 \leq 0$ because we are summing modulo 2, thus $-\mu p - \nu q = 0, 1$. These being said, now it must be evident that every term in left side of (5) has to be zero for the expression to hold. Therefore, if $\lambda_{\mu\nu} \neq 0$, then

$$(-1)^{-\mu p} = (-1)^{\nu q}, \qquad \forall\, \mu, \nu | \lambda_{\mu\nu} \neq 0. \tag{6}$$

The same expression holds for indices p', q'. Now we use this expression to evaluate

$$
\begin{aligned}
\tau_{m+m',n+n'} &= \frac{1}{2} \sum_{\mu,\nu=0}^{1} (-1)^{-\mu(p+p')-\nu(q+q')} \lambda_{\mu\nu} && \text{(7a)} \\
&= \frac{1}{2} \sum_{\mu,\nu|\lambda_{\mu\nu}\neq 0} (-1)^{\nu(q+q')-\nu(q+q')} \lambda_{\mu\nu} && \text{(7b)} \\
&= \frac{1}{2} \sum_{\mu,\nu|\lambda_{\mu\nu}\neq 0} \lambda_{\mu\nu} && \text{(7c)} \\
&= 1, && \text{(7d)}
\end{aligned}
$$

where (6) has been used in (7b) and the fact that Choi's eigenvalues sum 2 in (7c). □