# Quantum map for LMG

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First, let's see an equivalence between 3 different basis for the same Hilbert space

* A single spin

- Spin operators $\{\hat{J}_x, \hat{J}_y, \hat{J}_z\}$ are the generators of SU(2) (Lie groups)

$$\boxed{[\hat{J}_i, \hat{J}_k] = i \sum_{l} \epsilon_{ikl} \hat{J}_l}$$

- Eigenvalue equation

$$\hat{J}_z |J, m\rangle = m |J, m\rangle \quad ; \quad -J \leq m \leq J \quad \Rightarrow \dim(\mathcal{H}) = 2J+1$$

* A qudit

- Define the dimension of a single qudit $d = 2J+1$

* Symmetric States

- Consider a system of $N$ qubits with angular momentum $s = 1/2$
- The Pauli matrices fulfill $[\hat{\sigma}_i, \hat{\sigma}_k] = \sum_{l} \epsilon_{ikl} \hat{\sigma}_l$
- The state of a qubit is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ ; $|\psi\rangle \in \mathcal{H}$ ; $\dim(\mathcal{H}) = 2^1$
- Now, lets consider the following states in the Hilbert space of $N$ qubits

$$\dim(\mathcal{H}^N) = 2^N$$

The first $n$ qubits in the $|1\rangle$ state and the rest in the $|0\rangle$ states

$$\Rightarrow \quad |1_1 1_2 1_3 \dots 1_n 0_{n+1} 0_{n+2} \dots 0_N\rangle \in \mathcal{H}^N \quad ; \quad n \leq N$$

- For a fixed $n$, there are $p := \binom{N}{n}$ distinct permutations of the 1's but leave the total of 1's invariant
- Define the permutation invariant state $|N, n\rangle \equiv \frac{1}{\sqrt{p}} \sum_{k=1}^{p} \hat{P}_k |1, \dots, 1_n, 0_{n+1}, \dots 0_N\rangle$
- These states are called Dicke States

$N=3$ $\qquad N+1$ $\begin{cases} |3,0\rangle = |000\rangle \\ |3,1\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right) \\ |3,2\rangle = \frac{1}{\sqrt{3}}\left(|110\rangle + |011\rangle + |101\rangle\right) \\ |3,3\rangle = |111\rangle \end{cases}$

- For $N$ qubits, the Hilbert space dimension is $\dim(\mathcal{H}^N) = 2^N$, but the subspace of permutational invariant states is $\dim(\mathcal{H}^N_{Dicke}) = N+1$

- Define the collective angular momentum as

$$\tilde{J}_i \equiv 1/2 \sum_{k=1}^{N} \hat{\sigma}_i^{(k)} \qquad N=3 \Rightarrow \tilde{J}_x = 1/2\left(\hat{\sigma}_x^{(1)} \otimes I \otimes I + I \otimes \hat{\sigma}_x^{(2)} \otimes I + I \otimes I \otimes \hat{\sigma}_x^{(3)}\right)$$

- They obey the same commutation relation as the Pauli matrices

$$[\tilde{J}_i, \tilde{J}_k] = \sum_{\ell} \epsilon_{ik\ell} \tilde{J}_\ell$$

* As both basis span the same Hilbert space (dimension)

$$\tilde{J}_z |N,n\rangle = (N/2 - n)|N,n\rangle \Longleftrightarrow \vec{J}_z |J,m\rangle = m|J,m\rangle$$

$$\Rightarrow n = J - m \; ; \quad 0 \le n \le N, \quad -J \le m \le J$$

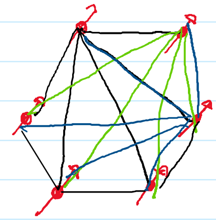
* Single spin of total angular momentum $\Longleftrightarrow$ Symmetric subspace of $N$ qubits $\Longleftrightarrow$ qudit of dimension $d$

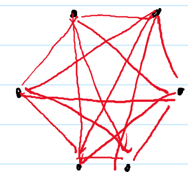
| Single spin | Symmetric subspace | qudit |
|---|---|---|
| $J = \frac{N}{2}$ | $N+1$ | $d = N+1$ |
| $-J \le m \le J$ | $0 \le n \le N$ | |

* There is a fourth representation which involves bosons (see Thesis)

* Spin chain with all to all interactions (2 body interactions only)

$$\hat{H} = \alpha \sum_{i=1}^{N} \sigma_z^{(i)} + \frac{2\beta}{J} \sum_{i,j} \hat{\sigma}_x^{(i)} \hat{\sigma}_x^{(j)}$$





$$\hat{H} = \alpha \hat{J}_z + \frac{\beta}{J} \hat{J}_x^{\,2} \qquad \text{(LMG - Lipkin model)}$$

$\hat{J}_z |J,m\rangle = m|J,m\rangle$ $\qquad$ $[\hat{J}_x, \hat{J}_y] = \hat{J}_z$

$\hat{J}^2 |J,m\rangle = J(J+1)|J,m\rangle$ $\qquad$ $\hat{J}_\pm \equiv \hat{J}_x \pm i\hat{J}_y$

$\underline{[\hat{H}, \hat{J}^2] = 0}$ , $\quad [\hat{H}, \hat{J}_z] = 0$ $\qquad$ $\hat{P} = \exp[i\pi \hat{J}_z] \Rightarrow [\hat{H}, \hat{P}] = 0$ $\quad \pm$ Parities (almost equivalent)

