Quantum map for LMG

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First, let's see an equivalence between 3 different basis for the same Hilbert space

* A single spin

- Spin operators $\{\hat{J}_x, \hat{J}_y, \hat{J}_z\}$ are the generators of SU(2) (Lie groups)

$$\boxed{[\hat{J}_i, \hat{J}_k] = i \sum_{l} \epsilon_{ikl} \hat{J}_l}$$

- Eigenvalue equation

$$\hat{J}_z |J, m\rangle = m |J, m\rangle \quad ; \quad -J \leq m \leq J \quad \Rightarrow \quad \dim(\mathcal{H}) = 2J+1$$

* A qudit

- Define the dimension of a single qudit $d = 2J+1$

* Symmetric States

- Consider a system of N qubits with angular momentum $s = 1/2$
- The Pauli matrices fulfill $[\hat{\sigma}_i, \hat{\sigma}_k] = \sum_{l} \epsilon_{ikl} \hat{\sigma}_l$
- The state of a qubit is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$; $|\psi\rangle \in \mathcal{H}$; $\dim(\mathcal{H}) = 2^1$
- Now, lets consider the following states in the Hilbert space of N qubits

$$\dim(\mathcal{H}^N) = 2^N$$

The first n qubits in the $|1\rangle$ state and the rest in the $|0\rangle$ states

$$\Rightarrow \quad |1_1 1_2 1_3 \ldots 1_n 0_{n+1} 0_{n+2} \cdots 0_N\rangle \in \mathcal{H}^N \quad ; \quad n \leq N$$

- For a fixed n, there are $p := \binom{N}{n}$ distinct permutations of the 1's but leave the total of 1's invariant
- Define the permutation invariant state $|N, n\rangle \equiv \frac{1}{\sqrt{p}} \sum_{k=1}^{p} \hat{P}_k |1, \ldots, 1_n, 0_{n+1}, \ldots 0_N\rangle$
- These states are called Dicke States

$N=3$

$N+1$
$$\begin{cases} |3,0\rangle = |000\rangle \\ |3,1\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right) \\ |3,2\rangle = \frac{1}{\sqrt{3}}\left(|110\rangle + |011\rangle + |101\rangle\right) \\ |3,3\rangle = |111\rangle \end{cases}$$

- For N qubits, the Hilbert space dimension is $\dim(\mathcal{H}^N) = 2^N$, but the subspace of permutational invariant states is $\dim(\mathcal{H}^N_{\text{Dicke}}) = N+1$

- Define the collective angular momentum as

$$\tilde{J}_i \equiv \tfrac{1}{2}\sum_{k=1}^{N} \hat{\sigma}_i^{(k)} \qquad N=3 \Rightarrow \tilde{J}_x = \tfrac{1}{2}\left(\hat{\sigma}_x^{(1)}\otimes I\otimes I + I\otimes\hat{\sigma}_x^{(2)}\otimes I + I\otimes I\otimes\hat{\sigma}_x^{(3)}\right)$$

- They obey the same commutation relation as the Pauli matrices

$$[\tilde{J}_i, \tilde{J}_k] = \sum_{\ell} \epsilon_{ik\ell}\,\tilde{J}_\ell$$

* As both basis span the same Hilbert space (dimension)

$$\tilde{J}_z|N,n\rangle = (N/2 - n)|N,n\rangle \iff \hat{J}_z|J,m\rangle = m|J,m\rangle$$

$$\Rightarrow n = J - m \;;\quad \begin{matrix} 0 \le n \le N \\ -J \le m \le J \end{matrix}$$

* Single spin of total angular momentum $\iff$ Symmetric subspace of N qobits $\iff$ qudit of dimension d

$J = \frac{N}{2}$ $\qquad\qquad$ $N+1$ $\qquad\qquad$ $d = N+1$

$-J \le m \le J$ $\qquad\qquad$ $0 \le n \le N$

* There is a fourth representation which involves bosons (see Thesis)

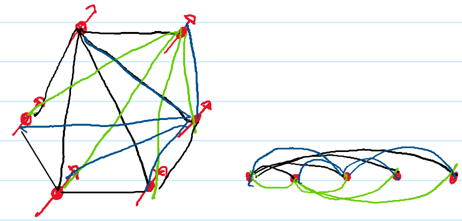
* Spin chain with all to all interactions (2 body interactions only)

$$\hat{H} = \alpha\sum_{i=1}^{N}\sigma_z^{(i)} + \frac{4\beta}{J}\sum_{i,j}\hat{\sigma}_x^{(i)}\hat{\sigma}_x^{(j)}$$