Symmetric Subspace

- The total Hilbert space has a dimension of $2^N$, but that space can be arrange in subspaces of constant angular momentum

$$\begin{pmatrix} (J) & & \\ & (J-1) & \\ & & \ddots \end{pmatrix}_{2^N}$$

1 — symmetric subspace of dimension $N+1$ ; $\frac{N}{2} = J$

- $\hat{H}$ is independent and conserves the total angular momentum
- 2 conserved quantities for a one degree of freedom

$\Rightarrow$ Regular system , but not analitically solvable and it has a positive Lyapunov exponent (instability)

$*$ Coherent state (spin, atomic, Bloch coherent state)

- Take the tensor product of a single qubit state $|\psi_i\rangle = \alpha|0\rangle + \beta|1\rangle$

$$|z(\alpha,\beta)\rangle = \prod_{i=1}^{N} |\psi_i\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) \otimes \cdots \otimes (\alpha|0\rangle + \beta|1\rangle)$$

$$|z(\alpha,\beta)\rangle \in \mathcal{H}^N_{\text{Dicke}}$$

- If you take the Bloch sphere parametrization

$$|z(\theta,\varphi)\rangle = \hat{R}(\theta,\varphi)|J,J\rangle \quad ; \quad |J,J\rangle \equiv |111\cdots1\rangle \text{ spin up state}$$

$$\hat{R}(\theta,\varphi) = \exp[i\varphi\hat{J}_z]\exp[i\theta\hat{J}_y]$$

## * Quantum map for the LMG

- The first requirement is the initial state, in order to begin (and stay) in the symmetric subspace, for a spin chain of N spins

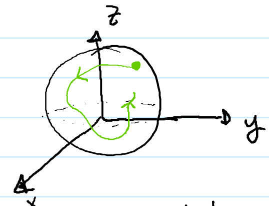
$$|\psi_T(0)\rangle = |\psi_S(0)\rangle \otimes |\psi_E(0)\rangle = |z(\theta,\varphi)\rangle = \underbrace{|z_i(\theta,\varphi)\rangle}_{1\text{ spin}} \otimes \underbrace{\prod_i^{N-1} |z_i(\theta,\varphi)\rangle}_{N-1\text{ spins}}$$

$$|z_i(\theta,\varphi)\rangle = \cos\theta/2\,|0\rangle + e^{i\varphi}\sin\theta/2\,|1\rangle$$

- The whole system evolves with an Unitary

$$|\psi_T(t)\rangle = \hat{U}(t)|\psi_T(0)\rangle = \hat{U}(t)|z(\theta,\varphi)\rangle \longrightarrow$$

$$\hat{U}(t) = \exp\left[-i\hat{H}_{LMG}\,t\right]$$




- The whole system evolves and always stays on the sphere (of radius J)

- The quantum maps then reads

$$\hat{\rho}_S(t) = Tr_E\left\{\hat{U}(t)|z\rangle\langle z|\,\hat{U}^\dagger(t)\right\} = \sum_j \hat{K}^j \hat{\rho}_S \hat{K}^{j\dagger}$$

$$\hat{\rho}_S(0) = |z_i(\theta,\varphi)\rangle\langle z_i(\theta,\varphi)|$$

$$\hat{K}^j = \langle j|\hat{U}|\psi_E(0)\rangle \quad ; \quad |\psi_E(0)\rangle = \prod_i^{N-1}|z_i(\theta,\varphi)\rangle$$

- We can track the evolution of $|\psi_T(t)\rangle$ over the sphere and the evolution of the system $\hat{\rho}_S(t)$ on the sphere too!

* The LMG model has a single point on the sphere $(\theta_c,\varphi_c)$ where the evolution is unstable (mimicks a lot of results of chaotic dynamics)

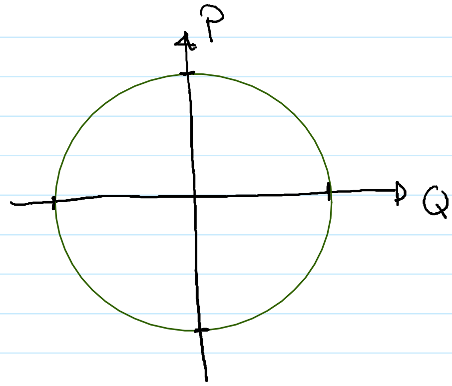
$\Longrightarrow$ The purity or the entanglement entropy in $(\theta_c,\varphi_c)$

$$S(t) = 1 - Tr\left(\hat{\rho}_S^2(t)\right)$$

* Stereographic projetion

- Instead of plotting the 3D sphere we can change (canonically) the coordinates

$$(\theta, \varphi) \longrightarrow (S_x, S_y, S_z) \longrightarrow (Q, P)$$




$$S_x = Q\sqrt{1 - \frac{Q^2+P^2}{4}} \qquad S_y = P\sqrt{1 - \frac{Q^2+P^2}{4}}$$

$$\boxed{S_z = \frac{Q^2+P^2}{2} - 1} \text{ QHO}$$

$$|z(Q,P)\rangle \equiv \frac{1}{(1+|z|^2)^J}\, e^{z\hat{J}_+} |J,-J\rangle$$

$$S_x^2 + S_y^2 + S_z^2 = 1$$

$$Q^2 + P^2 = 4$$

$$z = \frac{Q + iP}{\sqrt{4 - (Q^2+P^2)}}$$