$$\downarrow$$

$$\hat{H} = \alpha \hat{J}_z + \frac{\beta}{J} \hat{J}_x^{2} \quad \text{(LMG - Lipkin model)}$$

same strenght interaction



$$\hat{J}_z |J,m\rangle = m|J,m\rangle \qquad [\hat{J}_x, \hat{J}_y] = \hat{J}_z$$

$$\hat{J}^2 |J,m\rangle = J(J+1)|J,m\rangle \qquad \hat{J}_\pm \equiv \hat{J}_x \pm i\hat{J}_y$$

$$\underline{[\hat{H}, \hat{J}^2] = 0}, \quad [\hat{H}, \hat{J}_z] = 0 \qquad \hat{P} = \exp[i\pi \hat{J}_z] \Rightarrow [\hat{H}, \hat{P}] = 0$$

$\pm$ Parities (almost equivalent)

Symmetric Subspace

- The total Hilbert space has a dimension of 2^N, but that space can be arrange in subspaces of constant angular momentum

$$\begin{pmatrix} (J) & & \\ & (J-1) & \\ & & \ddots \end{pmatrix}_{2^N}$$

symmetric subspace of dimension $N+1$; $\frac{N}{2} = J$

- $\hat{H}$ is independent and conserves the total angular momentum
- 2 conserved quantities for a one degree of freedom

$\Rightarrow$ Regular system, but not analitically solvable and it has a positive Lyapunov exponent (instability)

* Coherent state (spin, atomic, Bloch coherent state)

- Take the tensor product of a single qubit state $|\psi_i\rangle = \alpha|0\rangle + \beta|1\rangle$

$$|z(\alpha,\beta)\rangle = \prod_{i=1}^{N} |\psi_i\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) \otimes \cdots \otimes (\alpha|0\rangle + \beta|1\rangle)$$

$$|z(\alpha,\beta)\rangle \in \mathcal{H}^N_{\text{Dicke}}$$

- If you take the Bloch sphere parametrization

$$|z(\theta,\varphi)\rangle = \hat{R}(\theta,\varphi)|J,J\rangle \quad ; \quad |J,J\rangle \equiv |111\cdots1\rangle \text{ spin up state}$$

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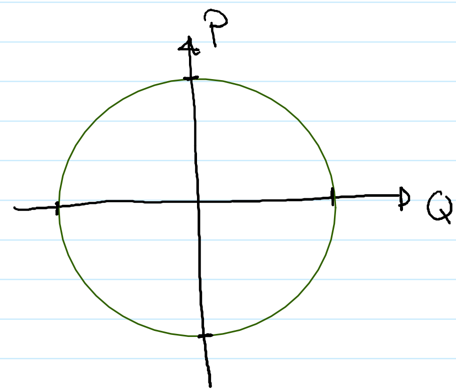
$$\hat{R}(\theta,\varphi) = \exp\left[i\varphi \hat{J}_z\right]\exp\left[i\theta \hat{J}_y\right]$$

→ Should I add the quantum circuit representation?

* Stereographic projection (to represent coherent states)

- Instead of plotting the 3D sphere we can change (canonically) the coordinates

$$(\theta,\varphi) \to (S_x, S_y, S_z) \to (Q,P)$$



$$S_x = Q\sqrt{1-\frac{Q^2+P^2}{4}} \qquad S_y = P\sqrt{1-\frac{Q^2+P^2}{4}}$$

$$\boxed{S_z = \frac{Q^2+P^2}{2} - 1}\ \text{QHO}$$

$$S_x^2+S_y^2+S_z^2 = 1$$

$$Q^2+P^2 = 4$$

$$|z(Q,P)\rangle \equiv \frac{1}{(1+|z|^2)^J}\, e^{z\hat{J}_+}\,|J,-J\rangle$$

$$z = \frac{Q+iP}{\sqrt{4-(Q^2+P^2)}}$$

* Quantum map for the LMG

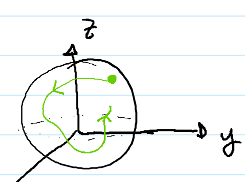
- The first requirement is the initial state in order to begin (and stay) in the symmetric subspace, for a spin chain of N spins

$$|\psi_T(0)\rangle = |\psi_S(0)\rangle \otimes |\psi_E(0)\rangle = |z(\theta,\varphi)\rangle = \underbrace{|z_i(\theta,\varphi)\rangle}_{1 \text{ spin}} \otimes \underbrace{\prod_i^{N-1} |z_i(\theta,\varphi)\rangle}_{N-1 \text{ spins}}$$

$$|z_i(\theta,\varphi)\rangle = \cos\theta/2\,|0\rangle + e^{i\varphi}\sin\theta/2\,|1\rangle$$

- The whole system evolves with an Unitary

$$|\psi_T(t)\rangle = \hat{U}(t)\,|\psi_T(0)\rangle = \hat{U}(t)\,|z(\theta,\varphi)\rangle \longrightarrow$$



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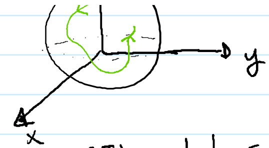

$$\hat{U}(t) = \exp\left[-i\,\hat{H}_{LMG}\,t\right]$$

- The whole system evolves and always stays on the sphere (of radius J)

- The quantum maps then reads

$$\hat{\rho}_S(t) = \mathrm{Tr}_E\left\{\hat{U}(t)\,|z\rangle\langle z|\,\hat{U}^\dagger(t)\right\} = \sum_j \hat{K}^j\,\hat{\rho}_S\,\hat{K}^{j\dagger}$$

$$\hat{\rho}_S(0) = |z_i(\theta,\varphi)\rangle\langle z_i(\theta,\varphi)|$$

$$\hat{K}^j = \langle j|\,\hat{U}\,|\psi_E(0)\rangle \;;\quad |\psi_E(0)\rangle = \prod_i^{N-1} |z_i(\theta,\varphi)\rangle$$

- We can track the evolution of $|\psi_T(t)\rangle$ over the sphere and the evolution of the system $\hat{\rho}_S(t)$ on the sphere too!

* The LMG model has a single point on the sphere (θ_c, φ_c) where the evolution is unstable (mimicks some results of chaotic dynamics)

* The purity or the entanglement entropy in (θ_c, φ_c)

$$S(t) = 1 - \mathrm{Tr}\left(\hat{\rho}_S^2(t)\right)$$

- The idea is that tracing over a subsystem which is originally in a maximally entangled state gives as a result the completely mixed state

  $\hookrightarrow$ The Dicke basis are already very entangled states as seen in the computational basis. So, independently of the regular/chaotic behavior of the model, making a partial trace generally produces mixed states

* Kraus operators

$$\hat{\rho}_S(t) = \mathrm{Tr}_E\left\{\hat{U}(t)\,|\psi_T\rangle\langle\psi_T|\,\hat{U}^\dagger(t)\right\} = \sum_j \hat{K}^j\,\hat{\rho}_S\,\hat{K}^{j\dagger}$$

$$|\psi_T\rangle = |z(\theta,\varphi)\rangle = |z_i(\theta,\varphi)\rangle \otimes \prod_i^{N-1} |z_i(\theta,\varphi)\rangle$$

$$\hat{U}(t) = \exp\left[-i\,\hat{H}_{LMG}\,t\right]$$

$$\hat{K}^j = \langle j|\,\hat{U}\,|\prod_{i=1}^{N-1} |z_i(\theta,\varphi)\rangle \;;\quad |j\rangle \in \mathcal{H}^{N-1}$$

$$\hat{K}^j_{\alpha\beta} = \langle\alpha|\,j|\,\hat{U}(t)\left(\prod^{N-1} |z_i(\theta,\varphi)\rangle\right)|\beta\rangle \;;\quad |\alpha\rangle, |\beta\rangle \in \mathcal{H}^1 = \{|0\rangle, |1\rangle\}$$

$$\hat{K}^{j}_{\alpha\beta} = \langle\alpha|\,j|\;\hat{U}(t)\Big(\underbrace{\prod^{N-1}|z_i(\theta,\varphi)\rangle\Big)|\beta\rangle}_{} \; ; \quad |\alpha\rangle,|\beta\rangle \in \mathcal{H}^{1} = \{|0\rangle,|1\rangle\}$$

This state does not belong to the symmetric subspace anymore

$$\hat{U}(t)\,|z_{N-1}(\theta,\varphi)\rangle\otimes|0\rangle$$

$$= \exp\left[-i\,a\,\hat{J}_z t - \frac{i\,b}{J}\hat{J}_x^{2}t\right]|z_i\rangle\otimes|z_i\rangle\otimes\cdots\otimes|z_i\rangle|0\rangle \; ; \quad |z_i\rangle = \cos\theta/2\,|0\rangle + e^{i\varphi}\,\mathrm{sen}\,\theta/2\,|1\rangle$$

$$\underset{t\ll 1}{\simeq} \exp\left[\frac{-i\,t\,b}{J}\hat{J}_x^{2}\right]\prod_i^{N}\exp\left[\frac{-ia}{2}\hat{\sigma}_z^{(i)}t\right]|z_i,z_i,\ldots,0\rangle \; ; \quad f(\hat{A})|a\rangle = f(a)|a\rangle$$